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# EXPLICIT BIREGULAR/BIRATIONAL GEOMETRY OF AFFINE THREEFOLDS: COMPLETIONS OF $\mathbb{A}^3$ INTO DEL PEZZO FIBRATIONS AND MORI CONIC BUNDLES

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

ABSTRACT. We study certain pencils  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  of del Pezzo surfaces generated by a smooth del Pezzo surface  $S$  of degree less or equal to 3 anti-canonically embedded into a weighted projective space  $\mathbb{P}$  and an appropriate multiple of a hyperplane  $H$ . Our main observation is that every minimal model program relative to the morphism  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  lifting  $\bar{f}$  on a suitable resolution  $\sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$  of its indeterminacies preserves the open subset  $\sigma^{-1}(\mathbb{P} \setminus H) \simeq \mathbb{A}^3$ . As an application, we obtain projective completions of  $\mathbb{A}^3$  into del Pezzo fibrations over  $\mathbb{P}^1$  of every degree less or equal to 4. We also obtain completions of  $\mathbb{A}^3$  into Mori conic bundles, whose restrictions to  $\mathbb{A}^3$  are twisted  $\mathbb{A}_*^1$ -fibrations over  $\mathbb{A}^2$ .

## INTRODUCTION

A threefold Mori fiber space is a mildly singular projective threefold  $X$  equipped with an extremal contraction  $\tau : X \rightarrow B$  over a lower dimensional normal projective variety  $B$ . More precisely,  $X$  has  $\mathbb{Q}$ -factorial terminal singularities,  $\tau$  has connected fibers, the anti-canonical divisor  $-K_X$  of  $X$  is ample on the fibers and the relative Picard number  $\rho(X/B) = \text{rk}(N_1(X)) - \text{rk}(N_1(B))$  is equal to 1. These fiber spaces are the possible outputs of Minimal Model Programs (MMP) ran from rational, or more generally uniruled, smooth projective threefolds and provide the natural higher dimensional analogues in this framework of the projective plane and the minimally ruled surfaces. Noting that rational minimally ruled surfaces  $\mathbb{F}_n$ ,  $n \geq 2$ ,  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathbb{P}^2$  are smooth projective completions of the affine plane  $\mathbb{A}^2$ , it is natural to ask which threefold Mori fiber spaces  $\tau : X \rightarrow B$  are projective completions of  $\mathbb{A}^3$  and, as a first step towards a potential geometric description of the structure of the automorphism group  $\text{Aut}(\mathbb{A}^3)$  of  $\mathbb{A}^3$  from the point of view of the Sarkisov Program [2], try to classify them up to birational isomorphisms preserving the inner open subset  $\mathbb{A}^3$ .

In the case  $\dim B = 0$ , Fano threefolds of Picard number 1 containing  $\mathbb{A}^3$  have received a lot of attention during the past decades: a complete classification is known in the smooth case (see e.g. [5] and the references therein) but the general picture in the singular case remains elusive. Much less seems to be known about completions of  $\mathbb{A}^3$  into “strict” Mori fiber spaces  $\tau : X \rightarrow B$ , where  $\dim B = 1, 2$ . There are two cases: del Pezzo fibrations when  $\dim B = 1$  and Mori conic bundles when  $\dim B = 2$ . Elementary examples of such completions are locally trivial projective bundles  $\tau : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \rightarrow \mathbb{P}^1$  and  $\tau : \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(m)) \rightarrow \mathbb{P}^2$  over  $\mathbb{P}^1$  and  $\mathbb{P}^2$ , which come respectively as projective models of linear projections from  $\mathbb{A}^3$  to  $\mathbb{A}^1$  and  $\mathbb{A}^2$ . But in general, there is no reason that the restriction to  $\mathbb{A}^3$  of the structure morphism  $\tau : X \rightarrow B$  of a completion into a strict Mori fiber space has general fibers isomorphic to affine spaces. For instance, since a smooth del Pezzo surface of degree  $d \leq 3$  with Picard number 1 is not rational [7], there cannot exist any completion of  $\mathbb{A}^3$  into a del Pezzo fibration  $\tau : X \rightarrow B = \mathbb{P}^1$  of degree  $d \leq 3$  whose restriction to  $\mathbb{A}^3$  is a fibration with generic fiber isomorphic to the affine plane  $\mathbb{A}^2$  over the function field of  $B$ .

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The main purpose of this article is to give examples of “twisted” completions of  $\mathbb{A}^3$  into strict Mori fiber spaces, that is completions  $\tau : X \rightarrow B$  for which the general fibers of the restriction of  $\tau$  to  $\mathbb{A}^3$  are not isomorphic to affine spaces. One strategy to construct such examples is to start from a regular function  $f : \mathbb{A}^3 \rightarrow \mathbb{A}^1$  with smooth rational general fibers which extends to a morphism  $\tilde{f}' : X' \rightarrow \mathbb{P}^1$  with smooth general fibers on a smooth projective threefold  $X'$  and to run a relative MMP  $\varphi : X' \dashrightarrow X$  over  $\mathbb{P}^1$ . The rationality of the fibers guarantees that the output  $\tilde{f} : X \rightarrow \mathbb{P}^1$  is either a del Pezzo fibration or factors through a Mori conic bundle  $\xi : X \rightarrow W$  over a normal projective surface  $W$ . The main obstacle is that there is no reason in general that a relative MMP  $\varphi : X' \dashrightarrow X$  preserves the open subset  $\mathbb{A}^3 \subset X'$ : such a process  $\varphi$  might contract divisors which are not supported on the boundary  $X' \setminus \mathbb{A}^3$ , inducing a nontrivial birational morphism between  $\mathbb{A}^3$  and its image by  $\varphi$  which, in this case is in general again affine, and even worse, small contractions might occur outside the boundary with the effect that the image of  $\mathbb{A}^3$  by  $\varphi$  is no longer affine. As a general fact, understanding the biregular geometry of an affine threefold via the birational geometry of its projective models requires to get some effective control on the birational maps appearing in MMP processes between these models. One solution in our situation is to consider functions  $f : \mathbb{A}^3 \rightarrow \mathbb{A}^1$  extending to fibrations  $\tilde{f}' : X' \rightarrow \mathbb{P}^1$  whose general fibers are already smooth del Pezzo surfaces. Here we can expect to gain more control on the possible horizontal divisors contracted by  $\varphi$  as well as on its flipping and flipped curves, and that the output will be in general a del Pezzo fibration, possibly of higher degree.

