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Finiteness of the space of n -cycles for a reduced $(n - 2)$ -concave complex space

Daniel Barlet

pour Yum-Tong Siu, avec mon amitié

Abstract. We show that for $n \geq 2$ the space of closed n -cycles in a strongly $(n - 2)$ -concave complex space has a natural structure of reduced complex space locally of finite dimension and represents the functor “analytic family of n -cycles” parametrized by Banach analytic sets.

Keywords. Closed n -cycles; strongly q -concave space; Hartogs figure; f -analytic family of cycles

2010 Mathematics Subject Classification. 32G13; 32G10; 32F10; 32D15

[Français]

Titre. Finitude de l’espace des n -cycles pour un espace complexe $(n - 2)$ -concave réduit

Résumé. Nous montrons que, pour $n \geq 2$, l’espace des n -cycles fermés dans un espace complexe fortement $(n - 2)$ -concave a une structure naturelle d’espace complexe réduit localement de dimension finie et que cet espace représente le foncteur “famille analytique de n -cycles” paramétrée par des ensembles analytiques banachiques.

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1. Introduction

The aim of this article is to show that in a reduced strongly $(n - 2)$ -concave¹ complex space Z with $n \geq 2$, the space of closed n -cycles is in a natural way endowed with a structure of a reduced complex space locally of finite dimension. With its tautological family of n -cycles it represents the functor “analytic family of n -cycles in Z ” and also the functor “f-analytic family of n -cycles in Z ” introduced in [Bar08] (see also [Bar13] and [Bar15]) parametrized by a Banach analytic set.

This answers a question asked to me by Y-T. Siu forty years ago.

I was able to solve this question thanks to the notion of f-analytic family introduced in *loc. cit.* and using the space $\mathcal{C}_n^f(Z)$ of finite type cycles with its natural topology.

We obtain the following results.

Theorem 1.1. *Let $n \geq 2$ be an integer. Let Z be a strongly $(n - 2)$ -concave reduced complex space of pure dimension $n + p$ that is to say admitting a \mathcal{C}^2 exhaustion function $\varphi : Z \rightarrow]0, 2]$ which is strongly $(n - 2)$ -convex outside the compact set $K := \varphi^{-1}([1, 2])$. For any $\alpha \in]0, 1[$ and any n -cycle X_0 in an open neighbourhood of the compact set $\varphi^{-1}([\alpha, 2])$ there exists $\beta \in]0, \alpha[$ such that, if $Z_\beta := \{z \in Z / \varphi(z) > \beta\}$, the cycle X_0 extends in a unique way to the open set Z_β and admits an open neighbourhood \mathcal{U} in the space $\mathcal{C}_n^f(Z_\beta)$ such that the ringed space defined by \mathcal{U} and the sheaf of holomorphic functions on \mathcal{U} is a (reduced) complex space locally of finite dimension.*

Recall that a holomorphic function $h : \mathcal{U} \rightarrow \mathbb{C}$ on an open set in $\mathcal{C}_n^f(Z_\beta)$ is a continuous function on \mathcal{U} such that for any holomorphic map $f : S \rightarrow \mathcal{U}$ (corresponding to an f-analytic family of n -cycles in Z , see *loc. cit.*) of a Banach analytic set S to \mathcal{U} the composed function $h \circ f$ is holomorphic.

¹ ↑ Our conventions will be precised below.

Theorem 1.2. *Consider the same situation as in the previous theorem, and let now $X_0 \in \mathcal{C}_n^f(Z)$ be a finite type n -cycle in Z . Then there exists $\beta \in]0, 1[$ and open neighbourhoods \mathcal{V} and \mathcal{U} respectively of X_0 in $\mathcal{C}_n^f(Z)$ and of $X_0 \cap Z_\beta$ in $\mathcal{C}_n^f(Z_\beta)$ such that the restriction map*

$$r : \mathcal{V} \rightarrow \mathcal{U}$$

is well defined and bi-holomorphic.

We are going to recall briefly the notion of f -analytic family of finite type n -cycles in a complex space Z .

Firstly the notion of an analytic family of n -cycles in a reduced complex space Z parametrized by a reduced complex space S is defined as follows (see² [Bar75, Chapter I, p. 33] or [BM14, Chapter IV, Section 3]), using the following notion of an adapted scale (see [BM14, Chapter IV, Section 2.1]).

Definition 1.3. We call $E := (U, B, j)$ a **n -scale on a complex space Z** when U and B are open relatively compact polydiscs respectively in \mathbb{C}^n and \mathbb{C}^p and where $j : Z_E \rightarrow W$ is a closed embedding of an open set Z_E in Z into an open neighbourhood W of $\bar{U} \times \bar{B}$ in \mathbb{C}^{n+p} . The open set Z_E is called the **domain** of E and the open set $j^{-1}(U \times B)$ the **center** of E . When X is a n -cycle in Z , the n -scale E is **adapted** to X when $|X| \cap j^{-1}(\bar{U} \times \partial B) = \emptyset$.

Note that when E is a n -scale adapted to a n -cycle X in Z , the projection of $U \times B$ on U restricted to $j_*(X \cap j^{-1}(U \times B))$ gives a finite proper map of degree $k \geq 0$ and the fibers of this map are classified by a holomorphic map $f : U \rightarrow \text{Sym}^k(B)$. In this case we shall say that f is the **classifying map** of the cycle X in the adapted scale E .

Definition 1.4. Let Z be a complex space and let $(X_s)_{s \in S}$ be a family of n -cycles in Z parametrized by a reduced complex space S . We shall say that this family is **analytic at a point $s_0 \in S$** if for any n -scale $E := (U, B, j)$ on Z which is adapted to the cycle X_{s_0} there exists an open neighbourhood S_0 of s_0 in S satisfying the following properties:

- i) For each $s \in S_0$ the scale E is adapted to X_s .
- ii) Assume that $k := \deg_E(X_{s_0})$. Then for each $s \in S_0$ we have $\deg_E(X_s) = k$.
- iii) There exists a holomorphic map $f : S_0 \times U \rightarrow \text{Sym}^k(B)$ such that for each $s \in S_0$ the restriction of f to $\{s\} \times U$ classifies the cycle X_s in the scale E .

It is easy to see that an analytic family of cycles has a “set theoretic” graph

$$|G| := \{(s, x) \in S \times Z \mid x \in |X_s|\}$$

which is a closed analytic subset in $S \times Z$ and that its projection on S has pure n -dimensional fibers (which are the supports of the cycles). When we have an analytic family $(X_s)_{s \in S}$ and when the projection of its graph $pr : |G| \rightarrow S$ is quasi-proper³ we shall say that $(X_s)_{s \in S}$ is a **f -analytic family** of (finite type) n -cycles in Z . Of course this condition implies that each cycle X_s is a finite type n -cycle (it means that each cycle admits only finitely many irreducible components) but this condition contains this fact in a local uniform manner on S .

In an analogous way, when we have an analytic family $(X_s)_{s \in S}$ and when the projection of its graph $pr : |G| \rightarrow S$ is proper, we shall say that it is a **proper analytic family of compact cycles in Z** .

The following corollary is of course the main result.

² ↑ See Chapter 3 Section 4 in [Bar75] for the case when S is a Banach analytic set.

³ ↑ This means, by definition, that for any $s_0 \in S$ there exists an open neighbourhood S_1 of s_0 in S and a compact set K in $|G|$ such that any irreducible component of any fiber $pr^{-1}(s)$ for any $s \in S_1$ meets K .

Corollary 1.5. *Consider the same situation as in the previous theorems. Then the ringed space given by the sheaf of holomorphic functions on $\mathcal{C}_n^f(Z)$ is a reduced complex space locally of finite dimension. Moreover, endowed with its tautological family of (finite type) n -cycles it represents the functor*

$$\begin{aligned} (\text{reduced complex spaces}) &\rightarrow (\text{sets}) \\ S &\mapsto \{\text{f-analytic family of } n\text{-cycles in } Z \text{ parametrized by } S\}. \end{aligned}$$

Remark. Let X be a (non empty) irreducible analytic subset of dimension n in a strongly $(n-2)$ -concave complex space Z as in Theorem 1.1. Let $x_0 \in X$ be a point in X where the supremum of the restriction of the exhaustion function φ to X is obtained. Then the point x_0 is in the compact set $K = \varphi^{-1}([1, 2])$ because the Levi form of φ at the point x_0 has at least n non positive eigenvalues. So any (non empty) irreducible analytic subset of dimension n in Z has to meet the compact set K . Then any n -cycle in Z is of finite type and any analytic family of n -cycles in Z has a quasi-proper graph so is a f-analytic family. This implies that, in the previous corollary, the obvious map⁴ $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$ is an isomorphism of ringed spaces and $\mathcal{C}_n^f(Z)$ represents also the functor “analytic family of n -cycles in Z ”.

Question. As it appears in the previous remark, we may expect the same result for $(n-1)$ -cycles under our assumption of strong $(n-2)$ -concavity. But our way to use Hartogs figures in the present article needs one more positive eigenvalue than one can expect. Is the result also true for $(n-1)$ -cycles under our hypothesis? It would probably be interesting, for instance, to have this kind of result for 1-cycles in a strongly 0-concave complex space.

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2. Hartogs figures

2.A. Banachization

For the analytic extension via Hartogs figures, the use of the Banach spaces $H(\bar{U}, \mathbb{C})$ of continuous functions, holomorphic inside on a compact polydisc \bar{U} is not well adapted. We shall use the Banach space $\mathcal{B}(U, \mathbb{C})$ of bounded holomorphic functions on U with the “sup” norm on U . Of course, $H(\bar{U}, \mathbb{C})$ is a closed Banach subspace in $\mathcal{B}(U, \mathbb{C})$.

Proposition 2.1. *Let U be a relatively compact polydisc in \mathbb{C}^n and let S be a Banach analytic set. Let $F : S \rightarrow \mathcal{B}(U, \mathbb{C})$ be a holomorphic map. Then the corresponding function $f : S \times U \rightarrow \mathbb{C}$ defined by $f(s, t) := F(s)[t]$ for (s, t) in $S \times U$ is holomorphic (and locally on S uniformly bounded on U).*

Conversely, if we have a holomorphic function $f : S \times U \rightarrow \mathbb{C}$ and an open polydisc $U'' \subset\subset U$, the associated map $F : S \rightarrow \mathcal{B}(U'', \mathbb{C})$, defined for (s, t) in $S \times U''$ by $F(s)[t] := f(s, t)$, is holomorphic.

Proof. The evaluation function $ev : \mathcal{B}(U, \mathbb{C}) \times U \rightarrow \mathbb{C}$ is holomorphic as one may easily see by differentiation. Then the function f associated to F is the composition of the holomorphic maps $F \times id_U$ and ev . So it is holomorphic.

The converse is consequence of the linear isometric inclusion of $H(\bar{U}'', \mathbb{C})$ in $\mathcal{B}(U'', \mathbb{C})$ as $F'' : S \rightarrow H(\bar{U}'', \mathbb{C})$ is holomorphic as soon as f is holomorphic (see [Bar75] or [BM]). \square

⁴ \uparrow The set $\mathcal{C}_n^{loc}(Z)$ is the set of all closed cycles of dimension n in the complex space Z ; its natural topology which is associated to the adapted scales is described in [BM14, Chapter IV]. In general, the inclusion map $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$ is continuous but is not a homeomorphism onto its image. See [Bar15] for the comparison between compact sets in these two spaces.

2.B. n -Hartogs figure on a complex space

For $\alpha \in (\mathbb{R}_+^*)^2$ let $M(\alpha)$ be the open set in \mathbb{C}^2 defined by

$$\begin{aligned} M(\alpha) &:= M^P(\alpha) \cup M^C(\alpha) \quad \text{with} \\ M^P(\alpha) &:= \{|t_1 - \alpha_1/2| < \alpha_1/4, \quad |t_2| < \alpha_2\} \\ M^C(\alpha) &:= \{|t_1| < \alpha_1, \quad \alpha_2/2 < |t_2| < \alpha_2\} \end{aligned}$$

and let also $\mathcal{M}(\alpha) := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i, i = 1, 2\}$.

For $\varepsilon > 0$ small enough, we define

$$\begin{aligned} M(\alpha)^\varepsilon &:= M^P(\alpha)^\varepsilon \cup M^C(\alpha)^\varepsilon \quad \text{with} \\ M^P(\alpha)^\varepsilon &:= \{|t_1 - \alpha_1/2| < \alpha_1/4 - \varepsilon/4, \quad |t_2| < \alpha_2 - \varepsilon\} \\ M^C(\alpha)^\varepsilon &:= \{|t_1| < \alpha_1 - \varepsilon, \quad \alpha_2/2 + \varepsilon/2 < |t_2| < \alpha_2 - \varepsilon\} \end{aligned}$$

and also $\mathcal{M}(\alpha)^\varepsilon := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i - \varepsilon, i = 1, 2\}$.

For a polydisc of radius R in \mathbb{C}^m we shall denote by P^ε the polydisc with same center and radius $R - \varepsilon$ for $0 < \varepsilon < R$.

Definition 2.2. Let $n \geq 2$ and $p \geq 1$ two integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space Z . We shall call $\mathcal{H} := (\mathcal{M}, M, B, j)$ a **n -Hartogs figure in Z relative to the boundary of Δ** , the following data

- an embedding j of an open set Z' in Δ' into an open set in \mathbb{C}^{n+p} ,
- open sets $\mathcal{M} \subset\subset \Delta'$ and $M \subset\subset \Delta$ relatively compact in Z' ,
- a relatively compact polydisc B in \mathbb{C}^p

such that there exists $\alpha \in (\mathbb{R}_+^*)^2$ and a relatively compact polydisc V in \mathbb{C}^{n-2} with the following conditions:

- i) The map j induces a closed embedding of \mathcal{M} in $\mathcal{M}(\alpha) \times V \times B$.
- ii) The map j induces a closed embedding of M in $M(\alpha) \times V \times B$.
- iii) We have $j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \subset \Delta$.

