



HAL
open science

ON THE RANGE OF THE CAMPANINO AND PETRITIS RANDOM WALK

Nadine Guillotin-Plantard, Françoise Pene

► **To cite this version:**

Nadine Guillotin-Plantard, Françoise Pene. ON THE RANGE OF THE CAMPANINO AND PETRITIS RANDOM WALK. 2015. hal-01183813

HAL Id: hal-01183813

<https://hal.science/hal-01183813>

Preprint submitted on 11 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE RANGE OF THE CAMPANINO AND PÉTRITIS RANDOM WALK

NADINE GUILLOTIN-PLANTARD AND FRANÇOISE PÈNE

ABSTRACT. We are interested in the behaviour of the range of the Campanino and Pétritis random walk [2], namely a simple random walk on the lattice \mathbb{Z}^2 with random orientations of the horizontal layers. We also study the range of random walks in random scenery, from which the asymptotic behaviour of the range of the first coordinate of the Campanino and Pétritis random walk can be deduced.

1. INTRODUCTION AND MAIN RESULTS

We consider the random walk on a randomly oriented lattice $M = (M_n)_n$ considered by Campanino and Pétritis [2]. It is a particular example of transient 2-dimensional random walk in random environment. We fix a $p \in (0, 1)$ corresponding to the probability for M to stay on the same horizontal line. The environment is given by a sequence $\epsilon = (\epsilon_k)_{k \in \mathbb{Z}}$ of i.i.d. (independent identically distributed) centered random variables with values in $\{\pm 1\}$ and defined on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$. Given ϵ , M is a closest-neighbours random walk on \mathbb{Z}^2 starting from 0 (i.e. $\mathbb{P}^\epsilon(M_0 = 0) = 1$) and with transition probabilities

$$\mathbb{P}^\epsilon(M_{n+1} = (x + \epsilon_y, y) | M_n = (x, y)) = p, \quad \mathbb{P}^\epsilon(M_{n+1} = (x, y \pm 1) | M_n = (x, y)) = \frac{1-p}{2}.$$

We will write \mathbb{P} for the annealed expectation, that is the integration of \mathbb{P}^ϵ with respect to \mathbb{P} . In the papers [7] and [3] respectively, a functional limit theorem and a local limit theorem were proved for the random walk M under the annealed measure \mathbb{P} . In this note we are interested in the asymptotic behaviour of the range \mathcal{R}_n of M , i.e. of the number of sites visited by M before time n :

$$\mathcal{R}_n := \#\{M_0, \dots, M_n\}.$$

Since we know (see [2, 3]) that M is transient for almost every environment ϵ , it is not surprising that \mathcal{R}_n has order n . More precisely we prove the following result.

Proposition 1. *The sequence $(\mathcal{R}_n/n)_n$ converges \mathbb{P} -almost surely to $\mathbb{P}[M_j \neq 0, \forall j \geq 1]$.*

We observe that the almost sure convergence result stated for the annealed probability \mathbb{P} implies directly the same convergence result for the quenched probability \mathbb{P}^ϵ for \mathbb{P} -almost every ϵ .

Since $\mathcal{R}_n \leq n + 1$, due to the Lebesgue dominated convergence theorem, we directly obtain the next result.

Corollary 2. *We have $\mathbb{E}[\mathcal{R}_n] \sim n\mathbb{P}[M_j \neq 0, \forall j \geq 1]$ and $\mathbb{E}^\epsilon[\mathcal{R}_n] \sim n\mathbb{P}[M_j \neq 0, \forall j \geq 1]$ for \mathbb{P} -almost every ϵ .*

2000 *Mathematics Subject Classification.* 60F05; 60G52.

Key words and phrases. Range; random walk in random scenery; local limit theorem; local time; stable process
This research was supported by the french ANR project MEMEMO2.

This last result contradicts the result expected by Le Ny in [11] for the behaviour of the quenched expectation. The main difficulty of this model is that M has stationary increments under the annealed probability \mathbb{P} and is a Markov chain under the quenched probability \mathbb{P}^ϵ for \mathbb{P} -almost every ϵ but M is not a Markov chain with stationary increments (neither for \mathbb{P} nor for \mathbb{P}^ϵ). This complicates seriously our study.

Remark 3. For \mathbb{P} -almost every ϵ , $(M_n)_n$ is a transient Markov chain with respect to \mathbb{P}^ϵ , hence $\mathbb{P}^\epsilon[M_j \neq 0, \forall j \geq 1] = 1/\sum_{n \geq 0} \mathbb{P}^\epsilon(M_n = 0)$ and $\mathbb{P}[M_j \neq 0, \forall j \geq 1] = \mathbb{E}\left[1/\sum_{n \geq 0} \mathbb{P}^\epsilon(M_n = 0)\right]$.