The functions we consider in this article are obtained as restrictions of pencils  $\mathcal{L}$  generated by a smooth del Pezzo surface  $S$  of degree 1, 2 or 3 anti-canonically embedded into a weighted projective 3-space  $\mathbb{P}$  and an appropriate multiple  $eH$  of a hyperplane  $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ . Namely,  $\mathbb{P} \setminus H$  is isomorphic to  $\mathbb{A}^3$  and  $f : \mathbb{A}^3 \rightarrow \mathbb{A}^1$  is the restriction of the rational map  $\tilde{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  defined by  $\mathcal{L}$ . For an appropriate class of resolutions  $\sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$  of  $\tilde{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  restricting to an isomorphism over  $\mathbb{P} \setminus H$  and for which  $\sigma^{-1}(H)$  induces an anti-canonical divisor on the generic fiber of the induced morphism  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$ , which we call *good resolutions*, we establish that every MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  relative to  $\tilde{f}$  restricts to an isomorphism between  $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \simeq \mathbb{A}^3$  and its image. The output  $\tilde{\mathbb{P}}'$  is then a compactification of  $\mathbb{A}^3$  either into a del Pezzo fibration  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  or into a Mori conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  over a certain normal projective surface  $q : W \rightarrow \mathbb{P}^1$ , and we characterize each possible type of output in terms of the structure of the base locus of  $\mathcal{L}$ . Our main result can be summarized as follows:

**Theorem.** *Let  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$  be the pencil generated by an anti-canonically embedded smooth del Pezzo surface  $S \subset \mathbb{P}$  of degree  $d \in \{1, 2, 3\}$  and a multiple of hyperplane  $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ , let  $\sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$  be a good resolution of the corresponding rational map  $\tilde{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ , and let  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  be a MMP relative to the induced morphism  $\tilde{f} = \tilde{f} \circ \sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$ . Then the induced morphism  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a projective completion of  $\mathbb{A}^3$  with  $\mathbb{Q}$ -factorial terminal singularities of one of the following types:*

- a) *If  $H \cap S$  is irreducible, then  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a del Pezzo fibration of degree  $d$ .*
- b) *If  $d = 2$  and  $H \cap S$  is reducible, then  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is del Pezzo fibration of degree  $d + 1 = 3$ .*
- c) *If  $H \cap S$  has three irreducible components, then  $\tilde{\mathbb{P}}'$  is a Mori conic bundle.*

In the case where the output  $\tilde{\mathbb{P}}'$  is a Mori conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$ , we establish further that the restriction of  $\xi$  to the inner  $\mathbb{A}^3$  is a *twisted  $\mathbb{A}_*^1$ -fibration*  $\xi_0 : \mathbb{A}^3 \rightarrow \mathbb{A}^2$ , that is, a flat fibration whose generic fiber is a nontrivial form of the punctured affine line  $\mathbb{A}_*^1$  over the function field of  $\mathbb{A}^2$ . This contrasts with the situation for  $\mathbb{A}^2$  for which no such type of  $\mathbb{A}_*^1$ -fibration can exist, essentially as a consequence of Tsen’s theorem and the factoriality of  $\mathbb{A}^2$  (see [8, Lemma 1.7.2]). We also provide a geometric interpretation of these fibrations in terms of the pair  $(S, H)$  initially chosen for the construction.

## 1. PENCILS OF DEL PEZZO SURFACES IN WEIGHTED PROJECTIVE SPACES

Recall that a smooth del Pezzo surface is a smooth projective surface  $S$  whose anti-canonical divisor  $-K_S$  is ample. The integer  $d = (-K_S^2) \in \{1, \dots, 9\}$  is called the degree of  $S$ . Every

such surface is either isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  or to the blow-up of the projective plane  $\mathbb{P}^2$  in 9 –  $d$  points in general position [7]. Anti-canonical models  $\text{Proj}_{\mathbb{C}}(\bigoplus_{m \geq 0} H^0(S, -mK_S))$  of smooth del Pezzo surfaces of degree  $\leq 3$  are naturally embedded as hypersurfaces in certain weighted projective spaces. Their properties are summarized by the following proposition:

**Proposition 1.**

1) *Every smooth del Pezzo surface of degree 3 is isomorphic to a smooth cubic surface in  $\mathbb{P}^3$ , and conversely every smooth cubic surface  $S$  in  $\mathbb{P}^3$  is a del Pezzo surface of degree 3. For every hyperplane  $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$ ,  $H|_S$  is a reduced anti-canonical divisor on  $S$  whose support is isomorphic to a plane cubic curve, either irreducible, or consisting of the union of a smooth conic  $C$  and a line  $\ell$  intersecting each other twice, either transversally in two distinct points or tangentially in a unique point, or consisting of three lines, either in general position or intersecting each other in a unique point, which is then an Eckardt point of  $S$ .*

2) *Every smooth del Pezzo surface of degree 2 is isomorphic to a smooth quartic hypersurface of the weighted projective space  $\mathbb{P}(1, 1, 1, 2)$ . Conversely, every smooth quartic  $S$  in  $\mathbb{P}(1, 1, 1, 2)$  is a del Pezzo surface of degree 2. For every  $H \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$ ,  $H|_S$  is a reduced anti-canonical divisor on  $S$  whose support is isomorphic either to an irreducible plane cubic curve, or to the union of two  $(-1)$ -curves on  $S$  intersecting each other twice, either transversally in two distinct points or tangentially in a unique point.*

3) *Every smooth del Pezzo surface of degree 1 is isomorphic to a smooth sextic hypersurface of the weighed projective space  $\mathbb{P}(1, 1, 2, 3)$ , and conversely every smooth sextic in  $\mathbb{P}(1, 1, 2, 3)$  is a del Pezzo surface of degree 1. For every  $H \in |\mathcal{O}_{\mathbb{P}(1,1,2,3)}(1)|$ ,  $H|_S$  is an irreducible and reduced anti-canonical divisor on  $S$  whose support is isomorphic to a plane cubic curve.*

**1.1. Pencils of del Pezzo surfaces of degree  $\leq 3$ .** In what follows, given an anti-canonically embedded smooth del Pezzo surface  $S$  of degree  $d \leq 3$  as in Proposition 1 above, we use the same notation  $\mathbb{P} = \text{Proj}(\mathbb{C}[x, y, z, w])$  to denote the ambient spaces  $\mathbb{P}^3$ ,  $\mathbb{P}(1, 1, 1, 2)$  and  $\mathbb{P}(1, 1, 2, 3)$  according to  $d = 3, 2$  and  $1$ , the variables  $x, y, z$  and  $w$  having degrees  $(1, 1, 1, 1)$ ,  $(1, 1, 1, 2)$  and  $(1, 1, 2, 3)$  respectively. The degree of  $S$  as a hypersurface of  $\mathbb{P}$  is denoted by  $e$ . It is equal to 3, 4 or 6 according to  $d = 3, 2$  and  $1$  respectively.

**Definition 2.** Let  $S \subset \mathbb{P}$  be a smooth del Pezzo surface of degree  $d \in \{1, 2, 3\}$  and let  $H \in |\mathcal{O}_{\mathbb{P}}(1)|$  be a hyperplane. We denote by  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$  the pencil generated by  $S$  and  $eH$  and by  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1 = \mathbb{P}(\mathcal{L}^*)$  the corresponding rational map.