Definition 2.3. Let $n \geq 2$ and $p \geq 1$ two integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space Z . Let $\mathcal{H} = (\mathcal{M}, M, B, j)$ be a n -Hartogs figure in Z relative to the boundary of Δ and let X_0 be a n -cycle in Δ . We shall say that \mathcal{H} is **adapted to X_0** when the following condition is satisfied :

$$j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \cap |X_0| = \emptyset. \quad (@)$$

Remarks.

1. Let \mathcal{H} be a n -Hartogs figure in Z relative to the boundary of Δ . If the open set $\Delta_1 \subset\subset \Delta'$ has a boundary $\partial\Delta_1$ near enough to $\partial\Delta$, then \mathcal{H} is again a n -Hartogs figure relative to the boundary of Δ_1 . For instance, if $\Delta := \{\varphi > 0\}$ where φ is a continuous proper function on Z , we may choose $\Delta_1 := \{\varphi > \varepsilon\}$ for $\varepsilon > 0$ small enough.
2. Note that the n -scale $E_{\mathcal{H}} := (M(\alpha) \times V, B, j)$ on Δ associated to \mathcal{H} is adapted to X_0 as soon as the n -Hartogs figure \mathcal{H} is adapted to X_0 .

3. If \tilde{X}_0 is a n -cycle in Δ' such that its restriction to Δ is equal to X_0 , the n -scale $E_{\tilde{\mathcal{H}}} := (\mathcal{M}(\alpha) \times V, B, j)$ on Δ' is adapted to \tilde{X}_0 if and only if the n -Hartogs figure \mathcal{H} is adapted to X_0 .

Note that the n -scale $E_{\tilde{\mathcal{H}}}$ is not a n -scale on Δ although the subset $\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B$ is contained in Δ .

Definition 2.4. In the situation above we define, for $\varepsilon > 0$ small enough, the n -Hartogs figure \mathcal{H}^ε on Δ as follows :

$$\mathcal{H}^\varepsilon := (\mathcal{M}^\varepsilon, M^\varepsilon, B, j)$$

where we use the notations

$$\mathcal{M}^\varepsilon := j^{-1}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon \times B) \quad \text{and also} \quad M^\varepsilon := j^{-1}(M(\alpha)^\varepsilon \times V^\varepsilon \times B).$$

It is obvious to see that when \mathcal{H} is a n -Hartogs figure relative to the boundary of Δ , then \mathcal{H}^ε is again a n -Hartogs figure relative to the boundary of Δ for all $\varepsilon > 0$ small enough.

Moreover, if \mathcal{H} is adapted to the n -cycle X_0 of Δ , the same is true for \mathcal{H}^ε for all $\varepsilon > 0$ small enough.

Lemma 2.5. *Let V be a relatively compact open polydisc in \mathbb{C}^q . The restriction map*

$$res : \mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha) \times V, \mathbb{C})$$

is a linear isometry of Banach spaces.

Note that the restriction map

$$res : \mathcal{B}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C})$$

induces also an isometry for all $\varepsilon > 0$ small enough.

Proof. Let $f(v, t_1, t_2) := \sum_{m \in \mathbb{Z}} a_m(v, t_1) \cdot t_2^m$ the Laurent expansion of the holomorphic function $f : M^C(\alpha) \times V \rightarrow \mathbb{C}$. The holomorphic functions a_m , $m \in \mathbb{Z}$ on the product of V by the disc $\{|t_1| < \alpha_1\}$ are given by the formula

$$a_m(v, t_1) := \frac{1}{2i\pi} \cdot \int_{|z|=r} f(v, t_1, z) \cdot \frac{dz}{z^{m+1}} \quad \text{with} \quad r \in]\alpha_2/2, \alpha_2[.$$

As the holomorphy of f on $M^P(\alpha) \times V$ implies that $a_m \equiv 0$ for each negative m on the open set $\{|t_1 - \alpha_1/2| < \alpha_1/4\} \times V$, we conclude that the functions a_m are identically zero for $m < 0$ and so f is holomorphic on $\mathcal{M}(\alpha) \times V$. This shows that the restriction map res is bijective (and also it is linear continuous) between the two Fréchet spaces $\mathcal{O}(\mathcal{M}(\alpha) \times V)$ and $\mathcal{O}(M(\alpha) \times V)$; so it is an isomorphism of Fréchet spaces.

Let us show that if f is in $\mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C})$, then $res(f)$, which belongs to the space $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$, has the same ‘sup’ norm. For this purpose fix $\varepsilon > 0$ small enough. As $\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ is a compact polydisc in $\mathcal{M}(\alpha) \times V$, the maximum of f on this compact is obtained at some point z in the distinguish boundary of it. But as z is also in the boundary of $M^C(\alpha)^\varepsilon \times \bar{V}^\varepsilon$, the desired equality follows. Conversely, if g is in $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$, its analytic extension f to $\mathcal{M}(\alpha) \times V$ will be bounded on the boundary of $\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon$ by the sup of g on $M^C(\alpha)^\varepsilon \times V^\varepsilon$. So we obtain the equality of the ‘sup’ norms for g and f respectively on $M(\alpha) \times V$ and $\mathcal{M}(\alpha) \times V$. \square

The Banach analytic set $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$. Recall that if $p \geq 1$ and $k \geq 1$ are integers, there exists a closed embedding (in fact given by a polynomial map) of $\text{Sym}^k(\mathbb{C}^p) := (\mathbb{C}^p)^k / \mathfrak{S}_k$ into the vector space $E(k) := \bigoplus_{h=1}^k S^h(\mathbb{C}^p)$ given by the elementary tensorial symmetric functions⁵. If U is an open relatively compact polydisc in \mathbb{C}^n , the subset $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ is closed and Banach analytic in the Banach space $\mathcal{B}(U, E(k))$. Indeed, if $Q : E(k) \rightarrow \mathbb{C}^N$ is a polynomial map such that $Q^{-1}(0) = \text{Sym}^k(\mathbb{C}^p)$, then the holomorphic map

$$\mathcal{Q} : \mathcal{B}(U, E(k)) \rightarrow \mathcal{B}(U, \mathbb{C}^N) \quad \text{defined by} \quad f \mapsto Q \circ f$$

satisfies $\mathcal{Q}^{-1}(0) = \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$.

Nevertheless, be aware that for an open set Ω in $E(k)$ the subset $\mathcal{B}(U, \Omega)$ of elements in $\mathcal{B}(U, E(k))$ taking their values in Ω is not, in general, open in $\mathcal{B}(U, E(k))$; so, for an open polydisc $B \subset \subset \mathbb{C}^p$, the subset $\mathcal{B}(U, \text{Sym}^k(B))$ is not open in $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ in general.

Remark. The obvious map $H(\bar{U}, E(k)) \rightarrow \mathcal{B}(U, E(k))$ is a closed (linear) isometry and induces a holomorphic inclusion map

$$i_U : H(\bar{U}, \text{Sym}^k(\mathbb{C}^p)) \hookrightarrow \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$$

and for all $\varepsilon > 0$ the restriction

$$r : \mathcal{B}(U, E(k)) \rightarrow H(\bar{U}^\varepsilon, E(k))$$

is a (linear and continuous) compact map which induces a holomorphic restriction map

$$\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p)) \rightarrow H(\bar{U}^\varepsilon, \text{Sym}^k(\mathbb{C}^p)).$$

This remark will allow us to use Lemma 2.5 with Banach analytic sets like $H(\bar{U}, \text{Sym}^k(B))$.

Notations. Let $k \in \mathbb{N}$ and let $U' \subset \subset U \subset \subset \mathbb{C}^n$ and $B \subset \subset \mathbb{C}^p$ be polydiscs. We shall note $\Sigma_{U, U'}(k)$ the Banach analytic set classifying the couples of an element in $H(\bar{U}, \text{Sym}^k(B))$ with its isotropy data on \bar{U}' . Recall that for a holomorphic map $f : U \rightarrow \text{Sym}^k(B)$ the isotropy data on the polydisc U' is the map

$$T(f) : U' \rightarrow F \otimes E'$$

where

$$F := \bigoplus_i \left(L(\Lambda^i(\mathbb{C}^n), \Lambda^i(\mathbb{C}^p)) \right) \quad \text{and} \quad E' := \bigoplus_{m=0}^{k-1} S^m(\mathbb{C}^p).$$

It corresponds to the collection of holomorphic maps

$$T_m^i(f) : U' \rightarrow L(\Lambda^i(\mathbb{C}^n), \Lambda^i(\mathbb{C}^p)) \otimes S^m(\mathbb{C}^p)$$

for all $i \in [1, \min(n, p)]$ and $m \in [0, k - 1]$ which are given, near a point in U' where the multiform graph X_f associated to f has local branches f_1, \dots, f_k , by the formula

$$T_m^i(f)(t) := \sum_{j=1}^k \Lambda^i(Df_j(t)) \otimes f_j(t)^m.$$

These maps are always holomorphic on all U and determine the trace map for the projection $\pi : X_f \rightarrow U$ of the holomorphic differential forms on $U \times B$.

The subset

$$\Sigma_{U, U'}(k) \subset H(\bar{U}, \text{Sym}^k(B)) \times H(\bar{U}', F \otimes E')$$

⁵ ↑ See for instance [BM14, Chapter I, §4].

is defined as the graph of the map $f \mapsto T(f) := \bigoplus_{i,m} T_m^i(f)$ which is not holomorphic in general. Nevertheless it is a Banach analytic subset and its natural projection

$$\Sigma_{U,U'}(k) \rightarrow H(\bar{U}, \text{Sym}^k(B))$$

is a holomorphic homeomorphism (see [Bar75, Chapter III, Proposition 2, p. 81] or [BM, Chapter V]).

The important point which motivates the introduction of this Banach analytic subset is the fact that, when S is a reduced complex space, an analytic family of multiform graphs given by a holomorphic map $f : S \times U \rightarrow \text{Sym}^k(B)$ will give an analytic family of cycles in $U \times B$ parametrized by S if and only if the corresponding isotropy data (given by the maps $T_m^i(f/S)$ on $S \times U$) are holomorphic on $S \times U$. This is the isotropy condition ; see [Bar75, Chapter II] or [BM14, Chapter IV, Section 5].

Our next result is the main tool for performing the analytic extension of n -cycles near a $(n - 2)$ -concave boundary.

Proposition 2.6. *Consider the open sets $\mathcal{M}(\alpha) \times V$ and $M(\alpha) \times V$ in \mathbb{C}^n . The inverse of the restriction map is a holomorphic isomorphism of analytic extension*

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)).$$

Composed with the restriction to the compact set $\bar{M}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ it sends the subset $H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$ into $H(\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon, \text{Sym}^k(B))$ for $\varepsilon > 0$ small enough.

Moreover, this holomorphic map induces a holomorphic map, again for $\varepsilon > 0$ small enough

$$\Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \longrightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

which factorizes the restriction map

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

through the restriction

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k).$$

Proof. Lemma 2.5 gives that the map

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, E(k)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, E(k))$$

is an isometry of Banach spaces. It is clear that its inverse sends $\mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$ into the Banach analytic subset $\mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$ and that if the map $f \in \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$ takes values in $\text{Sym}^k(B)$, the same is true for its restriction. This proves the first part of the proposition.

In order to prove the second part, it is enough to show that a holomorphic map of a Banach analytic set S with values in the subset

$$H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$$

which is isotropic on the product of S with any relatively compact subset in the open set $M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$ will have an analytic extension which will be isotropic on any relatively compact open set in $\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$. So it will be isotropic on the closure of the open set $\mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}$. \square

Proposition 2.7. *Let $n \geq 2$ and $p \geq 1$ be integers and let $U_1 \times B_1$ the product of two polydiscs with centers 0 respectively in \mathbb{C}^n and \mathbb{C}^p . Denote by $(t_1, \dots, t_n, x_1, \dots, x_p)$ coordinates on $U_1 \times B_1$. Let φ be a real valued function of class \mathcal{C}^2 on $U_1 \times B_1$, such that*

$$\varphi(t, x) = \text{Re}(t_1) + \sum_{i=1}^n \rho_i \cdot |t_i|^2 + \sum_{j=1}^p \sigma_j \cdot |x_j|^2 + o(\|(t, x)\|^2) \quad (@@)$$

where the real numbers $\rho_2, \sigma_1, \dots, \sigma_p$ are positive (so φ is $(n-2)$ -convex near $(0,0)$ and $d\varphi_{0,0} \neq 0$).

Let Δ be the open set $\{\varphi > 0\}$ in $U_1 \times B_1$ and let Δ' be an open neighbourhood of the compact set $\bar{\Delta}$ in \mathbb{C}^{n+p} . Let X_0 be a closed analytic subset of pure dimension n in Δ' such that each irreducible component of X_0 meets Δ and such that

$$|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}. \quad (*)$$

Then there exists $\alpha \in (\mathbb{R}_+^*)^2$ and polydiscs V and $B \subset\subset B_1$ with centers 0 respectively in \mathbb{C}^{n-2} and \mathbb{C}^p such that the following conditions are satisfied :

1. $\mathcal{M}(\alpha) \times V \times B \subset\subset \Delta'$;
2. $M(\alpha) \times V \times B \subset\subset \Delta$;
3. $\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B \subset \Delta$;
4. $|X_0| \cap (\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B) = \emptyset$ (this implies $|X_0| \cap (\overline{M(\alpha)} \times \bar{V} \times \partial B) = \emptyset$).