The Campanino and Pétritis random walk is closely related to Random Walks in Random Scenery (RWRS). This fact was first noticed in [7]. More precisely the first coordinate of the Campanino and Pétritis random walk can be viewed as a generalized RWRS, the second coordinate being a lazy random walk on \mathbb{Z} (see Section 5 of [3] for the details). The main point is that the range of the first coordinate of the Campanino and Pétritis random walk can easily be deduced from the following results about the range of random walks in random scenery. Let us recall the definition of the RWRS. Let $\xi := (\xi_y, y \in \mathbb{Z})$ and $X := (X_k, k \geq 1)$ be two independent sequences of independent identically distributed random variables taking their values in \mathbb{Z} . The sequence ξ is called the *random scenery*. The sequence X is the sequence of increments of the *random walk* $(S_n, n \geq 0)$ defined by $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$, for $n \geq 1$. The *random walk in random scenery* (RWRS) Z is then defined by

$$Z_0 := 0 \text{ and } \forall n \geq 1, Z_n := \sum_{k=1}^n \xi_{S_k}.$$

Denoting by $N_n(y)$ the local time of the random walk S :

$$N_n(y) := \#\{k = 1, \dots, n : S_k = y\},$$

it is straightforward to see that Z_n can be rewritten as $Z_n = \sum_y \xi_y N_n(y)$.

As in [9], the distribution of ξ_0 is assumed to belong to the normal domain of attraction of a strictly stable distribution \mathcal{S}_β of index $\beta \in (0, 2]$, with characteristic function ϕ given by

$$\phi(u) = e^{-|u|^\beta(A_1 + iA_2 \operatorname{sgn}(u))} \quad u \in \mathbb{R},$$

where $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \leq |\tan(\pi\beta/2)|$. When $\beta > 1$, this implies that $\mathbb{E}[\xi_0] = 0$. When $\beta = 1$, we assume the symmetry condition $\sup_{t>0} |\mathbb{E}[\xi_0 \mathbf{1}_{\{|\xi_0| \leq t\}}]| < +\infty$.

Concerning the random walk, the distribution of X_1 is assumed to belong to the normal basin of attraction of a stable distribution \mathcal{S}'_α with index $\alpha \in (0, 2]$, with characteristic function ψ given by

$$\psi(u) = e^{-|u|^\alpha(C_1 + iC_2 \operatorname{sgn}(u))} \quad u \in \mathbb{R},$$

where $0 < C_1 < \infty$ and $|C_1^{-1}C_2| \leq |\tan(\pi\alpha/2)|$. In the particular case where $\alpha = 1$, we assume that $C_2 = 0$. Moreover we assume that the additive group \mathbb{Z} is generated by the support of the distribution of X_1 .

Then the following weak convergences hold in the space of càdlàg real-valued functions defined on $[0, \infty)$ endowed with the Skorohod J_1 -topology :

$$\begin{aligned} & \left(n^{-\frac{1}{\alpha}} S_{[nt]} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (Y(t))_{t \geq 0}, \\ & \left(n^{-\frac{1}{\beta}} \sum_{k=0}^{[nx]} \xi_k \right)_{x \geq 0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (U(x))_{x \geq 0} \quad \text{and} \quad \left(n^{-\frac{1}{\beta}} \sum_{k=[-nx]}^1 \xi_k \right)_{x \geq 0} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (U(-x))_{x \geq 0} \end{aligned}$$

where $(U(x))_{x \geq 0}$, $(U(-x))_{x \geq 0}$ and $(Y(t))_{t \geq 0}$ are three independent Lévy processes such that $U(0) = 0$, $Y(0) = 0$, $Y(1)$ has distribution \mathcal{S}'_α , $U(1)$ and $U(-1)$ have distribution \mathcal{S}_β . We will denote by $(L_t(x))_{x \in \mathbb{R}, t \geq 0}$ a continuous version with compact support of the local time of the process $(Y(t))_{t \geq 0}$. Let us define

$$\delta := 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}.$$

In the case $\alpha \in (1, 2]$ and $\beta \in (0, 2]$, Kesten and Spitzer [9] proved the convergence in distribution of $(n^{-\delta} Z_{[nt]})_{t \geq 0}$, $n \geq 1$ (with respect to the J_1 -metric), to a process $\Delta = (\Delta_t)_{t \geq 0}$ defined in this case by

$$\Delta_t := \int_{\mathbb{R}} L_t(x) dU(x).$$

This process Δ is called Kesten-Spitzer process in the literature.

When $\alpha \in (0, 1)$ (when the random walk S is transient) and $\beta \in (0, 2] \setminus \{1\}$, $(n^{-\frac{1}{\beta}} Z_{[nt]})_{t \geq 0}$, $n \geq 1$ converges in distribution (with respect to the M_1 -metric), to $(\Delta_t := c_0 U_t)_{t \geq 0}$ for some $c_0 > 0$.

