



**HAL**  
open science

# Finiteness of the space of $n$ -cycles for a $n$ -concave complex space

Daniel Barlet

► **To cite this version:**

Daniel Barlet. Finiteness of the space of  $n$ -cycles for a  $n$ -concave complex space. 2015. hal-01184765v1

**HAL Id: hal-01184765**

**<https://hal.science/hal-01184765v1>**

Preprint submitted on 17 Aug 2015 (v1), last revised 11 Jul 2017 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Finiteness of the space of $n$ –cycles for a $n$ –concave complex space.

Daniel Barlet\*

14/08/15

*pour Yum-Tong Siu, avec mon amitié.*

**ABSTRACT.** We show that for  $n \geq 2$  the space of closed  $n$ –cycles in a strongly  $n$ –concave complex space has a natural structure of reduced complex space locally of finite dimension and represents the functor “analytic family of  $n$ –cycles” parametrized by banach analytic sets.

**AMS CLASSIFICATION 2010.** 32 G 13- 32 G 10- 32 F 10- 32 D 15

**KEY WORDS.** Closed  $n$ –cycles- Strongly  $n$ –convex space- Hartogs figure- f-analytic family of cycles.

---

\*Institut Elie Cartan, Algèbre et Géométrie,  
Université de Lorraine, CNRS UMR 7502 and Institut Universitaire de France.

# Contents

<b>1</b>	<b>Introduction.</b>	<b>2</b>
<b>2</b>	<b>Hartogs figures.</b>	<b>3</b>
2.1	Banachisation . . . . .	3
2.2	$n$ -Hartogs figure on a complex space. . . . .	4
2.3	$q$ -concave open sets. . . . .	11
<b>3</b>	<b>The extension and finiteness theorem.</b>	<b>14</b>
3.1	Some useful lemmas. . . . .	14
3.2	Ajusted scales. . . . .	16
3.3	Shrinkage. . . . .	21
3.4	Excellent family. . . . .	21
3.5	The extension and finiteness theorem. . . . .	23
<b>4</b>	<b>Finiteness of the space of <math>n</math>-cycles of a reduced strongly <math>n</math>-concave space (<math>n \geq 2</math>).</b>	<b>26</b>
4.1	The global extension theorem. . . . .	26
4.2	Some consequences. . . . .	27
4.3	An analytic extension criterion. . . . .	28
4.4	Proof of the theorem 1.0.2 and its corollary . . . . .	30
4.5	A compactness criterion for the connected components of the reduced complex space $\mathcal{C}_n^f(Z)$ . . . . .	32
<b>5</b>	<b>References.</b>	<b>33</b>

## 1 Introduction.

The aim of this article is to show that in a reduced strongly  $n$ -concave<sup>1</sup> complex space  $Z$  with  $n \geq 2$ , the space of closed  $n$ -cycles (not assumed compact) is in a natural way endowed with a structure of complex space locally of finite dimension. With its tautological family of  $n$ -cycles it represents the functor “analytic family of  $n$ -cycles in  $Z$ ” and also the functor “f-analytic family of  $n$ -cycles in  $Z$ ” introduced in [B.08] (see also [B.13] and [B.15-a]) parametrized by a banach analytic set.

This answers a question asked to me by Y-T. Siu forty years ago.

I was able to solve this question thanks to the notion of f-analytic family introduced in *loc. cit.* and using the space  $\mathcal{C}_n^f(Z)$  of finite type cycles with its natural topology. We obtain the following results.

**Theorem 1.0.1** *Let  $n \geq 2$  be an integer. let  $Z$  be a strongly  $n$ -concave<sup>2</sup> reduced complex space and let  $\varphi : Z \rightarrow ]0, 2]$  be its  $\mathcal{C}^2$  exhaustion function. So it is proper and*

---

<sup>1</sup>Our conventions will be precised below.

<sup>2</sup>See the section 2 for the definition.

strongly  $(n - 1)$ -convex outside the compact set  $K := \varphi^{-1}([1, 2])$ . For  $\alpha \in ]0, 1[$  and  $X_0$  a  $n$ -cycle in an open neighbourhood of the compact set  $\varphi^{-1}([\alpha, 2])$  there exists  $\beta \in ]0, \alpha[$  such that, if  $Z_\beta := \{z \in Z \mid \varphi(z) > \beta\}$ , the cycle  $X_0$  extends in an unique way to the open set  $Z_\beta$  and admits an open neighbourhood  $\mathcal{U}$  in the space  $\mathcal{C}_n^f(Z_\beta)$  such that the ringed space defined by  $\mathcal{U}$  and the sheaf of holomorphic functions on  $\mathcal{U}$  is a (reduced) complex space locally of finite dimension.

Recall that a holomorphic function  $h : \mathcal{U} \rightarrow \mathbb{C}$  on an open set in  $\mathcal{C}_n^f(Z_\beta)$  is a continuous function on  $\mathcal{U}$  such that for any holomorphic map  $f : S \rightarrow \mathcal{U}$  (corresponding to an  $f$ -analytic family of  $n$ -cycles in  $Z$ , see *loc. cit.*) of a banach analytic set  $S$  to  $\mathcal{U}$  the composed function  $h \circ f$  is holomorphic.

**Theorem 1.0.2** *Consider the same situation as in the previous theorem, and let now  $X_0 \in \mathcal{C}_n^f(Z)$  be a finite type  $n$ -cycle in  $Z$ . Then there exists  $\beta \in ]0, 1[$  and open neighbourhoods  $\mathcal{V}$  and  $\mathcal{U}$  respectively of  $X_0$  dans  $\mathcal{C}_n^f(Z)$  and of  $X_0 \cap Z_\beta$  dans  $\mathcal{C}_n^f(Z_\beta)$  such that the restriction map*

$$r : \mathcal{V} \rightarrow \mathcal{U}$$

*is well defined and bi-holomorphic.*

The following corollary is of course the main point.

**Corollary 1.0.3** *Consider the same situation as in the previous theorems. Then the ringed space given by the sheaf of holomorphic functions on  $\mathcal{C}_n^f(Z)$  is a reduced complex space locally of finite dimension. Moreover, endowed with its tautological family of  $n$ -cycles it represents the functor “ $f$ -analytic family of  $n$ -cycles in  $Z$ ” .*

In fact, we show that under the same hypothesis, the obvious map  $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$  is an isomorphism of ringed space, so that  $\mathcal{C}_n^f(Z)$  represents also the functor “analytic family of  $n$ -cycles in  $Z$ ” .

## 2 Hartogs figures.

### 2.1 Banachisation

For the analytic extension via Hartogs figures, the use of the Banach spaces  $H(\bar{U}, \mathbb{C})$  of continuous functions, holomorphic inside on a compact polydisc  $\bar{U}$  is not well adapted. We shall use the Banach space  $\mathcal{B}(U, \mathbb{C})$  of bounded holomorphic functions on  $U$  with the “sup” norm on  $U$ . Of course,  $H(\bar{U}, \mathbb{C})$  is a closed banach subspace in  $\mathcal{B}(U, \mathbb{C})$ .

**Proposition 2.1.1** *Let  $U$  be a relatively compact polydisc in  $\mathbb{C}^n$  and let  $S$  be a banach analytic set. Let  $F : S \rightarrow \mathcal{B}(U, \mathbb{C})$  be a holomorphic map. Then the corresponding function  $f : S \times U \rightarrow \mathbb{C}$  defined by  $f(s, t) := F(s)[t]$  for  $(s, t)$  in  $S \times U$  is holomorphic (and locally on  $S$  uniformly bounded on  $U$ ).*

*Conversely, if we have a holomorphic function  $f : S \times U \rightarrow \mathbb{C}$  and an open polydisc*

$U'' \subset\subset U$ , the the associated map  $F : S \rightarrow \mathcal{B}(U'', \mathbb{C})$ , defined for  $(s, t)$  in  $S \times U''$  by  $F(s)[t] := f(s, t)$ , is holomorphic.

PROOF. The evaluation function  $ev : \mathcal{B}(U, \mathbb{C}) \times U \rightarrow \mathbb{C}$  is holomorphic as one may easily see by differentiation. Then the function  $f$  associated to  $F$  is the composition of the holomorphic maps  $F \times id_U$  and  $ev$ . So it is holomorphic.

The converse is consequence of the linear isometric inclusion of  $H(\bar{U}'', \mathbb{C})$  in  $\mathcal{B}(U'', \mathbb{C})$  as  $F'' : S \rightarrow H(\bar{U}'', \mathbb{C})$  is holomorphic as soon as  $f$  is holomorphic (see [B.75] or [B-M 2]).  $\blacksquare$

## 2.2 $n$ -Hartogs figure on a complex space.

For  $\alpha \in (\mathbb{R}_+^*)^2$  let  $M(\alpha)$  be the open set in  $\mathbb{C}^2$  defined by

$$\begin{aligned} M(\alpha) &:= M^P(\alpha) \cup M^C(\alpha) \quad \text{with} \\ M^P(\alpha) &:= \{|t_1 - \alpha_1/2| < \alpha_1/4, \quad |t_2| < \alpha_2\} \\ M^C(\alpha) &:= \{|t_1| < \alpha_1, \quad \alpha_2/2 < |t_2| < \alpha_2\} \end{aligned}$$

and let also  $\mathcal{M}(\alpha) := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i, i = 1, 2\}$ .  
for  $\varepsilon > 0$  small enough. We define

$$\begin{aligned} M(\alpha)^\varepsilon &:= M^P(\alpha)^\varepsilon \cup M^C(\alpha)^\varepsilon \quad \text{with} \\ M^P(\alpha)^\varepsilon &:= \{|t_1 - \alpha_1/2| < \alpha_1/4 - \varepsilon/4, \quad |t_2| < \alpha_2 - \varepsilon\} \\ M^C(\alpha)^\varepsilon &:= \{|t_1| < \alpha_1 - \varepsilon, \quad \alpha_2/2 + \varepsilon/2 < |t_2| < \alpha_2 - \varepsilon\} \end{aligned}$$

and also  $\mathcal{M}(\alpha)^\varepsilon := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i - \varepsilon, i = 1, 2\}$ .

For a polydisc of radius  $R$  in  $\mathbb{C}^m$  we shall denote by  $P^\varepsilon$  the polydisc with same center and radius  $R - \varepsilon$  for  $0 < \varepsilon < R$ .

**Definition 2.2.1** Let  $n \geq 2$  and  $p \geq 1$  two integers and let  $\Delta \subset\subset \Delta'$  be two open sets in a reduced complex space  $Z$ . We shall call  $\mathcal{H} := (\mathcal{M}, M, B, j)$  a  $n$ -Hartogs figure in  $Z$  relative to the boundary of  $\Delta$ , the following data

- an embedding  $j$  of an open set  $Z'$  in  $\Delta'$  into an open set of  $\mathbb{C}^{n+p}$ ,
- open sets  $\mathcal{M} \subset\subset \Delta'$  and  $M \subset\subset \Delta$  relatively compact in  $Z'$ ,
- a relatively compact polydisc  $B$  in  $\mathbb{C}^p$

such that there exists  $\alpha \in (\mathbb{R}_+^*)^2$  and a relatively compact polydisc  $V$  in  $\mathbb{C}^{n-2}$  with the following conditions:

- i) The map  $j$  induces a closed embedding of  $\mathcal{M}$  in  $\mathcal{M}(\alpha) \times V \times B$ .

- ii) The map  $j$  induces a closed embedding of  $M$  in  $M(\alpha) \times V \times B$ .
- iii) We have  $j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \subset \Delta$ .

**Definition 2.2.2** Let  $n \geq 2$  and  $p \geq 1$  two integers and let  $\Delta \subset\subset \Delta'$  be two open sets in a reduced complex space  $Z$ . Let  $\mathcal{H} = (\mathcal{M}, M, B, j)$  be a  $n$ -Hartogs figure in  $Z$  relative to the boundary of  $\Delta$  and let  $X_0$  be a  $n$ -cycle in  $\Delta$ . We shall say that  $\mathcal{H}$  is adapted to  $X_0$  when the following condition is satisfied :

$$j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \cap |X_0| = \emptyset. \quad (@)$$

REMARKS.

1. Let  $\mathcal{H}$  be a  $n$ -Hartogs figure in  $Z$  relative to the boundary of  $\Delta$ . If the open set  $\Delta_1 \subset\subset \Delta'$  has a boundary  $\partial\Delta_1$  near enough to  $\partial\Delta$ , then  $\mathcal{H}$  is again relative to the boundary of  $\Delta_1$ . For instance, if  $\Delta := \{\varphi > 0\}$  where  $\varphi$  is a continuous proper function on  $Z$ , we may choose  $\Delta_1 := \{\varphi > \varepsilon\}$  for  $\varepsilon > 0$  small enough.
2. Note that the  $n$ -scale  $E_{\mathcal{H}} := (M(\alpha) \times V, B, j)$  on  $\Delta$  associated to  $\mathcal{H}$  is adapted to  $X_0$  as soon as the  $n$ -Hartogs figure  $\mathcal{H}$  is adapted to  $X_0$ .
3. If  $\tilde{X}_0$  is a  $n$ -cycle in  $\Delta'$  such that its restriction to  $\Delta$  is equal to  $X_0$ , the  $n$ -scale  $E_{\tilde{\mathcal{H}}} := (\mathcal{M}(\alpha) \times V, B, j)$  on  $\Delta'$  is adapted to  $\tilde{X}_0$  if and only if the  $n$ -Hartogs figure  $\mathcal{H}$  is adapted to  $X_0$ .

Note that the  $n$ -scale  $E_{\tilde{\mathcal{H}}}$  is not a  $n$ -scale on  $\Delta$  although the subset  $\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B$  is contained in  $\Delta$ .