Proof. Choose the polydisc $B \subset\subset B_1$ small enough in order that we have

$$X_0 \cap (\{0\} \times \bar{B}) \subset \{0\} \quad \text{and} \quad \{0\} \times \partial B \subset \Delta.$$

This is possible as we have $|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}$ and as φ is positive on a small enough punctured neighbourhood of the origin in the p -plane $\{t_1 = \dots = t_n = 0\} \times \mathbb{C}^p$. So we shall have

$$|X_0| \cap (\bar{W} \times \partial B) = \emptyset \quad \text{and} \quad \bar{W} \times \partial B \subset \Delta$$

for any small enough open neighbourhood W of the origin in \mathbb{C}^n . A immediate consequence is that Conditions 1, 3 and 4 will be satisfied as soon as α and V are small enough.

In order to check Condition 2, let us remark first that, up to choosing the real numbers ρ'_1 and ρ'_2 such that $\rho'_1 > |\rho_1|$, $\rho'_2 \in]0, \rho_2[$ and $r > \sup_{i \geq 3} |\rho_i|$, we obtain on $W \times B$ chosen small enough

$$\varphi(t, x) \geq \operatorname{Re}(t_1) - \rho'_1 \cdot |t_1|^2 + \rho'_2 \cdot |t_2|^2 - r \cdot \left(\sum_3^n |t_i|^2 \right) \quad (**)$$

with strict inequality as soon as $x \neq 0$. Then for

$$V_\varepsilon = \{(t_3, \dots, t_n) / \sum_3^n |t_i|^2 < \varepsilon^2\}$$

the following inequalities hold

$$\varphi(t, x) \geq \frac{1}{4} \alpha_1 - \rho'_1 \cdot \alpha_1^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^P(\alpha) \times V_\varepsilon \times B} \quad (1)$$

$$\varphi(t, x) \geq -\alpha_1 - \rho'_1 \cdot \alpha_1^2 + \frac{1}{4} \rho'_2 \cdot \alpha_2^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^C(\alpha) \times V_\varepsilon \times B} \quad (2)$$

for α and ε small enough in order that $\mathcal{M}(\alpha) \times V_\varepsilon$ is contained in W .

This allows to fix α_1, α_2 and ε .

Now we shall choose α_1 and ε smaller in order to satisfy the following conditions :

$$8\alpha_1 < \rho'_2 \cdot \alpha_2^2, \quad \alpha_1 < \frac{1}{8\rho'_1} \quad \text{and} \quad \varepsilon^2 < \frac{1}{8r} \alpha_1. \quad (3)$$

To obtain $M^P(\alpha) \times V_\varepsilon \times B \subset\subset \Delta$ it is enough to show that on $M^P(\alpha) \times V_\varepsilon \times B$ we have, if we let $\alpha_1 = u.\alpha_2$ and $\varepsilon^2 = v.\alpha_1 = uv.\alpha_2$

$$\frac{1}{4} > \rho'_1.\alpha_1 + r.v.$$

Indeed, as we assumed $\rho'_1.\alpha_1 < \frac{1}{8}$ and $r.\varepsilon^2 < \frac{1}{8}.\alpha_1$ (so $r.v < 1/8$) the first condition holds.

In order to satisfy $M^C(\alpha) \times V_\varepsilon \times B \subset\subset \Delta$ it is enough to show that on $M^C(\alpha) \times V_\varepsilon \times B$ we have

$$\frac{1}{4}\rho'_2.\alpha_2 > u + \rho'_1 u^2 \alpha_2 + r.uv.$$

But our condition implies

$$\rho'_1 u^2 \alpha_2 < \frac{1}{8}.u \quad r.uv < \frac{1}{8}.u$$

which gives $u + \rho'_1 u^2 .\alpha_2 + r.uv < 2.u$. The condition $8\alpha_1 < \rho'_2.\alpha_2^2$ which implies $2u < \frac{1}{4}\rho'_2.\alpha_2$, allows to conclude. \square

Remarks.

1. We only used Condition (*) for X_0 and Inequality (**) for φ in a neighbourhood of the origin in the proof above.
2. Sufficient conditions on $\varphi \in \mathcal{C}^2$ to satisfy (@@) are:

- i) The origin is not a critical point of φ .
- ii) The Levi form of φ at 0 has, at most, $(n - 2)$ non positive eigenvalues in the complex tangent hyperplane to the real hypersurface $\{\varphi(z) = 0\}$; the existence of real function $\varphi \in \mathcal{C}^2$ such that $\Delta = \{\varphi > 0\}$ and satisfying these two conditions is equivalent to the fact that the open set Δ has a strongly $(n - 2)$ -concave smooth boundary near the origin (see Definition 2.10 given below). Indeed, if φ is not critical at 0 and has a Levi form at 0 with, at most, $(n - 2)$ non positive eigenvalues in the complex hyperplane tangent to the real hypersurface $\{\varphi(z) = 0\}$, its order 2 Taylor expansion at the origin is written, in suitable local holomorphic coordinates (τ, x)

$$\varphi(\tau, x) = \operatorname{Re}(\tau_1) + \operatorname{Re}(Q(\tau, x)) + \sum_{i=1}^n \rho_i . |\tau_i|^2 + \sum_{j=1}^p \sigma_j . |x_j|^2 + o(\|(\tau, x)\|^2)$$

where Q is a holomorphic homogeneous degree 2 polynomial and where the real numbers ρ_i and $\sigma_j, j \in [1, p]$ are positive. Define new local holomorphic coordinates

$$t_1 := \tau_1 + Q(\tau, x), t_i := \tau_i \quad \text{for } i \in [2, n] \quad \text{and} \quad x_j := x_j \quad \text{for } j \in [1, p].$$

Then we obtain (@@).

3. Condition (*) implies that X_0 has no local irreducible component at 0 contained in the hyperplane $\{t_1 = 0\}$. In fact, as the coordinate t_1 is chosen in order to suppress the real part of the holomorphic homogeneous degree 2 term in the order 2 Taylor expansion of φ at 0 (see the previous remark), we want that no local irreducible component of X_0 at the origin is contained in the complex hypersurface $\tau_1 + Q(\tau, x) = 0$ locally defined near 0 for φ given. Then, as soon as the restriction $\varphi|_{X_0}$ has not 0 as a critical point, Condition (*) could be realized when the Levi form of φ has at most $(n - 2)$ non positive eigenvalues on the complex tangent hyperplane at the origin of the hypersurface $\{\varphi = 0\}$.

4. One may easily see that under our hypothesis, the cycle X_0 meets the open set Δ when it contains 0. Indeed, the analytic subset

$$\{t_1 = t_3 = \dots = t_n = 0\} \cap |X_0|$$

is nonempty, has dimension at least 1 and meets the complement of Δ only at the origin.

Of course, assuming that $0 \in X_0$, the proposition shows that, in fact, X_0 contains a branched covering of degree $k \geq 1$ of $M^P(\alpha) \times V_\varepsilon$ inside

$$M^P(\alpha) \times V_\varepsilon \times B \subset\subset \Delta.$$

5. In the situation of Proposition 2.7, for any continuous family $(X_s)_{s \in S}$ of n -cycles in Δ parametrized by a Banach analytic set S such that $X_{s_0} = X_0 \cap \Delta$, there exists an open neighbourhood S' of s_0 in S , such that for each $s \in S'$ Condition 4 remains true after analytic extension of the cycles (see Proposition 2.6), because, thanks to Condition 3, $\mathcal{M}(\alpha) \times \bar{V} \times \partial B$ is a compact subset in Δ .

2.C. q -concave open sets.

Definition 2.8. Let $\varphi : U \rightarrow \mathbb{R}$ be real valued \mathcal{C}^2 function on an open set U in \mathbb{C}^N . We shall say that φ is **strongly q -convex** when its Levi form at each point of U has at most q non positive eigenvalues.

So, with this definition a strongly 0-convex function is a strongly plurisubharmonic function.

Definition 2.9. Let $\varphi : Z \rightarrow \mathbb{R}$ a real valued \mathcal{C}^2 function on a reduced complex space Z . We shall say that φ is **strongly q -convex** if locally near each point of Z it can be induced by a \mathcal{C}^2 strongly q -convex function in a local embedding in an open set of an affine space.

Remark that a strongly q -convex function on an irreducible complex space of dimension at least equal to $q+1$ has no local maximum because there exists at any point a germ of curve on which the restriction of φ is strongly p.s.h.

Definition 2.10. Let Z be a reduced complex space and let Δ be a relatively compact open set in Z . We shall say that Δ has a **smooth \mathcal{C}^2 boundary** when for each point z in $\partial\Delta$ there exists a local holomorphic embedding $j : W \rightarrow U$ of an open neighbourhood W of z in an open set U of the Zariski tangent space of Z at z and an open set D with smooth \mathcal{C}^2 boundary in U such that $W \cap j^{-1}(\partial D) = W \cap \partial\Delta$.

We shall say that the open set $\Delta \subset Z$ with smooth \mathcal{C}^2 boundary is **strongly q -concave at a point** $z \in \partial\Delta$ if, in some local holomorphic embedding $j : W \rightarrow U$ of Z around z as above, one can define Δ in W as the subset $\{j \circ \varphi > 0\} \cap W$ where φ is a real valued \mathcal{C}^2 function on U such that

1. $d\varphi_{j(z)} \neq 0$ on the tangent space $T_{U,j(z)}$ of U at $j(z)$.
2. The restriction of the Levi form at $j(z)$ of φ to the complex hyperplane tangent at $j(z)$ to the real hypersurface $\{\varphi(x) = \varphi(j(z))\}$ in U has at most q non positive eigenvalues.

We shall say that Δ is **strongly q -concave** if Δ is strongly q -concave near each point in $\partial\Delta$.

Remark. Assume that Z is of pure dimension $q + p$. If the defining function φ of Δ satisfies Conditions 1 and 2 above, we can compose φ with a real strictly increasing (non critical) convex \mathcal{C}^2 function (this does not change the level sets $\{\varphi = \text{constant}\}$), in order that $c \circ \varphi$ is \mathcal{C}^2 strongly q -convex (and non critical) near z .

Conversely, if φ is a real valued \mathcal{C}^2 function which is strongly q -convex and not critical near a point $z \in Z$, the open set $\{\varphi(x) > \varphi(z)\}$ has a strongly q -concave boundary in a neighbourhood of z .

With this terminology, using the remarks above, we may give the following reformulation of Proposition 2.7:

Corollary 2.11. *Let $n \geq 2$ and $p \geq 1$ be integers, let Z be a reduced complex space of pure dimension $n + p$ and let $\Delta := \{\varphi > 0\}$ be an open set with \mathcal{C}^2 smooth boundary in Z . Let X_0 be a n -cycle in an open neighbourhood of a point $z \in \partial\Delta$ such that the function $\varphi|_{X_0}$ is not critical at z .*

Assume that Δ is strongly $(n - 2)$ -concave near z ; then there exists a n -Hartogs figure $\mathcal{H} := (\mathcal{M}, M, B, j)$ relative to the boundary of Δ , adapted to X_0 , and such that the point z lies in \mathcal{M} .

Proof. Using a local embedding of an open neighbourhood of z in an open set of the Zariski tangent space $T_{Z,z}$, it is enough to prove the corollary in the case where Z is an open set in $\mathbb{C}^{n+p'}$, with $p' \geq p$ an integer. As we may choose the function φ strongly $(n - 2)$ -convex such that $d\varphi_z \neq 0$ thanks to the previous remarks, we can choose local coordinates near z in order to be in the situation of Proposition 2.7 in the case $z \in X_0$, as we assumed that $\varphi|_{X_0}$ is not critical at z . In this case the proposition gives the result.

If z is not in X_0 , the same construction in an open neighbourhood of z with no limit point in X_0 allows to conclude, and in this case the degree of X_0 in the (adapted) scale $E_{\mathcal{H}}$ will be zero. \square

2.D. Convexity–concavity

In this paragraph we want to have a brief discussion about q -convexity and q -concavity.

Let us consider in an open set U of \mathbb{C}^{n+p} a \mathcal{C}^2 function $\varphi : U \rightarrow \mathbb{R}$ and a non critical zero z_0 on φ . So $\varphi(z_0) = 0$ and $d\varphi_{z_0} \neq 0$. Let $D := \{z \in U / \varphi(z) < 0\}$ and let H be the complex hyperplane tangent at z_0 to the real hypersurface $\{\varphi = 0\}$ which is smooth near z_0 .

Our terminology (Norguet–Siu convention, see [NS77]) is to say that the open set D is strongly q -convex near $z_0 \in \partial D$ if the restriction to H of the Levi form of φ at z_0 has at most q non positive eigenvalues.

Looking now at the same open set D but asking for some strong concavity condition, we write $D := \{z \in U / -\varphi(z) > 0\}$. Then we shall say that D is strongly q -concave at the point z_0 if the restriction to H of Levi form of $-\varphi$ at z_0 has at most q non positive eigenvalues.

If the signature of the restriction to H of the Levi form of φ at z_0 is given by $(p - 1)$ “plus” and n “minus” we see that that near z_0 our open set D will be strongly n -convex near z_0 and strongly $(p - 1)$ -concave near z_0 . So D will be strongly $(p - 1)$ -concave near z_0 if the function $-\varphi$ is strongly $(p - 1)$ -convex at the point z_0 .

