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The worst-case analysis of the Garey-Johnson Algorithm

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14 The Worst-case Analysis of the Garey-Johnson
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28 December 17, 2008
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31
32 **Abstract**
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34 The Garey-Johnson algorithm is a well known polynomial-time al-
35 gorithm constructing an optimal schedule for the maximum lateness
36 problem with unit execution time tasks, two parallel identical proces-
37 sors, precedence constraints and release times. The paper is concerned
38 with the worst-case analysis of a generalization of the Garey-Johnson
39 algorithm to the case of arbitrary number of processors. In contrast
40 to other algorithms for the maximum lateness problem, the tight per-
41 formance guarantee for the even number of processors differs from the
42 tight performance guarantee for the odd number of processors.
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48 *keywords:* scheduling, parallel machines, UET tasks, precedence con-
49 straints, maximum lateness
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1 Introduction

In this paper we consider the problem of scheduling a set $N=\{1,2,\dots,n\}$ of n tasks (jobs, operations) on $m > 1$ parallel identical processors (machines) subject to precedence constraints in the form of an anti-reflexive, anti-symmetric and transitive relation on N . If task i precedes task j , denoted $i \rightarrow j$, then the processing of task i must be completed before the processing of task j begins. Each processor can process only one task at a time, and each task can be processed by any processor. Once a processor begins executing a task, it processes this task until its completion (i.e. no preemptions are allowed). Each task j requires one unit of processor's time and its processing can commence only after the specified non-negative integer release time r_j .

Since no preemptions are allowed and all processors are identical, any schedule σ can be determined by specifying for each task j its completion time $C_j(\sigma)$ in such a way that

- $C_j(\sigma) \geq r_j + 1$, for all $j \in N$;
- not more than m tasks are assigned the same completion time;
- if $i \rightarrow j$, then $C_j(\sigma) \geq C_i(\sigma) + 1$.

The goal is to find a schedule that minimizes the criterion of maximum lateness

$$L_{max}(\sigma) = \max_{j \in N} [C_j(\sigma) - d_j], \quad (1)$$

where d_j is an integer due date associated with task j .

In the three-field notation (see for example [2]), the above problem is denoted by $P|prec, p_j = 1, r_j|L_{max}$, where the terms *prec* and r_j indicate

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10 the presence of precedence constraints and release times, and $p_j = 1$ reflects
11 the fact that all processing times are equal to one unit of time. If the partially
12 ordered set of tasks is an in-tree, then the term *prec* is replaced by *in-tree*.
13 Analogously, the term *out-tree* indicates that the partially ordered set of
14 tasks is an out-tree. If all due dates are equal to zero, the maximum lateness
15 problem becomes the makespan problem $P|prec, p_j = 1, r_j|C_{max}$ with the
16 criterion
17

$$C_{max}(\sigma) = \max_{j \in N} C_j(\sigma).$$

18 If the term r_j is omitted, then all tasks have the same release time of zero.
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20 It is well known that even $P|prec, p_j = 1|C_{max}$ is NP-hard in the strong
21 sense [8], [7]. Moreover, as has been shown in [11], the $P|prec, p_j = 1|C_{max}$
22 problem remains NP-hard in the strong sense even if the partially ordered
23 set of tasks is a bipartite graph. As far as the maximum lateness problem
24 is concerned, $P|out-tree, p_j = 1|L_{max}$ is also NP-hard in the strong sense
25 [3]. This implies the NP-hardness in the strong sense of $P|in-tree, p_j =$
26 $1, r_j|C_{max}$. These NP-hardness results boost the interest in the worst-case
27 performance of various approximation algorithms [1], [9], [10], [12].
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29 Notice that approximation analysis for L_{max} criteria can be done by
30 two ways. Let σ be a schedule obtained by an algorithm, and σ^* denote
31 the optimal schedule for a given instance. The first approach defines an
32 upper bound on $L_{max}(\sigma) - L_{max}(\sigma^*)$. Relative approximation seems to
33 be difficult since L_{max} criteria might have negative values. However, it is
34 straightforward that $P|prec, p_j = 1, r_j|L_{max}$ is equivalent to the problem
35 $P|prec, p_j = 1, r_j, q_j|G_{max}$, where each job j is associated with a release
36 time r_j and a positive tail q_j , such that for a given schedule σ , $G_{max}(\sigma) =$
37 $\max_{j \in N} C_j(\sigma) + q_j$. Indeed, from an instance of the L_{max} problem we define
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an instance of the G_{max} problem by setting $\forall j \in N, q_j = \max_{i \in N} d_i - d_j$ (and conversely $d_j = \max_{i \in N} q_i - q_j$), so that for a given schedule σ , $L_{max}(\sigma) = G_{max}(\sigma) - \max_{i \in N} d_i$. Hence G_{max} is always positive and a worst case approximation ratio can be found:

$$G_{max}(\sigma) \leq \alpha G_{max}(\sigma^*) - \beta$$

This will lead to an expression for L_{max} as:

$$L_{max}(\sigma) \leq \alpha L_{max}(\sigma^*) + (\alpha - 1) \max_{i \in N} d_i - \beta$$

All performance guarantee that can be found in the litterature for L_{max} , have this form and the best known ratio is $\alpha = 2 - \frac{2}{m}$ [9][12].

This paper is concerned with the Garey-Johnson algorithm [4]. Although the Garey-Johnson algorithm was originally developed for the $P2|prec, p_j = 1, r_j|L_{max}$ problem, i.e. the problem with only two processors, it can be generalized to the case of arbitrary number of processors. We will refer to this generalization by GJ-algorithm. Although the Garey-Johnson algorithm is among the most popular scheduling algorithms, its worst-case performance in the case of arbitrary number of processors has remained unknown for almost three decades. In what follows we analyze the worst-case performance of the GJ-algorithm.

2 GJ-algorithm

In order to define what GJ-algorithm is, we need to introduce the framework on which it relies. Let us consider an instance of $P|prec, p_j = 1|L_{max}$, defined by an acyclic graph G , and for any node i , an integer d_i the due-date of task i . Notice that in the rest of the paper, g, h, i, j will always denote tasks.

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10 Let us call $Decision(G, (d_i)_{i \in N})$ the following decision problem: does-it
11 exists a schedule which meets all due-dates (called in-time schedule)?
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13 The GJ-algorithm relies on an oracle \mathcal{O} which answers partially this
14 question: from original due-dates d_i , an algorithm \mathcal{A} computes consistent
15 modified due dates which should be met by any in-time schedule. If the
16 algorithm fails, then there is no in-time schedule, so that the oracle answers
17 “No”. Otherwise, the true answer is still unknown, but the oracle answers
18 “Yes”. Consistent due-dates are defined in the first subsection.
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23 Then, the GJ-algorithm will compute, by binary search, a minimum
24 value Δ^* such that the oracle \mathcal{O} answers yes to the problem $Decision(G, (\Delta^* +$
25 $d_i)_{i \in N})$. It is proven in subsection 2.2 that Δ^* is a lower bound on the op-
26 timal lateness for the initial problem.
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30 Finally, modified due-dates for the instance $(G, (\Delta^* + d_i)_{i \in N})$ are used
31 as priorities to build a schedule of the tasks with a list scheduling algorithm.
32 This part is described in subsection 2.3.
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37 2.1 Consistent due dates

38 If $i \rightarrow j$ and $r_i \geq r_j$, then the replacement of r_j by $r_i + 1$ does not affect
39 the feasibility of any schedule. Since one can recalculate all release times
40 in $O(n^2)$ operations, without loss of generality we will assume that $i \rightarrow j$
41 implies the inequality $r_i + 1 \leq r_j$. Without loss of generality we will also
42 assume that $\min_{j \in N} r_j = 0$.
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49 Let D_1, \dots, D_n be arbitrary nonnegative integers, then for any task i and
50 any two numbers s and d such that $S(i, s, d)$ will denote the set of all tasks
51 j with due date at most d ($D_j \leq d$) such that i precedes j according to G
52 ($i \rightarrow j$), or $r_j \geq s$.
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10 Intuitively if each task meets its due-date, then all tasks of $S(i, s, d)$
11 including i should end before d . Now assume that $r_i \leq s \leq D_i \leq d$. If
12 $D_i = s$ then obviously, if i ends exactly at its due-date, all tasks of $S(i, s, d)$
13 should be computed in the time interval $[s, d)$. Similarly, if $D_i > s$ and if i
14 ends at its due-date, all tasks of $S(i, s, d)$ and i should be performed in the
15 time interval $[s, d)$. The notion of consistency derives from this intuition:
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20 We will say that integers D_1, \dots, D_n are consistent if for every task j
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$$22 \quad r_j \leq D_j - 1, \quad (2)$$

