



Geometric Methods for the Study of Electrical  
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# I) Introduction

## Introduction

The methods developed by Gabriel Kron, for the tensor approach of networks (TAN), offers a great source of applications for the **geometrical and topological study of electrical networks**. The language of **tensorial analysis** is well adapted to the description of networks. Tools about **discrete combinatorial topology** are well adapted specifically to the study of graphs, for example, the **Euler-Poincaré characteristic**, the simplest invariant in topology is connected to a **formula obtained by Kron** connecting the nodes of the graphs, edges, mesh currents and node pairs. Extensions using tools of algebraic topology are possible in larger dimension. We thus find the node law and the law of mesh. The addition of differential geometry is used to connect discrete data obtained from circuit and continuous phenomena for example, transmission problems via antenna.

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## Keywords

Kron's formalism, tensorial analysis of networks, relativity.

## II) Definition of a tensor by Elie Cartan

### Cartan Definition

"Called tensor, a number system, analytically defining geometric (or physical) object with the property : by a change of cartesian coordinates, tensor components **change by linear transformations**. In addition the **coefficients** do not **depend** on the numerical values of these components but only on the two coordinate systems (and also **on the nature of the tensor**)

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### Example

If we apply this to the methodology proposed by Kron , and we look at the **tensor impedances**, for example **in the space of the edges**, a coordinate transformation to **express this tensor in the space of the mesh**, does **not transform the tensor**.

### III) N dimensional manifolds and Kron philosophy

#### Submanifold

A **submanifold** is the generalization of our familiar ideas on curves and surfaces **immersed in an ambient space** to objects of any size. Thus, a curve in Euclidean three-dimensional space is parameterized locally by one number, generally over time  $(x(t), y(t), z(t))$ . In the case of **two parameters**, e.g. the time and speed, it is possible to describe a **surface**.

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#### Manifolds, Configuration space

We can introduce a more general notion, the notion of **topological or differentiable manifold that makes no reference to an embedding** into ambient space. The notion of tensor is transportable to the formalism of differential geometry. A **configuration space**  $\mathcal{C}$  is a space whose points are the different states associated with an object during its evolution : an **ordinary rigid body** in Euclidean space, has a **configuration space 6 dimensional** : three for the position of the body and three for his direction ; An **electromagnetic system coupled with three meshes**, is a space of **three-dimensional configuration** : three intensities depending time.

### III) N dimensional manifolds and Kron philosophy

#### Topology

Why this configuration space  $\mathcal{C}$  is different from the continuous euclidian space  $\mathbb{R}^n$ , each "R" describing one degrees of freedom? : In the  $\mathbb{R}^n$  space, one **can pass** from any point to another, **continuous manner by translation**. Another way characterize the vector space  $\mathbb{R}^n$  is noted that any loop can be reduced to a point by continuous deformation : its **first homotopy group** (fundamental group) **is trivial**. However, a **more general configuration space**  $\mathcal{C}$  can has the **presence of holes**. these holes prevent an loop surrounding them, to be reduced to a point : it is said that this space at a **nontrivial homotopy**.

### III) N dimensional manifolds and Kron philosophy

*n*-manifold associated to a network

A **network**, and its cellular description, serves to **determine the vector of currents** in a selected space cell  $C$  and deduce the associated configuration in the form of a current vector. Therefore **how represent our manifold  $M$** ? we **can imagine** it as the **gluing** of different **local coordinate systems** (or "charts"), each chart being an open region. In this procedure will occur, changing cards that will connect the coordinates of a first system based on those a second in  $\mathbb{R}^n$ .



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#### Application

From the **viewpoint of Kron**, this arrangement is made in **electrical machines**. Coupling between coils? Specifically, the passage of the current expressions for angular position to the other is then, if we **equip associated manifold** with a **Riemannian structure** via Christoffel coefficients. We detail these points later. But we can already say that the justification of the tensor character of the analysis proposed by Kron based fundamentally on the specific behavior of magnetic couplings in electric machines.