In order that a \mathcal{C}^2 exhaustion function $\varphi : Z \rightarrow]0, 2]$ on a reduced complex space Z gives relatively compact q -concave subsets $Z_\alpha := \{\varphi(z) > \alpha\}$ for each $\alpha \in]0, 1[$ which is not critical for φ , we see that it is enough that the Levi form of φ at each point in $\varphi^{-1}(]0, 1[)$ has at most q non positive eigenvalues. That is to say that φ is strongly q -convex on this open set.

In order to reach the key situation given in Proposition 2.7 with a n -cycle, we need to dispose of a \mathcal{C}^2 -exhaustion $\varphi : Z \rightarrow]0, 2]$ which is $(n - 2)$ -strongly convex on the open set $\varphi^{-1}(]0, 1[)$. So we need to assume that $n \geq 2$.

2.E. Boxed Hartogs figures

Definition 2.12. Let $n \geq 2$ and $p \geq 1$ be integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space Z . Let $\mathcal{H} = (\mathcal{M}, M, B, j)$ and $\mathcal{H}' := (\mathcal{H}', M', B, j)$ be two n -Hartogs figures in Z relative to the boundary of Δ given by the same (local) embedding j and having the same polydisc $B \subset\subset \mathbb{C}^p$. We shall say that these two n -Hartogs figures are **boxed** when we can choose α, α', V, V' in Definition 2.2 in order to have

- $\mathcal{M}'(\alpha') \subset\subset \mathcal{M}(\alpha)$,
- $M'(\alpha') \subset\subset M(\alpha)$,
- $V' \subset\subset V$.

For instance, if $\varepsilon > 0$ is small enough, the n -Hartogs figures $(\mathcal{H}, \mathcal{H}^\varepsilon)$ are boxed (see Definition 2.4).

Proposition 2.13. Let $n \geq 2$ and $p \geq 1$ be integers, let Z be a reduced complex space of pure dimension $n+p$ and let $\Delta \subset\subset Z$ be an open set with smooth \mathcal{C}^2 boundary in Z which is strongly $(n-2)$ -concave. Assume that $\Delta := \{\varphi > 0\}$ and let X_0 be a n -cycle in an open neighbourhood Δ' of the compact set $\bar{\Delta}$, such that any irreducible component of X_0 meets Δ . Then there exists a finite family of boxed n -Hartogs figures $(\mathcal{H}'_a, \mathcal{H}_a)_{a \in A}$ relative to the boundary of Δ , such that the following conditions hold:

1. The open sets \mathcal{M}'_a for $a \in A$ cover the boundary $\partial\Delta$.
2. For each $a \in A$ the Hartogs figures \mathcal{H}_a and \mathcal{H}'_a are adapted to X_0 .
3. For each $a \in A$ any irreducible component of X_0 meeting \mathcal{M}_a meets the open set M'_a .
4. No **compact** irreducible component of $X_0 \cap \Delta$ meets the union of the compact sets $\bar{\mathcal{M}}_a$, $a \in A$.

Remark. Let X_0 be any n -cycle in Δ' . Choosing the open set Δ' small enough around the compact set $\bar{\Delta}$, we can assume that the cycle X_0 has only finitely many irreducible components in Δ' and that each of them which is not compact meets $\partial\Delta$ (see Remark 3 following Proposition 2.7).

Corollary 2.14. In the situation of Proposition 2.13, if we assume that the open set Δ' containing $\bar{\Delta}$ is small enough, any irreducible component Γ of the cycle X_0 in Δ' satisfies for all $a \in A$ and all $\eta > 0$ small enough:

$$\Gamma \cap (\mathcal{M}_a^\eta \times V_a^\eta \times B_a) = \text{prlgt}_a[\Gamma \cap (M_a \times V_a \times B_a)]$$

where $\text{prlgt}_a : H(\bar{M}_a \times \bar{V}_a, \text{Sym}^k(B_a)) \rightarrow H(\bar{\mathcal{M}}_a^\eta \times \bar{V}_a^\eta, \text{Sym}^k(B_a))$ is the holomorphic map of analytic extension built in Proposition 2.6.

Remark. In the situation of the previous corollary, choosing $\varepsilon > 0$ small enough, there exists, for each $a \in A$, a holomorphic extension map which lifts the map prlgt_a :

$$i\text{prlgt}_a : \Sigma_{M_a, M_a^\varepsilon}(k) \longrightarrow \Sigma_{\mathcal{M}_a^\eta, \mathcal{M}_a^{\eta+\varepsilon}}(k).$$

It allows to extend in this setting an analytic family of branched coverings in M_a parametrized by a Banach analytic set S and which is isotropic on $S \times M_a^\varepsilon$ to an analytic family of branched coverings in \mathcal{M}_a^η which is isotropic on $S \times \mathcal{M}_a^{\eta+\varepsilon}$.

Proof of Proposition 2.13. Corollary 2.11 and the remark following it implies the existence, for each $z \in \partial\Delta$ of a n -Hartogs figure \mathcal{H}_z relative to the boundary of Δ , contained in Δ' and satisfying the following properties:

- i) $z \in \mathcal{M}_z$;
- ii) \mathcal{H}_z is adapted to X_0 ;
- iii) each irreducible component of X_0 meeting \mathcal{M}_z meets the open set M_z ;
- iv) No compact irreducible component of $X_0 \cap \Delta$ meets $\bar{\mathcal{M}}_z$.

As the open sets \mathcal{M}_z cover the compact set $\partial\Delta$ we can find a finite sub-covering. Then the properties 1, 2, 3 and 4 are consequences of i), ii), iii) and iv) by letting $\mathcal{H}'_a := \mathcal{H}_a^\varepsilon$ and choosing $\varepsilon > 0$ small enough. \square

Proof of Corollary 2.14. Let Γ be an irreducible component of X_0 meeting \mathcal{M}_a for some $a \in A$. Then Γ meets M_a . As Γ does not meet $\bar{M}(\alpha)_a \times \bar{V}_a \times \partial B_a$ because \mathcal{H}_a is adapted to X_0 , the intersection $\Gamma \cap \bar{M}_a$ is the graph of an element $\gamma \in H(\bar{M}(\alpha)_a \times \bar{V}_a, \text{Sym}^{k_a}(B_a))$ with $k_a \in \mathbb{N}^*$ ⁶.

The closed analytic subset Y of the open set \mathcal{M}_a^η defined by $Y := \text{prlgt}_a[\Gamma \cap M_a]$ is not empty, of pure dimension n and is contained in Γ . So it is a union of irreducible components of $\Gamma \cap \mathcal{M}_a^\eta$. But it contains a non empty open set in each irreducible component of this branched covering. So these two analytic subsets coincide.

If an irreducible component of X_0 does not meet any \bar{M}_a it has to be compact and contained in Δ . In this case the desired equality is obvious. \square

3. The extension and finiteness theorem

3.A. Some useful lemmas

The version below of Sard's lemma is more or less classical.

Lemma 3.1. *Let Z be a reduced complex space and let $\varphi : Z \rightarrow \mathbb{R}$ be a real valued \mathcal{C}^1 function. Then the set of critical values of φ has Lebesgue measure 0.*

Proof. Firstly note that a point $z \in Z$ is critical for φ if, by definition, the differential of φ vanishes on $T_{Z,z}$, the Zariski tangent space of Z at z . Remember also that a complex space is, by definition, countable at infinity ; so Z and its singular locus have only countably many irreducible components. As a countable union of sets of measure 0 is again of measure 0, it is enough to prove the lemma when Z is irreducible. We shall prove the lemma by induction on the integer $\dim Z$. The case $\dim Z = 0$ is obvious. Assume the lemma true for $\dim Z \leq n - 1$ for some integer $n \geq 1$ and take an irreducible complex space Z of dimension n . The singular set S of Z has dimension at most $(n - 1)$, and for each irreducible component S_i of S the image of the critical set of $\varphi|_{S_i}$ has measure 0. So the critical set of $\varphi|_S$ is again of measure 0. But a critical point of φ which belongs to S is a critical point of $\varphi|_S$. So it is enough to show that the set of critical values of φ restricted to the complex connected manifold $Z \setminus S$ has measure 0. This is the classical Sard's lemma. \square

Lemma 3.2. *Let V be an open set and K be a compact set in $\bar{U} \times \bar{B}$. The subset \mathcal{V} in $H(\bar{U}, \text{Sym}^k(B))$ consisting of the X such that any irreducible component meeting K meets V is an open set in $H(\bar{U}, \text{Sym}^k(B))$.*

Proof. Let us clarify the meaning of an irreducible component of an element X in $H(\bar{U}, \text{Sym}^k(B))$: we call irreducible component of such a X the closure in $\bar{U} \times B$ of an irreducible component of the branched covering of U defined by the projection of $X \cap (U \times B)$ on U .

⁶ \uparrow as it exists some $(m, v) \in \bar{M}(\alpha)_a \times \bar{V}_a$ such that $\Gamma \cap (\{m, v\} \times B_a) \neq \emptyset$.

Let X_0 be such that any irreducible component of X_0 which meets K meets V , and assume that $(X_\nu)_{\nu \geq 1}$ is a sequence converging to X_0 such that for each $\nu \geq 1$ there exists an irreducible component Γ_ν of X_ν meeting K but not V . Passing to a subsequence, we may assume that the sequence $(\Gamma_\nu)_{\nu \geq 1}$ converges uniformly on any compact of $U \times B$ to a non empty n -cycle Γ with closure contained in X_0 and which is a branched covering of U . Then $\bar{\Gamma}$ meets K and not V . Indeed, if (t_0, x_0) would be in $\bar{\Gamma} \cap V$, there exists open neighbourhoods U_1 and B_1 of t_0 and x_0 respectively in \bar{U} and \bar{B} such that $U_1 \times B_1$ is contained in V . But then, as $U_2 := U_1 \cap U$ and $B_2 := B_1 \cap B$ are non empty open sets, for t_2 in U_2 the fibers of the Γ_ν at t_2 for ν big enough will meet $\{t_2\} \times B_2$ and so V . As at least one irreducible component of Γ meets K without meeting V and as its closure is an irreducible component of X_0 , this gives a contradiction. \square

Of course, in the case $V = \emptyset$, we get back the fact that the subset in $H(\bar{U}, \text{Sym}^k(B))$ of elements which do not meet K is open.

Lemma 3.3. *Let Z be a complex space and let $(\mathcal{U}_i)_{i \in I}$ be an open covering of Z . Assume that for each $i \in I$ a closed n -cycle X_i is given in \mathcal{U}_i . Assume that the following patching condition holds:*

$$\forall (i, j) \in I^2 \quad X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$$

as an equality of cycles in $\mathcal{U}_i \cap \mathcal{U}_j$. Then there exists a unique closed n -cycle X in Z such that for each $i \in I$ we have $X \cap \mathcal{U}_i = X_i$.

For the easy proof see [BM14, Chapter IV, Proposition 1.3.1].

The following variant will be used.

Lemma 3.4. (Variant) *In the situation of the previous lemma replace the patching condition by the following two conditions:*

1. *For each couple $(i, j) \in I^2$ an open subset $W_{i,j} \subset \subset \mathcal{U}_i \cap \mathcal{U}_j$ is given and we ask that $X_i \cap W_{i,j} = X_j \cap W_{i,j}$.*
2. *For each couple $(i, j) \in I^2$ we ask that any irreducible component of the cycle $X_i \cap \mathcal{U}_j$ meets the open set $W_{i,j}$.*

Then the conclusion is the same.

Proof. Let Γ be an irreducible component with multiplicity δ in the cycle $X_i \cap \mathcal{U}_j$. Let Γ' be the irreducible component of X_i which contains Γ , and put $X_i = X'_i + \delta \cdot \Gamma'$. Then Γ' meets $W_{i,j}$ and there exists a closed analytic subset Y of pure dimension n in $|X_j|$ such that its restriction to $W_{i,j}$ is equal to $\Gamma' \cap W_{i,j}$: indeed, Y is the union of the irreducible components of X_j containing a non empty open set in $\Gamma' \cap W_{i,j}$. Note that each of these irreducible components of X_j has multiplicity δ in the cycle X_j . Then put $X_j = X'_j + \delta \cdot Y$. We see that the cycles X'_i and X'_j respectively in \mathcal{U}_i and \mathcal{U}_j satisfy again the patching condition $X'_i \cap W_{i,j} = X'_j \cap W_{i,j}$.

This allows, for fixed (i, j) , to make a descending induction on the number (necessarily finite as $W_{i,j}$ is relatively compact) of irreducible components of $X_i \cap \mathcal{U}_j$, to show that the condition $X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$ holds. This reduces this lemma to the previous one. \square

3.B. Adjusted scales.

The definition of a scale adapted to a cycle is recalled in Definition 1.3.

Definition 3.5. 1. Let Z be a complex space. We shall call **adjusted n -scale on Z** , written down $\mathbb{E} := (U, U', U'', B, B'', j)$, the data of a n -scale on Z , $E := (U, B, j)$, with additional polydiscs $U'' \subset \subset U' \subset \subset U$ and $B'' \subset \subset B$. We call E the **underlying scale of the adjusted scale \mathbb{E}** .

2. We shall say that **the adjusted scale \mathbb{E} is adapted to a n -cycle X in Z** when we have

$$j^{-1}(\bar{U} \times (\bar{B} \setminus B'')) \cap |X| = \emptyset.$$

Note that this implies that the underlying scale E is adapted to X , but this condition is more restrictive.