When $\alpha = 1$ and $\beta \in (0, 2] \setminus \{1\}$, $(n^{-\frac{1}{\beta}} (\log n)^{\frac{1}{\beta}-1} Z_{[nt]})_{t \geq 0}$, $n \geq 1$ converges in distribution (with respect to the M_1 -metric), to $(\Delta_t := c_1 U_t)_{t \geq 0}$ for some $c_1 > 0$.

Hence in any of the cases considered above, $(Z_{[nt]}/a_n)_{t \geq 0}$ converges in distribution (with respect to the M_1 -metric) to some process Δ , with

$$a_n := \begin{cases} n^{1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}} & \text{if } \alpha \in (1, 2] \\ n^{\frac{1}{\beta}} (\log n)^{1-\frac{1}{\beta}} & \text{if } \alpha = 1 \\ n^{\frac{1}{\beta}} & \text{if } \alpha \in (0, 1). \end{cases}$$

We are interested in the asymptotic behaviour of the range \mathcal{R}_n of the RWRS Z , i.e. of the number of sites visited by Z before time n :

$$\mathcal{R}_n := \#\{Z_0, \dots, Z_n\}.$$

In the case when the RWRS is transient, we use the same argument as for $(M_n)_n$ and obtain the same kind of result.

Proposition 4. *Let $\alpha \in (0, 2]$ and $\beta \in (0, 1)$. Then, $(\mathcal{R}_n/n)_n$ converges \mathbb{P} -almost surely to $\mathbb{P}[Z_j \neq 0, \forall j \geq 1]$.*

For recurrent random walks in random scenery, we distinguish the easiest case when ξ_1 takes its values in $\{-1, 0, 1\}$. In that case, $\beta = 2$, U is the standard real Brownian motion,

$$a_n = \begin{cases} n^{1-\frac{1}{2\alpha}} & \text{if } \alpha \in (1, 2] \\ \sqrt{n \log n} & \text{if } \alpha = 1 \\ \sqrt{n} & \text{if } \alpha \in (0, 1) \end{cases}$$

and the limiting process Δ is either the Kesten-Spitzer process (case $\alpha \in (1, 2]$) or the real Brownian motion (case $\alpha \in (0, 1]$). Remark that in any case the limiting process is symmetric.

Proposition 5. *If $\alpha \in (0, 2]$ and if ξ_1 takes its values in $\{-1, 0, 1\}$. Then*

$$\frac{\mathcal{R}_n}{a_n} = \frac{\sup_{t \in [0,1]} Z_{[nt]} - \inf_{t \in [0,1]} Z_{[nt]} + 1}{a_n} \xrightarrow{\mathcal{L}} \sup_{t \in [0,1]} \Delta_t - \inf_{t \in [0,1]} \Delta_t$$

and

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} = 2 \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t \right].$$

We also study the asymptotic behaviour of the range of the first coordinate of the Campanino and Pétritis random walk. Let $\mathcal{R}_n^{(1)}$ be the number of vertical lines visited by $(M_k)_k$ up to time n , i.e.

$$\mathcal{R}_n^{(1)} := \#\{x \in \mathbb{Z} : \exists k = 0, \dots, n, \exists y \in \mathbb{Z} : M_k = (x, y)\}.$$

Let us recall that it has been shown in [7] that the first coordinate of $M_{[nt]}$ normalized by $n^{\frac{3}{4}}$ converges in distribution to $K_p \Delta_t^{(0)}$, where $K_p := \frac{p}{(1-p)^{\frac{1}{4}}}$ and where $\Delta^{(0)}$ is the Kesten-Spitzer process Δ with U and Y two independent standard Brownian motions.

Proposition 6 (Range of the first coordinate of the Campanino and Pétritis random walk). *$(\mathcal{R}_n^{(1)}/n^{\frac{3}{4}})_n$ converges in distribution to $K_p \left(\sup_{t \in [0,1]} \Delta_t^{(0)} - \inf_{t \in [0,1]} \Delta_t^{(0)} \right)$. Moreover*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n^{(1)}]}{n^{\frac{3}{4}}} = 2K_p \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t^{(0)} \right].$$

Since the second coordinate of the Campanino and Pétritis random walk is a true random walk, the asymptotic behaviour of its range is well known [10].

The range of RWRS in the general case $\beta \in (1, 2]$ is much more delicate. Indeed, the fact that \mathcal{R}_n is less than $\sup_{t \in [0,1]} Z_{[nt]} - \inf_{s \in [0,1]} Z_{[ns]} + 1$ will only provide an upper bound; we use a separate argument to obtain the lower bound insuring that \mathcal{R}_n has order a_n .

