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Finiteness of the space of n –cycles for a n –concave complex space.

Daniel Barlet*

14/08/15

pour Yum-Tong Siu, avec mon amitié.

ABSTRACT. We show that for $n \geq 2$ the space of closed n –cycles in a strongly n –concave complex space has a natural structure of reduced complex space locally of finite dimension and represents the functor “analytic family of n –cycles” parametrized by banach analytic sets.

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1 Introduction.

The aim of this article is to show that in a reduced strongly n -concave¹ complex space Z with $n \geq 2$, the space of closed n -cycles (not assumed compact) is in a natural way endowed with a structure of complex space locally of finite dimension. With its tautological family of n -cycles it represents the functor “analytic family of n -cycles in Z ” and also the functor “f-analytic family of n -cycles in Z ” introduced in [B.08] (see also [B.13] and [B.15-a]) parametrized by a banach analytic set.

This answers a question asked to me by Y-T. Siu forty years ago.

I was able to solve this question thanks to the notion of f-analytic family introduced in *loc. cit.* and using the space $\mathcal{C}_n^f(Z)$ of finite type cycles with its natural topology. We obtain the following results.

Theorem 1.0.1 *Let $n \geq 2$ be an integer. let Z be a strongly n -concave² reduced complex space and let $\varphi : Z \rightarrow]0, 2]$ be its \mathcal{C}^2 exhaustion function. So it is proper and*

¹Our conventions will be precised below.

²See the section 2 for the definition.

strongly $(n - 1)$ -convex outside the compact set $K := \varphi^{-1}([1, 2])$. For $\alpha \in]0, 1[$ and X_0 a n -cycle in an open neighbourhood of the compact set $\varphi^{-1}([\alpha, 2])$ there exists $\beta \in]0, \alpha[$ such that, if $Z_\beta := \{z \in Z \mid \varphi(z) > \beta\}$, the cycle X_0 extends in an unique way to the open set Z_β and admits an open neighbourhood \mathcal{U} in the space $\mathcal{C}_n^f(Z_\beta)$ such that the ringed space defined by \mathcal{U} and the sheaf of holomorphic functions on \mathcal{U} is a (reduced) complex space locally of finite dimension.

Recall that a holomorphic function $h : \mathcal{U} \rightarrow \mathbb{C}$ on an open set in $\mathcal{C}_n^f(Z_\beta)$ is a continuous function on \mathcal{U} such that for any holomorphic map $f : S \rightarrow \mathcal{U}$ (corresponding to an f -analytic family of n -cycles in Z , see *loc. cit.*) of a banach analytic set S to \mathcal{U} the composed function $h \circ f$ is holomorphic.

Theorem 1.0.2 *Consider the same situation as in the previous theorem, and let now $X_0 \in \mathcal{C}_n^f(Z)$ be a finite type n -cycle in Z . Then there exists $\beta \in]0, 1[$ and open neighbourhoods \mathcal{V} and \mathcal{U} respectively of X_0 dans $\mathcal{C}_n^f(Z)$ and of $X_0 \cap Z_\beta$ dans $\mathcal{C}_n^f(Z_\beta)$ such that the restriction map*

$$r : \mathcal{V} \rightarrow \mathcal{U}$$

is well defined and bi-holomorphic.

The following corollary is of course the main point.

Corollary 1.0.3 *Consider the same situation as in the previous theorems. Then the ringed space given by the sheaf of holomorphic functions on $\mathcal{C}_n^f(Z)$ is a reduced complex space locally of finite dimension. Moreover, endowed with its tautological family of n -cycles it represents the functor “ f -analytic family of n -cycles in Z ” .*

In fact, we show that under the same hypothesis, the obvious map $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z)$ is an isomorphism of ringed space, so that $\mathcal{C}_n^f(Z)$ represents also the functor “analytic family of n -cycles in Z ” .

2 Hartogs figures.

2.1 Banachisation

For the analytic extension via Hartogs figures, the use of the Banach spaces $H(\bar{U}, \mathbb{C})$ of continuous functions, holomorphic inside on a compact polydisc \bar{U} is not well adapted. We shall use the Banach space $\mathcal{B}(U, \mathbb{C})$ of bounded holomorphic functions on U with the “sup” norm on U . Of course, $H(\bar{U}, \mathbb{C})$ is a closed banach subspace in $\mathcal{B}(U, \mathbb{C})$.

Proposition 2.1.1 *Let U be a relatively compact polydisc in \mathbb{C}^n and let S be a banach analytic set. Let $F : S \rightarrow \mathcal{B}(U, \mathbb{C})$ be a holomorphic map. Then the corresponding function $f : S \times U \rightarrow \mathbb{C}$ defined by $f(s, t) := F(s)[t]$ for (s, t) in $S \times U$ is holomorphic (and locally on S uniformly bounded on U).*

Conversely, if we have a holomorphic function $f : S \times U \rightarrow \mathbb{C}$ and an open polydisc

$U'' \subset\subset U$, the the associated map $F : S \rightarrow \mathcal{B}(U'', \mathbb{C})$, defined for (s, t) in $S \times U''$ by $F(s)[t] := f(s, t)$, is holomorphic.

PROOF. The evaluation function $ev : \mathcal{B}(U, \mathbb{C}) \times U \rightarrow \mathbb{C}$ is holomorphic as one may easily see by differentiation. Then the function f associated to F is the composition of the holomorphic maps $F \times id_U$ and ev . So it is holomorphic.

The converse is consequence of the linear isometric inclusion of $H(\bar{U}'', \mathbb{C})$ in $\mathcal{B}(U'', \mathbb{C})$ as $F'' : S \rightarrow H(\bar{U}'', \mathbb{C})$ is holomorphic as soon as f is holomorphic (see [B.75] or [B-M 2]). \blacksquare

2.2 n -Hartogs figure on a complex space.

For $\alpha \in (\mathbb{R}_+^*)^2$ let $M(\alpha)$ be the open set in \mathbb{C}^2 defined by

$$\begin{aligned} M(\alpha) &:= M^P(\alpha) \cup M^C(\alpha) \quad \text{with} \\ M^P(\alpha) &:= \{|t_1 - \alpha_1/2| < \alpha_1/4, \quad |t_2| < \alpha_2\} \\ M^C(\alpha) &:= \{|t_1| < \alpha_1, \quad \alpha_2/2 < |t_2| < \alpha_2\} \end{aligned}$$

and let also $\mathcal{M}(\alpha) := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i, i = 1, 2\}$.
for $\varepsilon > 0$ small enough. We define

$$\begin{aligned} M(\alpha)^\varepsilon &:= M^P(\alpha)^\varepsilon \cup M^C(\alpha)^\varepsilon \quad \text{with} \\ M^P(\alpha)^\varepsilon &:= \{|t_1 - \alpha_1/2| < \alpha_1/4 - \varepsilon/4, \quad |t_2| < \alpha_2 - \varepsilon\} \\ M^C(\alpha)^\varepsilon &:= \{|t_1| < \alpha_1 - \varepsilon, \quad \alpha_2/2 + \varepsilon/2 < |t_2| < \alpha_2 - \varepsilon\} \end{aligned}$$

and also $\mathcal{M}(\alpha)^\varepsilon := \{(t_1, t_2) \in \mathbb{C}^2 / |t_i| < \alpha_i - \varepsilon, i = 1, 2\}$.

For a polydisc of radius R in \mathbb{C}^m we shall denote by P^ε the polydisc with same center and radius $R - \varepsilon$ for $0 < \varepsilon < R$.

Definition 2.2.1 Let $n \geq 2$ and $p \geq 1$ two integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space Z . We shall call $\mathcal{H} := (\mathcal{M}, M, B, j)$ a n -**Hartogs figure in Z relative to the boundary of Δ** , the following data

- an embedding j of an open set Z' in Δ' into an open set of \mathbb{C}^{n+p} ,
- open sets $\mathcal{M} \subset\subset \Delta'$ and $M \subset\subset \Delta$ relatively compact in Z' ,
- a relatively compact polydisc B in \mathbb{C}^p

such that there exists $\alpha \in (\mathbb{R}_+^*)^2$ and a relatively compact polydisc V in \mathbb{C}^{n-2} with the following conditions:

- i) The map j induces a closed embedding of \mathcal{M} in $\mathcal{M}(\alpha) \times V \times B$.

- ii) The map j induces a closed embedding of M in $M(\alpha) \times V \times B$.
- iii) We have $j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \subset \Delta$.

Definition 2.2.2 Let $n \geq 2$ and $p \geq 1$ two integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space Z . Let $\mathcal{H} = (\mathcal{M}, M, B, j)$ be a n -Hartogs figure in Z relative to the boundary of Δ and let X_0 be a n -cycle in Δ . We shall say that \mathcal{H} is adapted to X_0 when the following condition is satisfied :

$$j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \cap |X_0| = \emptyset. \quad (@)$$

REMARKS.

1. Let \mathcal{H} be a n -Hartogs figure in Z relative to the boundary of Δ . If the open set $\Delta_1 \subset\subset \Delta'$ has a boundary $\partial\Delta_1$ near enough to $\partial\Delta$, then \mathcal{H} is again relative to the boundary of Δ_1 . For instance, if $\Delta := \{\varphi > 0\}$ where φ is a continuous proper function on Z , we may choose $\Delta_1 := \{\varphi > \varepsilon\}$ for $\varepsilon > 0$ small enough.
2. Note that the n -scale $E_{\mathcal{H}} := (M(\alpha) \times V, B, j)$ on Δ associated to \mathcal{H} is adapted to X_0 as soon as the n -Hartogs figure \mathcal{H} is adapted to X_0 .
3. If \tilde{X}_0 is a n -cycle in Δ' such that its restriction to Δ is equal to X_0 , the n -scale $E_{\tilde{\mathcal{H}}} := (\mathcal{M}(\alpha) \times V, B, j)$ on Δ' is adapted to \tilde{X}_0 if and only if the n -Hartogs figure \mathcal{H} is adapted to X_0 .

Note that the n -scale $E_{\tilde{\mathcal{H}}}$ is not a n -scale on Δ although the subset $\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B$ is contained in Δ .

Definition 2.2.3 In the situation above we define, for $\varepsilon > 0$ small enough, the n -Hartogs figure \mathcal{H}^ε on Δ as follows :

$$\mathcal{H}^\varepsilon := (\mathcal{M}^\varepsilon, M^\varepsilon, B, j)$$

where we use the notations

$$\mathcal{M}^\varepsilon := j^{-1}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon \times B) \quad \text{and also} \quad M^\varepsilon := j^{-1}(M(\alpha)^\varepsilon \times V^\varepsilon \times B).$$

It is obvious to see that when \mathcal{H} is relative to the boundary of Δ , then \mathcal{H}^ε is again relative to the boundary of Δ for all $\varepsilon > 0$ small enough.

Moreover, if \mathcal{H} is adapted to the n -cycle X_0 of Δ , so it is for \mathcal{H}^ε for all $\varepsilon > 0$ small enough.

Lemma 2.2.4 Let V be a relatively compact open polydisc in \mathbb{C}^q . The restriction map

$$\text{res} : \mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha) \times V, \mathbb{C})$$

is a linear isometry of Banach spaces.

Note that the restriction map induces also an isometry

$$\text{res} : \mathcal{B}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C}) \rightarrow \mathcal{B}(M(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C})$$

for all $\varepsilon > 0$ small enough.

PROOF. Let $f(v, t_1, t_2) := \sum_{m \in \mathbb{Z}} a_m(v, t_1) \cdot t_2^m$ the Laurent expansion of the holomorphic function $f : M^C(\alpha) \times V \rightarrow \mathbb{C}$. The holomorphic functions $a_m, m \in \mathbb{Z}$ on the product of V by the disc $\{|t_1| < \alpha_1\}$ are given by the formula

$$a_m(v, t_1) := \frac{1}{2i\pi} \cdot \int_{|z|=r} f(v, t_1, z) \cdot \frac{dz}{z^{m+1}} \quad \text{with } r \in]\alpha_2/2, \alpha_2[.$$

As the holomorphy of f on $M^P(\alpha) \times V$ implies that $a_m \equiv 0$ for each negative m on the open set $\{|t_1 - \alpha_1/2| < \alpha_1/4\} \times V$, we conclude that the functions a_m are identically zero for $m < 0$ and so f is holomorphic on $\mathcal{M}(\alpha) \times V$. This shows that the restriction map res is bijective (and also it is linear continuous) between the two Frechet spaces $\mathcal{O}(\mathcal{M}(\alpha) \times V)$ and $\mathcal{O}(M(\alpha) \times V)$; so it is an isomorphism of Frechet spaces.