3. When the adjusted scale \mathbb{E} is adapted to the n -cycle X , we shall call **degree of X in \mathbb{E}** the degree of X in E .
4. We shall call **center of the adjusted scale**, written down $D(\mathbb{E})$, or more simply, $D(E)$, the open set $j^{-1}(U \times B)$ in Z which is also the center of the scale E .
5. We shall call **domain of isotropy of the adjusted scale**, written down $D'(\mathbb{E})$, the open set $j^{-1}(U' \times B)$ in Z .
6. We shall call **domain of patching of the adjusted scale**, written down $D''(\mathbb{E})$, the open set $j^{-1}(U'' \times B'')$ in Z .

Remarks.

1. The open set $D''(\mathbb{E})$ is relatively compact in $D'(\mathbb{E})$.
2. When a n -scale E is given, for any compact set K in $D(E)$, there exists an adjusted n -scale \mathbb{E} such that E is the underlying scale of \mathbb{E} and with $K \subset D''(\mathbb{E})$. Moreover, if E is adapted to a n -cycle X_0 in Z , we may choose \mathbb{E} in order that it is adapted to X_0 .
3. As for $X \in \mathcal{C}_n^{loc}(Z)$ the condition to avoid a given compact subset is open in $\mathcal{C}_n^{loc}(Z)$, when the adjusted scale \mathbb{E} is adapted to a cycle X_0 there exists an open neighbourhood, written down $\Omega_k(\mathbb{E})$, of X_0 in $\mathcal{C}_n^{loc}(Z)$ such that $\Omega_k(\mathbb{E})$ is the subset of all n -cycles X in Z for which \mathbb{E} is adapted and $\deg_{\mathbb{E}}(X) = k$ where $k := \deg_{\mathbb{E}}(X_0)$.

Let Z be a reduced complex space and let $\mathbb{E} = (U, U', U'', B, B'', j)$ be an adjusted scale on Z . For a given integer k consider the continuous map sending a branched covering in $H(\bar{U}, \text{Sym}^k(B''))$ to its isotropy data on \bar{U}' (for the notations see what follows Lemma 2.5)

$$T : H(\bar{U}, \text{Sym}^k(B'')) \rightarrow H(\bar{U}', F \otimes E').$$

The graph $\Sigma_{U, U'}(k)$ of this map is a Banach analytic set⁷, thanks to [Bar75, Proposition 2, p. 81] (see also [BM]).

The set of couples $(f, T(f))$ in $\Sigma_{U, U'}(k)$ for which the associated branched covering is contained in $j(Z \cap \bar{D}(\mathbb{E}))$ is a closed Banach analytic subset of $\Sigma_{U, U'}(k)$ being the pull-back by the projection of the subset of elements in $H(\bar{U}, \text{Sym}^k(B''))$ contained in $j(Z \cap \bar{D}(\mathbb{E}))$ which is a closed Banach analytic subset of $H(\bar{U}, \text{Sym}^k(B''))$ by Proposition 4, p. 27 of [Bar75] (see also [BM, Chapter V]).

Definition 3.6. We shall denote $\mathcal{G}_k(\mathbb{E})$ this Banach analytic set and we shall call it the **the k -th classifying space of the adjusted n -scale \mathbb{E} on Z** .

We have then a tautological family of n -cycles in the open set $D(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$. It is an analytic family of cycles in the open set $D'(\mathbb{E})$, in the sense of [Bar75], and the fact that, for $k \geq 1$, locally on $\mathcal{G}_k(\mathbb{E})$, any irreducible component of a branched covering in this family meets $\bar{U}'' \times \bar{B}''$ implies that we have a f -analytic family of cycles in $D'(\mathbb{E})$.

⁷ \uparrow homeomorphic to $H(\bar{U}, \text{Sym}^k(B''))$ via the projection!

Be careful that the tautological family of cycles on the open set $D(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$ is not, in general, an analytic family of n -cycles ; see the example of [Bar75, p. 83] (and also [BM14, Chapter IV]).

The next lemma is an obvious consequence of *loc. cit.*

Lemma 3.7. *Let \mathbb{E} be an adjusted n -scale on a reduced complex space Z and let k be an integer. The tautological family of n -cycles in the open set $D'(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$ has the following “almost universal” property:*

For any analytic family of n -cycles $(X_s)_{s \in S}$ in Z parametrized by a Banach analytic set S such that for each $s \in S$ the adjusted scale \mathbb{E} is adapted to X_s with $\deg_{\mathbb{E}}(X_s) = k$, there exists a unique holomorphic map

$$f : S \rightarrow \mathcal{G}_k(\mathbb{E})$$

such that the pull-back by f of the tautological family is the restriction to the open set $D'(\mathbb{E})$ of the given family.

Of course, conversely, such a holomorphic map gives a f -analytic family of n -cycles on the open set $D'(\mathbb{E})$.

Note that the pull-back family is in fact defined on the open set $D(\mathbb{E})$ but, as already noticed above, it may not be analytic outside $D'(\mathbb{E})$.

As a consequence of this “almost universal” property, we obtain that for any analytic family $(X_s)_{s \in S}$ of n -cycles in Z such that for a point $s_0 \in S$ the adjusted n -scale \mathbb{E} is adapted to the cycle X_{s_0} with $\deg_{\mathbb{E}}(X_{s_0}) = k$, there exists an open neighbourhood S' of s_0 in S such that the previous lemma applies for the family parametrized by S' . So we shall have a holomorphic classifying map $f : S' \rightarrow \mathcal{G}_k(\mathbb{E})$ in this situation.

We shall generalize now the concept of classifying space to the case of a finite family of adjusted n -scales.

Definition 3.8. Consider a reduced complex space Z and a finite family of adjusted n -scales $(\mathbb{E}_i)_{i \in I}$ on Z . Assume that they are adapted to a given finite type n -cycle \hat{X}_0 in Z . Assume that any irreducible component of \hat{X}_0 meets the open set $W'' := \bigcup_{i \in I} D''(\mathbb{E}_i)$.

We shall call **patching data for \hat{X}_0 associated to the family $(\mathbb{E}_i)_{i \in I}$** , written $\mathcal{R}((\mathbb{E}_i)_{i \in I}, F)$ or more simply \mathcal{R} when there is no ambiguity, a finite collection of n -scales $(F_{i,j,h})$ for $(i, j) \in I^2$, $i \neq j$, where h belongs to a finite set $H(i, j)$ for each couple $(i, j) \in I^2$, $i \neq j$, such that the following properties hold:

- i) $F_{i,j,h}$ is a n -scale on the open set $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.
- ii) The n -scales $F_{i,j,h}$ are adapted to \hat{X}_0 .

We shall say that the patching data \mathcal{R} are **complete** when the following condition also holds:

- iii) For each $i \neq j$ given, the union of domains of the scales $F_{i,j,h}$, $h \in H(i, j)$, covers the compact subset $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$ of $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

Notations.

1. In the sequel, when we shall consider a reduced complex space Z and a finite family of adjusted n -scales $(\mathbb{E}_i)_{i \in I}$, adapted to a finite type n -cycle \hat{X}_0 in Z , such that any irreducible component of \hat{X}_0 meets the open set

$$W'' := \bigcup_{i \in I} D''(\mathbb{E}_i),$$

we shall say that the family $(\mathbb{E}_i)_{i \in I}$ is **convenient** for \hat{X}_0 .

2. In this setting we shall use the following definitions :

- $W := \bigcup_{i \in I} D(\mathbb{E}_i)$;
- $W' := \bigcup_{i \in I} D'(\mathbb{E}_i)$;
- $W'' := \bigcup_{i \in I} D''(\mathbb{E}_i)$;
- $K := \bigcup_{i \in I} j_i^{-1}(\bar{U}_i \times (\bar{B}_i \setminus B_i''))$.
- When the family $(\mathbb{E}_i)_{i \in I}$ is convenient for a finite type n -cycle \hat{X}_0 , \mathcal{K} will be a compact neighbourhood of K disjoint from \hat{X}_0 .

3. For $\tilde{X} \in \prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$ we shall denote by X^i the closed cycle in $D(\mathbb{E}_i)$ associated to the i -th component of \tilde{X} .

Lemma 3.9. *Let $(\mathbb{E}_i)_{i \in I}$ be a finite family of adjusted n -scales on Z , convenient for a n -cycle \hat{X}_0 of finite type in Z , and let \mathcal{R} be some corresponding complete patching data. There exists an open neighbourhood \mathcal{V} of the image \tilde{X}_0 of \hat{X}_0 in the product $\prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$ such that for each $\tilde{X} \in \mathcal{V}$ we have the following properties:*

1. No X^i meets the compact set \mathcal{K} .
2. For each $i \in I$, any irreducible component of X^i meeting $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$ with $j \neq i$, meets the open set $\bigcup_h D(F_{i,j,h})$.
3. For each (i, j, h) the scale $F_{i,j,h}$ is adapted to X^i and X^j .
4. For each (i, j, h) we have $\deg_{F_{i,j,h}}(X^i) = \deg_{F_{i,j,h}}(X^j) = \deg_{F_{i,j,h}}(\hat{X}_0) = k_{i,j,h}$.

Proof. Conditions 1, 3 and 4 are clearly open. An easy consequence of Condition 1, of the inclusion ii) of Definition 3.8 and of Lemma 3.2 is that Condition 2 is also open. \square

For a n -scale $E := (U, B, j)$ on Z we shall abbreviate $H(\bar{U}, \text{Sym}^k(B))$ in $G_k(E)$.

When we consider a cycle X_0 in an adapted scale $E := (U, B, j)$ and when we dispose of a n -scale $F := (V, C, h)$ on $U \times B$, adapted to X_0 , where h is given by an isomorphism of an open set in $U \times B$ into some open neighbourhood of $\bar{V} \times \bar{C}$ in $\mathbb{C}^n \times \mathbb{C}^p$, we have a well-defined map of an open neighbourhood \mathcal{U} of X_0 is $H(\bar{U}, \text{Sym}^k(B))$ into $H(\bar{V}, \text{Sym}^l(C))$, where $l := \deg_F(X_0)$, sending $X \in \mathcal{U}$ to the multiform graph associated to $h_*(X)$ in the scale F . This is a consequence of the fact that the condition $X_0 \cap h^{-1}(\bar{V} \times \partial C) = \emptyset$ is open in $H(\bar{U}, \text{Sym}^k(B))$ and that the degree of X near enough X_0 in the adapted scale F will be equal to l .

Such a map, which will be called a **change of scale**, is not holomorphic in general but becomes holomorphic when we add the isotropy condition:

precisely, if $U' \subset\subset U$ and if $h^{-1}(\bar{V} \times \bar{C}) \subset U' \times B$, then the change of scale map

$$\Sigma_{U,U'} \rightarrow H(\bar{V}, \text{Sym}^l(C))$$

will be holomorphic (see Theorem 4, p. 66 in [Bar75]).

Definition 3.10. Let $(\mathbb{E}_i)_{i \in I}$ a finite family of adjusted n -scales on Z , convenient for a n -cycle \hat{X}_0 of finite type in Z , and let \mathcal{R} be some corresponding patching data. Let k_i be the degree of \hat{X}_0 in the adjusted scale \mathbb{E}_i . For each (i, j, h) , $i \neq j$ we have a couple of holomorphic maps

$$\prod_{\alpha \in I} \mathcal{G}_{k_\alpha}(\mathbb{E}_\alpha) \xrightarrow{\quad} G_{k_{i,j,h}}(F_{i,j,h})$$

obtained by the changes of scales $\mathbb{E}_i \rightarrow F_{i,j,h}$ and $\mathbb{E}_j \rightarrow F_{i,j,h}$, because, by construction, we have $D(F_{i,j,h}) \subset\subset D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

We shall denote by $S(\mathcal{R})$ the intersection of the kernels of these double maps⁸ with the open set \mathcal{V} built in Lemma 3.9. It is a Banach analytic set and we shall call it the **classifying space associated to $(\mathbb{E}_i)_{i \in I}, \hat{X}_0$ and \mathcal{R}** .

Remark that the patching data \mathcal{R} are not assumed to be complete in the previous definition.

Proposition 3.11. *Consider a finite family of adjusted n -scales which is convenient for the n -cycle X_0 in Z and let \mathcal{R} be some **complete** patching data associated. Keeping the previous notations we have for each $(X^i)_{i \in I} \in S(\mathcal{R})$ a unique n -cycle $X \in \mathcal{C}_n^f(W')$ such that $X \cap D'(\mathbb{E}_i) = X^i \cap D'(\mathbb{E}_i)$, $\forall i \in I$.*

Moreover, this defines a tautological family of cycles in W' which is a f -analytic family of cycles satisfying the following “almost universal” property:

For any analytic family of n -cycles $(X_s)_{s \in S}$ in an open neighbourhood of \bar{W} parametrized by a Banach analytic set S^9 such that for $s_0 \in S$ we have $X_{s_0} = X_0$ in a neighbourhood of \bar{W} , there exists an open neighbourhood S' of s_0 in S and a unique holomorphic map

$$f : S' \rightarrow S(\mathcal{R})$$

such that the pull-back by f of the tautological family parametrized by $S(\mathcal{R})$ is the restriction to the open set W' of the family $(X_s)_{s \in S'}$.

Proof. For each $\tilde{X} \in \mathcal{V}$ any X^i does not meet \mathcal{K} . As \mathcal{R} is complete, we may use Lemma 3.4 with $\mathcal{U}_i := D'(\mathbb{E}_i)$ and $W_{i,j} = \bigcup_h D(F_{i,j,h})$ to associate to \tilde{X} a finite type n -cycle X of the open set W' . The f -analyticity of the so defined family is obvious. The “almost universal” property is then clear. \square

Note that this proposition implies that the map $S(\mathcal{R}) \rightarrow \mathcal{C}_n^f(W')$ classifying the tautological family of n -cycles in W' is a holomorphic map.

3.C. Shrinkage.