1.1.1. A member  $S_{[\alpha:\beta]}$ ,  $[\alpha : \beta] \in \mathbb{P}^1$ , of  $\mathcal{L}$  is defined up to a linear transformation of  $\mathbb{P}$  by the vanishing of a weighted-homogeneous polynomial  $F \in \mathbb{C}[x, y, z, w]$  of degree  $e$  of the form

$$F = \beta s(x, y, z, w) - \alpha x^e,$$

where  $S$  and  $H$  are defined respectively by the vanishing of  $s(x, y, z, w)$  and  $x$ . The scheme-theoretic base locus of  $\mathcal{L}$  is equal to the closed subscheme of  $\mathbb{P}$  defined by the weighted-homogeneous ideal  $(s(x, y, z, w), x^e)$  of  $\mathbb{C}[x, y, z, w]$ . Its support is equal to  $H \cap S$ . With this description, the rational map  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  coincides with that defined by  $[x : y : z : w] \mapsto [s(x, y, z, w) : x^e]$ . The complement of  $H$  is isomorphic to  $\mathbb{A}^3$  with inhomogeneous coordinates  $Y = x^{-1}y$ ,  $Z = x^{-a}z$  and  $W = x^{-b}w$ , where  $(a, b) = (1, 1)$ ,  $(1, 2)$  and  $(2, 3)$  according to  $d = 3, 2$  and  $1$  respectively, and letting  $\infty = [1 : 0] = \bar{f}_*(H) \in \mathbb{P}^1$ , the restriction of  $\bar{f}$  to  $\mathbb{P} \setminus H$  coincides with the regular function

$$f : \mathbb{A}^3 = \mathbb{P} \setminus H \rightarrow \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\} \simeq \text{Spec}(\mathbb{C}[\lambda]), (X, Y, Z) \mapsto s(1, Y, Z, W).$$

The *generic member*  $S_{\eta}$  of  $\mathcal{L}$ , that is, the closure in  $\mathbb{P}_{\mathbb{C}(\lambda)} = \text{Proj}(\mathbb{C}(\lambda)[x, y, z, w])$  of the fiber of  $f$  over the generic point  $\eta$  of  $\mathbb{P}^1$ , is isomorphic to the projective surface over  $\mathbb{C}(\lambda)$  defined by the vanishing of weighted-homogeneous polynomial  $s(x, y, z, w) + \lambda x^e \in \mathbb{C}(\lambda)[x, y, z, w]$ . Since  $S$  is smooth, it follows from the Jacobian criterion that  $S_{\eta}$  is smooth, hence is a smooth del Pezzo surface of degree  $d$  defined over the function field  $\mathbb{C}(\lambda)$  of  $\mathbb{P}^1$ . This implies in particular that the general member of  $\mathcal{L}$  is a smooth del Pezzo surface of degree  $d$ . Some members of  $\mathcal{L}$  can be singular (see Example 4 below) but all members of  $\mathcal{L}$  except  $eH$  are integral schemes:

**Lemma 3.** *All members of  $\mathcal{L}$  except  $eH$  are irreducible and reduced.*

*Proof.* We consider each degree  $d = 3, 2, 1$  separately. If  $d = 3$  and  $S' \in \mathcal{L} \setminus \{S, 3H\}$  is either reducible or non reduced, then one of its irreducible components is necessarily a hyperplane, say  $H'$ , which is different from  $H$  as  $\mathcal{L}$  does not have any fixed component. So  $H' \cap S$  is distinct from  $H \cap S$ , hence is strictly contained in it as  $H \cap S$  coincides with the support of the base locus of  $\mathcal{L}$ . This is absurd in view of 1) in Proposition 1.

In the case  $d = 2$ , a member  $S' \in \mathcal{L} \setminus \{S, 4H\}$  which is either reducible or non reduced contains an irreducible component of degree one or two. In the first case, we would have again a hyperplane  $H' \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$  distinct from  $H$  for which  $H' \cap S$  is contained in  $H \cap S$ , which is absurd by virtue of 2) in Proposition 1. In the second case,  $S'$  would be the union of two irreducible quadric hypersurfaces  $Q_1$  and  $Q_2$  of  $\mathbb{P}(1, 1, 1, 2)$ , necessarily distinct from each other since otherwise every member of  $\mathcal{L}$  would be reducible. Since the restriction map

$$H^0(\mathbb{P}(1, 1, 1, 2), \mathcal{O}_{\mathbb{P}(1,1,1,2)}(2)) \xrightarrow{\sim} H^0(S, \mathcal{O}_{\mathbb{P}(1,1,1,2)}(2)|_S) \simeq H^0(S, \mathcal{O}_S(-2K_S))$$

is an isomorphism, both intersections  $Q_i \cap H$ ,  $i = 1, 2$  are strictly contained in  $H \cap S$ . Indeed, if  $Q_i \cap H = H \cap S$  then  $Q_i|_S = 2H|_S$  and then  $Q_i = 2H$  contradicting the irreducibility of  $Q_i$ . This implies in turn by virtue of 2) in Proposition 1 that  $Q_i|_S$  is supported on a  $(-1)$ -curve, which is absurd as  $Q_i|_S$  has non negative self-intersection.

Finally, if  $d = 1$  and  $S' \in \mathcal{L} \setminus \{S, 6H\}$  is not integral, then it contains an irreducible component  $P$  of degree 1, 2 or 3. Because of the isomorphisms

$$H^0(\mathbb{P}(1, 1, 2, 3), \mathcal{O}_{\mathbb{P}(1,1,2,3)}(j)) \xrightarrow{\sim} H^0(S, \mathcal{O}_{\mathbb{P}(1,1,2,3)}(j)|_S) \simeq H^0(S, \mathcal{O}_S(-jK_S)), \quad j = 1, 2, 3,$$

the same argument as in the previous case implies that  $P \cap S$  is strictly contained in  $H \cap S$ , which is absurd since the latter is irreducible by virtue of 3) in Proposition 1.  $\square$

**Example 4.** a) The sextics  $S_1$  and  $S_2$  in  $\mathbb{P}(1, 1, 2, 3) = \text{Proj}_{\mathbb{C}}(\mathbb{C}[x, y, z, w])$  defined respectively by the equations  $z^3 + w^2 + xy^5 = 0$  and  $z^3 + w^2 + x^2(x^3y + z^2) = 0$  are normal del Pezzo surfaces with a unique singular point of type  $E_8$  at  $p_1 = [1 : 0 : 0 : 0]$  and  $p_2 = [0 : 1 : 0 : 0]$  respectively. The general members of the pencils  $\bar{f}_i : \mathbb{P}(1, 1, 2, 3) \dashrightarrow \mathbb{P}^1$ ,  $i = 1, 2$ , generated respectively by  $S_1$  and  $6H_1$  where  $H_1 = \{x + by = 0\} \in |\mathcal{O}_{\mathbb{P}(1,1,2,3)}(1)|$ ,  $b \in \mathbb{C}$ , and  $S_2$  and  $6H_2$  where  $H_2 = \{ax + y = 0\} \in |\mathcal{O}_{\mathbb{P}(1,1,2,3)}(1)|$ ,  $a \in \mathbb{C}$ , are smooth del Pezzo surfaces of degree 1. The intersection  $S_1 \cap H_1$  is either a rational cuspidal cubic if  $b = 0$  or a smooth elliptic curve otherwise, while  $S_2 \cap H_2$  is either a nodal cubic if  $a = 0$  or a smooth elliptic curve otherwise.