Let us show that if f is in $\mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C})$, then $\text{res}(f)$, which belongs to the space $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$, has the same norm “sup”. For this purpose fix $\varepsilon > 0$ small enough. As $\mathcal{M}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ is a compact polydisc in $\mathcal{M}(\alpha) \times V$, the maximum of f on this compact is obtain at some point z in the distinguish boundary of it. But as z is also in the boundary of $M^C(\alpha)^\varepsilon \times \bar{V}^\varepsilon$, the desired equality follows.

Conversely, if g is in $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$, its analytic extension f to $\mathcal{M}(\alpha) \times V$ will be bounded on the boundary of $\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon$ by the sup of g on $M^C(\alpha)^\varepsilon \times V^\varepsilon$. So we obtain the equality of the “sup” norms for g and f respectively on $M(\alpha) \times V$ and $\mathcal{M}(\alpha) \times V$. ■

THE BANACH ANALYTIC SET $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$. Recall that if $p \geq 1$ and $k \geq 1$ are integers, there exists a closed embedding (in fact given by a polynomial map) of $\text{Sym}^k(\mathbb{C}^p) := (\mathbb{C}^p)^k / \mathfrak{S}_k$ into the vector space $E(k) := \bigoplus_{h=1}^k S^h(\mathbb{C}^p)$ given by the elementary tensorial symmetric functions³. If U is an open relatively compact polydisc in \mathbb{C}^n , the subset $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ is closed and banach analytic in the Banach space $\mathcal{B}(U, E(k))$. Indeed, if $Q : E(k) \rightarrow \mathbb{C}^N$ is a polynomial map such that $Q^{-1}(0) = \text{Sym}^k(\mathbb{C}^p)$, then the holomorphic map

$$\mathcal{Q} : \mathcal{B}(U, E(k)) \rightarrow \mathcal{B}(U, \mathbb{C}^N) \quad \text{defined by } f \mapsto Q \circ f$$

satisfies $\mathcal{Q}^{-1}(0) = \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$.

Nevertheless, be aware that for an open set Ω in $E(k)$ the subset $\mathcal{B}(U, \Omega)$ is generally not open in $\mathcal{B}(U, E(k))$ and so, for an open polydisc $B \subset\subset \mathbb{C}^p$, the subset

³see for instance [B-M 1] chapter I §4.

$\mathcal{B}(U, \text{Sym}^k(B))$ is not open in $\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$ in general.

The obvious map $H(\bar{U}, E(k)) \rightarrow \mathcal{B}(U, E(k))$ is a closed (linear) isometry and induces a holomorphic

$$i_U : H(\bar{U}, \text{Sym}^k(\mathbb{C}^p)) \hookrightarrow \mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p))$$

and for all $\varepsilon > 0$ the restriction

$$r : \mathcal{B}(U, E(k)) \rightarrow H(\bar{U}^\varepsilon, E(k))$$

is a (linear and continuous) compact map which induces a holomorphic map

$$\mathcal{B}(U, \text{Sym}^k(\mathbb{C}^p)) \rightarrow H(\bar{U}^\varepsilon, \text{Sym}^k(\mathbb{C}^p)).$$

NOTATIONS. Let $k \in \mathbb{N}$ and let $U' \subset\subset U \subset\subset \mathbb{C}^n$ and $B \subset\subset \mathbb{C}^p$ be polydiscs. We shall note $\Sigma_{U,U'}(k)$ the banach analytic set classifying the couples of an element in $H(\bar{U}, \text{Sym}^k(B))$ with its isotropy data on \bar{U}' . Recall that the natural projection $\Sigma_{U,U'}(k) \rightarrow H(\bar{U}, \text{Sym}^k(B))$ is a holomorphic homeomorphism (see [B.75] ch. III prop. 2 page 81 or [B-M 2] ch.V).

Our next result is the main tool for the analytic extension of n -cycles near a n -concave boundary.

Proposition 2.2.5 *Consider the open sets $\mathcal{M}(\alpha) \times V$ and $M(\alpha) \times V$ in \mathbb{C}^n . The inverse of the restriction map is a holomorphic isomorphism of analytic extension*

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p)).$$

Composed with the restriction to the compact set $\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ it sends the subset $H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$ in $H(\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon, \text{Sym}^k(B))$ for $\varepsilon > 0$ small enough.

Moreover, this holomorphic map induces a holomorphic map, again for $\varepsilon > 0$ small enough

$$\Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \longrightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

which factorizes the restriction map

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}, \mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}}(k)$$

through the restriction

$$\Sigma_{\mathcal{M}(\alpha) \times V, \mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k) \rightarrow \Sigma_{M(\alpha) \times V, M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}}(k).$$

PROOF. The lemma 2.2.4 gives that the map

$$\text{prlgt} : \mathcal{B}(M(\alpha) \times V, E(k)) \rightarrow \mathcal{B}(\mathcal{M}(\alpha) \times V, E(k))$$

is an isometry of Banach spaces. It is clear that its inverse sends the analytic subset $\mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$ in $\mathcal{B}(M(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$ and that if the map

$f \in \mathcal{B}(\mathcal{M}(\alpha) \times V, \text{Sym}^k(\mathbb{C}^p))$ takes values in $\text{Sym}^k(B)$, the same is true for its restriction. This proves the first part of the proposition.

In order to prove the second part, it is enough to show that a holomorphic map of a banach analytic set S with values in the subset

$$H(\bar{M}(\alpha) \times \bar{V}, \text{Sym}^k(B))$$

which is isotropic on the product of S with any relatively compact subset in the open set $M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$ will have an analytic extension which will be isotropic on any relatively compact open set in $\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$. So it will be isotropic on the closure of the open set $\mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}$. \blacksquare

Proposition 2.2.6 *Let $n \geq 2$ and $p \geq 1$ be integers and let $U_1 \times B_1$ the product of two polydiscs with centers 0 respectively in \mathbb{C}^n and \mathbb{C}^p . Note $(t_1, \dots, t_n, x_1, \dots, x_p)$ coordinates on $U_1 \times B_1$. Let φ be a real valued function of class \mathcal{C}^2 on $U_1 \times B_1$, such that*

$$\varphi(t, x) = \text{Re}(t_1) + \sum_{i=1}^n \rho_i \cdot |t_i|^2 + \sum_{j=1}^p \sigma_j \cdot |x_j|^2 + o(\|(t, x)\|^2) \quad (\text{@@})$$

where the real numbers $\rho_2, \sigma_1, \dots, \sigma_p$ are positive.

Let Δ be the open set $\{\varphi > 0\}$ in $U_1 \times B_1$ and let Δ' be an open neighbourhood of the compact set $\bar{\Delta}$ in \mathbb{C}^{n+p} . Let X_0 be a closed analytic subset of pure dimension n in Δ' such that each irreducible component of X_0 meets Δ and such that

$$|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}. \quad (*)$$

Then there exists $\alpha \in (\mathbb{R}_+^*)^2$ and polydiscs V and $B \subset\subset B_1$ with centers 0 respectively in \mathbb{C}^{n-2} and \mathbb{C}^p such that the following conditions are satisfied :

1. $\mathcal{M}(\alpha) \times V \times B \subset\subset \Delta'$;
2. $M(\alpha) \times V \times B \subset\subset \Delta$;
3. $\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B \subset \Delta$;
4. $|X_0| \cap (\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B) = \emptyset$ (this implies $|X_0| \cap (\overline{M(\alpha)} \times \bar{V} \times \partial B) = \emptyset$).

PROOF. Choose the polydisc $B \subset\subset B_1$ small enough in order that we have

$$X_0 \cap (\{0\} \times \bar{B}) \subset \{0\} \quad \text{and} \quad \{0\} \times \partial B \subset \Delta.$$

This is possible as we have $|X_0| \cap \{t_1 = \dots = t_n = 0\} \subset \{0\}$ and as φ is positive on a small enough punctured neighbourhood of the origin in the p -plane $\{t_1 = \dots = t_n = 0\} \times \mathbb{C}^p$. So we shall have

$$|X_0| \cap (\bar{W} \times \partial B) = \emptyset \quad \text{and} \quad \bar{W} \times \partial B \subset \Delta$$

for any small enough open neighbourhood W of the origin in \mathbb{C}^n . A immediate consequence is that conditions 1, 3 and 4 will be satisfied as soon as α and V are small enough.

In order to check the condition 2, let us remark first that, up to choose the real numbers $\rho'_1 > |\rho_1|$, $\rho'_2 \in]0, \rho_2[$ and $r > \sup_{i \geq 3} |\rho_i|$, we obtain on $W \times B$ chosen small enough

$$\varphi(t, x) \geq \operatorname{Re}(t_1) - \rho'_1 \cdot |t_1|^2 + \rho'_2 \cdot |t_2|^2 - r \cdot \left(\sum_3^n |t_i|^2 \right). \quad (**)$$

with strict inequality as soon as $x \neq 0$. Then for

$$V_\varepsilon = \{(t_3, \dots, t_n) / \sum_3^n |t_i|^2 < \varepsilon^2\}$$

the following inequalities hold

$$\varphi(t, x) \geq \frac{1}{4}\alpha_1 - \rho'_1 \cdot \alpha_1^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^P(\alpha) \times V_\varepsilon \times B} \quad (1)$$

$$\varphi(t, x) \geq -\alpha_1 - \rho'_1 \cdot \alpha_1^2 + \frac{1}{4}\rho'_2 \cdot \alpha_2^2 - r \cdot \varepsilon^2 \quad \text{on} \quad \overline{M^C(\alpha) \times V_\varepsilon \times B} \quad (2)$$

for α and ε small enough in order that $\mathcal{M}(\alpha) \times V_\varepsilon$ is contained in W .

This allows to fix α_1, α_2 and ε .

Now we shall choose α_1 and ε smaller in order to satisfy the following conditions :

$$8\alpha_1 < \rho'_2 \cdot \alpha_2^2 \quad , \quad \alpha_1 < \frac{1}{8\rho'_1} \quad \text{and} \quad \varepsilon^2 < \frac{1}{8r}\alpha_1. \quad (3)$$

To obtain $M^P(\alpha) \times V_\varepsilon \times B \subset \subset \Delta$ it is enough to show that on $M^P(\alpha) \times V_\varepsilon \times B$ we have, if we let $\alpha_1 = u \cdot \alpha_2$ and $\varepsilon^2 = v \cdot \alpha_1 = uv \cdot \alpha_2$

$$\frac{1}{4} > \rho'_1 \cdot \alpha_1 + r \cdot v.$$

Indeed, as we assumed $\rho'_1 \cdot \alpha_1 < \frac{1}{8}$ and $r \cdot \varepsilon^2 < \frac{1}{8} \cdot \alpha_1$ (so $r \cdot v < 1/8$) the first condition holds.

In order to satisfy $M^C(\alpha) \times V_\varepsilon \times B \subset \subset \Delta$ it is enough to show that on $M^C(\alpha) \times V_\varepsilon \times B$ we have

$$\frac{1}{4}\rho'_2 \cdot \alpha_2 > u + \rho'_1 u^2 \alpha_2 + r \cdot uv.$$

But our condition implies

$$\rho'_1 u^2 \alpha_2 < \frac{1}{8} \cdot u \quad r \cdot uv < \frac{1}{8} \cdot u$$

which gives $u + \rho'_1 u^2 \cdot \alpha_2 + r \cdot uv < 2 \cdot u$. The condition $8\alpha_1 < \rho'_2 \cdot \alpha_2^2$ which implies $2u < \frac{1}{4}\rho'_2 \cdot \alpha_2$, allows to conclude. ■

REMARKS.