**Definition 2.2.3** In the situation above we define, for  $\varepsilon > 0$  small enough, the  $n$ -Hartogs figure  $\mathcal{H}^\varepsilon$  on  $\Delta$  as follows :

$$\mathcal{H}^\varepsilon := (\mathcal{M}^\varepsilon, M^\varepsilon, B, j)$$

where we use the notations

$$\mathcal{M}^\varepsilon := j^{-1}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon \times B) \quad \text{and also} \quad M^\varepsilon := j^{-1}(M(\alpha)^\varepsilon \times V^\varepsilon \times B).$$

It is obvious to see that when  $\mathcal{H}$  is relative to the boundary of  $\Delta$ , then  $\mathcal{H}^\varepsilon$  is again relative to the boundary of  $\Delta$  for all  $\varepsilon > 0$  small enough.

Moreover, if  $\mathcal{H}$  is adapted to the  $n$ -cycle  $X_0$  of  $\Delta$ , so it is for  $\mathcal{H}^\varepsilon$  for all  $\varepsilon > 0$  small enough.

**Lemma 2.2.4** Let  $V$  be a relatively compact open polydisc in  $\mathbb{C}^q$ . The restriction map

$$\text{res} : \mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha) \times V, \mathbb{C})$$

is a linear isometry of Banach spaces.

Note that the restriction map induces also an isometry

$$\text{res} : \mathcal{B}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C})$$

for all  $\varepsilon > 0$  small enough.

PROOF. Let  $f(v, t_1, t_2) := \sum_{m \in \mathbb{Z}} a_m(v, t_1) \cdot t_2^m$  the Laurent expansion of the holomorphic function  $f : M^C(\alpha) \times V \rightarrow \mathbb{C}$ . The holomorphic functions  $a_m, m \in \mathbb{Z}$  on the product of  $V$  by the disc  $\{|t_1| < \alpha_1\}$  are given by the formula

$$a_m(v, t_1) := \frac{1}{2i\pi} \cdot \int_{|z|=r} f(v, t_1, z) \cdot \frac{dz}{z^{m+1}} \quad \text{with } r \in ]\alpha_2/2, \alpha_2[.$$

As the holomorphy of  $f$  on  $M^P(\alpha) \times V$  implies that  $a_m \equiv 0$  for each negative  $m$  on the open set  $\{|t_1 - \alpha_1/2| < \alpha_1/4\} \times V$ , we conclude that the functions  $a_m$  are identically zero for  $m < 0$  and so  $f$  is holomorphic on  $\mathcal{M}(\alpha) \times V$ . This shows that the restriction map  $\text{res}$  is bijective (and also it is linear continuous) between the two Frechet spaces  $\mathcal{O}(\mathcal{M}(\alpha) \times V)$  and  $\mathcal{O}(M(\alpha) \times V)$ ; so it is an isomorphism of Frechet spaces.

Let us show that if  $f$  is in  $\mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C})$ , then  $\text{res}(f)$ , which belongs to the space  $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$ , has the same norm “sup”. For this purpose fix  $\varepsilon > 0$  small enough. As  $\mathcal{M}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$  is a compact polydisc in  $\mathcal{M}(\alpha) \times V$ , the maximum of  $f$  on this compact is obtain at some point  $z$  in the distinguish boundary of it. But as  $z$  is also in the boundary of  $M^C(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ , the desired equality follows.

Conversely, if  $g$  is in  $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$ , its analytic extension  $f$  to  $\mathcal{M}(\alpha) \times V$  will be bounded on the boundary of  $\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon$  by the sup of  $g$  on  $M^C(\alpha)^\varepsilon \times V^\varepsilon$ . So we obtain the equality of the “sup” norms for  $g$  and  $f$  respectively on  $M(\alpha) \times V$  and  $\mathcal{M}(\alpha) \times V$ . ■

THE BANACH ANALYTIC SET  $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ . Recall that if  $p \geq 1$  and  $k \geq 1$  are integers, there exists a closed embedding ( in fact given by a polynomial map) of  $\text{Sym}^k(\mathbb{C}^p) := (\mathbb{C}^p)^k / \mathfrak{S}_k$  into the vector space  $E(k) := \bigoplus_{h=1}^k S^h(\mathbb{C}^p)$  given by the elementary tensorial symmetric functions<sup>3</sup>. If  $U$  is an open relatively compact polydisc in  $\mathbb{C}^n$ , the subset  $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$  is closed and banach analytic in the Banach space  $\mathcal{B}(U, E(k))$ . Indeed, if  $Q : E(k) \rightarrow \mathbb{C}^N$  is a polynomial map such that  $Q^{-1}(0) = \text{Sym}^k(\mathbb{C}^p)$ , then the holomorphic map

$$\mathcal{Q} : \mathcal{B}(U, E(k)) \rightarrow \mathcal{B}(U, \mathbb{C}^N) \quad \text{defined by } f \mapsto Q \circ f$$

satisfies  $\mathcal{Q}^{-1}(0) = \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ .

Nevertheless, be aware that for an open set  $\Omega$  in  $E(k)$  the subset  $\mathcal{B}(U, \Omega)$  is generally not open in  $\mathcal{B}(U, E(k))$  and so, for an open polydisc  $B \subset \subset \mathbb{C}^p$ , the subset

---

<sup>3</sup>see for instance [B-M 1] chapter I §4.

$\mathcal{B}(U, \text{Sym}^k(B))$  is not open in  $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$  in general.

The obvious map  $H(\bar{U}, E(k)) \rightarrow \mathcal{B}(U, E(k))$  is a closed (linear) isometry and induces a holomorphic

$$i_U : H(\bar{U}, \text{Sym}^k(\mathbb{C}^p)) \hookrightarrow \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$$

and for all  $\varepsilon > 0$  the restriction

$$r : \mathcal{B}(U, E(k)) \rightarrow H(\bar{U}^\varepsilon, E(k))$$

is a (linear and continuous) compact map which induces a holomorphic map

$$\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p)) \rightarrow H(\bar{U}^\varepsilon, \text{Sym}^k(\mathbb{C}^p)).$$

NOTATIONS. Let  $k \in \mathbb{N}$  and let  $U' \subset\subset U \subset\subset \mathbb{C}^n$  and  $B \subset\subset \mathbb{C}^p$  be polydiscs. We shall note  $\Sigma_{U,U'}(k)$  the banach analytic set classifying the couples of an element in  $H(\bar{U}, \text{Sym}^k(B))$  with its isotropy data on  $\bar{U}'$ . Recall that the natural projection  $\Sigma_{U,U'}(k) \rightarrow H(\bar{U}, \text{Sym}^k(B))$  is a holomorphic homeomorphism (see [B.75] ch. III prop. 2 page 81 or [B-M 2] ch.V).

Our next result is the main tool for the analytic extension of  $n$ -cycles near a  $n$ -concave boundary.

**Proposition 2.2.5** *Consider the open sets  $\mathcal{M}(\alpha) \times V$  and  $M(\alpha) \times V$  in  $\mathbb{C}^n$ . The inverse of the restriction map is a holomorphic isomorphism of analytic extension*

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)).$$

Composed with the restriction to the compact set  $\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$  it sends the subset  $H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$  in  $H(\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon, \text{Sym}^k(B))$  for  $\varepsilon > 0$  small enough.

Moreover, this holomorphic map induces a holomorphic map, again for  $\varepsilon > 0$  small enough

$$\Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \longrightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

which factorizes the restriction map

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

through the restriction

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k).$$

PROOF. The lemma 2.2.4 gives that the map

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, E(k)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, E(k))$$

is an isometry of Banach spaces. It is clear that its inverse sends the analytic subset  $\mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$  in  $\mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$  and that if the map

$f \in \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$  takes values in  $\text{Sym}^k(B)$ , the same is true for its restriction. This proves the first part of the proposition.

In order to prove the second part, it is enough to show that a holomorphic map of a banach analytic set  $S$  with values in the subset

$$H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$$

which is isotropic on the product of  $S$  with any relatively compact subset in the open set  $M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$  will have an analytic extension which will be isotropic on any relatively compact open set in  $\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$ . So it will be isotropic on the closure of the open set  $\mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}$ .  $\blacksquare$

**Proposition 2.2.6** *Let  $n \geq 2$  and  $p \geq 1$  be integers and let  $U_1 \times B_1$  the product of two polydiscs with centers 0 respectively in  $\mathbb{C}^n$  and  $\mathbb{C}^p$ . Note  $(t_1, \dots, t_n, x_1, \dots, x_p)$  coordinates on  $U_1 \times B_1$ . Let  $\varphi$  be a real valued function of class  $\mathcal{C}^2$  on  $U_1 \times B_1$ , such that*

$$\varphi(t, x) = \text{Re}(t_1) + \sum_{i=1}^n \rho_i \cdot |t_i|^2 + \sum_{j=1}^p \sigma_j \cdot |x_j|^2 + o(\|(t, x)\|^2) \quad (\text{@@})$$

where the real numbers  $\rho_2, \sigma_1, \dots, \sigma_p$  are positive.

Let  $\Delta$  be the open set  $\{\varphi > 0\}$  in  $U_1 \times B_1$  and let  $\Delta'$  be an open neighbourhood of the compact set  $\bar{\Delta}$  in  $\mathbb{C}^{n+p}$ . Let  $X_0$  be a closed analytic subset of pure dimension  $n$  in  $\Delta'$  such that each irreducible component of  $X_0$  meets  $\Delta$  and such that

$$|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}. \quad (*)$$

Then there exists  $\alpha \in (\mathbb{R}_+^*)^2$  and polydiscs  $V$  and  $B \subset\subset B_1$  with centers 0 respectively in  $\mathbb{C}^{n-2}$  and  $\mathbb{C}^p$  such that the following conditions are satisfied :

1.  $\mathcal{M}(\alpha) \times V \times B \subset\subset \Delta'$ ;
2.  $M(\alpha) \times V \times B \subset\subset \Delta$ ;
3.  $\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B \subset \Delta$ ;
4.  $|X_0| \cap (\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B) = \emptyset$  (this implies  $|X_0| \cap (\overline{M(\alpha)} \times \bar{V} \times \partial B) = \emptyset$ ).

PROOF. Choose the polydisc  $B \subset\subset B_1$  small enough in order that we have

$$X_0 \cap (\{0\} \times \bar{B}) \subset \{0\} \quad \text{and} \quad \{0\} \times \partial B \subset \Delta.$$

This is possible as we have  $|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}$  and as  $\varphi$  is positive on a small enough punctured neighbourhood of the origin in the  $p$ -plane  $\{t_1 = \dots = t_n = 0\} \times \mathbb{C}^p$ . So we shall have

$$|X_0| \cap (\bar{W} \times \partial B) = \emptyset \quad \text{and} \quad \bar{W} \times \partial B \subset \Delta$$

for any small enough open neighbourhood  $W$  of the origin in  $\mathbb{C}^n$ . A immediate consequence is that conditions 1, 3 and 4 will be satisfied as soon as  $\alpha$  and  $V$  are small enough.

In order to check the condition 2, let us remark first that, up to choose the real numbers  $\rho'_1 > |\rho_1|$ ,  $\rho'_2 \in ]0, \rho_2[$  and  $r > \sup_{i \geq 3} |\rho_i|$ , we obtain on  $W \times B$  choosen small enough

$$\varphi(t, x) \geq \operatorname{Re}(t_1) - \rho'_1 \cdot |t_1|^2 + \rho'_2 \cdot |t_2|^2 - r \cdot \left( \sum_3^n |t_i|^2 \right). \quad (**)$$

with strict inequality as soon as  $x \neq 0$ . Then for

$$V_\varepsilon = \{(t_3, \dots, t_n) / \sum_3^n |t_i|^2 < \varepsilon^2\}$$

the following inequalities hold

$$\varphi(t, x) \geq \frac{1}{4}\alpha_1 - \rho'_1 \cdot \alpha_1^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^P(\alpha) \times V_\varepsilon \times B} \quad (1)$$

$$\varphi(t, x) \geq -\alpha_1 - \rho'_1 \cdot \alpha_1^2 + \frac{1}{4}\rho'_2 \cdot \alpha_2^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^C(\alpha) \times V_\varepsilon \times B} \quad (2)$$

for  $\alpha$  and  $\varepsilon$  small enough in order that  $\mathcal{M}(\alpha) \times V_\varepsilon$  is contained in  $W$ .

This allows to fix  $\alpha_1, \alpha_2$  and  $\varepsilon$ .

Now we shall choose  $\alpha_1$  and  $\varepsilon$  smaller in order to satisfy the following conditions :

$$8\alpha_1 < \rho'_2 \cdot \alpha_2^2 \quad , \quad \alpha_1 < \frac{1}{8\rho'_1} \quad \text{and} \quad \varepsilon^2 < \frac{1}{8r}\alpha_1. \quad (3)$$

To obtain  $M^P(\alpha) \times V_\varepsilon \times B \subset \subset \Delta$  it is enough to show that on  $M^P(\alpha) \times V_\varepsilon \times B$  we have, if we let  $\alpha_1 = u \cdot \alpha_2$  and  $\varepsilon^2 = v \cdot \alpha_1 = uv \cdot \alpha_2$

$$\frac{1}{4} > \rho'_1 \cdot \alpha_1 + r \cdot v.$$

Indeed, as we assumed  $\rho'_1 \cdot \alpha_1 < \frac{1}{8}$  and  $r \cdot \varepsilon^2 < \frac{1}{8} \cdot \alpha_1$  (so  $r \cdot v < 1/8$ ) the first condition holds.