## IV) The electric Kron machine, tensor according to Lynn

### Tensors machines

Here we develop the arguments presented by **Lynn** in his book "Tensors in electrical engineering : The equation of an electric machine is developed here as a generalization the **matrix equation** attached to a **set of fixed coils**. . The equations of a machine best simple, are then :

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$$v_e = R_{ea}i^a + \frac{d}{dt}(L_{ea}i^a) \quad (1)$$
$$\frac{d}{dt}(L_{ea}i^a) = L_{ea} \frac{di^a}{dt} + i^a \frac{\partial L_{ea}}{\partial \theta} \frac{d\theta}{dt}$$

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$$v_e = R_{ea}i^a + L_{ea} \frac{\delta i^a}{\delta t} \quad (2)$$

## IV) The electric Kron machine, tensor according to Lynn covariant derivative, riemannian connection

One vector,  $\vec{A}$  can be expressed in terms of its contravariant coordinates :  
 $\vec{A} = A^q \vec{a}_q$ . The **differential of the vector**  $\vec{A}$  is then :

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$$\frac{\partial \vec{A}}{\partial x^j} = \frac{\partial A^i}{\partial x^j} \vec{a}_i + A^i \frac{\partial \vec{a}_j}{\partial x^j} \vec{a}_i \quad (4)$$

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and with metric  $g_{ij} = \vec{a}_i \cdot \vec{a}_j$  :

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial a^i}{\partial x^k} \vec{a}_j + \frac{\partial \vec{a}_j}{\partial x^k} \vec{a}_i \quad (5)$$



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Metric compatibility

We have metric compatibility :

$$[ij, k] = \frac{1}{2} \left( \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \quad (7)$$

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### Variational interpretation

By introducing the **Lagrangian of a free particle** moving in a Riemannian space :  $T = \frac{1}{2}mv^2$ , noting,  $l = g_{ab}dx^a dx^b$ , If we **add a dissipative energy** (friction, electrical resistivity) and solving the **Euler-Lagrange equation** :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^c}\right) - \frac{\partial T}{\partial x^c} = f_c \quad (8)$$

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we find :

$$f_c = \frac{1}{2}m\left(\frac{\partial g_{cb}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}\right)\dot{x}^a \dot{x}^b + mg_{cb}\ddot{x}^b \quad (9)$$

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### Application to Electric Machines

**Electrokinetic analogy Kron machines** gives, by replacing the speed by the strength of the current, the Lagrangian can be written :

$$T = \frac{1}{2} L_{\mu\nu} i^\mu i^\nu \quad (10)$$

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The equation of the **electric machine** is then given immediately from (9) :

$$v_c = R_{ca} i^a + L_{ca} \frac{di^a}{dt} + [ab, c] i^a i^b \quad (11)$$

# Topology : Euler Characteristic

## Contents of a topological graph :

Topologically, a graph is a fairly simple structure, it can be planar or not, and it is composed of "Lego" very simple basis, **trees** and **cycles** (or circuits), as it is oriented or not, the trees virtue to retract one point, a non-cycle. A suitable invariant, is the Euler characteristic Poincaré

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## Euler Characteristic

We can distinguish different kind of graph with a coarse invariant : the **Euler Poincaré** characteristic that counts the number of vertices minus the number of edges.



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## Euler Characteristic

We can distinguish different kind of graph with a coarse invariant : the **Euler Poincaré** characteristic that counts the number of vertices minus the number of edges.

## Segmentation

If two graphs are **homeomorphic**, or one is **retract** from each other, they have **the same** Euler characteristic

# Invariant for a graph

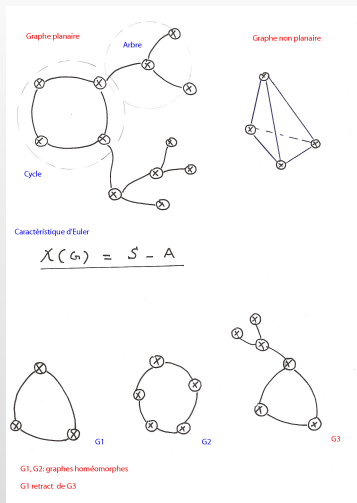


FIGURE:

# Euler Characteristic from Kron Point of view

## Some topological extensions of tensor analysis on network

Basically, Kron, on **electrical circuits**, proposes an extension of the tensor calculus. He adds some concepts of **combinatorial topology**. A circuit, often forming a graph, planar in the simplest case, There is an adaptation of the famous Euler Poincaré formula. In the case of cellular complex  $\mathcal{C}$  of two dimensions, this relation well know give :

$$\chi(\mathcal{C}) = \#face - \#edges + \#vertices \quad (12)$$

This quantity is a very crude invariant that differentiates some topological surfaces. In the case of electrical networks, Kron proposed the following formula that will reflect the **topology of the electrical circuits** (Kron relation) :

$$\begin{aligned} M &= B - N + R \\ P &= N - R \end{aligned} \quad (13)$$

# Euler Characteristic from Kron Point of view

## example

For example in the following figure, we have :