b) The quartic surface  $S = \{w^2 + yz^3 + xy^3 = 0\}$  in  $\mathbb{P}(1, 1, 1, 2) = \text{Proj}_{\mathbb{C}}(\mathbb{C}[x, y, z, w])$  is a normal del Pezzo surface with a unique singular point of type  $E_7$  at  $[1 : 0 : 0 : 0]$ . The general members of the pencil  $f : \mathbb{P}(1, 1, 1, 2) \dashrightarrow \mathbb{P}^1$  generated by  $S$  and  $4H$  where  $H = \{x + ay + bz = 0\} \in |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(1)|$ ,  $a, b \in \mathbb{C}$  are smooth del Pezzo surfaces of degree 2. The intersection of  $H$  with  $S$  is either a cuspidal cubic if  $b = 0$  or a smooth elliptic curve otherwise.

c) The cubic surfaces  $S_1(\lambda) = \{x^3 + w(\lambda x^2 + y^2 + wz) = 0\}$ ,  $\lambda \in \mathbb{C}$ , and  $S_2 = \{xyz + y^3 + w^2z = 0\}$  in  $\mathbb{P}^3$  are normal del Pezzo surfaces respectively with a unique singularity of type  $E_6$  at  $[0 : 0 : 1 : 0]$  and a pair of singular points  $[1 : 0 : 0 : 0]$  and  $[0 : 0 : 1 : 0]$  of types  $A_1$  and  $A_5$ . The general members of the pencils  $\bar{f}_i : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ ,  $i = 1, 2$ , generated respectively by  $S_1(\lambda)$  and  $3H_1$ , where  $H_1 = \{z = 0\}$ , and by  $S_2$  and  $3H_2$ , where  $H_2 = \{x + z = 0\}$ , are smooth cubic surfaces.

## 2. GOOD RESOLUTIONS AND RELATIVE MMPS

In this section, we introduce particular resolutions  $\sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$  of the indeterminacies of the rational map  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  associated to a pencil as in Definition 2 above. These have the property to restrict to isomorphisms over the open subset  $\mathbb{A}^3 = \mathbb{P} \setminus H$ , and we show that every MMP process  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  relative to the induced morphism  $\tilde{f} = \bar{f} \circ \sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  again preserves  $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \simeq \mathbb{P} \setminus H$ , inducing an isomorphism between  $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \simeq \mathbb{P} \setminus H$  and  $\tilde{\mathbb{P}}' \setminus \varphi_*(\sigma^{-1}(H))$ .

**2.1. Good resolutions of del Pezzo pencils.** Let  $S \subset \mathbb{P}$  be a smooth del Pezzo surface of degree  $d \leq 3$ , let  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$  be the pencil generated by  $S$  and  $eH$  for some  $H \in |\mathcal{O}_{\mathbb{P}}(1)|$  and let  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  be the corresponding rational map as in Definition 2. Similarly as in § 1.1.1, we let  $\infty = \bar{f}_*(H) \in \mathbb{P}^1$ .

**Definition 5.** A *good resolution* of  $\bar{f}$  is a triple  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  consisting of a projective threefold  $\tilde{\mathbb{P}}$ , a birational morphism  $\sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$  and a morphism  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  satisfying the following properties:

a) The diagram

$$\begin{array}{ccc} \tilde{\mathbb{P}} & \xrightarrow{\sigma} & \mathbb{P} \\ \tilde{f} \downarrow & & \downarrow \bar{f} \\ \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 \end{array}$$

commutes.

b)  $\tilde{\mathbb{P}}$  has at most  $\mathbb{Q}$ -factorial terminal singularities and is smooth outside  $\tilde{f}^{-1}(\infty)$ .

c)  $\sigma : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$  is a sequence of blow-ups whose successive centers lie above the base locus of  $\mathcal{L}$ , inducing an isomorphism  $\tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \xrightarrow{\sim} \mathbb{P} \setminus H$ , and whose restriction to every closed fiber of  $\tilde{f}$  except  $\tilde{f}^{-1}(\infty)$  is an isomorphism onto its image.

2.1.1. It follows from the definition that all irreducible divisors in the exceptional locus  $\text{Exc}(\sigma)$  of a good resolution  $\sigma$  that are vertical for  $\tilde{f}$  are contained in  $\tilde{f}^{-1}(\infty)$ . Furthermore, since the restriction of  $\sigma$  to the generic fiber of  $\tilde{f}$  is an isomorphism onto the generic member of  $\mathcal{L}$ ,  $\text{Exc}(\sigma)$  contains exactly as many irreducible horizontal divisors as there are irreducible components in  $H \cap S$ . Indeed, there is a one to one correspondence between irreducible horizontal divisors in  $\text{Exc}(\sigma)$  and irreducible components of the intersection of  $\sigma^{-1}(H)$  with the generic fiber of  $\tilde{f}$ . By assumption, the latter is isomorphic to the smooth del Pezzo surface  $S_\eta$  of degree  $d$  in  $\mathbb{P}_{\mathbb{C}(\lambda)}$  with equation  $s(x, y, z, w) - \lambda x^e = 0$  (see § 1.1.1), and the definition of  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  implies that it intersects  $\sigma^{-1}(H)$  along the curve  $D_\eta \simeq (H \cap S) \times_{\text{Spec}(\mathbb{C})} \text{Spec}(\mathbb{C}(\lambda))$  with equation  $s(0, y, z, w) = 0$  in  $\text{Proj}(\mathbb{C}(\lambda)[y, z, w])$ . In particular,  $D_\eta$  is an anti-canonical divisor on  $S_\eta$  with the same number of irreducible components as  $H \cap S$ , all them being defined over  $\mathbb{C}(\lambda)$ . Note also that the intersection of  $\sigma^{-1}(H)$  with a closed fiber  $\tilde{f}^{-1}(c)$  distinct from  $\tilde{f}^{-1}(\infty)$  is isomorphic to the intersection of  $H$  with the corresponding member  $\sigma(\tilde{f}^{-1}(c))$  of  $\mathcal{L}$ .

A good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  of  $f : \mathbb{P} \dashrightarrow \mathbb{P}^1$  always exists. For instance, let  $\tau : X \rightarrow \mathbb{P}$  be the blow-up of scheme-theoretic base locus of  $\mathcal{L}$ . Then  $X$  is isomorphic to the hypersurface in  $\mathbb{P} \times \text{Proj}(\mathbb{C}[\alpha, \beta])$  defined by the weighted bi-homogeneous equation  $\beta s(x, y, z, w) - \alpha \beta x^e = 0$ , and we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\tau} & \mathbb{P} \\ & \searrow & \downarrow \bar{f} \\ \pi = \text{pr}_2|_X & & \mathbb{P}^1 \end{array}$$

The morphism  $\tau$  restricts on each fiber of  $\pi$  to an isomorphism onto the corresponding member of  $\mathcal{L}$  and  $X \setminus \tau^{-1}(H) \simeq \mathbb{P} \setminus H$ . Furthermore, since  $S$  is smooth, it follows from the Jacobian criterion that  $X$  is smooth outside  $\pi^{-1}(\infty)$ . Letting  $\tau_1 : \tilde{\mathbb{P}} \rightarrow X$  be any resolution of the singularities of  $X$ , the triple  $(\tilde{\mathbb{P}}, \tau \circ \tau_1, \pi \circ \tau_1)$  is a good resolution of  $\bar{f}$  for which  $\tilde{\mathbb{P}}$  is even smooth.

