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# Finiteness of the space of $n$ -cycles for a reduced $(n - 2)$ -concave complex space

Daniel Barlet

*pour Yum-Tong Siu, avec mon amitié*

**Abstract.** We show that for  $n \geq 2$  the space of closed  $n$ -cycles in a strongly  $(n - 2)$ -concave complex space has a natural structure of reduced complex space locally of finite dimension and represents the functor “analytic family of  $n$ -cycles” parametrized by Banach analytic sets.

**Keywords.** Closed  $n$ -cycles; strongly  $q$ -concave space; Hartogs figure;  $f$ -analytic family of cycles

**2010 Mathematics Subject Classification.** 32G13; 32G10; 32F10; 32D15

[Français]

**Titre.** Finitude de l’espace des  $n$ -cycles pour un espace complexe  $(n - 2)$ -concave réduit

**Résumé.** Nous montrons que, pour  $n \geq 2$ , l’espace des  $n$ -cycles fermés dans un espace complexe fortement  $(n - 2)$ -concave a une structure naturelle d’espace complexe réduit localement de dimension finie et que cet espace représente le foncteur “famille analytique de  $n$ -cycles” paramétrée par des ensembles analytiques banachiques.

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## 1. Introduction

The aim of this article is to show that in a reduced strongly  $(n - 2)$ -concave<sup>1</sup> complex space  $Z$  with  $n \geq 2$ , the space of closed  $n$ -cycles is in a natural way endowed with a structure of a reduced complex space locally of finite dimension. With its tautological family of  $n$ -cycles it represents the functor “analytic family of  $n$ -cycles in  $Z$ ” and also the functor “f-analytic family of  $n$ -cycles in  $Z$ ” introduced in [Bar08] (see also [Bar13] and [Bar15]) parametrized by a Banach analytic set.

This answers a question asked to me by Y-T. Siu forty years ago.

I was able to solve this question thanks to the notion of f-analytic family introduced in *loc. cit.* and using the space  $\mathcal{C}_n^f(Z)$  of finite type cycles with its natural topology.

We obtain the following results.

**Theorem 1.1.** *Let  $n \geq 2$  be an integer. Let  $Z$  be a strongly  $(n - 2)$ -concave reduced complex space of pure dimension  $n + p$  that is to say admitting a  $\mathcal{C}^2$  exhaustion function  $\varphi : Z \rightarrow ]0, 2]$  which is strongly  $(n - 2)$ -convex outside the compact set  $K := \varphi^{-1}([1, 2])$ . For any  $\alpha \in ]0, 1[$  and any  $n$ -cycle  $X_0$  in an open neighbourhood of the compact set  $\varphi^{-1}([\alpha, 2])$  there exists  $\beta \in ]0, \alpha[$  such that, if  $Z_\beta := \{z \in Z / \varphi(z) > \beta\}$ , the cycle  $X_0$  extends in a unique way to the open set  $Z_\beta$  and admits an open neighbourhood  $\mathcal{U}$  in the space  $\mathcal{C}_n^f(Z_\beta)$  such that the ringed space defined by  $\mathcal{U}$  and the sheaf of holomorphic functions on  $\mathcal{U}$  is a (reduced) complex space locally of finite dimension.*

Recall that a holomorphic function  $h : \mathcal{U} \rightarrow \mathbb{C}$  on an open set in  $\mathcal{C}_n^f(Z_\beta)$  is a continuous function on  $\mathcal{U}$  such that for any holomorphic map  $f : S \rightarrow \mathcal{U}$  (corresponding to an f-analytic family of  $n$ -cycles in  $Z$ , see *loc. cit.*) of a Banach analytic set  $S$  to  $\mathcal{U}$  the composed function  $h \circ f$  is holomorphic.

<sup>1</sup> ↑ Our conventions will be precised below.

**Theorem 1.2.** *Consider the same situation as in the previous theorem, and let now  $X_0 \in \mathcal{C}_n^f(Z)$  be a finite type  $n$ -cycle in  $Z$ . Then there exists  $\beta \in ]0, 1[$  and open neighbourhoods  $\mathcal{V}$  and  $\mathcal{U}$  respectively of  $X_0$  in  $\mathcal{C}_n^f(Z)$  and of  $X_0 \cap Z_\beta$  in  $\mathcal{C}_n^f(Z_\beta)$  such that the restriction map*

$$r : \mathcal{V} \rightarrow \mathcal{U}$$

*is well defined and bi-holomorphic.*

We are going to recall briefly the notion of  $f$ -analytic family of finite type  $n$ -cycles in a complex space  $Z$ .

Firstly the notion of an analytic family of  $n$ -cycles in a reduced complex space  $Z$  parametrized by a reduced complex space  $S$  is defined as follows (see<sup>2</sup> [Bar75, Chapter I, p. 33] or [BM14, Chapter IV, Section 3]), using the following notion of an adapted scale (see [BM14, Chapter IV, Section 2.1]).

**Definition 1.3.** We call  $E := (U, B, j)$  a  **$n$ -scale on a complex space  $Z$**  when  $U$  and  $B$  are open relatively compact polydiscs respectively in  $\mathbb{C}^n$  and  $\mathbb{C}^p$  and where  $j : Z_E \rightarrow W$  is a closed embedding of an open set  $Z_E$  in  $Z$  into an open neighbourhood  $W$  of  $\bar{U} \times \bar{B}$  in  $\mathbb{C}^{n+p}$ . The open set  $Z_E$  is called the **domain** of  $E$  and the open set  $j^{-1}(U \times B)$  the **center** of  $E$ . When  $X$  is a  $n$ -cycle in  $Z$ , the  $n$ -scale  $E$  is **adapted** to  $X$  when  $|X| \cap j^{-1}(\bar{U} \times \partial B) = \emptyset$ .

Note that when  $E$  is a  $n$ -scale adapted to a  $n$ -cycle  $X$  in  $Z$ , the projection of  $U \times B$  on  $U$  restricted to  $j_*(X \cap j^{-1}(U \times B))$  gives a finite proper map of degree  $k \geq 0$  and the fibers of this map are classified by a holomorphic map  $f : U \rightarrow \text{Sym}^k(B)$ . In this case we shall say that  $f$  is the **classifying map** of the cycle  $X$  in the adapted scale  $E$ .

**Definition 1.4.** Let  $Z$  be a complex space and let  $(X_s)_{s \in S}$  be a family of  $n$ -cycles in  $Z$  parametrized by a reduced complex space  $S$ . We shall say that this family is **analytic at a point  $s_0 \in S$**  if for any  $n$ -scale  $E := (U, B, j)$  on  $Z$  which is adapted to the cycle  $X_{s_0}$  there exists an open neighbourhood  $S_0$  of  $s_0$  in  $S$  satisfying the following properties:

- i) For each  $s \in S_0$  the scale  $E$  is adapted to  $X_s$ .
- ii) Assume that  $k := \deg_E(X_{s_0})$ . Then for each  $s \in S_0$  we have  $\deg_E(X_s) = k$ .
- iii) There exists a holomorphic map  $f : S_0 \times U \rightarrow \text{Sym}^k(B)$  such that for each  $s \in S_0$  the restriction of  $f$  to  $\{s\} \times U$  classifies the cycle  $X_s$  in the scale  $E$ .

It is easy to see that an analytic family of cycles has a “set theoretic” graph

$$|G| := \{(s, x) \in S \times Z \mid x \in |X_s|\}$$

which is a closed analytic subset in  $S \times Z$  and that its projection on  $S$  has pure  $n$ -dimensional fibers (which are the supports of the cycles). When we have an analytic family  $(X_s)_{s \in S}$  and when the projection of its graph  $pr : |G| \rightarrow S$  is quasi-proper<sup>3</sup> we shall say that  $(X_s)_{s \in S}$  is a  **$f$ -analytic family** of (finite type)  $n$ -cycles in  $Z$ . Of course this condition implies that each cycle  $X_s$  is a finite type  $n$ -cycle (it means that each cycle admits only finitely many irreducible components) but this condition contains this fact in a local uniform manner on  $S$ .

In an analogous way, when we have an analytic family  $(X_s)_{s \in S}$  and when the projection of its graph  $pr : |G| \rightarrow S$  is proper, we shall say that it is a **proper analytic family of compact cycles in  $Z$** .

The following corollary is of course the main result.

<sup>2</sup> ↑ See Chapter 3 Section 4 in [Bar75] for the case when  $S$  is a Banach analytic set.

<sup>3</sup> ↑ This means, by definition, that for any  $s_0 \in S$  there exists an open neighbourhood  $S_1$  of  $s_0$  in  $S$  and a compact set  $K$  in  $|G|$  such that any irreducible component of any fiber  $pr^{-1}(s)$  for any  $s \in S_1$  meets  $K$ .

**Corollary 1.5.** *Consider the same situation as in the previous theorems. Then the ringed space given by the sheaf of holomorphic functions on  $\mathcal{C}_n^f(Z)$  is a reduced complex space locally of finite dimension. Moreover, endowed with its tautological family of (finite type)  $n$ -cycles it represents the functor*

$$\begin{aligned} (\text{reduced complex spaces}) &\rightarrow (\text{sets}) \\ S &\mapsto \{\text{f-analytic family of } n\text{-cycles in } Z \text{ parametrized by } S\}. \end{aligned}$$

**Remark.** Let  $X$  be a (non empty) irreducible analytic subset of dimension  $n$  in a strongly  $(n - 2)$ -concave complex space  $Z$  as in Theorem 1.1. Let  $x_0 \in X$  be a point in  $X$  where the supremum of the restriction of the exhaustion function  $\varphi$  to  $X$  is obtained. Then the point  $x_0$  is in the compact set  $K = \varphi^{-1}([1, 2])$  because the Levi form of  $\varphi$  at the point  $x_0$  has at least  $n$  non positive eigenvalues. So any (non empty) irreducible analytic subset of dimension  $n$  in  $Z$  has to meet the compact set  $K$ . Then any  $n$ -cycle in  $Z$  is of finite type and any analytic family of  $n$ -cycles in  $Z$  has a quasi-proper graph so is a f-analytic family. This implies that, in the previous corollary, the obvious map<sup>4</sup>  $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$  is an isomorphism of ringed spaces and  $\mathcal{C}_n^f(Z)$  represents also the functor “analytic family of  $n$ -cycles in  $Z$ ”.

**Question.** As it appears in the previous remark, we may expect the same result for  $(n - 1)$ -cycles under our assumption of strong  $(n - 2)$ -concavity. But our way to use Hartogs figures in the present article needs one more positive eigenvalue than one can expect. Is the result also true for  $(n - 1)$ -cycles under our hypothesis? It would probably be interesting, for instance, to have this kind of result for 1-cycles in a strongly 0-concave complex space.

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## 2. Hartogs figures

### 2.A. Banachization

For the analytic extension via Hartogs figures, the use of the Banach spaces  $H(\bar{U}, \mathbb{C})$  of continuous functions, holomorphic inside on a compact polydisc  $\bar{U}$  is not well adapted. We shall use the Banach space  $\mathcal{B}(U, \mathbb{C})$  of bounded holomorphic functions on  $U$  with the “sup” norm on  $U$ . Of course,  $H(\bar{U}, \mathbb{C})$  is a closed Banach subspace in  $\mathcal{B}(U, \mathbb{C})$ .

**Proposition 2.1.** *Let  $U$  be a relatively compact polydisc in  $\mathbb{C}^n$  and let  $S$  be a Banach analytic set. Let  $F : S \rightarrow \mathcal{B}(U, \mathbb{C})$  be a holomorphic map. Then the corresponding function  $f : S \times U \rightarrow \mathbb{C}$  defined by  $f(s, t) := F(s)[t]$  for  $(s, t)$  in  $S \times U$  is holomorphic (and locally on  $S$  uniformly bounded on  $U$ ).*

*Conversely, if we have a holomorphic function  $f : S \times U \rightarrow \mathbb{C}$  and an open polydisc  $U'' \subset\subset U$ , the associated map  $F : S \rightarrow \mathcal{B}(U'', \mathbb{C})$ , defined for  $(s, t)$  in  $S \times U''$  by  $F(s)[t] := f(s, t)$ , is holomorphic.*

*Proof.* The evaluation function  $ev : \mathcal{B}(U, \mathbb{C}) \times U \rightarrow \mathbb{C}$  is holomorphic as one may easily see by differentiation. Then the function  $f$  associated to  $F$  is the composition of the holomorphic maps  $F \times id_U$  and  $ev$ . So it is holomorphic.

The converse is consequence of the linear isometric inclusion of  $H(\bar{U}'', \mathbb{C})$  in  $\mathcal{B}(U'', \mathbb{C})$  as  $F'' : S \rightarrow H(\bar{U}'', \mathbb{C})$  is holomorphic as soon as  $f$  is holomorphic (see [Bar75] or [BM]).  $\square$

<sup>4</sup>  $\uparrow$  The set  $\mathcal{C}_n^{loc}(Z)$  is the set of all closed cycles of dimension  $n$  in the complex space  $Z$ ; its natural topology which is associated to the adapted scales is described in [BM14, Chapter IV]. In general, the inclusion map  $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$  is continuous but is not a homeomorphism onto its image. See [Bar15] for the comparison between compact sets in these two spaces.

## 2.B. $n$ -Hartogs figure on a complex space

For  $\alpha \in (\mathbb{R}_+^*)^2$  let  $M(\alpha)$  be the open set in  $\mathbb{C}^2$  defined by

$$\begin{aligned} M(\alpha) &:= M^P(\alpha) \cup M^C(\alpha) \quad \text{with} \\ M^P(\alpha) &:= \{|t_1 - \alpha_1/2| < \alpha_1/4, \quad |t_2| < \alpha_2\} \\ M^C(\alpha) &:= \{|t_1| < \alpha_1, \quad \alpha_2/2 < |t_2| < \alpha_2\} \end{aligned}$$

and let also  $\mathcal{M}(\alpha) := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i, i = 1, 2\}$ .

For  $\varepsilon > 0$  small enough, we define

$$\begin{aligned} M(\alpha)^\varepsilon &:= M^P(\alpha)^\varepsilon \cup M^C(\alpha)^\varepsilon \quad \text{with} \\ M^P(\alpha)^\varepsilon &:= \{|t_1 - \alpha_1/2| < \alpha_1/4 - \varepsilon/4, \quad |t_2| < \alpha_2 - \varepsilon\} \\ M^C(\alpha)^\varepsilon &:= \{|t_1| < \alpha_1 - \varepsilon, \quad \alpha_2/2 + \varepsilon/2 < |t_2| < \alpha_2 - \varepsilon\} \end{aligned}$$

and also  $\mathcal{M}(\alpha)^\varepsilon := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i - \varepsilon, i = 1, 2\}$ .

For a polydisc of radius  $R$  in  $\mathbb{C}^m$  we shall denote by  $P^\varepsilon$  the polydisc with same center and radius  $R - \varepsilon$  for  $0 < \varepsilon < R$ .

**Definition 2.2.** Let  $n \geq 2$  and  $p \geq 1$  two integers and let  $\Delta \subset\subset \Delta'$  be two open sets in a reduced complex space  $Z$ . We shall call  $\mathcal{H} := (\mathcal{M}, M, B, j)$  a  **$n$ -Hartogs figure in  $Z$  relative to the boundary of  $\Delta$** , the following data

- an embedding  $j$  of an open set  $Z'$  in  $\Delta'$  into an open set in  $\mathbb{C}^{n+p}$ ,
- open sets  $\mathcal{M} \subset\subset \Delta'$  and  $M \subset\subset \Delta$  relatively compact in  $Z'$ ,
- a relatively compact polydisc  $B$  in  $\mathbb{C}^p$

such that there exists  $\alpha \in (\mathbb{R}_+^*)^2$  and a relatively compact polydisc  $V$  in  $\mathbb{C}^{n-2}$  with the following conditions:

- i) The map  $j$  induces a closed embedding of  $\mathcal{M}$  in  $\mathcal{M}(\alpha) \times V \times B$ .
- ii) The map  $j$  induces a closed embedding of  $M$  in  $M(\alpha) \times V \times B$ .
- iii) We have  $j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \subset \Delta$ .