In order to satisfy  $M^C(\alpha) \times V_\varepsilon \times B \subset \subset \Delta$  it is enough to show that on  $M^C(\alpha) \times V_\varepsilon \times B$  we have

$$\frac{1}{4}\rho'_2 \cdot \alpha_2 > u + \rho'_1 u^2 \alpha_2 + r \cdot uv.$$

But our condition implies

$$\rho'_1 u^2 \alpha_2 < \frac{1}{8} \cdot u \quad r \cdot uv < \frac{1}{8} \cdot u$$

which gives  $u + \rho'_1 u^2 \cdot \alpha_2 + r \cdot uv < 2 \cdot u$ . The condition  $8\alpha_1 < \rho'_2 \cdot \alpha_2^2$  which implies  $2u < \frac{1}{4}\rho'_2 \cdot \alpha_2$ , allows to conclude. ■

## REMARKS.

1. We only used the condition (\*) for  $X_0$  and the inequality (\*\*) for  $\varphi$  in a neighbourhood of the origin in the proof above.
2. Sufficient conditions on  $\varphi \in \mathcal{C}^2$  to satisfy (@@) are:

- i) The origin is not a critical point of  $\varphi$ .
- ii) The Levi form of  $\varphi$  at 0 has, at most,  $(n-1)$  non positive eigenvalues in the complex tangent hyperplane to the real hypersurface  $\{\varphi(z) = 0\}$ ; the existence of  $\varphi \in \mathcal{C}^2$  such that  $\Delta = \{\varphi > 0\}$  and satisfying these two conditions is equivalent to the fact that the open set  $\Delta$  has a strongly  $n$ -concave (see the definition 2.3.3 given below) smooth boundary near the origin. Indeed, if  $\varphi$  is not critical at 0 and has a Levi form at 0 with, at most,  $(n-1)$  non positive eigenvalues values in the complex hyperplane tangent to the real hypersurface  $\{\varphi(z) = 0\}$ , its order 2 Taylor expansion at the origin is written, in suitable local holomorphic coordinates  $(\tau, x)$

$$\varphi(\tau, x) = \operatorname{Re}(\tau_1) + \operatorname{Re}(Q(\tau, x)) + \sum_{i=1}^n \rho_i \cdot |\tau_i|^2 + \sum_{j=1}^p \sigma_j \cdot |x_j|^2 + o(\|(\tau, x)\|^2)$$

where  $Q$  is a holomorphic homogeneous degree 2 polynomial and where the real numbers  $\rho_2$  and  $\sigma_j, j \in [1, p]$  are positive. Define new local holomorphic coordinates

$$t_1 := \tau_1 + Q(\tau, x), t_i := \tau_i \quad \text{for } i \in [2, n] \quad \text{and} \quad x_j := x_j \quad \text{for } j \in [1, p].$$

Then we obtain (@@).

3. The condition (\*) implies that  $X_0$  has no local irreducible component at 0 contained in the hyperplane  $\{t_1 = 0\}$ . In fact, as the coordinate  $t_1$  is chosen in order to suppress the real part of the holomorphic homogeneous degree 2 term in the order 2 Taylor expansion of  $\varphi$  at 0 (see the previous remark), we want that no local irreducible component of  $X_0$  at the origin is contained in the complex hypersurface  $\tau_1 + Q(\tau, x) = 0$  locally defined near 0 for  $\varphi$  given. Then, as soon as the restriction  $\varphi|_{X_0}$  has not 0 as a critical point, the condition (\*) could be realized when  $\varphi$  is strongly  $(n-1)$ -convex near 0.
4. One may easily see that under our hypothesis, the cycle  $X_0$  meets the open set  $\Delta$  when it contains 0. Indeed, the analytic subset

$$\{t_1 = t_3 = \cdots = t_n = 0\} \cap |X_0|$$

is non empty, has dimension at least 1 and meets the complement of  $\Delta$  only at the origin.

Of course, assuming that  $0 \in X_0$ , the proposition shows that, in fact,  $X_0$  contains a branched covering of degree  $k \geq 1$  of  $M^P(\alpha) \times V_\varepsilon$  inside

$$M^P(\alpha) \times V_\varepsilon \times B \subset \subset \Delta.$$

5. In the situation of the proposition 2.2.6, for any continuous family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $\Delta$  parametrized by a banach analytic set  $S$  such that  $X_{s_0} = X_0 \cap \Delta$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$ , such that for each  $s \in S'$  the condition 4 remains true after analytic extension of the cycles (see the proposition 2.2.5), because, thanks to the condition 3,  $\mathcal{M}(\alpha) \times \bar{V} \times \partial B$  is a compact subset in  $\Delta$ .  $\square$

### 2.3 $q$ -concave open sets.

**Definition 2.3.1** Let  $\varphi : U \rightarrow \mathbb{R}$  be real valued  $\mathcal{C}^2$  function on an open set  $U$  in  $\mathbb{C}^N$ . We shall say that  $\varphi$  is **strongly  $q$ -convex** when its Levi form at each point of  $U$  has at most  $q$  non positive eigenvalues.

So, with this definition a strongly 0-convex function is a strongly plurisubharmonic function.

**Definition 2.3.2** Let  $\varphi : Z \rightarrow \mathbb{R}$  a real valued  $\mathcal{C}^2$  function on a reduced complex space  $Z$ . We shall say that  $\varphi$  is **strongly  $q$ -convex** if locally near each point of  $Z$  it can be induced by a  $\mathcal{C}^2$  strongly  $q$ -convex function in a local embedding in an open set of an affine space.

Remark that a strongly  $q$ -convex function on an irreducible complex space of dimension at least equal to  $q + 1$  has no local maximum because there exists at any point a germ of curve on which the restriction of  $\varphi$  is strongly p.s.h.

**Definition 2.3.3** Let  $Z$  be a reduced complex space and let  $\Delta$  be a relatively compact open set in  $Z$ . We shall say that  $\Delta$  has a **smooth  $\mathcal{C}^2$  boundary** when for each point  $z$  in  $\partial\Delta$  there exists a local smooth embedding of an open neighbourhood  $W$  of  $z$  in an open set  $U$  of an affine space and an open set  $D$  with smooth  $\mathcal{C}^2$  boundary in  $U$  such that  $\Delta = W \cap D$ .

We shall say that the open set  $\Delta \subset Z$  with smooth  $\mathcal{C}^2$  boundary is **strongly  $q$ -concave in a point**  $z \in \partial\Delta$  if, in some local smooth embedding of  $Z$  around  $z$ , one can define  $\Delta$  in  $Z$  near  $z$  as the subset  $\{\varphi > 0\} \cap Z$  where  $\varphi$  is a real valued  $\mathcal{C}^2$  function such that

1.  $d\varphi_z \neq 0$  on the Zariski tangent space  $T_{Z,z}$  de  $Z$  en  $z$ .
2. The restriction of the Levi form at  $z$  of  $\varphi$  to the complex hyperplane tangent at  $z$  to the real hypersurface  $\{\varphi(x) = \varphi(z)\}$  has at most  $(q - 1)$  non positive eigenvalues.

We shall say that  $\Delta$  is **strongly  $q$ -concave** if  $\Delta$  is strongly  $q$ -concave near each point in  $\partial\Delta$ .

REMARK. Up to compose  $\varphi$  with a real strictly increasing (non critical) convex  $\mathcal{C}^2$  function (this does not change the level sets  $\{\varphi = \text{constant}\}$ ), it is possible to assume that  $c \circ \varphi$  is  $\mathcal{C}^2$  strongly  $(q - 1)$ -convex (and non critical) near  $z$ . Conversely, if  $\varphi$  is a real valued  $\mathcal{C}^2$  function which is strongly  $(q - 1)$ -convex and not critical near a point  $z \in Z$ , the open set  $\{\varphi(x) > \varphi(z)\}$  has a strongly  $q$ -concave boundary in a neighbourhood of  $z$ .

With this terminology, using the remarks above, we may give the following reformulation of the proposition 2.2.6:

**Corollary 2.3.4** *Let  $n \geq 2$  and  $p \geq 1$  be integers, let  $Z$  be a reduced complex space of pure dimension  $n + p$  and let  $\Delta := \{\varphi > 0\}$  be an open set with  $\mathcal{C}^2$  smooth boundary in  $Z$ . Let  $X_0$  be a  $n$ -cycle in an open neighbourhood of a point  $z \in \partial\Delta$  such that the function  $\varphi|_{X_0}$  is not critical at  $z$ .*

*Assume that  $\Delta$  is strongly  $n$ -concave near  $z$ ; then there exists a  $n$ -Hartogs figure  $\mathcal{H} := (\mathcal{M}, M, B, j)$  relative to the boundary of  $\Delta$ , adapted to  $X_0$ , and such that the point  $z$  lies in  $\mathcal{M}$ . ■*

PROOF. Using a local embedding of an open neighbourhood of  $z$  in an open set of the Zariski tangent space  $T_{Z,z}$ , it is enough to prove the corollary in the case where  $Z$  is an open set in  $\mathbb{C}^{n+p'}$ , with  $p' \geq p$  an integer. As we may choose the function  $\varphi$  strongly  $(n - 1)$ -convex such that  $d\varphi_z \neq 0$  thanks to the previous remarks, we can choose local coordinates near  $z$  in order to be in the situation of the proposition 2.2.6 in the case  $z \in X_0$ , as we assumed that  $\varphi|_{X_0}$  is not critical at  $z$ . In this case the proposition gives the result.

If  $z$  is not in  $X_0$ , the same construction in an open neighbourhood of  $z$  with no limit point in  $X_0$  allows to conclude, and in this case the degree of  $X_0$  in the (adapted) scale  $E_{\mathcal{H}}$  will be zero. ■

**Definition 2.3.5** *Let  $n \geq 2$  and  $p \geq 1$  be integers and let  $\Delta \subset\subset \Delta'$  be two open sets in a reduced complex space  $Z$ . Let  $\mathcal{H} = (\mathcal{M}, M, B, j)$  and  $\mathcal{H}' := (\mathcal{H}', M', B, j)$  be two  $n$ -Hartogs figures in  $Z$  relative to the boundary of  $\Delta$  given by the same (local) embedding  $j$  and having the same polydisc  $B \subset\subset \mathbb{C}^p$ . We shall say that these two  $n$ -Hartogs figures are **boxed** when we have  $\mathcal{M}'(\alpha') \subset\subset \mathcal{M}(\alpha)$ ,  $M'(\alpha') \subset\subset M(\alpha)$  and  $V' \subset\subset V$ .*

For instance, if  $\varepsilon > 0$  is small enough, the  $n$ -Hartogs figures  $(\mathcal{H}, \mathcal{H}^\varepsilon)$  are boxed (see the definition 2.2.3).

**Proposition 2.3.6** *Let  $n \geq 2$  and  $p \geq 1$  be integers, let  $Z$  be a reduced complex space of pure dimension  $n + p$  and let  $\Delta \subset\subset Z$  be an open set with smooth  $\mathcal{C}^2$  boundary in  $Z$  which is strongly  $n$ -concave. Assume that  $\Delta := \{\varphi > 0\}$  and let  $X_0$  be a  $n$ -cycle in an open neighbourhood  $\Delta'$  of the compact set  $\bar{\Delta}$ , such that any irreducible component of  $X_0$  meets  $\Delta$ . Then there exists a finite family of*

boxed  $n$ -Hartogs figures  $(\mathcal{H}'_a, \mathcal{H}_a)_{a \in A}$  relative to the boundary of  $\Delta$ , such that the following conditions hold:

1. The  $\mathcal{M}'_a$  for  $a \in A$  cover the boundary  $\partial\Delta$ .
2. For each  $a \in A$  the  $\mathcal{H}_a$  and  $\mathcal{H}'_a$  are adapted to  $X_0$ .
3. For each  $a \in A$  any irreducible component of  $X_0$  meeting  $\mathcal{M}_a$  meets the open set  $M'_a$ .
4. No **compact** irreducible component of  $X_0 \cap \Delta$  meets the union of the compact sets  $\bar{\mathcal{M}}_a, a \in A$ .

REMARK. Let  $X_0$  be any  $n$ -cycle in  $\Delta'$ . Up to shrink the open set  $\Delta'$  around the compact set  $\bar{\Delta}$ , we can assume that the cycle  $X_0$  has only finitely many irreducible components in  $\Delta'$  and that each of them is not compact and meets  $\bar{\Delta}$ . Note that, in this situation, any irreducible component of  $X_0$  meets  $\Delta$ , because the strong  $n$ -concavity of  $\partial\Delta$  (see the remark 3 following the proposition 2.2.6).

**Corollary 2.3.7** *In the situation of the proposition 2.3.6, if we assume that the open set  $\Delta'$  contains  $\bar{\Delta}$ , any irreducible component  $\Gamma$  of the cycle  $X_0$  in  $\Delta'$  satisfies for all  $a \in A$  and all  $\eta > 0$  small enough:*

$$\Gamma \cap (\mathcal{M}_a^\eta \times V_a^\eta \times B_a) = \text{prlgt}_a[\Gamma \cap (M_a \times V_a \times B_a)]$$

where  $\text{prlgt}_a : H(\bar{M}_a \times \bar{V}_a, \text{Sym}^k(B_a)) \rightarrow H(\bar{\mathcal{M}}_a^\eta \times \bar{V}_a^\eta, \text{Sym}^k(B_a))$  is the holomorphic map of analytic extension built in the proposition 2.2.5.

REMARK. In the situation of the previous corollary there exists, for each  $a \in A$ , a holomorphic extension map which lifts the map  $\text{prlgt}_a$  :

$$i\text{prlgt}_a : \Sigma_{M_a, IM_a}(k) \longrightarrow \Sigma_{\mathcal{M}_a^\eta, I\mathcal{M}_a^\eta}(k)$$

and it allows to extend in this setting an analytic family of branched coverings in  $M_a$  parametrized by a banach analytic set  $S$  and which is isotropic on  $S \times M'_a$  to an analytic family of branched coverings in  $\mathcal{M}_a^\eta$  which is isotropic on  $S \times \mathcal{M}'^\eta_a$ .

PROOF OF THE PROPOSITION 2.3.6. The corollary 2.3.4 and the remark following it implies the existence, for each  $z \in \partial\Delta$  of a  $n$ -Hartogs figure  $\mathcal{H}_z$  relative to the boundary of  $\Delta$ , contained in  $\Delta'$  and satisfying the following properties:

- i)  $z \in \mathcal{M}_z$ ;
- ii)  $\mathcal{H}_z$  is adapted to  $X_0$  ;
- iii) each irreducible component of  $X_0$  meeting  $\mathcal{M}_z$  meets the open set  $M_z$ ;

iv) No compact irreducible component of  $X_0 \cap \Delta$  meets  $\bar{\mathcal{M}}_z$ .

