On the stabilizability of discrete-time switched linear systems: novel conditions and comparisons

Mirko Fiacchini, Antoine Girard, Marc Jungers

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***Abstract*—In this paper we deal with the stabilizability property for discrete-time switched linear systems. A recent necessary and sufficient characterization of stabilizability, based on set theory, is considered as the reference for comparing the computation-oriented sufficient conditions. The classical BMI conditions based on Lyapunov-Metzler inequalities are considered and extended. Novel LMI conditions for stabilizability, derived from the geometric ones, are presented that permit to combine generality with computational affordability. For the different conditions, the geometrical interpretations are provided and the induced stabilizing switching laws are given. The relations and the implications between the stabilizability conditions are analyzed to infer and compare their conservatism and their complexity.**

***Index Terms*—Switched systems, stabilizability, LMI.**

I. Introduction

Switched systems are characterized by dynamics that may change along the time among a finite number of possible dynamical behaviors, see [1]. Each behavior is determined by a mode and the active one is selected by means of a function of time, referred to as switching law. The interest that such kind of systems rose in the last decades relies in their capability of modelling complex real systems, as embedded or networked, and also for the theoretical issues involved. Their dynamical properties, in fact, are often not intuitive nor trivial and the problems related to analysis and control design may result rather challenging, also for linear switched system, see [2].

The problem of stability or stabilizability, depending on the assumption on the switching law, of linear switched system attracted many research efforts, see the overview [3] and the monograph [4]. Conditions for stability, that is when the switching law is considered as an exogenous signal, have been proposed: for instance, the joint spectral radius approach [5]; the polyhedral Lyapunov functions [6], [7] and the path-dependent switched Lyapunov ones [8] or the variational approach [9].

In this paper we are considering the problem of stabilizability of switched linear systems, namely the condition under which a switching law can be designed for the system to be asymptotically stable. Many approaches presented in literature resort to Lyapunov stability theory to achieve this aim, in the context of continuous-time [10], [11], discrete-time [12]–[15] and also, more recently, for distributed parameter systems [16]–[19]. The decision problem of answering on the stabilizability of a discrete-time switched autonomous linear system is related to NP-hard problems. For instance, it has been proved that the quadratic stabilizability is a NP-hard problem, see the discussion provided in [20], [21] and the recent overview [22].

Sufficient conditions for stabilizability have been provided in literature, mainly based on min-switching policies, introduced in [23], developed in [1], [24] and leading to Lyapunov-Metzler inequalities [15], [25]. The fact that the existence of a min-switching control law is necessary and sufficient for exponential stabilizability has been claimed in [4]. In the same work, as well as in [26], it has been proved that the stabilizability of a switched system does not imply the existence of a convex Lyapunov function. Thus, for characterizing stabilizability, the attention might be directed to more complex, nonconvex in general, Lyapunov functions, for instance to composite quadratic functions, as suggested in [4], [15]. A general necessary and sufficient condition, based on set theory, for the stabilizability of discrete-time switched linear systems has appeared recently in [27]. Nevertheless, this condition might result to be often computationally unaffordable, as it requires to check whether some particular set is contained in the union of others. On the other hand, such computational complexity appears to be inherent to the problem itself, then avoidable only at the price of introducing some conservatism.

The first main objective of this paper, whose preliminary version is [28], is to propose new conditions for stabilizability of discrete-time switched linear systems which could conjugate computational affordability with generality. We are focusing in particular on conditions in form of matrix inequalities, leading possibly to convex optimization problems, [29], [30]. Moreover, we provide geometrical and numerical insights on different stabilizability conditions to quantify their conservatism and the relations between them and with the necessary and sufficient ones. We proved the implications between the conditions, which permit to get a clear picture of their relations, their conservatism and their complexity.

The paper is organized as follows: Section II presents the problem of stabilizability of switched systems. Section III provides the analysis of the Lyapunov-Metzler approach and proposes its generalizations. In Section IV a novel condition, in LMI form, is given and analyzed. The relations between different stabilizability conditions are provided in Section V. Numerical examples are presented in Section VI and Section

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VII draws some conclusions.

Notation: Given $n \in \mathbb{N}$, define $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. Given $\alpha \in \mathbb{R}^n$, α_i denote its i-th element; given $\pi \in \mathbb{R}^{n\times m}$, π_{ij} is the entry of i-th row and j-th column. Given $\Omega \subseteq \mathbb{R}^n$ define the interior of Ω as $\text{int}(\Omega)$. Given $P \in \mathbb{R}^{n\times n}$ with $P > 0$ denote with $\mathscr{E}(P) = \left\{x \in \mathbb{R}^n : \ x^T P x \leq 1\right\}$, the related ellipsoid. The i-th element of a finite set of matrices is denoted, with slight abuse, as A_i. The spectral radius of $A \in \mathbb{R}^{n\times n}$ is $\rho(A)$ and A is Schur if $\rho(A) < 1$. Given $a \in \mathbb{R}$, the maximal integer smaller than or equal to a is $\lfloor a \rfloor$.

II. Problem statement

We consider the problem of stabilizability of autonomous discrete-time switched linear system of the form

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}_q$ is the switching law and $\{A_i\}_{i\in\mathbb{N}_q}$, with $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathbb{N}_q$. Given the initial state x_0 and a switching law $\sigma(\cdot)$, we denote with $x_N^\sigma(x_0)$ the state of the system (1) at time N starting from x_0 by applying the switching law $\sigma(\cdot)$. In some cases σ can be a function of the state, for instance in the case of switching control law, as shown later. The following assumption, not necessary, is supposed to hold throughout the paper for simplicity.

Assumption 1: All the matrices A_i, with $i \in \mathbb{N}_q$, are invertible and non-Schur.

Notice that no loss of generality is entailed by Assumption 1. In fact, the presence of a Schur matrix A_i would make the problem trivial while assuming the matrices invertible simplifies the technical developments and proofs, although these could be extended considering matrix inversion in the set-valued sense.

The following notations are employed in the paper:

- $\mathscr{I} = \mathbb{N}_q$: finite set of switching modes.
- $\mathscr{I}^k = \prod_{j=1}^k \mathscr{I}$: all the possible sequences of modes of length k.
- $\mathscr{I}^{[M:N]} = \bigcup_{k=M}^N \mathscr{I}^k$: all the possible sequences of modes of length from M to N.
- $\bar{N} = \sum_{k=1}^N q^k$: given $N \in \mathbb{N}$, number of elements $i \in \mathscr{I}^{[1:N]}$. Analogous definition for $\bar{M}$.
- Given $i = (i_1, \ldots, i_k)$ such that $i \in \mathscr{I}^{[1:N]}$ and a set Ω, define:

$$\begin{aligned} &\mathbb{A}_i = \textstyle\prod_{j=1}^k A_{i_j} = A_{i_k} \cdots A_{i_1}, \\ &\Omega_i = \Omega_i(\Omega) = \{x \in \mathbb{R}^n : \ \mathbb{A}_i x \in \Omega\}, \\ &\mathscr{B}_i = \{x \in \mathbb{R}^n : \ x^T \mathbb{A}_i^T \mathbb{A}_i x \leq 1\}, \end{aligned}$$

  and then $\mathscr{B}_i = \Omega_i(\mathscr{B})$ with $\mathscr{B} = \{x \in \mathbb{R}^n : \ x^T x \leq 1\}$. The dependence of Ω_i on Ω is omitted when clear from the context.
- $\mathscr{M}_N$: set of Metzler matrices of dimension N, i.e. matrices $\pi \in \mathbb{R}^{N\times N}$ whose elements are nonnegative and $\sum_{j=1}^N \pi_{ji} = 1$ for all $i \in \mathbb{N}_N$.

In this paper we will refer to the property of global exponential stabilizability, defined below, simply as stabilizability.

Definition 2: The system (1) is globally exponentially stabilizable if there are $c \geq 0$ and $\lambda \in [0,1)$ and, for all $x \in \mathbb{R}^n$, there exists a switching law $\sigma : \mathbb{N} \to \mathbb{N}_q$, such that

$$\|x_k^\sigma(x)\| \leq c\lambda^k \|x\|, \quad \forall k \in \mathbb{N}. \tag{2}$$

A periodic switching law is given by $\sigma(k) = i_{p(k)}$ and

$$p(k) = k - M\lfloor k/M \rfloor + 1,$$

with $M \in \mathbb{N}$ and $i \in \mathscr{I}^M$, which means that the sequence of modes given by i repeats cyclically in time. One issue that will be treated in this paper concerns the stabilizability through periodic switching law, i.e. conditions under which system (1) is stabilized by means of a periodic $\sigma(\cdot)$. This property, formalized below, will be referred to as periodic stabilizability.

Definition 3: The system (1) is periodic stabilizable if there exist a periodic switching law $\sigma : \mathbb{N} \to \mathbb{N}_q$, $c \geq 0$ and $\lambda \in [0,1)$ such that (2) holds for all $x \in \mathbb{R}^n$.

Notice that for stabilizability the switching function might be state-dependent, hence a state feedback, whereas for having periodic stabilizability the switching law must be independent on the state. The following direct result is given with no need of proof. The reader is referred to [4] for an analogous results and its proof.

Lemma 4: The system (1) is periodic stabilizable if and only if there exists $M \in \mathbb{N}$ and $i \in \mathscr{I}^M$ such that $\mathbb{A}_i$ is Schur.