Definition 3.12. Let Z be a reduced complex space and let $\mathbb{E} := (U, U', U'', B, B'', j)$ be an adjusted n -scale on Z . For any real $\tau > 0$ small enough we shall denote by \mathbb{E}^τ the adjusted n -scale on Z defined as $\mathbb{E}^\tau := (U^\tau, U'^\tau, U''^\tau, B, B'', j)$. We shall call \mathbb{E}^τ the τ -shrinkage of \mathbb{E} .

Recall that for a polydisc P of radius R , P^τ is the polydisc with same center and radius $R - \tau$. The definitions of $\mathcal{M}(\alpha)^\tau$, $M(\alpha)^\tau$ are given in the section 2.B.

Remarks.

1. By definition, the shrinkage of \mathbb{E} does not change the embedding j and the polydiscs U'', B, B'' .
2. As j is a closed embedding of an open set in Z in an open neighbourhood of the compact set $\bar{U} \times \bar{B}$, it is clear that for any given adjusted n -scale \mathbb{E} on Z , there exists a real $\varepsilon > 0$ (depending on \mathbb{E}) such that for any $\tau \in]0, \varepsilon[$, \mathbb{E}^τ is again an adjusted n -scale on Z .
3. If \mathbb{E} is an adjusted scale adapted to the n -cycle X_0 in Z , for τ small enough (depending on \mathbb{E} and X_0), the adjusted n -scale \mathbb{E}^τ remains adapted to X_0 and we shall have also $\deg_{E^\tau}(X_0) = \deg_E(X_0)$.

⁸ \uparrow The kernel of a double map $f, g : A \rightarrow B$ is the pull-back of the diagonal in $B \times B$ by the map $(f, g) : A \rightarrow B \times B$.

⁹ \uparrow Recall that, by definition, this means that for any n -scale $E := (U, B, j)$ on Z , adapted to some $X_{s_0}, s_0 \in S$, with $\deg_E(X_{s_0}) = k$, we have an open neighbourhood S_0 of s_0 in S and a classifying map for the corresponding family of branched coverings $f : S_0 \times U \rightarrow \text{Sym}^k(B)$ which is **isotropic**.

4. If \mathbb{E} is an adjusted scale adapted to the n -cycle X_0 in Z , there exists an open neighbourhood \mathcal{U} of X_0 in $\mathcal{C}_n^{loc}(Z)$ and a real $\varepsilon > 0$ such that for any $X \in \mathcal{U}$ and any $\tau \in]0, \varepsilon[$, the adjusted scale \mathbb{E}^τ remains adapted to X with again $\deg_{E^\tau}(X) = \deg_E(X_0)$.
5. In the situation of the previous proposition 3.11, we may, for $\tau > 0$ small enough, keep the same patching data \mathcal{R} on the finite family $(\mathbb{E}_i^\tau)_{i \in I}$ of adjusted n -scales; if it was complete, it remains complete and if it was convenient¹⁰ for the n -cycle X_0 , it remains convenient for the n -cycle X_0 . Then we have a holomorphic restriction map

$$S(\mathcal{R}) \rightarrow S^\tau(\mathcal{R})$$

where $S^\tau(\mathcal{R})$ is the classifying space associated to the family $(\mathbb{E}^\tau)_{i \in I}$, and this map is induced by a **finite product of linear (continuous) compact maps**. This last point is crucial for the finiteness results.

3.D. Excellent family.

In order to avoid that our notations become too heavy we shall introduce the following conventions when \mathcal{H} is a n -Hartogs figure in \mathbb{C}^{n+p} ; we shall put, using the notations introduced above

$$U := M(\alpha) \times V, \quad U' := M(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad U'' := M(\alpha)^{\varepsilon''} \times V^{\varepsilon''}, \quad B'' := B^{\varepsilon''}$$

where $\varepsilon' > 0$ is small enough, and where $0 < \varepsilon'' < \varepsilon'$. The choices of ε' and ε'' will be precised when they are useful. We shall associate to \mathcal{H} the adjusted n -scale on Δ given by:

$$\mathbb{E}_{\mathcal{H}} := (U, U', U'', B, B'', j).$$

We shall put also

$$\tilde{U} := \mathcal{M}(\alpha) \times V, \quad \tilde{U}' := \mathcal{M}(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad \tilde{U}'' := \mathcal{M}(\alpha)^{\varepsilon''} \times V^{\varepsilon''}$$

the holomorphy envelopes respectively of U , U' and U'' . Then we shall have a family of adjusted n -scales on Δ' , written down:

$$\mathbb{E}_{\tilde{\mathcal{H}}}^\eta := (\tilde{U}^\eta, \tilde{U}'^\eta, \tilde{U}''^\eta, B, B'', j) \quad \text{with} \quad \tilde{U}^\eta := \mathcal{M}(\alpha)^\eta \times V^\eta, \quad \tilde{U}'^\eta := \mathcal{M}(\alpha)^{\varepsilon'+\eta} \times V^{\varepsilon'+\eta},$$

where η is a non negative real number, small enough (for $\eta = 0$ we shall simply write $\mathbb{E}_{\tilde{\mathcal{H}}}$).

Then we shall have the isotropic classifying spaces

$$\mathcal{G}_k(\mathcal{H}) := \Sigma_{U, U'}(k) \quad \text{and also} \quad \mathcal{G}_k^\eta(\tilde{\mathcal{H}}) := \Sigma_{\tilde{U}^\eta, \tilde{U}'^\eta}(k).$$

Proposition 2.6 gives a holomorphic analytic extension map $prlgt^\eta$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}_k(\tilde{\mathcal{H}}) & \xrightarrow{res^\eta} & \mathcal{G}_k^\eta(\tilde{\mathcal{H}}) \\ res \downarrow & \nearrow prlgt^\eta & \\ \mathcal{G}_k(\mathcal{H}) & & \end{array}$$

Definition 3.13. Let Z be a reduced complex space of pure dimension $n + p$, let $\Delta \subset\subset Z$ be a strongly $(n - 2)$ -concave open set in Z and \tilde{X}_0 a n -cycle in an open neighbourhood Δ' of Δ in Z . We shall say that a finite family $(\mathcal{H}_a)_{a \in A}$ of n -Hartogs figures relative to the boundary of Δ is **excellent** for the cycle \tilde{X}_0 when the following conditions hold, where we write $\tilde{\mathbb{E}}_a$ and \mathbb{E}_a the adjusted scales respectively on Δ' and Δ associated to the n -Hartogs figure \mathcal{H}_a :

¹⁰ ↑ See the notations following Definition 3.8.

1. The adjusted scales $\tilde{\mathbb{E}}_a$ and \mathbb{E}_a are adapted to the cycle \tilde{X}_0 .
2. We may choose the patching domains of the adjusted scales $(\tilde{\mathbb{E}}_a)_{a \in A}$ in order that the union

$$\tilde{D}''(A) := \bigcup_{a \in A} j_a^{-1}(\tilde{U}''_a \times B''_a)$$

contains the compact set $\partial\Delta$.

3. There exists a finite family of adjusted n -scales $(\mathbb{E}_b)_{b \in B}$ on Δ , adapted to \tilde{X}_0 , such that the finite families $(\mathbb{E}_c)_{c \in A \cup B}$ and $(\tilde{\mathbb{E}}_c)_{c \in A \cup B}$ are convenient for \tilde{X}_0 , where we put $\tilde{\mathbb{E}}_b = \mathbb{E}_b$ for $b \in B$. Moreover we ask that the union $D''(B)$ of the patching domains of the $(\mathbb{E}_b)_{b \in B}$ covers the compact set $\Delta \setminus \tilde{D}''(A)$ of Δ ; so $\tilde{D}''(A) \cup D''(B)$ in an open set containing $\bar{\Delta}$.

As a consequence, the union $\tilde{D}'(A)^\eta \cup D'(B)$ of the isotropy domains will cover $\bar{\Delta}$ for $\eta > 0$ small enough.

Proposition 3.14. (Existence of excellent families) *Let Z be a reduced complex space of pure dimension $n + p$, $\Delta \subset\subset Z$ be a strongly $(n - 2)$ -concave open set with smooth boundary and \tilde{X}_0 a n -cycle in an open neighbourhood $\Delta' \subset Z$ of $\bar{\Delta}$ such that any irreducible component of \tilde{X}_0 meets Δ .*

Then there exists a finite family $(\mathcal{H}_a)_{a \in A}$ of n -Hartogs figures relative to the boundary of Δ which is excellent for the cycle \tilde{X}_0 .

Proof. First we use Proposition 2.13 to cover $\partial\Delta$ by a finite family of n -Hartogs figures relative to the boundary of Δ adapted to the cycle \tilde{X}_0 such that Conditions 1 and 2 hold. Then we build a finite family of adjusted n -scales $(\mathbb{E}_b)_{b \in B}$ on Δ , adapted to \tilde{X}_0 in order that Condition 3 holds. \square

3.E. The extension and finiteness theorem.

The next theorem will be crucial in the proof of Theorem 1.1.

Theorem 3.15. *Let Z be a reduced complex space of pure dimension $n + p$, where $n \geq 2$, $p \geq 1$. Assume that there exists a \mathcal{C}^2 exhaustion $\varphi : Z \rightarrow]0, 2]$ which is strongly $(n - 2)$ -convex on the open set $\varphi^{-1}(]0, 1[)$ and let $\Delta := \{x \in Z / \varphi(x) > \alpha\}$ for some $\alpha \in]0, 1[$ which is not a critical value for φ . Let \tilde{X}_0 a closed n -cycle of an open neighbourhood Δ' of $\bar{\Delta}$ in Z such that any irreducible component of \tilde{X}_0 meets Δ . Then there exists an open neighbourhood Δ'' of $\bar{\Delta}$ in Δ' and a \mathfrak{f} -analytic family $(\tilde{X}_\xi)_{\xi \in \Xi}$ of n -cycles in Δ'' parametrized by a reduced complex space Ξ (of finite dimension) such that $X_{\xi_0} = \tilde{X}_0 \cap \Delta''$ and such that Ξ is isomorphic to an open neighbourhood of $\tilde{X}_0 \cap \Delta''$ in $C_n^{\mathfrak{f}}(\Delta'')$. It satisfies the following universal property:*

For any \mathfrak{f} -analytic family $(X_s)_{s \in S}$ of n -cycles in Δ parametrized by a Banach analytic set S and such that its value at some $s_0 \in S$ is equal to $\tilde{X}_0 \cap \Delta$, there exists an open neighbourhood S' of s_0 in S and an unique holomorphic map

$$h : S' \rightarrow \Xi$$

satisfying the equality $X_s = \tilde{X}_{h(s)} \cap \Delta$ for each $s \in S'$.

Proof. Note that Δ is a relatively compact open set in Z with a \mathcal{C}^2 boundary which is strongly $(n - 2)$ -concave.

Begin by covering the compact set $\partial\Delta$ by an excellent finite family $(\mathcal{H}_a)_{a \in A}$ of n -Hartogs figures for Δ adapted to the cycle \tilde{X}_0 . Choose then open sets $\Delta_1 \subset\subset \Delta \subset\subset \Delta'' \subset\subset \Delta'$, such that the following properties hold, where we use the notations introduced above for the finite family of the adjusted scales $(\tilde{\mathbb{E}}_a)_{a \in A}$ associated to $(\mathcal{H}_a)_{a \in A}$:

- i) $\bar{\Delta}'' \setminus \Delta_1 \subset \bigcup_{a \in A} W''(\tilde{\mathbb{E}}_a)$.
- ii) $\bigcup_{a \in A} W(\tilde{\mathbb{E}}_a) \subset \subset \Delta'$.
- iii) $\bigcup_{a \in A} \bar{W}(\mathbb{E}_a) \subset \Delta$.
- iv) $K := \bigcup_{a \in A} j_a^{-1}(\bar{U}_a \times (\bar{B}_a \setminus B''_a)) \subset \Delta_1$

It is easy to fulfill these conditions for Δ_1 and Δ'' near enough to Δ , as, by assumption, the subsets $\bar{W}(\mathbb{E}_a)$ and $j_a^{-1}(\bar{U}_a \times \bar{B}_a \setminus B''_a)$ are compact in Δ , and as the union of the open sets $W''(\tilde{\mathbb{E}}_a)$ contains $\partial\Delta$.

Note that Condition iv) allows to choose the compact neighbourhood \mathcal{K} of K inside Δ_1 .

Choose now a convenient finite family $(\mathbb{E}_b)_{b \in B}$ of adjusted n -scales on Δ , adapted to \tilde{X}_0 in order that the open set $\bigcup_{b \in B} W''(\mathbb{E}_b)$ contains $\bar{\Delta}_1$.

Put $\tilde{\mathbb{E}}_b = \mathbb{E}_b$ for $b \in B$, and define $C := A \cup B$. The family of adjusted scales $(\mathbb{E}_c)_{c \in C}$ in Δ is then convenient for $\tilde{X}_0 \cap \Delta_1$, up to choosing the patching domains big enough. Fix some complete patching data \mathcal{R} associated to the family $(\mathbb{E}_c)_{c \in C}$.

The family $(\tilde{\mathbb{E}}_c)_{c \in C}$ of adjusted scales in Δ' is convenient for $\tilde{X}_0 \cap \Delta''$ if we choose the patching domains big enough. Consider now some complete patching data for this family of the form $\mathcal{R} \cup \mathcal{R}''$, that is to say containing the patching scales already in \mathcal{R} . Define the following Banach analytic sets:

1. S_0 is the classifying space of the family $(\tilde{\mathbb{E}}_c)_{c \in C}$, the degrees being those of \tilde{X}_0 in the various scales adapted to the cycle \tilde{X}_0 , with the patching conditions defined by \mathcal{R} . Note that \mathcal{R} is not complete in general.
2. S_+ is the classifying space of the family $(\tilde{\mathbb{E}}_c)_{c \in C}$ with the patching conditions defined by $\mathcal{R} \cup \mathcal{R}''$.
3. S_- is the classifying space of the family $(\mathbb{E}_c)_{c \in C}$, with the (complete) patching conditions defined by \mathcal{R} .