As the open sets  $\mathcal{M}_z$  cover the compact set  $\partial\Delta$  we can find a finite sub-cover. Then the properties 1, 2, 3 and 4 are consequences of i), ii), iii) and iv) by letting  $\mathcal{H}' := \mathcal{H}^\varepsilon$  and by choosing  $\varepsilon > 0$  small enough. ■

PROOF OF THE COROLLARY 2.3.7. Let  $\Gamma$  be an irreducible component of  $X_0$  meeting  $\mathcal{M}_a$  for some  $a \in A$ . Then  $\Gamma$  meets  $M_a$ . As  $\Gamma$  does not meet  $\bar{M}(\alpha)_a \times \bar{V}_a \times \partial B_a$  because  $\mathcal{H}_a$  is adapted to  $X_0$ , the intersection  $\Gamma \cap \bar{M}_a$  is the graph of an element  $\gamma \in H(\bar{M}(\alpha)_a \times \bar{V}_a, \text{Sym}^{k_a}(B_a))$  with  $k_a \in \mathbb{N}^*$ <sup>4</sup>.

The closed analytic subset  $Y$  of the open set  $\mathcal{M}_a^n$  defined by analytic extension,  $Y := \text{pr}lgt_a[\Gamma \cap M_a]$ , is not empty, of pure dimension  $n$  and is contained in  $\Gamma$ . So it is an union of irreducible components of  $\Gamma \cap \mathcal{M}_a^n$ . But it contains a non empty open set in each irreducible component of this branched covering. So these two analytic subsets coincide.

If an irreducible component of  $X_0$  does not meet any  $\bar{M}_a$  it has to be compact and contained in  $\Delta$ . In this case the desired equality is obvious. ■

### 3 The extension and finiteness theorem.

#### 3.1 Some useful lemmas.

The Sard's lemma below is more or less classical.

**Lemma 3.1.1** *Let  $Z$  be a reduced complex space and let  $\varphi : Z \rightarrow \mathbb{R}$  be a real valued  $\mathcal{C}^1$  function. Then the set of critical values of  $\varphi$  has Lebesgue measure 0.*

PROOF. First note that a point  $z \in Z$  is critical for  $\varphi$  if, by definition, the differential of  $\varphi$  vanishes on  $T_{Z,z}$ , the Zariski tangent space of  $Z$  at  $z$ . Remember also that a complex space is, by definition, countable at infinity ; so  $Z$  and its singular locus have only countably many irreducible components. As a countable union of sets of measure 0 is again of measure 0, it is enough to prove the lemma when  $Z$  is irreducible. We shall prove the lemma by induction on the integer  $\dim Z$ . The case  $\dim Z = 0$  is obvious. Assume the lemma true for  $\dim Z \leq n - 1$  for some integer  $n \geq 1$  and take an irreducible complex space  $Z$  of dimension  $n$ . The singular set  $S$  of  $Z$  has dimension at most  $(n - 1)$ , and for each irreducible component  $S_i$  of  $S$  the image of the critical set of  $\varphi|_{S_i}$  has measure 0. So the critical set of  $\varphi|_S$  is again of measure 0. But a critical point of  $\varphi$  which belongs to  $S$  is a critical point of  $\varphi|_S$ . So it is enough to show that the set of critical values of  $\varphi$  restricted to the complex connected manifold  $Z \setminus S$  has measure 0. This is the classical Sard's lemma. ■

---

<sup>4</sup>as it exists some  $(m, v) \in \bar{M}(\alpha)_a \times \bar{V}_a$  such that  $\Gamma \cap (\{m, v\} \times B_a) \neq \emptyset$ .

**Lemma 3.1.2** *Let  $V$  be an open set and  $K$  be a compact set in  $\bar{U} \times \bar{B}$ . The subset  $\mathcal{V}$  in  $H(\bar{U}, \text{Sym}^k(B))$  consisting of the  $X$  such that any irreducible component meeting  $K$  meets  $V$  is an open set in  $H(\bar{U}, \text{Sym}^k(B))$ .*

PROOF. First let us clarify the meaning of an irreducible component of  $X$  in  $H(\bar{U}, \text{Sym}^k(B))$  : we call irreducible component of such a  $X$  the closure in  $\bar{U} \times B$  of an irreducible component of the branched covering of  $U$  defined by the projection of  $X \cap (U \times B)$  on  $U$ .

Let  $X_0$  be such that any irreducible component of  $X_0$  which meets  $K$  meets  $V$ , and assume that  $(X_\nu)_{\nu \geq 1}$  is a sequence converging to  $X_0$  such that for each  $\nu \geq 1$  there exists an irreducible component  $\Gamma_\nu$  of  $X_\nu$  meeting  $K$  but not  $V$ . Up to pass to a subsequence, we may assume that the sequence  $(\Gamma_\nu)_{\nu \geq 1}$  converges uniformly on any compact of  $U \times B$  to a non empty  $n$ -cycle  $\Gamma$  with closure contained in  $X_0$  and which is a branched covering of  $U$ . Then  $\bar{\Gamma}$  meets  $K$  and not  $V$ . Indeed, if  $(t_0, x_0)$  would be in  $\bar{\Gamma} \cap V$ , there exists open neighbourhoods  $U_1$  and  $B_1$  of  $t_0$  et  $x_0$  respectively in  $\bar{U}$  and  $\bar{B}$  such that  $U_1 \times B_1$  is contained in  $V$ . But then, as  $U_2 := U_1 \cap U$  and  $B_2 := B_1 \cap B$  are non empty open sets, and for  $t_2$  in  $U_2$  the fibers of the  $\Gamma_\nu$  at  $t_2$  for  $\nu$  big enough will meet  $\{t_2\} \times B_2$  and so  $V$ . As at least one irreducible component of  $\Gamma$  meets  $K$  without meeting  $V$  and as its closure is an irreducible component of  $X_0$ , this gives a contradiction. ■

Of course, in the case  $V = \emptyset$  we get back the fact that the subset of  $X \in H(\bar{U}, \text{Sym}^k(B))$  which do not meet  $K$  is open.

**Lemma 3.1.3** *Let  $Z$  be a complex space and let  $(\mathcal{U}_i)_{i \in I}$  be an open covering of  $Z$ . Assume that for each  $i \in I$  a closed  $n$ -cycle  $X_i$  is given in  $\mathcal{U}_i$ . Assume that the following patching condition holds :*

$$\forall (i, j) \in I^2 \quad X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$$

*as an equality of cycles in  $\mathcal{U}_i \cap \mathcal{U}_j$ .*

*Then there exists a unique closed  $n$ -cycle  $X$  in  $Z$  such that for each  $i \in I$  we have  $X \cap \mathcal{U}_i = X_i$ .*

For the easy proof see [B.75] or [B-M 2].

The following variant will be used.

**Lemma 3.1.4 (Variant.)** *In the situation of the previous lemma replace the patching condition by the following two conditions :*

1. *For each couple  $(i, j) \in I^2$  an open subset  $W_{i,j} \subset \subset \mathcal{U}_i \cap \mathcal{U}_j$  is given and we ask that  $X_i \cap W_{i,j} = X_j \cap W_{i,j}$ .*
2. *For each couple  $(i, j) \in I^2$  we ask that any irreducible component of the cycle  $X_i \cap \mathcal{U}_j$  meets the open set  $W_{i,j}$ .*

*Then the conclusion is the same.*

PROOF. Let  $\Gamma$  be an irreducible component with multiplicity  $\delta$  in the cycle  $X_i \cap \mathcal{U}_j$ . Let  $\Gamma'$  be the irreducible component of  $X_i$  which contains  $\Gamma$ , and put  $X_i = X'_i + \delta\Gamma'$ . Then  $\Gamma'$  meets  $W_{i,j}$  and there exists a closed analytic subset  $Y$  of pure dimension  $n$  in  $|X_j|$  such that its restriction to  $W_{i,j}$  is equal to  $\Gamma' \cap W_{i,j}$ : indeed,  $Y$  is the union of the irreducible components of  $X_j$  containing a non empty open set in  $\Gamma' \cap W_{i,j}$ . Note that each of these irreducible components of  $X_j$  has multiplicity  $\delta$  in the cycle  $X_j$ . Then put  $X_j = X'_j + \delta Y$ . We see that the cycles  $X'_i$  and  $X'_j$  respectively in  $\mathcal{U}_i$  and  $\mathcal{U}_j$  satisfy again the patching condition  $X'_i \cap W_{i,j} = X'_j \cap W_{i,j}$ .

This allows, for fixed  $(i, j)$ , to make a descending induction on the number (necessarily finite as  $W_{i,j}$  is relatively compact) of irreducible components of  $X_i \cap \mathcal{U}_j$ , to show that the condition  $X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$  holds. This reduces this lemma to the previous one.  $\blacksquare$

### 3.2 Adjusted scales.

**Definition 3.2.1** 1. Let  $Z$  be a complex space. We shall call **adjusted  $n$ -scale on  $Z$** , written down  $\mathbb{E} := (U, U', U'', B, B'', j)$ , the data of a  $n$ -scale on  $Z$ ,  $E := (U, B, j)$ , with additional polydiscs  $U'' \subset\subset U' \subset\subset U$  and  $B'' \subset\subset B$ . We call  $E$  the **underlying scale of the adjusted scale  $\mathbb{E}$** .

2. We shall say that **the adjusted scale  $\mathbb{E}$  is adapted to a  $n$ -cycle  $X$  in  $Z$**  when we have

$$j^{-1}(\bar{U} \times (\bar{B} \setminus B'')) \cap |X| = \emptyset.$$

Note that this implies that the underlying scale  $E$  is adapted to  $X$ , but this condition is more restrictive.

3. When the adjusted scale  $\mathbb{E}$  is adapted to the  $n$ -cycle  $X$ , we shall call **degree of  $X$  in  $\mathbb{E}$**  the degree of  $X$  in  $E$ .

4. We shall call **domain of the adjusted scale**, written down  $D(\mathbb{E})$ , or more simply,  $D(E)$ , the open set  $j^{-1}(U \times B)$  in  $Z$  which is also the domain of the scale  $E$ .

5. We shall call **domain of isotropy of the adjusted scale**, written down  $D'(\mathbb{E})$ , the open set  $j^{-1}(U' \times B)$  in  $Z$ .

6. We shall call **domain of patching of the adjusted scale**, written down  $D''(\mathbb{E})$ , the open set  $j^{-1}(U'' \times B'')$  in  $Z$ .

REMARKS.

1. The open set  $D''(\mathbb{E})$  is relatively compact in  $D'(\mathbb{E})$ .

2. When a  $n$ -scale  $E$  is given, for any compact set  $K$  in  $D(E)$ , there exists an adjusted  $n$ -scale  $\mathbb{E}$  such  $E$  is the underlying scale of  $\mathbb{E}$  and with  $K \subset D'(\mathbb{E})$ . Moreover, if  $E$  is adapted to a  $n$ -cycle  $X_0$  in  $Z$ , we may choose  $\mathbb{E}$  in order that it is adapted to  $X_0$ .
3. As for  $X \in \mathcal{C}_n^{loc}(Z)$  the condition to avoid a compact subset is open in  $\mathcal{C}_n^{loc}(Z)$ , when the adjusted scale  $\mathbb{E}$  is adapted to a cycle  $X_0$  there exists an open neighbourhood, written down  $\Omega_k(\mathbb{E})$ , of  $X_0$  in  $\mathcal{C}_n^{loc}(Z)$  such that  $\Omega_k(\mathbb{E})$  is the subset of all  $n$ -cycles  $X$  in  $Z$  for which  $\mathbb{E}$  is adapted and  $\deg_{\mathbb{E}}(X) = k$  where  $k := \deg_{\mathbb{E}}(X_0)$ .

Let  $Z$  be a reduced complex space and let  $\mathbb{E}$  be an adjusted scale on  $Z$ . For a given integer  $k$  consider the continuous map sending a branched covering in  $H(\bar{U}, Sym^k(B''))$  to its isotropy data on  $\bar{U}'$

$$T : H(\bar{U}, Sym^k(B'')) \rightarrow H(\bar{U}', F \otimes E').$$

The graph  $\Sigma_{U,U'}(k)$  of this map is a banach analytic set<sup>5</sup>, thanks to [B.75], proposition 2 p.81 (see also [B-M 2]).

The set of couples  $(f, T(f))$  in  $\Sigma_{U,U'}(k)$  for which the associated branched covering is contained in  $j(Z)$  is a closed banach analytic subset of  $\Sigma_{U,U'}(k)$  being the pull-back by the projection of the subset of elements in  $H(\bar{U}, Sym^k(B''))$  contained in  $j(Z)$  which is a closed banach analytic subset of  $H(\bar{U}, Sym^k(B''))$  by the proposition 4 p.27 of [B.75] (see also [B-M 2] ch. V).

**Definition 3.2.2** We shall denote  $\mathcal{G}_k(\mathbb{E})$  this banach analytic set and we shall call it the **the  $k$ -th classifying space of the adjusted  $n$ -scale  $\mathbb{E}$  on  $Z$** .

We have then a tautological family of  $n$ -cycles in the open set  $D(\mathbb{E})$  parametrized by  $\mathcal{G}_k(\mathbb{E})$ . It is an analytic family of cycles in the open set  $D'(\mathbb{E})$ , in the sense of [B.75], and the fact that, for  $k \geq 1$ , locally on  $\mathcal{G}_k(\mathbb{E})$ , any irreducible component of a branched covering in this family meets  $\bar{U}'' \times \bar{B}''$  implies that we have a f-analytic family of cycles in  $D'(\mathbb{E})$ .

Be careful that the tautological family of cycles on the open set  $D(\mathbb{E})$  parametrized by  $\mathcal{G}_k(\mathbb{E})$  is not, in general, an analytic family of  $n$ -cycles ; see the example of [B.75] p.83 (and also [B-M 1] ch.IV).

The next lemma is an obvious consequence of *loc. cit.*

**Lemma 3.2.3** Let  $\mathbb{E}$  be an adjusted  $n$ -scale on a reduced complex space  $Z$  and let  $k$  be an integer. The tautological family of  $n$ -cycles in the open set  $D'(\mathbb{E})$  parametrized by  $\mathcal{G}_k(\mathbb{E})$  has the following “almost universal” property:

<sup>5</sup>homeomorphic to  $H(\bar{U}, Sym^k(B''))$  via the projection !