We recall hereafter the main results proposed in [27] on the stabilizability of switched linear systems (1). These results are based on the Algorithm 1 in [27] that basically consists in computing the successive pre-images of a C*-set $\Omega \subseteq \mathbb{R}^n$ with respect to all the possible modes. A C*-set $\Omega \subseteq \mathbb{R}^n$ is a compact, star-convex set containing the origin in its interior. The stabilizability of the system (1) is equivalent to the fact that the algorithm ends with a finite number of steps. The Theorem 1 in [27], recalled below, provides a geometric necessary and sufficient condition for stabilizability of the system (1).

Theorem 5 ([27]): Let Ω be a C*-set. The switched system (1) is stabilizable if and only if there exists $N \in \mathbb{N}$ such that

$$\Omega \subseteq \text{int}\Big(\bigcup_{i\in\mathscr{I}^{[1:N]}} \Omega_i\Big). \tag{3}$$

To have a geometrical hint of Theorem 5, recall that Ω_i is the preimage of Ω through $\mathbb{A}_i$, that is the set of states which reach Ω by applying the switching sequence $i \in \mathscr{I}^{[1:N]}$. Thus, the switched system (1) is stabilizable if and only if the union of all the preimages of Ω related to sequences of length smaller or equal than N covers Ω, with N finite. This means that after N step at most, all the points in the union of preimages are driven in Ω by an appropriate switching sequence.

Since the stabilizability property is not dependent on the choice of the initial C*-set Ω, focusing on the case $\Omega = \mathscr{B}$ and ellipsoidal pre-images entails no loss of generality, see [27]. Then condition (3) can be replaced by

$$\mathscr{B} \subseteq \text{int}\Big(\bigcup_{i\in\mathscr{I}^{[1:N]}} \mathscr{B}_i\Big), \tag{4}$$

for what concerns stabilizability, although the value N might depend on the choice of Ω.

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The set inclusions (3) or (4) are the stopping conditions of the algorithm and then must be numerically checked at every step. The main computational issue is that determining if a C*-set Ω is included into the interior of the union of some C*-sets is very complex in general, also in the case of ellipsoidal sets where it relates to quantifier elimination over real closed fields [31]. On the other hand, the condition given by Theorem 5 provides an exact characterization of the complexity inherent to the problem of stabilizing a switched linear system.

The objective of this paper is to consider alternative conditions for stabilizability, taken from the literature and novel ones, to provide geometrical and numerical insights and analyze their conservatism by comparison with the necessary and sufficient one given in Theorem 5.

III. Lyapunov-Metzler BMI conditions

The condition we are considering first is related to the Lyapunov-Metzler inequality: it is sufficient and given by a set of BMI inequalities involving the Metzler matrices.

Theorem 6 ([15]): If there exist $P_i > 0$, with $i \in \mathscr{I}$, and $\pi \in \mathscr{M}_q$ such that

$$A_i^T \left(\sum_{j=1}^{q} \pi_{ji} P_j \right) A_i - P_i < 0, \qquad \forall i \in \mathscr{I}, \tag{5}$$

holds, then the switched system (1) is stabilizable.

As proved in the paper [15], the satisfaction of condition (5) implies that the homogeneous function induced by the set $\bigcup_{i\in\mathscr{I}} \mathscr{E}(P_i)$ is a control Lyapunov function for the system.

A first relation between the Lyapunov-Metzler condition (13) and the geometric one (3) is provided below. The following lemma is functional for this purpose.

Lemma 7: Given $P_i > 0$, $i \in \mathbb{N}_m$, the set defined by

$$\Gamma = \bigcup_{\substack{\pi_i \geq 0, \\ \pi_1 + \cdots + \pi_m = 1}} \mathscr{E}\left(\sum_{i\in\mathbb{N}_m} \pi_i P_i \right) \tag{6}$$

is such that

$$\Gamma = \bigcup_{i\in\mathbb{N}_m} \mathscr{E}(P_i). \tag{7}$$

Proof: The equality (7) is satisfied if and only if the following conditions

$$\bigcup_{i\in\mathbb{N}_m} \mathscr{E}(P_i) \subseteq \Gamma \subseteq \bigcup_{i\in\mathbb{N}_m} \mathscr{E}(P_i) \tag{8}$$

hold. The first inclusion in (8) is trivially proved by noticing that for $\pi_j = 1$ and the other coefficients $\pi_k = 0$ for all $k \neq j$, then $\mathscr{E}\left(\sum_{i\in\mathbb{N}_m} \pi_i P_i\right) = \mathscr{E}(P_j)$.

Consider the second inclusion in (8) and suppose that $x \in \Gamma$. Then, from the definition of Γ, we have that there exist $\pi_i^* \in [0,1]$ such that $\sum_{i\in\mathbb{N}_m} \pi_i^* = 1$ Finally, while classical Lyapunov-Metzler inequalities can be adapted to handle the co-design of switching laws and continuous feedback gains (est-ce vrai ? donner une ref ?), the extension of the other conditions presented in this paper for handling that problem is not straightforward and requires further research.and

$$\sum_{i\in\mathbb{N}_m} \pi_i^* x^T P_i x \leq 1. \tag{9}$$

All the terms $x^T P_i x$ being non-negative, it yields

$$\min_{i\in\mathbb{N}_m} x^T P_i x \leq \sum_{i\in\mathbb{N}_m} \pi_i^* x^T P_i x \leq 1. \tag{10}$$

That leads to the existence of $i^* \in \mathbb{N}_m$ such that $x \in \mathscr{E}(P_{i^*})$ and finally $x \in \bigcup_{i\in\mathbb{N}_m} \mathscr{E}(P_i)$. ∎

We prove now that the satisfaction of the Lyapunov-Metzler inequalities (5) implies that the necessary and sufficient condition given by Theorem 5 holds for the particular case of $\Omega = \bigcup_{i\in\mathscr{I}} A_i \mathscr{E}(P_i)$ and $N = 1$.

Theorem 8: If the Lyapunov-Metzler condition (5) holds then (3) holds with $N = 1$ and $\Omega = \bigcup_{i\in\mathscr{I}} A_i \mathscr{E}(P_i)$.

Proof: If (5) holds then for all $x \in \mathscr{E}(P_i)$, i.e. such that $x^T P_i x \leq 1$, we have that

$$x^T A_i^T \left(\sum_{j\in\mathscr{I}} \pi_{ji} P_j \right) A_i x < x^T P_i x \leq 1$$

which implies $A_i x \in \text{int}(\Gamma)$, for all $i \in \mathscr{I}$, from Lemma 7 with $\Gamma = \bigcup_{i\in\mathscr{I}} \mathscr{E}(P_i)$. Thus the condition (5) implies that $A_i \mathscr{E}(P_i) \subseteq \text{int}(\Gamma)$ and then

$$\Omega = \bigcup_{i\in\mathscr{I}} A_i \mathscr{E}(P_i) \subseteq \text{int}(\Gamma) = \text{int}\left(\bigcup_{i\in\mathscr{I}} \mathscr{E}(P_i) \right). \tag{11}$$

Now, by definition we have that

$$\begin{aligned} \Omega_i &= \{x \in \mathbb{R}^n : A_i x \in \Omega\} = \{x \in \mathbb{R}^n : A_i x \in \bigcup_{j\in\mathscr{I}} A_j \mathscr{E}(P_j)\} \\ &\supseteq \{x \in \mathbb{R}^n : A_i x \in A_i \mathscr{E}(P_i)\} = \mathscr{E}(P_i), \end{aligned} \tag{12}$$

for all $i \in \mathscr{I}$. From (11) and (12), condition (3) holds with $N = 1$. ∎

Theorem 8 provides a geometrical meaning of the Lyapunov-Metzler condition and a first relation with the necessary and sufficient condition for stabilizability given in Theorem 5.

A. Lyapunov-Metzler conditions for bimodal switched systems

Consider the bimodal switched systems, that is such that $q = 2$. Given $i \in \mathscr{I}$, we analyze the geometrical meaning of the Lyapunov-Metzler condition, that is

$$A_i^T (\pi_{1i} P_1 + \pi_{2i} P_2) A_i - P_i < 0, \quad \forall i \in \{1,2\}, \tag{13}$$

with $\pi_{1i} + \pi_{2i} = 1$ and $\pi_{1i}, \pi_{2i} \geq 0$. Define

$$\begin{aligned} &\mathscr{E}_1 = \mathscr{E}(P_1) = \left\{x \in \mathbb{R}^n : \ x^T P_1 x \leq 1\right\}, \\ &\mathscr{E}_2 = \mathscr{E}(P_2) = \left\{x \in \mathbb{R}^n : \ x^T P_2 x \leq 1\right\}, \\ &\Theta = \bigcup_{\substack{\pi_1,\pi_2 \geq 0 \\ \pi_1+\pi_2=1}} \mathscr{E}(\pi_1 P_1 + \pi_2 P_2) = \{x \in \mathbb{R}^n : \exists \pi_1, \pi_2 \geq 0 \\ &\qquad \text{s.t. } \pi_1 + \pi_2 = 1, \ x^T(\pi_1 P_1 + \pi_2 P_2) x \leq 1\right\}. \end{aligned} \tag{14}$$

Using the results from Lemma 7 with $m = 2$, we can now provide relations between the conditions of the type (13) and the set inclusion, for $i \in \{1,2\}$. First we prove that all the ellipsoids contained in the union of $\mathscr{E}_1$ and $\mathscr{E}_2$ are contained in one of the ellipsoids parameterized by all the π_1 and π_2 such that $\pi_1 + \pi_2 = 1$.