**Definition 2.3.** Let  $n \geq 2$  and  $p \geq 1$  two integers and let  $\Delta \subset\subset \Delta'$  be two open sets in a reduced complex space  $Z$ . Let  $\mathcal{H} = (\mathcal{M}, M, B, j)$  be a  $n$ -Hartogs figure in  $Z$  relative to the boundary of  $\Delta$  and let  $X_0$  be a  $n$ -cycle in  $\Delta$ . We shall say that  $\mathcal{H}$  is **adapted to  $X_0$**  when the following condition is satisfied :

$$j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \cap |X_0| = \emptyset. \quad (@)$$

### Remarks.

1. Let  $\mathcal{H}$  be a  $n$ -Hartogs figure in  $Z$  relative to the boundary of  $\Delta$ . If the open set  $\Delta_1 \subset\subset \Delta'$  has a boundary  $\partial\Delta_1$  near enough to  $\partial\Delta$ , then  $\mathcal{H}$  is again a  $n$ -Hartogs figure relative to the boundary of  $\Delta_1$ . For instance, if  $\Delta := \{\varphi > 0\}$  where  $\varphi$  is a continuous proper function on  $Z$ , we may choose  $\Delta_1 := \{\varphi > \varepsilon\}$  for  $\varepsilon > 0$  small enough.
2. Note that the  $n$ -scale  $E_{\mathcal{H}} := (M(\alpha) \times V, B, j)$  on  $\Delta$  associated to  $\mathcal{H}$  is adapted to  $X_0$  as soon as the  $n$ -Hartogs figure  $\mathcal{H}$  is adapted to  $X_0$ .

3. If  $\tilde{X}_0$  is a  $n$ -cycle in  $\Delta'$  such that its restriction to  $\Delta$  is equal to  $X_0$ , the  $n$ -scale  $E_{\tilde{\mathcal{H}}} := (\mathcal{M}(\alpha) \times V, B, j)$  on  $\Delta'$  is adapted to  $\tilde{X}_0$  if and only if the  $n$ -Hartogs figure  $\mathcal{H}$  is adapted to  $X_0$ .

Note that the  $n$ -scale  $E_{\tilde{\mathcal{H}}}$  is not a  $n$ -scale on  $\Delta$  although the subset  $\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B$  is contained in  $\Delta$ .

**Definition 2.4.** In the situation above we define, for  $\varepsilon > 0$  small enough, the  $n$ -Hartogs figure  $\mathcal{H}^\varepsilon$  on  $\Delta$  as follows :

$$\mathcal{H}^\varepsilon := (\mathcal{M}^\varepsilon, M^\varepsilon, B, j)$$

where we use the notations

$$\mathcal{M}^\varepsilon := j^{-1}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon \times B) \quad \text{and also} \quad M^\varepsilon := j^{-1}(M(\alpha)^\varepsilon \times V^\varepsilon \times B).$$

It is obvious to see that when  $\mathcal{H}$  is a  $n$ -Hartogs figure relative to the boundary of  $\Delta$ , then  $\mathcal{H}^\varepsilon$  is again a  $n$ -Hartogs figure relative to the boundary of  $\Delta$  for all  $\varepsilon > 0$  small enough.

Moreover, if  $\mathcal{H}$  is adapted to the  $n$ -cycle  $X_0$  of  $\Delta$ , the same is true for  $\mathcal{H}^\varepsilon$  for all  $\varepsilon > 0$  small enough.

**Lemma 2.5.** *Let  $V$  be a relatively compact open polydisc in  $\mathbb{C}^q$ . The restriction map*

$$res : \mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha) \times V, \mathbb{C})$$

*is a linear isometry of Banach spaces.*

Note that the restriction map

$$res : \mathcal{B}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C})$$

induces also an isometry for all  $\varepsilon > 0$  small enough.

*Proof.* Let  $f(v, t_1, t_2) := \sum_{m \in \mathbb{Z}} a_m(v, t_1) \cdot t_2^m$  the Laurent expansion of the holomorphic function  $f : M^C(\alpha) \times V \rightarrow \mathbb{C}$ . The holomorphic functions  $a_m$ ,  $m \in \mathbb{Z}$  on the product of  $V$  by the disc  $\{|t_1| < \alpha_1\}$  are given by the formula

$$a_m(v, t_1) := \frac{1}{2i\pi} \cdot \int_{|z|=r} f(v, t_1, z) \cdot \frac{dz}{z^{m+1}} \quad \text{with} \quad r \in ]\alpha_2/2, \alpha_2[.$$

As the holomorphy of  $f$  on  $M^P(\alpha) \times V$  implies that  $a_m \equiv 0$  for each negative  $m$  on the open set  $\{|t_1 - \alpha_1/2| < \alpha_1/4\} \times V$ , we conclude that the functions  $a_m$  are identically zero for  $m < 0$  and so  $f$  is holomorphic on  $\mathcal{M}(\alpha) \times V$ . This shows that the restriction map  $res$  is bijective (and also it is linear continuous) between the two Fréchet spaces  $\mathcal{O}(\mathcal{M}(\alpha) \times V)$  and  $\mathcal{O}(M(\alpha) \times V)$ ; so it is an isomorphism of Fréchet spaces.

Let us show that if  $f$  is in  $\mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C})$ , then  $res(f)$ , which belongs to the space  $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$ , has the same ‘sup’ norm. For this purpose fix  $\varepsilon > 0$  small enough. As  $\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$  is a compact polydisc in  $\mathcal{M}(\alpha) \times V$ , the maximum of  $f$  on this compact is obtained at some point  $z$  in the distinguish boundary of it. But as  $z$  is also in the boundary of  $M^C(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ , the desired equality follows. Conversely, if  $g$  is in  $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$ , its analytic extension  $f$  to  $\mathcal{M}(\alpha) \times V$  will be bounded on the boundary of  $\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon$  by the sup of  $g$  on  $M^C(\alpha)^\varepsilon \times V^\varepsilon$ . So we obtain the equality of the ‘sup’ norms for  $g$  and  $f$  respectively on  $M(\alpha) \times V$  and  $\mathcal{M}(\alpha) \times V$ .  $\square$

**The Banach analytic set**  $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ . Recall that if  $p \geq 1$  and  $k \geq 1$  are integers, there exists a closed embedding (in fact given by a polynomial map) of  $\text{Sym}^k(\mathbb{C}^p) := (\mathbb{C}^p)^k / \mathfrak{S}_k$  into the vector space  $E(k) := \bigoplus_{h=1}^k S^h(\mathbb{C}^p)$  given by the elementary tensorial symmetric functions<sup>5</sup>. If  $U$  is an open relatively compact polydisc in  $\mathbb{C}^n$ , the subset  $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$  is closed and Banach analytic in the Banach space  $\mathcal{B}(U, E(k))$ . Indeed, if  $Q : E(k) \rightarrow \mathbb{C}^N$  is a polynomial map such that  $Q^{-1}(0) = \text{Sym}^k(\mathbb{C}^p)$ , then the holomorphic map

$$\mathcal{Q} : \mathcal{B}(U, E(k)) \rightarrow \mathcal{B}(U, \mathbb{C}^N) \quad \text{defined by} \quad f \mapsto Q \circ f$$

satisfies  $\mathcal{Q}^{-1}(0) = \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ .

Nevertheless, be aware that for an open set  $\Omega$  in  $E(k)$  the subset  $\mathcal{B}(U, \Omega)$  of elements in  $\mathcal{B}(U, E(k))$  taking their values in  $\Omega$  is not, in general, open in  $\mathcal{B}(U, E(k))$ ; so, for an open polydisc  $B \subset \subset \mathbb{C}^p$ , the subset  $\mathcal{B}(U, \text{Sym}^k(B))$  is not open in  $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$  in general.

**Remark.** The obvious map  $H(\bar{U}, E(k)) \rightarrow \mathcal{B}(U, E(k))$  is a closed (linear) isometry and induces a holomorphic inclusion map

$$i_U : H(\bar{U}, \text{Sym}^k(\mathbb{C}^p)) \hookrightarrow \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$$

and for all  $\varepsilon > 0$  the restriction

$$r : \mathcal{B}(U, E(k)) \rightarrow H(\bar{U}^\varepsilon, E(k))$$

is a (linear and continuous) compact map which induces a holomorphic restriction map

$$\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p)) \rightarrow H(\bar{U}^\varepsilon, \text{Sym}^k(\mathbb{C}^p)).$$

This remark will allow us to use Lemma 2.5 with Banach analytic sets like  $H(\bar{U}, \text{Sym}^k(B))$ .

**Notations.** Let  $k \in \mathbb{N}$  and let  $U' \subset \subset U \subset \subset \mathbb{C}^n$  and  $B \subset \subset \mathbb{C}^p$  be polydiscs. We shall note  $\Sigma_{U, U'}(k)$  the Banach analytic set classifying the couples of an element in  $H(\bar{U}, \text{Sym}^k(B))$  with its isotropy data on  $\bar{U}'$ . Recall that for a holomorphic map  $f : U \rightarrow \text{Sym}^k(B)$  the isotropy data on the polydisc  $U'$  is the map

$$T(f) : U' \rightarrow F \otimes E'$$

where

$$F := \bigoplus_i \left( L(\Lambda^i(\mathbb{C}^n), \Lambda^i(\mathbb{C}^p)) \right) \quad \text{and} \quad E' := \bigoplus_{m=0}^{k-1} S^m(\mathbb{C}^p).$$

It corresponds to the collection of holomorphic maps

$$T_m^i(f) : U' \rightarrow L(\Lambda^i(\mathbb{C}^n), \Lambda^i(\mathbb{C}^p)) \otimes S^m(\mathbb{C}^p)$$

for all  $i \in [1, \min(n, p)]$  and  $m \in [0, k - 1]$  which are given, near a point in  $U'$  where the multiform graph  $X_f$  associated to  $f$  has local branches  $f_1, \dots, f_k$ , by the formula

$$T_m^i(f)(t) := \sum_{j=1}^k \Lambda^i(Df_j(t)) \otimes f_j(t)^m.$$

These maps are always holomorphic on all  $U$  and determine the trace map for the projection  $\pi : X_f \rightarrow U$  of the holomorphic differential forms on  $U \times B$ .

The subset

$$\Sigma_{U, U'}(k) \subset H(\bar{U}, \text{Sym}^k(B)) \times H(\bar{U}', F \otimes E')$$

<sup>5</sup> ↑ See for instance [BM14, Chapter I, §4].



is defined as the graph of the map  $f \mapsto T(f) := \bigoplus_{i,m} T_m^i(f)$  which is not holomorphic in general. Nevertheless it is a Banach analytic subset and its natural projection

$$\Sigma_{U,U'}(k) \rightarrow H(\bar{U}, \text{Sym}^k(B))$$

is a holomorphic homeomorphism (see [Bar75, Chapter III, Proposition 2, p. 81] or [BM, Chapter V]).

The important point which motivates the introduction of this Banach analytic subset is the fact that, when  $S$  is a reduced complex space, an analytic family of multiform graphs given by a holomorphic map  $f : S \times U \rightarrow \text{Sym}^k(B)$  will give an analytic family of cycles in  $U \times B$  parametrized by  $S$  if and only if the corresponding isotropy data (given by the maps  $T_m^i(f/S)$  on  $S \times U$ ) are holomorphic on  $S \times U$ . This is the isotropy condition ; see [Bar75, Chapter II] or [BM14, Chapter IV, Section 5].

Our next result is the main tool for performing the analytic extension of  $n$ -cycles near a  $(n - 2)$ -concave boundary.

**Proposition 2.6.** *Consider the open sets  $\mathcal{M}(\alpha) \times V$  and  $M(\alpha) \times V$  in  $\mathbb{C}^n$ . The inverse of the restriction map is a holomorphic isomorphism of analytic extension*

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)).$$

Composed with the restriction to the compact set  $\bar{M}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$  it sends the subset  $H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$  into  $H(\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon, \text{Sym}^k(B))$  for  $\varepsilon > 0$  small enough.

Moreover, this holomorphic map induces a holomorphic map, again for  $\varepsilon > 0$  small enough

$$\Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \longrightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

which factorizes the restriction map

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

through the restriction

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k).$$

*Proof.* Lemma 2.5 gives that the map

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, E(k)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, E(k))$$

is an isometry of Banach spaces. It is clear that its inverse sends  $\mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$  into the Banach analytic subset  $\mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$  and that if the map  $f \in \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$  takes values in  $\text{Sym}^k(B)$ , the same is true for its restriction. This proves the first part of the proposition.

In order to prove the second part, it is enough to show that a holomorphic map of a Banach analytic set  $S$  with values in the subset

$$H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$$

which is isotropic on the product of  $S$  with any relatively compact subset in the open set  $M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$  will have an analytic extension which will be isotropic on any relatively compact open set in  $\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$ . So it will be isotropic on the closure of the open set  $\mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}$ .  $\square$

**Proposition 2.7.** *Let  $n \geq 2$  and  $p \geq 1$  be integers and let  $U_1 \times B_1$  the product of two polydiscs with centers 0 respectively in  $\mathbb{C}^n$  and  $\mathbb{C}^p$ . Denote by  $(t_1, \dots, t_n, x_1, \dots, x_p)$  coordinates on  $U_1 \times B_1$ . Let  $\varphi$  be a real valued function of class  $\mathcal{C}^2$  on  $U_1 \times B_1$ , such that*

$$\varphi(t, x) = \text{Re}(t_1) + \sum_{i=1}^n \rho_i \cdot |t_i|^2 + \sum_{j=1}^p \sigma_j \cdot |x_j|^2 + o(\|(t, x)\|^2) \quad (@@)$$

where the real numbers  $\rho_2, \sigma_1, \dots, \sigma_p$  are positive (so  $\varphi$  is  $(n-2)$ -convex near  $(0,0)$  and  $d\varphi_{0,0} \neq 0$ ).

Let  $\Delta$  be the open set  $\{\varphi > 0\}$  in  $U_1 \times B_1$  and let  $\Delta'$  be an open neighbourhood of the compact set  $\bar{\Delta}$  in  $\mathbb{C}^{n+p}$ . Let  $X_0$  be a closed analytic subset of pure dimension  $n$  in  $\Delta'$  such that each irreducible component of  $X_0$  meets  $\Delta$  and such that

$$|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}. \quad (*)$$

Then there exists  $\alpha \in (\mathbb{R}_+^*)^2$  and polydiscs  $V$  and  $B \subset\subset B_1$  with centers 0 respectively in  $\mathbb{C}^{n-2}$  and  $\mathbb{C}^p$  such that the following conditions are satisfied :

1.  $\mathcal{M}(\alpha) \times V \times B \subset\subset \Delta'$ ;
2.  $M(\alpha) \times V \times B \subset\subset \Delta$ ;
3.  $\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B \subset \Delta$ ;
4.  $|X_0| \cap (\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B) = \emptyset$  (this implies  $|X_0| \cap (\overline{M(\alpha)} \times \bar{V} \times \partial B) = \emptyset$ ).