Four physical nodes :  $n_1 \dots n_4$  :  $N = 4$

Five branches :  $b_1 \dots b_5$  :  $B = 5$

Three meshes :  $m_1 \dots m_3$  :  $M = 3$

Two networks  $R_1, R_2$  :  $R = 2$

Two nodes pair :  $P = 2$

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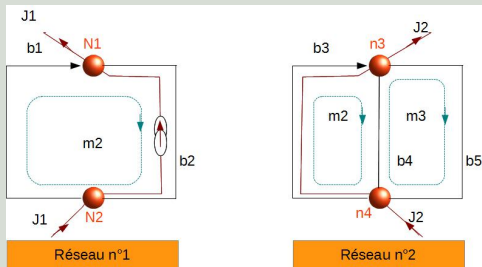


FIGURE: Simple example

# Euler Characteristic from Kron Point of view

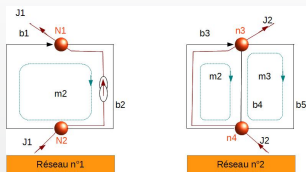


FIGURE: 6 Elements of the chosen topology

## Topology choice for the first network

we choosing arbitrarily on our first network, the node 1 as an initial reference, we depart of this Node worm node 2, we have an return of Node 2 to Node 1. We construct by this return , the first couple P1 who will wear the current source  $J1$ , and will be in final, our current injected in the first network coming from the second network.we verify the relationship for node pair :

$$P = N - R = 4 - 2 = 2$$

and meshes :

$$M = B - N + R = 5 - 4 + 2 = 3.$$

# Connectivity : toward algebraic topology

## k-connectivity

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## Complete graph

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## Homotopy

Poincaré sets, new invariants thanks to its **Homotopy theory** is based on the concept of related and adapts to topological spaces more general

# Algebraic topology

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Algebraic Topology **refines** the search for invariants and rests on two pillars of homotopy, homology. Its formalization, gives homological algebra homotopy

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## homotopy

Poincaré invented the **fundamental group** and the groups of higher homotopies, this allows to transpose at the continuous topological spaces notion of  $k$ -connectedness "discrete" graphs.

## homology

Poincaré invented at the same time that detects homology to the **Default** a cycle to be the boundary of a domain.

# Homotopy groups

## n-loop

one loop is a map :  $I/\partial I \simeq S^1 \rightarrow X$  de simily une  $n$  - loop is a map :  $I^n/\partial I^n \simeq S^n \rightarrow X$  and define  $\pi_n(X)$  n-th higher homotopy group and test  **$n$ -connectivity**

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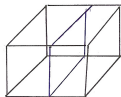
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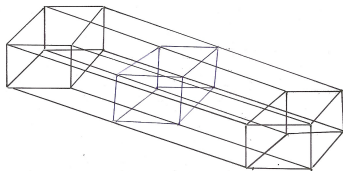
## Long exact sequence of a fibration

Given a fibration  $E \rightarrow X$  and  $F$  the fiber type that has the long sequence  
 $\rightarrow \pi_n(F, f_0) \rightarrow \pi_n(E, f_0) \rightarrow \pi_n(X, x_0) \xrightarrow{\partial} \pi_{n-1}(F, f_0) \dots \rightarrow \pi_0(E, f_0) \rightarrow 0$

# Spheres connectivity



Pour disconnecter une 2-sphère: retrancher une 1-sphère (cercle)



Pour disconnecter une 3-sphère: retrancher une 2-sphère

FIGURE:



# Singular homology

## Simplex, singular simplex

Name  $r$ -simplexe de  $\mathbb{R}^n$  the set denoted  $\sigma_r$  et défini par :

$$\sigma_r = \{x \in \mathbb{R}^n / x = \sum_{i=0}^r c_i p_i, c_i \geq 0, \sum_{i=0}^r c_i = 1\}$$

It is assumed that **the simplexes are oriented**. Define singular  $p$ -simplex a map  $\sigma$  from standard simplex of dimension  $p$  to the topological space  $X$  :

$$\sigma : \Delta_p \rightarrow X$$

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## Singular complex

we can then define the group of  $p$ -chains, and an boundary operator on a simplex :  $\partial$  with  $\partial \circ \partial = 0$  we obtain a **singular** complex :

$$\dots \longrightarrow \mathcal{C}_n(X) \xrightarrow{\partial_n} \mathcal{C}_{n-1}(X) \xrightarrow{\partial_{n-1}} \mathcal{C}_{n-2}(X) \dots \xrightarrow{\partial_1} \mathcal{C}_0(X) \longrightarrow 0$$