- For any analytic family of  $n$ -cycles  $(X_s)_{s \in S}$  in  $Z$  parametrized by a banach analytic set  $S$  such that for each  $s \in S$  the adjusted scale  $\mathbb{E}$  is adapted to  $X_s$  with  $\deg_{\mathbb{E}}(X_s) = k$ , there exists an unique holomorphic map

$$f : S \rightarrow \mathcal{G}_k(\mathbb{E})$$

such that the pull-back by  $f$  of the tautological family is the restriction to the open set  $D'(\mathbb{E})$  of the given family.

Of course, conversely, a holomorphic map gives a f-analytic family of  $n$ -cycles on the open set  $D'(\mathbb{E})$ .

Note that the pull-back family is in fact defined on the open set  $D(\mathbb{E})$  but, as already noticed above, it may not be analytic outside  $D'(\mathbb{E})$ .

As a consequence of this “almost universal” property, we obtain that for any analytic family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $Z$  such that for a point  $s_0 \in S$  the adjusted  $n$ -scale  $\mathbb{E}$  is adapted to the cycle  $X_{s_0}$  with  $\deg_{\mathbb{E}}(X_{s_0}) = k$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  such that the previous lemma applies for the family parametrized by  $S'$ . So we shall have a holomorphic classifying map  $f : S' \rightarrow \mathcal{G}_k(\mathbb{E})$  in this situation.

We shall generalize now the concept of classifying space to a finite family of adjusted  $n$ -scales.

**Definition 3.2.4** Consider a reduced complex space  $Z$  and a finite family of adjusted  $n$ -scales  $(\mathbb{E}_i)_{i \in I}$  on  $Z$ . Assume that they are adapted to a given finite type  $n$ -cycle  $\hat{X}_0$  in  $Z$ . Assume that any irreducible component of  $\hat{X}_0$  meets the open set  $W'' := \cup_{i \in I} D''(\mathbb{E}_i)$ .

We shall call **patching data for  $\hat{X}_0$  associated to the family  $(\mathbb{E}_i)_{i \in I}$** , written down  $\mathcal{R}((\mathbb{E}_i)_{i \in I}, F)$  or more simply  $\mathcal{R}$  when there is no ambiguity, a finite collection of  $n$ -scales  $(F_{i,j,h})$  for  $(i,j) \in I^2, i \neq j$ , where  $h$  belongs to a finite set  $H(i,j)$  for each couple  $(i,j) \in I^2, i \neq j$ , such that the following properties hold:

- i)  $F_{i,j,h}$  is a  $n$ -scale on the open set  $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$ .
- ii) The  $n$ -scales  $F_{i,j,h}$  are adapted to  $\hat{X}_0$ .

We shall say that the patching data  $\mathcal{R}$  are **complete** when, moreover, the following condition also holds :

- iii) For each  $i \neq j$  given, the union of domains of the scales  $F_{i,j,h}, h \in H(i,j)$ , covers the compact subset  $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$  of  $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$ .

## NOTATIONS.

1. In the sequel, when we shall consider a reduced complex space  $Z$  and a finite family of adjusted  $n$ -scales  $(\mathbb{E}_i)_{i \in I}$ , adapted to a finite type  $n$ -cycle  $\hat{X}_0$  in  $Z$ , such that any irreducible component of  $\hat{X}_0$  meets the open set

$$W'' := \cup_{i \in I} D''(\mathbb{E}_i),$$

we shall say that the family  $(\mathbb{E}_i)_{i \in I}$  is **convenient** for  $\hat{X}_0$ .

2. In this setting we shall use the following definitions :

- $W := \cup_{i \in I} D(\mathbb{E}_i)$  ;
- $W' := \cup_{i \in I} D'(\mathbb{E}_i)$  ;
- $W'' := \cup_{i \in I} D''(\mathbb{E}_i)$  ;
- $K := \cup_{i \in I} j_i^{-1}(\bar{U}_i \times (\bar{B}_i \setminus B_i''))$ .
- When the family  $(\mathbb{E}_i)_{i \in I}$  is convenient for a finite type  $n$ -cycle  $\hat{X}_0$ ,  $\mathcal{K}$  will be a compact neighbourhood of  $K$  disjoint from  $\hat{X}_0$ .

3. For  $\tilde{X} \in \prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$  we shall denote  $X^i$  the closed cycle in  $D(\mathbb{E}_i)$  associated to the  $i$ -th component of  $\tilde{X}$ .

**Lemma 3.2.5** *Let  $(\mathbb{E}_i)_{i \in I}$  a finite family of adjusted  $n$ -scales on  $Z$ , convenient for a  $n$ -cycle  $\hat{X}_0$  of finite type in  $Z$ , and let  $\mathcal{R}$  be some corresponding complete patching data. There exists an open neighbourhood  $\mathcal{V}$  of the image  $\tilde{X}_0$  of  $\hat{X}_0$  in the product  $\prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$  such that for each  $\tilde{X} \in \mathcal{V}$  we have the following properties:*

1. No  $X^i$  meets the compact set  $\mathcal{K}$ .
2. For each  $i \in I$ , any irreducible component of  $X^i$  meeting  $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$  with  $j \neq i$ , meets the open set  $\cup_h D(F_{i,j,h})$ .
3. For each  $(i, j, h)$  the scale  $F_{i,j,h}$  is adapted to  $X^i$  and  $X^j$ .
4. For each  $(i, j, h)$  we have  $\deg_{F_{i,j,h}}(X^i) = \deg_{F_{i,j,h}}(X^j) = \deg_{F_{i,j,h}}(\hat{X}_0) = k_{i,j,h}$ .

PROOF. The conditions 1, 3 and 4 are clearly open. An easy consequence of the condition 1, of the inclusion ii) of the definition 3.2.4 and of the lemma 3.1.2 is that the condition 2 is also open. ■

For a  $n$ -scale  $E := (U, B, j)$  on  $Z$  we shall abbreviate  $H(\bar{U}, \text{Sym}^k(B))$  in  $G_k(E)$ .

**Definition 3.2.6** Let  $(\mathbb{E}_i)_{i \in I}$  a finite family of adjusted  $n$ -scales on  $Z$ , convenient for a  $n$ -cycle  $\hat{X}_0$  of finite type in  $Z$ , and let  $\mathcal{R}$  be some corresponding patching data. Let  $k_i$  be the degree of  $\hat{X}_0$  in the adjusted scale  $\mathbb{E}_i$ . For each  $(i, j, h), i \neq j$  we have a couple of holomorphic maps

$$\prod_{\alpha \in I} \mathcal{G}_{k_\alpha}(\mathbb{E}_\alpha) \rightrightarrows G_{k_{i,j,h}}(F_{i,j,h})$$

obtained by the change of projections  $\mathbb{E}_i \rightarrow F_{i,j,h}$  and  $\mathbb{E}_j \rightarrow F_{i,j,h}$ , because, by construction, we have  $D(F_{i,j,h}) \subset\subset D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$ .

We shall note  $S(\mathcal{R})$  the intersection of the kernels of these double maps<sup>6</sup> with the open set  $\mathcal{V}$  built in the lemma 3.2.5. It is a banach analytic set and we shall call it the **classifying space associated to**  $(\mathbb{E}_i)_{i \in I}, \hat{X}_0$  **and**  $\mathcal{R}$ .

Remark that the patching data  $\mathcal{R}$  are not assumed to be complete in the previous definition.

**Proposition 3.2.7** Consider a finite family of adjusted  $n$ -scales which is convenient for the  $n$ -cycle  $X_0$  in  $Z$  and let  $\mathcal{R}$  be some **complete** patching data associated. Keeping the previous notations we have for each  $(X^i)_{i \in I} \in S(\mathcal{R})$  an unique  $n$ -cycle  $X \in \mathcal{C}_n^f(W')$  such that  $X \cap D'(\mathbb{E}_i) = X^i \cap D'(\mathbb{E}_i), \forall i \in I$ .

Moreover, this defines a tautological family of cycles in  $W'$  which is a  $f$ -analytic family of cycles satisfying the following “almost universal”property :

- For any analytic family of  $n$ -cycles  $(X_s)_{s \in S}$  in an open neighbourhood of  $\bar{W}$  parametrized by a banach analytic set  $S'$  such that for  $s_0 \in S$  we have  $X_{s_0} = X_0$  in a neighbourhood of  $\bar{W}$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  and an unique holomorphic map

$$f : S' \rightarrow S(\mathcal{R})$$

such that the pull-back by  $f$  of the tautological family parametrized by  $S(\mathcal{R})$  is the restriction to the open set  $W'$  of the family  $(X_s)_{s \in S'}$ .

PROOF. For each  $\tilde{X} \in \mathcal{V}$  any  $X^i$  does not meet  $\mathcal{K}$ . As  $\mathcal{R}$  is complete, we may use the lemma 3.1.4 with  $\mathcal{U}_i := D'(\mathbb{E}_i)$  et  $W_{i,j} = \cup_h D(F_{i,j,h})$  to associate to  $\tilde{X}$  a finite type  $n$ -cycle  $X$  of the open set  $W'$ . The  $f$ -analyticity of the so defined family is obvious. The “almost universal”property is then clear.  $\blacksquare$

<sup>6</sup>The kernel of a double map  $f, g : A \rightarrow B$  is the pull-back of the diagonal in  $B \times B$  by the map  $(f, g) : A \rightarrow B \times B$ .

<sup>7</sup>Recall that, by definition, this means that for any  $n$ -scale  $E := (U, B, j)$  on  $Z$ , adapted to some  $X_{s_0}, s_0 \in S$ , with  $\deg_E(X_{s_0}) = k$ , we have an open neighbourhood  $S_0$  of  $s_0$  in  $S$  and a classifying map for the corresponding family of branched coverings  $f : S_0 \times U \rightarrow \text{Sym}^k(B)$  which is **isotropic**.

### 3.3 Shrinkage.

**Definition 3.3.1** *Let  $Z$  be a reduced complex space and let  $\mathbb{E} := (U, U', U'', B, B'', j)$  be an adjusted  $n$ -scale on  $Z$ . For any real  $\tau > 0$  small enough we shall note  $\mathbb{E}^\tau$  the adjusted  $n$ -scale on  $Z$  defined as  $\mathbb{E}^\tau := (U^\tau, U'^\tau, U'', B, B'', j)$ . We shall call  $\mathbb{E}^\tau$  the  $\tau$ -shrinkage of  $\mathbb{E}$ . Recall that for a polydisc  $P$  of radius  $R$ ,  $P^\tau$  is the polydisc with same center and radius  $R - \tau$ . The definitions of  $\mathcal{M}(\alpha)^\tau, M(\alpha)^\tau$  are given in the section 2.2.*

REMARKS.

1. By definition, the shrinkage of  $\mathbb{E}$  does not change the polydiscs  $U'', B, B''$ .
2. As  $j$  is a closed embedding of an open set in  $Z$  in an open neighbourhood of the compact set  $\bar{U} \times \bar{B}$ , it is clear that for any given adjusted  $n$ -scale  $\mathbb{E}$  on  $Z$ , there exists a real  $\varepsilon > 0$  (depending on  $\mathbb{E}$ ) such that for any  $\tau \in ]0, \varepsilon[$ ,  $\mathbb{E}^\tau$  is again an adjusted  $n$ -scale on  $Z$ .
3. If  $\mathbb{E}$  is adapted to the  $n$ -cycle  $X_0$  in  $Z$ , for  $\tau$  small enough (depending on  $\mathbb{E}$  and  $X_0$ ), the adjusted  $n$ -scale  $\mathbb{E}^\tau$  remains adapted to  $X_0$  and we shall have also  $\deg_{\mathbb{E}^\tau}(X_0) = \deg_{\mathbb{E}}(X_0)$ .
4. If  $\mathbb{E}$  is adapted to the  $n$ -cycle  $X_0$  in  $Z$ , there exists an open neighbourhood  $\mathcal{U}$  of  $X_0$  in  $\mathcal{C}_n^{loc}(Z)$  and a real  $\varepsilon > 0$  such that for any  $X \in \mathcal{U}$  and any  $\tau \in ]0, \varepsilon[$ , the adjusted scale  $\mathbb{E}^\tau$  remains adapted to  $X$  with again  $\deg_{\mathbb{E}^\tau}(X) = \deg_{\mathbb{E}}(X_0)$ .
5. In the situation of the previous proposition 3.2.7, we may, for  $\tau > 0$  small enough, keep the same patching data  $\mathcal{R}$  on the finite family  $(\mathbb{E}_i^\tau)_{i \in I}$  of adjusted  $n$ -scales; if it was complete, it remains complete and if it was convenient for the  $n$ -cycle  $X_0$ , it remains convenient for the  $n$ -cycle  $X_0$ . Then we have a holomorphic restriction map

$$S(\mathcal{R}) \rightarrow S^\tau(\mathcal{R})$$

which is induced by a finite product of linear (continuous) compact maps.  $\square$

### 3.4 Excellent family.

In order not to make our notations too heavy we shall introduce the following conventions when  $\mathcal{H}$  is a  $n$ -Hartogs figure in  $\mathbb{C}^{n+p}$ ; we shall put, using the notations introduced above

$$U := M(\alpha) \times V, \quad U' := M(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad U'' := M(\alpha)^{\varepsilon''} \times V^{\varepsilon''}, \quad B'' := B^{\varepsilon'}$$

where  $\varepsilon' > 0$  is small enough, and where  $0 < \varepsilon'' < \varepsilon'$ . The choices of  $\varepsilon'$  and  $\varepsilon''$  will be precised when they are useful. We shall associate to  $\mathcal{H}$  the adjusted  $n$ -scale on  $\Delta$  given by :

$$\mathbb{E}_{\mathcal{H}} := (U, U', U'', B, B'', j).$$

We shall put also

$$\tilde{U} := \mathcal{M}(\alpha) \times V, \quad \tilde{U}' := \mathcal{M}(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad \tilde{U}'' := \mathcal{M}(\alpha)^{\varepsilon''} \times V^{\varepsilon''}$$

the holomorphy envelopes respectively of  $U, U'$  and  $U''$ . Then we shall have a family of adjusted  $n$ -scales on  $\Delta'$ , written down:

$$\mathbb{E}_{\tilde{\mathcal{H}}}^{\eta} := (\tilde{U}^{\eta}, \tilde{U}'^{\eta}, \tilde{U}''^{\eta}, B, B'', j) \quad \text{with } \tilde{U}^{\eta} := \mathcal{M}(\alpha)^{\eta} \times V^{\eta}, \quad \tilde{U}'^{\eta} := \mathcal{M}(\alpha)^{\varepsilon'+\eta} \times V^{\varepsilon'+\eta},$$

where  $\eta$  is a positive real number, small enough (for  $\eta = 0$  we shall simply write  $\mathbb{E}_{\tilde{\mathcal{H}}}$ ).