Proposition 7. *Let $\alpha \in (0, 2]$ and $\beta \in (1, 2]$. Then*

$$0 < \liminf_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \leq \limsup_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} < \infty.$$

We actually prove that $\limsup_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \leq \mathbb{E}[\sup_{t \in [0,1]} \Delta_t - \inf_{t \in [0,1]} \Delta_t]$. The question whether $\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} = \mathbb{E}[\sup_{t \in [0,1]} \Delta_t - \inf_{t \in [0,1]} \Delta_t]$ or not is still open.

The paper is organized as follows. Section 2 provides the proof of Propositions 1 and 4. Section 3 is devoted to the proof of Propositions 5, 6 and 7.

2. BEHAVIOUR OF THE RANGE IN TRANSIENT CASES

Let (Ω, μ, T) be an ergodic probability dynamical system and let $f : \Omega \rightarrow \mathbb{Z}^d$ be a measurable function. We consider the process $(M_n)_{n \geq 0}$ defined by $M_n = \sum_{k=0}^{n-1} f \circ T^k$ for $n \geq 1$ and $M_0 = 0$. Now we assume that $\sum_{n \geq 0} \mathbb{P}(M_n = 0) < +\infty$, so $(M_n)_n$ is transient. Let \mathcal{R}_n be the range of $(M_n)_n$, that is $\mathcal{R}_n := \#\{M_0, \dots, M_n\}$.

Proposition 8. *Assume that $\mathbb{P}(M_n = 0) = O(n^{-\theta})$ for some $\theta > 1$. Then $\lim_{n \rightarrow +\infty} \mathcal{R}_n/n = \mu(M_j \neq 0, \forall j \geq 1)$, μ -almost surely.*

Proof. It is worth noting that

$$\mathcal{R}_n = 1 + \sum_{k=0}^{n-1} \mathbf{1}_{\{M_{k+j} \neq M_k, \forall j=1, \dots, n-k\}}.$$

Indeed $M_{k+j} \neq M_k, \forall j = 1, \dots, n-k$ means that the site M_k visited at time k is not visited again before time n . We define now

$$\mathcal{R}'_n := 1 + \sum_{k=0}^{n-1} \mathbf{1}_{\{M_{k+j} - M_k \neq 0, \forall j \geq 1\}}.$$

We first prove the almost sure convergence of $(\mathcal{R}'_n/n)_n$. To this end, we observe that \mathcal{R}'_n can be rewritten

$$\mathcal{R}'_n = 1 + \sum_{k=0}^{n-1} 1_{\{M_j \neq 0, \forall j \geq 1\}} \circ T^k.$$

By ergodicity of T , $(\mathcal{R}'_n/n)_n$ converges almost surely to $\mathbb{P}[M_j \neq 0, \forall j \geq 1]$. Now let us estimate $\mathcal{R}_n - \mathcal{R}'_n$. We have

$$\begin{aligned} \|\mathcal{R}_n - \mathcal{R}'_n\|_1 &= \mathbb{E}[\mathcal{R}_n - \mathcal{R}'_n] \\ &\leq \sum_{k=0}^{n-1} \mathbb{P}(\exists j \geq n-k, M_{k+j} - M_k = 0) \\ &\leq \sum_{k=0}^{n-1} \mathbb{P}(\exists j \geq n-k, M_j = 0) \\ &\leq \sum_{k=1}^n \sum_{j \geq k} \mathbb{P}(M_j = 0) \\ &\leq \sum_{k=1}^n \sum_{j \geq k} C j^{-\theta} \\ &= \begin{cases} O(n^{2-\theta}) & \text{when } 1 < \theta < 2 \\ O(\log n) & \text{when } \theta = 2 \\ O(1) & \text{when } \theta > 2 \end{cases} \end{aligned}$$

using the stationarity of the increments of $(M_n)_n$. Hence, when $1 < \theta < 2$, $\|(\mathcal{R}_n - \mathcal{R}'_n)/n\|_1 = O(n^{1-\theta})$. Let $\gamma > 0$ be such that $\gamma(\theta - 1) > 1$. Due to the Borel-Cantelli Lemma, $((\mathcal{R}_{k^\gamma} - \mathcal{R}'_{k^\gamma})/k^\gamma)_k$ to 0, and so $(\mathcal{R}_{k^\gamma}/k^\gamma)_k$ converges almost surely to $\mathbb{P}[M_j \neq 0, \forall j \geq 1]$. To conclude, we use the increase of $(\mathcal{R}_n)_n$ which gives that

$$\frac{\mathcal{R}_{\lfloor n^{\frac{1}{\gamma}} \rfloor}^\gamma}{n} \leq \frac{\mathcal{R}_n}{n} \leq \frac{\mathcal{R}_{\lceil n^{\frac{1}{\gamma}} \rceil}^\gamma}{n}.$$

We conclude by noticing that $\lceil n^{\frac{1}{\gamma}} \rceil^\gamma \sim n$ and $\lfloor n^{\frac{1}{\gamma}} \rfloor^\gamma \sim n$.