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## Singular homology

The  $p$  - *th* singular homology group is given by :  $H_p(X) = Z_p(X)/B_p(X)$ .  
Read default for a cycle to be an boundary

# The homology and Euler characteristic Poincaré

## Simplex, Simplex singular

Taking the real coefficients, **homology groups** : becomes **vector spaces**  
so :

$$\dim(\mathcal{C}_i(X)) = \dim(\text{Ker}\partial_i) + \dim(\text{Im}\partial_i)$$

$$\beta_i = \dim(H_i) = \dim(\text{ker}\partial_i) - \dim(\text{Im}\partial_{i+1}) \text{ (Betti numbers)}$$

The dimension of the vector space  $\mathcal{C}_i(X)$  is equal, according to  $i$  independent in the triangulation of  $X$  :

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$$\dim(\mathcal{C}_i(X)) = \dim(\text{Ker } \partial_i) + \dim(\text{Im } \partial_i)$$

$$\beta_i = \dim(H_i) = \dim(\text{ker } \partial_i) - \dim(\text{Im } \partial_{i+1}) \text{ (Betti numbers)}$$

The dimension of the vector space  $\mathcal{C}_i(X)$  is equal, according to  $i$  independent in the triangulation of  $X$  :

## Alternating sum of the Betti numbers

we deduce (case of a surface  $X$ ) :

$$\beta_0 - \beta_1 + \beta_2 = \dim(\mathcal{C}_0(X)) - \dim(\mathcal{C}_1(X)) + \dim(\mathcal{C}_2(X)) \text{ soit :}$$

$$\beta_0 - \beta_1 + \beta_2 = S - A + F = \chi(X)$$

# The homology and Euler characteristic Poincaré

## Simplex, Simplex singular

Taking the real coefficients, **homology groups** : becomes **vector spaces**  
so :

$$\dim(\mathcal{C}_i(X)) = \dim(\text{Ker } \partial_i) + \dim(\text{Im } \partial_i)$$

$$\beta_i = \dim(H_i) = \dim(\text{ker } \partial_i) - \dim(\text{Im } \partial_{i+1}) \text{ (Betti numbers)}$$

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## Finer invariants

It follows that the Betti numbers represented **less coarse invariants**.

## V-4) Tensorial analysis of network and algebraic topology

### Electrical networks and homology

We can define a complex chain directly **suited** to the study of **electrical networks**. We can define the **dual spaces associated**. As in algebraic topology, the definition of homology and cohomology spaces is possible. A re-interpretation of Poincaré duality, allows to recover Kirchhoff's laws.

# Geometrical and algebraic introduction

Cellular complex  $\mathcal{T}$  with geometrical objects :

vertices  $s \in \mathcal{T}^0$

edges  $a \in \mathcal{T}^1$

faces  $f \in \mathcal{T}^2$

volums  $k \in \mathcal{T}^3$

*etc!*

Spaces of **chains** : formal vector space  $\mathcal{T}_j$   
generated by the previous components :



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*etc!*

Spaces of **chains** : formal vector space  $\mathcal{T}_j$   
generated by the previous components :

$j = 0$  : vertices basis  $|s\rangle \in \mathcal{T}_0$

$j = 1$  : edges basis  $|a\rangle \in \mathcal{T}_1$

$j = 2$  : faces basis  $|f\rangle \in \mathcal{T}_2$

$j = 3$  : volums basis  $|k\rangle \in \mathcal{T}_3$

$$\mathcal{T}_j = \sum_{\sigma \in \mathcal{T}^j} \alpha_\sigma |\sigma\rangle$$

## Geometrical and algebraic introduction (ii)

We distinguish

the set  $\mathcal{T}^j$  of geometrical objects of dimension  $j$   
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So we introduce space of **chaînes**  $\mathcal{T} = \bigoplus_{j \in \mathbb{N}} \mathcal{T}_j$

for applications, space  $\mathcal{T}$  is finite dimension

The boundary **boundary** operator  $\partial$

linear

correspond at geometrical intuition

Boundary of an edge is a set of two vertices

Boundary of triangular face is a set of three edges

*etc.*

## Geometrical and algebraic introduction (iii)

We have

$$\begin{aligned}\partial_{1/2} : \mathcal{T}_0 &\longrightarrow \{0\} \\ \partial_{j+1/2} : \mathcal{T}_{j+1} &\longrightarrow \mathcal{T}_j \quad \text{if } j \geq 1\end{aligned}$$

thus by linearity  $\partial : \mathcal{T} \longrightarrow \mathcal{T}$

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Propriété Fundamental property :