**2.2. Basic properties of relative MMPs ran from good resolutions.** Let  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  be a good resolution of the rational map  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  associated to a pencil  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$  as above. Recall [6, 3.31] that a MMP  $\varphi : \tilde{\mathbb{P}}_0 = \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}' = \tilde{\mathbb{P}}_n$  relative to  $\tilde{f}_0 = \tilde{f} : \tilde{\mathbb{P}}_0 \rightarrow \mathbb{P}^1$  consists of a finite sequence  $\varphi = \varphi_n \circ \dots \circ \varphi_1$  of birational maps

$$\begin{array}{ccc} \tilde{\mathbb{P}}_{k-1} & \xrightarrow{\varphi_k} & \tilde{\mathbb{P}}_k \\ \tilde{f}_{k-1} \downarrow & & \downarrow \tilde{f}_k \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array} \quad k = 1, \dots, n,$$

where each  $\varphi_k$  is associated to an extremal ray  $R_{k-1}$  of the closure  $\overline{NE}(\tilde{\mathbb{P}}_{k-1}/\mathbb{P}^1)$  of the relative cone of curves of  $\tilde{\mathbb{P}}_{k-1}$  over  $\mathbb{P}^1$ . Each of these birational maps  $\varphi_k$  is either of divisorial contraction or a flip whose flipping and flipped curves are contained in the fibers of  $\tilde{f}_{k-1}$  and  $\tilde{f}_k$  respectively. Letting  $\Delta_0 = \sigma^{-1}(H)$  and  $\Delta_k = (\varphi_k)_*(\Delta_{k-1})$  for every  $k = 1, \dots, n$ , the next result asserts in particular that every relative MMP ran from a good resolution of  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  preserves the open subset  $\sigma^{-1}(\mathbb{P} \setminus H) \simeq \mathbb{P} \setminus H \simeq \mathbb{A}^3$ .

**Proposition 6.** *Let  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$  be as above and let  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  be any good resolution of the corresponding rational map  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ . Then every MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  relative to  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  restricts to an isomorphism  $\mathbb{A}^3 \simeq \tilde{\mathbb{P}} \setminus \sigma^{-1}(H) \xrightarrow{\sim} \tilde{\mathbb{P}}' \setminus \varphi_*(\sigma^{-1}(H))$ . More precisely, the following hold at each intermediate step:*

- a) *The threefold  $\tilde{\mathbb{P}}_k$  is smooth outside  $\tilde{f}_k^{-1}(\infty)$ ,*
- b) *The birational map  $\varphi_k : \tilde{\mathbb{P}}_{k-1} \dashrightarrow \tilde{\mathbb{P}}_k$  restricts to an isomorphism  $\tilde{\mathbb{P}}_{k-1} \setminus \Delta_{k-1} \rightarrow \tilde{\mathbb{P}}_k \setminus \Delta_k$ ,*
- c) *The restriction of  $\varphi_k$  to a general closed fiber of  $\tilde{f}_k$  is either an isomorphism onto its image, or the contraction of finitely many disjoint  $(-1)$ -curves.*

*Proof.* Since by virtue of Lemma 3, all members of  $\mathcal{L}$  except  $eH$  are irreducible and reduced, the fact that  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  is a good resolution guarantees that all fibers of  $\tilde{f}_0$  except maybe  $\tilde{f}_0^{-1}(\infty)$  are irreducible and reduced. This implies in turn that the divisors contracted by  $\varphi : \tilde{\mathbb{P}}_0 \dashrightarrow \tilde{\mathbb{P}}_n$  are either irreducible components of  $\tilde{f}_0^{-1}(\infty)$  or are horizontal for  $\tilde{f}_0$ . Let  $\varphi_0 = \text{id}_{\tilde{\mathbb{P}}_0}$ . If  $\varphi_k$ ,  $k \geq 1$ , is the contraction of a divisor  $E_{k-1} \subset \tilde{\mathbb{P}}_{k-1}$  onto a curve  $B_k \subset \tilde{\mathbb{P}}_k$ , then by the previous observation,  $E$  is either an irreducible component of  $\tilde{f}_{k-1}^{-1}(\infty)$  or is horizontal for  $\tilde{f}_{k-1}$ . In the second case,  $E_{k-1}$  is the proper transform in  $\tilde{\mathbb{P}}_{k-1}$  of an irreducible divisor  $E \subset \tilde{\mathbb{P}}_0$ , which is necessarily contained in the support of  $\Delta_0$ . Indeed, by induction hypothesis, the restriction  $\varphi_{k-1} \circ \dots \circ \varphi_1 \circ \varphi_0 : S_{c,0} = \tilde{f}_0^{-1}(c) \rightarrow S_{c,k-1} = \tilde{f}_{k-1}^{-1}(c)$  to a general closed fiber of  $\tilde{f}_0$  is either an isomorphism or a sequence of contractions of  $(-1)$ -curves. Since  $E_{k-1} \cap S_{c,k-1}$  consists of a disjoint union of  $(-1)$ -curves, it follows that  $E \cap S_{c,0}$  is a curve  $C$  on  $S_{c,0}$  that can be contracted to a finite number of smooth points, hence consists of a disjoint union of  $(-1)$ -curves because  $S_{c,0}$  is a smooth del Pezzo surface. But on the other hand, if  $E$  were not  $\sigma$ -exceptional, the hypothesis that  $\sigma$  maps  $S_{c,0}$  isomorphically onto its image in  $\mathbb{P}$  would imply that the proper transform  $\sigma_*E$  of  $E$  in  $\mathbb{P}$  is an ample divisor intersecting  $\sigma(S_{c,0})$  along the curve  $\sigma(C)$  which is absurd as  $\sigma(C)$  consists again of a disjoint union of  $(-1)$ -curves. Thus  $E$  is contained in  $\Delta_0$  and hence  $E_{k-1}$  is contained in  $\Delta_{k-1}$ . Furthermore, since  $\tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty)$  is smooth by hypothesis, it follows that  $\tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$  is still smooth along  $B_k \setminus (B_k \cap \tilde{f}_k^{-1}(\infty))$ . More precisely,  $B_k \setminus (B_k \cap \tilde{f}_k^{-1}(\infty))$  is smooth and

$$\varphi_k|_{\tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty)} : \tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty) \rightarrow \tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$$

coincides with the blow-up of  $\tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$  along  $B_k \setminus (B_k \cap \tilde{f}_k^{-1}(\infty))$  [3]. Finally, the restriction of  $\varphi_k$  to a general closer fiber of  $\tilde{f}_{k-1}$  is either an isomorphism onto its image, or the contraction of finitely many disjoint  $(-1)$ -curves, in particular its image by  $\varphi_k$  is again a smooth del Pezzo surface. Otherwise, if  $\varphi_k$  is a flip, then since its flipping curves must pass through a singular point of  $\tilde{\mathbb{P}}_{k-1}$  [1, 14.6.4], they are contained in  $\tilde{f}_{k-1}^{-1}(\infty)$ . The flipped curves of  $\varphi_k$  are thus contained in  $\tilde{f}_k^{-1}(\infty)$  and  $\varphi_k$  restricts to an isomorphism between  $\tilde{\mathbb{P}}_{k-1} \setminus \tilde{f}_{k-1}^{-1}(\infty)$  and  $\tilde{\mathbb{P}}_k \setminus \tilde{f}_k^{-1}(\infty)$ , which is thus again smooth.  $\square$