Then we get a holomorphic extension map

$$\alpha : S_- \rightarrow S_0$$

deduced from the extension maps in the n -Hartogs figures $(\mathcal{H}_a)_{a \in A}$.

By definition S_+ is a closed Banach analytic subset of S_0 as it is defined in S_0 by the patching conditions given by \mathcal{R}'' . Then put $\Xi := \alpha^{-1}(S_+)$. So we have a holomorphic map $\alpha : \Xi \rightarrow S_+$. We want to show the following claim:

Claim. There exists a holomorphic map $\beta : S_+ \rightarrow \Xi$ satisfying the two properties:

1. We have $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$ in a neighbourhood of the points defined by \tilde{X}_0 respectively in S_+ and Ξ .
2. The holomorphic map β is the composition of a holomorphic map induced by a linear (continuous) compact map and a holomorphic map.

As the open set $\bigcup_{c \in C} W''(\tilde{\mathbb{E}}_c)$ contains Δ'' , there is, on S_+ , a tautological family of n -cycles which is f -analytic on Δ'' . Then the ‘‘almost universal’’ property of S_- gives a holomorphic map $\tilde{\beta} : S_+ \rightarrow S_-$ which factorizes via the closed Banach analytic subset $\Xi \subset S_-$. Let us show that the holomorphic map $\beta : S_+ \rightarrow \Xi$ deduced from this factorization satisfies the two properties of the claim.

First we have $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$ respectively in S_+ and Ξ , by definition of Ξ and β (see Proposition 2.6).

To see the second property, consider $\tau > 0$ small enough and let us show that the holomorphic map induced by the linear compact (by Vitali’s theorem) restriction $r^\tau : S_+ \rightarrow S_+^\tau$ factorizes β , where

S_+^τ is the classifying space corresponding to the τ -shrinkage $(\tilde{\mathbb{E}}_c^\tau)_{c \in C}$ of the family $(\tilde{\mathbb{E}}_c)_{c \in C}$ with the patching data $\mathcal{R} \cup \mathcal{R}''$. Indeed, for τ small enough, the tautological family of S_+^τ is still f-analytic on Δ and the ‘‘almost universal’’ property of S_- gives again a holomorphic map $\tilde{\beta}^\tau : S_+^\tau \rightarrow S_-$. And we have $\beta = \tilde{\beta}^\tau \circ r^\tau$ proving our assertion. In fact, the map $\tilde{\beta}^\tau$ takes its values in Ξ because the τ -shrinkage does not change the patching data deduced from $\mathcal{R} \cup \mathcal{R}''$ for τ small enough.

We conclude that the Banach analytic set Ξ has finite dimension thanks to the finiteness lemma of [Bar75, p. 8] (see also [BM, Chapter V]). Moreover, it is isomorphic to S_+ which is also of finite dimension.

The holomorphic isomorphism $\beta : S_+ \rightarrow \Xi$ factorizes via an open neighbourhood of $X_0 \cap \Delta''$ in $\mathcal{C}_n^f(\Delta'')$ because S_+ parametrizes a f-analytic family of cycles in Δ'' , and because a cycle X near enough to $X_0 \cap \Delta''$ defines an element in $\Xi \subset S_-$ as it satisfies the patching conditions $\mathcal{R} \cup \mathcal{R}''$. This shows that we can identify the Banach analytic set Ξ (which is a reduced finite dimensional complex space) with an open neighbourhood of $X_0 \cap \Delta''$ in $\mathcal{C}_n^f(\Delta'')$.

The universal property is obvious. \square

Note that it is not restrictive to choose $\Delta'' := \{\varphi > \alpha_1\}$ for some $\alpha_1 \in]0, \alpha[$, near enough to α .

Remarks.

1. The reduced complex space (of finite dimension) Ξ built in the previous theorem parametrizes a f-analytic family of n -cycles in the open set Δ'' which is an open neighbourhood of $\tilde{\Delta}$. So, when we have a f-analytic family $(X_s)_{s \in S}$ of n -cycles in $\tilde{\Delta}$ parametrized by a Banach analytic set S and such that its value at some $s_0 \in S$ is equal to $\tilde{X}_0 \cap \Delta$ we can extend each cycle X_s , $s \in S'$, to a n -cycle \tilde{X}_s in Δ'' in order that the family $(\tilde{X}_s)_{s \in S'}$ is f-analytic in Δ'' , with the condition that each irreducible component of \tilde{X}_s meets Δ and with the equality $\tilde{X}_s \cap \Delta = X_s$ for each $s \in S'$.
2. We shall see later on that Ξ is also (isomorphic to) an open neighbourhood of the point $\tilde{X}_0 \cap \Delta$ in $\mathcal{C}_n^f(\Delta)$.

4. Finiteness of the space of n -cycles of a reduced strongly $(n - 2)$ -concave space ($n \geq 2$).

4.A. The global extension theorem.

First we have to complete our terminology.

Definition 4.1. We shall say that a reduced complex space Z is **strongly q -concave**, where $q \geq 0$ is a natural integer, if there exists a real valued \mathcal{C}^2 exhaustion function on Z , $\varphi : Z \rightarrow]0, 2]$, which is strongly q -convex outside the compact set $K := \varphi^{-1}([1, 2])$.

In the sequel, when we consider a reduced complex space Z which is assumed to be q -concave, we shall always assume implicitly that we have chosen such an exhaustion φ . For instance, any reduced compact complex space is strongly q -concave for any $q \geq 0$.

When Z is strongly q -concave irreducible and non compact of dimension at least $q + 1$, the function φ achieves its maximum at a point in K . So we shall have $\varphi(Z) =]0, u]$ with $u \geq 1$.

Theorem 4.2. *Let $n \geq 2$ be an integer. Let Z be a reduced complex space which is strongly $(n - 2)$ -concave. Let $\alpha \in]0, 1[$ and let X be a finite type n -cycle in the open set $Z_\alpha := \{z \in Z \mid \varphi(z) > \alpha\}$. Then there exists a unique n -cycle \tilde{X} in $\mathcal{C}_n^f(Z)$ such that $\tilde{X} \cap Z_\alpha = X$.*

The proof will use the following remark.

Remark. Consider a closed irreducible analytic subset Γ of dimension n in Z . As the restriction $\varphi|_{\Gamma}$ of the exhaustion φ to Γ must reach its maximum (as φ is continuous and proper), this maximum cannot be obtained at a point in which φ is strongly $(n - 2)$ -convex. So we have $\Gamma \cap \varphi^{-1}[1, 2] \neq \emptyset$.

Proof. The uniqueness of \tilde{X} is a consequence of the previous remark: if \tilde{X}_i , $i = 1, 2$, are in $\mathcal{C}_n^f(Z)$ and satisfy $\tilde{X}_i \cap Z_\alpha = X$, then any irreducible component Γ_1 of \tilde{X}_1 has to meet Z_α and so has to contain an open set in \tilde{X}_2 . Then it has to be an irreducible component of \tilde{X}_2 and its multiplicity in \tilde{X}_1 and \tilde{X}_2 must coincide. So $\tilde{X}_1 = \tilde{X}_2$.

To show that this cycle exists, consider first the case where X is compact in Z_α . Then $\tilde{X} := X$ is a solution. So it enough to consider the case where X is irreducible and non compact. Thanks to [ST71, Theorem 8.3], for each $z \in \partial Z_\alpha$ there exists an open set U_z and an unique closed analytic set X_z in $Z_\alpha \cup U_z$ of pure dimension n such that $X_z \cap Z_\alpha = X$. Choose a finite set of points z_1, \dots, z_N and open sets $U'_i \subset\subset U_i := U_{z_i}$ such that the union of the U'_i covers the compact set ∂Z_α . Let $\Omega := Z_\alpha \cup (\bigcup_{i=1}^N U'_i)$ and put

$$X_1 := (X \cup (\bigcup_{i=1}^N X_{z_i})) \cap \Omega.$$

Let us show that X_1 is a closed analytic subset in Ω . Consider $z \in \Omega$. If z is in Z_α we have $X_1 = X$ in a neighbourhood of z and the assertion is clear. If not, either z is not in any $\partial U'_i$ and X_1 is the union of the X_{z_i} in a neighbourhood of z and the assertion is clear, or z is in $\partial U'_{j_1}, \dots, \partial U'_{j_k}$ for j_1, \dots, j_k in $[1, N]$. As the set X_{j_h} is closed and analytic in $Z_\alpha \cup U_{j_h}$, then X_1 is again the union of the X_{z_i} near z in Ω , and the assertion is proved.

So in this situation there exists a real positive $\beta < \alpha$ such that $Z_\beta \subset \Omega$. Let X_2 be the irreducible component of $X_1 \cap Z_\beta$ which contains X ; then X_2 is a closed irreducible analytic subset of Z_β such that $X_2 \cap Z_\alpha = X$.

Now let

$$\gamma := \inf\{\beta \leq \alpha / \exists X_\beta \text{ irreducible } n\text{-cycle of } Z_\beta \text{ such that } X_\beta \cap Z_\alpha = X\}.$$

Then what we obtained above shows that we have $\gamma < \alpha$, and, applying the same arguments to the cycle X_γ defined on Z_γ via the cover of Z_γ by the $Z_\beta, \beta > \gamma$ in which we already built an irreducible n -cycle X_β extending X , we conclude that $\gamma = 0$ and that there exists an (unique) irreducible n -cycle \tilde{X} in Z extending X . \square

4.B. Some consequences.

We shall give first some easy consequences of the fact that the reduced complex space Z is strongly n -concave.

Proposition 4.3. *Let $n \geq 2$ be an integer and let Z be a reduced complex space which is strongly $(n - 2)$ -concave. Then the natural map $j : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$ is a homeomorphism. Moreover, for each $\alpha \in]0, 1[$ the restriction map*

$$res_\alpha : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$$

is well defined and is also a homeomorphism.

Proof. Let us prove first that any n -cycle X in Z has finitely many irreducible components. As this implies the same result for each Z_α for $\alpha \in]0, 1[$, this will imply the fact that the restriction map res_α is well defined, and then bijective as a consequence of Theorem 4.2.

As the family of irreducible components of a n -cycle is locally finite, only finitely many irreducible components of X can meet the compact set $K := \varphi^{-1}([1, 2])$. But we have seen in the remark following the previous theorem that **any** irreducible component of X must meet K . So X is a finite type cycle.

To show the continuity of res_α^{-1} it is then enough to prove that j is a homeomorphism which is an easy consequence of the lemma below.

Lemma 4.4. *Let Z be a reduced complex space and let $(X_\nu)_{\nu \geq 0}$ be a sequence of n -cycles in Z converging in $\mathcal{C}_n^{loc}(Z)$ to a cycle Y . Assume that there exists a relatively compact open set Ω in Z such that any irreducible component of each X_ν and of Y meets Ω . Then all these cycles are of finite type and the sequence converges to Y in $\mathcal{C}_n^f(Z)$.*

Proof of Lemma 4.4. First if $Y = \emptyset$, for $\nu \gg 1$ the cycle X_ν will be disjoint from the compact set $K := \bar{\Omega}$, and this implies that X_ν is the empty cycle. So the conclusion holds in this case. If Y is not empty, let U be a relatively compact open set in Z meeting each irreducible component of Y . We have to show, by definition of the topology of $\mathcal{C}_n^f(Z)$, that for $\nu \gg 1$ each irreducible component of X_ν meets U . If it is not the case, passing to a subsequence, we may assume that for each ν there exists an irreducible component Γ_ν of X_ν disjoint from U . As, passing again to a subsequence, we may assume that the sequence (Γ_ν) converges in $\mathcal{C}_n^{loc}(Z)$ to a cycle Γ , we shall have $|\Gamma| \subset |Y|$ and $|\Gamma| \cap U = \emptyset$. To conclude, it is enough to show that Γ is not the empty cycle, as any irreducible component of Γ is also an irreducible component of Y and then meets U by hypothesis. As each Γ_ν is not empty, it has to meet $K = \bar{\Omega}$. This implies that Γ also meets K and so is not empty. This contradicts our assumption. \square

End of the proof of Proposition 4.3. We have proved that j , and then also each $j_\alpha : \mathcal{C}_n^f(Z_\alpha) \rightarrow \mathcal{C}_n^{loc}(Z_\alpha)$ for $\alpha \in]0, 1[$, is a holomorphic homeomorphism. To conclude the proof we have to show the continuity of res_α^{-1} , and this reduces to prove that if the sequence (X_ν) of $\mathcal{C}_n^{loc}(Z)$ is such that the sequence $(X_\nu \cap Z_\alpha)$ converges in $\mathcal{C}_n^{loc}(Z_\alpha)$, then it converges in $\mathcal{C}_n^{loc}(Z)$. Let $Y_\alpha \in \mathcal{C}_n^{loc}(Z_\alpha)$ be the limit of this sequence in $\mathcal{C}_n^{loc}(Z_\alpha)$ and let $Y \in \mathcal{C}_n^{loc}(Z)$ be the cycle extending it. Let A be the set of $\beta \in]0, \alpha]$ such that the sequence $(X_\nu \cap Z_\beta)$ converges in $\mathcal{C}_n^{loc}(Z_\beta)$ to $Y \cap Z_\beta$. Then α is in A so A is not empty. Put $\gamma := \inf A$. Theorem 3.15 implies that $\gamma = 0$ and we obtain also the convergence in any $\mathcal{C}_n^{loc}(Z_\beta)$, for any $\beta > 0$; this gives the convergence in $\mathcal{C}_n^{loc}(Z)$, as, by definition, a n -scale on Z is also a n -scale on Z_β for $\beta > 0$ small enough. \square

4.C. An analytic extension criterion.