The boundary of the boundary is reduced to zero :  $\partial \circ \partial = 0$

which comes to write  $\text{im } \partial_{j+1/2} \subset \text{ker } \partial_{j-1/2}$

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Definition

$j$  i-th homology space :  $H_j \equiv \text{ker } \partial_{j-1/2} / \text{im } \partial_{j+1/2}$

## Geometrical and algebraic introduction (iv)

Duality : space of co-chain of degree  $j$

$\mathcal{T}_j^*$  : dual of space  $\mathcal{T}_j$  of chains of degree  $j$  :

So  $\langle \sigma, s \rangle$  is well define for  $s \in \mathcal{T}_0$  and  $\sigma \in \mathcal{T}_0^*$

as well  $\langle \alpha, a \rangle$  with  $a \in \mathcal{T}_1$  and  $\alpha \in \mathcal{T}_1^*$

and for  $\langle \Phi, f \rangle$  with  $f \in \mathcal{T}_1$  and  $\Phi \in \mathcal{T}_1^*$

Cobord

polar operateur  $\partial^\circ$  of the boundary operator  $\partial$

By definition, for  $\varphi \in \mathcal{T}_j^*$  et  $g \in \mathcal{T}_{j+1}$

$$\langle \partial^\circ \varphi, g \rangle \equiv \langle \varphi, \partial g \rangle$$

we obtain

$$\partial^\circ : \mathcal{T}_j^* \longrightarrow \mathcal{T}_{j+1}^*$$

The boundary  $\partial$  make **reduce** the dimension

while the coboundary  $\partial^\circ$  make it **grow** !!



# Geometrical and algebraic introduction (v)

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# Writing Kirchhoff's laws

The electrical current  $I$  is defined by the edges

$$I \in \mathcal{T}_1$$

The electrical potential  $V$  acts on the vertices

$$V \in \mathcal{T}_0^*$$

of cellular complex

# Writing Kirchhoff's laws

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of cellular complex

The electrical potential  $V$  acts on the vertices

$$V \in \mathcal{T}_0^*$$

The duality product  $\langle V, \partial I \rangle$  therefore makes sense.

By assumption of Kirchhoff's laws, this expression is null :

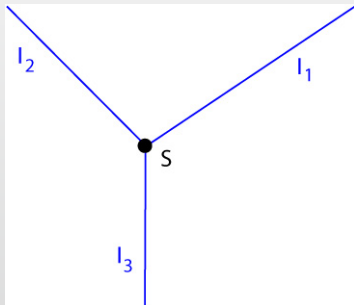
$$\langle V, \partial I \rangle = 0.$$

## Writing Kirchhoff's laws (ii)

This condition is expressed by taking  $V$  arbitrary :  
one expresses thus the **node law** :  
the **sum** (algebraic!) **of currents**  
which lead to a vertex given **is null**.

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which lead to a vertex given **is null**.



$$I_1 + I_2 + I_3 = 0$$

## Writing Kirchhoff's laws (iii)

Duality of the node law :

$$\langle \partial^0 V, I \rangle = 0.$$

We calculate this coboundary , with la convention :

$$\partial |a\rangle = \sum_{s \in \mathcal{T}^0} \partial_{s a} |s\rangle$$

Thus after an elementary calculation, for  $\sigma \in \mathcal{T}^0$  :

$$\partial^0 \langle \sigma^* | \equiv \sum_{b \in \mathcal{T}^1} \partial_{\sigma b} \langle b^* |,$$

and adding a \* means the passage to the dual basis.

Alors pour  $V = \sum_{s \in \mathcal{T}^0} V_s \langle s^* |$  arbitrary, we have

$$\partial^0 V = \sum_{a \in \mathcal{T}^1} \left( \sum_{s \in \mathcal{T}^0} \partial_{s a} V_s \right) \langle a^* |$$

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$$\partial^0 V = \sum_{a \in \mathcal{T}^1} \left( \sum_{s \in \mathcal{T}^0} \partial_{s a} V_s \right) \langle a^* |$$

which allows to introduce the **potential difference** :

$$U_a \equiv \sum_{s \in \mathcal{T}^0} \partial_{s a} V_s, \quad a \in \mathcal{T}^1$$

## Writing Kirchhoff's laws (iv)

So we have  $\partial^0 V = \sum_{a \in \mathcal{T}^1} U_a \langle a^* |$

$$\text{with } U_a \equiv \sum_{s \in \mathcal{T}^0} \partial_{s,a} V_s \quad \text{for } a \in \mathcal{T}^1$$

We introduce a closed circuit  $\gamma$ .