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25 And for any task i , and any integers s, d linked to i by the following relation:
26

$$27 \quad r_i \leq s \leq D_i \leq d, \quad (3)$$

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30 we have the following property:
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$$32 \quad \left\{ \begin{array}{l} \text{either } |S(i, s, d)| < m(d - s) \\ \text{or } |S(i, s, d)| = m(d - s) \text{ and } D_i = s \end{array} \right. \quad (4)$$

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37 It is easy to see that this definition is equivalent to the definition of
38 consistency given in [4]. Indeed, according to [4], integers D_1, \dots, D_n are
39 consistent if the inequality (2) holds for every $j \in N$, and for any task i and
40 any two integers s and d satisfying (3), the inequality
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$$45 \quad |S(i, s, d)| \geq m(d - s) \quad (5)$$

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48 implies

$$49 \quad D_i \leq d - \left\lceil \frac{|S(i, s, d)|}{m} \right\rceil. \quad (6)$$

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52 The equivalence of both definitions follows from the fact that the inequal-
53 ities $|S(i, s, d)| > m(d - s)$ and (6) contradict (3), and that the equality
54 $|S(i, s, d)| = m(d - s)$ together with (6) and (3) implies $D_i = s$.
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Example: Let us consider the problem instance pictured in figure 1. And assume that $m = 7$. Let us consider the upper-left most node and call it a . Notice that $r_a = 0, d_a = 1$. Let us consider $s = 1, d = 2$. The set $S(a, 1, 2)$ comprises all nodes of the second row, since they all have a release date not less than 1 and a due date not more than 2. As there are $9 > 7$ nodes in the second row, the due-dates are not consistent.

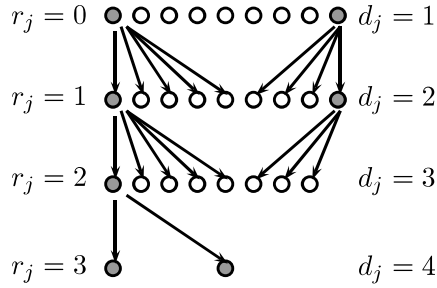


Figure 1: An example with inconsistent due-dates for $m = 7$.

Lemma 1 *If integers D_1, \dots, D_n are consistent and $i \rightarrow j$, then $D_i \leq D_j - 1$.*

Proof

Suppose that $i \rightarrow j$ and $D_i \geq D_j$. Since D_1, \dots, D_n are consistent, $r_i < D_i$, and analogously to [4], $s = d = D_i$ satisfy (3). For these s and d , $j \in S(i, s, d)$ and hence $|S(i, s, d)| > m(d - s)$, which contradicts the definition of consistency. \square

2.2 Δ -modified due dates

Let Δ be an arbitrary integer. We will say that integers D_1, \dots, D_n are Δ -modified due dates if they are consistent and $D_i \leq d_i + \Delta$ for all $i \in N$.

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10 Example: Let us consider again the problem instance pictured in figure
11 1. Setting $D_j = d_j + 1 = i + 2$ for jobs j of row i (rows are indexed from 0
12 to 4) defines 1-modified due dates. One can easily check the new due-dates
13 are consistent.
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16 As has been shown in [4], there exists an algorithm which for any given
17 Δ in $O(n^3)$ operations either calculates Δ -modified due dates or establishes
18 that such due dates do not exist at all. The idea of this algorithm is based
19 on the definition of consistency in [4] and can be outlined as follows. In
20 order to compute for a given Δ a set of Δ -modified due dates we first set
21 $D_j = d_j + \Delta$ for all $j \in N$. If (2) holds for all $j \in N$ and the inequality
22 (6) holds for all triples (i, s, d) satisfying (3) and (5), integers D_1, \dots, D_n
23 themselves are Δ -modified due dates. If some j does not satisfy (2), the
24 desired set of Δ -modified due dates does not exist at all. Suppose that
25 $D_j \geq r_j + 1$ for all $j \in N$, but for some triple (i, s, d) satisfying (3) and (5)
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$$28 \quad D_i > d - \left\lceil \frac{|S(i, s, d)|}{m} \right\rceil. \quad (7)$$

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30 It is easy to see that if Δ -modified due dates exist, then the due date asso-
31 ciated with i is not greater than the right-hand side of the inequality (7).
32

33 Hence we set

$$34 \quad D_i = d - \left\lceil \frac{|S(i, s, d)|}{m} \right\rceil$$

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36 and again check (2) and the inequality (6) for all triples (i, s, d) satisfying
37 (3) and (5). At each such iteration we either conclude that the desired set of
38 Δ -modified due dates does not exist, or establish that the current integers
39 D_1, \dots, D_n are Δ -modified due dates, or reduce value of some D_i . The
40 result that this procedure terminates in $O(n^3)$ operations is based on three
41 observations. First, that it suffices to consider only values of d coinciding
42 with one of the current integers D_i . Second, that for a given d it suffices
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to consider only values of s coinciding with one of release times or d itself. Third, that if the procedure is structured as three nested loops, where the outer loop selects d in decreasing order, the next loop selects i , and for fixed d and i the inner loop selects s in increasing order, each triple cannot appear in more than one iteration.

The following lemma shows that the existence of a schedule that meets due dates $d_1 + \Delta, \dots, d_n + \Delta$ implies the existence of Δ -modified due dates.

Lemma 2 *Let σ be an arbitrary schedule and $D_j = C_j(\sigma)$, for all $j \in N$, then D_1, \dots, D_n are consistent.*

Proof

Since σ is a feasible schedule, $r_j \leq C_j(\sigma) - 1 = D_j - 1$ for all $j \in N$. Suppose that some triple (i, s, d) satisfies (3) and the equality $|S(i, s, d)| = m(d - s)$. Then at least $(d - s) + 1$ time units are required to complete all tasks constituting the set $S(i, s, d) \cup \{i\}$. Since $C_j(\sigma) = D_j \leq d$ for each $j \in S(i, s, d) \cup \{i\}$, there exists a task $j \in S(i, s, d) \cup \{i\}$ such that $C_j(\sigma) \leq s$. From the definition of $S(i, s, d)$, either $i = j$ or $i \rightarrow j$. Therefore $D_i = C_i(\sigma) \leq s$, and by (3), $D_i = s$.

Now suppose that some triple (i, s, d) satisfies (3) and the inequality $|S(i, s, d)| > m(d - s)$. Then

$$d - \left\lceil \frac{|S(i, s, d)|}{m} \right\rceil \leq d - \frac{|S(i, s, d)|}{m} < s,$$

and since s and d are integers,

$$d - \left\lceil \frac{|S(i, s, d)|}{m} \right\rceil + 1 \leq s.$$

Because at least $\left\lceil \frac{|S(i, s, d)|}{m} \right\rceil$ time units are required to complete $|S(i, s, d)|$

tasks, there exists a task $j \in S(i, s, d)$ such that

$$C_j(\sigma) \leq d - \left\lceil \frac{|S(i, s, d)|}{m} \right\rceil + 1.$$

Since for this task $C_j(\sigma) \leq s$, $r_j < s$, and by the definition of $S(i, s, d)$, $i \rightarrow j$. Hence, $D_i = C_i(\sigma) < s$, which contradicts (3). \square

We will say that schedule σ is active if there is no schedule σ' such that $C_j(\sigma') \leq C_j(\sigma)$ for all $j \in N$ and at least one of these inequalities is strict. In what follows, for any integer t , we will refer to the time interval $[t - 1, t]$ also as a time slot t .