The aim of this paragraph is to prove the following analytic extension result.

Theorem 4.5. *Let Z be a complex space and n an integer. Consider a f -continuous¹¹ family $(X_s)_{s \in S}$ of n -cycles of finite type in Z parametrized by a reduced complex space S . Fix a point s_0 in S and assume that the open set Z' in Z meets all irreducible components of X_{s_0} and such that the family of cycles $(X_s \cap Z')_{s \in S}$ is analytic at s_0 . Then there exists an open neighbourhood S_0 of s_0 in S such that the family $(X_s)_{s \in S_0}$ is f -analytic.*

The hypotheses translated in terms of classifying maps means that we have a continuous map $\varphi : S \rightarrow \mathcal{C}_n^f(Z)$ such that the composed map $r \circ \varphi$ is holomorphic at s_0 , where $r : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z')$ is the restriction map.

Then the theorem says that there exists an open neighbourhood S_0 of s_0 in S such that the map φ is holomorphic on S_0 . Note that, as r is holomorphic¹², the hypothesis that φ is holomorphic at s_0 is a necessary condition.

This result is not true in general if we take for S a non smooth Banach analytic set which is not of finite dimension (locally). The reader may find a counter-exemple with an isolated singularity in [BM, Chapter V].

The key point for the proof of the previous theorem is the following analytic extension result.

¹¹ \uparrow This means that we have a continuous family of finite type n -cycles such that its graph is quasi-proper over S . This is equivalent to the continuity of the classifying map $\varphi : S \rightarrow \mathcal{C}_n^f(Z)$ of this family.

¹² \uparrow in the sense that for any holomorphic map $\psi : T \rightarrow \mathcal{C}_n^f(Z)$ of a reduced complex space T the composed map $r \circ \psi$ is holomorphic.

Proposition 4.6. *Let S a reduced complex space and let $\emptyset \neq U_1 \subset U_2$ be two polydiscs in \mathbb{C}^n . Let $f : S \times U_2 \rightarrow \mathbb{C}$ a continuous function, holomorphic on $\{s\} \times U_2$ for each $s \in S$. Assume moreover that the restriction of f to $S \times U_1$ is holomorphic. Then f is holomorphic on $S \times U_2$.*

Proof of Proposition 4.6. First consider the case S smooth. As the question is local on S it is enough to consider the case where S is an open set in \mathbb{C}^m . Fix an open relatively compact polydisc P in S . The function f defines a map $F : U_2 \rightarrow \mathcal{C}^0(\bar{P}, \mathbb{C})$, where we write down $\mathcal{C}^0(\bar{P}, \mathbb{C})$ the Banach space of continuous functions on \bar{P} , via the formula $F(t)[s] = f(s, t)$ for $t \in U_2$ and $s \in \bar{P}$. First we shall show that the map F is holomorphic.

Let $U \subset\subset U_2$ be a polydisc. For any fix $s \in S$ we have

$$\frac{\partial f}{\partial t_i}(s, t) = \frac{1}{(2i\pi)^n} \cdot \int_{\partial \partial U} f(s, \tau) \cdot \frac{d\tau_1 \wedge \cdots \wedge d\tau_n}{(\tau_1 - t_1) \cdots (\tau_i - t_i)^2 \cdots (\tau_n - t_n)} \quad \forall t \in U \quad \forall i \in [1, n].$$

where $t := (t_1, \dots, t_n)$ are coordinates on \mathbb{C}^n . This Cauchy formula shows that F is \mathbb{C} -differentiable and its differential at the point $t \in U$ is given by

$$h \mapsto \sum_{i=1}^n F_i(t) \cdot h_i, \quad h \in \mathbb{C}^n,$$

where F_i is associated to the function

$$(s, t) \mapsto \frac{\partial f}{\partial t_i}(s, t) \quad i \in [1, n]$$

which is holomorphic for each fixed $s \in S$ thanks to the Cauchy formula above.

Let $H(\bar{P}, \mathbb{C})$ the closed subspace of $\mathcal{C}^0(\bar{P}, \mathbb{C})$ of functions which are holomorphic on P . Our hypothesis implies that the restriction of F to the non empty open set U_1 takes its values in this subspace. Let us show that this is also true on U_2 . Assume that there exists $t_0 \in U_2$ such that $F(t_0)$ is not in $H(\bar{P}, \mathbb{C})$. Thanks to the *Hahn-Banach theorem* we can find a continuous linear form λ on $\mathcal{C}^0(\bar{P}, \mathbb{C})$, vanishing on $H(\bar{P}, \mathbb{C})$, and such that $\lambda(F(t_0)) \neq 0$. But the function $t \mapsto \lambda(F(t))$ is holomorphic on U_2 and vanishes on U_1 ; this contradicts $\lambda(F(t_0)) \neq 0$. So F takes values in $H(\bar{P}, \mathbb{C})$ and f is holomorphic on $S \times U_2$ when S is a complex manifold.

The case where S is a weakly normal complex space follows immediately.

When S is a general reduced complex space, the function f is meromorphic and continuous on $S \times U_2$ and holomorphic on $S \times U_1$. So the closed analytic subset $Y \subset S \times U_2$ along which f may not be holomorphic has no interior point in each $\{s\} \times U_2$. The analytic extension criterion of [BM14, Chapter IV, Criterion 3.1.7] allows to conclude. \square

Proof of Theorem 4.5. Let $|G| \subset S \times Z$ be the graph of the f -continuous family $(X_s)_{s \in S}$ and let A be the set of points in $(\sigma, \zeta) \in |G|$ admitting an open neighbourhood $S_\sigma \times Z_\zeta$ in $S \times Z$ such that the family of cycles $(X_s \cap Z_\zeta)_{s \in S_\sigma}$ is analytic. Remark that, thanks to our hypothesis, the open set A in $|G|$ meets every irreducible component of $\{s_0\} \times |X_{s_0}|$.

Assume to begin that there exists a smooth point z_0 of $|X_{s_0}|$ in the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$. Choose a n -scale $E := (U, B, j)$ on Z which is adapted to X_{s_0} and satisfying :

$$\begin{aligned} \deg_E(|X_{s_0}|) &= 1, & j_*(X_{s_0}) &= k \cdot (U \times \{0\}) \\ z_0 &\in j^{-1}(U \times B), & j(z_0) &:= (t_0, 0). \end{aligned}$$

It is clear that such a n -scale exists as z_0 is a smooth point in $|X_{s_0}|$. Let S_1 be a sufficiently small open neighbourhood of s_0 in S and let $f : S_1 \times U \rightarrow \text{Sym}^k(B)$ be the (continuous) classifying map for the family $(X_s)_{s \in S_1}$ in the scale E . As $j^{-1}(U \times \{0\})$ meets A , there exists a non empty polydisc $U_2 \subset U$ such that $U_2 \times \{0\}$ is contained in A . Then we may apply Proposition 4.6 to each scalar

component of f in order to obtain that f is holomorphic on $S_1 \times U$. Moreover, as the same argument applies to any linear projection of $U \times B$ to U near enough the vertical one; this implies that f is an isotropic map, up to shrinking slightly U . This contradicts the fact that the point (s_0, z_0) is in the boundary of the open set $A \cap (\{s_0\} \times |X_{s_0}|)$ of $|X_{s_0}|$.

If the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$ is contained in the singular set of $|X_{s_0}|$, we may apply the analytic extension criterion of [BM14, Chapter IV, Criterion 3.1.7], and we obtain directly that A contains $|X_{s_0}|$. So in any case the family of cycles $(X_s)_{s \in S}$ is analytic at s_0 . As the graph $|G|$ is, by assumption, quasi-proper on S , it is enough to use the next proposition (which is proved in [Bar15, Proposition 2.2.3]) to conclude.

Proposition 4.7. *Let Z and S be reduced complex spaces and let $(X_s)_{s \in S}$ be a f -continuous family of n -cycles in Z . Assume that this family is analytic in $s_0 \in S$. Then there exists an open neighbourhood S' of s_0 in S such that the family $(X_s)_{s \in S'}$ is a f -analytic family of n -cycles in Z .*

4.D. Proof of Theorem 1.2 and its corollary

We shall begin by a lemma which will give the case where the n -cycle X_0 is compact.

Lemma 4.8. *Let Z be a strongly $(n - 2)$ -concave reduced complex space. Then $\mathcal{C}_n(Z)$ is open in $\mathcal{C}_n^f(Z)$.*

Proof. Let X_0 be a compact cycle in Z . There exists $\alpha \in]0, 1[$ such that X_0 is contained in $Z_\alpha = \{x \in Z / \varphi(x) > \alpha\}$. So X_0 does not meet the compact set $\varphi^{-1}(\{\alpha\})$. This is an open condition in $\mathcal{C}_n^f(Z)$. And as any irreducible n -dimensional analytic subset in Z has to meet K , if it does not meet $\varphi^{-1}(\{\alpha\})$ it is contained in Z_α (by connectedness). Then any $X \in \mathcal{C}_n^f(Z)$ which is near enough X_0 is contained in Z_α so is compact. \square

Note that under the hypothesis of the previous lemma, $\mathcal{C}_n(Z)$ is not closed in $\mathcal{C}_n^f(Z)$ in general, as one can see taking $Z := \mathbb{P}_N \setminus \{0\}$ and considering the set of hyperplanes in \mathbb{P}_N .

As we already know from [Bar75] that $\mathcal{C}_n(Z)$ is a reduced complex space, Theorem 1.2 and its corollary are proved near a compact n -cycle in Z .

The case where X_0 is not compact. Of course we are in the case where Z is not compact. Fix X_0 a non compact n -cycle in $\mathcal{C}_n^f(Z)$ and choose an $\alpha \in]0, 1[$ which is not a critical value of φ and of the restriction of φ to $|X_0|$; this is possible thanks to Sard's Lemma 3.1 and the fact that $\varphi(|X_0|)$ contains $]0, 1[$.

Consider now the reduced complex space Ξ constructed in Theorem 3.15. It is an open neighbourhood of $X_0 \cap \Delta''$ in $\mathcal{C}_n^f(\Delta'')$ and we may assume that $\Delta'' := Z_\beta$ for some $\beta < \alpha$ very near α . Due to Proposition 4.3 it is homeomorphic to an open neighbourhood \mathcal{V} of X_0 in $\mathcal{C}_n^f(Z)$. So the restriction map

$$res_0 : \mathcal{V} \rightarrow \Xi$$

is holomorphic, bijective and is a homeomorphism. Now the continuity of res_0^{-1} and the finiteness of Ξ allow to apply Theorem 4.5 because we already know that the tautological family of cycles parametrized by Ξ is f -analytic on the open set Δ'' which is an open set which meets any irreducible component of each cycle in this family (because $K \subset Z_\beta$). Then res_0^{-1} is holomorphic and so res_0 is an isomorphism of Banach analytic sets. \square

4.E. A compactness criterion for the connected components of the reduced complex space $\mathcal{C}_n^f(Z)$ when Z is strongly $(n-2)$ -concave

For a reduced complex space Z which is compact, the compactness of the connected components of $\mathcal{C}_n(Z)$ is a consequence of the existence of a \mathcal{C}^1 $2n$ -differential form on Z which is d -closed and such that its (n, n) part is positive definite in the Lelong sense. Indeed, this gives that the volume (computed with this (n, n) part) of the n -cycles is constant on connected components. The result follows then from Bishop's theorem [Bis64] (see [BM14, Chapter IV] for details).

In the case of a non compact strongly $(n-2)$ -concave reduced complex space Z we have the following analogous result :

Proposition 4.9. *Let Z be a reduced complex space which is strongly $(n-2)$ -concave. Assume that there exists on Z a \mathcal{C}^1 $2n$ -differential form ω which is d -closed with compact support and such that its (n, n) part is positive definite in the Lelong sense in a neighbourhood of $K := \varphi^{-1}([1, 2])$, and everywhere non negative in the Lelong sense. Then the connected components of $\mathcal{C}_n^f(Z)$ are compact.*

Proof. For $\alpha < 1$ near enough to 1 and for any continuous hermitian metric h on Z there exists a constant C such that the following inequality holds:

$$\text{vol}_h(X \cap Z_\alpha) \leq C \cdot \int_X \omega \quad \text{for any cycle } X \in \mathcal{C}_n^f(Z).$$

As the function $X \mapsto \int_X \omega$ is locally constant on $\mathcal{C}_n^f(Z)$ because $d\omega = 0$ (the direct image of ω as a current is d -closed, so locally constant at smooth points of $\mathcal{C}_n^f(Z)$, and this current is a continuous function on $\mathcal{C}_n^f(Z)$ thanks to Proposition IV 2.3.1 of *loc. cit.*), we have a uniform bound for the volume of $X \cap Z_\alpha$ for X in a given connected component of $\mathcal{C}_n^f(Z)$. This implies that the closure of the image of this connected component in $\mathcal{C}_n^f(Z_\alpha)$ is compact, thanks to Bishop's theorem (see [BM14, Chapter IV, Theorem 2.7.20]). But the restriction map $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$ is a homeomorphism by Proposition 4.3, so the image of a connected component is closed and then compact. \square

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