Lemma 9: Given $P, P_1, P_2 > 0$ and the sets defined in (14), the inclusion

$$\mathscr{E}(P) \subseteq \text{int}(\Theta), \tag{15}$$

holds if and only if there exist $\pi_1, \pi_2 \geq 0$ such that $\pi_1 + \pi_2 = 1$ and

$$\pi_1 P_1 + \pi_2 P_2 < P. \tag{16}$$

Proof: We assume without loss of generality that $\mathscr{E}(P) \nsubseteq \mathscr{E}_1$ (the case $\mathscr{E}(P) \nsubseteq \mathscr{E}_2$ is symmetric). Sufficiency consists in proving that (16) implies (15). Notice that (16) is equivalent to

$$x^T(\pi_1 P_1 + \pi_2 P_2)x < x^T P x,$$

for all $x \in \mathbb{R}^n$ and if $x \in \mathscr{E}(P)$ then $x \in \text{int}(\Theta)$, from the definitions (14), and (15) holds.

For necessity, let us assume that $\mathscr{E}(P) \subseteq \text{int}(\mathscr{E}_1 \cup \mathscr{E}_2)$ and notice that $\text{int}(\mathscr{E}_1 \cup \mathscr{E}_2) = \text{int}(\mathscr{E}_1) \cup \text{int}(\mathscr{E}_2)$ for $\mathscr{E}_1$ and $\mathscr{E}_2$ as in (14). Then, for all $x \in \mathscr{E}(P)$, such that $x \notin \text{int}(\mathscr{E}_1)$, it must hold $x \in \text{int}(\mathscr{E}_2)$. This can be reformulated as follows:

$$\forall x \in \mathbb{R}^n,\ \left(x^T P x \leq 1 \text{ and } x^T P_1 x \geq 1\right) \implies x^T P_2 x < 1.$$

Let $y \in \mathbb{R}^n$, $y \neq 0$, such that $y^T(P_1 - P)y \geq 0$. Let $x = y/\sqrt{y^T P y}$, then $x^T P x = 1$ and

$$x^T P_1 x = \frac{y^T P_1 y}{y^T P y} \geq 1.$$

Then, it follows that we must have $x^T P_2 x < 1$ and therefore $y^T(P - P_2)y > 0$. Summarizing, we have shown

$$\forall y \in \mathbb{R}^n,\ y \neq 0,\ \left(y^T(P_1 - P)y \geq 0\right) \implies y^T(P - P_2)y > 0.$$

Since $\mathscr{E}(P) \nsubseteq \mathscr{E}_1$, it follows that there exists y_0 such that $y_0^T(P_1 - P)y_0 > 0$. Then, it follows from the S-lemma that there exists $\tau \geq 0$ such that

$$P - P_2 > \tau(P_1 - P).$$

Then, it follows that $\pi_1 P_1 + \pi_2 P_2 < P$ with $\pi_1 = \tau/(1+\tau)$ and $\pi_2 = 1/(1+\tau)$. ∎

The result stated in Lemma 9 is not true for Θ given by more than two ellipsoids, see for instance the counter-example given in Example 17. In general, the LMI condition analogous to (16) is only sufficient for the set inclusion. To have a hint of this statement, recall that the S-lemma, used in Lemma 9, gives equivalent conditions if applied to two quadratic functions, but only a sufficient one for an higher number of them.

Using the Lemma 9, we can provide the geometrical interpretation of the Lyapunov-Metzler condition for $q = 2$.

Proposition 10: For $q = 2$, the Lyapunov-Metzler condition (13) is equivalent to $A_1\mathscr{E}(P_1) \cup A_2\mathscr{E}(P_2) \subseteq \text{int}\left(\mathscr{E}(P_1) \cup \mathscr{E}(P_2)\right)$.

Proof: Following the lines of the proof of Theorem 8 and from Lemma 9, condition (13) is equivalent to $A_1\mathscr{E}(P_1) \subseteq \text{int}\left(\mathscr{E}(P_1) \cup \mathscr{E}(P_2)\right)$ and $A_2\mathscr{E}(P_2) \subseteq \text{int}\left(\mathscr{E}(P_1) \cup \mathscr{E}(P_2)\right)$ and then the result holds. ∎

Thus, the Lyapunov-Metzler condition for $q = 2$ is equivalent to the existence of a contractive set formed by two ellipsoids, and then to a Lyapunov function given by the pointwise minimum of two quadratic functions. Notice that Proposition 10 provides for the case of $q = 2$ a result stronger than the one of Theorem 8. In fact, for the general case of $q \in \mathbb{N}$ the Lyapunov-Metzler condition is just sufficient for $\bigcup_{i \in \mathscr{I}} A_i\mathscr{E}(P_i) \subseteq \text{int}\left(\bigcup_{i \in \mathscr{I}} \mathscr{E}(P_i)\right)$ to hold (see the proof of Theorem 8) whereas it is also necessary for $q = 2$.

A relation between the Lyapunov-Metzler condition (13) and the geometric one (3) follows.

Proposition 11: For $q = 2$, the existence of $Q \in \mathbb{R}^{n \times n}$, positive definite, such that (3) holds with $N = 1$ and $\Omega = \mathscr{E}(Q)$ implies that the Lyapunov-Metzler condition (13) holds with $P_1 = A_1^T Q A_1$ and $P_2 = A_2^T Q A_2$.

Proof: The satisfaction of (3) with $N = 1$ and $\Omega = \mathscr{E}(Q)$ is equivalent, from Lemma 9, to the existence of $p \in [0, 1]$ such that

$$pA_1^T Q A_1 + (1-p)A_2^T Q A_2 < Q,$$

and then to

$$\begin{cases} A_1^T\left(pA_1^T Q A_1 + (1-p)A_2^T Q A_2\right)A_1 < A_1^T Q A_1, \\ A_2^T\left(pA_1^T Q A_1 + (1-p)A_2^T Q A_2\right)A_2 < A_2^T Q A_2, \end{cases}$$

that is (13) with $P_i = A_i^T Q A_i$, $\pi_{1i} = p$ and $\pi_{2i} = 1 - p$ for $i = \{1, 2\}$. ∎

Proposition 11 states that the existence of an ellipsoidal set Ω such that the geometric condition (3) holds after one step is sufficient for the satisfaction of Lyapunov-Metzler condition (13).

B. Generalized Lyapunov-Metzler conditions

An interesting issue is whether the Lyapunov-Metzler condition is necessary and sufficient for stabilizability. From the results presented in the following sections, one can infer that the Lyapunov-Metzler condition (13) is only sufficient for stabilizability, in general.

A direct generalization of the Lyapunov-Metzler condition can be given, by removing the unnecessary link between the number of ellipsoids (and matrices P_i) and the system modes.

Proposition 12: If there exist $M \in \mathbb{N}$ and $P_i > 0$, with $i \in \mathscr{I}^{[1:M]}$, and $\pi \in \mathscr{M}_{\bar{M}}$ such that

$$\mathbb{A}_i^T\left(\sum_{j \in \mathscr{I}^{[1:M]}} \pi_{ji} P_j\right)\mathbb{A}_i - P_i < 0, \qquad \forall i \in \mathscr{I}^{[1:M]}, \tag{17}$$

holds, then the switched system (1) is stabilizable.

Proof: The proof is analogous to that one of the classical Lyapunov-Metzler condition, see [15]. ∎

Proposition 12 extends the Lyapunov-Metzler condition providing a more general one. Notice in fact that Theorem 6 is recovered for $M = 1$. An interesting issue is the relation with the necessary and sufficient condition for stabilizability, as well as with other ones.

Remark 13: The condition (17) can be interpreted in terms of the classical Lyapunov-Metzler condition (5) by considering the switched system obtained by defining one fictitious mode for every matrix $\mathbb{A}_i$ with $i \in \mathscr{I}^{[1:M]}$. Thus, testing the generalized Lyapunov-Metzler condition is equivalent to check the classical one for a system whose modes are related to every possible sequence of the original system (1), of length M or less.

Another possible extension of the classical Lyapunov-Metzler condition follows. The idea is to maintain the sequence length in 1 but increase the number of ellipsoids involved.

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Proposition 14: If for every $i \in \mathscr{I}$ there exist a set of indices $\mathscr{K}_i = \mathbb{N}_{h_i}$, with $h_i \in \mathbb{N}$; a set of matrices $P_k^{(i)} > 0$, with $k \in \mathscr{K}_i$, and there are $\pi_{m,k}^{(p,i)} \in [0,1]$, satisfying

$$\sum_{p\in\mathscr{I}} \sum_{m\in\mathscr{K}_p} \pi_{m,k}^{(p,i)} = 1, \tag{18}$$

for all $k \in \mathscr{K}_i$, such that

$$A_i^T \left(\sum_{p\in\mathscr{I}} \sum_{m\in\mathscr{K}_p} \pi_{m,k}^{(p,i)} P_m^{(p)} \right) A_i - P_k^{(i)} < 0, \quad \forall i \in \mathscr{I},\ \forall k \in \mathscr{K}_i, \tag{19}$$

holds, then the switched system (1) is stabilizable.

Geometrically, Proposition 14 provides a condition under which there exists a C*-set composed by a finite number of ellipsoids that is contractive. Namely, the condition is sufficient for the existence of a set of ellipsoids, determined by $P_k^{(i)}$ with $k \in \mathscr{K}_i$, associated to every mode i, whose image through A_i is mapped inside the C*-set. Thus, the induced homogeneous function is a control Lyapunov function. Notice that the classical Lyapunov-Metzler condition, i.e. Theorem 6, is a particular case of Proposition 14, with the restriction $h_i = 1$ for all $i \in \mathscr{I}$.