The cases $\theta = 2$ and $\theta > 2$ can be handled in a similar way. \square

Proof of Proposition 1. Let us consider $\Omega := \{-1, 1\}^{\mathbb{Z}} \times \{-1, 0, 1\}^{\mathbb{Z}}$ and the transformation T on Ω given by $T((\epsilon_k)_k, (\omega_k)_k) = ((\epsilon_{k+\omega_0})_k, (\omega_{k+1})_k)$. This transformation preserves the probability measure $\mu := (\frac{\delta_1 + \delta_{-1}}{2})^{\otimes \mathbb{Z}} \otimes (p\delta_0 + \frac{1-p}{2}\delta_1 + \frac{1-p}{2}\delta_{-1})^{\otimes \mathbb{Z}}$ and is ergodic (see for instance [8], p.162). We also set $f((\epsilon_k)_k, (\omega_k)_k) = (\epsilon_0, 0)$ if $\omega_0 = 0$, $f((\epsilon_k)_k, (\omega_k)_k) = (0, \omega_0)$ otherwise.

We observe that $(M_j)_{j \geq 1}$ has the same distribution under \mathbb{P} as $(\sum_{k=0}^{j-1} f \circ T^k)_{j \geq 1}$ under μ . We conclude by Proposition 8 since we know from [3] that $\mathbb{P}(M_n = 0) = O(n^{-\theta})$ with $\theta = 5/4$. \square

Proof of Proposition 4. We consider $\Omega := \mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}}$ and the transformation T on Ω given by $T((\alpha_k)_k, (\epsilon_k)_k) = ((\alpha_{k+1})_k, (\epsilon_{k+\alpha_0})_k)$. This transformation preserves the probability measure $\mu := (\mathbb{P}_{S_1})^{\otimes \mathbb{Z}} \otimes (\mathbb{P}_{\xi_1})^{\otimes \mathbb{Z}}$. This time we set $f((\alpha_k)_k, (\epsilon_k)_k) = \epsilon_0$. With these choices, $(Z_j)_{j \geq 1}$ has the same distribution under \mathbb{P} as $(\sum_{k=1}^j f \circ T^k)_{j \geq 1}$ under μ . Again we conclude thanks to Proposition 8, to the ergodicity of T (see for instance [8], p.162) and to the local limit theorems established in [3] (Theorems 1 and 2) and [4] (Theorem 3). \square

3. RANGE OF RECURRENT RANDOM WALKS IN RANDOM SCENERY

In this section we prove Propositions 5, 6 and 7. We write $M_n^{(1)}$ for the first coordinate of the Campanino and Pétritis random walk M_n .

For Propositions 5, 6, we observe that $\mathcal{R}_n = \max_{0 \leq k \leq n} Z_k - \min_{0 \leq k \leq n} Z_k + 1$ and $\mathcal{R}_n^{(1)} = \max_{0 \leq k \leq n} M_k^{(1)} - \min_{0 \leq k \leq n} M_k^{(1)} + 1$ whereas for Proposition 7, we only have $\mathcal{R}_n \leq \max_{0 \leq k \leq n} Z_k - \min_{0 \leq k \leq n} Z_k + 1$. Hence the convergence of the means in Propositions 5 and 6 and the upper bound in Proposition 7 will come from lemmas 9 and 10 below. Let us start by the convergence in distribution.

Proof of the convergences in distribution. Due to the convergence for the M_1 -topology of $((a_n^{-1} Z_{[nt]})_t)_n$ to $(\Delta_t)_t$ as n goes to infinity, we know (see Section 12.3 in [13]) that $(a_n^{-1}(\max_{0 \leq k \leq n} Z_k - \min_{0 \leq k \leq n} Z_k))_n$ converges in distribution to $\sup_{t \in [0,1]} \Delta_t - \inf_{s \in [0,1]} \Delta_s$ as n goes to infinity.

Due to [7], $((M_{[nt]}^{(1)}/n^{\frac{3}{4}})_t)_n$ converges in distribution to $(K_p \Delta_t^{(0)})_t$ in the Skorohod space endowed with the J_1 -metric. Hence $(n^{-\frac{3}{4}}(\max_{k=0, \dots, n} M_k^{(1)} - \min_{\ell=0, \dots, n} M_\ell^{(1)}))_n$ converges in distribution to $K_p(\sup_{t \in [0,1]} \Delta_t^{(0)} - \inf_{s \in [0,1]} \Delta_s^{(0)})$. \square

Lemma 9 (RWRS). *Assume $\beta > 1$, then*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\max_{k=0, \dots, n} Z_k]}{a_n} = \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t \right].$$

Lemma 10 (First coordinate of the Campanino and Pétritis random walk).