Lemma 3 *If a schedule σ is active, then $C_j(\sigma) - r_j \leq n$ for all $j \in N$.*

Proof

Suppose that there exists a task j such that $C_j(\sigma) - r_j > n$. Then all m processors are idle in at least one time slot t satisfying the inequalities $r_j < t < C_j(\sigma)$. Because σ is active, this implies the existence of a task g such that $g \rightarrow j$ and $t < C_g(\sigma) < C_j(\sigma)$. Among all these tasks g select a task with the smallest completion time. Let it be task i . Since $i \rightarrow j$, $r_i < r_j < t$ and task i can be processed in the time slot t without changing completion times of all other tasks which contradicts the fact that σ is active. \square

The following lemma establishes upper and lower bounds on the optimal value of the criterion.

Lemma 4 *Let σ^* be an optimal schedule and Δ^* be the minimal Δ allowing Δ -modified due dates. Then*

$$\max_{i \in N} (r_i - d_i) < \Delta^* \leq L_{max}(\sigma^*) \leq n + \max_{i \in N} (r_i - d_i).$$

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10 **Proof**

11 If $\Delta = r_j - d_j$ for some $j \in N$, then Δ -modified due dates D_1, \dots, D_n do
12 not exist, because in this case
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$$14 \quad D_j \leq d_j + \Delta = r_j,$$

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17 which contradicts the inequalities $D_j \geq r_j + 1$. Hence,
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$$19 \quad \max_{i \in N} (r_i - d_i) < \Delta^*.$$

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22 Without loss of generality we assume that σ^* is active, because otherwise
23 σ^* can be replaced by an active schedule σ' such that $C_j(\sigma') \leq C_j(\sigma^*)$, for all
24 $j \in N$, and hence also optimal. Consider any task j satisfying the equality
25 $C_j(\sigma^*) - d_j = L_{max}(\sigma^*)$. Then using lemma 3 we have
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$$31 \quad L_{max}(\sigma^*) = C_j(\sigma^*) - r_j + r_j - d_j \leq n + r_j - d_j \leq n + \max_{i \in N} (r_i - d_i).$$

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34 To complete the proof, we observe that $C_j(\sigma^*) \leq d_j + L_{max}(\sigma^*)$, for all
35 $j \in N$, and therefore by lemma 2 the integers $C_1(\sigma^*), C_2(\sigma^*), \dots, C_n(\sigma^*)$
36 are $L_{max}(\sigma^*)$ -modified due dates. Hence, $\Delta^* \leq L_{max}(\sigma^*)$. \square
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41 2.3 GJ-algorithm for an arbitrary number of processors

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44 The algorithm considered in this paper is a straightforward generalization
45 of that presented in [4] for the two-processor case. Both algorithms use
46 as a subroutine the so called list algorithm (see for example [2]). The list
47 algorithm assumes that the tasks are arranged in a list \mathcal{L} and constructs an
48 active schedule $\sigma_{\mathcal{L}}$ iteratively from time $t = 0$, computing at each time t
49 the set F_t of feasible jobs at time t according to the partial schedule of the
50 interval $[0, t - 1]$, schedule all of them at time t if $|F_t| \leq m$, otherwise choose
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among F_t the m jobs with lowest index in \mathcal{L} and schedule them at t . t is then updated to the next time slot in which a job is feasible.

As in lemma 4, let Δ^* be the minimal Δ allowing Δ -modified due dates.

GJ-algorithm

1. Set $\Delta_L = \max_{j \in N}(r_j - d_j)$ and $\Delta_U = n + \max_{j \in N}(r_j - d_j)$. Using the binary search, compute Δ^* and Δ^* -modified due dates.
2. Construct a list schedule for a list where tasks are arranged in the non-decreasing order of there Δ^* -modified due dates.

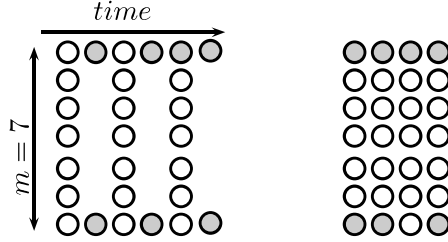
Figure 2 illustrates the schedule produced by GJ-algorithm applied to the graph of figure 1 on 7 processors, with $\Delta^* = 1$ in comparison to the optimal schedule.

Recall that, for a given Δ , the number of operations needed to compute Δ -modified due dates or to determine that these due dates do not exist is $O(n^3)$. The selection of Δ_U and Δ_L is justified by lemma 4. So, the calculation of Δ^* and the corresponding Δ^* -modified due dates can be accomplished in $O(n^3 \log_2 n)$ operations.

In what follows, σ^* will denote a schedule minimizing the maximum lateness, D_1, \dots, D_n will denote Δ^* -modified due dates calculated in accord with the first step of the GJ-algorithm, and $\bar{\sigma}$ will denote a schedule constructed by this algorithm.

3 Decomposition procedure

In order to analyze the performance of the GJ-schedule, we use a decomposition procedure which will build a sequence j_0, \dots, j_q of jobs and bounds on D_{j_k} that will be useful in the next sections.



GJ Schedule: $L_{max}(\bar{\sigma}) = 2$ Optimal schedule: $L_{max}(\sigma^*) = 1$

Figure 2: GJ and optimal schedule for the example of figure 1.

Decomposition of schedules is a classical tool to analyze their worst-case performance, see for example [2],[1] for makespan minimization analysis. However, with each algorithm a specific decomposition is needed to derive the performance bound.

For any integers t and D , denote by $\delta(t, D)$ the set of all tasks j such that $C_j(\bar{\sigma}) = t$ and $D_j \leq D$.

Lemma 5 For any task j and any integers D and t , satisfying

$$D_j \leq D, \quad r_j < t < C_q(\bar{\sigma}) \quad \text{and} \quad |\delta(t, D)| < m,$$

there exists a task $h \in \delta(t, D - 1)$ such that $h \rightarrow j$.

Proof

Since the schedule $\bar{\sigma}$ is constructed in accord with the list algorithm and since $|\delta(t, D)| < m$ and $r_j < t$, there exists a task i such that $C_i(\bar{\sigma}) \geq t$ and $i \rightarrow j$. Among all such tasks i select a task with the smallest completion time. Let it be task h . Lemma 1 implies that $D_h \leq D_j - 1 \leq D - 1$. Hence the lemma holds if $C_h(\bar{\sigma}) = t$. In order to prove this equality, assume that $C_h(\bar{\sigma}) > t$. The relation $h \rightarrow j$ implies $r_h < r_j$, and since the schedule $\bar{\sigma}$ was constructed in accord with the list algorithm, there exists a task f such

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10 that $C_f(\bar{\sigma}) \geq t$ and $f \rightarrow h$. Then by transitivity $f \rightarrow j$, which contradicts
11 the selection of h . \square
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13 Further exploring the structure of $\bar{\sigma}$, we observe that
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$$15 \max_{j \in N} [C_j(\bar{\sigma}) - D_j] \geq \max_{j \in N} [C_j(\bar{\sigma}) - (d_j + \Delta^*)] = L_{max}(\bar{\sigma}) - \Delta^*$$

16 and by lemma 4,
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$$18 \geq L_{max}(\bar{\sigma}) - L_{max}(\sigma^*) \geq 0.$$

19 Hence, if
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$$21 \max_{j \in N} [C_j(\bar{\sigma}) - D_j] = 0,$$

22 then $\bar{\sigma}$ is an optimal schedule for the original problem. Since our goal is the
23 worst-case analysis of the GJ-algorithm, we will assume that there exists a
24 task g such that
25

$$26 C_g(\bar{\sigma}) - D_g > 0. \tag{8}$$

27 The following procedure, which will be referred to as a decomposition
28 procedure, constructs for any task g , satisfying (8), a sequence of tasks
29 $j_0 = g, \dots, j_{l(g)}$ and the sequence of corresponding sets of tasks $M^0, \dots,$
30 $M^{l(g)}$. Suppose that the sequence j_0, \dots, j_i and the corresponding sequence
31 of sets M^0, \dots, M^{k-1} have been already constructed. Obviously, if $k = 0$,
32 no sets have been constructed yet. Let t be an integer such that $t < C_{j_k}(\bar{\sigma})$
33 and $|\delta(t, D_{j_k})| < m$. Observe that both inequalities hold for example for
34 $t = 0$. Among all such t select the largest one and denote it by τ . Then
35

$$36 M^k = \cup_{\tau < t \leq C_{j_k}(\bar{\sigma})} \delta(t, D_{j_k}).$$

37 If $r_h \geq \tau$ for all $h \in M^k$, then the procedure terminates with $l(g) = k$. If
38 $r_h < \tau$ for at least one $h \in M^k$, then according to lemma 5 $\delta(\tau, D_{j_k} - 1) \neq \emptyset$.
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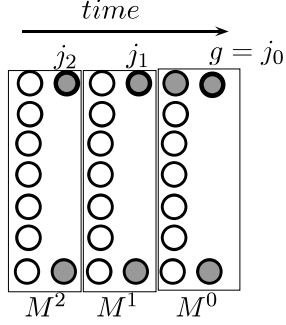


Figure 3: Decomposition of the GJ-schedule of figure 2.