### 3. OUTPUTS OF RELATIVE MMPS

Since a general member of a pencil  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$  as in Definition 2 above is a rational surface, the output  $\tilde{\mathbb{P}}'$  of a relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  ran from a good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  of the corresponding rational map  $\bar{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$  is a Mori fiber space  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$ . More precisely,  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is either a del Pezzo fibration with relative Picard number 1, or a Mori conic bundle over a certain normal projective surface  $W$ , say  $\tilde{f}' = q \circ \xi : \tilde{\mathbb{P}}' \rightarrow W \rightarrow \mathbb{P}^1$  where  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  is a flat morphism of relative

Picard number 1, with connected fibers and such that  $-K_{\tilde{\mathbb{P}}'}$  is relatively ample. In each case, it follows from Proposition 6 that  $\tilde{\mathbb{P}}'$  is a projective completion of  $\mathbb{A}^3$  with at most  $\mathbb{Q}$ -factorial terminal singularities. The following theorem shows in particular that except maybe in the case where  $d = 3$  and  $H \cap S$  consists of two irreducible components, the nature of  $\tilde{\mathbb{P}}'$  depends only on the base locus of  $\mathcal{L}$ . In particular, it depends neither on the chosen good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  nor on the relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ .

**Theorem 7.** *Let  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}}(e)|$  be the pencil generated by a smooth del Pezzo surface  $S \subset \mathbb{P}$  of degree  $d \in \{1, 2, 3\}$  and  $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ , let  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  be a good resolution of the corresponding rational map  $\tilde{f} : \mathbb{P} \dashrightarrow \mathbb{P}^1$ , and let  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  be a relative MMP. Then the following hold:*

- a) *If  $H \cap S$  is irreducible, then  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a del Pezzo fibration of degree  $d$ .*
- b) *If  $d = 2$  and  $H \cap S$  is reducible, then  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a del Pezzo fibration of degree  $d + 1 = 3$ .*
- c) *If  $H \cap S$  has three irreducible components, then  $\tilde{\mathbb{P}}'$  is a Mori conic bundle.*

*Proof.* If  $H \cap S$  is irreducible then  $\sigma^{-1}(H)$  has a unique horizontal irreducible component, whose intersection with the generic fiber  $S_\eta$  of  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  is an irreducible anti-canonical divisor with self-intersection  $d$ . So with the notation of section 2.2 and Proposition 6, it follows that at each intermediate step  $\varphi_k : \tilde{\mathbb{P}}_{k-1} \dashrightarrow \tilde{\mathbb{P}}_k$  of  $\varphi$ , the intersection of  $\Delta_{k-1}$  with the generic fiber of  $\tilde{f}_{k-1} : \tilde{\mathbb{P}}_{k-1} \rightarrow \mathbb{P}^1$  is an irreducible curve with non negative self-intersection, which is therefore not contracted by  $\varphi_k$ . So  $\varphi$  does not contract the unique horizontal irreducible component of  $\sigma^{-1}(H)$ . It follows that  $\varphi$  restricts to an isomorphism between the generic fibers of  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  and  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$ , the former being a smooth del Pezzo surface of degree  $d$  over the function field  $\mathbb{C}(\lambda)$  of  $\mathbb{P}^1$  by virtue of § 2.1.1. On the other hand, Lemma 8 below implies that  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  cannot be a Mori conic bundle, and so  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a del Pezzo fibration of degree  $d$ .

If  $d = 2$  and  $H \cap S$  is reducible, then  $\sigma^{-1}(H)$  consists of two horizontal irreducible components, and its intersection with the generic fiber  $S_\eta$  of  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  is a reduced anti-canonical divisor whose support consists of the union of two  $(-1)$ -curves  $C_1$  and  $C_2$  defined over  $\mathbb{C}(\lambda)$  intersecting each other twice, either with multiplicity 2 at a unique  $\mathbb{C}(\lambda)$ -rational point, or transversally at a pair of distinct  $\mathbb{C}(\lambda)$ -rational points, or at unique point whose residue field is a quadratic extension of  $\mathbb{C}(\lambda)$  (see 2.1.1). These two curves being independent in the Néron-Severi group of  $S_\eta$ , the Picard number  $\rho(S_\eta)$  is bigger or equal to 2. If  $\varphi$  does not contract any horizontal component of  $\sigma^{-1}(H)$  then  $\varphi$  restricts to an isomorphism between  $S_\eta$  and the generic fiber  $S'_\eta$  of  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$ . Since  $\rho(S'_\eta) = \rho(S_\eta) \geq 2$ , this implies that  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a Mori conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  over a normal projective surface  $q : W \rightarrow \mathbb{P}^1$ . Furthermore, the general fibers of  $\tilde{f}'$  being rational, so are the general fibers of  $q$ , implying that  $q : W \rightarrow \mathbb{P}^1$  is a  $\mathbb{P}^1$ -fibration. Restricting  $\xi$  over the generic point  $\eta$  of  $\mathbb{P}^1$ , we obtain a Mori conic bundle  $\xi_\eta : S'_\eta \rightarrow W_\eta \simeq \mathbb{P}_{\mathbb{C}(\lambda)}^1$  defined over  $\mathbb{C}(\lambda)$ . Letting  $C'_1$  and  $C'_2$  be the images of  $C_1$  and  $C_2$  respectively in  $S'_\eta$ , we have  $-K_{S'_\eta} \sim C'_1 + C'_2$  and since  $(-K_{S'_\eta} \cdot \ell) = 2$  for every general  $\mathbb{C}(\lambda)$ -rational fiber  $\ell$  of  $\xi_\eta$ , it follows that either  $C'_1$  and  $C'_2$  are both sections of  $\xi_\eta$  or, up to a permutation, that  $C'_1$  is a 2-section of  $\xi_\eta$  while  $C'_2$  is contained in a fiber. The second possibility is excluded because a Mori conic bundle over  $\mathbb{P}_{\mathbb{C}(\lambda)}^1$  does not contain any  $(-1)$ -curve defined over  $\mathbb{C}(\lambda)$  in its closed fibers. In the first case, since the relative Picard number  $\rho(S'_\eta/\mathbb{P}_{\mathbb{C}(\lambda)}^1)$  is equal to 1, we would have  $C'_2 \sim C'_1 + a\ell$  for some  $a \in \mathbb{Q}$  such that  $2 = C'_1 \cdot C'_2 = (C'_1)^2 + a = -1 + a$  and  $-1 = (C'_2)^2 = (C'_1)^2 + 2a = -1 + 2a$ , which is absurd. So  $\varphi$  contracts at least one of the two horizontal irreducible components of  $\sigma^{-1}(H)$ , say the one intersecting  $S_\eta$  along  $C_1$ . Letting  $\varphi_k : \tilde{\mathbb{P}}_{k-1} \dashrightarrow \tilde{\mathbb{P}}_k$  be the intermediate step of  $\varphi$  at which this contraction occurs, the induced morphism  $\varphi_{k,\eta} : S_{k-1,\eta} \rightarrow S_{k,\eta}$  between the generic fibers of  $\tilde{f}_{k-1} : \tilde{\mathbb{P}}_{k-1} \rightarrow \mathbb{P}^1$  and  $\tilde{f}_k : \tilde{\mathbb{P}}_k \rightarrow \mathbb{P}^1$  coincides with the contraction of  $C_1$ . So  $S_{k,\eta}$  is a smooth del Pezzo surface of degree 3 defined over  $\mathbb{C}(\lambda)$ , which intersects the proper transform  $\Delta_k$  of  $\sigma^{-1}(H)$  along the image of  $C_2$ . The latter being an irreducible  $\mathbb{C}(\lambda)$ -rational curve with self-intersection 3, the same argument as in the previous case implies that the corresponding horizontal irreducible component of  $\Delta_k$  cannot be contracted at any further step  $\varphi_{k'}, k' \geq k + 1$ , of  $\varphi$ . So  $\varphi$  contracts exactly one irreducible component of  $\sigma^{-1}(H)$