IV. LMI SUFFICIENT CONDITION

The main drawback of the necessary and sufficient set-inclusion condition for stabilizability is, as already stated, its inherent complexity. On the other hand, the Lyapunov-Metzler-based approach leads to a more practical BMI sufficient condition. Nevertheless, the complexity could be still computationally prohibitive, see [32]. Our next aim is to formulate an alternative condition that could be checked by convex optimization algorithms.

Theorem 15: The switched system (1) is stabilizable if there exist $N \in \mathbb{N}$ and $\eta \in \mathbb{R}^{\bar{N}}$ such that $\eta \geq 0$, $\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i = 1$ and

$$\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I. \tag{20}$$

Proof: The result follows directly from the fact that (20) implies (4), as a consequence of Lemma 7. ■

An interesting issue is whether the sufficient condition for stabilizability given in Theorem 15 is also necessary. One particular case in which the LMI condition is guaranteed to have a solution, provided the switched system (1) is stabilizable, follows.

Corollary 16: If there exist $N \in \mathbb{N}$ and $i_1, i_2 \in \mathscr{I}^{[1:N]}$ such that $\mathscr{B} \subseteq \text{int}(\mathscr{B}_{i_1} \cup \mathscr{B}_{i_2})$ then there is $\eta \in [0,1]$ such that

$$\eta \mathbb{A}_{i_1}^T \mathbb{A}_{i_1} + (1-\eta) \mathbb{A}_{i_2}^T \mathbb{A}_{i_2} < I.$$

Proof: The property is a consequence of Lemma 9. ■

The condition presented in Theorem 15 is just sufficient unless there exists, among the $\mathscr{B}_i$, two ellipsoids containing $\mathscr{B}$ in their union, see Corollary 16. This is proved by the following counter-example.

Example 17: The aim of this illustrative example is to show a case for which the inclusion condition (4) is satisfied with $N = 1$, but there is not a finite value of $\hat{N} \in \mathbb{N}$ for which condition (20) holds. Consider the three modes given by the matrices

$$A_1 = AR(0), \quad A_2 = AR\left(\frac{2\pi}{3}\right), \quad A_3 = AR\left(\frac{-2\pi}{3}\right),$$

where

$$A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{21}$$

with $a = 0.6$. Set $\Omega = \mathscr{B}$. By geometric inspection of Figure 1, condition (4) holds at the first step, i.e. for $N = 1$. On the other hand, A_i are such that $\det(A_i^T A_i) = a^2 a^{-2} = 1$ and $\text{tr}(A_i^T A_i) = a^2 + a^{-2} = 3.1378$ while the determinant and trace of the matrix defining $\mathscr{B}$ are 1 and 2, respectively. Notice that $a^2 + a^{-2} > 2$ for every a different from 1 or -1 and $a^2 + a^{-2} = 2$ otherwise.

For every N and every $\mathscr{B}_i$ with $i \in \mathscr{I}^{[1:N]}$, the related $\mathbb{A}_i$ is such that $\det(\mathbb{A}_i^T \mathbb{A}_i) = 1$ and $\text{tr}(\mathbb{A}_i^T \mathbb{A}_i) \geq 2$. Notice that, for all the matrices $Q > 0$ in $\mathbb{R}^{2\times 2}$ such that $\det(Q) = 1$, then $\text{tr}(Q) \geq 2$ and $\text{tr}(Q) = 2$ if and only if $Q = I$, since the determinant is the product of the eigenvalues and the trace its sum. Thus, for every subset of the ellipsoids $\mathscr{B}_i$, determined by a subset of indices $K \subseteq \mathscr{I}^{[1:N]}$, we have that

$$\sum_{i\in K} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I,$$

cannot hold, since either $\text{tr}(\mathbb{A}_i^T \mathbb{A}_i) > 2$ or $\mathbb{A}_i^T \mathbb{A}_i = I$.

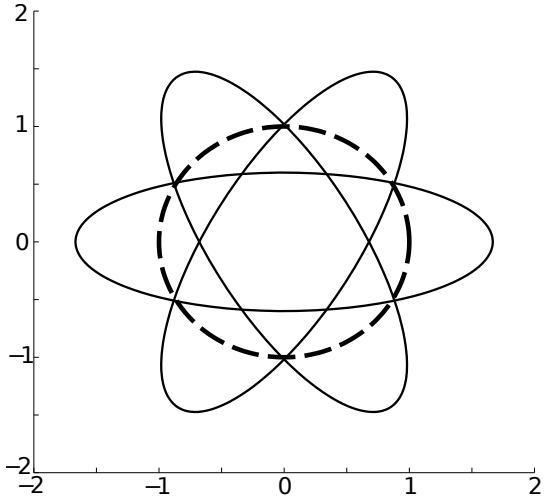

Fig. 1. Sets $\mathscr{B}_1$ $\mathscr{B}_2$ $\mathscr{B}_3$ (solid) and $\Omega = \mathscr{B}$ (dashed).

Thus the LMI condition (20) is sufficient but not necessary.

A. LMI-based control Lyapunov functions

The LMI condition (20) can be interpreted in terms of control Lyapunov functions and can be used to derive the controller synthesis techniques. Let us assume that (20) holds, then there exists $\mu \in [0,1)$ such that

$$\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i \leq \mu^2 I. \tag{22}$$

Also, for all $x \in \mathbb{R}^n$, it holds

$$\min_{i\in\mathscr{I}^{[1:N]}} (x^T \mathbb{A}_i^T \mathbb{A}_i x) \leq \mu^2 x^T x. \tag{23}$$

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We can now describe the stabilizing control strategy. The controller does not necessarily select at each time step $k \in \mathbb{N}$ which input should be applied. This is done only at given instant $\{k_p\}_{p\in\mathbb{N}}$ with $k_0 = 0$, and $k_p < k_{p+1} \leq k_p + N$, for all $p \in \mathbb{N}$. At time k_p, the controller selects the sequence of inputs to be applied up to step $k_{p+1} - 1$. The instant k_{p+1} is also determined by the controller at time k_p. More precisely, the controller acts as follows for all $p \in \mathbb{N}$, let

$$i_p = \arg \min_{i\in\mathscr{I}^{[1:N]}} (x_{k_p}^T \mathbb{A}_i^T \mathbb{A}_i x_{k_p}). \tag{24}$$

Then, the next instant k_{p+1} is given by

$$k_{p+1} = k_p + l(i_p), \tag{25}$$

with $l(i_p)$ length of i_p, and the controller applies the sequence of inputs

$$\sigma_{k_p+j-1} = i_{p,j}, \quad \forall j \in \{1,\dots,l(i_p)\}. \tag{26}$$

Theorem 18: Let us assume that (20) holds, and consider the control strategy given by (24), (25), (26). Then, for all $x_0 \in \mathbb{R}^n$, for all $k \in \mathbb{N}$,

$$\|x_k\| \leq \mu^{k/N-1} L^{N-1} \|x_0\| \tag{27}$$

where $L \geq \|A_i\|$, for all $i \in \mathscr{I}$. Then, the controlled switched system is globally exponentially stable.

Proof: Using the proposed control strategy, we have $x_{k_{p+1}} = \mathbb{A}_{i_p} x_{k_p}$ for all $p \in \mathbb{N}$. Then, it follows from (23) and (24) that $\|x_{k_{p+1}}\| \leq \mu \|x_{k_p}\|$ and thus for all $p \in \mathbb{N}$, $\|x_{k_p}\| \leq \mu^p \|x_{k_0}\|$. Moreover, since $k_{p+1} - k_p \leq N$ and $L \geq 1$ from Assumption 1, we have for all $p \in \mathbb{N}$:

$$\|x_k\| \leq L^{k-k_p} \|x_{k_p}\| \leq \mu^p L^{N-1} \|x_0\|, \ \forall k \in \{k_p,\dots,k_{p+1}-1\}. \tag{28}$$

Now let $k \in \mathbb{N}$, and let $p \in \mathbb{N}$ be such that $k \in \{k_p,\dots,k_{p+1}-1\}$ then necessarily $p \geq \lfloor k/N \rfloor \geq k/N - 1$. Then (27) follows from (28). ■

From Theorem 18, the LMI condition (20) implies that the switched systems with the switching rule given by (24), (25), (26) is globally exponentially stable. Nevertheless, neither the Euclidean norm of x nor the function $\min_{i\in\mathscr{I}^{[1:N]}} (x^T \mathbb{A}_i^T \mathbb{A}_i x)$ are monotonically decreasing along the trajectories. On the other hand a positive definite homogeneous nonconvex function decreasing at every step can be inferred for a different switching rule.

Proposition 19: Given the switched system (1), suppose there exist $N \in \mathbb{N}$ and $\eta \in \mathbb{R}^{\bar{N}}$ such that $\eta \geq 0$, $\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i = 1$ and (20) hold. Then there is $\lambda \in [0,1)$ such that the function

$$V(x) = \min_{i\in\mathscr{I}^{[1:N]}} (x^T \lambda^{-n_i} \mathbb{A}_i^T \mathbb{A}_i x), \tag{29}$$

where n_i is the length of $i \in \mathscr{I}^{[1:N]}$, satisfies $V(A_{\sigma(x)}x) \leq \lambda V(x)$ for all $x \in \mathbb{R}^n$, with

$$i^*(x) = \arg \min_{i\in\mathscr{I}^{[1:N]}} (x^T \lambda^{-n_i} \mathbb{A}_i^T \mathbb{A}_i x), \tag{30}$$

and $\sigma(x) = i_1^*(x)$.