*Proof.* Choose the polydisc  $B \subset\subset B_1$  small enough in order that we have

$$X_0 \cap (\{0\} \times \bar{B}) \subset \{0\} \quad \text{and} \quad \{0\} \times \partial B \subset \Delta.$$

This is possible as we have  $|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}$  and as  $\varphi$  is positive on a small enough punctured neighbourhood of the origin in the  $p$ -plane  $\{t_1 = \dots = t_n = 0\} \times \mathbb{C}^p$ . So we shall have

$$|X_0| \cap (\bar{W} \times \partial B) = \emptyset \quad \text{and} \quad \bar{W} \times \partial B \subset \Delta$$

for any small enough open neighbourhood  $W$  of the origin in  $\mathbb{C}^n$ . A immediate consequence is that Conditions 1, 3 and 4 will be satisfied as soon as  $\alpha$  and  $V$  are small enough.

In order to check Condition 2, let us remark first that, up to choosing the real numbers  $\rho'_1$  and  $\rho'_2$  such that  $\rho'_1 > |\rho_1|$ ,  $\rho'_2 \in ]0, \rho_2[$  and  $r > \sup_{i \geq 3} |\rho_i|$ , we obtain on  $W \times B$  chosen small enough

$$\varphi(t, x) \geq \operatorname{Re}(t_1) - \rho'_1 \cdot |t_1|^2 + \rho'_2 \cdot |t_2|^2 - r \cdot \left( \sum_3^n |t_i|^2 \right) \quad (**)$$

with strict inequality as soon as  $x \neq 0$ . Then for

$$V_\varepsilon = \{(t_3, \dots, t_n) / \sum_3^n |t_i|^2 < \varepsilon^2\}$$

the following inequalities hold

$$\varphi(t, x) \geq \frac{1}{4} \alpha_1 - \rho'_1 \cdot \alpha_1^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^P(\alpha) \times V_\varepsilon \times B} \quad (1)$$

$$\varphi(t, x) \geq -\alpha_1 - \rho'_1 \cdot \alpha_1^2 + \frac{1}{4} \rho'_2 \cdot \alpha_2^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^C(\alpha) \times V_\varepsilon \times B} \quad (2)$$

for  $\alpha$  and  $\varepsilon$  small enough in order that  $\mathcal{M}(\alpha) \times V_\varepsilon$  is contained in  $W$ .

This allows to fix  $\alpha_1, \alpha_2$  and  $\varepsilon$ .

Now we shall choose  $\alpha_1$  and  $\varepsilon$  smaller in order to satisfy the following conditions :

$$8\alpha_1 < \rho'_2 \cdot \alpha_2^2, \quad \alpha_1 < \frac{1}{8\rho'_1} \quad \text{and} \quad \varepsilon^2 < \frac{1}{8r} \alpha_1. \quad (3)$$

To obtain  $M^P(\alpha) \times V_\varepsilon \times B \subset\subset \Delta$  it is enough to show that on  $M^P(\alpha) \times V_\varepsilon \times B$  we have, if we let  $\alpha_1 = u.\alpha_2$  and  $\varepsilon^2 = v.\alpha_1 = uv.\alpha_2$

$$\frac{1}{4} > \rho'_1.\alpha_1 + r.v.$$

Indeed, as we assumed  $\rho'_1.\alpha_1 < \frac{1}{8}$  and  $r.\varepsilon^2 < \frac{1}{8}.\alpha_1$  (so  $r.v < 1/8$ ) the first condition holds.

In order to satisfy  $M^C(\alpha) \times V_\varepsilon \times B \subset\subset \Delta$  it is enough to show that on  $M^C(\alpha) \times V_\varepsilon \times B$  we have

$$\frac{1}{4}\rho'_2.\alpha_2 > u + \rho'_1 u^2 \alpha_2 + r.uv.$$

But our condition implies

$$\rho'_1 u^2 \alpha_2 < \frac{1}{8}.u \quad r.uv < \frac{1}{8}.u$$

which gives  $u + \rho'_1 u^2.\alpha_2 + r.uv < 2.u$ . The condition  $8\alpha_1 < \rho'_2.\alpha_2^2$  which implies  $2u < \frac{1}{4}\rho'_2.\alpha_2$ , allows to conclude.  $\square$

### Remarks.

1. We only used Condition (\*) for  $X_0$  and Inequality (\*\*) for  $\varphi$  in a neighbourhood of the origin in the proof above.
2. Sufficient conditions on  $\varphi \in \mathcal{C}^2$  to satisfy (@@) are:

- i) The origin is not a critical point of  $\varphi$ .
- ii) The Levi form of  $\varphi$  at 0 has, at most,  $(n - 2)$  non positive eigenvalues in the complex tangent hyperplane to the real hypersurface  $\{\varphi(z) = 0\}$ ; the existence of real function  $\varphi \in \mathcal{C}^2$  such that  $\Delta = \{\varphi > 0\}$  and satisfying these two conditions is equivalent to the fact that the open set  $\Delta$  has a strongly  $(n - 2)$ -concave smooth boundary near the origin (see Definition 2.10 given below). Indeed, if  $\varphi$  is not critical at 0 and has a Levi form at 0 with, at most,  $(n - 2)$  non positive eigenvalues in the complex hyperplane tangent to the real hypersurface  $\{\varphi(z) = 0\}$ , its order 2 Taylor expansion at the origin is written, in suitable local holomorphic coordinates  $(\tau, x)$

$$\varphi(\tau, x) = \operatorname{Re}(\tau_1) + \operatorname{Re}(Q(\tau, x)) + \sum_{i=1}^n \rho_i.|\tau_i|^2 + \sum_{j=1}^p \sigma_j.|x_j|^2 + o(\|(\tau, x)\|^2)$$

where  $Q$  is a holomorphic homogeneous degree 2 polynomial and where the real numbers  $\rho_i$  and  $\sigma_j, j \in [1, p]$  are positive. Define new local holomorphic coordinates

$$t_1 := \tau_1 + Q(\tau, x), t_i := \tau_i \quad \text{for } i \in [2, n] \quad \text{and} \quad x_j := x_j \quad \text{for } j \in [1, p].$$

Then we obtain (@@).

3. Condition (\*) implies that  $X_0$  has no local irreducible component at 0 contained in the hyperplane  $\{t_1 = 0\}$ . In fact, as the coordinate  $t_1$  is chosen in order to suppress the real part of the holomorphic homogeneous degree 2 term in the order 2 Taylor expansion of  $\varphi$  at 0 (see the previous remark), we want that no local irreducible component of  $X_0$  at the origin is contained in the complex hypersurface  $\tau_1 + Q(\tau, x) = 0$  locally defined near 0 for  $\varphi$  given. Then, as soon as the restriction  $\varphi|_{X_0}$  has not 0 as a critical point, Condition (\*) could be realized when the Levi form of  $\varphi$  has at most  $(n - 2)$  non positive eigenvalues on the complex tangent hyperplane at the origin of the hypersurface  $\{\varphi = 0\}$ .

4. One may easily see that under our hypothesis, the cycle  $X_0$  meets the open set  $\Delta$  when it contains 0. Indeed, the analytic subset

$$\{t_1 = t_3 = \dots = t_n = 0\} \cap |X_0|$$

is nonempty, has dimension at least 1 and meets the complement of  $\Delta$  only at the origin.

Of course, assuming that  $0 \in X_0$ , the proposition shows that, in fact,  $X_0$  contains a branched covering of degree  $k \geq 1$  of  $M^P(\alpha) \times V_\varepsilon$  inside

$$M^P(\alpha) \times V_\varepsilon \times B \subset\subset \Delta.$$

5. In the situation of Proposition 2.7, for any continuous family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $\Delta$  parametrized by a Banach analytic set  $S$  such that  $X_{s_0} = X_0 \cap \Delta$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$ , such that for each  $s \in S'$  Condition 4 remains true after analytic extension of the cycles (see Proposition 2.6), because, thanks to Condition 3,  $\mathcal{M}(\alpha) \times \bar{V} \times \partial B$  is a compact subset in  $\Delta$ .

## 2.C. $q$ -concave open sets.

**Definition 2.8.** Let  $\varphi : U \rightarrow \mathbb{R}$  be real valued  $\mathcal{C}^2$  function on an open set  $U$  in  $\mathbb{C}^N$ . We shall say that  $\varphi$  is **strongly  $q$ -convex** when its Levi form at each point of  $U$  has at most  $q$  non positive eigenvalues.

So, with this definition a strongly 0-convex function is a strongly plurisubharmonic function.

**Definition 2.9.** Let  $\varphi : Z \rightarrow \mathbb{R}$  a real valued  $\mathcal{C}^2$  function on a reduced complex space  $Z$ . We shall say that  $\varphi$  is **strongly  $q$ -convex** if locally near each point of  $Z$  it can be induced by a  $\mathcal{C}^2$  strongly  $q$ -convex function in a local embedding in an open set of an affine space.

Remark that a strongly  $q$ -convex function on an irreducible complex space of dimension at least equal to  $q+1$  has no local maximum because there exists at any point a germ of curve on which the restriction of  $\varphi$  is strongly p.s.h.

**Definition 2.10.** Let  $Z$  be a reduced complex space and let  $\Delta$  be a relatively compact open set in  $Z$ . We shall say that  $\Delta$  has a **smooth  $\mathcal{C}^2$  boundary** when for each point  $z$  in  $\partial\Delta$  there exists a local holomorphic embedding  $j : W \rightarrow U$  of an open neighbourhood  $W$  of  $z$  in an open set  $U$  of the Zariski tangent space of  $Z$  at  $z$  and an open set  $D$  with smooth  $\mathcal{C}^2$  boundary in  $U$  such that  $W \cap j^{-1}(\partial D) = W \cap \partial\Delta$ .

We shall say that the open set  $\Delta \subset Z$  with smooth  $\mathcal{C}^2$  boundary is **strongly  $q$ -concave at a point**  $z \in \partial\Delta$  if, in some local holomorphic embedding  $j : W \rightarrow U$  of  $Z$  around  $z$  as above, one can define  $\Delta$  in  $W$  as the subset  $\{j \circ \varphi > 0\} \cap W$  where  $\varphi$  is a real valued  $\mathcal{C}^2$  function on  $U$  such that

1.  $d\varphi_{j(z)} \neq 0$  on the tangent space  $T_{U,j(z)}$  of  $U$  at  $j(z)$ .
2. The restriction of the Levi form at  $j(z)$  of  $\varphi$  to the complex hyperplane tangent at  $j(z)$  to the real hypersurface  $\{\varphi(x) = \varphi(j(z))\}$  in  $U$  has at most  $q$  non positive eigenvalues.

We shall say that  $\Delta$  is **strongly  $q$ -concave** if  $\Delta$  is strongly  $q$ -concave near each point in  $\partial\Delta$ .

**Remark.** Assume that  $Z$  is of pure dimension  $q + p$ . If the defining function  $\varphi$  of  $\Delta$  satisfies Conditions 1 and 2 above, we can compose  $\varphi$  with a real strictly increasing (non critical) convex  $\mathcal{C}^2$  function (this does not change the level sets  $\{\varphi = \text{constant}\}$ ), in order that  $c \circ \varphi$  is  $\mathcal{C}^2$  strongly  $q$ -convex (and non critical) near  $z$ .

Conversely, if  $\varphi$  is a real valued  $\mathcal{C}^2$  function which is strongly  $q$ -convex and not critical near a point  $z \in Z$ , the open set  $\{\varphi(x) > \varphi(z)\}$  has a strongly  $q$ -concave boundary in a neighbourhood of  $z$ .

With this terminology, using the remarks above, we may give the following reformulation of Proposition 2.7:

**Corollary 2.11.** *Let  $n \geq 2$  and  $p \geq 1$  be integers, let  $Z$  be a reduced complex space of pure dimension  $n + p$  and let  $\Delta := \{\varphi > 0\}$  be an open set with  $\mathcal{C}^2$  smooth boundary in  $Z$ . Let  $X_0$  be a  $n$ -cycle in an open neighbourhood of a point  $z \in \partial\Delta$  such that the function  $\varphi|_{X_0}$  is not critical at  $z$ .*

*Assume that  $\Delta$  is strongly  $(n - 2)$ -concave near  $z$ ; then there exists a  $n$ -Hartogs figure  $\mathcal{H} := (\mathcal{M}, M, B, j)$  relative to the boundary of  $\Delta$ , adapted to  $X_0$ , and such that the point  $z$  lies in  $\mathcal{M}$ .*

*Proof.* Using a local embedding of an open neighbourhood of  $z$  in an open set of the Zariski tangent space  $T_{Z,z}$ , it is enough to prove the corollary in the case where  $Z$  is an open set in  $\mathbb{C}^{n+p'}$ , with  $p' \geq p$  an integer. As we may choose the function  $\varphi$  strongly  $(n - 2)$ -convex such that  $d\varphi_z \neq 0$  thanks to the previous remarks, we can choose local coordinates near  $z$  in order to be in the situation of Proposition 2.7 in the case  $z \in X_0$ , as we assumed that  $\varphi|_{X_0}$  is not critical at  $z$ . In this case the proposition gives the result.

If  $z$  is not in  $X_0$ , the same construction in an open neighbourhood of  $z$  with no limit point in  $X_0$  allows to conclude, and in this case the degree of  $X_0$  in the (adapted) scale  $E_{\mathcal{H}}$  will be zero.  $\square$

## 2.D. Convexity–concavity

In this paragraph we want to have a brief discussion about  $q$ -convexity and  $q$ -concavity.

Let us consider in an open set  $U$  of  $\mathbb{C}^{n+p}$  a  $\mathcal{C}^2$  function  $\varphi : U \rightarrow \mathbb{R}$  and a non critical zero  $z_0$  on  $\varphi$ . So  $\varphi(z_0) = 0$  and  $d\varphi_{z_0} \neq 0$ . Let  $D := \{z \in U / \varphi(z) < 0\}$  and let  $H$  be the complex hyperplane tangent at  $z_0$  to the real hypersurface  $\{\varphi = 0\}$  which is smooth near  $z_0$ .

Our terminology (Norguet–Siu convention, see [NS77]) is to say that the open set  $D$  is strongly  $q$ -convex near  $z_0 \in \partial D$  if the restriction to  $H$  of the Levi form of  $\varphi$  at  $z_0$  has at most  $q$  non positive eigenvalues.

Looking now at the same open set  $D$  but asking for some strong concavity condition, we write  $D := \{z \in U / -\varphi(z) > 0\}$ . Then we shall say that  $D$  is strongly  $q$ -concave at the point  $z_0$  if the restriction to  $H$  of Levi form of  $-\varphi$  at  $z_0$  has at most  $q$  non positive eigenvalues.

If the signature of the restriction to  $H$  of the Levi form of  $\varphi$  at  $z_0$  is given by  $(p - 1)$  “plus” and  $n$  “minus” we see that that near  $z_0$  our open set  $D$  will be strongly  $n$ -convex near  $z_0$  and strongly  $(p - 1)$ -concave near  $z_0$ . So  $D$  will be strongly  $(p - 1)$ -concave near  $z_0$  if the function  $-\varphi$  is strongly  $(p - 1)$ -convex at the point  $z_0$ .

In order that a  $\mathcal{C}^2$  exhaustion function  $\varphi : Z \rightarrow ]0, 2]$  on a reduced complex space  $Z$  gives relatively compact  $q$ -concave subsets  $Z_\alpha := \{\varphi(z) > \alpha\}$  for each  $\alpha \in ]0, 1[$  which is not critical for  $\varphi$ , we see that it is enough that the Levi form of  $\varphi$  at each point in  $\varphi^{-1}(]0, 1[)$  has at most  $q$  non positive eigenvalues. That is to say that  $\varphi$  is strongly  $q$ -convex on this open set.