and the generic fiber  $S'_\eta$  is isomorphic to the image of  $S_\eta$  by the contraction of the corresponding  $(-1)$ -curve defined over  $\mathbb{C}(\lambda)$ . Thus  $S'_\eta$  is a smooth del Pezzo surface of degree 3 defined over  $\mathbb{C}(\lambda)$ . We deduce again from Lemma 8 that  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  cannot be a Mori conic bundle, and so  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a del Pezzo fibration of degree 3.

Finally, if  $d = 3$  and  $H \cap S$  has three irreducible components, then the intersection of  $\sigma^{-1}(H)$  with  $S_\eta$  is a reduced anti-canonical divisor on  $S_\eta$  whose support consists of the union of three  $(-1)$ -curves  $C_1, C_2$  and  $C_3$  defined over  $\mathbb{C}(\lambda)$  and intersecting each other transversally at  $\mathbb{C}(\lambda)$ -rational points. If  $\varphi$  does not contract any horizontal irreducible component of  $\sigma^{-1}(H)$ , then it induces an isomorphism between  $S_\eta$  and the generic fiber  $S'_\eta$  of  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$ . The latter is thus a smooth del Pezzo surface of degree 3 defined over  $\mathbb{C}(\lambda)$  and having the sum  $C'_1 + C'_2 + C'_3$  of the images of the  $C_i$ 's as an anti-canonical divisor. The Picard number of  $S'_\eta$  is thus strictly bigger than one, and so  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is again a Mori conic bundle, restricting over the generic point  $\eta$  of  $\mathbb{P}^1$  to a Mori conic bundle  $\xi_\eta : S'_\eta \rightarrow \mathbb{P}^1_{\mathbb{C}(\lambda)}$  defined over  $\mathbb{C}(\lambda)$ . Since  $(-K_{S'_\eta} \cdot \ell) = 2$  for every general  $\mathbb{C}(\lambda)$ -rational fiber  $\ell$  of  $\xi_\eta$ , either two of the  $C'_i$  are sections of  $\xi_\eta$  and the third one is contained in a fiber or one of the  $C'_i$  is a 2-section of  $\xi_\eta$  and the two other ones are contained in a fiber. In each case, there would exist a closed fiber of  $\xi_\eta : S'_\eta \rightarrow \mathbb{P}^1_{\mathbb{C}(\lambda)}$  containing a  $(-1)$ -curve defined over  $\mathbb{C}(\lambda)$ , which is impossible. So  $\varphi$  contracts at least one horizontal irreducible component of  $\sigma^{-1}(H)$ , say the one intersecting  $S_\eta$  along  $C_1$ . The proper transforms of  $C_2$  and  $C_3$  in the image of  $S_\eta$  by the induced contraction are 0-curves intersecting each other twice at  $\mathbb{C}(\lambda)$ -rational points. The same argument as in the previous case implies that no other horizontal irreducible component of  $\sigma^{-1}(H)$  is contracted by  $\varphi$ . So  $S'_\eta$  is isomorphic to the image of  $S_\eta$  by the contraction of  $C_1$ , hence is a smooth del Pezzo surface of degree 4 defined over  $\mathbb{C}(\lambda)$ , having the sum  $C'_2 + C'_3$  of the images of  $C_2$  and  $C_3$  as an anti-canonical divisor. The Picard number  $\rho(S'_\eta)$  is thus bigger or equal to 2 and so,  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is necessarily a Mori conic bundle.  $\square$

In the proof of Theorem 7 above, we used the following criterion for the output of a relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  to be a Mori conic bundle:

**Lemma 8.** *With the notation above, let  $r \in \{1, 2, 3\}$  and  $h_\varphi \in \{0, 1\}$  be the number of irreducible components of  $H \cap S$  and the number of horizontal irreducible component of  $\sigma^{-1}(H)$  contracted by  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ . If  $\tilde{\mathbb{P}}'$  is a Mori conic bundle  $\tilde{f}' = q \circ \xi : \tilde{\mathbb{P}}' \rightarrow W \rightarrow \mathbb{P}^1$ , then  $r = h_\varphi + 2$ .*

*Proof.* We first observe that the inverse image by  $\xi$  of every irreducible curve  $C \subset W$  is again irreducible. Indeed, assuming on the contrary that  $\xi^{-1}(C)$  has at least two irreducible components  $F_1$  and  $F_2$  such that  $F_1 \cap F_2 \neq \emptyset$ , we can choose an irreducible curve  $\ell_1 \subset F_1$  whose class  $[\ell_1]$  in  $\overline{NE}(\tilde{\mathbb{P}}')$  belongs to the extremal ray giving rise to  $\xi$  and such that  $\ell_1 \cap F_2 \neq \emptyset$ . Then for a general fiber  $\ell$  of  $\xi$ , we have by definition  $[\ell] = a[\ell_1]$  for some  $a > 0$ , but since  $\ell$  is disjoint from  $F_2$ , this would lead to the contradiction  $0 = F_2 \cdot \ell = aF_2 \cdot \ell_1 > 0$ . Since all fibers of  $\tilde{f}'$  except maybe  $(\tilde{f}')^{-1}(\infty)$  are irreducible and rational, it follows that  $q : W \rightarrow \mathbb{P}^1$  is a  $\mathbb{P}^1$ -fibration with  $\eta^{-1}(\infty)$  as a unique possibly reducible fiber. In particular, the Picard number  $\rho(W)$  is equal to  $\nu_\infty + 1$ , where  $\nu_\infty$  denotes the number of irreducible components of  $\eta^{-1}(\infty)$ , which by the previous observation is equal to the number of irreducible components of  $(\tilde{f}')^{-1}(\infty)$ . Since  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  is a good resolution, the number of horizontal irreducible components of  $\sigma^{-1}(H)$  is equal to  $r$ . So the Picard number  $\rho(\tilde{\mathbb{P}})$  of  $\tilde{\mathbb{P}}$  is equal to  $\rho(\mathbb{P}) + r + e_v = 1 + r + e_v$ , where  $e_v$  denote the number of vertical exceptional divisors of  $\sigma$ , all of them being contained in  $\tilde{f}^{-1}(\infty)$  (see § 2.1.1). We obtain

$$\nu_\infty + 1 = \rho(W) = \rho(\tilde{\mathbb{P}}') - 1 = 1 + r + e_v - h_\varphi - v_\varphi - 1 = (1 + e_v - v_\varphi) + (r - h_\varphi) - 1 = \nu_\infty + (r - h_\varphi) - 1$$

where  $v_\varphi$  denotes the number of vertical component of  $\sigma^{-1}(H)$  contracted by  $\varphi$ . So  $r = h_\varphi + 2$ .  $\square$