Proof: From (20), there exists $\mu \in [0,1)$ such that (22) holds and then, posing $\lambda = \mu^{2/N}$, it follows that

$$I \geq \sum_{i\in\mathscr{I}^{[1:N]}} \eta_i \mu^{-2} \mathbb{A}_i^T \mathbb{A}_i = \sum_{i\in\mathscr{I}^{[1:N]}} \eta_i \lambda^{-N} \mathbb{A}_i^T \mathbb{A}_i \geq \sum_{i\in\mathscr{I}^{[1:N]}} \eta_i \lambda^{-n_i} \mathbb{A}_i^T \mathbb{A}_i, \tag{31}$$

since $\lambda^N \leq \lambda^k$, and then $\lambda^{-N} \geq \lambda^{-k}$ for all $k \leq N$. If $n_{i^*(x)} \geq 2$, one has

$$\begin{aligned} V(x) &= x^T \lambda^{-n_{i^*}} \mathbb{A}_{i^*}^T \mathbb{A}_{i^*} x = \lambda^{-1} (A_{i_1^*} x)^T \Big(\lambda^{-n_{i^*}+1} \mathbb{A}_{j(i^*)}^T \mathbb{A}_{j(i^*)} \Big) A_{i_1^*} x \\ &\geq \min_{j\in\mathscr{I}^{[1:N-1]}} \lambda^{-1} (A_{i_1^*} x)^T \Big(\lambda^{-n_j} \mathbb{A}_j^T \mathbb{A}_j \Big) A_{i_1^*} x \\ &\geq \min_{j\in\mathscr{I}^{[1:N]}} \lambda^{-1} (A_{i_1^*} x)^T \Big(\lambda^{-n_j} \mathbb{A}_j^T \mathbb{A}_j \Big) A_{i_1^*} x = \lambda^{-1} V(A_{i_1^*} x), \end{aligned}$$

where we employed the notation $i^*(x) = i^* = (i_1^*, j(i^*))$. If $n_{i^*(x)} = 1$ then

$$\begin{aligned} V(x) &= x^T \lambda^{-1} A_{i^*}^T A_{i^*} x = \lambda^{-1} (A_{i^*} x)^T I \, A_{i^*} x \\ &\geq \lambda^{-1} (A_{i^*} x)^T \Big(\sum_{j\in\mathscr{I}^{[1:N]}} \eta_j \lambda^{-n_j} \mathbb{A}_j^T \mathbb{A}_j \Big) A_{i^*} x \\ &\geq \min_{j\in\mathscr{I}^{[1:N]}} \lambda^{-1} (A_{i^*} x)^T \Big(\lambda^{-n_j} \mathbb{A}_j^T \mathbb{A}_j \Big) A_{i^*} x = \lambda^{-1} V(A_{i^*} x), \end{aligned}$$

from (31), and therefore, $V(A_{i_1^*} x) \leq \lambda V(x)$ for all $x \in \mathbb{R}^n$. ■

Remark 20: If the LMI (20) has a solution, then there exists a scalar $\mu \in [0,1)$, such that (22) is verified. The value of μ induces straightforwardly the rate of convergence λ for the Lyapunov function (29). Thus one might solve the optimization problem $\min_{\mu^2,\eta} \mu^2$ subject to (22), to get higher convergence rate.

Notice that, although $V(x)$ defined in (29) is not homogeneous of order one, its square root is so, as for the set-induced Lyapunov functions given in [27].

B. LMI-condition and periodic stabilizability

Another interesting implication that follows from the Example 17 concerns the stabilizability through periodic switching sequences.

Proposition 21: The existence of a stabilizing periodic switching law is sufficient but not necessary for the stabilizability of the system (1).

Proof: Sufficiency is trivial. We infer that necessity does not hold by proving that the system given in Example 17 is stabilizable but there is not a stabilizing periodic switching law. Suppose, by contradiction, that the periodic switching law characterized by $i^* \in \mathscr{I}^M$, with M finite, stabilizes the system of Example 17. This implies that the Euclidean norm of x is decreasing after Mm steps, for an appropriate $m \in \mathbb{N}$. This is equivalent to condition $(\mathbb{A}_{i^*}^m)^T \mathbb{A}_{i^*}^m < I$, which implies the satisfaction of the LMI condition (20) with $N = Mm$ and $\eta_i = 0$ for all $i \neq I^* = (i^*,\dots,i^*)$ and $\eta_{I^*} = 1$. But this has been proved to be impossible for the system of Example 17, which is nonetheless stabilizable. ■

As in the proof of Proposition 21 we used the fact that the existence of a stabilizing periodic switching law implies the satisfaction of the LMI condition, one might wonder if there exists an equivalence relation between periodic stabilizability and condition (20). The answer is provided below.

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Theorem 22: A stabilizing periodic switching law for the system (1) exists if and only if condition (20) holds.

Proof: The fact that the existence of a stabilizing periodic switching law implies satisfaction of (20) is direct, see the proof of Proposition 21. To prove the reverse, suppose that the LMI condition (20) holds. Then it follows that

$$\sum_{i\in\mathscr{I}^{[1:N]}}\sum_{j\in\mathscr{I}^{[1:N]}}\eta_i\eta_j\mathbb{A}_i^T\mathbb{A}_j^T\mathbb{A}_j\mathbb{A}_i\leq\lambda^2 I,$$

with $\lambda\in[0,1)$ and, more generally, for all $q\in\mathbb{N}$,

$$\sum_{I\in\mathscr{I}^{[q,Nq]}}\eta_I\,\mathbb{A}_I^T\mathbb{A}_I\leq\lambda^q I,$$

where for every $I=(i_1,\ldots,i_q)\in\mathscr{I}^{[q,Nq]}$ we define

$$\eta_I=\prod_{k\in\mathbb{N}_q}\eta_{i_k},\qquad \mathbb{A}_I=\prod_{k\in\mathbb{N}_q}\mathbb{A}_{i_k}.$$

From the linearity of the trace and the fact that $\sum_{I\in\mathscr{I}^{[q,Nq]}}\eta_I=1$, we have that

$$\begin{aligned}\min_{I\in\mathscr{I}^{[q,Nq]}}\mathrm{tr}(\mathbb{A}_I^T\mathbb{A}_I)&=\sum_{I\in\mathscr{I}^{[q,Nq]}}\eta_I\min_{I\in\mathscr{I}^{[q,Nq]}}\mathrm{tr}(\mathbb{A}_I^T\mathbb{A}_I)\\&\leq\sum_{I\in\mathscr{I}^{[q,Nq]}}\eta_I\,\mathrm{tr}(\mathbb{A}_I^T\mathbb{A}_I)=\mathrm{tr}\Big(\sum_{I\in\mathscr{I}^{[q,Nq]}}\eta_I\,\mathbb{A}_I^T\mathbb{A}_I\Big)\leq\lambda^q n,\end{aligned}$$

since $\mathrm{tr}(\mathbb{A}_I^T\mathbb{A}_I)>0$ for all $I\in\mathscr{I}^{[q,Nq]}$. Thus, for q big enough, for which $\lambda^q n<1$, there exists a $I^*\in\mathscr{I}^{[q,Nq]}$ such that $\mathrm{tr}(\mathbb{A}_{I^*}^T\mathbb{A}_{I^*})<1$, which implies that $\mathbb{A}_{I^*}$ is Schur. Then the periodic switching law characterized by the sequence I^* stabilizes the system (20). ∎

Notice that, although periodic stabilizability and condition (20) are equivalent from the stabilizability point of view, the computational aspects and the resulting controls are different. Indeed, the first consists of an eigenvalue test for a number of matrices exponential in M, see Lemma 4, while condition (20) is an LMI that grows exponentially with N. On the other hand, M is always greater or equal than N, much greater in general. Finally, notice that the periodic law is in open loop whereas (20) leads to a state-dependent switching law.

V. Stabilizability conditions relations

In what follows we characterize the relations between the different stabilizability conditions presented and recalled so far. First, we provide a relation with the generalized Lyapunov-Metzler condition (17). Recall that the Lyapunov-Metzler condition regards nonconvex sets and sequences of length one (possibly of extended systems) whereas the LMI one concerns quadratic Lyapunov functions and switching control sequences. It can be proved that the LMI sufficient condition (20) holds if and only if the generalized Lyapunov-Metzler one can be satisfied.

Theorem 23: There exist $M\in\mathbb{N}$, $P_i>0$, with $i\in\mathscr{I}^{[1:M]}$, and $\pi\in\mathscr{M}_{\bar{M}}$ such that (17) holds if and only if there exists $N\in\mathbb{N}$ and $\eta\in\mathbb{R}^{\bar{N}}$ such that $\eta\geq 0$, $\sum_{i\in\mathscr{I}^{[1:N]}}\eta_i=1$ and (20) holds.