We test the relation  $\langle \partial^0 V, I \rangle = 0$  for  $I = I_0 \sum_{a \in \gamma} |a\rangle$

$$\text{Thus } \langle \partial^0 V, I \rangle = I_0 \sum_{a \in \mathcal{T}^1} \sum_{b \in \gamma} U_a \langle a^* | b \rangle = I_0 \sum_{a \in \mathcal{T}^1} U_a$$



## Writing Kirchhoff's laws (iv)

So we have  $\partial^0 V = \sum_{a \in \mathcal{T}^1} U_a \langle a^* |$

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We introduce a closed circuit  $\gamma$ .

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expression of the **mesh law** :  $\sum_{a \in \gamma} U_a = 0$

the sum of the potential differences  
along a closed circuit is zero.

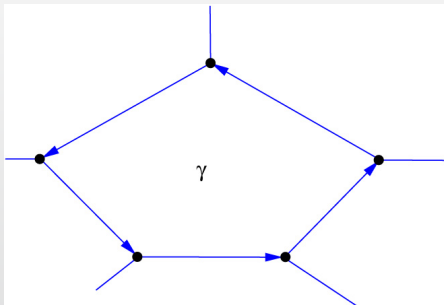
# Writing Kirchhoff's laws (v)

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# V) Some applications of tensor calculus to electrical machines

## two mesh interaction

We consider two circuits made each of one resistor and one capacitor. The network is shown fig.1.

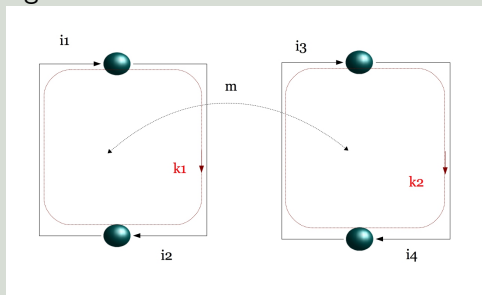


FIGURE: Simple example

## V-1) Two mesh interaction

We can define the metric **in the edge space**, i.e. the impedance functions of each edge (we mean by impedance the generalized operator giving the relation between a current in an edge and the voltage dropped across). Seeing this metric, the nature of each edge appears clearly. Edges are numbered from 1 to 4, the edge space dimension is 4. For example we describe the impedance tensor (metric) as :

$$z_{ab} = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & \frac{1}{C_2 \rho} & 0 & 0 \\ 0 & 0 & \frac{1}{C_3 \rho} & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} \quad (14)$$

$\rho$  is the Laplace's operator. For the moment, we don't have cross talked between the edges.

## V-1) Two mesh interaction

It could be possible, some functions may have been added to translate interactions between them. Making the bilinear transformation we obtain :

$$z_{\mu\nu} = L_{\mu}^a z_{ab} L_{\nu}^b = \begin{bmatrix} R_1 + \frac{1}{C_{2p}} & 0 \\ 0 & \frac{1}{C_{3p}} + R_4 \end{bmatrix} \quad (15)$$

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Now we can add some inductances values associated with each circuit, coming from their loops :

$$z_{\mu\nu} = L_{\mu}^a z_{ab} L_{\nu}^b = \begin{bmatrix} R_1 + \frac{1}{C_{2p}} + L_1 p & 0 \\ 0 & \frac{1}{C_{3p}} + R_4 + L_2 p \end{bmatrix} \quad (16)$$



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And finally, add the interaction through mutual inductance  $m$  between the two loops (we don't care here of the exact formulation of these elements. Our study is abstract and we theorize the problem) :

$$z_{\mu\nu} = L_{\mu}^a z_{ab} L_{\nu}^b = \begin{bmatrix} R_1 + \frac{1}{C_{2p}} + L_1 p & -m_{12} p \\ -m_{21} p & \frac{1}{C_{3p}} + R_4 + L_2 p \end{bmatrix} \quad (17)$$

# V-1) Two mesh interaction

## two mesh interaction

We consider two circuits made each of one resistor and one capacitor. The network is shown fig.1.