Then we shall have the isotropic classifying spaces  $\mathcal{G}_k(\mathcal{H}) := \Sigma_{U, U'}(k)$  and also  $\mathcal{G}_k^{\eta}(\tilde{\mathcal{H}}) := \Sigma_{\tilde{U}^{\eta}, \tilde{U}'^{\eta}}(k)$ . The proposition 2.2.5 gives a holomorphic analytic extension map such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}_k(\tilde{\mathcal{H}}) & \xrightarrow{\text{res}^{\eta}} & \mathcal{G}_k^{\eta}(\tilde{\mathcal{H}}) \\ \text{res} \downarrow & \nearrow \text{prlgt}^{\eta} & \\ \mathcal{G}_k(\mathcal{H}) & & \end{array}$$

**Definition 3.4.1** *Let  $Z$  be a reduced complex space of pure dimension  $n + p$ , let  $\Delta \subset\subset Z$  be a strongly  $n$ -concave open set in  $Z$  and  $\tilde{X}_0$  a  $n$ -cycle in an open neighbourhood  $\Delta'$  of  $\bar{\Delta}$  in  $Z$ . We shall say that a finite family  $(\mathcal{H}_a)_{a \in A}$  of  $n$ -Hartogs figures relative to the boundary of  $\Delta$  is **excellent** for the cycle  $\tilde{X}_0$  when the following conditions hold, where we write  $\tilde{\mathbb{E}}_a$  and  $\mathbb{E}_a$  the adjusted scales respectively on  $\Delta'$  and  $\Delta$  associated to the  $n$ -Hartogs figure  $\mathcal{H}_a$  :*

1. *The adjusted scales  $\tilde{\mathbb{E}}_a$  and  $\mathbb{E}_a$  are adapted to the cycle  $\tilde{X}_0$ .*
2. *We may choose the patching domains of the adjusted scales  $(\tilde{\mathbb{E}}_a)_{a \in A}$  in order that the union*

$$\tilde{D}''(A) := \cup_{a \in A} j_a^{-1}(\tilde{U}''_a \times B''_a)$$

*contains the compact set  $\partial\Delta$ .*

3. *There exists a finite family of adjusted  $n$ -scales  $(\mathbb{E}_b)_{b \in B}$  on  $\Delta$ , adapted to  $\tilde{X}_0$ , such that the finite families  $(\mathbb{E}_c)_{c \in A \cup B}$  and  $(\tilde{\mathbb{E}}_c)_{c \in A \cup B}$  are convenient for  $\tilde{X}_0$ , where we put  $\tilde{\mathbb{E}}_b = \mathbb{E}_b$  for  $b \in B$ .*

*Moreover we ask that the union  $D''(B)$  of the patching domains of the  $(\mathbb{E}_b)_{b \in B}$  covers the compact set  $\Delta \setminus \tilde{D}''(A)$  of  $\Delta$ ; so  $\tilde{D}''(A) \cup D''(B)$  in an open set containing  $\bar{\Delta}$ .*

As a consequence, the union  $\tilde{D}'(A)^{\eta} \cup D'(B)$  of the isotropy domains will cover  $\bar{\Delta}$  for  $\eta > 0$  small enough.

**Proposition 3.4.2 (Existence of an excellent family.)** *Let  $Z$  be a reduced complex space of pure dimension  $n + p$ ,  $\Delta \subset\subset Z$  be a strongly  $n$ -concave open set with smooth boundary and  $\tilde{X}_0$  a  $n$ -cycle in an open neighbourhood  $\Delta' \subset Z$  of  $\bar{\Delta}$  such that any irreducible component of  $\tilde{X}_0$  meets  $\Delta$ . Then there exists a finite family  $(\mathcal{H}_a)_{a \in A}$  of  $n$ -Hartogs figures relative to the boundary of  $\Delta$  which is excellent for the cycle  $\tilde{X}_0$ .*

PROOF. First we use the proposition 2.3.6 to cover  $\partial\Delta$  by a finite family of  $n$ -Hartogs figures relative to the boundary of  $\Delta$  adapted to the cycle  $\tilde{X}_0$  such that the conditions 1 and 2 hold. Then we build a finite family of adjusted  $n$ -scales  $(\mathbb{E}_b)_{b \in B}$  on  $\Delta$ , adapted to  $\tilde{X}_0$  in order that the condition 3 holds.  $\blacksquare$

### 3.5 The extension and finiteness theorem.

The next theorem will allow to prove the theorem 1.0.1.

**Theorem 3.5.1** *Let  $Z$  be a reduced complex space of pure dimension  $n + p$ , where  $n \geq 2, p \geq 1$ . Let  $\Delta$  be a relatively compact open set with  $\mathcal{C}^2$  boundary in  $Z$  which is strongly  $n$ -concave. Let  $\tilde{X}_0$  a closed  $n$ -cycle of an open neighbourhood  $\Delta'$  of  $\bar{\Delta}$  in  $Z$  such that any irreducible component of  $\tilde{X}_0$  meets  $\Delta$ . Then there exists an open neighbourhood  $\Delta''$  of  $\bar{\Delta}$  in  $\Delta'$  and a  $f$ -analytic family  $(\tilde{X}_\xi)_{\xi \in \Xi}$  of  $n$ -cycles in  $\Delta''$  parametrized by a reduced complex space  $\Xi$  (of finite dimension) such that  $X_{\xi_0} = \tilde{X}_0 \cap \Delta''$  and such that  $\Xi$  is an open neighbourhood of  $\tilde{X}_0 \cap \Delta''$  in  $\mathcal{C}_n^f(\Delta'')$ . It satisfies the following universal property:*

- For any  $f$ -analytic family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $\Delta$  parametrized by a banach analytic set  $S$  and such that its value at some  $s_0 \in S$  is equal to  $\tilde{X}_0 \cap \Delta$ , there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  and an unique holomorphic map

$$h : S' \rightarrow \Xi$$

satisfying the equality  $X_s = \tilde{X}_{h(s)} \cap \Delta$  for each  $s \in S'$

PROOF. Assume that  $\Delta := \{\varphi > \alpha\}, 0 < \alpha < 1$ , where  $\varphi : Z \rightarrow ]0, 2]$  is an  $\mathcal{C}^2$  exhaustion of  $Z$  which is strongly  $(n - 1)$ -convex on the open set  $\varphi^{-1}(]0, 1[)$ . Up to replace  $\Delta$  by  $\{\varphi > \alpha'\}, \alpha' < \alpha, \alpha'$  very close to  $\alpha$ , we may assume, thanks to the Sard' lemma 3.1.1, that  $\varphi$  and  $\varphi|_{\tilde{X}_0}$  have no critical point on the compact set  $\tilde{X}_0 \cap \partial\Delta$ .

Begin by cover the compact set  $\partial\Delta$  by an excellent finite family  $(\mathcal{H}_a)_{a \in A}$  of  $n$ -Hartogs figures for  $\Delta$  adapted to the cycle  $\tilde{X}_0$ . Choose then open sets  $\Delta_1 \subset\subset \Delta \subset\subset \Delta'' \subset\subset \Delta'$ , such that the following properties hold, where we use the notations introduced above for the finite family of the adjusted scales  $(\tilde{\mathbb{E}}_a)_{a \in A}$  associated to  $(\mathcal{H}_a)_{a \in A}$ :

- i)  $\bar{\Delta}'' \setminus \Delta_1 \subset \cup_{a \in A} W''(\tilde{\mathbb{E}}_a)$ .
- ii)  $\cup_{a \in A} \bar{W}(\tilde{\mathbb{E}}_a) \subset \Delta'$ .
- iii)  $\cup_{a \in A} \bar{W}(\mathbb{E}_a) \subset \Delta$ .
- iv)  $K := \cup_{a \in A} j_a^{-1}(\bar{U}_a \times (\bar{B}_a \setminus B''_a)) \subset \Delta_1$

It is easy to fulfill these conditions for  $\Delta_1$  and  $\Delta''$  near enough to  $\Delta$ , as, by assumption, the subsets  $\bar{W}(\mathbb{E}_a)$  and  $j_a^{-1}(\bar{U}_a \times \bar{B}_a \setminus B''_a)$  are compact in  $\Delta$ , and as the union of the open sets  $W''(\tilde{\mathbb{E}}_a)$  contains  $\partial\Delta$ .

Note that the condition iv) allows to choose le compact neighbourhood  $\mathcal{K}$  of  $K$  inside  $\Delta_1$ .

Choose now a convenient finite family  $(\mathbb{E}_b)_{b \in B}$  of adjusted  $n$ -scales on  $\Delta$ , adapted to  $\tilde{X}_0$  in order that the open set  $\cup_{b \in B} W''(\mathbb{E}_b)$  contains  $\bar{\Delta}_1$ .

Put  $\tilde{\mathbb{E}}_b = \mathbb{E}_b$  for  $b \in B$ , and define  $C := A \cup B$ . The family of adjusted scales  $(\mathbb{E}_c)_{c \in C}$  in  $\Delta$  is then convenient for  $\tilde{X}_0 \cap \Delta_1$ , up to choose the patching domains big enough. Fix some complete patching data  $\mathcal{R}$  associated to the family  $(\mathbb{E}_c)_{c \in C}$ . The family  $(\tilde{\mathbb{E}}_c)_{c \in C}$  of adjusted scales in  $\Delta'$  is convenient for  $\tilde{X}_0 \cap \Delta''$  if we choose the patching domains big enough. Consider now some complete patching data for this family of the form  $\mathcal{R} \cup \mathcal{R}''$ , that is to say containing the patching scales already in  $\mathcal{R}$ . Define the following banach analytic sets:

1.  $S_0$  is the classifying space of the family  $(\tilde{\mathbb{E}}_c)_{c \in C}$ , the degrees being these of  $\tilde{X}_0$  in the various scales adapted to the cycle  $\tilde{X}_0$ , with the patching conditions defined by  $\mathcal{R}$ . Note that  $\mathcal{R}$  is not complete in general.
2.  $S_+$  is the classifying space of the family  $(\tilde{\mathbb{E}}_c)_{c \in C}$  with the patching conditions defined by  $\mathcal{R} \cup \mathcal{R}''$ .
3.  $S_-$  is the classifying space of the family  $(\mathbb{E}_c)_{c \in C}$ , with the (complete) patching conditions defined by  $\mathcal{R}$ .

Then we get a holomorphic extension map

$$\alpha : S_- \rightarrow S_0$$

deduced from the extension maps in the  $n$ -Hartogs figures  $(\mathcal{H}_a)_{a \in A}$ .

By definition  $S_+$  is a closed banach analytic subset of  $S_0$  as it is defined in  $S_0$  by the patching conditions given by  $\mathcal{R}''$ . Then put  $\Xi := \alpha^{-1}(S_+)$ . So we have a holomorphic map  $\alpha : \Xi \rightarrow S_+$ . We want to show the following claim:

CLAIM. There exists a holomorphic map  $\beta : S_+ \rightarrow \Xi$  satisfying the two properties:

1. We have  $\alpha \circ \beta = Id$  and  $\beta \circ \alpha = Id$  in a neighbourhood of the point defined by  $\tilde{X}_0$  respectively in  $S_+$  and  $\Xi$ .

2. The holomorphic map  $\beta$  is the composition of a holomorphic map induced by a linear (continuous) compact map and a holomorphic map.

As the open set  $\cup_{c \in C} W''(\tilde{\mathbb{E}}_c)$  contains  $\Delta''$ , there is, on  $S_+$ , a tautological family of  $n$ -cycles which is  $f$ -analytic on  $\Delta''$ . Then the “almost universal” property of  $S_-$  gives a holomorphic map  $\tilde{\beta} : S_+ \rightarrow S_-$  which factorizes via the closed banach analytic subset  $\Xi \subset S_-$ . Let us show that the holomorphic map  $\beta : S_+ \rightarrow \Xi$  deduced from this factorization satisfies the two properties of the claim.

First we have  $\alpha \circ \beta = Id$  and  $\beta \circ \alpha = Id$  respectively in  $S_+$  and  $\Xi$ , by definition of  $\Xi$  and  $\beta$ .

To see the second property, consider  $\tau > 0$  small enough and let us show that the holomorphic map induced by the linear compact (by Vitali’ theorem) restriction  $r^\tau : S_+ \rightarrow S_+^\tau$  factorizes  $\beta$ , where  $S_+^\tau$  is the classifying space corresponding to the  $\tau$ -shrinkage  $(\tilde{\mathbb{E}}_c^\tau)_{c \in C}$  of the family  $(\tilde{\mathbb{E}}_c)_{c \in C}$  with the patching data  $\mathcal{R} \cup \mathcal{R}''$ . Indeed, for  $\tau$  small enough, the tautological family of  $S_+^\tau$  is still  $f$ -analytic on  $\Delta$  and the “almost universal” property of  $S_-$  gives again a holomorphic map  $\tilde{\beta}^\tau : S_+^\tau \rightarrow S_-$ . And we have  $\beta = \tilde{\beta}^\tau \circ r^\tau$  proving our assertion. In fact, the map  $\tilde{\beta}^\tau$  takes its values in  $\Xi$  because the  $\tau$ -shrinkage does not change the patching data deduced from  $\mathcal{R} \cup \mathcal{R}''$  for  $\tau$  small enough.

We conclude that the banach analytic set  $\Xi$  has finite dimension thanks to the finiteness lemma of [B.75] (see also [B-M 2] ch.V). Moreover, it is isomorphic to  $S_+$  which is also of finite dimension.