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\max_{k=0, \dots, n} M_k^{(1)}]}{n^{\frac{3}{4}}} = K_p \mathbb{E} \left[\sup_{t \in [0,1]} \Delta_t^{(0)} \right].$$

Proof of Lemma 9. As explained above, we know that $(a_n^{-1} \max_{0 \leq k \leq n} Z_k)_n$ converges in distribution to $\sup_{t \in [0,1]} \Delta_t$ as n goes to infinity. Now let us prove that this sequence is uniformly integrable. To this end we will use the fact that, conditionally to the walk S , the increments of $(Z_n)_n$ are centered and positively associated. Let $\beta' \in (1, \beta)$ be fixed. Due to Theorem 2.1 of [6], there exists some constant $c_{\beta'} > 0$ such that

$$\begin{aligned} \mathbb{E} \left[\left| \max_{j=0, \dots, n} Z_j \right|^{\beta'} \middle| S \right] &\leq \mathbb{E} \left[\max_{j=0, \dots, n} |Z_j|^{\beta'} \middle| S \right] \\ &\leq c_{\beta'} \mathbb{E} \left[|Z_n|^{\beta'} \middle| S \right] \end{aligned}$$

so

$$\begin{aligned} \mathbb{E} \left[\left| \max_{j=0, \dots, n} Z_j \right|^{\beta'} \right] &= \mathbb{E} \left[\mathbb{E} \left[\left| \max_{j=0, \dots, n} Z_j \right|^{\beta'} \middle| S \right] \right] \\ &\leq c_{\beta'} \mathbb{E} \left[|Z_n|^{\beta'} \right]. \end{aligned}$$

It remains now to prove that $\mathbb{E}[|Z_n|^{\beta'}] = O(a_n^{\beta'})$.

Let us first consider the easiest case when the random scenery is square integrable that is $\beta = 2$, then we take $\beta' = 2$ in the above computations and observe that $\mathbb{E}[|Z_n|^2] = \mathbb{E}[\xi_0^2] \mathbb{E}[V_n]$, where V_n

is the number of self-intersections up to time n of the random walk S , i.e. $V_n = \sum_x (N_n(x))^2 = \sum_{i,j=1}^n \mathbf{1}_{S_i=S_j}$. Usual computations (see Lemma 2.3 in [1]) give that

$$\mathbb{E}[V_n] = \sum_{i,j=1}^n \mathbb{P}(S_{i-j} = 0) \sim c'(a_n)^2$$

and the result follows.

When $\beta \in (1, 2)$, let us define $V_n(\beta)$ as follows

$$V_n(\beta) := \sum_{y \in \mathbb{Z}} (N_n(y))^\beta.$$

Due to Lemma 2 of [12],

$$(1) \quad \mathbb{E} \left[|Z_n|^{\beta'} \right] = \frac{\Gamma(\beta' + 1)}{\pi} \sin \left(\frac{\pi \beta'}{2} \right) \int_{\mathbb{R}} \frac{1 - \operatorname{Re}(\varphi_{Z_n}(t))}{|t|^{\beta'+1}} dt,$$

where φ_{Z_n} stands for the characteristic function of Z_n , which is given by

$$(2) \quad \forall t \in \mathbb{R}, \quad \varphi_{Z_n}(t) := \mathbb{E}[e^{itZ_n}] = \mathbb{E} \left[\mathbb{E} \left[e^{itZ_n} \mid (S_k)_k \right] \right] = \mathbb{E} \left[\prod_{y \in \mathbb{Z}} \varphi_\xi(tN_n(y)) \right].$$

Due to our assumptions on ξ , we know that $1 - \varphi_\xi(u) = |u|^\beta (A_1 + iA_2 \operatorname{sgn}(u))(1 + o(1))$ as u goes to 0. Let $A, B > 0$ be such that $|1 - \varphi_\xi(u)| < B|u|^\beta$ for every real number u satisfying $|u| < A$. Hence, for every t such that $|t| < A(V_n(\beta))^{-\frac{1}{\beta}}$, we have $|tN_n(y)| \leq A$ and so $|1 - \varphi_\xi(tN_n(y))| \leq B|t|^\beta (N_n(y))^\beta$ and

$$\begin{aligned} \left| 1 - \operatorname{Re} \left(\prod_{y \in \mathbb{Z}} \varphi_\xi(tN_n(y)) \right) \right| &\leq \left| 1 - \left(\prod_{y \in \mathbb{Z}} \varphi_\xi(tN_n(y)) \right) \right| \\ &\leq \sum_{y \in \mathbb{Z}} |1 - \varphi_\xi(tN_n(y))| \\ &\leq B|t|^\beta V_n(\beta). \end{aligned}$$