In order to reach the key situation given in Proposition 2.7 with a  $n$ -cycle, we need to dispose of a  $\mathcal{C}^2$ -exhaustion  $\varphi : Z \rightarrow ]0, 2]$  which is  $(n - 2)$ -strongly convex on the open set  $\varphi^{-1}(]0, 1[)$ . So we need to assume that  $n \geq 2$ .

## 2.E. Boxed Hartogs figures

**Definition 2.12.** Let  $n \geq 2$  and  $p \geq 1$  be integers and let  $\Delta \subset\subset \Delta'$  be two open sets in a reduced complex space  $Z$ . Let  $\mathcal{H} = (\mathcal{M}, M, B, j)$  and  $\mathcal{H}' := (\mathcal{H}', M', B, j)$  be two  $n$ -Hartogs figures in  $Z$  relative to the boundary of  $\Delta$  given by the same (local) embedding  $j$  and having the same polydisc  $B \subset\subset \mathbb{C}^p$ . We shall say that these two  $n$ -Hartogs figures are **boxed** when we can choose  $\alpha, \alpha', V, V'$  in Definition 2.2 in order to have

- $\mathcal{M}'(\alpha') \subset\subset \mathcal{M}(\alpha)$ ,
- $M'(\alpha') \subset\subset M(\alpha)$ ,
- $V' \subset\subset V$ .

For instance, if  $\varepsilon > 0$  is small enough, the  $n$ -Hartogs figures  $(\mathcal{H}, \mathcal{H}^\varepsilon)$  are boxed (see Definition 2.4).

**Proposition 2.13.** Let  $n \geq 2$  and  $p \geq 1$  be integers, let  $Z$  be a reduced complex space of pure dimension  $n+p$  and let  $\Delta \subset\subset Z$  be an open set with smooth  $\mathcal{C}^2$  boundary in  $Z$  which is strongly  $(n-2)$ -concave. Assume that  $\Delta := \{\varphi > 0\}$  and let  $X_0$  be a  $n$ -cycle in an open neighbourhood  $\Delta'$  of the compact set  $\bar{\Delta}$ , such that any irreducible component of  $X_0$  meets  $\Delta$ . Then there exists a finite family of boxed  $n$ -Hartogs figures  $(\mathcal{H}'_a, \mathcal{H}_a)_{a \in A}$  relative to the boundary of  $\Delta$ , such that the following conditions hold:

1. The open sets  $\mathcal{M}'_a$  for  $a \in A$  cover the boundary  $\partial\Delta$ .
2. For each  $a \in A$  the Hartogs figures  $\mathcal{H}_a$  and  $\mathcal{H}'_a$  are adapted to  $X_0$ .
3. For each  $a \in A$  any irreducible component of  $X_0$  meeting  $\mathcal{M}_a$  meets the open set  $M'_a$ .
4. No **compact** irreducible component of  $X_0 \cap \Delta$  meets the union of the compact sets  $\bar{\mathcal{M}}_a$ ,  $a \in A$ .

**Remark.** Let  $X_0$  be any  $n$ -cycle in  $\Delta'$ . Choosing the open set  $\Delta'$  small enough around the compact set  $\bar{\Delta}$ , we can assume that the cycle  $X_0$  has only finitely many irreducible components in  $\Delta'$  and that each of them which is not compact meets  $\partial\Delta$  (see Remark 3 following Proposition 2.7).

**Corollary 2.14.** In the situation of Proposition 2.13, if we assume that the open set  $\Delta'$  containing  $\bar{\Delta}$  is small enough, any irreducible component  $\Gamma$  of the cycle  $X_0$  in  $\Delta'$  satisfies for all  $a \in A$  and all  $\eta > 0$  small enough:

$$\Gamma \cap (\mathcal{M}_a^\eta \times V_a^\eta \times B_a) = \text{prlgt}_a[\Gamma \cap (M_a \times V_a \times B_a)]$$

where  $\text{prlgt}_a : H(\bar{M}_a \times \bar{V}_a, \text{Sym}^k(B_a)) \rightarrow H(\bar{\mathcal{M}}_a^\eta \times \bar{V}_a^\eta, \text{Sym}^k(B_a))$  is the holomorphic map of analytic extension built in Proposition 2.6.

**Remark.** In the situation of the previous corollary, choosing  $\varepsilon > 0$  small enough, there exists, for each  $a \in A$ , a holomorphic extension map which lifts the map  $\text{prlgt}_a$ :

$$i\text{prlgt}_a : \Sigma_{M_a, M_a^\varepsilon}(k) \longrightarrow \Sigma_{\mathcal{M}_a^\eta, \mathcal{M}_a^{\eta+\varepsilon}}(k).$$

It allows to extend in this setting an analytic family of branched coverings in  $M_a$  parametrized by a Banach analytic set  $S$  and which is isotropic on  $S \times M_a^\varepsilon$  to an analytic family of branched coverings in  $\mathcal{M}_a^\eta$  which is isotropic on  $S \times \mathcal{M}_a^{\eta+\varepsilon}$ .

*Proof of Proposition 2.13.* Corollary 2.11 and the remark following it implies the existence, for each  $z \in \partial\Delta$  of a  $n$ -Hartogs figure  $\mathcal{H}_z$  relative to the boundary of  $\Delta$ , contained in  $\Delta'$  and satisfying the following properties:

- i)  $z \in \mathcal{M}_z$ ;
- ii)  $\mathcal{H}_z$  is adapted to  $X_0$  ;
- iii) each irreducible component of  $X_0$  meeting  $\mathcal{M}_z$  meets the open set  $M_z$ ;
- iv) No compact irreducible component of  $X_0 \cap \Delta$  meets  $\bar{\mathcal{M}}_z$ .

As the open sets  $\mathcal{M}_z$  cover the compact set  $\partial\Delta$  we can find a finite sub-covering. Then the properties 1, 2, 3 and 4 are consequences of i), ii), iii) and iv) by letting  $\mathcal{H}'_a := \mathcal{H}_a^\varepsilon$  and choosing  $\varepsilon > 0$  small enough.  $\square$

*Proof of Corollary 2.14.* Let  $\Gamma$  be an irreducible component of  $X_0$  meeting  $\mathcal{M}_a$  for some  $a \in A$ . Then  $\Gamma$  meets  $M_a$ . As  $\Gamma$  does not meet  $\bar{M}(\alpha)_a \times \bar{V}_a \times \partial B_a$  because  $\mathcal{H}_a$  is adapted to  $X_0$ , the intersection  $\Gamma \cap \bar{M}_a$  is the graph of an element  $\gamma \in H(\bar{M}(\alpha)_a \times \bar{V}_a, \text{Sym}^{k_a}(B_a))$  with  $k_a \in \mathbb{N}^*$  <sup>6</sup>.

The closed analytic subset  $Y$  of the open set  $\mathcal{M}_a^\eta$  defined by  $Y := \text{prlgt}_a[\Gamma \cap M_a]$  is not empty, of pure dimension  $n$  and is contained in  $\Gamma$ . So it is a union of irreducible components of  $\Gamma \cap \mathcal{M}_a^\eta$ . But it contains a non empty open set in each irreducible component of this branched covering. So these two analytic subsets coincide.

If an irreducible component of  $X_0$  does not meet any  $\bar{M}_a$  it has to be compact and contained in  $\Delta$ . In this case the desired equality is obvious.  $\square$

### 3. The extension and finiteness theorem

#### 3.A. Some useful lemmas

The version below of Sard's lemma is more or less classical.

**Lemma 3.1.** *Let  $Z$  be a reduced complex space and let  $\varphi : Z \rightarrow \mathbb{R}$  be a real valued  $\mathcal{C}^1$  function. Then the set of critical values of  $\varphi$  has Lebesgue measure 0.*

*Proof.* Firstly note that a point  $z \in Z$  is critical for  $\varphi$  if, by definition, the differential of  $\varphi$  vanishes on  $T_{Z,z}$ , the Zariski tangent space of  $Z$  at  $z$ . Remember also that a complex space is, by definition, countable at infinity ; so  $Z$  and its singular locus have only countably many irreducible components. As a countable union of sets of measure 0 is again of measure 0, it is enough to prove the lemma when  $Z$  is irreducible. We shall prove the lemma by induction on the integer  $\dim Z$ . The case  $\dim Z = 0$  is obvious. Assume the lemma true for  $\dim Z \leq n - 1$  for some integer  $n \geq 1$  and take an irreducible complex space  $Z$  of dimension  $n$ . The singular set  $S$  of  $Z$  has dimension at most  $(n - 1)$ , and for each irreducible component  $S_i$  of  $S$  the image of the critical set of  $\varphi|_{S_i}$  has measure 0. So the critical set of  $\varphi|_S$  is again of measure 0. But a critical point of  $\varphi$  which belongs to  $S$  is a critical point of  $\varphi|_S$ . So it is enough to show that the set of critical values of  $\varphi$  restricted to the complex connected manifold  $Z \setminus S$  has measure 0. This is the classical Sard's lemma.  $\square$

**Lemma 3.2.** *Let  $V$  be an open set and  $K$  be a compact set in  $\bar{U} \times \bar{B}$ . The subset  $\mathcal{V}$  in  $H(\bar{U}, \text{Sym}^k(B))$  consisting of the  $X$  such that any irreducible component meeting  $K$  meets  $V$  is an open set in  $H(\bar{U}, \text{Sym}^k(B))$ .*

*Proof.* Let us clarify the meaning of an irreducible component of an element  $X$  in  $H(\bar{U}, \text{Sym}^k(B))$ : we call irreducible component of such a  $X$  the closure in  $\bar{U} \times B$  of an irreducible component of the branched covering of  $U$  defined by the projection of  $X \cap (U \times B)$  on  $U$ .

<sup>6</sup>  $\uparrow$  as it exists some  $(m, v) \in \bar{M}(\alpha)_a \times \bar{V}_a$  such that  $\Gamma \cap (\{m, v\} \times B_a) \neq \emptyset$ .

Let  $X_0$  be such that any irreducible component of  $X_0$  which meets  $K$  meets  $V$ , and assume that  $(X_\nu)_{\nu \geq 1}$  is a sequence converging to  $X_0$  such that for each  $\nu \geq 1$  there exists an irreducible component  $\Gamma_\nu$  of  $X_\nu$  meeting  $K$  but not  $V$ . Passing to a subsequence, we may assume that the sequence  $(\Gamma_\nu)_{\nu \geq 1}$  converges uniformly on any compact of  $U \times B$  to a non empty  $n$ -cycle  $\Gamma$  with closure contained in  $X_0$  and which is a branched covering of  $U$ . Then  $\bar{\Gamma}$  meets  $K$  and not  $V$ . Indeed, if  $(t_0, x_0)$  would be in  $\bar{\Gamma} \cap V$ , there exists open neighbourhoods  $U_1$  and  $B_1$  of  $t_0$  and  $x_0$  respectively in  $\bar{U}$  and  $\bar{B}$  such that  $U_1 \times B_1$  is contained in  $V$ . But then, as  $U_2 := U_1 \cap U$  and  $B_2 := B_1 \cap B$  are non empty open sets, for  $t_2$  in  $U_2$  the fibers of the  $\Gamma_\nu$  at  $t_2$  for  $\nu$  big enough will meet  $\{t_2\} \times B_2$  and so  $V$ . As at least one irreducible component of  $\Gamma$  meets  $K$  without meeting  $V$  and as its closure is an irreducible component of  $X_0$ , this gives a contradiction.  $\square$

Of course, in the case  $V = \emptyset$ , we get back the fact that the subset in  $H(\bar{U}, \text{Sym}^k(B))$  of elements which do not meet  $K$  is open.

**Lemma 3.3.** *Let  $Z$  be a complex space and let  $(\mathcal{U}_i)_{i \in I}$  be an open covering of  $Z$ . Assume that for each  $i \in I$  a closed  $n$ -cycle  $X_i$  is given in  $\mathcal{U}_i$ . Assume that the following patching condition holds:*

$$\forall (i, j) \in I^2 \quad X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$$

*as an equality of cycles in  $\mathcal{U}_i \cap \mathcal{U}_j$ . Then there exists a unique closed  $n$ -cycle  $X$  in  $Z$  such that for each  $i \in I$  we have  $X \cap \mathcal{U}_i = X_i$ .*

For the easy proof see [BM14, Chapter IV, Proposition 1.3.1].

The following variant will be used.

**Lemma 3.4. (Variant)** *In the situation of the previous lemma replace the patching condition by the following two conditions:*

1. *For each couple  $(i, j) \in I^2$  an open subset  $W_{i,j} \subset\subset \mathcal{U}_i \cap \mathcal{U}_j$  is given and we ask that  $X_i \cap W_{i,j} = X_j \cap W_{i,j}$ .*
2. *For each couple  $(i, j) \in I^2$  we ask that any irreducible component of the cycle  $X_i \cap \mathcal{U}_j$  meets the open set  $W_{i,j}$ .*

*Then the conclusion is the same.*

*Proof.* Let  $\Gamma$  be an irreducible component with multiplicity  $\delta$  in the cycle  $X_i \cap \mathcal{U}_j$ . Let  $\Gamma'$  be the irreducible component of  $X_i$  which contains  $\Gamma$ , and put  $X_i = X'_i + \delta \cdot \Gamma'$ . Then  $\Gamma'$  meets  $W_{i,j}$  and there exists a closed analytic subset  $Y$  of pure dimension  $n$  in  $|X_j|$  such that its restriction to  $W_{i,j}$  is equal to  $\Gamma' \cap W_{i,j}$ : indeed,  $Y$  is the union of the irreducible components of  $X_j$  containing a non empty open set in  $\Gamma' \cap W_{i,j}$ . Note that each of these irreducible components of  $X_j$  has multiplicity  $\delta$  in the cycle  $X_j$ . Then put  $X_j = X'_j + \delta \cdot Y$ . We see that the cycles  $X'_i$  and  $X'_j$  respectively in  $\mathcal{U}_i$  and  $\mathcal{U}_j$  satisfy again the patching condition  $X'_i \cap W_{i,j} = X'_j \cap W_{i,j}$ .

This allows, for fixed  $(i, j)$ , to make a descending induction on the number (necessarily finite as  $W_{i,j}$  is relatively compact) of irreducible components of  $X_i \cap \mathcal{U}_j$ , to show that the condition  $X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$  holds. This reduces this lemma to the previous one.  $\square$

### 3.B. Adjusted scales.

The definition of a scale adapted to a cycle is recalled in Definition 1.3.

**Definition 3.5.** 1. Let  $Z$  be a complex space. We shall call **adjusted  $n$ -scale on  $Z$** , written down  $\mathbb{E} := (U, U', U'', B, B'', j)$ , the data of a  $n$ -scale on  $Z$ ,  $E := (U, B, j)$ , with additional polydiscs  $U'' \subset\subset U' \subset\subset U$  and  $B'' \subset\subset B$ . We call  $E$  the **underlying scale of the adjusted scale  $\mathbb{E}$** .



2. We shall say that **the adjusted scale  $\mathbb{E}$  is adapted to a  $n$ -cycle  $X$  in  $Z$**  when we have

$$j^{-1}(\bar{U} \times (\bar{B} \setminus B'')) \cap |X| = \emptyset.$$

Note that this implies that the underlying scale  $E$  is adapted to  $X$ , but this condition is more restrictive.

3. When the adjusted scale  $\mathbb{E}$  is adapted to the  $n$ -cycle  $X$ , we shall call **degree of  $X$  in  $\mathbb{E}$**  the degree of  $X$  in  $E$ .
4. We shall call **center of the adjusted scale**, written down  $D(\mathbb{E})$ , or more simply,  $D(E)$ , the open set  $j^{-1}(U \times B)$  in  $Z$  which is also the center of the scale  $E$ .
5. We shall call **domain of isotropy of the adjusted scale**, written down  $D'(\mathbb{E})$ , the open set  $j^{-1}(U' \times B)$  in  $Z$ .
6. We shall call **domain of patching of the adjusted scale**, written down  $D''(\mathbb{E})$ , the open set  $j^{-1}(U'' \times B'')$  in  $Z$ .