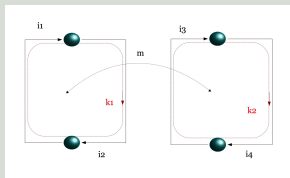


FIGURE: Simple example

## V-1) Two mesh interaction

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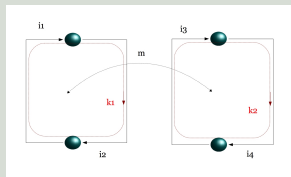


FIGURE: Simple example

We see that the various steps in the problem construction follow the natural minding of an engineer. That's a very interesting side of the approach. Another fact is **that the mesh space** dimension is here only 2, two times lower than the edge one.

## V-2) Lorentz mesh

### Lorentz Mesh and special relativity

Relativistic transformation of a cell is studied. We place ourselves in the approximation  $qv = 1$ , either by confusing current and electrical pulse. Consider four branches as shown following picture :

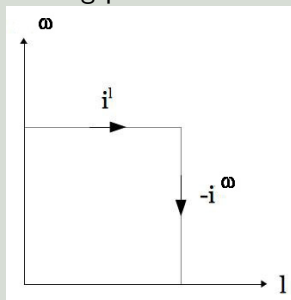


FIGURE: Lorentz mesh

## V-2) Lorentz mesh

### Lorentz Mesh and special relativity

It is assumed that the **speeds are not negligible** compared to the speed of light : The four branches of the square mesh are numbered from 1 to 4  
The first horizontal leg (at the top) is written as if the mesh moves with the velocity  $V$  in the direction  $l$  :

## V-2) Lorentz mesh

### Lorentz Mesh and special relativity

It is assumed that the **speeds are not negligible** compared to the speed of light : The four branches of the square mesh are numbered from 1 to 4. The first horizontal leg (at the top) is written as if the mesh moves with the velocity  $V$  in the direction  $l$  :

$$qv^{1\{i\}(l)} \rightarrow q \frac{v^{1\{V\}(l)} + V}{\gamma(1 + \beta c^{-1} v^{1\{V\}(l)})} \quad (18)$$
$$\beta = Vc^{-1}, \gamma = \sqrt{1 - \beta^2}^{-1}$$

## V-2) Lorentz mesh

The mesh bottom :

$$qv^{3\{i\}(l)} \rightarrow q \frac{v^{3\{V\}(l)} + V}{\gamma(1 + \beta c^{-1} v^{1\{V\}(l)})} \quad (19)$$

## V-2) Lorentz mesh

The mesh bottom :

$$qv^{3\{i\}(l)} \rightarrow q \frac{v^{3\{V\}(l)} + V}{\gamma(1 + \beta c^{-1} v^{1\{V\}(l)})} \quad (19)$$

Vertical legs :

$$qv^{k\{i\}(\omega)} \rightarrow q \frac{v^{k\{V\}(\omega)}}{\gamma(1 + \beta c^{-1} v^{1\{V\}(l)})} \quad (20)$$

$k = 2, 4$



## V-2) Lorentz mesh

The mesh bottom :

$$q_{V^3\{i\}(l)} \rightarrow q \frac{v^3\{V\}(l) + V}{\gamma(1 + \beta c^{-1} v^1\{V\}(l))} \quad (19)$$

Vertical legs :

$$q_{V^k\{i\}(\omega)} \rightarrow q \frac{v^k\{V\}(\omega)}{\gamma(1 + \beta c^{-1} v^1\{V\}(l))} \quad (20)$$

$k = 2, 4$

Calculate the transformation applied to the spatial components of the product resistance - current. for resistance aligned along **the x axis parallel to the relative velocity  $\mathbf{V}$** , we have :

$$z = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \quad (21)$$

## V-2) Lorentz mesh

### Transformation of the resistance

starting from its classical expression becomes :

$$R = \frac{1}{\sigma} \frac{x}{yz} \rightarrow \frac{1}{\gamma\sigma} \frac{x}{yz} = \frac{R}{\gamma} \quad (22)$$

## V-2) Lorentz mesh

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by **contraction of the length** and for the current we have this time, using 4-current density :

$$I = SJ \rightarrow S\gamma(J' + \beta c\rho') \quad (23)$$

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The  $S$  section being perpendicular to the displacement undergoes no contraction.  $RI$  product becomes :

$$RI \rightarrow RI' + R\beta S c\rho' = RI' + RI^e \quad (24)$$

## V-2) Lorentz mesh

### Relativistic effects

$I_e$  could be called "the drive current." The transition from one referential to another is therefore Consequently a change in the metric :

$$\begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & I_e/I \end{pmatrix} \right] \quad (25)$$