1. We only used the condition (*) for X_0 and the inequality (**) for φ in a neighbourhood of the origin in the proof above.
2. Sufficient conditions on $\varphi \in \mathcal{C}^2$ to satisfy (@@) are:

- i) The origin is not a critical point of φ .
- ii) The Levi form of φ at 0 has, at most, $(n - 1)$ non positive eigenvalues in the complex tangent hyperplane to the real hypersurface $\{\varphi(z) = 0\}$; the existence of $\varphi \in \mathcal{C}^2$ such that $\Delta = \{\varphi > 0\}$ and satisfying these two conditions is equivalent to the fact that the open set Δ has a strongly n -concave (see the definition 2.3.3 given below) smooth boundary near the origin. Indeed, if φ is not critical at 0 and has a Levi form at 0 with, at most, $(n - 1)$ non positive eigenvalues values in the complex hyperplane tangent to the real hypersurface $\{\varphi(z) = 0\}$, its order 2 Taylor expansion at the origin is written, in suitable local holomorphic coordinates (τ, x)

$$\varphi(\tau, x) = Re(\tau_1) + Re(Q(\tau, x)) + \sum_{i=1}^n \rho_i \cdot |\tau_i|^2 + \sum_{j=1}^p \sigma_j \cdot |x_j|^2 + o(\|(\tau, x)\|^2)$$

where Q is a holomorphic homogeneous degree 2 polynomial and where the real numbers ρ_2 and $\sigma_j, j \in [1, p]$ are positive. Define new local holomorphic coordinates

$$t_1 := \tau_1 + Q(\tau, x), t_i := \tau_i \quad \text{for } i \in [2, n] \quad \text{and} \quad x_j := x_j \quad \text{for } j \in [1, p].$$

Then we obtain (@@).

3. The condition (*) implies that X_0 has no local irreducible component at 0 contained in the hyperplane $\{t_1 = 0\}$. In fact, as the coordinate t_1 is chosen in order to suppress the real part of the holomorphic homogeneous degree 2 term in the order 2 Taylor expansion of φ at 0 (see the previous remark), we want that no local irreducible component of X_0 at the origin is contained in the complex hypersurface $\tau_1 + Q(\tau, x) = 0$ locally defined near 0 for φ given. Then, as soon as the restriction $\varphi|_{X_0}$ has not 0 as a critical point, the condition (*) could be realized when φ is strongly $(n - 1)$ -convex near 0.
4. One may easily see that under our hypothesis, the cycle X_0 meets the open set Δ when it contains 0. Indeed, the analytic subset

$$\{t_1 = t_3 = \dots = t_n = 0\} \cap |X_0|$$

is non empty, has dimension at least 1 and meets the complement of Δ only at the origin.

Of course, assuming that $0 \in X_0$, the proposition shows that, in fact, X_0 contains a branched covering of degree $k \geq 1$ of $M^P(\alpha) \times V_\varepsilon$ inside

$$M^P(\alpha) \times V_\varepsilon \times B \subset \subset \Delta.$$

5. In the situation of the proposition 2.2.6, for any continuous family $(X_s)_{s \in S}$ of n -cycles in Δ parametrized by a banach analytic set S such that $X_{s_0} = X_0 \cap \Delta$, there exists an open neighbourhood S' of s_0 in S , such that for each $s \in S'$ the condition 4 remains true after analytic extension of the cycles (see the proposition 2.2.5), because, thanks to the condition 3, $\mathcal{M}(\alpha) \times \bar{V} \times \partial B$ is a compact subset in Δ . \square

2.3 q -concave open sets.

Definition 2.3.1 Let $\varphi : U \rightarrow \mathbb{R}$ be real valued \mathcal{C}^2 function on an open set U in \mathbb{C}^N . We shall say that φ is **strongly q -convex** when its Levi form at each point of U has at most q non positive eigenvalues.

So, with this definition a strongly 0-convex function is a strongly plurisubharmonic function.

Definition 2.3.2 Let $\varphi : Z \rightarrow \mathbb{R}$ a real valued \mathcal{C}^2 function on a reduced complex space Z . We shall say that φ is **strongly q -convex** if locally near each point of Z it can be induced by a \mathcal{C}^2 strongly q -convex function in a local embedding in an open set of an affine space.

Remark that a strongly q -convex function on an irreducible complex space of dimension at least equal to $q + 1$ has no local maximum because there exists at any point a germ of curve on which the restriction of φ is strongly p.s.h.

Definition 2.3.3 Let Z be a reduced complex space and let Δ be a relatively compact open set in Z . We shall say that Δ has a **smooth \mathcal{C}^2 boundary** when for each point z in $\partial\Delta$ there exists a local smooth embedding of an open neighbourhood W of z in an open set U of an affine space and an open set D with smooth \mathcal{C}^2 boundary in U such that $\Delta = W \cap D$.

We shall say that the open set $\Delta \subset Z$ with smooth \mathcal{C}^2 boundary is **strongly q -concave in a point** $z \in \partial\Delta$ if, in some local smooth embedding of Z around z , one can define Δ in Z near z as the subset $\{\varphi > 0\} \cap Z$ where φ is a real valued \mathcal{C}^2 function such that

1. $d\varphi_z \neq 0$ on the Zariski tangent space $T_{Z,z}$ de Z en z .
2. The restriction of the Levi form at z of φ to the complex hyperplane tangent at z to the real hypersurface $\{\varphi(x) = \varphi(z)\}$ has at most $(q - 1)$ non positive eigenvalues.

We shall say that Δ is **strongly q -concave** if Δ is strongly q -concave near each point in $\partial\Delta$.

REMARK. Up to compose φ with a real strictly increasing (non critical) convex \mathcal{C}^2 function (this does not change the level sets $\{\varphi = \text{constant}\}$), it is possible to assume that $c \circ \varphi$ is \mathcal{C}^2 strongly $(q - 1)$ -convex (and non critical) near z . Conversely, if φ is a real valued \mathcal{C}^2 function which is strongly $(q - 1)$ -convex and not critical near a point $z \in Z$, the open set $\{\varphi(x) > \varphi(z)\}$ has a strongly q -concave boundary in a neighbourhood of z .

With this terminology, using the remarks above, we may give the following reformulation of the proposition 2.2.6:

Corollary 2.3.4 *Let $n \geq 2$ and $p \geq 1$ be integers, let Z be a reduced complex space of pure dimension $n + p$ and let $\Delta := \{\varphi > 0\}$ be an open set with \mathcal{C}^2 smooth boundary in Z . Let X_0 be a n -cycle in an open neighbourhood of a point $z \in \partial\Delta$ such that the function $\varphi|_{X_0}$ is not critical at z .*

Assume that Δ is strongly n -concave near z ; then there exists a n -Hartogs figure $\mathcal{H} := (\mathcal{M}, M, B, j)$ relative to the boundary of Δ , adapted to X_0 , and such that the point z lies in \mathcal{M} . ■

PROOF. Using a local embedding of an open neighbourhood of z in an open set of the Zariski tangent space $T_{Z,z}$, it is enough to prove the corollary in the case where Z is an open set in $\mathbb{C}^{n+p'}$, with $p' \geq p$ an integer. As we may choose the function φ strongly $(n - 1)$ -convex such that $d\varphi_z \neq 0$ thanks to the previous remarks, we can choose local coordinates near z in order to be in the situation of the proposition 2.2.6 in the case $z \in X_0$, as we assumed that $\varphi|_{X_0}$ is not critical at z . In this case the proposition gives the result.

If z is not in X_0 , the same construction in an open neighbourhood of z with no limit point in X_0 allows to conclude, and in this case the degree of X_0 in the (adapted) scale $E_{\mathcal{H}}$ will be zero. ■

Definition 2.3.5 *Let $n \geq 2$ and $p \geq 1$ be integers and let $\Delta \subset \subset \Delta'$ be two open sets in a reduced complex space Z . Let $\mathcal{H} = (\mathcal{M}, M, B, j)$ and $\mathcal{H}' := (\mathcal{H}', M', B, j)$ be two n -Hartogs figures in Z relative to the boundary of Δ given by the same (local) embedding j and having the same polydisc $B \subset \subset \mathbb{C}^p$. We shall say that these two n -Hartogs figures are **boxed** when we have $\mathcal{M}'(\alpha') \subset \subset \mathcal{M}(\alpha)$, $M'(\alpha') \subset \subset M(\alpha)$ and $V' \subset \subset V$.*

For instance, if $\varepsilon > 0$ is small enough, the n -Hartogs figures $(\mathcal{H}, \mathcal{H}^\varepsilon)$ are boxed (see the definition 2.2.3).

Proposition 2.3.6 *Let $n \geq 2$ and $p \geq 1$ be integers, let Z be a reduced complex space of pure dimension $n + p$ and let $\Delta \subset \subset Z$ be an open set with smooth \mathcal{C}^2 boundary in Z which is strongly n -concave. Assume that $\Delta := \{\varphi > 0\}$ and let X_0 be a n -cycle in an open neighbourhood Δ' of the compact set $\bar{\Delta}$, such that any irreducible component of X_0 meets Δ . Then there exists a finite family of*

boxed n -Hartogs figures $(\mathcal{H}'_a, \mathcal{H}_a)_{a \in A}$ relative to the boundary of Δ , such that the following conditions hold:

1. The \mathcal{M}'_a for $a \in A$ cover the boundary $\partial\Delta$.
2. For each $a \in A$ the \mathcal{H}_a and \mathcal{H}'_a are adapted to X_0 .
3. For each $a \in A$ any irreducible component of X_0 meeting \mathcal{M}_a meets the open set M'_a .
4. No **compact** irreducible component of $X_0 \cap \Delta$ meets the union of the compact sets $\bar{\mathcal{M}}_a, a \in A$.

REMARK. Let X_0 be any n -cycle in Δ' . Up to shrink the open set Δ' around the compact set $\bar{\Delta}$, we can assume that the cycle X_0 has only finitely many irreducible components in Δ' and that each of them is not compact and meets $\bar{\Delta}$. Note that, in this situation, any irreducible component of X_0 meets Δ , because the strong n -concavity of $\partial\Delta$ (see the remark 3 following the proposition 2.2.6).

Corollary 2.3.7 *In the situation of the proposition 2.3.6, if we assume that the open set Δ' contains $\bar{\Delta}$, any irreducible component Γ of the cycle X_0 in Δ' satisfies for all $a \in A$ and all $\eta > 0$ small enough:*

$$\Gamma \cap (\mathcal{M}_a^\eta \times V_a^\eta \times B_a) = \text{prlgt}_a[\Gamma \cap (M_a \times V_a \times B_a)]$$

where $\text{prlgt}_a : H(\bar{M}_a \times \bar{V}_a, \text{Sym}^k(B_a)) \rightarrow H(\bar{\mathcal{M}}_a^\eta \times \bar{V}_a^\eta, \text{Sym}^k(B_a))$ is the holomorphic map of analytic extension built in the proposition 2.2.5.

REMARK. In the situation of the previous corollary there exists, for each $a \in A$, a holomorphic extension map which lifts the map prlgt_a :

$$i\text{prlgt}_a : \Sigma_{M_a, IM_a}(k) \longrightarrow \Sigma_{\mathcal{M}_a^\eta, I\mathcal{M}_a^\eta}(k)$$

and it allows to extend in this setting an analytic family of branched coverings in M_a parametrized by a banach analytic set S and which is isotropic on $S \times M'_a$ to an analytic family of branched coverings in \mathcal{M}_a^η which is isotropic on $S \times \mathcal{M}'^\eta_a$.

PROOF OF THE PROPOSITION 2.3.6. The corollary 2.3.4 and the remark following it implies the existence, for each $z \in \partial\Delta$ of a n -Hartogs figure \mathcal{H}_z relative to the boundary of Δ , contained in Δ' and satisfying the following properties:

- i) $z \in \mathcal{M}_z$;
- ii) \mathcal{H}_z is adapted to X_0 ;
- iii) each irreducible component of X_0 meeting \mathcal{M}_z meets the open set M_z ;

iv) No compact irreducible component of $X_0 \cap \Delta$ meets $\bar{\mathcal{M}}_z$.