#### Remarks.

1. The open set  $D''(\mathbb{E})$  is relatively compact in  $D'(\mathbb{E})$ .
2. When a  $n$ -scale  $E$  is given, for any compact set  $K$  in  $D(E)$ , there exists an adjusted  $n$ -scale  $\mathbb{E}$  such that  $E$  is the underlying scale of  $\mathbb{E}$  and with  $K \subset D''(\mathbb{E})$ . Moreover, if  $E$  is adapted to a  $n$ -cycle  $X_0$  in  $Z$ , we may choose  $\mathbb{E}$  in order that it is adapted to  $X_0$ .
3. As for  $X \in \mathcal{C}_n^{loc}(Z)$  the condition to avoid a given compact subset is open in  $\mathcal{C}_n^{loc}(Z)$ , when the adjusted scale  $\mathbb{E}$  is adapted to a cycle  $X_0$  there exists an open neighbourhood, written down  $\Omega_k(\mathbb{E})$ , of  $X_0$  in  $\mathcal{C}_n^{loc}(Z)$  such that  $\Omega_k(\mathbb{E})$  is the subset of all  $n$ -cycles  $X$  in  $Z$  for which  $\mathbb{E}$  is adapted and  $\deg_{\mathbb{E}}(X) = k$  where  $k := \deg_{\mathbb{E}}(X_0)$ .

Let  $Z$  be a reduced complex space and let  $\mathbb{E} = (U, U', U'', B, B'', j)$  be an adjusted scale on  $Z$ . For a given integer  $k$  consider the continuous map sending a branched covering in  $H(\bar{U}, \text{Sym}^k(B''))$  to its isotropy data on  $\bar{U}'$  (for the notations see what follows Lemma 2.5)

$$T : H(\bar{U}, \text{Sym}^k(B'')) \rightarrow H(\bar{U}', F \otimes E').$$

The graph  $\Sigma_{U, U'}(k)$  of this map is a Banach analytic set<sup>7</sup>, thanks to [Bar75, Proposition 2, p. 81] (see also [BM]).

The set of couples  $(f, T(f))$  in  $\Sigma_{U, U'}(k)$  for which the associated branched covering is contained in  $j(Z \cap \bar{D}(\mathbb{E}))$  is a closed Banach analytic subset of  $\Sigma_{U, U'}(k)$  being the pull-back by the projection of the subset of elements in  $H(\bar{U}, \text{Sym}^k(B''))$  contained in  $j(Z \cap \bar{D}(\mathbb{E}))$  which is a closed Banach analytic subset of  $H(\bar{U}, \text{Sym}^k(B''))$  by Proposition 4, p. 27 of [Bar75] (see also [BM, Chapter V]).

**Definition 3.6.** We shall denote  $\mathcal{G}_k(\mathbb{E})$  this Banach analytic set and we shall call it the **the  $k$ -th classifying space of the adjusted  $n$ -scale  $\mathbb{E}$  on  $Z$** .

We have then a tautological family of  $n$ -cycles in the open set  $D(\mathbb{E})$  parametrized by  $\mathcal{G}_k(\mathbb{E})$ . It is an analytic family of cycles in the open set  $D'(\mathbb{E})$ , in the sense of [Bar75], and the fact that, for  $k \geq 1$ , locally on  $\mathcal{G}_k(\mathbb{E})$ , any irreducible component of a branched covering in this family meets  $\bar{U}'' \times \bar{B}''$  implies that we have a  $f$ -analytic family of cycles in  $D'(\mathbb{E})$ .

<sup>7</sup>  $\uparrow$  homeomorphic to  $H(\bar{U}, \text{Sym}^k(B''))$  via the projection!

Be careful that the tautological family of cycles on the open set  $D(\mathbb{E})$  parametrized by  $\mathcal{G}_k(\mathbb{E})$  is not, in general, an analytic family of  $n$ -cycles ; see the example of [Bar75, p. 83] (and also [BM14, Chapter IV]).

The next lemma is an obvious consequence of *loc. cit.*

**Lemma 3.7.** *Let  $\mathbb{E}$  be an adjusted  $n$ -scale on a reduced complex space  $Z$  and let  $k$  be an integer. The tautological family of  $n$ -cycles in the open set  $D'(\mathbb{E})$  parametrized by  $\mathcal{G}_k(\mathbb{E})$  has the following “almost universal” property:*

*For any analytic family of  $n$ -cycles  $(X_s)_{s \in S}$  in  $Z$  parametrized by a Banach analytic set  $S$  such that for each  $s \in S$  the adjusted scale  $\mathbb{E}$  is adapted to  $X_s$  with  $\deg_{\mathbb{E}}(X_s) = k$ , there exists a unique holomorphic map*

$$f : S \rightarrow \mathcal{G}_k(\mathbb{E})$$

*such that the pull-back by  $f$  of the tautological family is the restriction to the open set  $D'(\mathbb{E})$  of the given family.*

Of course, conversely, such a holomorphic map gives a  $f$ -analytic family of  $n$ -cycles on the open set  $D'(\mathbb{E})$ .

Note that the pull-back family is in fact defined on the open set  $D(\mathbb{E})$  but, as already noticed above, it may not be analytic outside  $D'(\mathbb{E})$ .

As a consequence of this “almost universal” property, we obtain that for any analytic family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $Z$  such that for a point  $s_0 \in S$  the adjusted  $n$ -scale  $\mathbb{E}$  is adapted to the cycle  $X_{s_0}$  with  $\deg_{\mathbb{E}}(X_{s_0}) = k$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  such that the previous lemma applies for the family parametrized by  $S'$ . So we shall have a holomorphic classifying map  $f : S' \rightarrow \mathcal{G}_k(\mathbb{E})$  in this situation.

We shall generalize now the concept of classifying space to the case of a finite family of adjusted  $n$ -scales.

**Definition 3.8.** Consider a reduced complex space  $Z$  and a finite family of adjusted  $n$ -scales  $(\mathbb{E}_i)_{i \in I}$  on  $Z$ . Assume that they are adapted to a given finite type  $n$ -cycle  $\hat{X}_0$  in  $Z$ . Assume that any irreducible component of  $\hat{X}_0$  meets the open set  $W'' := \bigcup_{i \in I} D''(\mathbb{E}_i)$ .

We shall call **patching data for  $\hat{X}_0$  associated to the family  $(\mathbb{E}_i)_{i \in I}$** , written  $\mathcal{R}((\mathbb{E}_i)_{i \in I}, F)$  or more simply  $\mathcal{R}$  when there is no ambiguity, a finite collection of  $n$ -scales  $(F_{i,j,h})$  for  $(i, j) \in I^2$ ,  $i \neq j$ , where  $h$  belongs to a finite set  $H(i, j)$  for each couple  $(i, j) \in I^2$ ,  $i \neq j$ , such that the following properties hold:

- i)  $F_{i,j,h}$  is a  $n$ -scale on the open set  $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$ .
- ii) The  $n$ -scales  $F_{i,j,h}$  are adapted to  $\hat{X}_0$ .

We shall say that the patching data  $\mathcal{R}$  are **complete** when the following condition also holds:

- iii) For each  $i \neq j$  given, the union of domains of the scales  $F_{i,j,h}$ ,  $h \in H(i, j)$ , covers the compact subset  $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$  of  $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$ .

### Notations.

1. In the sequel, when we shall consider a reduced complex space  $Z$  and a finite family of adjusted  $n$ -scales  $(\mathbb{E}_i)_{i \in I}$ , adapted to a finite type  $n$ -cycle  $\hat{X}_0$  in  $Z$ , such that any irreducible component of  $\hat{X}_0$  meets the open set

$$W'' := \bigcup_{i \in I} D''(\mathbb{E}_i),$$

we shall say that the family  $(\mathbb{E}_i)_{i \in I}$  is **convenient** for  $\hat{X}_0$ .

2. In this setting we shall use the following definitions :

- $W := \bigcup_{i \in I} D(\mathbb{E}_i)$  ;
- $W' := \bigcup_{i \in I} D'(\mathbb{E}_i)$  ;
- $W'' := \bigcup_{i \in I} D''(\mathbb{E}_i)$  ;
- $K := \bigcup_{i \in I} j_i^{-1}(\bar{U}_i \times (\bar{B}_i \setminus B_i''))$ .
- When the family  $(\mathbb{E}_i)_{i \in I}$  is convenient for a finite type  $n$ -cycle  $\hat{X}_0$ ,  $\mathcal{K}$  will be a compact neighbourhood of  $K$  disjoint from  $\hat{X}_0$ .

3. For  $\tilde{X} \in \prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$  we shall denote by  $X^i$  the closed cycle in  $D(\mathbb{E}_i)$  associated to the  $i$ -th component of  $\tilde{X}$ .

**Lemma 3.9.** *Let  $(\mathbb{E}_i)_{i \in I}$  be a finite family of adjusted  $n$ -scales on  $Z$ , convenient for a  $n$ -cycle  $\hat{X}_0$  of finite type in  $Z$ , and let  $\mathcal{R}$  be some corresponding complete patching data. There exists an open neighbourhood  $\mathcal{V}$  of the image  $\tilde{X}_0$  of  $\hat{X}_0$  in the product  $\prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$  such that for each  $\tilde{X} \in \mathcal{V}$  we have the following properties:*

1. No  $X^i$  meets the compact set  $\mathcal{K}$ .
2. For each  $i \in I$ , any irreducible component of  $X^i$  meeting  $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$  with  $j \neq i$ , meets the open set  $\bigcup_h D(F_{i,j,h})$ .
3. For each  $(i, j, h)$  the scale  $F_{i,j,h}$  is adapted to  $X^i$  and  $X^j$ .
4. For each  $(i, j, h)$  we have  $\deg_{F_{i,j,h}}(X^i) = \deg_{F_{i,j,h}}(X^j) = \deg_{F_{i,j,h}}(\hat{X}_0) = k_{i,j,h}$ .

*Proof.* Conditions 1, 3 and 4 are clearly open. An easy consequence of Condition 1, of the inclusion ii) of Definition 3.8 and of Lemma 3.2 is that Condition 2 is also open.  $\square$

For a  $n$ -scale  $E := (U, B, j)$  on  $Z$  we shall abbreviate  $H(\bar{U}, \text{Sym}^k(B))$  in  $G_k(E)$ .

When we consider a cycle  $X_0$  in an adapted scale  $E := (U, B, j)$  and when we dispose of a  $n$ -scale  $F := (V, C, h)$  on  $U \times B$ , adapted to  $X_0$ , where  $h$  is given by an isomorphism of an open set in  $U \times B$  into some open neighbourhood of  $\bar{V} \times \bar{C}$  in  $\mathbb{C}^n \times \mathbb{C}^p$ , we have a well-defined map of an open neighbourhood  $\mathcal{U}$  of  $X_0$  is  $H(\bar{U}, \text{Sym}^k(B))$  into  $H(\bar{V}, \text{Sym}^l(C))$ , where  $l := \deg_F(X_0)$ , sending  $X \in \mathcal{U}$  to the multiform graph associated to  $h_*(X)$  in the scale  $F$ . This is a consequence of the fact that the condition  $X_0 \cap h^{-1}(\bar{V} \times \partial C) = \emptyset$  is open in  $H(\bar{U}, \text{Sym}^k(B))$  and that the degree of  $X$  near enough  $X_0$  in the adapted scale  $F$  will be equal to  $l$ .

Such a map, which will be called a **change of scale**, is not holomorphic in general but becomes holomorphic when we add the isotropy condition:

precisely, if  $U' \subset\subset U$  and if  $h^{-1}(\bar{V} \times \bar{C}) \subset U' \times B$ , then the change of scale map

$$\Sigma_{U,U'} \rightarrow H(\bar{V}, \text{Sym}^l(C))$$

will be holomorphic (see Theorem 4, p. 66 in [Bar75]).

**Definition 3.10.** Let  $(\mathbb{E}_i)_{i \in I}$  a finite family of adjusted  $n$ -scales on  $Z$ , convenient for a  $n$ -cycle  $\hat{X}_0$  of finite type in  $Z$ , and let  $\mathcal{R}$  be some corresponding patching data. Let  $k_i$  be the degree of  $\hat{X}_0$  in the adjusted scale  $\mathbb{E}_i$ . For each  $(i, j, h)$ ,  $i \neq j$  we have a couple of holomorphic maps

$$\prod_{\alpha \in I} \mathcal{G}_{k_\alpha}(\mathbb{E}_\alpha) \xrightarrow{\quad} G_{k_{i,j,h}}(F_{i,j,h})$$

obtained by the changes of scales  $\mathbb{E}_i \rightarrow F_{i,j,h}$  and  $\mathbb{E}_j \rightarrow F_{i,j,h}$ , because, by construction, we have  $D(F_{i,j,h}) \subset\subset D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$ .

We shall denote by  $S(\mathcal{R})$  the intersection of the kernels of these double maps<sup>8</sup> with the open set  $\mathcal{V}$  built in Lemma 3.9. It is a Banach analytic set and we shall call it the **classifying space associated to  $(\mathbb{E}_i)_{i \in I}, \hat{X}_0$  and  $\mathcal{R}$** .

Remark that the patching data  $\mathcal{R}$  are not assumed to be complete in the previous definition.

**Proposition 3.11.** *Consider a finite family of adjusted  $n$ -scales which is convenient for the  $n$ -cycle  $X_0$  in  $Z$  and let  $\mathcal{R}$  be some **complete** patching data associated. Keeping the previous notations we have for each  $(X^i)_{i \in I} \in S(\mathcal{R})$  a unique  $n$ -cycle  $X \in \mathcal{C}_n^f(W')$  such that  $X \cap D'(\mathbb{E}_i) = X^i \cap D'(\mathbb{E}_i)$ ,  $\forall i \in I$ .*

*Moreover, this defines a tautological family of cycles in  $W'$  which is a  $f$ -analytic family of cycles satisfying the following “almost universal” property:*

*For any analytic family of  $n$ -cycles  $(X_s)_{s \in S}$  in an open neighbourhood of  $\bar{W}$  parametrized by a Banach analytic set  $S^9$  such that for  $s_0 \in S$  we have  $X_{s_0} = X_0$  in a neighbourhood of  $\bar{W}$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  and a unique holomorphic map*

$$f : S' \rightarrow S(\mathcal{R})$$

*such that the pull-back by  $f$  of the tautological family parametrized by  $S(\mathcal{R})$  is the restriction to the open set  $W'$  of the family  $(X_s)_{s \in S'}$ .*

*Proof.* For each  $\tilde{X} \in \mathcal{V}$  any  $X^i$  does not meet  $\mathcal{K}$ . As  $\mathcal{R}$  is complete, we may use Lemma 3.4 with  $\mathcal{U}_i := D'(\mathbb{E}_i)$  and  $W_{i,j} = \bigcup_h D(F_{i,j,h})$  to associate to  $\tilde{X}$  a finite type  $n$ -cycle  $X$  of the open set  $W'$ . The  $f$ -analyticity of the so defined family is obvious. The “almost universal” property is then clear.  $\square$

Note that this proposition implies that the map  $S(\mathcal{R}) \rightarrow \mathcal{C}_n^f(W')$  classifying the tautological family of  $n$ -cycles in  $W'$  is a holomorphic map.