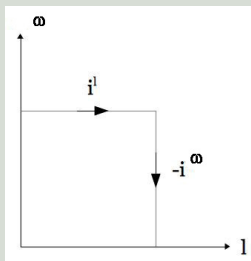


FIGURE: Lorentz mesh

## V-3) Curvature applied to electromagnetic field

### Recall about curvature

Recall that in a manifold equipped with a **linear connection**, parallel Transport of a vector along a parallelogram of following a broken geodesics, is used to define the torsion and curvature. If **in addition** the connection is the **Levi-Civita**, the **connection is without torsion**, the parallelogram describe a closed curve, it is by this means that we define the Riemann curvature tensor,

$$R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})Z \quad (26)$$

We have in locate coordinates :

$$R_{ij}^k = \frac{\partial \Gamma_{j,l}^k}{\partial x^i}(p) - \frac{\partial \Gamma_{i,l}^k}{\partial x^j}(p) + \Gamma_{i,s}^k(p) \Gamma_{j,l}^s(p) - \Gamma_{j,s}^k(p) \Gamma_{i,l}^s(p) \quad (27)$$

# Parallel transport

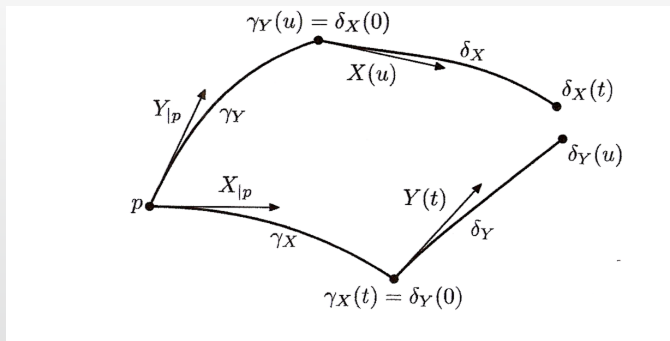


FIGURE: Parallel transport

## V-3) Curvature applied to electromagnetic field

### Electromagnetic analogy

It was speculated that the scale of a network of gravitational curvature was negligible. We can ask if that would be printed to the reluctances is done? Considering the **presence of masses magnetic** this curvature can be calculated initially with a similar approach that for gravitational fields starting on the assumption that we consider that the magnetic energy outcome of these masses that these are then weighted. We know that the computation for geodesic space-time Einstein determine the trajectories of the photons so far field. It will therefore no need to repeat this calculation for the actual chords.



## V-3) Curvature applied to electromagnetic field

### Electromagnetic analogy

Let **magnetic energy**  $2W_H = \mu_{ab}H^aH^b$ . By differentiating each component of this energy is found :  $PH = \mu_{ab}\dot{H}^a\dot{H}^b = \mu_{ab}h^ah^b$ . The root of this amount refers to a emf. This emf is a work of the field on any curved path. It can therefore integrate for express the action integral  $S$  of the field between two points  $A$  and  $B$  (this also means following the flow or reluctances lines) : Let  $S$  the action for this :

$$S = \int_A^B d\lambda \sqrt{\mu_{ab}h^ah^b} \quad (28)$$

## V-3) Curvature applied to electromagnetic field

### Electromagnetic analogy

The Euler equation for this set up is :






$$\frac{d}{d\lambda} \left( \frac{\partial \mathcal{L}}{\partial h^\alpha} - \frac{\partial \mathcal{L}}{\partial H^\alpha} \right) = e_\alpha \quad (29)$$

Under these conditions, **the tenseur of Riemann curvature applied to the electromagnetic field** are given in local coordinates is given by :

$$R_{ij}^k = \frac{\partial \Gamma_{j,l}^k}{\partial H^i}(p) - \frac{\partial \Gamma_{i,l}^k}{\partial H^j}(p) + \Gamma_{i,s}^k(p) \Gamma_{j,l}^s(p) - \Gamma_{j,s}^k(p) \Gamma_{i,l}^s(p) \quad (30)$$

## Conclusion

The generalized interaction terms under the tensorial analysis of networks invented by Kron in 1939 allows to take into account many kind of coupling. Various applications was made using these principles in information[2], guided waves, cavities[3], power choppers[4], etc. Each time it gives very efficient and optimized modelling giving fast and accurate results. Next step could be to apply the approach for numerical schematic. It allows to mix quite easily integral, PEEC and GTD in a common FEM for a numerical tool in Maxwell field computation. But more than anything, the global technique gives a very powerful tool to analyze theoretically the problems of engineers, even in non linear cases[5].

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