As the open sets \mathcal{M}_z cover the compact set $\partial\Delta$ we can find a finite sub-cover. Then the properties 1, 2, 3 and 4 are consequences of i), ii), iii) and iv) by letting $\mathcal{H}' := \mathcal{H}^\varepsilon$ and by choosing $\varepsilon > 0$ small enough. ■

PROOF OF THE COROLLARY 2.3.7. Let Γ be an irreducible component of X_0 meeting \mathcal{M}_a for some $a \in A$. Then Γ meets M_a . As Γ does not meet $\bar{M}(\alpha)_a \times \bar{V}_a \times \partial B_a$ because \mathcal{H}_a is adapted to X_0 , the intersection $\Gamma \cap \bar{M}_a$ is the graph of an element $\gamma \in H(\bar{M}(\alpha)_a \times \bar{V}_a, \text{Sym}^{k_a}(B_a))$ with $k_a \in \mathbb{N}^*$ ⁴.

The closed analytic subset Y of the open set \mathcal{M}_a^n defined by analytic extension, $Y := \text{pr}lgt_a[\Gamma \cap M_a]$, is not empty, of pure dimension n and is contained in Γ . So it is an union of irreducible components of $\Gamma \cap \mathcal{M}_a^n$. But it contains a non empty open set in each irreducible component of this branched covering. So these two analytic subsets coincide.

If an irreducible component of X_0 does not meet any \bar{M}_a it has to be compact and contained in Δ . In this case the desired equality is obvious. ■

3 The extension and finiteness theorem.

3.1 Some useful lemmas.

The Sard's lemma below is more or less classical.

Lemma 3.1.1 *Let Z be a reduced complex space and let $\varphi : Z \rightarrow \mathbb{R}$ be a real valued \mathcal{C}^1 function. Then the set of critical values of φ has Lebesgue measure 0.*

PROOF. First note that a point $z \in Z$ is critical for φ if, by definition, the differential of φ vanishes on $T_{Z,z}$, the Zariski tangent space of Z at z . Remember also that a complex space is, by definition, countable at infinity ; so Z and its singular locus have only countably many irreducible components. As a countable union of sets of measure 0 is again of measure 0, it is enough to prove the lemma when Z is irreducible. We shall prove the lemma by induction on the integer $\dim Z$. The case $\dim Z = 0$ is obvious. Assume the lemma true for $\dim Z \leq n - 1$ for some integer $n \geq 1$ and take an irreducible complex space Z of dimension n . The singular set S of Z has dimension at most $(n - 1)$, and for each irreducible component S_i of S the image of the critical set of $\varphi|_{S_i}$ has measure 0. So the critical set of $\varphi|_S$ is again of measure 0. But a critical point of φ which belongs to S is a critical point of $\varphi|_S$. So it is enough to show that the set of critical values of φ restricted to the complex connected manifold $Z \setminus S$ has measure 0. This is the classical Sard's lemma. ■

⁴as it exists some $(m, v) \in \bar{M}(\alpha)_a \times \bar{V}_a$ such that $\Gamma \cap (\{m, v\} \times B_a) \neq \emptyset$.

Lemma 3.1.2 *Let V be an open set and K be a compact set in $\bar{U} \times \bar{B}$. The subset \mathcal{V} in $H(\bar{U}, \text{Sym}^k(B))$ consisting of the X such that any irreducible component meeting K meets V is an open set in $H(\bar{U}, \text{Sym}^k(B))$.*

PROOF. First let us clarify the meaning of an irreducible component of X in $H(\bar{U}, \text{Sym}^k(B))$: we call irreducible component of such a X the closure in $\bar{U} \times B$ of an irreducible component of the branched covering of U defined by the projection of $X \cap (U \times B)$ on U .

Let X_0 be such that any irreducible component of X_0 which meets K meets V , and assume that $(X_\nu)_{\nu \geq 1}$ is a sequence converging to X_0 such that for each $\nu \geq 1$ there exists an irreducible component Γ_ν of X_ν meeting K but not V . Up to pass to a subsequence, we may assume that the sequence $(\Gamma_\nu)_{\nu \geq 1}$ converges uniformly on any compact of $U \times B$ to a non empty n -cycle Γ with closure contained in X_0 and which is a branched covering of U . Then $\bar{\Gamma}$ meets K and not V . Indeed, if (t_0, x_0) would be in $\bar{\Gamma} \cap V$, there exists open neighbourhoods U_1 and B_1 of t_0 et x_0 respectively in \bar{U} and \bar{B} such that $U_1 \times B_1$ is contained in V . But then, as $U_2 := U_1 \cap U$ and $B_2 := B_1 \cap B$ are non empty open sets, and for t_2 in U_2 the fibers of the Γ_ν at t_2 for ν big enough will meet $\{t_2\} \times B_2$ and so V . As at least one irreducible component of Γ meets K without meeting V and as its closure is an irreducible component of X_0 , this gives a contradiction. ■

Of course, in the case $V = \emptyset$ we get back the fact that the subset of $X \in H(\bar{U}, \text{Sym}^k(B))$ which do not meet K is open.

Lemma 3.1.3 *Let Z be a complex space and let $(\mathcal{U}_i)_{i \in I}$ be an open covering of Z . Assume that for each $i \in I$ a closed n -cycle X_i is given in \mathcal{U}_i . Assume that the following patching condition holds :*

$$\forall (i, j) \in I^2 \quad X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$$

as an equality of cycles in $\mathcal{U}_i \cap \mathcal{U}_j$.

Then there exists an unique closed n -cycle X in Z such that for each $i \in I$ we have $X \cap \mathcal{U}_i = X_i$.

For the easy proof see [B.75] or [B-M 2].

The following variant will be used.

Lemma 3.1.4 (Variant.) *In the situation of the previous lemma replace the patching condition by the following two conditions :*

1. *For each couple $(i, j) \in I^2$ an open subset $W_{i,j} \subset \subset \mathcal{U}_i \cap \mathcal{U}_j$ is given and we ask that $X_i \cap W_{i,j} = X_j \cap W_{i,j}$.*
2. *For each couple $(i, j) \in I^2$ we ask that any irreducible component of the cycle $X_i \cap \mathcal{U}_j$ meets the open set $W_{i,j}$.*

Then the conclusion is the same.

PROOF. Let Γ be an irreducible component with multiplicity δ in the cycle $X_i \cap \mathcal{U}_j$. Let Γ' be the irreducible component of X_i which contains Γ , and put $X_i = X'_i + \delta\Gamma'$. Then Γ' meets $W_{i,j}$ and there exists a closed analytic subset Y of pure dimension n in $|X_j|$ such that its restriction to $W_{i,j}$ is equal to $\Gamma' \cap W_{i,j}$: indeed, Y is the union of the irreducible components of X_j containing a non empty open set in $\Gamma' \cap W_{i,j}$. Note that each of these irreducible components of X_j has multiplicity δ in the cycle X_j . Then put $X_j = X'_j + \delta Y$. We see that the cycles X'_i and X'_j respectively in \mathcal{U}_i and \mathcal{U}_j satisfy again the patching condition $X'_i \cap W_{i,j} = X'_j \cap W_{i,j}$.

This allows, for fixed (i, j) , to make a descending induction on the number (necessarily finite as $W_{i,j}$ is relatively compact) of irreducible components of $X_i \cap \mathcal{U}_j$, to show that the condition $X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$ holds. This reduces this lemma to the previous one. \blacksquare

3.2 Adjusted scales.

Definition 3.2.1 1. Let Z be a complex space. We shall call **adjusted n -scale on Z** , written down $\mathbb{E} := (U, U', U'', B, B'', j)$, the data of a n -scale on Z , $E := (U, B, j)$, with additional polydiscs $U'' \subset\subset U' \subset\subset U$ and $B'' \subset\subset B$. We call E the **underlying scale of the adjusted scale \mathbb{E}** .

2. We shall say that **the adjusted scale \mathbb{E} is adapted to a n -cycle X in Z** when we have

$$j^{-1}(\bar{U} \times (\bar{B} \setminus B'')) \cap |X| = \emptyset.$$

Note that this implies that the underlying scale E is adapted to X , but this condition is more restrictive.

3. When the adjusted scale \mathbb{E} is adapted to the n -cycle X , we shall call **degree of X in \mathbb{E}** the degree of X in E .

4. We shall call **domain of the adjusted scale**, written down $D(\mathbb{E})$, or more simply, $D(E)$, the open set $j^{-1}(U \times B)$ in Z which is also the domain of the scale E .

5. We shall call **domain of isotropy of the adjusted scale**, written down $D'(\mathbb{E})$, the open set $j^{-1}(U' \times B)$ in Z .

6. We shall call **domain of patching of the adjusted scale**, written down $D''(\mathbb{E})$, the open set $j^{-1}(U'' \times B'')$ in Z .

REMARKS.

1. The open set $D''(\mathbb{E})$ is relatively compact in $D'(\mathbb{E})$.

2. When a n -scale E is given, for any compact set K in $D(E)$, there exists an adjusted n -scale \mathbb{E} such E is the underlying scale of \mathbb{E} and with $K \subset D'(\mathbb{E})$. Moreover, if E is adapted to a n -cycle X_0 in Z , we may choose \mathbb{E} in order that it is adapted to X_0 .
3. As for $X \in \mathcal{C}_n^{loc}(Z)$ the condition to avoid a compact subset is open in $\mathcal{C}_n^{loc}(Z)$, when the adjusted scale \mathbb{E} is adapted to a cycle X_0 there exists an open neighbourhood, written down $\Omega_k(\mathbb{E})$, of X_0 in $\mathcal{C}_n^{loc}(Z)$ such that $\Omega_k(\mathbb{E})$ is the subset of all n -cycles X in Z for which \mathbb{E} is adapted and $\deg_{\mathbb{E}}(X) = k$ where $k := \deg_{\mathbb{E}}(X_0)$.

Let Z be a reduced complex space and let \mathbb{E} be an adjusted scale on Z . For a given integer k consider the continuous map sending a branched covering in $H(\bar{U}, Sym^k(B''))$ to its isotropy data on \bar{U}'

$$T : H(\bar{U}, Sym^k(B'')) \rightarrow H(\bar{U}', F \otimes E').$$

The graph $\Sigma_{U,U'}(k)$ of this map is a banach analytic set⁵, thanks to [B.75], proposition 2 p.81 (see also [B-M 2]).

The set of couples $(f, T(f))$ in $\Sigma_{U,U'}(k)$ for which the associated branched covering is contained in $j(Z)$ is a closed banach analytic subset of $\Sigma_{U,U'}(k)$ being the pull-back by the projection of the subset of elements in $H(\bar{U}, Sym^k(B''))$ contained in $j(Z)$ which is a closed banach analytic subset of $H(\bar{U}, Sym^k(B''))$ by the proposition 4 p.27 of [B.75] (see also [B-M 2] ch. V).

Definition 3.2.2 We shall denote $\mathcal{G}_k(\mathbb{E})$ this banach analytic set and we shall call it the **the k -th classifying space of the adjusted n -scale \mathbb{E} on Z** .

We have then a tautological family of n -cycles in the open set $D(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$. It is an analytic family of cycles in the open set $D'(\mathbb{E})$, in the sense of [B.75], and the fact that, for $k \geq 1$, locally on $\mathcal{G}_k(\mathbb{E})$, any irreducible component of a branched covering in this family meets $\bar{U}'' \times \bar{B}''$ implies that we have a f-analytic family of cycles in $D'(\mathbb{E})$.

Be careful that the tautological family of cycles on the open set $D(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$ is not, in general, an analytic family of n -cycles ; see the example of [B.75] p.83 (and also [B-M 1] ch.IV).

The next lemma is an obvious consequence of *loc. cit.*

Lemma 3.2.3 Let \mathbb{E} be an adjusted n -scale on a reduced complex space Z and let k be an integer. The tautological family of n -cycles in the open set $D'(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$ has the following “almost universal” property:

⁵homeomorphic to $H(\bar{U}, Sym^k(B''))$ via the projection !

- For any analytic family of n -cycles $(X_s)_{s \in S}$ in Z parametrized by a banach analytic set S such that for each $s \in S$ the adjusted scale \mathbb{E} is adapted to X_s with $\deg_{\mathbb{E}}(X_s) = k$, there exists an unique holomorphic map

$$f : S \rightarrow \mathcal{G}_k(\mathbb{E})$$

such that the pull-back by f of the tautological family is the restriction to the open set $D'(\mathbb{E})$ of the given family.