The universal property is then obvious. ■

#### REMARKS.

1. The reduced complex space (of finite dimension)  $\Xi$  built in the previous theorem parametrizes a  $f$ -analytic family of  $n$ -cycles in the open set  $\Delta''$  which is an open neighbourhood of  $\bar{\Delta}$ . So, when we have a  $f$ -analytic family  $(X_s)_{s \in S}$  of  $n$ -cycles in  $\Delta$  parametrized by a banach analytic set  $S$  and such that its value at some  $s_0 \in S$  is equal to  $\tilde{X}_0 \cap \Delta$  we can extend each cycle  $X_s, s \in S'$ , to a  $n$ -cycle  $\tilde{X}_s$  in  $\Delta''$  in order that the family  $(\tilde{X}_s)_{s \in S'}$  is  $f$ -analytic in  $\Delta''$ , with the condition that each irreducible component of  $\tilde{X}_s$  meets  $\Delta$  and with the equality  $\tilde{X}_s \cap \Delta = X_s$  for each  $s \in S'$ .
2. We shall see later on that  $\Xi$  is also (isomorphic to) an open neighbourhood of the point  $\tilde{X}_0 \cap \Delta$  of  $\mathcal{C}_n^f(\Delta)$ .

## 4 Finiteness of the space of $n$ -cycles of a reduced strongly $n$ -concave space ( $n \geq 2$ ).

### 4.1 The global extension theorem.

First we have to complete our terminology.

**Definition 4.1.1** *We shall say that a reduced complex space  $Z$  is **strongly  $q$ -concave**, where  $q$  is a positive integer, if there exists a real valued  $\mathcal{C}^2$  exhaustion function on  $Z$ ,  $\varphi : Z \rightarrow ]0, 2]$ , which is strongly  $(q - 1)$ -convex outside the compact set  $K := \varphi^{-1}([1, 2])$ .*

For instance, any reduced compact complex space is strongly  $q$ -concave for any  $q \geq 1$ .

When  $Z$  is strongly  $q$ -concave irreducible and non compact of dimension at least  $q + 1$ , the function  $\varphi$  achieves its maximum at a point in which it cannot be strongly  $(q - 1)$ -convex. So we shall have  $\varphi(Z) = ]0, u]$  with  $u \geq 1$ .

In the sequel we shall consider mainly the closed  $q$ -cycles of  $Z$  in the case where  $Z$  is strongly  $q$ -concave and non compact, with  $\dim Z \geq q + 1$ .

**Theorem 4.1.2** *Let  $n \geq 2$  be an integer. Let  $Z$  be a reduced complex space which is strongly  $n$ -concave. Let  $\alpha \in ]0, 1[$  and let  $X$  be a finite type  $n$ -cycle in the open set  $Z_\alpha := \{z \in Z / \varphi(z) > \alpha\}$ . Then there exists a unique  $n$ -cycle  $\tilde{X}$  in  $\mathcal{C}_n^f(Z)$  such that  $\tilde{X} \cap Z_\alpha = X$ .*

**PROOF.** First we shall prove uniqueness. If a non empty  $n$ -cycle  $\tilde{X}$  in  $Z$  satisfies  $\tilde{X} \cap Z_\alpha = \emptyset$  with  $\alpha < 1$ , the maximum of  $\varphi$  on the set  $|\tilde{X}|$  is obtained at a point where  $\varphi$  is strongly  $(n - 1)$ -convex. This is a contradiction. So the uniqueness of the cycle  $\tilde{X}$  is proved.

To show that this cycle exists, consider first the case where  $X$  is compact in  $Z_\alpha$ . Then  $\tilde{X} := X$  is a solution. So it enough to consider the case where  $X$  is irreducible and non compact. Thanks to the theorem (8.3) of [S-T], for each  $z \in \partial Z_\alpha$  there exists an open set  $U_z$  and an unique closed analytic set  $X_z$  in  $Z_\alpha \cup U_z$  of pure dimension  $n$  such that  $X_z \cap Z_\alpha = X$ . Choose a finite set of points  $z_1, \dots, z_N$  and open sets  $U'_i \subset\subset U_i := U_{z_i}$  such that the union of the  $U'_i$  covers the compact set  $\partial Z_\alpha$ . Let  $\Omega := Z_\alpha \cup (\cup_{i=1}^N U'_i)$  and put

$$X_1 := (X \cup (\cup_{i=1}^N X_{z_i})) \cap \Omega.$$

Let us show that  $X_1$  is a closed analytic subset in  $\Omega$ .

Consider  $z \in \Omega$ . If  $z$  is in  $Z_\alpha$  we have  $X_1 = X$  in a neighbourhood of  $z$  and the assertion is clear. If not, either  $z$  is not in any  $\partial U'_i$  and  $X_1$  is the union of the  $X_{z_i}$  in a neighbourhood of  $z$  and the assertion is clear, or  $z$  is in  $\partial U'_{j_1}, \dots, \partial U'_{j_k}$  for  $j_1, \dots, j_k$  in  $[1, N]$ . As the set  $X_{j_h}$  is closed and analytic in  $Z_\alpha \cup U_{j_h}$ , then  $X_1$  is again the union of the  $X_{z_i}$  near  $z$  in  $\Omega$ , and the assertion is proved.

So in this situation there exists a real positive  $\beta < \alpha$  such that  $Z_\beta \subset \Omega$ . Let  $X_2$  be the irreducible component of  $X_1 \cap Z_\beta$  which contains  $X$ ; then  $X_2$  is a closed irreducible analytic subset of  $Z_\beta$  such that  $X_2 \cap Z_\alpha = X$ .

Now let

$$\gamma := \inf\{\beta \leq \alpha / \exists X_\beta \text{ irreducible } n\text{-cycle of } Z_\beta \text{ such that } X_\beta \cap Z_\alpha = X\}.$$

Then what we obtained above shows that we have  $\gamma < \alpha$ , and, applying the same arguments to the cycle  $X_\gamma$  defined on  $Z_\gamma$  via the cover of  $Z_\gamma$  by the  $Z_\beta, \beta > \gamma$  in which we already built an irreducible  $n$ -cycle  $X_\beta$  extending  $X$ , we conclude that  $\gamma = 0$  and that there exists an (unique) irreducible  $n$ -cycle  $\tilde{X}$  in  $Z$  extending  $X$ . ■

## 4.2 Some consequences.

We shall give first some easy consequences of the fact that the reduced complex space  $Z$  is strongly  $n$ -concave.

**Proposition 4.2.1** *Let  $n \geq 2$  be an integer and let  $Z$  be a reduced strongly  $n$ -concave complex space. Then the natural map  $j : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{\text{loc}}(Z)$  is a homeomorphism. Moreover, for each  $\alpha \in ]0, 1[$  the restriction map*

$$\text{res}_\alpha : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$$

*is well defined and is also a homeomorphism.*

**PROOF.** Let us prove first that any  $n$ -cycle  $X$  in  $Z$  has finitely many irreducible components. As this implies the same result for each  $Z_\alpha$  for  $\alpha \in ]0, 1[$ , this will imply the fact that the restriction map  $\text{res}_\alpha$  is well defined, and then bijective as a consequence of the theorem 4.1.2.

Consider a closed irreducible analytic subset  $\Gamma$  of dimension  $n$  in  $Z$ . As the restriction  $\varphi|_\Gamma$  of the exhaustion  $\varphi$  to  $\Gamma$  must reach its maximum (as  $\varphi$  is continuous and proper), this maximum cannot be obtained at a point in which  $\varphi$  is strongly  $(n-1)$ -convex. So we have  $\Gamma \cap \varphi^{-1}[1, 2] \neq \emptyset$ . As the family of irreducible components of a  $n$ -cycle is locally finite, only finitely many irreducible components of  $X$  can meet the compact set  $K := \varphi^{-1}([1, 2])$ . But we have seen that **any** irreducible component of  $X$  must meet  $K$ . So  $X$  is a finite type cycle.

To show the continuity of  $\text{res}_\alpha$  it is then enough to prove that  $j$  is a homeomorphism which is an easy consequence of the lemma below. ■

**Lemma 4.2.2** *Let  $Z$  be a reduced complex space and let  $(X_\nu)_{\nu \geq 0}$  be a sequence of  $n$ -cycles in  $Z$  converging in  $\mathcal{C}_n^{\text{loc}}(Z)$  to a cycle  $Y$ . Assume that there exists a relatively compact open set  $\Omega$  in  $Z$  such that any irreducible component of each  $X_\nu$  and of  $Y$  meets  $\Omega$ . Then all these cycles are of finite type and the sequence converges to  $Y$  in  $\mathcal{C}_n^f(Z)$ .*

PROOF. First if  $Y = \emptyset$ , for  $\nu \gg 1$  the cycle  $X_\nu$  will be disjoint from the compact set  $K := \bar{\Omega}$ , and this implies that  $X_\nu$  is the empty cycle. So the conclusion holds in this case. If  $Y$  is not empty, let  $U$  be a relatively compact open set in  $Z$  meeting each irreducible component of  $Y$ . We have to show, by definition of the topology of  $\mathcal{C}_n^f(Z)$ , that for  $\nu \gg 1$  each irreducible component of  $X_\nu$  meets  $U$ . If it is not the case, up to pass to a subsequence, we may assume that for each  $\nu$  there exists an irreducible component  $\Gamma_\nu$  of  $X_\nu$  disjoint from  $U$ . As, again up to pass to a subsequence, we may assume that the sequence  $(\Gamma_\nu)$  converges in  $\mathcal{C}_n^{loc}(Z)$  to a cycle  $\Gamma$ , we shall have  $|\Gamma| \subset |Y|$  and  $|\Gamma| \cap U = \emptyset$ . To conclude, it is enough to show that  $\Gamma$  is not the empty cycle, as any irreducible component of  $\Gamma$  is also an irreducible component of  $Y$  and then meets  $U$  by hypothesis. As each  $\Gamma_\nu$  is not empty, it has to meet  $K = \bar{\Omega}$ . This implies that  $\Gamma$  also meets  $K$  and so is not empty. This contradicts our assumption. ■

END OF THE PROOF OF THE PROPOSITION 4.2.1. We have proved that  $j$ , and then also each  $j_\alpha : \mathcal{C}_n^f(Z_\alpha) \rightarrow \mathcal{C}_n^{loc}(Z_\alpha)$  for  $\alpha \in ]0, 1[$ , is a holomorphic homeomorphism. To conclude the proof we have to show the continuity of  $res_\alpha^{-1}$ , and this reduces to prove that if the sequence  $(X_\nu)$  of  $\mathcal{C}_n^{loc}(Z)$  is such that the sequence  $(X_\nu \cap Z_\alpha)$  converges in  $\mathcal{C}_n^{loc}(Z_\alpha)$ , then it converges in  $\mathcal{C}_n^{loc}(Z)$ . Let  $Y_\alpha \in \mathcal{C}_n^{loc}(Z_\alpha)$  be the limit of this sequence in  $\mathcal{C}_n^{loc}(Z_\alpha)$  and let  $Y \in \mathcal{C}_n^{loc}(Z)$  be the cycle extending it. Let  $A$  be the set of  $\beta \in ]0, \alpha]$  such that the sequence  $(X_\nu \cap Z_\beta)$  converges in  $\mathcal{C}_n^{loc}(Z_\beta)$  to  $Y \cap Z_\beta$ . Then  $\alpha$  is in  $A$  so  $A$  is not empty. Put  $\gamma := \inf A$ . The theorem 3.5.1 implies that  $\gamma = 0$  and we obtain also the convergence in any  $\mathcal{C}_n^{loc}(Z_\beta)$ , for any  $\beta > 0$ ; this gives the convergence in  $\mathcal{C}_n^{loc}(Z)$ , as, by definition, a  $n$ -scale on  $Z$  is also a  $n$ -scale on  $Z_\beta$  for  $\beta > 0$  small enough. ■

### 4.3 An analytic extension criterion.

The aim of this paragraph is to prove the following analytic extension result.

**Theorem 4.3.1** *Let  $Z$  be a complex space and  $n$  an integer. Consider a  $f$ -continuous family  $(X_s)_{s \in S}$  of  $n$ -cycles of finite type in  $Z$  parametrized by a reduced complex space  $S$ . Fix a point  $s_0$  in  $S$  and assume that the open set  $Z'$  in  $Z$  meets all irreducible components of  $X_{s_0}$  and such that the family of cycles  $(X_s \cap Z')_{s \in S}$  is analytic at  $s_0$ . Then there exists an open neighbourhood  $S_0$  of  $s_0$  in  $S$  such that the family  $(X_s)_{s \in S_0}$  is  $f$ -analytic.*

The hypotheses translated in term of classifying maps means that we have a continuous map  $\varphi : S \rightarrow \mathcal{C}_n^f(Z)$  such that the composed map  $r \circ \varphi$  is holomorphic at  $s_0$ , where  $r : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z')$  is the restriction map.

Then the theorem says that there exists an open neighbourhood  $S_0$  of  $s_0$  in  $S$  such that the map  $\varphi$  is holomorphic on  $S_0$ . Note that, as  $r$  is holomorphic<sup>8</sup>, the hypoth-

---

<sup>8</sup>in the sense that for any holomorphic map  $\psi : T \rightarrow \mathcal{C}_n^f(Z)$  of a reduced complex space  $T$  the composed map  $r \circ \psi$  is holomorphic.

esis that  $\varphi$  is holomorphic at  $s_0$  is a necessary condition.

This result is not true in general if we take for  $S$  a non smooth banach analytic set which is not of finite dimension (locally). The reader may find a counter-exemple with an isolated singularity in [B-M 2] ch.V.

The key point for the proof of the previous theorem is the following analytic extension result.