### 3.C. Shrinkage.

**Definition 3.12.** Let  $Z$  be a reduced complex space and let  $\mathbb{E} := (U, U', U'', B, B'', j)$  be an adjusted  $n$ -scale on  $Z$ . For any real  $\tau > 0$  small enough we shall denote by  $\mathbb{E}^\tau$  the adjusted  $n$ -scale on  $Z$  defined as  $\mathbb{E}^\tau := (U^\tau, U'^\tau, U''^\tau, B, B'', j)$ . We shall call  $\mathbb{E}^\tau$  the  $\tau$ -shrinkage of  $\mathbb{E}$ .

Recall that for a polydisc  $P$  of radius  $R$ ,  $P^\tau$  is the polydisc with same center and radius  $R - \tau$ . The definitions of  $\mathcal{M}(\alpha)^\tau$ ,  $M(\alpha)^\tau$  are given in the section 2.B.

#### Remarks.

1. By definition, the shrinkage of  $\mathbb{E}$  does not change the embedding  $j$  and the polydiscs  $U'', B, B''$ .
2. As  $j$  is a closed embedding of an open set in  $Z$  in an open neighbourhood of the compact set  $\bar{U} \times \bar{B}$ , it is clear that for any given adjusted  $n$ -scale  $\mathbb{E}$  on  $Z$ , there exists a real  $\varepsilon > 0$  (depending on  $\mathbb{E}$ ) such that for any  $\tau \in ]0, \varepsilon[$ ,  $\mathbb{E}^\tau$  is again an adjusted  $n$ -scale on  $Z$ .
3. If  $\mathbb{E}$  is an adjusted scale adapted to the  $n$ -cycle  $X_0$  in  $Z$ , for  $\tau$  small enough (depending on  $\mathbb{E}$  and  $X_0$ ), the adjusted  $n$ -scale  $\mathbb{E}^\tau$  remains adapted to  $X_0$  and we shall have also  $\deg_{E^\tau}(X_0) = \deg_E(X_0)$ .

<sup>8</sup>  $\uparrow$  The kernel of a double map  $f, g : A \rightarrow B$  is the pull-back of the diagonal in  $B \times B$  by the map  $(f, g) : A \rightarrow B \times B$ .

<sup>9</sup>  $\uparrow$  Recall that, by definition, this means that for any  $n$ -scale  $E := (U, B, j)$  on  $Z$ , adapted to some  $X_{s_0}, s_0 \in S$ , with  $\deg_E(X_{s_0}) = k$ , we have an open neighbourhood  $S_0$  of  $s_0$  in  $S$  and a classifying map for the corresponding family of branched coverings  $f : S_0 \times U \rightarrow \text{Sym}^k(B)$  which is **isotropic**.

4. If  $\mathbb{E}$  is an adjusted scale adapted to the  $n$ -cycle  $X_0$  in  $Z$ , there exists an open neighbourhood  $\mathcal{U}$  of  $X_0$  in  $\mathcal{C}_n^{loc}(Z)$  and a real  $\varepsilon > 0$  such that for any  $X \in \mathcal{U}$  and any  $\tau \in ]0, \varepsilon[$ , the adjusted scale  $\mathbb{E}^\tau$  remains adapted to  $X$  with again  $\deg_{E^\tau}(X) = \deg_E(X_0)$ .
5. In the situation of the previous proposition 3.11, we may, for  $\tau > 0$  small enough, keep the same patching data  $\mathcal{R}$  on the finite family  $(\mathbb{E}_i^\tau)_{i \in I}$  of adjusted  $n$ -scales; if it was complete, it remains complete and if it was convenient<sup>10</sup> for the  $n$ -cycle  $X_0$ , it remains convenient for the  $n$ -cycle  $X_0$ . Then we have a holomorphic restriction map

$$S(\mathcal{R}) \rightarrow S^\tau(\mathcal{R})$$

where  $S^\tau(\mathcal{R})$  is the classifying space associated to the family  $(\mathbb{E}_i^\tau)_{i \in I}$ , and this map is induced by a **finite product of linear (continuous) compact maps**. This last point is crucial for the finiteness results.

### 3.D. Excellent family.

In order to avoid that our notations become too heavy we shall introduce the following conventions when  $\mathcal{H}$  is a  $n$ -Hartogs figure in  $\mathbb{C}^{n+p}$ ; we shall put, using the notations introduced above

$$U := M(\alpha) \times V, \quad U' := M(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad U'' := M(\alpha)^{\varepsilon''} \times V^{\varepsilon''}, \quad B'' := B^{\varepsilon''}$$

where  $\varepsilon' > 0$  is small enough, and where  $0 < \varepsilon'' < \varepsilon'$ . The choices of  $\varepsilon'$  and  $\varepsilon''$  will be precised when they are useful. We shall associate to  $\mathcal{H}$  the adjusted  $n$ -scale on  $\Delta$  given by:

$$\mathbb{E}_{\mathcal{H}} := (U, U', U'', B, B'', j).$$

We shall put also

$$\tilde{U} := \mathcal{M}(\alpha) \times V, \quad \tilde{U}' := \mathcal{M}(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad \tilde{U}'' := \mathcal{M}(\alpha)^{\varepsilon''} \times V^{\varepsilon''}$$

the holomorphy envelopes respectively of  $U$ ,  $U'$  and  $U''$ . Then we shall have a family of adjusted  $n$ -scales on  $\Delta'$ , written down:

$$\mathbb{E}_{\tilde{\mathcal{H}}}^\eta := (\tilde{U}^\eta, \tilde{U}'^\eta, \tilde{U}''^\eta, B, B'', j) \quad \text{with} \quad \tilde{U}^\eta := \mathcal{M}(\alpha)^\eta \times V^\eta, \quad \tilde{U}'^\eta := \mathcal{M}(\alpha)^{\varepsilon'+\eta} \times V^{\varepsilon'+\eta},$$

where  $\eta$  is a non negative real number, small enough (for  $\eta = 0$  we shall simply write  $\mathbb{E}_{\tilde{\mathcal{H}}}$ ).

Then we shall have the isotropic classifying spaces

$$\mathcal{G}_k(\mathcal{H}) := \Sigma_{U, U'}(k) \quad \text{and also} \quad \mathcal{G}_k^\eta(\tilde{\mathcal{H}}) := \Sigma_{\tilde{U}^\eta, \tilde{U}'^\eta}(k).$$

Proposition 2.6 gives a holomorphic analytic extension map  $prlgt^\eta$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}_k(\tilde{\mathcal{H}}) & \xrightarrow{res^\eta} & \mathcal{G}_k^\eta(\tilde{\mathcal{H}}) \\ res \downarrow & \nearrow prlgt^\eta & \\ \mathcal{G}_k(\mathcal{H}) & & \end{array}$$

**Definition 3.13.** Let  $Z$  be a reduced complex space of pure dimension  $n + p$ , let  $\Delta \subset\subset Z$  be a strongly  $(n - 2)$ -concave open set in  $Z$  and  $\tilde{X}_0$  a  $n$ -cycle in an open neighbourhood  $\Delta'$  of  $\Delta$  in  $Z$ . We shall say that a finite family  $(\mathcal{H}_a)_{a \in A}$  of  $n$ -Hartogs figures relative to the boundary of  $\Delta$  is **excellent** for the cycle  $\tilde{X}_0$  when the following conditions hold, where we write  $\tilde{\mathbb{E}}_a$  and  $\mathbb{E}_a$  the adjusted scales respectively on  $\Delta'$  and  $\Delta$  associated to the  $n$ -Hartogs figure  $\mathcal{H}_a$  :

<sup>10</sup> ↑ See the notations following Definition 3.8.

1. The adjusted scales  $\tilde{\mathbb{E}}_a$  and  $\mathbb{E}_a$  are adapted to the cycle  $\tilde{X}_0$ .
2. We may choose the patching domains of the adjusted scales  $(\tilde{\mathbb{E}}_a)_{a \in A}$  in order that the union

$$\tilde{D}''(A) := \bigcup_{a \in A} j_a^{-1}(\tilde{U}''_a \times B''_a)$$

contains the compact set  $\partial\Delta$ .

3. There exists a finite family of adjusted  $n$ -scales  $(\mathbb{E}_b)_{b \in B}$  on  $\Delta$ , adapted to  $\tilde{X}_0$ , such that the finite families  $(\mathbb{E}_c)_{c \in A \cup B}$  and  $(\tilde{\mathbb{E}}_c)_{c \in A \cup B}$  are convenient for  $\tilde{X}_0$ , where we put  $\tilde{\mathbb{E}}_b = \mathbb{E}_b$  for  $b \in B$ . Moreover we ask that the union  $D''(B)$  of the patching domains of the  $(\mathbb{E}_b)_{b \in B}$  covers the compact set  $\Delta \setminus \tilde{D}''(A)$  of  $\Delta$ ; so  $\tilde{D}''(A) \cup D''(B)$  in an open set containing  $\bar{\Delta}$ .

As a consequence, the union  $\tilde{D}'(A)^\eta \cup D'(B)$  of the isotropy domains will cover  $\bar{\Delta}$  for  $\eta > 0$  small enough.

**Proposition 3.14. (Existence of excellent families)** *Let  $Z$  be a reduced complex space of pure dimension  $n + p$ ,  $\Delta \subset\subset Z$  be a strongly  $(n - 2)$ -concave open set with smooth boundary and  $\tilde{X}_0$  a  $n$ -cycle in an open neighbourhood  $\Delta' \subset Z$  of  $\bar{\Delta}$  such that any irreducible component of  $\tilde{X}_0$  meets  $\Delta$ .*

*Then there exists a finite family  $(\mathcal{H}_a)_{a \in A}$  of  $n$ -Hartogs figures relative to the boundary of  $\Delta$  which is excellent for the cycle  $\tilde{X}_0$ .*

*Proof.* First we use Proposition 2.13 to cover  $\partial\Delta$  by a finite family of  $n$ -Hartogs figures relative to the boundary of  $\Delta$  adapted to the cycle  $\tilde{X}_0$  such that Conditions 1 and 2 hold. Then we build a finite family of adjusted  $n$ -scales  $(\mathbb{E}_b)_{b \in B}$  on  $\Delta$ , adapted to  $\tilde{X}_0$  in order that Condition 3 holds.  $\square$

### 3.E. The extension and finiteness theorem.

The next theorem will be crucial in the proof of Theorem 1.1.

**Theorem 3.15.** *Let  $Z$  be a reduced complex space of pure dimension  $n + p$ , where  $n \geq 2$ ,  $p \geq 1$ . Assume that there exists a  $\mathcal{C}^2$  exhaustion  $\varphi : Z \rightarrow ]0, 2]$  which is strongly  $(n - 2)$ -convex on the open set  $\varphi^{-1}(]0, 1[)$  and let  $\Delta := \{x \in Z / \varphi(x) > \alpha\}$  for some  $\alpha \in ]0, 1[$  which is not a critical value for  $\varphi$ . Let  $\tilde{X}_0$  a closed  $n$ -cycle of an open neighbourhood  $\Delta'$  of  $\bar{\Delta}$  in  $Z$  such that any irreducible component of  $\tilde{X}_0$  meets  $\Delta$ . Then there exists an open neighbourhood  $\Delta''$  of  $\bar{\Delta}$  in  $\Delta'$  and a  $\mathfrak{f}$ -analytic family  $(\tilde{X}_\xi)_{\xi \in \Xi}$  of  $n$ -cycles in  $\Delta''$  parametrized by a reduced complex space  $\Xi$  (of finite dimension) such that  $X_{\xi_0} = \tilde{X}_0 \cap \Delta''$  and such that  $\Xi$  is isomorphic to an open neighbourhood of  $\tilde{X}_0 \cap \Delta''$  in  $C_n^{\mathfrak{f}}(\Delta'')$ . It satisfies the following universal property:*

*For any  $\mathfrak{f}$ -analytic family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $\Delta$  parametrized by a Banach analytic set  $S$  and such that its value at some  $s_0 \in S$  is equal to  $\tilde{X}_0 \cap \Delta$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  and an unique holomorphic map*

$$h : S' \rightarrow \Xi$$

*satisfying the equality  $X_s = \tilde{X}_{h(s)} \cap \Delta$  for each  $s \in S'$ .*

*Proof.* Note that  $\Delta$  is a relatively compact open set in  $Z$  with a  $\mathcal{C}^2$  boundary which is strongly  $(n - 2)$ -concave.

Begin by covering the compact set  $\partial\Delta$  by an excellent finite family  $(\mathcal{H}_a)_{a \in A}$  of  $n$ -Hartogs figures for  $\Delta$  adapted to the cycle  $\tilde{X}_0$ . Choose then open sets  $\Delta_1 \subset\subset \Delta \subset\subset \Delta'' \subset\subset \Delta'$ , such that the following properties hold, where we use the notations introduced above for the finite family of the adjusted scales  $(\tilde{\mathbb{E}}_a)_{a \in A}$  associated to  $(\mathcal{H}_a)_{a \in A}$ :

- i)  $\bar{\Delta}'' \setminus \Delta_1 \subset \bigcup_{a \in A} W''(\tilde{\mathbb{E}}_a)$ .
- ii)  $\bigcup_{a \in A} W(\tilde{\mathbb{E}}_a) \subset \subset \Delta'$ .
- iii)  $\bigcup_{a \in A} \bar{W}(\mathbb{E}_a) \subset \Delta$ .
- iv)  $K := \bigcup_{a \in A} j_a^{-1}(\bar{U}_a \times (\bar{B}_a \setminus B''_a)) \subset \Delta_1$

It is easy to fulfill these conditions for  $\Delta_1$  and  $\Delta''$  near enough to  $\Delta$ , as, by assumption, the subsets  $\bar{W}(\mathbb{E}_a)$  and  $j_a^{-1}(\bar{U}_a \times \bar{B}_a \setminus B''_a)$  are compact in  $\Delta$ , and as the union of the open sets  $W''(\tilde{\mathbb{E}}_a)$  contains  $\partial\Delta$ .

Note that Condition iv) allows to choose the compact neighbourhood  $\mathcal{K}$  of  $K$  inside  $\Delta_1$ .

Choose now a convenient finite family  $(\mathbb{E}_b)_{b \in B}$  of adjusted  $n$ -scales on  $\Delta$ , adapted to  $\tilde{X}_0$  in order that the open set  $\bigcup_{b \in B} W''(\mathbb{E}_b)$  contains  $\bar{\Delta}_1$ .

Put  $\tilde{\mathbb{E}}_b = \mathbb{E}_b$  for  $b \in B$ , and define  $C := A \cup B$ . The family of adjusted scales  $(\mathbb{E}_c)_{c \in C}$  in  $\Delta$  is then convenient for  $\tilde{X}_0 \cap \Delta_1$ , up to choosing the patching domains big enough. Fix some complete patching data  $\mathcal{R}$  associated to the family  $(\mathbb{E}_c)_{c \in C}$ .

The family  $(\tilde{\mathbb{E}}_c)_{c \in C}$  of adjusted scales in  $\Delta'$  is convenient for  $\tilde{X}_0 \cap \Delta''$  if we choose the patching domains big enough. Consider now some complete patching data for this family of the form  $\mathcal{R} \cup \mathcal{R}''$ , that is to say containing the patching scales already in  $\mathcal{R}$ . Define the following Banach analytic sets:

1.  $S_0$  is the classifying space of the family  $(\tilde{\mathbb{E}}_c)_{c \in C}$ , the degrees being these of  $\tilde{X}_0$  in the various scales adapted to the cycle  $\tilde{X}_0$ , with the patching conditions defined by  $\mathcal{R}$ . Note that  $\mathcal{R}$  is not complete in general.
2.  $S_+$  is the classifying space of the family  $(\tilde{\mathbb{E}}_c)_{c \in C}$  with the patching conditions defined by  $\mathcal{R} \cup \mathcal{R}''$ .
3.  $S_-$  is the classifying space of the family  $(\mathbb{E}_c)_{c \in C}$ , with the (complete) patching conditions defined by  $\mathcal{R}$ .