Of course, conversely, a holomorphic map gives a f-analytic family of n -cycles on the open set $D'(\mathbb{E})$.

Note that the pull-back family is in fact defined on the open set $D(\mathbb{E})$ but, as already noticed above, it may not be analytic outside $D'(\mathbb{E})$.

As a consequence of this “almost universal” property, we obtain that for any analytic family $(X_s)_{s \in S}$ of n -cycles in Z such that for a point $s_0 \in S$ the adjusted n -scale \mathbb{E} is adapted to the cycle X_{s_0} with $\deg_{\mathbb{E}}(X_{s_0}) = k$, there exists an open neighbourhood S' of s_0 in S such that the previous lemma applies for the family parametrized by S' . So we shall have a holomorphic classifying map $f : S' \rightarrow \mathcal{G}_k(\mathbb{E})$ in this situation.

We shall generalize now the concept of classifying space to a finite family of adjusted n -scales.

Definition 3.2.4 Consider a reduced complex space Z and a finite family of adjusted n -scales $(\mathbb{E}_i)_{i \in I}$ on Z . Assume that they are adapted to a given finite type n -cycle \hat{X}_0 in Z . Assume that any irreducible component of \hat{X}_0 meets the open set $W'' := \cup_{i \in I} D''(\mathbb{E}_i)$.

We shall call **patching data for \hat{X}_0 associated to the family $(\mathbb{E}_i)_{i \in I}$** , written down $\mathcal{R}((\mathbb{E}_i)_{i \in I}, F)$ or more simply \mathcal{R} when there is no ambiguity, a finite collection of n -scales $(F_{i,j,h})$ for $(i,j) \in I^2, i \neq j$, where h belongs to a finite set $H(i,j)$ for each couple $(i,j) \in I^2, i \neq j$, such that the following properties hold:

i) $F_{i,j,h}$ is a n -scale on the open set $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

ii) The n -scales $F_{i,j,h}$ are adapted to \hat{X}_0 .

We shall say that the patching data \mathcal{R} are **complete** when, moreover, the following condition also holds :

iii) For each $i \neq j$ given, the union of domains of the scales $F_{i,j,h}, h \in H(i,j)$, covers the compact subset $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$ of $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

NOTATIONS.

1. In the sequel, when we shall consider a reduced complex space Z and a finite family of adjusted n -scales $(\mathbb{E}_i)_{i \in I}$, adapted to a finite type n -cycle \hat{X}_0 in Z , such that any irreducible component of \hat{X}_0 meets the open set

$$W'' := \cup_{i \in I} D''(\mathbb{E}_i),$$

we shall say that the family $(\mathbb{E}_i)_{i \in I}$ is **convenient** for \hat{X}_0 .

2. In this setting we shall use the following definitions :

- $W := \cup_{i \in I} D(\mathbb{E}_i)$;
- $W' := \cup_{i \in I} D'(\mathbb{E}_i)$;
- $W'' := \cup_{i \in I} D''(\mathbb{E}_i)$;
- $K := \cup_{i \in I} j_i^{-1}(\bar{U}_i \times (\bar{B}_i \setminus B_i''))$.
- When the family $(\mathbb{E}_i)_{i \in I}$ is convenient for a finite type n -cycle \hat{X}_0 , \mathcal{K} will be a compact neighbourhood of K disjoint from \hat{X}_0 .

3. For $\tilde{X} \in \prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$ we shall denote X^i the closed cycle in $D(\mathbb{E}_i)$ associated to the i -th component of \tilde{X} .

Lemma 3.2.5 *Let $(\mathbb{E}_i)_{i \in I}$ a finite family of adjusted n -scales on Z , convenient for a n -cycle \hat{X}_0 of finite type in Z , and let \mathcal{R} be some corresponding complete patching data. There exists an open neighbourhood \mathcal{V} of the image \tilde{X}_0 of \hat{X}_0 in the product $\prod_{i \in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$ such that for each $\tilde{X} \in \mathcal{V}$ we have the following properties:*

1. No X^i meets the compact set \mathcal{K} .
2. For each $i \in I$, any irreducible component of X^i meeting $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$ with $j \neq i$, meets the open set $\cup_h D(F_{i,j,h})$.
3. For each (i, j, h) the scale $F_{i,j,h}$ is adapted to X^i and X^j .
4. For each (i, j, h) we have $\deg_{F_{i,j,h}}(X^i) = \deg_{F_{i,j,h}}(X^j) = \deg_{F_{i,j,h}}(\hat{X}_0) = k_{i,j,h}$.

PROOF. The conditions 1, 3 and 4 are clearly open. An easy consequence of the condition 1, of the inclusion ii) of the definition 3.2.4 and of the lemma 3.1.2 is that the condition 2 is also open. ■

For a n -scale $E := (U, B, j)$ on Z we shall abbreviate $H(\bar{U}, \text{Sym}^k(B))$ in $G_k(E)$.

Definition 3.2.6 Let $(\mathbb{E}_i)_{i \in I}$ a finite family of adjusted n -scales on Z , convenient for a n -cycle \hat{X}_0 of finite type in Z , and let \mathcal{R} be some corresponding patching data. Let k_i be the degree of \hat{X}_0 in the adjusted scale \mathbb{E}_i . For each $(i, j, h), i \neq j$ we have a couple of holomorphic maps

$$\prod_{\alpha \in I} \mathcal{G}_{k_\alpha}(\mathbb{E}_\alpha) \rightrightarrows G_{k_{i,j,h}}(F_{i,j,h})$$

obtained by the change of projections $\mathbb{E}_i \rightarrow F_{i,j,h}$ and $\mathbb{E}_j \rightarrow F_{i,j,h}$, because, by construction, we have $D(F_{i,j,h}) \subset\subset D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

We shall note $S(\mathcal{R})$ the intersection of the kernels of these double maps⁶ with the open set \mathcal{V} built in the lemma 3.2.5. It is a banach analytic set and we shall call it the **classifying space associated to** $(\mathbb{E}_i)_{i \in I}, \hat{X}_0$ **and** \mathcal{R} .

Remark that the patching data \mathcal{R} are not assumed to be complete in the previous definition.

Proposition 3.2.7 Consider a finite family of adjusted n -scales which is convenient for the n -cycle X_0 in Z and let \mathcal{R} be some **complete** patching data associated. Keeping the previous notations we have for each $(X^i)_{i \in I} \in S(\mathcal{R})$ an unique n -cycle $X \in \mathcal{C}_n^f(W')$ such that $X \cap D'(\mathbb{E}_i) = X^i \cap D'(\mathbb{E}_i), \forall i \in I$.

Moreover, this defines a tautological family of cycles in W' which is a f -analytic family of cycles satisfying the following “almost universal”property :

- For any analytic family of n -cycles $(X_s)_{s \in S}$ in an open neighbourhood of \bar{W} parametrized by a banach analytic set S^7 such that for $s_0 \in S$ we have $X_{s_0} = X_0$ in a neighbourhood of \bar{W} , there exists an open neighbourhood S' of s_0 in S and an unique holomorphic map

$$f : S' \rightarrow S(\mathcal{R})$$

such that the pull-back by f of the tautological family parametrized by $S(\mathcal{R})$ is the restriction to the open set W' of the family $(X_s)_{s \in S'}$.

PROOF. For each $\tilde{X} \in \mathcal{V}$ any X^i does not meet \mathcal{K} . As \mathcal{R} is complete, we may use the lemma 3.1.4 with $\mathcal{U}_i := D'(\mathbb{E}_i)$ et $W_{i,j} = \cup_h D(F_{i,j,h})$ to associate to \tilde{X} a finite type n -cycle X of the open set W' . The f -analyticity of the so defined family is obvious. The “almost universal”property is then clear. \blacksquare

⁶The kernel of a double map $f, g : A \rightarrow B$ is the pull-back of the diagonal in $B \times B$ by the map $(f, g) : A \rightarrow B \times B$.

⁷Recall that, by definition, this means that for any n -scale $E := (U, B, j)$ on Z , adapted to some $X_{s_0}, s_0 \in S$, with $\deg_E(X_{s_0}) = k$, we have an open neighbourhood S_0 of s_0 in S and a classifying map for the corresponding family of branched coverings $f : S_0 \times U \rightarrow \text{Sym}^k(B)$ which is **isotropic**.

3.3 Shrinkage.

Definition 3.3.1 *Let Z be a reduced complex space and let $\mathbb{E} := (U, U', U'', B, B'', j)$ be an adjusted n -scale on Z . For any real $\tau > 0$ small enough we shall note \mathbb{E}^τ the adjusted n -scale on Z defined as $\mathbb{E}^\tau := (U^\tau, U'^\tau, U'', B, B'', j)$. We shall call \mathbb{E}^τ the τ -shrinkage of \mathbb{E} . Recall that for a polydisc P of radius R , P^τ is the polydisc with same center and radius $R - \tau$. The definitions of $\mathcal{M}(\alpha)^\tau, M(\alpha)^\tau$ are given in the section 2.2.*

REMARKS.

1. By definition, the shrinkage of \mathbb{E} does not change the polydiscs U'', B, B'' .
2. As j is a closed embedding of an open set in Z in an open neighbourhood of the compact set $\bar{U} \times \bar{B}$, it is clear that for any given adjusted n -scale \mathbb{E} on Z , there exists a real $\varepsilon > 0$ (depending on \mathbb{E}) such that for any $\tau \in]0, \varepsilon[$, \mathbb{E}^τ is again an adjusted n -scale on Z .
3. If \mathbb{E} is adapted to the n -cycle X_0 in Z , for τ small enough (depending on \mathbb{E} and X_0), the adjusted n -scale \mathbb{E}^τ remains adapted to X_0 and we shall have also $\deg_{\mathbb{E}^\tau}(X_0) = \deg_{\mathbb{E}}(X_0)$.
4. If \mathbb{E} is adapted to the n -cycle X_0 in Z , there exists an open neighbourhood \mathcal{U} of X_0 in $\mathcal{C}_n^{loc}(Z)$ and a real $\varepsilon > 0$ such that for any $X \in \mathcal{U}$ and any $\tau \in]0, \varepsilon[$, the adjusted scale \mathbb{E}^τ remains adapted to X with again $\deg_{\mathbb{E}^\tau}(X) = \deg_{\mathbb{E}}(X_0)$.
5. In the situation of the previous proposition 3.2.7, we may, for $\tau > 0$ small enough, keep the same patching data \mathcal{R} on the finite family $(\mathbb{E}_i^\tau)_{i \in I}$ of adjusted n -scales; if it was complete, it remains complete and if it was convenient for the n -cycle X_0 , it remains convenient for the n -cycle X_0 . Then we have a holomorphic restriction map

$$S(\mathcal{R}) \rightarrow S^\tau(\mathcal{R})$$

which is induced by a finite product of linear (continuous) compact maps. \square

3.4 Excellent family.

In order not to make our notations too heavy we shall introduce the following conventions when \mathcal{H} is a n -Hartogs figure in \mathbb{C}^{n+p} ; we shall put, using the notations introduced above

$$U := M(\alpha) \times V, \quad U' := M(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad U'' := M(\alpha)^{\varepsilon''} \times V^{\varepsilon''}, \quad B'' := B^{\varepsilon'}$$

where $\varepsilon' > 0$ is small enough, and where $0 < \varepsilon'' < \varepsilon'$. The choices of ε' and ε'' will be precised when they are useful. We shall associate to \mathcal{H} the adjusted n -scale on Δ given by :

$$\mathbb{E}_{\mathcal{H}} := (U, U', U'', B, B'', j).$$

We shall put also

$$\tilde{U} := \mathcal{M}(\alpha) \times V, \quad \tilde{U}' := \mathcal{M}(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad \tilde{U}'' := \mathcal{M}(\alpha)^{\varepsilon''} \times V^{\varepsilon''}$$

the holomorphy envelopes respectively of U, U' and U'' . Then we shall have a family of adjusted n -scales on Δ' , written down:

$$\mathbb{E}_{\tilde{\mathcal{H}}}^{\eta} := (\tilde{U}^{\eta}, \tilde{U}'^{\eta}, \tilde{U}''^{\eta}, B, B'', j) \quad \text{with } \tilde{U}^{\eta} := \mathcal{M}(\alpha)^{\eta} \times V^{\eta}, \quad \tilde{U}'^{\eta} := \mathcal{M}(\alpha)^{\varepsilon'+\eta} \times V^{\varepsilon'+\eta},$$

where η is a positive real number, small enough (for $\eta = 0$ we shall simply write $\mathbb{E}_{\tilde{\mathcal{H}}}$).