If $|\delta(\tau, D_{j_k} - 1)| = 1$, then the procedure terminates with $l(g) = k$. In this case, the task constituting the set $\delta(\tau, D_{j_k} - 1)$ will be denoted by $a(g)$. If $|\delta(\tau, D_{j_k} - 1)| \geq 2$, then choose as j_{k+1} any task h satisfying

$$D_h = \max_{i \in \delta(\tau, D_{j_k} - 1)} D_i$$

and start a new iteration by constructing the set M^{k+1} .

Figure 3 illustrates the decomposition, in a case where $a(g)$ does not exist. The following lemmas express the properties of the decomposition, firstly by defining lower bounds on release times of tasks of subsets M^k (lemmas 8 and 7) and then a lower bound on the due-date of g (lemma 8).

Lemma 6 *If task g satisfies (8) and the decomposition procedure cannot determine $a(g)$, then*

$$\min_{i \in \bigcup_{k=0}^{l(g)} M^k} r_i = \min_{i \in M^{l(g)}} C_i(\bar{\sigma}) - 1.$$

Proof

Since $a(g)$ does not exist, according to the decomposition procedure

$$\min_{i \in M^{l(g)}} r_i = \min_{i \in M^{l(g)}} C_i(\bar{\sigma}) - 1. \quad (9)$$

Suppose that for some $k < l(g)$ there exists $j \in M^k$ such that

$$r_j < \min_{i \in M^{l(g)}} C_i(\bar{\sigma}) - 1.$$

Then by lemma 5 there exists task $h \in M^{l(g)}$ such that $h \rightarrow j$. Since $h \rightarrow j$ implies $r_h < r_j$,

$$r_h < \min_{i \in M^{l(g)}} C_i(\bar{\sigma}) - 1,$$

which contradicts (9). \square

Lemma 7 *Let g satisfy (8) and let, in the corresponding decomposition, $j \in M^k$, then*

$$r_j \geq \begin{cases} r_{a(g)} + l(g) - k + 1 & \text{if } a(g) \text{ exists} \\ l(g) - k & \text{if } a(g) \text{ does not exist} \end{cases} \quad (10)$$

Proof

Suppose that $a(g)$ exists and $k = l(g)$. If

$$r_j < \min_{i \in M^{l(g)}} C_i(\bar{\sigma}) - 1,$$

then according to the decomposition procedure and lemma 5, $a(g) \rightarrow j$.

Hence, $r_j \geq r_{a(g)} + 1$ and the lemma holds. If

$$r_j \geq \min_{i \in M^{l(g)}} C_i(\bar{\sigma}) - 1,$$

then the same result follows from the observation that

$$\min_{i \in M^{l(g)}} C_i(\bar{\sigma}) - 1 = C_{a(g)}(\bar{\sigma}) \geq r_{a(g)} + 1.$$

Since $\min_{u \in N} r_u = 0$, the lemma also holds if $a(g)$ does not exist.

Suppose that the lemma holds for any $k > w$ for some nonnegative integer $w < l(g)$. Let $k = w$, then by the assumption, for any $u \in$

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10 $\delta(C_{j_{k+1}}(\bar{\sigma}), D_{j_k} - 1)$

$$r_u \geq \begin{cases} r_{a(g)} + l(g) - (k+1) + 1 & \text{if } a(g) \text{ exists} \\ l(g) - (k+1) & \text{if } a(g) \text{ does not exist} \end{cases}$$

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16 If $r_j < C_{j_{k+1}}(\bar{\sigma})$, then by lemma 5 there exists $h \in \delta(C_{j_{k+1}}(\bar{\sigma}), D_{j_k} - 1)$ such
17 that $h \rightarrow j$ and therefore

$$r_j \geq C_h(\bar{\sigma}) \geq r_h + 1 \geq \begin{cases} r_{a(g)} + l(g) - k + 1 & \text{if } a(g) \text{ exists} \\ l(g) - k & \text{if } a(g) \text{ does not exist} \end{cases}$$

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24 If $r_j \geq C_{j_{k+1}}(\bar{\sigma})$, then (10) follows from $r_j \geq C_{j_{k+1}}(\bar{\sigma}) \geq r_{j_{k+1}} + 1$. Hence,
25 the lemma holds for any k . \square

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27
28 **Lemma 8** *If task g satisfies (8), then*

$$D_g \geq \begin{cases} D_{a(g)} + l(g) + 1 & \text{if } a(g) \text{ exists} \\ l(g) + 2 & \text{if } a(g) \text{ does not exist} \end{cases}$$

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34 **Proof**

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36 If $a(g)$ exists, then the inequality

$$D_g \geq D_{a(g)} + l(g) + 1$$

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41 follows from the fact that according to the decomposition procedure $D_{j_k} \geq$
42 $D_{j_{k+1}} + 1$ for all $0 \leq k \leq l(g)$.

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45 Suppose that $a(g)$ does not exist, then by lemma 7 and the condition (2)
46 of consistency, $D_g \geq l(g) + 1$. Suppose that the lemma does not hold, i.e.
47 suppose that $D_g = l(g) + 1$. Then, taking into account that according to the
48 decomposition procedure $D_{j_k} \geq D_{j_{k+1}} + 1$ for all $0 \leq k \leq l(g)$, $D_{j_{l(g)}} = 1$
49 and $r_{j_{l(g)}} = 0$. If $|M^k| \leq m$ for all $0 \leq k \leq l(g)$, then all tasks constituting
50 each M^k are processed in the same time slot, and therefore $C_g(\bar{\sigma}) = l(g) + 1$
51 which contradicts (8).
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10 Let k be the largest l among all l satisfying $|M^l| > m$. In order to show
11 that $k = l(g)$, assume that $k < l(g)$. Then, for each $k < l \leq l(g)$, all tasks
12 constituting M^l are processed in the same time slot $C_{j_l}(\bar{\sigma}) = l(g) - l + 1$.
13 Hence,
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$$15 \min_{h \in M^k} C_h(\bar{\sigma}) - 1 = C_{j_{k+1}}(\bar{\sigma}) = l(g) - k.$$

16 This by virtue of lemma 7 implies that for any $j \in M^k$
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$$19 r_j \geq \min_{h \in M^k} C_h(\bar{\sigma}) - 1,$$

20 which contradicts the decomposition procedure which should terminate after
21 the construction of set M^k . Hence, $k = l(g)$. Let $s = 0 = r_{j_{l(g)}}$ and
22 $d = 1 = D_{j_{l(g)}}$. Then, since $|\delta(C_{j_{l(g)}}(\bar{\sigma}), D_{j_{l(g)}})| \geq 2$,
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$$25 |S(j_{l(g)}, s, d)| \geq |M^{l(g)} - \{j_{l(g)}\}| > m(d - s)$$

26 which contradicts the fact that the Δ^* -modified due dates are consistent. \square
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33 4 Completion times in $\bar{\sigma}$ and Δ^* -modified due dates