Proof: First we prove that satisfaction of (17) implies the existence of N such that (20) holds. Suppose that for appropriate P_i, with $i\in\mathscr{I}^{[1:M]}$, and $\pi\in\mathscr{M}_{\bar{M}}$, (17) holds or, equivalently, that there exists $\lambda\in[0,1)$ such that

$$\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}P_j\Big)\mathbb{A}_m\leq\lambda P_m,\quad\forall m\in\mathscr{I}^{[1:M]}.$$

Let us choose an arbitrary $m\in\mathscr{I}^{[1:M]}$. We have

$$\begin{aligned}&\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}\mathbb{A}_j^T\Big(\sum_{k\in[1:M]}\pi_{kj}P_k\Big)\mathbb{A}_j\Big)\mathbb{A}_m\\&\qquad\leq\lambda\,\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}P_j\Big)\mathbb{A}_m\leq\lambda^2P_m,\end{aligned}$$

which is equivalent to

$$\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}\sum_{k\in\mathscr{I}^{[1:M]}}\pi_{kj}\Big(\mathbb{A}_j^TP_k\mathbb{A}_j\Big)\Big)\mathbb{A}_m\leq\lambda^2P_m.$$

From $\lambda<1$ and Assumption 1, we have that, for every $m\in\mathscr{I}^{[1:M]}$ there exists $s(m)=s\in\mathbb{N}$ such that $\lambda^sP_m<\mathbb{A}_m^T\mathbb{A}_m$ and then

$$\mathbb{A}_m^T\Big(\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_s\in\mathscr{I}^{[1:M]}}\pi_{i_si_{s-1}}\Big(\mathbb{A}_i^TP_{i_s}\mathbb{A}_i\Big)\Big)A_m\leq\lambda^sP_m<\mathbb{A}_m^T\mathbb{A}_m,$$

with $i=(i_1,\ldots,i_{s-1})\in\mathscr{I}^{[1:M(s-1)]}$. Since there is no loss of generality, assume that $I\leq P_k$ for all $k\in\mathscr{I}^{[1:M]}$ and then

$$\begin{aligned}&\mathbb{A}_m^T\Big(\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_{s-1}\in\mathscr{I}^{[1:M]}}\pi_{i_{s-1}i_{s-2}}\Big(\mathbb{A}_i^T\mathbb{A}_i\Big)\Big)\mathbb{A}_m\\&\leq\mathbb{A}_m^T\Big(\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_s\in\mathscr{I}^{[1:M]}}\pi_{i_si_{s-1}}\Big(\mathbb{A}_i^TP_{i_s}\mathbb{A}_i\Big)\Big)\mathbb{A}_m<\mathbb{A}_m^T\mathbb{A}_m,\end{aligned}$$

that implies

$$\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_{s-1}\in\mathscr{I}^{[1:M]}}\pi_{i_{s-1}i_{s-2}}\Big(\mathbb{A}_i^T\mathbb{A}_i\Big)<I,\qquad(32)$$

from Assumption 1. Denoting for every $i\in\mathscr{I}^{[1:M(s-1)]}$ the parameter $\eta_i=\pi_{i_1m}\pi_{i_2i_1}\ldots\pi_{i_{s-1}i_{s-2}}$ it can be proved that $0\leq\eta_i\leq 1$ and $\sum_{i\in\mathscr{I}^{[1:M(s-1)]}}\eta_i=1$ and (32) is equivalent to (20).

We prove now that the satisfaction of the LMI condition (20) with appropriate N and η implies that the generalized Lyapunov-Metzler one is satisfied with adequate M. From (20) one has

$$\mathbb{A}_i^T\Big(\sum_{j\in\mathscr{I}^{[1:N]}}\eta_j\mathbb{A}_j^T\mathbb{A}_j\Big)\mathbb{A}_i<\mathbb{A}_i^T\mathbb{A}_i,\quad\forall i\in\mathscr{I}^{[1:N]},$$

which is equivalent to (17) with $P_j=\mathbb{A}_j^T\mathbb{A}_j$ and $\pi_{ji}=\eta_j$, for all $i,j\in\mathscr{I}^{[1:N]}$ and $M=N$. ∎

Remark 24: Notice that, in the first part of proof of Theorem 23, there is a dependence on the index $m\in\mathscr{I}^{[1:M]}$. In reality, for every other possible index, the stabilizability result would be the same. The only difference would be the length s, that depends on m, and the values of the parameters η_i, that should be written as dependent on m.

We would like to point out that, even though (17) and (20) are equivalent, they generally hold for different values of M and N, with $N\geq M$, from the proof of Theorem 23.

We prove hereafter that the condition for stabilizability given by Theorem 15 and Proposition 14 are equivalent. It is,

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nevertheless, worth recalling that the former is in LMI form whereas the latter involves BMIs.

Theorem 25: For every $i \in \mathscr{I}$ there exist: the indices $\mathscr{K}_i = \mathbb{N}_{h_i}$, with $h_i \in \mathbb{N}$; the matrices $P_k^{(i)} > 0$, with $k \in \mathscr{K}_i$; and $\pi_{m,k}^{(p,i)} \in [0,1]$, satisfying (18) for all $k \in \mathscr{K}_i$, such that (19) holds if and only if there exists $N \in \mathbb{N}$ and $\eta \in \mathbb{R}^{\bar{N}}$ such that $\eta \geq 0$, $\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i \geq 1$ and (20) holds.

Proof: We prove first that Theorem 15 implies Proposition 14. Suppose then that (20) is satisfied for appropriate N and η. From (20) it follows that there exists $\mu \in [0,1)$ such that

$$\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i Q_i \leq I, \tag{33}$$

where $Q_i = \mu^{n_i} \mathbb{A}_i^T \mathbb{A}_i$, for all $i \in \mathscr{I}^{[1:N]}$, with $n_i \in \mathbb{N}_N$ the length of i (and then also the number of matrices composing $\mathbb{A}_i$). By definition, Q_i are such that, for all $i \in \mathscr{I}^{[1:N]}$ of length greater than 1, we have:

$$Q_i = \mu A_{i_1}^T Q_{j(i)} A_{i_1}, \quad \Longrightarrow \quad A_{i_1}^T Q_{j(i)} A_{i_1} = \mu^{-1} Q_i < Q_i, \tag{34}$$

where $j(i) \in \mathscr{I}^{[1:N-1]}$ is such that $i = (i_1, j(i))$. If the length of i is 1 then we have that $Q_i = \mu A_i^T A_i$ and thus

$$A_i^T \left(\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i Q_i \right) A_i \leq A_i^T A_i = \mu^{-1} Q_i < Q_i, \tag{35}$$

from (33). Finally defining for every mode $m \in \mathscr{I}$ the sets of matrices

$$\left\{ P_k^{(m)} \right\}_{k \in \mathscr{K}_m} = \{ \mu^{n_i} \mathbb{A}_i^T \mathbb{A}_i \in \mathbb{R}^{n \times n} : \ \forall i \in \mathscr{I}^{[1:N]} \text{ s.t. } i_1 = m \}$$

with adequate $\mathscr{K}_m$, condition (19) holds from (34) and (35).

The fact that Proposition 14 implies Theorem 15 can be proved with reasonings analogous to those of Theorem 23. ■

Therefore, allowing to employ a set of ellipsoids $P_k^{(i)}$ for every mode $i \in \mathscr{I}$, leads to BMI Lyapunov-Metzler-like conditions equivalent to the LMI one. Nevertheless, it can be proved that such equivalence is lost for the classical Lyapunov-Metzler condition, i.e. if one considers a single matrix P_i for each mode, see (5). This result is illustrated in the following. We first introduce a lemma functional to that purpose.

Lemma 26: Let A_1, A_2 non-Schur. If there exists $P_1 > 0$, $P_2 > 0$, $\alpha, \beta \in (0,1)$ such that

$$A_1^T(\alpha P_1 + (1-\alpha)P_2)A_1 < P_1 \tag{36}$$

$$A_2^T(\beta P_1 + (1-\beta)P_2)A_2 < P_2 \tag{37}$$

then the matrix A_α given by

$$A_\alpha = \alpha A_1 + (1-\alpha)\frac{\sqrt{\rho(A_2)^2 - 1}}{\rho(A_2)} A_2 A_1 \tag{38}$$

is Schur.

Proof: From (37), it follows that $(1-\beta)A_2^T P_2 A_2 < P_2$ which gives $1-\beta < \frac{1}{\rho(A_2)^2}$ and $\beta > \frac{\rho(A_2)^2-1}{\rho(A_2)^2}$. Then, (37) gives

$$\frac{\rho(A_2)^2 - 1}{\rho(A_2)^2} A_2^T P_1 A_2 < \beta A_2^T P_1 A_2 < P_2.$$

Substituting in (36) yields

$$\alpha A_1^T P_1 A_1 + (1-\alpha)\frac{\rho(A_2)^2 - 1}{\rho(A_2)^2} A_1^T A_2^T P_1 A_2 A_1 < P_1.$$

Since $P_1 > 0$, it follows by convexity that $A_\alpha^T P_1 A_\alpha < P_1$, and then A_α is Schur. ■

An example follows which shows that the LMI condition (20) can hold for a system while the Lyapunov-Metzler one does not.

Example 27: Consider the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad A_2 = 4\sqrt{2}R(\pi/4) = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}. \tag{39}$$

As proved in Lemma 30 in appendix, $\rho(A_\alpha) > 1$ for all $\alpha \in (0,1)$, see also Figure 2. Therefore, from Lemma 26, the Lyapunov-Metzler inequalities (36) and (37) cannot hold. One

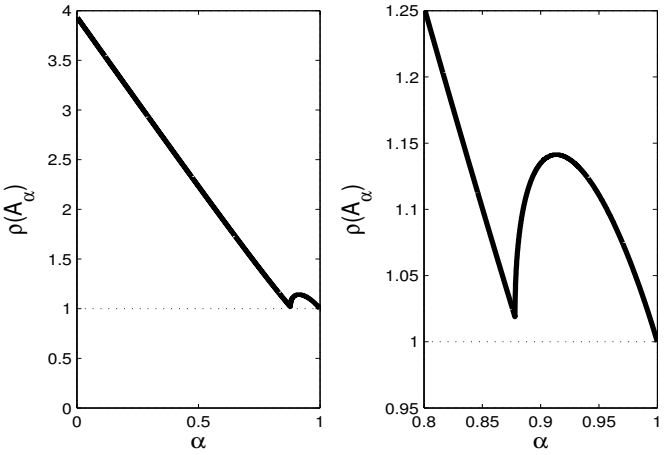

Fig. 2. Spectral radius of A_α.

can also check that

$$A_1^{11} A_2^2 = \begin{bmatrix} 0 & -32 \\ 1/64 & 0 \end{bmatrix}$$

whose spectral radius is $\sqrt{2}/2$ and therefore the system is periodic stabilizable and LMI condition (20) holds. Actually, the LMI condition (20) holds for sequences of length smaller than or equal to 8. Consider now the matrices A_1 and $A_1^6 A_2^2$ and define

$$M_1 = A_1^T A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad M_2 = (A_1^6 A_2^2)^T A_1^6 A_2^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1024 \end{bmatrix}.$$

It can be verified that the matrix

$$M_\alpha = (1-\alpha)M_1 + \alpha M_2 = \begin{bmatrix} \frac{4-3\alpha}{4} & 0 \\ 0 & \frac{1+4095\alpha}{4} \end{bmatrix}$$

is such that $M_\alpha < I$, for all $\alpha \in (0, 3/4095)$. Then, there exists a solution for condition (20) to be satisfied with $N = 8$.