Then we get a holomorphic extension map

$$\alpha : S_- \rightarrow S_0$$

deduced from the extension maps in the  $n$ -Hartogs figures  $(\mathcal{H}_a)_{a \in A}$ .

By definition  $S_+$  is a closed Banach analytic subset of  $S_0$  as it is defined in  $S_0$  by the patching conditions given by  $\mathcal{R}''$ . Then put  $\Xi := \alpha^{-1}(S_+)$ . So we have a holomorphic map  $\alpha : \Xi \rightarrow S_+$ . We want to show the following claim:

**Claim.** There exists a holomorphic map  $\beta : S_+ \rightarrow \Xi$  satisfying the two properties:

1. We have  $\alpha \circ \beta = Id$  and  $\beta \circ \alpha = Id$  in a neighbourhood of the points defined by  $\tilde{X}_0$  respectively in  $S_+$  and  $\Xi$ .
2. The holomorphic map  $\beta$  is the composition of a holomorphic map induced by a linear (continuous) compact map and a holomorphic map.

As the open set  $\bigcup_{c \in C} W''(\tilde{\mathbb{E}}_c)$  contains  $\Delta''$ , there is, on  $S_+$ , a tautological family of  $n$ -cycles which is  $f$ -analytic on  $\Delta''$ . Then the ‘‘almost universal’’ property of  $S_-$  gives a holomorphic map  $\tilde{\beta} : S_+ \rightarrow S_-$  which factorizes via the closed Banach analytic subset  $\Xi \subset S_-$ . Let us show that the holomorphic map  $\beta : S_+ \rightarrow \Xi$  deduced from this factorization satisfies the two properties of the claim.

First we have  $\alpha \circ \beta = Id$  and  $\beta \circ \alpha = Id$  respectively in  $S_+$  and  $\Xi$ , by definition of  $\Xi$  and  $\beta$  (see Proposition 2.6).

To see the second property, consider  $\tau > 0$  small enough and let us show that the holomorphic map induced by the linear compact (by Vitali’s theorem) restriction  $r^\tau : S_+ \rightarrow S_+^\tau$  factorizes  $\beta$ , where

$S_+^\tau$  is the classifying space corresponding to the  $\tau$ -shrinkage  $(\tilde{\mathbb{E}}_c^\tau)_{c \in C}$  of the family  $(\tilde{\mathbb{E}}_c)_{c \in C}$  with the patching data  $\mathcal{R} \cup \mathcal{R}''$ . Indeed, for  $\tau$  small enough, the tautological family of  $S_+^\tau$  is still f-analytic on  $\Delta$  and the ‘‘almost universal’’ property of  $S_-$  gives again a holomorphic map  $\tilde{\beta}^\tau : S_+^\tau \rightarrow S_-$ . And we have  $\beta = \tilde{\beta}^\tau \circ r^\tau$  proving our assertion. In fact, the map  $\tilde{\beta}^\tau$  takes its values in  $\Xi$  because the  $\tau$ -shrinkage does not change the patching data deduced from  $\mathcal{R} \cup \mathcal{R}''$  for  $\tau$  small enough.

We conclude that the Banach analytic set  $\Xi$  has finite dimension thanks to the finiteness lemma of [Bar75, p. 8] (see also [BM, Chapter V]). Moreover, it is isomorphic to  $S_+$  which is also of finite dimension.

The holomorphic isomorphism  $\beta : S_+ \rightarrow \Xi$  factorizes via an open neighbourhood of  $X_0 \cap \Delta''$  in  $\mathcal{C}_n^f(\Delta'')$  because  $S_+$  parametrizes a f-analytic family of cycles in  $\Delta''$ , and because a cycle  $X$  near enough to  $X_0 \cap \Delta''$  defines an element in  $\Xi \subset S_-$  as it satisfies the patching conditions  $\mathcal{R} \cup \mathcal{R}''$ . This shows that we can identify the Banach analytic set  $\Xi$  (which is a reduced finite dimensional complex space) with an open neighbourhood of  $X_0 \cap \Delta''$  in  $\mathcal{C}_n^f(\Delta'')$ .

The universal property is obvious.  $\square$

Note that it is not restrictive to choose  $\Delta'' := \{\varphi > \alpha_1\}$  for some  $\alpha_1 \in ]0, \alpha[$ , near enough to  $\alpha$ .

### Remarks.

1. The reduced complex space (of finite dimension)  $\Xi$  built in the previous theorem parametrizes a f-analytic family of  $n$ -cycles in the open set  $\Delta''$  which is an open neighbourhood of  $\tilde{\Delta}$ . So, when we have a f-analytic family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $\tilde{\Delta}$  parametrized by a Banach analytic set  $S$  and such that its value at some  $s_0 \in S$  is equal to  $\tilde{X}_0 \cap \Delta$  we can extend each cycle  $X_s$ ,  $s \in S'$ , to a  $n$ -cycle  $\tilde{X}_s$  in  $\Delta''$  in order that the family  $(\tilde{X}_s)_{s \in S'}$  is f-analytic in  $\Delta''$ , with the condition that each irreducible component of  $\tilde{X}_s$  meets  $\Delta$  and with the equality  $\tilde{X}_s \cap \Delta = X_s$  for each  $s \in S'$ .
2. We shall see later on that  $\Xi$  is also (isomorphic to) an open neighbourhood of the point  $\tilde{X}_0 \cap \Delta$  in  $\mathcal{C}_n^f(\Delta)$ .

## 4. Finiteness of the space of $n$ -cycles of a reduced strongly $(n - 2)$ -concave space ( $n \geq 2$ ).

### 4.A. The global extension theorem.

First we have to complete our terminology.

**Definition 4.1.** We shall say that a reduced complex space  $Z$  is **strongly  $q$ -concave**, where  $q \geq 0$  is a natural integer, if there exists a real valued  $\mathcal{C}^2$  exhaustion function on  $Z$ ,  $\varphi : Z \rightarrow ]0, 2]$ , which is strongly  $q$ -convex outside the compact set  $K := \varphi^{-1}([1, 2])$ .

In the sequel, when we consider a reduced complex space  $Z$  which is assumed to be  $q$ -concave, we shall always assume implicitly that we have chosen such an exhaustion  $\varphi$ . For instance, any reduced compact complex space is strongly  $q$ -concave for any  $q \geq 0$ .

When  $Z$  is strongly  $q$ -concave irreducible and non compact of dimension at least  $q + 1$ , the function  $\varphi$  achieves its maximum at a point in  $K$ . So we shall have  $\varphi(Z) = ]0, u]$  with  $u \geq 1$ .

**Theorem 4.2.** *Let  $n \geq 2$  be an integer. Let  $Z$  be a reduced complex space which is strongly  $(n - 2)$ -concave. Let  $\alpha \in ]0, 1[$  and let  $X$  be a finite type  $n$ -cycle in the open set  $Z_\alpha := \{z \in Z \mid \varphi(z) > \alpha\}$ . Then there exists a unique  $n$ -cycle  $\tilde{X}$  in  $\mathcal{C}_n^f(Z)$  such that  $\tilde{X} \cap Z_\alpha = X$ .*

The proof will use the following remark.



**Remark.** Consider a closed irreducible analytic subset  $\Gamma$  of dimension  $n$  in  $Z$ . As the restriction  $\varphi|_{\Gamma}$  of the exhaustion  $\varphi$  to  $\Gamma$  must reach its maximum (as  $\varphi$  is continuous and proper), this maximum cannot be obtained at a point in which  $\varphi$  is strongly  $(n - 2)$ -convex. So we have  $\Gamma \cap \varphi^{-1}[1, 2] \neq \emptyset$ .

*Proof.* The uniqueness of  $\tilde{X}$  is a consequence of the previous remark: if  $\tilde{X}_i$ ,  $i = 1, 2$ , are in  $\mathcal{C}_n^f(Z)$  and satisfy  $\tilde{X}_i \cap Z_\alpha = X$ , then any irreducible component  $\Gamma_1$  of  $\tilde{X}_1$  has to meet  $Z_\alpha$  and so has to contain an open set in  $\tilde{X}_2$ . Then it has to be an irreducible component of  $\tilde{X}_2$  and its multiplicity in  $\tilde{X}_1$  and  $\tilde{X}_2$  must coincide. So  $\tilde{X}_1 = \tilde{X}_2$ .

To show that this cycle exists, consider first the case where  $X$  is compact in  $Z_\alpha$ . Then  $\tilde{X} := X$  is a solution. So it enough to consider the case where  $X$  is irreducible and non compact. Thanks to [ST71, Theorem 8.3], for each  $z \in \partial Z_\alpha$  there exists an open set  $U_z$  and an unique closed analytic set  $X_z$  in  $Z_\alpha \cup U_z$  of pure dimension  $n$  such that  $X_z \cap Z_\alpha = X$ . Choose a finite set of points  $z_1, \dots, z_N$  and open sets  $U'_i \subset \subset U_i := U_{z_i}$  such that the union of the  $U'_i$  covers the compact set  $\partial Z_\alpha$ . Let  $\Omega := Z_\alpha \cup (\bigcup_{i=1}^N U'_i)$  and put

$$X_1 := (X \cup (\bigcup_{i=1}^N X_{z_i})) \cap \Omega.$$

Let us show that  $X_1$  is a closed analytic subset in  $\Omega$ . Consider  $z \in \Omega$ . If  $z$  is in  $Z_\alpha$  we have  $X_1 = X$  in a neighbourhood of  $z$  and the assertion is clear. If not, either  $z$  is not in any  $\partial U'_i$  and  $X_1$  is the union of the  $X_{z_i}$  in a neighbourhood of  $z$  and the assertion is clear, or  $z$  is in  $\partial U'_{j_1}, \dots, \partial U'_{j_k}$  for  $j_1, \dots, j_k$  in  $[1, N]$ . As the set  $X_{j_h}$  is closed and analytic in  $Z_\alpha \cup U_{j_h}$ , then  $X_1$  is again the union of the  $X_{z_i}$  near  $z$  in  $\Omega$ , and the assertion is proved.

So in this situation there exists a real positive  $\beta < \alpha$  such that  $Z_\beta \subset \Omega$ . Let  $X_2$  be the irreducible component of  $X_1 \cap Z_\beta$  which contains  $X$ ; then  $X_2$  is a closed irreducible analytic subset of  $Z_\beta$  such that  $X_2 \cap Z_\alpha = X$ .

Now let

$$\gamma := \inf\{\beta \leq \alpha / \exists X_\beta \text{ irreducible } n\text{-cycle of } Z_\beta \text{ such that } X_\beta \cap Z_\alpha = X\}.$$

Then what we obtained above shows that we have  $\gamma < \alpha$ , and, applying the same arguments to the cycle  $X_\gamma$  defined on  $Z_\gamma$  via the cover of  $Z_\gamma$  by the  $Z_\beta, \beta > \gamma$  in which we already built an irreducible  $n$ -cycle  $X_\beta$  extending  $X$ , we conclude that  $\gamma = 0$  and that there exists an (unique) irreducible  $n$ -cycle  $\tilde{X}$  in  $Z$  extending  $X$ .  $\square$

#### 4.B. Some consequences.

We shall give first some easy consequences of the fact that the reduced complex space  $Z$  is strongly  $n$ -concave.

**Proposition 4.3.** *Let  $n \geq 2$  be an integer and let  $Z$  be a reduced complex space which is strongly  $(n - 2)$ -concave. Then the natural map  $j : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$  is a homeomorphism. Moreover, for each  $\alpha \in ]0, 1[$  the restriction map*

$$res_\alpha : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$$

*is well defined and is also a homeomorphism.*

*Proof.* Let us prove first that any  $n$ -cycle  $X$  in  $Z$  has finitely many irreducible components. As this implies the same result for each  $Z_\alpha$  for  $\alpha \in ]0, 1[$ , this will imply the fact that the restriction map  $res_\alpha$  is well defined, and then bijective as a consequence of Theorem 4.2.

As the family of irreducible components of a  $n$ -cycle is locally finite, only finitely many irreducible components of  $X$  can meet the compact set  $K := \varphi^{-1}([1, 2])$ . But we have seen in the remark following the previous theorem that **any** irreducible component of  $X$  must meet  $K$ . So  $X$  is a finite type cycle.

To show the continuity of  $res_\alpha^{-1}$  it is then enough to prove that  $j$  is a homeomorphism which is an easy consequence of the lemma below.

**Lemma 4.4.** *Let  $Z$  be a reduced complex space and let  $(X_\nu)_{\nu \geq 0}$  be a sequence of  $n$ -cycles in  $Z$  converging in  $\mathcal{C}_n^{loc}(Z)$  to a cycle  $Y$ . Assume that there exists a relatively compact open set  $\Omega$  in  $Z$  such that any irreducible component of each  $X_\nu$  and of  $Y$  meets  $\Omega$ . Then all these cycles are of finite type and the sequence converges to  $Y$  in  $\mathcal{C}_n^f(Z)$ .*

*Proof of Lemma 4.4.* First if  $Y = \emptyset$ , for  $\nu \gg 1$  the cycle  $X_\nu$  will be disjoint from the compact set  $K := \bar{\Omega}$ , and this implies that  $X_\nu$  is the empty cycle. So the conclusion holds in this case. If  $Y$  is not empty, let  $U$  be a relatively compact open set in  $Z$  meeting each irreducible component of  $Y$ . We have to show, by definition of the topology of  $\mathcal{C}_n^f(Z)$ , that for  $\nu \gg 1$  each irreducible component of  $X_\nu$  meets  $U$ . If it is not the case, passing to a subsequence, we may assume that for each  $\nu$  there exists an irreducible component  $\Gamma_\nu$  of  $X_\nu$  disjoint from  $U$ . As, passing again to a subsequence, we may assume that the sequence  $(\Gamma_\nu)$  converges in  $\mathcal{C}_n^{loc}(Z)$  to a cycle  $\Gamma$ , we shall have  $|\Gamma| \subset |Y|$  and  $|\Gamma| \cap U = \emptyset$ . To conclude, it is enough to show that  $\Gamma$  is not the empty cycle, as any irreducible component of  $\Gamma$  is also an irreducible component of  $Y$  and then meets  $U$  by hypothesis. As each  $\Gamma_\nu$  is not empty, it has to meet  $K = \bar{\Omega}$ . This implies that  $\Gamma$  also meets  $K$  and so is not empty. This contradicts our assumption.  $\square$

*End of the proof of Proposition 4.3.* We have proved that  $j$ , and then also each  $j_\alpha : \mathcal{C}_n^f(Z_\alpha) \rightarrow \mathcal{C}_n^{loc}(Z_\alpha)$  for  $\alpha \in ]0, 1[$ , is a holomorphic homeomorphism. To conclude the proof we have to show the continuity of  $res_\alpha^{-1}$ , and this reduces to prove that if the sequence  $(X_\nu)$  of  $\mathcal{C}_n^{loc}(Z)$  is such that the sequence  $(X_\nu \cap Z_\alpha)$  converges in  $\mathcal{C}_n^{loc}(Z_\alpha)$ , then it converges in  $\mathcal{C}_n^{loc}(Z)$ . Let  $Y_\alpha \in \mathcal{C}_n^{loc}(Z_\alpha)$  be the limit of this sequence in  $\mathcal{C}_n^{loc}(Z_\alpha)$  and let  $Y \in \mathcal{C}_n^{loc}(Z)$  be the cycle extending it. Let  $A$  be the set of  $\beta \in ]0, \alpha]$  such that the sequence  $(X_\nu \cap Z_\beta)$  converges in  $\mathcal{C}_n^{loc}(Z_\beta)$  to  $Y \cap Z_\beta$ . Then  $\alpha$  is in  $A$  so  $A$  is not empty. Put  $\gamma := \inf A$ . Theorem 3.15 implies that  $\gamma = 0$  and we obtain also the convergence in any  $\mathcal{C}_n^{loc}(Z_\beta)$ , for any  $\beta > 0$ ; this gives the convergence in  $\mathcal{C}_n^{loc}(Z)$ , as, by definition, a  $n$ -scale on  $Z$  is also a  $n$ -scale on  $Z_\beta$  for  $\beta > 0$  small enough.  $\square$

#### 4.C. An analytic extension criterion.

The aim of this paragraph is to prove the following analytic extension result.