34 In this section we derive bounds on C_g from the decomposition procedure
35 defined in the previous section (lemmas 10 and 11.)
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37 It is convenient to introduce the following notation:
38

$$39 \alpha(m) = \begin{cases} \frac{2}{m+1} & \text{if } m \text{ is odd} \\ \frac{2}{m} & \text{if } m \text{ is even} \end{cases}.$$

40 **Lemma 9** For any positive integer p
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$$43 \left\lceil \frac{2p-1}{m} \right\rceil \geq \alpha(m)p.$$

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10 **Proof**

11 Let m be even. Since $2p - 1$ is odd, $\frac{2p-1}{m}$ cannot be integer, and therefore

$$12 \quad \left\lceil \frac{2p-1}{m} \right\rceil = \left\lceil \frac{2p-1}{m} + \frac{1}{m} \right\rceil \geq \frac{2p}{m} = \alpha(m)p.$$

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16 If m is odd and $\frac{2p-1}{m} < 1$, then $\frac{2p-1}{m} + \frac{1}{m} < 1$, and therefore

$$17 \quad \left\lceil \frac{2p-1}{m} \right\rceil = \left\lceil \frac{2p-1}{m} + \frac{1}{m} \right\rceil > \frac{2p}{m} > \alpha(m)p.$$

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21 Finally, let m be odd and $\frac{2p-1}{m} \geq 1$, then $\frac{2p-1}{m} \geq \frac{2p}{m+1}$, and therefore

$$22 \quad \left\lceil \frac{2p-1}{m} \right\rceil \geq \left\lceil \frac{2p}{m+1} \right\rceil \geq \frac{2p}{m+1} = \alpha(m)p$$

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27 which completes the proof. \square

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30 **Lemma 10** *If task g satisfies (8) and $a(g)$ does not exist, then*

$$31 \quad C_g(\bar{\sigma}) \leq [1 - \alpha(m)][l(g) + 2] + D_g - [1 - \alpha(m)].$$

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35 **Proof**

36 From the construction of $M^{l(g)}$, there exists a task $h \in M^{l(g)}$ such that

$$37 \quad r_h = \min_{j \in M^{l(g)}} C_j(\bar{\sigma}) - 1.$$

38 Let $s = r_h$ and $d = D_g$, then by lemma 6 and the fact that $D_j \leq D_g$, for all
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42 $j \in \cup_{i=0}^{l(g)} M^i$,

$$43 \quad r_h = s < D_h \leq d \quad \text{and} \quad (\cup_{k=0}^{l(g)} M^k - \{h\}) \subseteq S(h, s, d).$$

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47 If $|S(h, s, d)| = m(d - s)$, then by the consistency of the Δ^* -modified
48 due dates $D_h = s$, which contradicts the inequality $s < D_h$. Therefore
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10 t , satisfying the inequalities $r_h < t < C_g(\bar{\sigma})$, contains exactly m tasks from
11 $\cup_{k=0}^{l(g)} M^k$. Consequently,

$$12 \quad m[C_g(\bar{\sigma}) - r_h - l(g) - 1] + 2l(g) \leq |S(h, s, d)| < m(d - s) = m(D_g - r_h).$$

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16 Hence

$$17 \quad C_g(\bar{\sigma}) < l(g) + 1 - \frac{2l(g)}{m} + D_g. \quad (11)$$

18 If m is even, then since $2l(g)$ is also even and (11) is a strict inequality,

$$19 \quad C_g(\bar{\sigma}) \leq l(g) + 1 - \frac{2l(g)}{m} + D_g - \frac{2}{m}.$$

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25 Consequently,

$$26 \quad C_g(\bar{\sigma}) \leq l(g) + 2 - \frac{2[l(g) + 2]}{m} + D_g - 1 + \frac{2}{m} = [1 - \alpha(m)][l(g) + 2] + D_g - [1 - \alpha(m)].$$

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31 If m is odd, then from (11)

$$32 \quad C_g(\bar{\sigma}) < l(g) + 1 - \frac{2l(g)}{m + 1} + D_g,$$

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36 and because both $2l(g)$ and $m + 1$ are even

$$37 \quad C_g(\bar{\sigma}) \leq l(g) + 1 - \frac{2l(g)}{m + 1} + D_g - \frac{2}{m + 1}.$$

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42 Hence

$$43 \quad C_g(\bar{\sigma}) \leq l(g) + 2 - \frac{2[l(g) + 2]}{m + 1} + D_g - 1 + \frac{2}{m + 1} = [1 - \alpha(m)][l(g) + 2]$$

$$44 \quad + D_g - [1 - \alpha(m)]$$

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49 which completes the proof. \square

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53 **Lemma 11** *If task g satisfies (8) and $a(g)$ exists, then*

$$54 \quad C_g(\bar{\sigma}) - C_{a(g)}(\bar{\sigma}) \leq [1 - \alpha(m)][l(g) + 1] + D_g - \min[C_{a(g)}(\bar{\sigma}), D_{a(g)}].$$

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10 **Proof**

11 According to the decomposition procedure $D_{j_i} \leq D_{j_{i-1}} - 1$ for all $1 \leq i \leq$
12 $l(g) + 1$. Adding all these inequalities, we have

$$13 \quad D_{a(g)} \leq D_{j_0} - l(g) - 1,$$

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17 which gives

$$18 \quad D_g - \min[C_{a(g)}(\bar{\sigma}), D_{a(g)}] \geq D_g - D_{a(g)} \geq l(g) + 1. \quad (12)$$

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21 Because the Δ^* -modified due dates are consistent, $r_{a(g)} < D_{a(g)}$. Let $d = D_g$
22 and $s = \min[C_{a(g)}(\bar{\sigma}), D_{a(g)}]$, then

$$23 \quad r_{a(g)} < s \leq D_{a(g)} < d.$$

24
25
26 By lemma 5 and the decomposition procedure, for any $j \in \cup_{k=0}^{l(g)} M^k$ ei-
27 ther $r_j \geq C_{a(g)} \geq s$ or $a(g) \rightarrow j$. Moreover, by the decomposition procedure,
28 $D_j \leq D_g = d$ for all $j \in \cup_{k=0}^{l(g)} M^k$. Hence

$$29 \quad \cup_{k=0}^{l(g)} M^k \subseteq S(a(g), s, d).$$

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31
32 On the other hand, each time slot $C_{j_k}(\bar{\sigma})$, where $1 \leq k \leq l(g)$, contains
33 at least two tasks from M^k , and any other time slot t , satisfying the in-
34 equalities $C_{a(g)}(\bar{\sigma}) < t < C_g(\bar{\sigma})$, contains exactly m tasks from $\cup_{k=0}^{l(g)} M^k$.
35 Consequently,

$$36 \quad |S(a(g), s, d)| \geq m[C_g(\bar{\sigma}) - C_{a(g)}(\bar{\sigma}) - l(g) - 1] + 2l(g) + 1. \quad (13)$$

37
38
39 If $|S(a(g), s, d)| < m(d - s)$, then taking into account that d and s are
40 integers, we have

$$41 \quad \left\lceil \frac{|S(a(g), s, d)|}{m} \right\rceil \leq d - s = D_g - \min[C_{a(g)}(\bar{\sigma}), D_{a(g)}].$$

If $|S(a(g), s, d)| = m(d - s)$, then

$$\min[C_{a(g)}(\bar{\sigma}), D_{a(g)}] = s = d - \left\lceil \frac{|S(a(g), s, d)|}{m} \right\rceil = D_g - \left\lceil \frac{|S(a(g), s, d)|}{m} \right\rceil.$$

Therefore in both cases

$$D_g - \min[C_{a(g)}(\bar{\sigma}), D_{a(g)}] \geq \left\lceil \frac{|S(a(g), s, d)|}{m} \right\rceil$$

and using (13) and lemma 9

$$\begin{aligned} &\geq C_g(\bar{\sigma}) - C_{a(g)}(\bar{\sigma}) - l(g) - 1 + \left\lceil \frac{2l(g) + 1}{m} \right\rceil = C_g(\bar{\sigma}) - C_{a(g)}(\bar{\sigma}) \\ &- l(g) - 1 + \left\lceil \frac{2[l(g) + 1] - 1}{m} \right\rceil \geq C_g(\bar{\sigma}) - C_{a(g)}(\bar{\sigma}) - [1 - \alpha(m)][l(g) + 1]. \end{aligned}$$

Hence

$$C_g(\bar{\sigma}) - C_{a(g)}(\bar{\sigma}) \leq [1 - \alpha(m)][l(g) + 1] + D_g - \min[C_{a(g)}(\bar{\sigma}), D_{a(g)}].$$

which completes the proof. \square

5 Performance guarantees

We now have enough material to prove the performance guarantee of GJ algorithm. In the last subsection, we show that the bounds are tight.