Hence

$$(3) \quad \begin{aligned} \int_{|t| < A(V_n(\beta))^{-\frac{1}{\beta}}} \frac{|1 - \operatorname{Re} \left(\prod_{y \in \mathbb{Z}} \varphi_\xi(tN_n(y)) \right)|}{|t|^{\beta'+1}} dt &\leq \int_{|t| < A(V_n(\beta))^{-\frac{1}{\beta}}} \frac{B|t|^\beta V_n(\beta)}{|t|^{\beta'+1}} dt \\ &\leq BV_n(\beta) \int_{|t| < A(V_n(\beta))^{-\frac{1}{\beta}}} |t|^{\beta-\beta'-1} dt \\ &\leq \frac{2A^{\beta-\beta'} B}{\beta - \beta'} (V_n(\beta))^{\frac{\beta'}{\beta}}. \end{aligned}$$

Moreover

$$(4) \quad \begin{aligned} \int_{|t| \geq A(V_n(\beta))^{-\frac{1}{\beta}}} \frac{|1 - \operatorname{Re} \left(\prod_{y \in \mathbb{Z}} \varphi_\xi(tN_n(y)) \right)|}{|t|^{\beta'+1}} dt &\leq 2 \int_{|t| \geq A(V_n(\beta))^{-\frac{1}{\beta}}} |t|^{-\beta'-1} dt \\ &\leq 4\beta' A^{-\beta'} (V_n(\beta))^{\frac{\beta'}{\beta}}. \end{aligned}$$

Putting together (1), (2), (3) and (4), we obtain that there exists some constant $C > 0$ such that for every n

$$\mathbb{E}[|Z_n|^{\beta'}] \leq C \mathbb{E} \left[(V_n(\beta))^{\frac{\beta'}{\beta}} \right].$$

If $\alpha > 1$, due to Lemma 3.3 of [5], we know that $\mathbb{E}[V_n(\beta)] = O\left(a_n^\beta\right)$ and so

$$(5) \quad \mathbb{E}\left[(V_n(\beta))^{\frac{\beta'}{\beta}}\right] = O\left(a_n^{\beta'}\right).$$

If $\alpha \in (0, 1]$, using Hölder's inequality, we have

$$\mathbb{E}[V_n(\beta)] \leq \mathbb{E}[R_n]^{1-\frac{\beta}{2}} \mathbb{E}[V_n]^{\frac{\beta}{2}}.$$

Now if $\alpha = 1$, we know that $\mathbb{E}[R_n] \sim c \frac{n}{\log n}$ (see for instance Theorem 6.9, page 398 in [10]) and $\mathbb{E}[V_n] \sim cn \log n$ so $\mathbb{E}[V_n(\beta)] = O\left(a_n^\beta\right)$ with $a_n = n^{\frac{1}{\beta}}(\log n)^{1-\frac{1}{\beta}}$. In the case $\alpha \in (0, 1)$, the random walk is transient and the expectations of R_n and V_n behaves as n , we deduce that $\mathbb{E}[V_n(\beta)] = O\left(a_n^\beta\right)$ with $a_n = n^{\frac{1}{\beta}}$.

We conclude that

$$\lim_{n \rightarrow +\infty} \mathbb{E}\left[\max_{j=0, \dots, n} \frac{Z_j}{a_n}\right] = \mathbb{E}\left[\max_{t \in [0, 1]} \Delta_t\right].$$

□

Proof of Lemma 10. We know that $(n^{-\frac{3}{4}} \max_{k=0, \dots, n} M_k^{(1)})_n$ converges in distribution to $K_p \sup_{t \in [0, 1]} \Delta_t^{(0)}$. To conclude, it is enough to prove that this sequence is uniformly integrable. To this end we will prove that it is bounded in L^2 .

Recall that the second coordinate of the Campanino and Pétritis random walk is a random walk. Let us write it $(S_n)_n$. Observe that

$$M_n^{(1)} := \sum_{k=1}^n \varepsilon_{S_k} \mathbb{1}_{\{S_k = S_{k-1}\}} = \sum_{y \in \mathbb{Z}} \varepsilon_y \tilde{N}_n(y),$$

with $\tilde{N}_n(y) := \#\{k = 1, \dots, n : S_k = S_{k-1} = y\}$. Observe that \tilde{N} is measurable with respect to the random walk S and that $0 \leq \tilde{N}_n(y) \leq N_n(y)$.