**Theorem 4.5.** *Let  $Z$  be a complex space and  $n$  an integer. Consider a  $f$ -continuous<sup>11</sup> family  $(X_s)_{s \in S}$  of  $n$ -cycles of finite type in  $Z$  parametrized by a reduced complex space  $S$ . Fix a point  $s_0$  in  $S$  and assume that the open set  $Z'$  in  $Z$  meets all irreducible components of  $X_{s_0}$  and such that the family of cycles  $(X_s \cap Z')_{s \in S}$  is analytic at  $s_0$ . Then there exists an open neighbourhood  $S_0$  of  $s_0$  in  $S$  such that the family  $(X_s)_{s \in S_0}$  is  $f$ -analytic.*

The hypotheses translated in terms of classifying maps means that we have a continuous map  $\varphi : S \rightarrow \mathcal{C}_n^f(Z)$  such that the composed map  $r \circ \varphi$  is holomorphic at  $s_0$ , where  $r : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z')$  is the restriction map.

Then the theorem says that there exists an open neighbourhood  $S_0$  of  $s_0$  in  $S$  such that the map  $\varphi$  is holomorphic on  $S_0$ . Note that, as  $r$  is holomorphic<sup>12</sup>, the hypothesis that  $\varphi$  is holomorphic at  $s_0$  is a necessary condition.

This result is not true in general if we take for  $S$  a non smooth Banach analytic set which is not of finite dimension (locally). The reader may find a counter-exemple with an isolated singularity in [BM, Chapter V].

The key point for the proof of the previous theorem is the following analytic extension result.

<sup>11</sup>  $\uparrow$  This means that we have a continuous family of finite type  $n$ -cycles such that its graph is quasi-proper over  $S$ . This is equivalent to the continuity of the classifying map  $\varphi : S \rightarrow \mathcal{C}_n^f(Z)$  of this family.

<sup>12</sup>  $\uparrow$  in the sense that for any holomorphic map  $\psi : T \rightarrow \mathcal{C}_n^f(Z)$  of a reduced complex space  $T$  the composed map  $r \circ \psi$  is holomorphic.

**Proposition 4.6.** *Let  $S$  a reduced complex space and let  $\emptyset \neq U_1 \subset U_2$  be two polydiscs in  $\mathbb{C}^n$ . Let  $f : S \times U_2 \rightarrow \mathbb{C}$  a continuous function, holomorphic on  $\{s\} \times U_2$  for each  $s \in S$ . Assume moreover that the restriction of  $f$  to  $S \times U_1$  is holomorphic. Then  $f$  is holomorphic on  $S \times U_2$ .*

*Proof of Proposition 4.6.* First consider the case  $S$  smooth. As the question is local on  $S$  it is enough to consider the case where  $S$  is an open set in  $\mathbb{C}^m$ . Fix an open relatively compact polydisc  $P$  in  $S$ . The function  $f$  defines a map  $F : U_2 \rightarrow \mathcal{C}^0(\bar{P}, \mathbb{C})$ , where we write down  $\mathcal{C}^0(\bar{P}, \mathbb{C})$  the Banach space of continuous functions on  $\bar{P}$ , via the formula  $F(t)[s] = f(s, t)$  for  $t \in U_2$  and  $s \in \bar{P}$ . First we shall show that the map  $F$  is holomorphic.

Let  $U \subset\subset U_2$  be a polydisc. For any fix  $s \in S$  we have

$$\frac{\partial f}{\partial t_i}(s, t) = \frac{1}{(2i\pi)^n} \cdot \int_{\partial\partial U} f(s, \tau) \cdot \frac{d\tau_1 \wedge \cdots \wedge d\tau_n}{(\tau_1 - t_1) \cdots (\tau_i - t_i)^2 \cdots (\tau_n - t_n)} \quad \forall t \in U \quad \forall i \in [1, n].$$

where  $t := (t_1, \dots, t_n)$  are coordinates on  $\mathbb{C}^n$ . This Cauchy formula shows that  $F$  is  $\mathbb{C}$ -differentiable and its differential at the point  $t \in U$  is given by

$$h \mapsto \sum_{i=1}^n F_i(t) \cdot h_i, \quad h \in \mathbb{C}^n,$$

where  $F_i$  is associated to the function

$$(s, t) \mapsto \frac{\partial f}{\partial t_i}(s, t) \quad i \in [1, n]$$

which is holomorphic for each fixed  $s \in S$  thanks to the Cauchy formula above.

Let  $H(\bar{P}, \mathbb{C})$  the closed subspace of  $\mathcal{C}^0(\bar{P}, \mathbb{C})$  of functions which are holomorphic on  $P$ . Our hypothesis implies that the restriction of  $F$  to the non empty open set  $U_1$  takes its values in this subspace. Let us show that this is also true on  $U_2$ . Assume that there exists  $t_0 \in U_2$  such that  $F(t_0)$  is not in  $H(\bar{P}, \mathbb{C})$ . Thanks to the *Hahn-Banach theorem* we can find a continuous linear form  $\lambda$  on  $\mathcal{C}^0(\bar{P}, \mathbb{C})$ , vanishing on  $H(\bar{P}, \mathbb{C})$ , and such that  $\lambda(F(t_0)) \neq 0$ . But the function  $t \mapsto \lambda(F(t))$  is holomorphic on  $U_2$  and vanishes on  $U_1$ ; this contradicts  $\lambda(F(t_0)) \neq 0$ . So  $F$  takes values in  $H(\bar{P}, \mathbb{C})$  and  $f$  is holomorphic on  $S \times U_2$  when  $S$  is a complex manifold.

The case where  $S$  is a weakly normal complex space follows immediately.

When  $S$  is a general reduced complex space, the function  $f$  is meromorphic and continuous on  $S \times U_2$  and holomorphic on  $S \times U_1$ . So the closed analytic subset  $Y \subset S \times U_2$  along which  $f$  may not be holomorphic has no interior point in each  $\{s\} \times U_2$ . The analytic extension criterion of [BM14, Chapter IV, Criterion 3.1.7] allows to conclude.  $\square$

*Proof of Theorem 4.5.* Let  $|G| \subset S \times Z$  be the graph of the  $f$ -continuous family  $(X_s)_{s \in S}$  and let  $A$  be the set of points in  $(\sigma, \zeta) \in |G|$  admitting an open neighbourhood  $S_\sigma \times Z_\zeta$  in  $S \times Z$  such that the family of cycles  $(X_s \cap Z_\zeta)_{s \in S_\sigma}$  is analytic. Remark that, thanks to our hypothesis, the open set  $A$  in  $|G|$  meets every irreducible component of  $\{s_0\} \times |X_{s_0}|$ .

Assume to begin that there exists a smooth point  $z_0$  of  $|X_{s_0}|$  in the boundary of  $A \cap (\{s_0\} \times |X_{s_0}|)$ . Choose a  $n$ -scale  $E := (U, B, j)$  on  $Z$  which is adapted to  $X_{s_0}$  and satisfying :

$$\begin{aligned} \deg_E(|X_{s_0}|) &= 1, & j_*(X_{s_0}) &= k \cdot (U \times \{0\}) \\ z_0 &\in j^{-1}(U \times B), & j(z_0) &:= (t_0, 0). \end{aligned}$$

It is clear that such a  $n$ -scale exists as  $z_0$  is a smooth point in  $|X_{s_0}|$ . Let  $S_1$  be a sufficiently small open neighbourhood of  $s_0$  in  $S$  and let  $f : S_1 \times U \rightarrow \text{Sym}^k(B)$  be the (continuous) classifying map for the family  $(X_s)_{s \in S_1}$  in the scale  $E$ . As  $j^{-1}(U \times \{0\})$  meets  $A$ , there exists a non empty polydisc  $U_2 \subset U$  such that  $U_2 \times \{0\}$  is contained in  $A$ . Then we may apply Proposition 4.6 to each scalar

component of  $f$  in order to obtain that  $f$  is holomorphic on  $S_1 \times U$ . Moreover, as the same argument applies to any linear projection of  $U \times B$  to  $U$  near enough the vertical one; this implies that  $f$  is an isotropic map, up to shrinking slightly  $U$ . This contradicts the fact that the point  $(s_0, z_0)$  is in the boundary of the open set  $A \cap (\{s_0\} \times |X_{s_0}|)$  of  $|X_{s_0}|$ .

If the boundary of  $A \cap (\{s_0\} \times |X_{s_0}|)$  is contained in the singular set of  $|X_{s_0}|$ , we may apply the analytic extension criterion of [BM14, Chapter IV, Criterion 3.1.7], and we obtain directly that  $A$  contains  $|X_{s_0}|$ . So in any case the family of cycles  $(X_s)_{s \in S}$  is analytic at  $s_0$ . As the graph  $|G|$  is, by assumption, quasi-proper on  $S$ , it is enough to use the next proposition (which is proved in [Bar15, Proposition 2.2.3]) to conclude.

**Proposition 4.7.** *Let  $Z$  and  $S$  be reduced complex spaces and let  $(X_s)_{s \in S}$  be a  $f$ -continuous family of  $n$ -cycles in  $Z$ . Assume that this family is analytic in  $s_0 \in S$ . Then there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  such that the family  $(X_s)_{s \in S'}$  is a  $f$ -analytic family of  $n$ -cycles in  $Z$ .*

#### 4.D. Proof of Theorem 1.2 and its corollary

We shall begin by a lemma which will give the case where the  $n$ -cycle  $X_0$  is compact.

**Lemma 4.8.** *Let  $Z$  be a strongly  $(n - 2)$ -concave reduced complex space. Then  $\mathcal{C}_n(Z)$  is open in  $\mathcal{C}_n^f(Z)$ .*

*Proof.* Let  $X_0$  be a compact cycle in  $Z$ . There exists  $\alpha \in ]0, 1[$  such that  $X_0$  is contained in  $Z_\alpha = \{x \in Z / \varphi(x) > \alpha\}$ . So  $X_0$  does not meet the compact set  $\varphi^{-1}(\{\alpha\})$ . This is an open condition in  $\mathcal{C}_n^f(Z)$ . And as any irreducible  $n$ -dimensional analytic subset in  $Z$  has to meet  $K$ , if it does not meet  $\varphi^{-1}(\{\alpha\})$  it is contained in  $Z_\alpha$  (by connectedness). Then any  $X \in \mathcal{C}_n^f(Z)$  which is near enough  $X_0$  is contained in  $Z_\alpha$  so is compact.  $\square$

Note that under the hypothesis of the previous lemma,  $\mathcal{C}_n(Z)$  is not closed in  $\mathcal{C}_n^f(Z)$  in general, as one can see taking  $Z := \mathbb{P}_N \setminus \{0\}$  and considering the set of hyperplanes in  $\mathbb{P}_N$ .

As we already know from [Bar75] that  $\mathcal{C}_n(Z)$  is a reduced complex space, Theorem 1.2 and its corollary are proved near a compact  $n$ -cycle in  $Z$ .

*The case where  $X_0$  is not compact.* Of course we are in the case where  $Z$  is not compact. Fix  $X_0$  a non compact  $n$ -cycle in  $\mathcal{C}_n^f(Z)$  and choose an  $\alpha \in ]0, 1[$  which is not a critical value of  $\varphi$  and of the restriction of  $\varphi$  to  $|X_0|$ ; this is possible thanks to Sard's Lemma 3.1 and the fact that  $\varphi(|X_0|)$  contains  $]0, 1[$ .

Consider now the reduced complex space  $\Xi$  constructed in Theorem 3.15. It is an open neighbourhood of  $X_0 \cap \Delta''$  in  $\mathcal{C}_n^f(\Delta'')$  and we may assume that  $\Delta'' := Z_\beta$  for some  $\beta < \alpha$  very near  $\alpha$ . Due to Proposition 4.3 it is homeomorphic to an open neighbourhood  $\mathcal{V}$  of  $X_0$  in  $\mathcal{C}_n^f(Z)$ . So the restriction map

$$res_0 : \mathcal{V} \rightarrow \Xi$$

is holomorphic, bijective and is a homeomorphism. Now the continuity of  $res_0^{-1}$  and the finiteness of  $\Xi$  allow to apply Theorem 4.5 because we already know that the tautological family of cycles parametrized by  $\Xi$  is  $f$ -analytic on the open set  $\Delta''$  which is an open set which meets any irreducible component of each cycle in this family (because  $K \subset Z_\beta$ ). Then  $res_0^{-1}$  is holomorphic and so  $res_0$  is an isomorphism of Banach analytic sets.  $\square$

#### 4.E. A compactness criterion for the connected components of the reduced complex space $\mathcal{C}_n^f(Z)$ when $Z$ is strongly $(n-2)$ -concave

For a reduced complex space  $Z$  which is compact, the compactness of the connected components of  $\mathcal{C}_n(Z)$  is a consequence of the existence of a  $\mathcal{C}^1$   $2n$ -differential form on  $Z$  which is  $d$ -closed and such that its  $(n, n)$  part is positive definite in the Lelong sense. Indeed, this gives that the volume (computed with this  $(n, n)$  part) of the  $n$ -cycles is constant on connected components. The result follows then from Bishop's theorem [Bis64] (see [BM14, Chapter IV] for details).

In the case of a non compact strongly  $(n-2)$ -concave reduced complex space  $Z$  we have the following analogous result :

**Proposition 4.9.** *Let  $Z$  be a reduced complex space which is strongly  $(n-2)$ -concave. Assume that there exists on  $Z$  a  $\mathcal{C}^1$   $2n$ -differential form  $\omega$  which is  $d$ -closed with compact support and such that its  $(n, n)$  part is positive definite in the Lelong sense in a neighbourhood of  $K := \varphi^{-1}([1, 2])$ , and everywhere non negative in the Lelong sense. Then the connected components of  $\mathcal{C}_n^f(Z)$  are compact.*

*Proof.* For  $\alpha < 1$  near enough to 1 and for any continuous hermitian metric  $h$  on  $Z$  there exists a constant  $C$  such that the following inequality holds:

$$\text{vol}_h(X \cap Z_\alpha) \leq C \cdot \int_X \omega \quad \text{for any cycle } X \in \mathcal{C}_n^f(Z).$$

As the function  $X \mapsto \int_X \omega$  is locally constant on  $\mathcal{C}_n^f(Z)$  because  $d\omega = 0$  (the direct image of  $\omega$  as a current is  $d$ -closed, so locally constant at smooth points of  $\mathcal{C}_n^f(Z)$ , and this current is a continuous function on  $\mathcal{C}_n^f(Z)$  thanks to Proposition IV 2.3.1 of *loc. cit.*), we have a uniform bound for the volume of  $X \cap Z_\alpha$  for  $X$  in a given connected component of  $\mathcal{C}_n^f(Z)$ . This implies that the closure of the image of this connected component in  $\mathcal{C}_n^f(Z_\alpha)$  is compact, thanks to Bishop's theorem (see [BM14, Chapter IV, Theorem 2.7.20]). But the restriction map  $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$  is a homeomorphism by Proposition 4.3, so the image of a connected component is closed and then compact.  $\square$

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