The implications between the stabilizability conditions are summarized in the diagram in Figure 3. Remark that, compared to the classical Lyapunov-Metzler inequalities, the LMI condition concerns a convex problem and it is less conservative. On the other hand, the dimension of the LMI problem might be consistently higher than the BMI one.

Finally, notice that the LMI condition (20) leads to a state-dependent switching law. The direct extension to the case of output-based switching design, treated for instance in [33], is not straightforward and requires further research. Nevertheless, since the LMI condition and the periodic stabilizability are

equivalent, if (20) has a solution then an open-loop stabilizing switching sequence can be designed, and no output is necessary to stabilize the system.

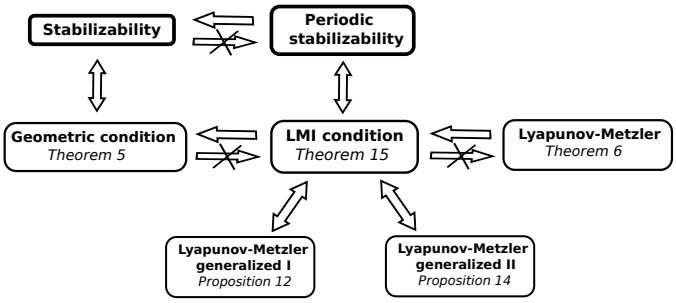

Fig. 3. Implications diagram of stabilizability conditions.

VI. Numerical examples

Two numerical examples are treated here in detailed.

Example 28: Consider the system (1) with $q = 2$, $n = 2$, $x_0 = [-3,3]^T$ and the non-Schur matrices

$$A_1 = 1.01R\left(\tfrac{\pi}{5}\right), \qquad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix}.$$

Four different stabilizing switching laws are designed and compared. In particular we consider the geometric condition given in Theorem 5, which proves the stabilizability of the systems; the min-switching strategy (24)-(26) related to a solution of the LMI condition (20); the switching control law given in Proposition 19 and the periodic switching law, that exists from Theorem 22.

As noticed in [15], [27], for systems with $q = 2$ the Lyapunov-Metzler inequalities become two linear matrix inequalities once two parameters, both contained in $[0,1]$, are fixed. Such LMIs have been checked for this example to be infeasible on a grid of these two parameters, with step of 0.01. It is then reasonable to conclude that the Lyapunov-Metzler inequalities are infeasible for this numerical example. Notice that, the Lyapunov-Metzler inequalities being bilinear, there is no generic numerical method to solve them when a solution exists, except the gridding approach. Furthermore it is evident that the computational complexity is unmanageable as q increases. Recall moreover that, to circumvent the conservatism proper of the classical Lyapunov-Metzler inequalities with respect to the LMI condition (20), one should increase the problem dimension, see Proposition 12 and 14. Therefore, employing Lyapunov-Metzler inequalities to prove stabilizability might often be computationally intractable, also for systems with few modes.

First, an iterative procedure is applied to determine $N \in \mathbb{N}$ such that (4) is satisfied. The result is that (4) holds with $N = 5$ and then the homogeneous function induced by the set represented in Figure 4 is a control Lyapunov function and the related min-switching rule is a stabilizing law. The state evolution and the switching law are depicted in Figure 5.

The LMI condition (20) is solved with $N = 7$ and the min-switching law (24)-(26) is applied to the system at first. The

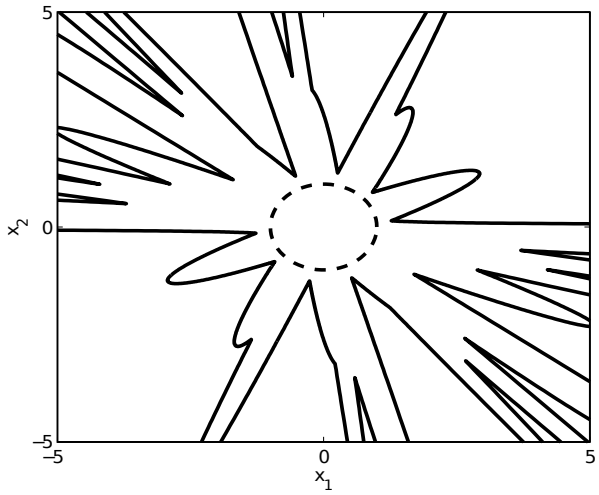

Fig. 4. Sets $\mathscr{B}$ (dashed) and $\bigcup_{i\in\mathscr{I}^{[1:N]}} \mathscr{B}_i$ (solid).

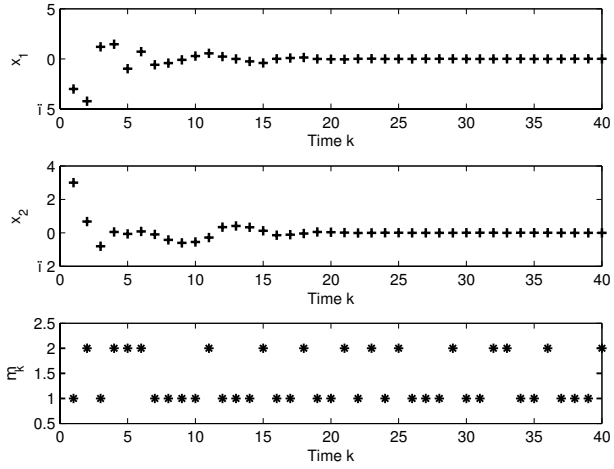

Fig. 5. State evolution and switching control induced by the geometric condition (4).

control results in the concatenation of elements of $\mathscr{I}^{[1,7]}$, respectively of lengths $\{7,6,5,7,7,\ldots\}$. The time-varying length of the switching subsequences is a consequence of the state dependence of the min-switching strategy. The resulting behavior is depicted in Figure 6. Then, the control law defined in Proposition 19, namely (30) with $\lambda = 0.9661$, is applied and the result is shown in Figure 7. The value of λ is obtained by solving the optimization problem described in Remark 20.

The periodic switching law of length $M = 4$ is then obtained, by searching the shorter sequence of switching modes which yields a Schur matrix $\mathbb{A}_i$. The resulting evolution is represented in Figure 8.

Finally a comparison between the different switching laws is provided in Figure 9, where the time-evolution of the Euclidean distance of the state from the origin is depicted. Recall that, although every switching rule entails the exponential decreasing of an homogeneous function, each law is induced by different sets, potentially nonconvex. For this reason we choose the Euclidean norm as a common measure to compare the convergence performances. Notice that, from Figure 9, the higher convergence rate seems to be obtained for the geometric

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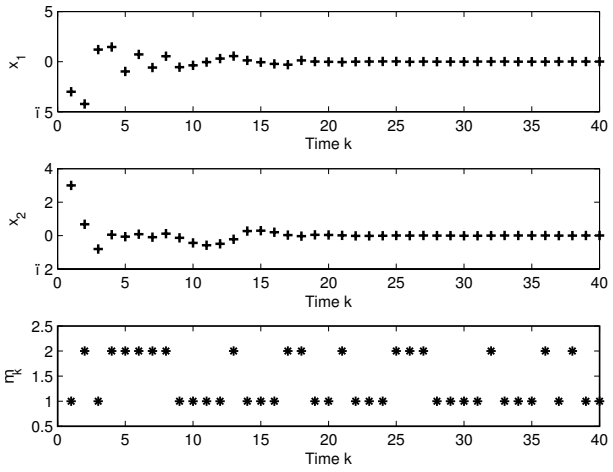

Fig. 6. State evolution and min-switching control (24)-(26).

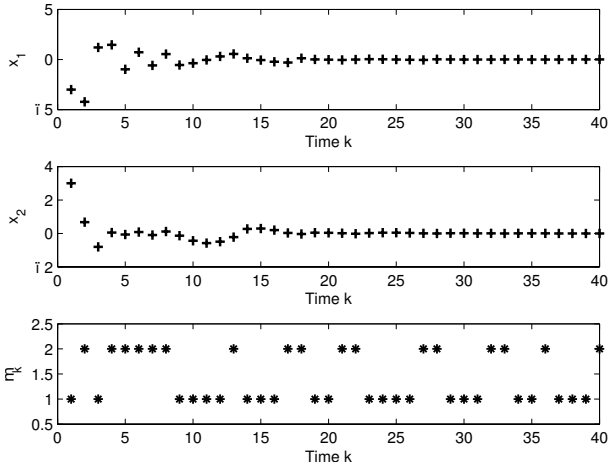

Fig. 7. State evolution and min-switching control (30).