5.1 Worst case performance analysis

In order to use the lemma of the previous sections, we need a second decomposition procedure, which will produce a sequence of jobs g_0, \dots, g_k with interesting properties.

Suppose that $L_{max}(\bar{\sigma}) > L_{max}(\sigma^*)$. Let g_0 be any task satisfying the equality

$$C_{g_0}(\bar{\sigma}) - d_{g_0} = L_{max}(\bar{\sigma}). \quad (14)$$

By the definition of Δ^* -modified due dates and lemma 4

$$L_{max}(\bar{\sigma}) = C_{g_0}(\bar{\sigma}) - d_{g_0} \leq C_{g_0}(\bar{\sigma}) - (D_{g_0} - \Delta^*) \leq C_{g_0}(\bar{\sigma}) - D_{g_0} + L_{max}(\sigma^*),$$

which implies $C_{g_0}(\bar{\sigma}) - D_{g_0} > 0$. Using the decomposition procedure and starting with g_0 we can construct a sequence of tasks as follows:

- if $a(g_0)$ - given by the first decomposition of the previous section - does not exist, then this sequence contains only one task g_0 ;
- if $a(g_0)$ exists, then repeatedly applying the decomposition procedure, we get a sequence of tasks g_0, \dots, g_k such that $C_{g_i}(\bar{\sigma}) - D_{g_i} > 0$ and $g_{i+1} = a(g_i)$ for all $0 \leq i < k$, and either $C_{g_k}(\bar{\sigma}) - D_{g_k} \leq 0$ or $C_{g_k}(\bar{\sigma}) - D_{g_k} > 0$ but $a(g_k)$ does not exist.

Theorem 1 For any $m \geq 2$

$$L_{max}(\bar{\sigma}) - L_{max}(\sigma^*) \leq [1 - \alpha(m)][1 + \max_{j \in N} r_j]. \quad (15)$$

Proof

Suppose that $a(g_0)$ does not exist. Then taking into account lemma 10, lemma 7 and lemma 4,

$$\begin{aligned} C_{g_0}(\bar{\sigma}) &\leq [1 - \alpha(m)][l(g_0) + 1] + D_{g_0} \leq [1 - \alpha(m)][r_{g_0} + 1] + d_{g_0} + \Delta^* \\ &\leq [1 - \alpha(m)][1 + \max_{j \in N} r_j] + d_{g_0} + L_{max}(\sigma^*) \end{aligned}$$

which by virtue of (14) implies (15).

Suppose that $k \geq 1$, $C_{g_k}(\bar{\sigma}) - D_{g_k} > 0$ and $a(g_k)$ does not exist. Then by lemma 11 and lemma 7, for all $0 \leq i \leq k - 1$,

$$C_{g_i}(\bar{\sigma}) - C_{g_{i+1}}(\bar{\sigma}) \leq [1 - \alpha(m)][l(g_i) + 1] + D_{g_i} - \min[C_{g_{i+1}}(\bar{\sigma}), D_{g_{i+1}}]$$

$$= [1 - \alpha(m)][l(g_i) + 1] + D_{g_i} - D_{g_{i+1}} \leq [1 - \alpha(m)][r_{g_i} - r_{g_{i+1}}] + D_{g_i} - D_{g_{i+1}}.$$

Adding inequalities

$$C_{g_i}(\bar{\sigma}) - C_{g_{i+1}}(\bar{\sigma}) \leq [1 - \alpha(m)][r_{g_i} - r_{g_{i+1}}] + D_{g_i} - D_{g_{i+1}} \quad (16)$$

for all $0 \leq i \leq k - 1$, we have

$$C_{g_0}(\bar{\sigma}) - C_{g_k}(\bar{\sigma}) \leq [1 - \alpha(m)][r_{g_0} - r_{g_k}] + D_{g_0} - D_{g_k}. \quad (17)$$

On the other hand, by lemma 10 and lemma 7

$$C_{g_k}(\bar{\sigma}) \leq [1 - \alpha(m)][l(g_k) + 1] + D_{g_k} \leq [1 - \alpha(m)][r_{g_k} + 1] + D_{g_k}.$$

This together with (17) and lemma 4 gives

$$\begin{aligned} C_{g_0}(\bar{\sigma}) &\leq [1 - \alpha(m)][r_{g_0} + 1] + D_{g_0} \leq [1 - \alpha(m)][1 + \max_{j \in N} r_j] + d_{g_0} + \Delta^* \\ &\leq [1 - \alpha(m)][1 + \max_{j \in N} r_j] + d_{g_0} + L_{max}(\sigma^*) \end{aligned}$$

which implies (15).

Suppose that $k \geq 1$, $C_{g_k}(\bar{\sigma}) - D_{g_k} \leq 0$. If $k \geq 2$, then by adding inequalities (16) for all $0 \leq i \leq k - 2$, we obtain

$$C_{g_0}(\bar{\sigma}) - C_{g_{k-1}}(\bar{\sigma}) \leq [1 - \alpha(m)][r_{g_0} - r_{g_{k-1}}] + D_{g_0} - D_{g_{k-1}}. \quad (18)$$

It is easy to see that (18) also holds when $k = 1$. On the other hand, by lemma 11, the fact that $C_{g_k}(\bar{\sigma}) - D_{g_k} \leq 0$ and lemma 7,

$$\begin{aligned} C_{g_{k-1}}(\bar{\sigma}) - C_{g_k}(\bar{\sigma}) &\leq [1 - \alpha(m)][l(g_{k-1}) + 1] + D_{g_{k-1}} - \min[C_{g_k}(\bar{\sigma}), D_{g_k}] \\ &\leq [1 - \alpha(m)][r_{g_{k-1}} - r_{g_k}] + D_{g_{k-1}} - C_{g_k}(\bar{\sigma}) \end{aligned}$$

which together with (18) and lemma 4 gives

$$C_{g_0}(\bar{\sigma}) \leq [1 - \alpha(m)][r_{g_0} - r_{g_k}] + D_{g_0} \leq [1 - \alpha(m)][1 + \max_{j \in N} r_j] + d_{g_0} + \Delta^*$$

$$\leq [1 - \alpha(m)][1 + \max_{j \in N} r_j] + d_{g_0} + L_{max}(\sigma^*)$$

which implies (15). \square

Theorem 2 For any $m \geq 2$

$$L_{max}(\bar{\sigma}) \leq [2 - \alpha(m)]L_{max}(\sigma^*) + [1 - \alpha(m)] \max_{v \in N} d_v - [1 - \alpha(m)]. \quad (19)$$

Proof

Let j be an arbitrary task. Since $r_j \geq 0$ and therefore $C_j(\sigma^*) \geq 1$,

$$L_{max}(\sigma^*) \geq C_j(\sigma^*) - d_j \geq 1 - \max_{v \in N} d_v.$$

Hence, $L_{max}(\sigma^*) + \max_{v \in N} d_v \geq 1$, and by virtue of $1 - \alpha(m) \geq 0$

$$[1 - \alpha(m)][L_{max}(\sigma^*) + \max_{v \in N} d_v] - [1 - \alpha(m)] \geq 0.$$

Hence, if $L_{max}(\bar{\sigma}) = L_{max}(\sigma^*)$, then (19) holds.