Then we shall have the isotropic classifying spaces $\mathcal{G}_k(\mathcal{H}) := \Sigma_{U, U'}(k)$ and also $\mathcal{G}_k^{\eta}(\tilde{\mathcal{H}}) := \Sigma_{\tilde{U}^{\eta}, \tilde{U}'^{\eta}}(k)$. The proposition 2.2.5 gives a holomorphic analytic extension map such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}_k(\tilde{\mathcal{H}}) & \xrightarrow{\text{res}^{\eta}} & \mathcal{G}_k^{\eta}(\tilde{\mathcal{H}}) \\ \text{res} \downarrow & \nearrow \text{prlgt}^{\eta} & \\ \mathcal{G}_k(\mathcal{H}) & & \end{array}$$

Definition 3.4.1 *Let Z be a reduced complex space of pure dimension $n + p$, let $\Delta \subset\subset Z$ be a strongly n -concave open set in Z and \tilde{X}_0 a n -cycle in an open neighbourhood Δ' of $\bar{\Delta}$ in Z . We shall say that a finite family $(\mathcal{H}_a)_{a \in A}$ of n -Hartogs figures relative to the boundary of Δ is **excellent** for the cycle \tilde{X}_0 when the following conditions hold, where we write $\tilde{\mathbb{E}}_a$ and \mathbb{E}_a the adjusted scales respectively on Δ' and Δ associated to the n -Hartogs figure \mathcal{H}_a :*

1. *The adjusted scales $\tilde{\mathbb{E}}_a$ and \mathbb{E}_a are adapted to the cycle \tilde{X}_0 .*
2. *We may choose the patching domains of the adjusted scales $(\tilde{\mathbb{E}}_a)_{a \in A}$ in order that the union*

$$\tilde{D}''(A) := \cup_{a \in A} j_a^{-1}(\tilde{U}''_a \times B''_a)$$

contains the compact set $\partial\Delta$.

3. *There exists a finite family of adjusted n -scales $(\mathbb{E}_b)_{b \in B}$ on Δ , adapted to \tilde{X}_0 , such that the finite families $(\mathbb{E}_c)_{c \in A \cup B}$ and $(\tilde{\mathbb{E}}_c)_{c \in A \cup B}$ are convenient for \tilde{X}_0 , where we put $\tilde{\mathbb{E}}_b = \mathbb{E}_b$ for $b \in B$.*

Moreover we ask that the union $D''(B)$ of the patching domains of the $(\mathbb{E}_b)_{b \in B}$ covers the compact set $\Delta \setminus \tilde{D}''(A)$ of Δ ; so $\tilde{D}''(A) \cup D''(B)$ in an open set containing $\bar{\Delta}$.

As a consequence, the union $\tilde{D}'(A)^{\eta} \cup D'(B)$ of the isotropy domains will cover $\bar{\Delta}$ for $\eta > 0$ small enough.

Proposition 3.4.2 (Existence of an excellent family.) *Let Z be a reduced complex space of pure dimension $n + p$, $\Delta \subset\subset Z$ be a strongly n -concave open set with smooth boundary and \tilde{X}_0 a n -cycle in an open neighbourhood $\Delta' \subset Z$ of $\bar{\Delta}$ such that any irreducible component of \tilde{X}_0 meets Δ . Then there exists a finite family $(\mathcal{H}_a)_{a \in A}$ of n -Hartogs figures relative to the boundary of Δ which is excellent for the cycle \tilde{X}_0 .*

PROOF. First we use the proposition 2.3.6 to cover $\partial\Delta$ by a finite family of n -Hartogs figures relative to the boundary of Δ adapted to the cycle \tilde{X}_0 such that the conditions 1 and 2 hold. Then we build a finite family of adjusted n -scales $(\mathbb{E}_b)_{b \in B}$ on Δ , adapted to \tilde{X}_0 in order that the condition 3 holds. \blacksquare

3.5 The extension and finiteness theorem.

The next theorem will allow to prove the theorem 1.0.1.

Theorem 3.5.1 *Let Z be a reduced complex space of pure dimension $n + p$, where $n \geq 2, p \geq 1$. Let Δ be a relatively compact open set with \mathcal{C}^2 boundary in Z which is strongly n -concave. Let \tilde{X}_0 a closed n -cycle of an open neighbourhood Δ' of $\bar{\Delta}$ in Z such that any irreducible component of \tilde{X}_0 meets Δ . Then there exists an open neighbourhood Δ'' of $\bar{\Delta}$ in Δ' and a f -analytic family $(\tilde{X}_\xi)_{\xi \in \Xi}$ of n -cycles in Δ'' parametrized by a reduced complex space Ξ (of finite dimension) such that $X_{\xi_0} = \tilde{X}_0 \cap \Delta''$ and such that Ξ is an open neighbourhood of $\tilde{X}_0 \cap \Delta''$ in $\mathcal{C}_n^f(\Delta'')$. It satisfies the following universal property:*

- For any f -analytic family $(X_s)_{s \in S}$ of n -cycles in Δ parametrized by a banach analytic set S and such that its value at some $s_0 \in S$ is equal to $\tilde{X}_0 \cap \Delta$, there exists an open neighbourhood S' of s_0 in S and an unique holomorphic map

$$h : S' \rightarrow \Xi$$

satisfying the equality $X_s = \tilde{X}_{h(s)} \cap \Delta$ for each $s \in S'$

PROOF. Assume that $\Delta := \{\varphi > \alpha\}, 0 < \alpha < 1$, where $\varphi : Z \rightarrow]0, 2]$ is an \mathcal{C}^2 exhaustion of Z which is strongly $(n - 1)$ -convex on the open set $\varphi^{-1}(]0, 1[)$. Up to replace Δ by $\{\varphi > \alpha'\}, \alpha' < \alpha, \alpha'$ very close to α , we may assume, thanks to the Sard' lemma 3.1.1, that φ and $\varphi|_{\tilde{X}_0}$ have no critical point on the compact set $\tilde{X}_0 \cap \partial\Delta$.

Begin by cover the compact set $\partial\Delta$ by an excellent finite family $(\mathcal{H}_a)_{a \in A}$ of n -Hartogs figures for Δ adapted to the cycle \tilde{X}_0 . Choose then open sets $\Delta_1 \subset\subset \Delta \subset\subset \Delta'' \subset\subset \Delta'$, such that the following properties hold, where we use the notations introduced above for the finite family of the adjusted scales $(\tilde{\mathbb{E}}_a)_{a \in A}$ associated to $(\mathcal{H}_a)_{a \in A}$:

- i) $\bar{\Delta}'' \setminus \Delta_1 \subset \cup_{a \in A} W''(\tilde{\mathbb{E}}_a)$.
- ii) $\cup_{a \in A} \bar{W}(\tilde{\mathbb{E}}_a) \subset \Delta'$.
- iii) $\cup_{a \in A} \bar{W}(\mathbb{E}_a) \subset \Delta$.
- iv) $K := \cup_{a \in A} j_a^{-1}(\bar{U}_a \times (\bar{B}_a \setminus B''_a)) \subset \Delta_1$

It is easy to fulfill these conditions for Δ_1 and Δ'' near enough to Δ , as, by assumption, the subsets $\bar{W}(\mathbb{E}_a)$ and $j_a^{-1}(\bar{U}_a \times \bar{B}_a \setminus B''_a)$ are compact in Δ , and as the union of the open sets $W''(\tilde{\mathbb{E}}_a)$ contains $\partial\Delta$.

Note that the condition iv) allows to choose le compact neighbourhood \mathcal{K} of K inside Δ_1 .

Choose now a convenient finite family $(\mathbb{E}_b)_{b \in B}$ of adjusted n -scales on Δ , adapted to \tilde{X}_0 in order that the open set $\cup_{b \in B} W''(\mathbb{E}_b)$ contains $\bar{\Delta}_1$.

Put $\tilde{\mathbb{E}}_b = \mathbb{E}_b$ for $b \in B$, and define $C := A \cup B$. The family of adjusted scales $(\mathbb{E}_c)_{c \in C}$ in Δ is then convenient for $\tilde{X}_0 \cap \Delta_1$, up to choose the patching domains big enough. Fix some complete patching data \mathcal{R} associated to the family $(\mathbb{E}_c)_{c \in C}$. The family $(\tilde{\mathbb{E}}_c)_{c \in C}$ of adjusted scales in Δ' is convenient for $\tilde{X}_0 \cap \Delta''$ if we choose the patching domains big enough. Consider now some complete patching data for this family of the form $\mathcal{R} \cup \mathcal{R}''$, that is to say containing the patching scales already in \mathcal{R} . Define the following banach analytic sets:

1. S_0 is the classifying space of the family $(\tilde{\mathbb{E}}_c)_{c \in C}$, the degrees being these of \tilde{X}_0 in the various scales adapted to the cycle \tilde{X}_0 , with the patching conditions defined by \mathcal{R} . Note that \mathcal{R} is not complete in general.
2. S_+ is the classifying space of the family $(\tilde{\mathbb{E}}_c)_{c \in C}$ with the patching conditions defined by $\mathcal{R} \cup \mathcal{R}''$.
3. S_- is the classifying space of the family $(\mathbb{E}_c)_{c \in C}$, with the (complete) patching conditions defined by \mathcal{R} .

Then we get a holomorphic extension map

$$\alpha : S_- \rightarrow S_0$$

deduced from the extension maps in the n -Hartogs figures $(\mathcal{H}_a)_{a \in A}$.

By definition S_+ is a closed banach analytic subset of S_0 as it is defined in S_0 by the patching conditions given by \mathcal{R}'' . Then put $\Xi := \alpha^{-1}(S_+)$. So we have a holomorphic map $\alpha : \Xi \rightarrow S_+$. We want to show the following claim:

CLAIM. There exists a holomorphic map $\beta : S_+ \rightarrow \Xi$ satisfying the two properties:

1. We have $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$ in a neighbourhood of the point defined by \tilde{X}_0 respectively in S_+ and Ξ .

2. The holomorphic map β is the composition of a holomorphic map induced by a linear (continuous) compact map and a holomorphic map.

As the open set $\cup_{c \in C} W''(\tilde{\mathbb{E}}_c)$ contains Δ'' , there is, on S_+ , a tautological family of n -cycles which is f -analytic on Δ'' . Then the “almost universal” property of S_- gives a holomorphic map $\tilde{\beta} : S_+ \rightarrow S_-$ which factorizes via the closed banach analytic subset $\Xi \subset S_-$. Let us show that the holomorphic map $\beta : S_+ \rightarrow \Xi$ deduced from this factorization satisfies the two properties of the claim.

First we have $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$ respectively in S_+ and Ξ , by definition of Ξ and β .

To see the second property, consider $\tau > 0$ small enough and let us show that the holomorphic map induced by the linear compact (by Vitali’ theorem) restriction $r^\tau : S_+ \rightarrow S_+^\tau$ factorizes β , where S_+^τ is the classifying space corresponding to the τ -shrinkage $(\tilde{\mathbb{E}}_c^\tau)_{c \in C}$ of the family $(\tilde{\mathbb{E}}_c)_{c \in C}$ with the patching data $\mathcal{R} \cup \mathcal{R}''$. Indeed, for τ small enough, the tautological family of S_+^τ is still f -analytic on Δ and the “almost universal” property of S_- gives again a holomorphic map $\tilde{\beta}^\tau : S_+^\tau \rightarrow S_-$. And we have $\beta = \tilde{\beta}^\tau \circ r^\tau$ proving our assertion. In fact, the map $\tilde{\beta}^\tau$ takes its values in Ξ because the τ -shrinkage does not change the patching data deduced from $\mathcal{R} \cup \mathcal{R}''$ for τ small enough.

We conclude that the banach analytic set Ξ has finite dimension thanks to the finiteness lemma of [B.75] (see also [B-M 2] ch.V). Moreover, it is isomorphic to S_+ which is also of finite dimension.