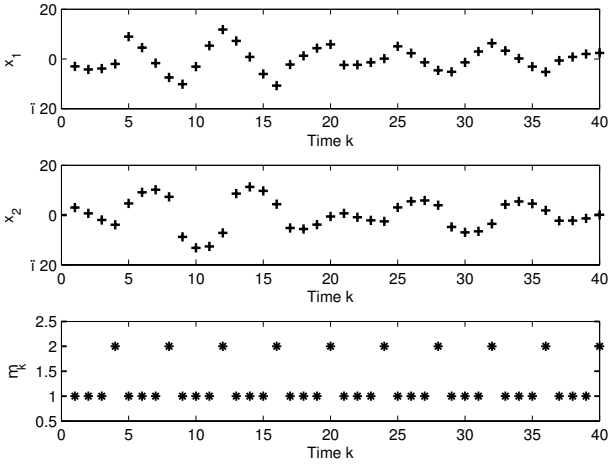

Fig. 8. State evolution and periodic switching control with $M = 4$.

approach, which is reasonable since it has been proved to be given by the less conservative stabilizability condition. Furthermore, the lower convergence speed is provided for this example by the periodic switching rule, which might reflect the fact that such switching rule does not employ the information on the state but depends only on time. Finally, the min-switching rules induced by the LMI condition (20) provide an average performance, due to their state-feedback nature on one side and to the conservatism with respect to the geometric condition related law, on the other.

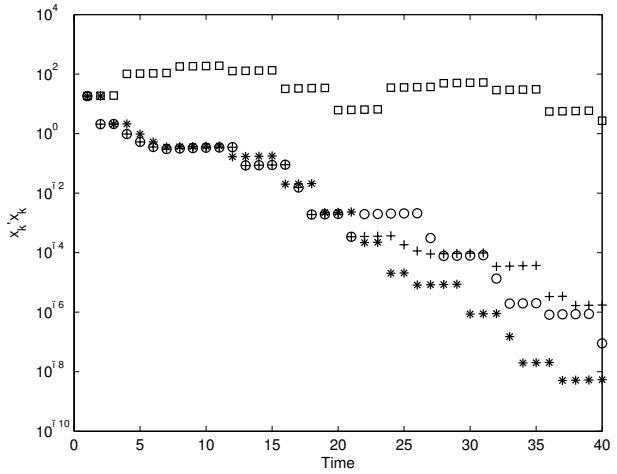

Fig. 9. Comparison between the evolution of the Euclidean norm of the state for the different switching laws: induced by geometric condition (4) (star); min-switching law (24)-(26) (cross); min-switching control (30) (circle) and periodic rule in (square).

Example 29: Let us revisit the 3D numerical example [27, Example 7], that is $q = 2$ and $n = 3$ with

$$A_1 = \begin{bmatrix} 1.2 & 0 & 0 \\ -1 & 0.8 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & -0.6 & -2 \\ 0 & 0 & -1.2 \end{bmatrix}.$$

A_1 and A_2 are not Schur, but the product A_1A_2 is Schur and induces the existence of a 2-periodic stabilization law. The geometric condition terminates at the third step (see [27]). By applying Theorem 15, the LMI (20) is infeasible for $N = 1, 2, 3$ but feasible for $N = 4$, then the dimension of η is given by $\bar{N} = \bar{4} = 30$ (see η depicted in Figure 10). The state evolution is depicted in Figure 11. It is noteworthy that the solution obtained by the LMI solver is not unique. Guided by the three main values of the weighting vector η, another solution of LMI (20) is given by

$$0.3460\mathbb{A}^T_{(1;1;2;1)}\mathbb{A}_{(1;1;2;1)} + 0.1753\mathbb{A}^T_{(1;1;2;2)}\mathbb{A}_{(1;1;2;2)} + 0.4787\mathbb{A}^T_{(1;2;1;2)}\mathbb{A}_{(1;2;1;2)} < I_3.$$

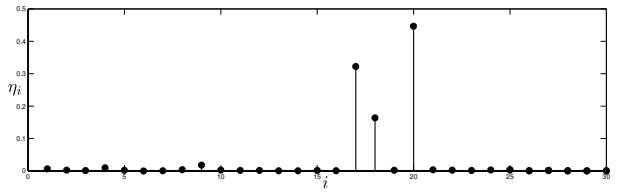

Fig. 10. A vector $\eta \in \mathbb{R}^{30}$ solution of the LMI (20) for Example 29.

VII. Conclusion

In this paper we provide a characterization of the relations and implications of different conditions, new and known ones,

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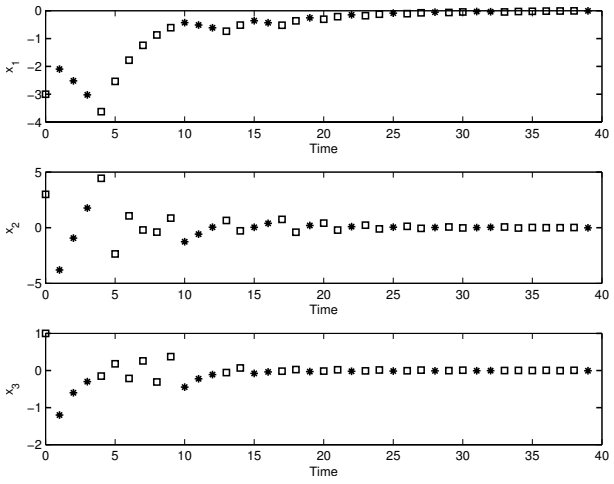

Fig. 11. State evolution induced by Theorem 15. The star denotes mode 1, the square mode 2.

for stabilizability of switched linear systems. A comparison in terms of conservatism and complexity is presented. Extensions to novel conditions and new computational methods for testing stabilizability are the objectives of our current and future work.

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APPENDIX

Lemma 30: The matrix A_α defined in (38) for the matrices (39) is not Schur for any $\alpha \in [0,1]$.

Proof: By introducing $\varepsilon = \frac{\sqrt{31}}{2\sqrt{2}}$, we have

$$A_\alpha = \begin{bmatrix} \alpha + 2(1-\alpha)\varepsilon & (\alpha-1)\varepsilon \\ 2(1-\alpha)\varepsilon & \frac{\alpha}{2} + (1-\alpha)\varepsilon \end{bmatrix}.$$

The characteristic polynomial $\chi_\alpha(\lambda)$ associated with the matrix A_α is given by

$$\chi_\alpha(\lambda) = \lambda^2 - \lambda\left(\frac{3\alpha}{2} + 3(1-\alpha)\varepsilon\right) + \frac{\alpha^2}{2} + 2\alpha(1-\alpha)\varepsilon + 4(1-\alpha)^2\varepsilon^2.$$

The discriminant Δ of the quadratic polynomial $\chi_\alpha(\lambda)$ is a quadratic form with respect to the parameter α:

$$\Delta(\alpha) = \alpha^2\left(\frac{1}{4} - 7\varepsilon^2 - \varepsilon\right) + \alpha(14\varepsilon^2 + \varepsilon) - 7\varepsilon^2,$$

whose own discriminant is equal to $31 > 0$. The quadratic form $\Delta(\alpha)$ admits thus two real roots. By introducing

$$\alpha^* = \frac{-(14\varepsilon^2+\varepsilon) \pm 2\sqrt{2}\varepsilon}{1/2 - 14\varepsilon^2 - 2\varepsilon} \equiv 0.87,$$

we have the two cases:

- If $\alpha \in [0, \alpha^*]$, then $\Delta(\alpha) \leq 0$ and $\chi_\alpha(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ has two complex conjugate roots. By induction,

  $$\mathrm{Re}(\lambda_i) = \frac{3}{4}(1-2\varepsilon)\alpha + \frac{3}{2}\varepsilon \geq \frac{3}{4}(1-2\varepsilon)\alpha^* + \frac{3}{2}\varepsilon > 1,$$

  because $(1-2\varepsilon) < 0$.
- If $\alpha \in (\alpha^*, 1]$, then $\Delta(\alpha) > 0$ and $\chi_\alpha(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ has two real roots. Without loss of generality, let us choose $\lambda_1(\alpha) \leq \lambda_2(\alpha)$, allowing to write:

  $$\lambda_2(\alpha) = \frac{3}{4}\alpha + \frac{3}{2}(1-\alpha)\varepsilon + \frac{1}{2}\sqrt{\Delta(\alpha)},$$

  whose second derivative is

  $$\frac{\mathrm{d}^2\lambda_2(\alpha)}{\mathrm{d}^2\alpha} = \frac{1}{4}\left(\frac{\frac{\mathrm{d}^2\Delta(\alpha)}{\mathrm{d}\alpha^2}\sqrt{\Delta(\alpha)} - \frac{1}{2}\left(\frac{\mathrm{d}\Delta(\alpha)}{\mathrm{d}\alpha}\right)^2 \frac{1}{\sqrt{\Delta(\alpha)}}}{\Delta(\alpha)} \right)$$

  and then it has the sign of

  $$2\frac{\mathrm{d}^2\Delta(\alpha)}{\mathrm{d}\alpha^2}\Delta(\alpha) - \left(\frac{\mathrm{d}\Delta(\alpha)}{\mathrm{d}\alpha}\right)^2 = -8\varepsilon^2 < 0.$$

  Thus $\lambda_2(\alpha)$ is a strictly concave function for $\alpha \in [\alpha^*, 1]$. Finally since $\lambda_2(\alpha^*) > 1$ and $\lambda_2(1) = 1$, we conclude that A_α is not Schur on this interval.

■

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