Suppose that $L_{max}(\bar{\sigma}) > L_{max}(\sigma^*)$. Let g_0 be a task, satisfying the equality

$$C_{g_0}(\bar{\sigma}) - d_{g_0} = L_{max}(\bar{\sigma}),$$

and let g_0, \dots, g_k be the sequence of tasks build according to the second decomposition procedure, such that $C_{g_i}(\bar{\sigma}) - D_{g_i} > 0$ and $g_{i+1} = a(g_i)$ for all $0 \leq i < k$, and either $C_{g_k}(\bar{\sigma}) - D_{g_k} \leq 0$ or $C_{g_k}(\bar{\sigma}) - D_{g_k} > 0$ but $a(g_k)$ does not exist.

If $C_{g_k}(\bar{\sigma}) - D_{g_k} \leq 0$, then $k \geq 1$. For all $0 \leq i \leq k - 1$, by lemma 11 and lemma 8

$$\begin{aligned} C_{g_i}(\bar{\sigma}) - C_{g_{i+1}}(\bar{\sigma}) &\leq [1 - \alpha(m)][l(g_i) + 1] + D_{g_i} - \min[C_{g_{i+1}}(\bar{\sigma}), D_{g_{i+1}}] \\ &\leq [1 - \alpha(m)][D_{g_i} - D_{g_{i+1}}] + D_{g_i} - \min[C_{g_{i+1}}(\bar{\sigma}), D_{g_{i+1}}] \end{aligned}$$

$$\begin{aligned}
&= [2 - \alpha(m)]D_{g_i} - [1 - \alpha(m)]D_{g_{i+1}} - \min[C_{g_{i+1}}(\bar{\sigma}), D_{g_{i+1}}] \\
&\leq [2 - \alpha(m)]\{D_{g_i} - \min[C_{g_{i+1}}(\bar{\sigma}), D_{g_{i+1}}]\}.
\end{aligned}$$

Adding all these inequalities and taking into account that $C_{g_k}(\bar{\sigma}) \leq D_{g_k}$ and $C_{g_{i+1}}(\bar{\sigma}) > D_{g_{i+1}}$ if $i + 1 < k$, we have

$$C_{g_0}(\bar{\sigma}) - C_{g_k}(\bar{\sigma}) \leq [2 - \alpha(m)]\{D_{g_0} - C_{g_k}(\bar{\sigma})\},$$

and since $C_{g_k}(\bar{\sigma}) \geq 1$,

$$C_{g_0}(\bar{\sigma}) \leq [2 - \alpha(m)]D_{g_0} - [1 - \alpha(m)]. \quad (20)$$

Suppose that $C_{g_k}(\bar{\sigma}) - D_{g_k} > 0$, then $a(g_k)$ does not exist. By lemma 10 and lemma 8

$$C_{g_k}(\bar{\sigma}) \leq [2 - \alpha(m)]D_{g_k} - [1 - \alpha(m)]. \quad (21)$$

If $k = 0$, then (21) coincides with (20). Let $k \geq 1$. For all $0 \leq i \leq k - 1$, by lemma 11 and the fact that $C_{g_{i+1}}(\bar{\sigma}) > D_{g_{i+1}}$

$$C_{g_i}(\bar{\sigma}) - C_{g_{i+1}}(\bar{\sigma}) \leq [2 - \alpha(m)]\{D_{g_i} - D_{g_{i+1}}\},$$

Adding all these inequalities, we have

$$C_{g_0}(\bar{\sigma}) - C_{g_k}(\bar{\sigma}) \leq [2 - \alpha(m)]\{D_{g_0} - D_{g_k}\},$$

which together with (21) gives (20).

Using (20),

$$L_{max}(\bar{\sigma}) = C_{g_0}(\bar{\sigma}) - d_{g_0} \leq [2 - \alpha(m)]D_{g_0} - [1 - \alpha(m)] - d_{g_0}$$

and by the definition of Δ^* -modified due dates and lemma 4

$$\begin{aligned}
&\leq [2 - \alpha(m)](d_{g_0} + \Delta^*) - [1 - \alpha(m)] - d_{g_0} \\
&\leq [2 - \alpha(m)]L_{max}(\sigma^*) + [1 - \alpha(m)] \max_{j \in N} d_j - [1 - \alpha(m)]
\end{aligned}$$

which completes the proof. \square

5.2 Tightness

In order to show that, for any $m \geq 5$, (15) and (19) are asymptotically tight, we will consider graphs $G_{x,m}$, each comprising $m \cdot x \cdot u_m$ nodes which form $x \cdot u_m$ rows, where x is a positive integer and

$$u_m = \begin{cases} \frac{m+1}{2} & \text{if } m \text{ is odd} \\ \frac{m}{2} & \text{if } m \text{ is even} \end{cases} \quad (22)$$

The rows of nodes are numbered from 0 to $x \cdot u_m - 1$ (see figure 4), and

$$[\text{the number of nodes in row } i] = \begin{cases} m+2 & \text{if } i \bmod u_m \leq u_m - 3 \\ m+1 & \text{if } i \bmod u_m = u_m - 2 \\ 2 & \text{if } i \bmod u_m = u_m - 1 \end{cases}$$

Each graph $G_{x,m}$ represents a partially ordered set of tasks, where each node represents a task and the arcs represent precedence constraints. All tasks corresponding to row i have the same release time equals i and the same due-date equals $d_i = i + 2$. We will use the following notation:

- If $i \bmod u_m \leq u_m - 3$ then the $m + 2$ nodes, constituting row i , will be denoted by $a_i^1, a_i^2, b_i^1, \dots, b_i^m$.
- If $i \bmod u_m = u_m - 2$ then the $m + 1$ nodes of row i will be denoted by $a_i^1, b_i^1, \dots, b_i^m$.
- If $i \bmod u_m = u_m - 1$ then the two nodes, constituting row i , will be denoted by a_i^1 and a_i^2 .

In figure 4, nodes a_i^1 and a_i^2 are shaded. Only nodes a_i^1 and a_i^2 have successors (see figure 4):

- If $m \geq 7$ and $i \bmod u_m < u_m - 3$, then a_i^1 precedes $a_{i+1}^1, b_{i+1}^1, \dots, b_{i+1}^{u_m}$ and a_i^2 precedes $a_{i+1}^2, b_{i+1}^{u_m+1}, \dots, b_{i+1}^m$.

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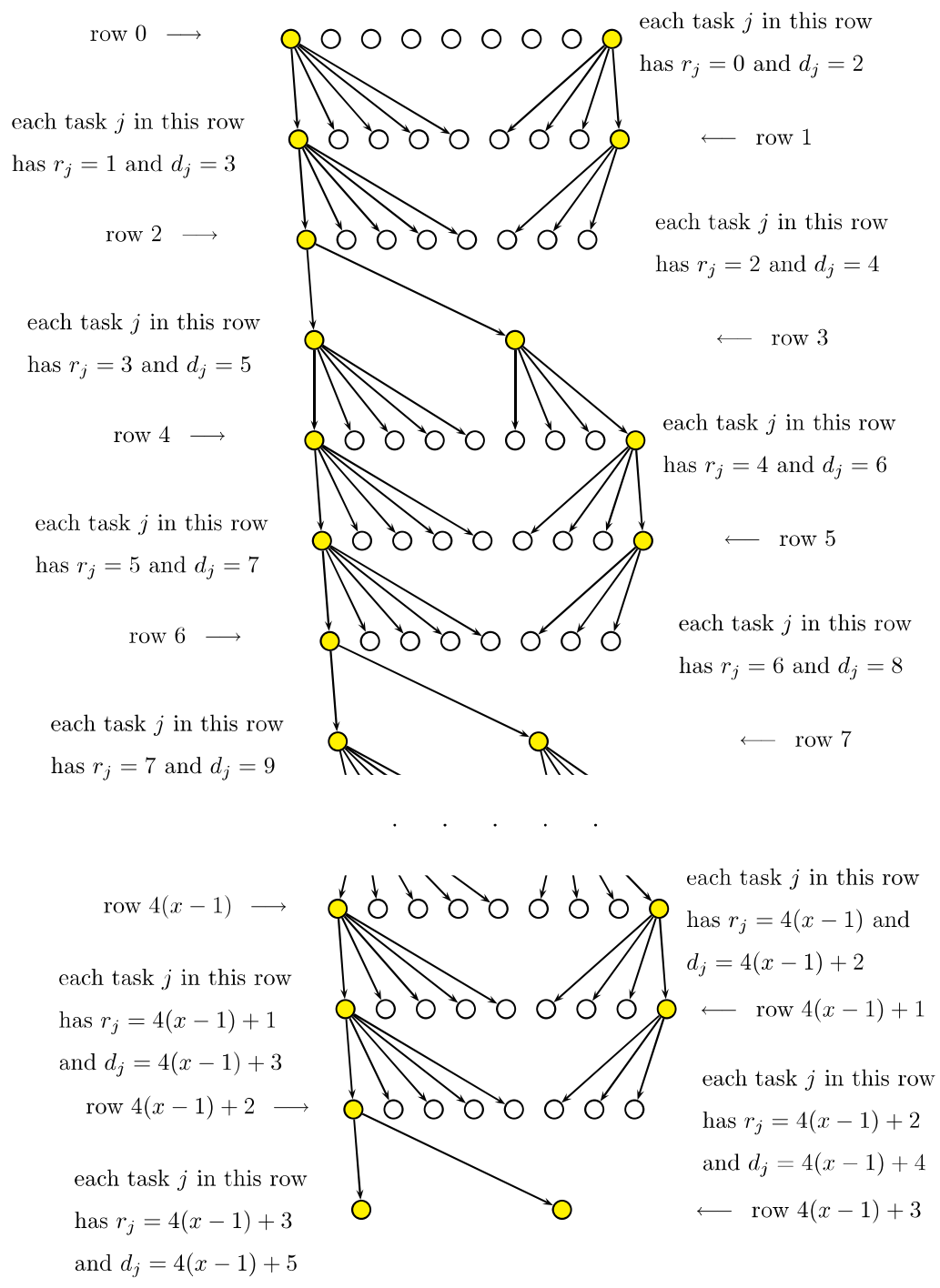