The universal property is then obvious. ■

REMARKS.

1. The reduced complex space (of finite dimension) Ξ built in the previous theorem parametrizes a f -analytic family of n -cycles in the open set Δ'' which is an open neighbourhood of $\bar{\Delta}$. So, when we have a f -analytic family $(X_s)_{s \in S}$ of n -cycles in Δ parametrized by a banach analytic set S and such that its value at some $s_0 \in S$ is equal to $\tilde{X}_0 \cap \Delta$ we can extend each cycle $X_s, s \in S'$, to a n -cycle \tilde{X}_s in Δ'' in order that the family $(\tilde{X}_s)_{s \in S'}$ is f -analytic in Δ'' , with the condition that each irreducible component of \tilde{X}_s meets Δ and with the equality $\tilde{X}_s \cap \Delta = X_s$ for each $s \in S'$.
2. We shall see later on that Ξ is also (isomorphic to) an open neighbourhood of the point $\tilde{X}_0 \cap \Delta$ of $\mathcal{C}_n^f(\Delta)$.

4 Finiteness of the space of n -cycles of a reduced strongly n -concave space ($n \geq 2$).

4.1 The global extension theorem.

First we have to complete our terminology.

Definition 4.1.1 *We shall say that a reduced complex space Z is **strongly q -concave**, where q is a positive integer, if there exists a real valued \mathcal{C}^2 exhaustion function on Z , $\varphi : Z \rightarrow]0, 2]$, which is strongly $(q - 1)$ -convex outside the compact set $K := \varphi^{-1}([1, 2])$.*

For instance, any reduced compact complex space is strongly q -concave for any $q \geq 1$.

When Z is strongly q -concave irreducible and non compact of dimension at least $q + 1$, the function φ achieves its maximum at a point in which it cannot be strongly $(q - 1)$ -convex. So we shall have $\varphi(Z) =]0, u]$ with $u \geq 1$.

In the sequel we shall consider mainly the closed q -cycles of Z in the case where Z is strongly q -concave and non compact, with $\dim Z \geq q + 1$.

Theorem 4.1.2 *Let $n \geq 2$ be an integer. Let Z be a reduced complex space which is strongly n -concave. Let $\alpha \in]0, 1[$ and let X be a finite type n -cycle in the open set $Z_\alpha := \{z \in Z / \varphi(z) > \alpha\}$. Then there exists a unique n -cycle \tilde{X} in $\mathcal{C}_n^f(Z)$ such that $\tilde{X} \cap Z_\alpha = X$.*

PROOF. First we shall prove uniqueness. If a non empty n -cycle \tilde{X} in Z satisfies $\tilde{X} \cap Z_\alpha = \emptyset$ with $\alpha < 1$, the maximum of φ on the set $|\tilde{X}|$ is obtained at a point where φ is strongly $(n - 1)$ -convex. This is a contradiction. So the uniqueness of the cycle \tilde{X} is proved.

To show that this cycle exists, consider first the case where X is compact in Z_α . Then $\tilde{X} := X$ is a solution. So it enough to consider the case where X is irreducible and non compact. Thanks to the theorem (8.3) of [S-T], for each $z \in \partial Z_\alpha$ there exists an open set U_z and an unique closed analytic set X_z in $Z_\alpha \cup U_z$ of pure dimension n such that $X_z \cap Z_\alpha = X$. Choose a finite set of points z_1, \dots, z_N and open sets $U'_i \subset\subset U_i := U_{z_i}$ such that the union of the U'_i covers the compact set ∂Z_α . Let $\Omega := Z_\alpha \cup (\cup_{i=1}^N U'_i)$ and put

$$X_1 := (X \cup (\cup_{i=1}^N X_{z_i})) \cap \Omega.$$

Let us show that X_1 is a closed analytic subset in Ω .

Consider $z \in \Omega$. If z is in Z_α we have $X_1 = X$ in a neighbourhood of z and the assertion is clear. If not, either z is not in any $\partial U'_i$ and X_1 is the union of the X_{z_i} in a neighbourhood of z and the assertion is clear, or z is in $\partial U'_{j_1}, \dots, \partial U'_{j_k}$ for j_1, \dots, j_k in $[1, N]$. As the set X_{j_h} is closed and analytic in $Z_\alpha \cup U_{j_h}$, then X_1 is again the union of the X_{z_i} near z in Ω , and the assertion is proved.

So in this situation there exists a real positive $\beta < \alpha$ such that $Z_\beta \subset \Omega$. Let X_2 be the irreducible component of $X_1 \cap Z_\beta$ which contains X ; then X_2 is a closed irreducible analytic subset of Z_β such that $X_2 \cap Z_\alpha = X$.

Now let

$$\gamma := \inf\{\beta \leq \alpha / \exists X_\beta \text{ irreducible } n\text{-cycle of } Z_\beta \text{ such that } X_\beta \cap Z_\alpha = X\}.$$

Then what we obtained above shows that we have $\gamma < \alpha$, and, applying the same arguments to the cycle X_γ defined on Z_γ via the cover of Z_γ by the $Z_\beta, \beta > \gamma$ in which we already built an irreducible n -cycle X_β extending X , we conclude that $\gamma = 0$ and that there exists an (unique) irreducible n -cycle \tilde{X} in Z extending X . ■

4.2 Some consequences.

We shall give first some easy consequences of the fact that the reduced complex space Z is strongly n -concave.

Proposition 4.2.1 *Let $n \geq 2$ be an integer and let Z be a reduced strongly n -concave complex space. Then the natural map $j : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{\text{loc}}(Z)$ is a homeomorphism. Moreover, for each $\alpha \in]0, 1[$ the restriction map*

$$\text{res}_\alpha : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$$

is well defined and is also a homeomorphism.

PROOF. Let us prove first that any n -cycle X in Z has finitely many irreducible components. As this implies the same result for each Z_α for $\alpha \in]0, 1[$, this will imply the fact that the restriction map res_α is well defined, and then bijective as a consequence of the theorem 4.1.2.

Consider a closed irreducible analytic subset Γ of dimension n in Z . As the restriction $\varphi|_\Gamma$ of the exhaustion φ to Γ must reach its maximum (as φ is continuous and proper), this maximum cannot be obtained at a point in which φ is strongly $(n-1)$ -convex. So we have $\Gamma \cap \varphi^{-1}[1, 2] \neq \emptyset$. As the family of irreducible components of a n -cycle is locally finite, only finitely many irreducible components of X can meet the compact set $K := \varphi^{-1}([1, 2])$. But we have seen that **any** irreducible component of X must meet K . So X is a finite type cycle.

To show the continuity of res_α it is then enough to prove that j is a homeomorphism which is an easy consequence of the lemma below. ■

Lemma 4.2.2 *Let Z be a reduced complex space and let $(X_\nu)_{\nu \geq 0}$ be a sequence of n -cycles in Z converging in $\mathcal{C}_n^{\text{loc}}(Z)$ to a cycle Y . Assume that there exists a relatively compact open set Ω in Z such that any irreducible component of each X_ν and of Y meets Ω . Then all these cycles are of finite type and the sequence converges to Y in $\mathcal{C}_n^f(Z)$.*

PROOF. First if $Y = \emptyset$, for $\nu \gg 1$ the cycle X_ν will be disjoint from the compact set $K := \bar{\Omega}$, and this implies that X_ν is the empty cycle. So the conclusion holds in this case. If Y is not empty, let U be a relatively compact open set in Z meeting each irreducible component of Y . We have to show, by definition of the topology of $\mathcal{C}_n^f(Z)$, that for $\nu \gg 1$ each irreducible component of X_ν meets U . If it is not the case, up to pass to a subsequence, we may assume that for each ν there exists an irreducible component Γ_ν of X_ν disjoint from U . As, again up to pass to a subsequence, we may assume that the sequence (Γ_ν) converges in $\mathcal{C}_n^{loc}(Z)$ to a cycle Γ , we shall have $|\Gamma| \subset |Y|$ and $|\Gamma| \cap U = \emptyset$. To conclude, it is enough to show that Γ is not the empty cycle, as any irreducible component of Γ is also an irreducible component of Y and then meets U by hypothesis. As each Γ_ν is not empty, it has to meet $K = \bar{\Omega}$. This implies that Γ also meets K and so is not empty. This contradicts our assumption. ■

END OF THE PROOF OF THE PROPOSITION 4.2.1. We have proved that j , and then also each $j_\alpha : \mathcal{C}_n^f(Z_\alpha) \rightarrow \mathcal{C}_n^{loc}(Z_\alpha)$ for $\alpha \in]0, 1[$, is a holomorphic homeomorphism. To conclude the proof we have to show the continuity of res_α^{-1} , and this reduces to prove that if the sequence (X_ν) of $\mathcal{C}_n^{loc}(Z)$ is such that the sequence $(X_\nu \cap Z_\alpha)$ converges in $\mathcal{C}_n^{loc}(Z_\alpha)$, then it converges in $\mathcal{C}_n^{loc}(Z)$. Let $Y_\alpha \in \mathcal{C}_n^{loc}(Z_\alpha)$ be the limit of this sequence in $\mathcal{C}_n^{loc}(Z_\alpha)$ and let $Y \in \mathcal{C}_n^{loc}(Z)$ be the cycle extending it. Let A be the set of $\beta \in]0, \alpha]$ such that the sequence $(X_\nu \cap Z_\beta)$ converges in $\mathcal{C}_n^{loc}(Z_\beta)$ to $Y \cap Z_\beta$. Then α is in A so A is not empty. Put $\gamma := \inf A$. The theorem 3.5.1 implies that $\gamma = 0$ and we obtain also the convergence in any $\mathcal{C}_n^{loc}(Z_\beta)$, for any $\beta > 0$; this gives the convergence in $\mathcal{C}_n^{loc}(Z)$, as, by definition, a n -scale on Z is also a n -scale on Z_β for $\beta > 0$ small enough. ■

4.3 An analytic extension criterion.

The aim of this paragraph is to prove the following analytic extension result.

Theorem 4.3.1 *Let Z be a complex space and n an integer. Consider a f -continuous family $(X_s)_{s \in S}$ of n -cycles of finite type in Z parametrized by a reduced complex space S . Fix a point s_0 in S and assume that the open set Z' in Z meets all irreducible components of X_{s_0} and such that the family of cycles $(X_s \cap Z')_{s \in S}$ is analytic at s_0 . Then there exists an open neighbourhood S_0 of s_0 in S such that the family $(X_s)_{s \in S_0}$ is f -analytic.*

The hypotheses translated in term of classifying maps means that we have a continuous map $\varphi : S \rightarrow \mathcal{C}_n^f(Z)$ such that the composed map $r \circ \varphi$ is holomorphic at s_0 , where $r : \mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^{loc}(Z')$ is the restriction map.

Then the theorem says that there exists an open neighbourhood S_0 of s_0 in S such that the map φ is holomorphic on S_0 . Note that, as r is holomorphic⁸, the hypoth-

⁸in the sense that for any holomorphic map $\psi : T \rightarrow \mathcal{C}_n^f(Z)$ of a reduced complex space T the composed map $r \circ \psi$ is holomorphic.

esis that φ is holomorphic at s_0 is a necessary condition.

This result is not true in general if we take for S a non smooth banach analytic set which is not of finite dimension (locally). The reader may find a counter-exemple with an isolated singularity in [B-M 2] ch.V.

The key point for the proof of the previous theorem is the following analytic extension result.

Proposition 4.3.2 *Let S a reduced complex space and let $\emptyset \neq U_1 \subset U_2$ be two polydiscs in \mathbb{C}^n . Let $f : S \times U_2 \rightarrow \mathbb{C}$ a continuous function, holomorphic on $\{s\} \times U_2$ for each $s \in S$. Assume moreover that the restriction of f to $S \times U_1$ is holomorphic. Then f is holomorphic on $S \times U_2$.*

PROOF OF THE PROPOSITION 4.3.2. First consider the case S smooth. As the question is local on S it is enough to consider the case where S is an open set in \mathbb{C}^m . Fix an open relatively compact polydisc P in S . The function f defines a map $F : U_2 \rightarrow \mathcal{C}^0(\bar{P}, \mathbb{C})$, where we write down $\mathcal{C}^0(\bar{P}, \mathbb{C})$ the Banach space of continuous functions on \bar{P} , via the formula $F(t)[s] = f(s, t)$ for $t \in U_2$ and $s \in \bar{P}$. First we shall show that the map F is holomorphic.