**Proposition 4.3.2** *Let  $S$  a reduced complex space and let  $\emptyset \neq U_1 \subset U_2$  be two polydiscs in  $\mathbb{C}^n$ . Let  $f : S \times U_2 \rightarrow \mathbb{C}$  a continuous function, holomorphic on  $\{s\} \times U_2$  for each  $s \in S$ . Assume moreover that the restriction of  $f$  to  $S \times U_1$  is holomorphic. Then  $f$  is holomorphic on  $S \times U_2$ .*

**PROOF OF THE PROPOSITION 4.3.2.** First consider the case  $S$  smooth. As the question is local on  $S$  it is enough to consider the case where  $S$  is an open set in  $\mathbb{C}^m$ . Fix an open relatively compact polydisc  $P$  in  $S$ . The function  $f$  defines a map  $F : U_2 \rightarrow \mathcal{C}^0(\bar{P}, \mathbb{C})$ , where we write down  $\mathcal{C}^0(\bar{P}, \mathbb{C})$  the Banach space of continuous functions on  $\bar{P}$ , via the formula  $F(t)[s] = f(s, t)$  for  $t \in U_2$  and  $s \in \bar{P}$ . First we shall show that the map  $F$  is holomorphic.

Let  $U \subset\subset U_2$  be a polydisc. For any fix  $s \in S$  we have

$$\frac{\partial f}{\partial t_i}(s, t) = \frac{1}{(2i\pi)^n} \int_{\partial\partial U} f(s, \tau) \cdot \frac{d\tau_1 \wedge \cdots \wedge d\tau_n}{(\tau_1 - t_1) \cdots (\tau_i - t_i)^2 \cdots (\tau_n - t_n)} \quad \forall t \in U \quad \forall i \in [1, n].$$

where  $t := (t_1, \dots, t_n)$  are coordinates on  $\mathbb{C}^n$ . This Cauchy formula shows that  $F$  is  $\mathbb{C}$ -différentiable and its differential at the point  $t \in U$  is given by

$$h \mapsto \sum_{i=1}^n F_i(t) \cdot h_i, \quad h \in \mathbb{C}^n,$$

where  $F_i$  is associated to the function

$$(s, t) \mapsto \frac{\partial f}{\partial t_i}(s, t) \quad i \in [1, n]$$

which is holomorphic for each fixed  $s \in S$  thanks to the Cauchy formula above.

Let  $H(\bar{P}, \mathbb{C})$  the closed subspace of  $\mathcal{C}^0(\bar{P}, \mathbb{C})$  of functions which are holomorphic on  $P$ . Our hypothesis implies that the restriction of  $F$  to the non empty open set  $U_1$  takes its values in this subspace. Let us show that this is also true on  $U_2$ . Assume that there exists  $t_0 \in U_2$  such that  $F(t_0)$  is not in  $H(\bar{P}, \mathbb{C})$ . Thanks to the *Hahn-Banach theorem* we can find a continuous linear form  $\lambda$  on  $\mathcal{C}^0(\bar{P}, \mathbb{C})$ , vanishing on  $H(\bar{P}, \mathbb{C})$ , and such that  $\lambda(F(t_0)) \neq 0$ . But the function  $t \mapsto \lambda(F(t))$  is holomorphic on  $U_2$  and vanishes on  $U_1$ ; this contradicts  $\lambda(F(t_0)) \neq 0$ . So  $F$  takes values in  $H(\bar{P}, \mathbb{C})$  and  $f$  is holomorphic on  $S \times U_2$  when  $S$  is a complex manifold.

The case where  $S$  is a weakly normal complex space follows immediately.

When  $S$  is a general reduced complex space, the function  $f$  is meromorphic and

continuous on  $S \times U_2$  and holomorphic on  $S \times U_1$ . So the the closed analytic subset  $Y \subset S \times U_2$  along which  $f$  may not be holomorphic has no interior point in each  $\{s\} \times U_2$ . The analytic extension criterion of [B-M 1] ch. IV, 3.1.7 allows to conclude. ■

**PROOF OF THE THEOREM 4.3.1.** Let  $|G| \subset S \times Z$  be the graph of the  $f$ -continuous family  $(X_s)_{s \in S}$  and let  $A$  the set of points in  $(\sigma, \zeta) \in |G|$  admitting an open neighbourhood  $S_\sigma \times Z_\zeta$  in  $S \times Z$  such that the family of cycles  $(X_s \cap Z_\zeta)_{s \in S_\sigma}$  is analytic. Remark that, thanks to our hypothesis, the open set  $A$  in  $|G|$  meets every irreducible component of  $\{s_0\} \times |X_{s_0}|$ .

Assume to begin that there exists a smooth point  $z_0$  of  $|X_{s_0}|$  in the boundary of  $A \cap (\{s_0\} \times |X_{s_0}|)$ . Choose a  $n$ -scale  $E := (U, B, j)$  on  $Z$  which is adapted to  $X_{s_0}$  and satisfying :

$$\begin{aligned} \deg_E(|X_{s_0}|) &= 1 \\ j_*(X_{s_0}) &= k \cdot (U \times \{0\}) \\ z_0 &\in j^{-1}(U \times B) \\ j(z_0) &:= (t_0, 0). \end{aligned}$$

It is clear that such a  $n$ -scale exists as  $z_0$  is a smooth point in  $|X_{s_0}|$ . Let  $S_1$  be a sufficiently small open neighbourhood of  $s_0$  in  $S$  and let  $f : S_1 \times U \rightarrow \text{Sym}^k(B)$  be the (continuous) classifying map for the family  $(X_s)_{s \in S_1}$  in the scale  $E$ . As  $j^{-1}(U \times \{0\})$  meets  $A$ , there exists a non empty polydisc  $U_2 \subset U$  such that  $U_2 \times \{0\}$  is contained in  $A$ . Then we may apply the proposition 4.3.2 to each scalar component of  $f$  in order to obtain that  $f$  is holomorphic on  $S_1 \times U$ . Moreover, as the same argument applies to any linear projection of  $U \times B$  to  $U$  near enough the vertical one; this implies that  $f$  is an isotropic map, up to shrink slightly  $U$ . This contradicts the fact that the point  $(s_0, z_0)$  is in the boundary of the open set  $A \cap (\{s_0\} \times |X_{s_0}|)$  of  $|X_{s_0}|$ . If the boundary of  $A \cap (\{s_0\} \times |X_{s_0}|)$  is contained in the singular set of  $|X_{s_0}|$ , we may apply the analytic extension criterion of [B-M 1] ch. IV, 3.1.7, and we obtain directly that  $A$  contains  $|X_{s_0}|$ . So in any case the family of cycles  $(X_s)_{s \in S}$  is analytic at  $s_0$ . As the graph  $|G|$  is, by assumption, quasi-proper on  $S$ , it is enough to use the next proposition (which is proved in [B.15-a] proposition 2.2.3) to conclude. ■

**Proposition 4.3.3** *Let  $Z$  and  $S$  be reduced complex spaces and let  $(X_s)_{s \in S}$  be a  $f$ -continuous family of  $n$ -cycles in  $Z$ . Assume that this family is analytic in  $s_0 \in S$ . Then there exists an open neighbourhood  $S'$  of  $s_0$  in  $S$  such that the family  $(X_s)_{s \in S'}$  is a  $f$ -analytic family of  $n$ -cycles in  $Z$ .*

#### 4.4 Proof of the theorem 1.0.2 and its corollary

We shall begin by a lemma which will give the case where the  $n$ -cycle  $X_0$  is compact.

**Lemma 4.4.1** *Let  $Z$  be a strongly  $n$ -concave reduced complex space. Then  $\mathcal{C}_n(Z)$  is open and closed in  $\mathcal{C}_n^f(Z)$ .*

PROOF. When  $Z$  is compact the assertion is obvious. So we may assume that  $Z$  is not compact and that  $\varphi(Z) = ]0, 2]$ . If an irreducible component  $\Gamma$  of a  $n$ -cycle  $X$  meets  $Z_\alpha := \{z \in Z / \varphi(z) > \alpha\}$  with  $\alpha < 1$ , this component is non compact in  $Z$ : indeed, if  $\Gamma$  is compact the function  $\varphi$  reach its minimum on it at a point where  $\varphi$  is strongly  $(n - 1)$ -convex and this gives a contradiction. So any compact cycle in  $Z$  is contained in the compact set  $\varphi^{-1}([1, 2])$  and conversely any cycle contained in this compact set is compact. So  $\mathcal{C}_n(Z)$  is open in  $\mathcal{C}_n^f(Z)$  because it is the subset of cycles which do not meet the compact set  $\varphi^{-1}([1/3, 2/3])$ . It is closed because its complement is the (open) subset of cycles meeting the open set  $\{z \in Z / \varphi(z) < 1\}$ . ■

So for such a  $Z$ ,  $\mathcal{C}_n(Z)$  is a union of connected components of  $\mathcal{C}_n^f(Z)$ .

We already know from [B.75] that it is a reduced complex space. The theorem 1.0.2 and its corollary are proved for these connected components.

THE CASE WHERE  $X_0$  IS NOT COMPACT. Fix  $X_0$  a non compact  $n$ -cycle in  $\mathcal{C}_n^f(Z)$  and choose an  $\alpha \in ]0, 1[$  which is not a critical value of  $\varphi$  and of the restriction of  $\varphi$  to  $|X_0|$ ; this is possible thanks to the Sard' lemma 3.1.1 and the fact that  $\varphi(|X_0|)$  contains  $]0, 1[$ . Then the constructions done in the proof of the theorem 3.5.1 give an open neighbourhood  $\mathcal{U}$  of  $X_0 \cap Z_\alpha$  in  $\mathcal{C}_n^f(Z_\alpha)$  and the theorem 4.1.2 gives an open neighbourhood  $\mathcal{V}$  of  $X_0$  in  $\mathcal{C}_n^f(Z)$  such that the restriction map

$$res : \mathcal{V} \rightarrow \mathcal{U}$$

is holomorphic and bijective. Indeed, it is enough to define  $\mathcal{V}$  by asking that all the adjusted scales used to build  $\mathcal{U}$  are adapted to a cycle in  $\mathcal{V}$  with the same degrees than  $X_0$  and adding the open condition not to meet the compact set  $\mathcal{K}$  to have the bijectivity.

We want to show now that the family of finite type  $n$ -cycles in  $Z$  parametrized by  $\mathcal{U}$  via the inverse of the map  $res$  is f-continuous. Remark that we have already a fix compact set  $\bar{\Delta}_1$  meeting any irreducible component of any cycle in this family. So it is enough to prove the continuity of this family in  $\mathcal{C}_n^{loc}(Z)$ . This is a consequence of the proposition 4.2.1. We obtain then the f-analyticity of this family using the theorem 4.3.1 as  $\mathcal{U}$  is a reduced complex space by the theorem 3.5.1. The holomorphic classifying map for this family is the map  $res$  which is then an isomorphism. This implies that  $\mathcal{C}_n^f(Z)$  is a reduced complex space in a neighbourhood of each of its points. ■

#### 4.5 A compactness criterion for the connected components of the reduced complex space $\mathcal{C}_n^f(Z)$ .

For a reduced complex space  $Z$  which is compact, the compactness of the connected components of  $\mathcal{C}_n(Z)$  is a consequence of the existence of a  $\mathcal{C}^1$   $2n$ -differential form on  $Z$  which is  $d$ -closed and such that its  $(n, n)$  part is positive definite in the Lelong sense. Indeed, this gives that the volume (computed with this  $(n, n)$  part) of the  $n$ -cycles is constant on connected components. The result follows then from Bishop's theorem [Bish.64].

In the case of a non compact strongly  $n$ -concave reduced complex space  $Z$  we have the following analogous result :

**Proposition 4.5.1** *Let  $Z$  be a reduced complex space which is strongly  $n$ -concave. Assume that there exists on  $Z$  a  $\mathcal{C}^1$   $2n$ -differential form  $\omega$  which is  $d$ -closed with compact support and such that its  $(n, n)$  part is positive definite in the Lelong sense in a neighbourhood of  $K := \varphi^{-1}([1, 2])$ , and which is everywhere non negative in the Lelong sense. Then the connected components of  $\mathcal{C}_n^f(Z)$  are compact.*

**PROOF.** For  $\alpha < 1$  near enough to 1 and for any continuous hermitian metric  $h$  on  $Z$  there exists a constant  $C$  such that the following inequality holds:

$$vol_h(X \cap Z_\alpha) \leq C \cdot \int_X \omega \quad \text{for any cycle } X \in \mathcal{C}_n^f(Z).$$

As the function  $X \mapsto \int_X \omega$  is locally constant on  $\mathcal{C}_n^f(Z)$  because  $d\omega = 0$  (see [B-M 1] ch.IV), we have an uniform bound for the volume of  $X \cap Z_\alpha$  for  $X$  in a given connected component of  $\mathcal{C}_n^f(Z)$ . This implies that the closure of the image of this connected component in  $\mathcal{C}_n^f(Z_\alpha)$  is compact, thanks to Bishop's theorem (see [Bish.64] or [B-M 1] ch.IV). But the restriction map  $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$  is a homeomorphism by the proposition 4.2.1, so the image of a connected component is closed and then compact. ■

## 5 References.

- [B.75] Barlet, D. *Espace analytique réduit* ... Séminaire F.Norguet 1974-1975, L.N. 482 (1975), p.1-158.
- [B.08] Barlet, D. *Reparamétrisation universelle de familles  $f$ -analytiques de cycles et théorème de  $f$ -aplatissement géométrique* Comment. Math. Helv. 83 n°4 (2008) p.869-888.
- [B.13] Barlet D. *Quasi-proper meromorphic equivalence relations*, Math. Z. (2013), vol. 273, p. 461-484.
- [B.15-a] Barlet, D. *Strongly quasi-proper maps and the  $f$ -flattening theorem*, arXiv: 1504.01579 (math.CV)
- [B-M 1] Barlet, D. et Magnusson, J. *Cycles analytiques complexes I : théorèmes de préparation des cycles*, Cours Spécialisés 22, Société Mathématique de France 2014.
- [B-M 2] Barlet, D. et Magnusson, J. *Cycles analytiques complexes II*, livre en préparation.
- [Bish.64] Bishop, E. *Conditions for the analyticity of certain sets*, Mich. Math. J. 11, (1964), p. 289-304.
- [S-T] Siu, Y.T. and Trautman, G. *Gap-Sheaves and Extensions of Coherent Analytic Subsheaves* Lecture Notes 172 (1971) Springer Verlag.