Conditionally to the walk S , the increments of $(M_n^{(1)})_n$ are centered and positively associated. It follows from Theorem 2.1 of [6] that

$$\begin{aligned} \mathbb{E}\left[\left|\max_{j=0, \dots, n} M_j^{(1)}\right|^2 \middle| S\right] &\leq c_2 \mathbb{E}\left[|M_n^{(1)}|^2 \middle| S\right] \\ &\leq c_2 \sum_{y \in \mathbb{Z}} (\tilde{N}_n(y))^2 \leq c_2 V_n, \end{aligned}$$

where again $V_n = \sum_{y \in \mathbb{Z}} (N_n(y))^2$. Therefore

$$\mathbb{E}\left[\left|\max_{j=0, \dots, n} M_j^{(1)}\right|^2\right] \leq c_2 \mathbb{E}[V_n].$$

Again the result follows from the fact that $\mathbb{E}[V_n] \sim c'n^{\frac{3}{2}}$. □

Proof of the lower bound of Proposition 7. Let $\mathcal{N}_n(x) := \#\{k = 1, \dots, n : Z_k = x\}$. Applying the Cauchy-Schwarz inequality to $n = \sum_x \mathcal{N}_n(x) \mathbf{1}_{\{\mathcal{N}_n(x) > 0\}}$, we obtain

$$n^2 \leq \sum_y \mathbf{1}_{\{\mathcal{N}_n(y) > 0\}} \sum_x (\mathcal{N}_n(x))^2 = \mathcal{R}_n \mathcal{V}_n,$$

with $\mathcal{V}_n = \sum_x (\mathcal{N}_n(x))^2 = \sum_{i,j=1}^n \mathbf{1}_{\{Z_i=Z_j\}}$ the number of self-intersections of Z up to time n and so using Jensen's inequality,

$$\frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \geq \frac{n^2}{a_n} \mathbb{E}[(\mathcal{V}_n)^{-1}] \geq \frac{n^2}{a_n} \mathbb{E}[\mathcal{V}_n]^{-1}.$$

Moreover, using the local limit theorems for the RWRS proved in [3, 4],

$$\mathbb{E}[\mathcal{V}_n] = n + 2 \sum_{1 \leq i < j \leq n} \mathbb{P}(Z_{j-i} = 0) \sim C' \frac{n^2}{a_n}.$$

Hence

$$\liminf_{n \rightarrow +\infty} \frac{\mathbb{E}[\mathcal{R}_n]}{a_n} \geq \frac{1}{C'} > 0.$$

□

REFERENCES

- [1] E. Bolthausen, *A central limit theorem for two-dimensional random walks in random sceneries*. Ann. Probab. **17** (1989), No. 1, 108–115.
- [2] M. Campanino and D. Pétritis, *Random walks on randomly oriented lattices*. Mark. Proc. Relat. Fields (2003), 9, 391–412.
- [3] F. Castell, N. Guillotin-Plantard, F. Pène and Br. Schapira, *A local limit theorem for random walks in random scenery and on randomly oriented lattices*. Annals of Probability (2011), Vol. 39, No 6, 2079–2118.
- [4] F. Castell, N. Guillotin-Plantard and F. Pène, *Limit theorems for one and two-dimensional random walks in random scenery*. Annales de l'Institut Henri Poincaré - Probabilités et Statistiques (2013), Vol. 49, No 2, 506–528.
- [5] C. Dombry and N. Guillotin-Plantard, *Discrete approximation of a stable self-similar stationary increments process* Bernoulli (2009), Vol. 15, No 1, 195–222.
- [6] X. Gong, *Maximal ϕ -inequalities for demimartingales*, J. Inequal. Appl. 2011, 2011:59, 10 pp.
- [7] N. Guillotin-Plantard and A. Le Ny *A functional limit theorem for a 2d- random walk with dependent marginals* Electronic Communications in Probability (2008), Vol. **13**, 337–351.
- [8] S. Kalikow and R. McCutcheon, *An outline of ergodic theory* Cambridge University Press (2010).
- [9] H. Kesten and F. Spitzer, *A limit theorem related to a new class of self-similar processes*. Z. Wahrsch. Verw. Gebiete (1979) 50:5–25.
- [10] J.F. Le Gall and J. Rosen, *The range of stable random walks*. Ann. Probab. **19** (1991), 650–705.
- [11] A. Le Ny, *Range of a Transient 2d-Random Walk* (2011) ArXiv:1111.0877.
- [12] B. Von Bahr and C-G. Esseen, *Inequalities for the rth Absolute Moment of a Sum of Random Variables*, $1 \leq r \leq 2$. Ann. Math. Statist. Vol. 36, No 1 (1965), 299–303.
- [13] W. Whitt, *Stochastic process limits*. Springer series in Operations research, Springer Verlag, New York, 2002.

INSTITUT CAMILLE JORDAN, CNRS UMR 5208, UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, 43, BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE, FRANCE.

E-mail address: nadine.guillotin@univ-lyon1.fr

UNIVERSITÉ DE BREST AND IUF, LMBA, UMR CNRS 6205, 29238 BREST CEDEX, FRANCE

E-mail address: francoise.pene@univ-brest.fr