Figure 4: Graph $G_{x,m}$ for $m = 7$.

- If $i \bmod u_m = u_m - 3$, then a_i^1 precedes $a_{i+1}^1, b_{i+1}^1, \dots, b_{i+1}^{u_m}$ and a_i^2 precedes $b_{i+1}^{u_m+1}, \dots, b_{i+1}^m$.
- If $i \bmod u_m = u_m - 2$, then a_i^1 precedes a_{i+1}^1 and a_{i+1}^2 .
- If $i \bmod u_m = u_m - 1$ and $i < k \cdot u_m - 1$, then a_i^1 precedes $a_{i+1}^1, b_{i+1}^1, \dots, b_{i+1}^{u_m}$ and a_i^2 precedes $a_{i+1}^2, b_{i+1}^{u_m+1}, \dots, b_{i+1}^m$.

Although the graph presented in [13] and $G_{x,5}$ have the same structure, the example in [13] was developed for a different algorithm and the GJ-algorithm constructs for this example an optimal schedule. The distinct feature of the example presented in this paper is the assignment of release times and due dates in such a way that ensures the consistency of these due dates.

Lemma 12 *For any $m \geq 5$ and any x , the due dates corresponding to $G_{x,m}$ are consistent.*

Proof

In order to prove this lemma, we must study the sets $S(j, s, d)$. As it is long and tedious, please refer to the technical report [5] for a detailed proof. \square

For any $G_{x,m}$, consider a list where tasks are arranged in a nondecreasing order of due dates and for each i , where $i \bmod u_m \neq u_m - 2$, all b tasks of this row are listed before the a tasks of the same row. Let $\sigma_{x,m}$ be the corresponding list schedule and let $\sigma_{x,m}^*$ be an optimal schedule for the maximum lateness problem specified by the graph $G_{x,m}$. It is easy to see that

$$\max_{j \in N} d_j = u_m x + 1, \quad L_{max}(\sigma_{x,m}^*) = 0, \quad \max_{j \in N} r_j = u_m x - 1$$

and

$$L_{max}(\sigma_{x,m}) = (2u_m - 1)x - u_mx - 1 = u_mx - x - 1.$$

Indeed, GJ-Schedule as well as optimal schedule can be deduced from the example of figure 2 by concatenating schedules of u_m consecutive rows.

Taking into account that $\alpha(m) = \frac{1}{u_m}$,

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{L_{max}(\sigma_{x,m}) - L_{max}(\sigma_{x,m}^*)}{\max_{j \in N} r_j + 1} &= \lim_{x \rightarrow +\infty} \frac{u_mx - x - 1}{u_mx - 1 + 1} \\ &= \lim_{x \rightarrow +\infty} \frac{u_mx - u_m\alpha(m)x}{u_mx} = 1 - \alpha(m), \end{aligned}$$

and (15) is asymptotically tight. Analogously,

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{[2 - \alpha(m)]L_{max}(\sigma_{x,m}^*) + [1 - \alpha(m)]\max_{j \in N} d_j - [1 - \alpha(m)]}{L_{max}(\sigma_{x,m})} \\ &= \lim_{x \rightarrow +\infty} \frac{[1 - \alpha(m)](u_mx + 1) - [1 - \alpha(m)]}{u_mx - x - 2} \\ &= \lim_{x \rightarrow +\infty} \frac{[1 - \alpha(m)](u_mx + 1)}{u_mx - x} = \lim_{x \rightarrow +\infty} \frac{u_mx - x + [1 - \alpha(m)]}{u_mx - x} = 1, \end{aligned}$$

and (19) is asymptotically tight.

For $m = 3$ and $m = 4$ the proof of asymptotical tightness is similar to that for $m \geq 5$ and is based on the following graphs $H_{x,m}$. Each graph $H_{x,m}$ has $x \cdot u_m$ rows of nodes, numbered from 0 to $x \cdot u_m - 1$, where u_m is specified by (22) and

$$[\text{number of nodes in row } i] = \begin{cases} m + 1 & \text{if } i \bmod u_m = 0 \\ 2 & \text{if } i \bmod u_m = 1 \end{cases}$$

For arbitrary $H_{x,m}$, let $a_i^1, b_i^1, \dots, b_i^m$ be nodes constituting row i such that $i \bmod u_m = 0$, and let a_{i+1}^1 and a_{i+1}^2 be the only nodes of row i such that $i \bmod u_m = 1$. For any row i ,

- if $i \bmod u_m = 0$, then a_i^1 precedes both a_{i+1}^1 and a_{i+1}^2 ;
- if $i \bmod m = 1$ and $i \leq u_m x - 3$, then a_i^1 precedes $a_{i+1}^1, b_{i+1}^1, \dots, b_{i+1}^{u_m}$ and a_i^2 precedes $b_{i+1}^{u_m+1}, \dots, b_{i+1}^m$.

All tasks corresponding to row i have the same release time equals i and the same due-date equals $d_i = i + 2$.

6 Conclusion

In this paper we generalized the Garey-Johnson algorithm, originally designed to solve optimally $P2|prec, p_j = 1, r_j|L_{max}$ to find solutions of the problem $P|prec, p_j = 1, r_j|L_{max}$. We analysed its worst case behavior and provided a tight bound on the performance ratio which equals the best known ratio for this problem for even number of processors, and is slightly worse for odd number of processors. The ideas behind Garey-Johnson algorithm are commonly used to solve or find bounds for very different scheduling problems like RCPSP, with so called energy bounds, preemptive scheduling, scheduling with communication delays, scheduling on uniform processors. The structure of the analysis presented in this paper could probably be used in these cases for algorithms based on the same ideas and to derive tight bounds on their worst case performance ratio.

Preemption deserves a particular attention since not much tight bounds have been proven for the worst case performance of scheduling algorithms when preemption is allowed, and since Garey and Johnson derived an algorithm that solves optimally $P2|prec, pmtn|L_{max}$. A first insight of such a study is given in [6].

Existing approximation algorithm for $P|prec, p_j = 1, r_i|L_{max}$ all have a

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10 worst case ratio that tends to 2 when the number of processors grows to
11 infinity. Hence it is a real challenge to design a polynomial algorithm that
12 passes under this threshold. We proved that Garey-Johnson algorithm is
13 not the one, however, its analysis could help the design of new algorithms.
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16 17 18 **References** 19

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