Let $U \subset\subset U_2$ be a polydisc. For any fix $s \in S$ we have

$$\frac{\partial f}{\partial t_i}(s, t) = \frac{1}{(2i\pi)^n} \int_{\partial\partial U} f(s, \tau) \cdot \frac{d\tau_1 \wedge \dots \wedge d\tau_n}{(\tau_1 - t_1) \dots (\tau_i - t_i)^2 \dots (\tau_n - t_n)} \quad \forall t \in U \quad \forall i \in [1, n].$$

where $t := (t_1, \dots, t_n)$ are coordinates on \mathbb{C}^n . This Cauchy formula shows that F is \mathbb{C} -différentiable and its differential at the point $t \in U$ is given by

$$h \mapsto \sum_{i=1}^n F_i(t) \cdot h_i, \quad h \in \mathbb{C}^n,$$

where F_i is associated to the function

$$(s, t) \mapsto \frac{\partial f}{\partial t_i}(s, t) \quad i \in [1, n]$$

which is holomorphic for each fixed $s \in S$ thanks to the Cauchy formula above.

Let $H(\bar{P}, \mathbb{C})$ the closed subspace of $\mathcal{C}^0(\bar{P}, \mathbb{C})$ of functions which are holomorphic on P . Our hypothesis implies that the restriction of F to the non empty open set U_1 takes its values in this subspace. Let us show that this is also true on U_2 . Assume that there exists $t_0 \in U_2$ such that $F(t_0)$ is not in $H(\bar{P}, \mathbb{C})$. Thanks to the *Hahn-Banach theorem* we can find a continuous linear form λ on $\mathcal{C}^0(\bar{P}, \mathbb{C})$, vanishing on $H(\bar{P}, \mathbb{C})$, and such that $\lambda(F(t_0)) \neq 0$. But the function $t \mapsto \lambda(F(t))$ is holomorphic on U_2 and vanishes on U_1 ; this contradicts $\lambda(F(t_0)) \neq 0$. So F takes values in $H(\bar{P}, \mathbb{C})$ and f is holomorphic on $S \times U_2$ when S is a complex manifold.

The case where S is a weakly normal complex space follows immediately.

When S is a general reduced complex space, the function f is meromorphic and

continuous on $S \times U_2$ and holomorphic on $S \times U_1$. So the the closed analytic subset $Y \subset S \times U_2$ along which f may not be holomorphic has no interior point in each $\{s\} \times U_2$. The analytic extension criterion of [B-M 1] ch. IV, 3.1.7 allows to conclude. ■

PROOF OF THE THEOREM 4.3.1. Let $|G| \subset S \times Z$ be the graph of the f -continuous family $(X_s)_{s \in S}$ and let A the set of points in $(\sigma, \zeta) \in |G|$ admitting an open neighbourhood $S_\sigma \times Z_\zeta$ in $S \times Z$ such that the family of cycles $(X_s \cap Z_\zeta)_{s \in S_\sigma}$ is analytic. Remark that, thanks to our hypothesis, the open set A in $|G|$ meets every irreducible component of $\{s_0\} \times |X_{s_0}|$.

Assume to begin that there exists a smooth point z_0 of $|X_{s_0}|$ in the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$. Choose a n -scale $E := (U, B, j)$ on Z which is adapted to X_{s_0} and satisfying :

$$\begin{aligned} \deg_E(|X_{s_0}|) &= 1 \\ j_*(X_{s_0}) &= k \cdot (U \times \{0\}) \\ z_0 &\in j^{-1}(U \times B) \\ j(z_0) &:= (t_0, 0). \end{aligned}$$

It is clear that such a n -scale exists as z_0 is a smooth point in $|X_{s_0}|$. Let S_1 be a sufficiently small open neighbourhood of s_0 in S and let $f : S_1 \times U \rightarrow \text{Sym}^k(B)$ be the (continuous) classifying map for the family $(X_s)_{s \in S_1}$ in the scale E . As $j^{-1}(U \times \{0\})$ meets A , there exists a non empty polydisc $U_2 \subset U$ such that $U_2 \times \{0\}$ is contained in A . Then we may apply the proposition 4.3.2 to each scalar component of f in order to obtain that f is holomorphic on $S_1 \times U$. Moreover, as the same argument applies to any linear projection of $U \times B$ to U near enough the vertical one; this implies that f is an isotropic map, up to shrink slightly U . This contradicts the fact that the point (s_0, z_0) is in the boundary of the open set $A \cap (\{s_0\} \times |X_{s_0}|)$ of $|X_{s_0}|$. If the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$ is contained in the singular set of $|X_{s_0}|$, we may apply the analytic extension criterion of [B-M 1] ch. IV, 3.1.7, and we obtain directly that A contains $|X_{s_0}|$. So in any case the family of cycles $(X_s)_{s \in S}$ is analytic at s_0 . As the graph $|G|$ is, by assumption, quasi-proper on S , it is enough to use the next proposition (which is proved in [B.15-a] proposition 2.2.3) to conclude. ■

Proposition 4.3.3 *Let Z and S be reduced complex spaces and let $(X_s)_{s \in S}$ be a f -continuous family of n -cycles in Z . Assume that this family is analytic in $s_0 \in S$. Then there exists an open neighbourhood S' of s_0 in S such that the family $(X_s)_{s \in S'}$ is a f -analytic family of n -cycles in Z .*

4.4 Proof of the theorem 1.0.2 and its corollary

We shall begin by a lemma which will give the case where the n -cycle X_0 is compact.

Lemma 4.4.1 *Let Z be a strongly n -concave reduced complex space. Then $\mathcal{C}_n(Z)$ is open and closed in $\mathcal{C}_n^f(Z)$.*

PROOF. When Z is compact the assertion is obvious. So we may assume that Z is not compact and that $\varphi(Z) =]0, 2]$. If an irreducible component Γ of a n -cycle X meets $Z_\alpha := \{z \in Z / \varphi(z) > \alpha\}$ with $\alpha < 1$, this component is non compact in Z : indeed, if Γ is compact the function φ reach its minimum on it at a point where φ is strongly $(n - 1)$ -convex and this gives a contradiction. So any compact cycle in Z is contained in the compact set $\varphi^{-1}([1, 2])$ and conversely any cycle contained in this compact set is compact. So $\mathcal{C}_n(Z)$ is open in $\mathcal{C}_n^f(Z)$ because it is the subset of cycles which do not meet the compact set $\varphi^{-1}([1/3, 2/3])$. It is closed because its complement is the (open) subset of cycles meeting the open set $\{z \in Z / \varphi(z) < 1\}$. ■

So for such a Z , $\mathcal{C}_n(Z)$ is a union of connected components of $\mathcal{C}_n^f(Z)$.

We already know from [B.75] that it is a reduced complex space. The theorem 1.0.2 and its corollary are proved for these connected components.

THE CASE WHERE X_0 IS NOT COMPACT. Fix X_0 a non compact n -cycle in $\mathcal{C}_n^f(Z)$ and choose an $\alpha \in]0, 1[$ which is not a critical value of φ and of the restriction of φ to $|X_0|$; this is possible thanks to the Sard' lemma 3.1.1 and the fact that $\varphi(|X_0|)$ contains $]0, 1[$. Then the constructions done in the proof of the theorem 3.5.1 give an open neighbourhood \mathcal{U} of $X_0 \cap Z_\alpha$ in $\mathcal{C}_n^f(Z_\alpha)$ and the theorem 4.1.2 gives an open neighbourhood \mathcal{V} of X_0 in $\mathcal{C}_n^f(Z)$ such that the restriction map

$$res : \mathcal{V} \rightarrow \mathcal{U}$$

is holomorphic and bijective. Indeed, it is enough to define \mathcal{V} by asking that all the adjusted scales used to build \mathcal{U} are adapted to a cycle in \mathcal{V} with the same degrees than X_0 and adding the open condition not to meet the compact set \mathcal{K} to have the bijectivity.

We want to show now that the family of finite type n -cycles in Z parametrized by \mathcal{U} via the inverse of the map res is f-continuous. Remark that we have already a fix compact set $\bar{\Delta}_1$ meeting any irreducible component of any cycle in this family. So it is enough to prove the continuity of this family in $\mathcal{C}_n^{loc}(Z)$. This is a consequence of the proposition 4.2.1. We obtain then the f-analyticity of this family using the theorem 4.3.1 as \mathcal{U} is a reduced complex space by the theorem 3.5.1. The holomorphic classifying map for this family is the map res which is then an isomorphism. This implies that $\mathcal{C}_n^f(Z)$ is a reduced complex space in a neighbourhood of each of its points. ■

4.5 A compactness criterion for the connected components of the reduced complex space $\mathcal{C}_n^f(Z)$.

For a reduced complex space Z which is compact, the compactness of the connected components of $\mathcal{C}_n(Z)$ is a consequence of the existence of a \mathcal{C}^1 $2n$ -differential form on Z which is d -closed and such that its (n, n) part is positive definite in the Lelong sense. Indeed, this gives that the volume (computed with this (n, n) part) of the n -cycles is constant on connected components. The result follows then from Bishop's theorem [Bish.64].

In the case of a non compact strongly n -concave reduced complex space Z we have the following analogous result :

Proposition 4.5.1 *Let Z be a reduced complex space which is strongly n -concave. Assume that there exists on Z a \mathcal{C}^1 $2n$ -differential form ω which is d -closed with compact support and such that its (n, n) part is positive definite in the Lelong sense in a neighbourhood of $K := \varphi^{-1}([1, 2])$, and which is everywhere non negative in the Lelong sense. Then the connected components of $\mathcal{C}_n^f(Z)$ are compact.*

PROOF. For $\alpha < 1$ near enough to 1 and for any continuous hermitian metric h on Z there exists a constant C such that the following inequality holds:

$$vol_h(X \cap Z_\alpha) \leq C \cdot \int_X \omega \quad \text{for any cycle } X \in \mathcal{C}_n^f(Z).$$

As the function $X \mapsto \int_X \omega$ is locally constant on $\mathcal{C}_n^f(Z)$ because $d\omega = 0$ (see [B-M 1] ch.IV), we have an uniform bound for the volume of $X \cap Z_\alpha$ for X in a given connected component of $\mathcal{C}_n^f(Z)$. This implies that the closure of the image of this connected component in $\mathcal{C}_n^f(Z_\alpha)$ is compact, thanks to Bishop's theorem (see [Bish.64] or [B-M 1] ch.IV). But the restriction map $\mathcal{C}_n^f(Z) \rightarrow \mathcal{C}_n^f(Z_\alpha)$ is a homeomorphism by the proposition 4.2.1, so the image of a connected component is closed and then compact. ■

5 References.

- [B.75] Barlet, D. *Espace analytique réduit* ... Séminaire F.Norguet 1974-1975, L.N. 482 (1975), p.1-158.
- [B.08] Barlet, D. *Reparamétrisation universelle de familles f -analytiques de cycles et théorème de f -aplatissement géométrique* Comment. Math. Helv. 83 n°4 (2008) p.869-888.
- [B.13] Barlet D. *Quasi-proper meromorphic equivalence relations*, Math. Z. (2013), vol. 273, p. 461-484.
- [B.15-a] Barlet, D. *Strongly quasi-proper maps and the f -flattening theorem*, arXiv: 1504.01579 (math.CV)
- [B-M 1] Barlet, D. et Magnusson, J. *Cycles analytiques complexes I : théorèmes de préparation des cycles*, Cours Spécialisés 22, Société Mathématique de France 2014.
- [B-M 2] Barlet, D. et Magnusson, J. *Cycles analytiques complexes II*, livre en préparation.
- [Bish.64] Bishop, E. *Conditions for the analyticity of certain sets*, Mich. Math. J. 11, (1964), p. 289-304.
- [S-T] Siu, Y.T. and Trautman, G. *Gap-Sheaves and Extensions of Coherent Analytic Subsheaves* Lecture Notes 172 (1971) Springer Verlag.