3.1. The remaining case where  $d = 3$  and  $H \cap S$  has two irreducible components is more intricate. Here given a good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  of the rational map  $\bar{f} : \mathbb{P} = \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ , the intersection of  $\sigma^{-1}(H)$  with the generic fiber  $S_\eta$  of  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  is a reduced anti-canonical divisor whose support consists of the union of a  $(-1)$ -curve  $C_1$  and of a 0-curve  $C_2$  both defined over  $\mathbb{C}(\lambda)$ . The same

argument as in the proof of Theorem 7 for the case  $d = 2$  with  $H \cap S$  reducible implies that a relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  can contract at most one horizontal component of  $\sigma^{-1}(H)$ , namely the one intersecting  $S_\eta$  along  $C_1$ . If this component is indeed contracted by  $\varphi$ , then the image of  $S_\eta$  by the induced birational morphism is a smooth del Pezzo surface of degree 4 defined over  $\mathbb{C}(\lambda)$  and the output  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is a del Pezzo fibration of degree 4 by virtue of Lemma 8. Otherwise, if  $\varphi$  does not contract any horizontal irreducible component of  $\sigma^{-1}(H)$  then  $\varphi$  restricts to an isomorphism between  $S_\eta$  and the generic fiber  $S'_\eta$  of  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$ . Since  $C_1 + C_2$  is an anti-canonical divisor on  $S_\eta$ ,  $\rho(S_\eta) \geq 2$  and so  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  is necessarily a Mori conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  over a normal projective surface  $q : W \rightarrow \mathbb{P}^1$ , whose restriction over the generic point  $\eta$  of  $\mathbb{P}^1$  is a Mori conic bundle  $\xi_\eta : S'_\eta \rightarrow W_\eta \simeq \mathbb{P}_{\mathbb{C}(\lambda)}^1$  defined over  $\mathbb{C}(\lambda)$ . Since  $(-K_{S'_\eta} \cdot \ell) = 2$  for every general  $\mathbb{C}(\lambda)$ -rational fiber  $\ell$  of  $\xi_\eta$  and  $C_1$  is a  $(-1)$ -curve defined over  $\mathbb{C}(\lambda)$ , hence cannot be contained in a fiber of  $\xi_\eta$ , the only possibilities are that either  $C_1$  and  $C_2$  are both sections of  $\xi_\eta$  or that  $C_1$  is a 2-section of  $\xi_\eta$  while  $C_2$  is a full fiber of it. Similarly as in the case  $d = 2$  in the proof of Theorem 7 above, the first possibility is excluded by the fact that  $\rho(S'_\eta/\mathbb{P}_{\mathbb{C}(\lambda)}^1) = 1$ : indeed, we would have  $C_2 \sim C_1 + a\ell$  for some  $a \in \mathbb{Q}$  satisfying simultaneously the identities  $0 = C_2^2 = C_1^2 + 2a = -1 + 2a$  and  $2 = C_2 \cdot C_1 = C_1^2 + a = -1 + a$ , which is impossible. But in contrast with the case  $d = 2$ , the second possibility cannot be excluded. Actually a smooth cubic surface  $S'_\eta \subset \mathbb{P}_{\mathbb{C}(\lambda)}^3$  containing a  $(-1)$ -curve  $C_1$  defined over  $\mathbb{C}(\lambda)$  always admit a conic bundle structure  $\pi : S'_\eta \rightarrow \mathbb{P}_{\mathbb{C}(\lambda)}^1$  with five degenerate fibers, defined by the mobile part of the restriction to  $S'_\eta$  of the pencil of hyperplanes in  $\mathbb{P}_{\mathbb{C}(\lambda)}^3$  containing  $C_1$ .

So in contrast with the other cases, this suggests that the nature of the output  $\tilde{\mathbb{P}}'$  might depend on the chosen good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  and on the relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ . Partial results on the structure of  $\tilde{\mathbb{P}}'$  can be obtained by a more careful study of relative MMPs ran from particular explicit good resolutions  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$ , but a complete discussion would lead us far beyond the intended aim of this article. The following result, which we mention without proof referring the reader to the forthcoming paper [4] for the detail, asserts the existence of relative MMPs whose outputs are del Pezzo fibrations of degree 4. In contrast, we do not know examples for which the output is a Mori conic bundle (see also Remark 12 below).

**Proposition 9.** *Let  $S \subset \mathbb{P}^3$  be a smooth cubic surface, let  $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$  be a hyperplane intersecting  $S$  along the union of a line and smooth conic, let  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$  be the pencil generated by  $S$  and  $3H$  and let  $\tilde{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  be the corresponding rational map. Then there exists a good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  and a MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  relative to  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  whose output is a del Pezzo fibration  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  of degree 4.*

#### 4. MORI CONIC BUNDLES AND TWISTED $\mathbb{A}_*^1$ -FIBRATIONS

In this section, we investigate more closely the case where a relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  ran from a good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  terminates with a Mori conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  over a normal projective surface  $W$ . According to Theorem 7 and § 3.1, this occurs for all pencils  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$  generated by a smooth cubic surface  $S \subset \mathbb{P}^3$  and three times a hyperplane  $H \subset \mathbb{P}^3$  such that  $H \cap S$  consists of three lines, and possibly for pencils for which  $H \cap S$  consists of a line and smooth conic when  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  does not contract any horizontal irreducible component of  $\sigma^{-1}(H)$ .

**Theorem 10.** *Let  $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$  be a pencil as above and let  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  be a relative MMP ran from good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  of the corresponding rational map  $\tilde{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  whose output is a Mori conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  over a normal projective surface  $q : W \rightarrow \mathbb{P}^1$ . Then there exists an open subset  $U \subset W$  isomorphic to  $\mathbb{A}^2$  such that the induced morphism  $\xi_0 = \xi \circ \varphi \circ \sigma^{-1} : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \rightarrow W$  factors through a twisted  $\mathbb{A}_*^1$ -fibration over  $U$ .*

*Proof.* Recall that by virtue of Proposition 6, the composition  $\varphi \circ \sigma^{-1} : \mathbb{P}^3 \setminus H \rightarrow \tilde{\mathbb{P}}' \setminus \varphi_*(\sigma^{-1}(H))$  is an isomorphism. As observed in the proof of Lemma 8,  $q : W \rightarrow \mathbb{P}^1$  is a  $\mathbb{P}^1$ -fibration with  $\eta^{-1}(\infty)$

as a unique possibly reducible fiber, where  $\infty = \bar{f}_*(H)$ . So the restriction of  $q$  over  $\mathbb{P}^1 \setminus \{\infty\}$  is isomorphic to the trivial bundle  $\mathbb{P}^1 \setminus \{\infty\} \times \mathbb{P}^1$ . The union of all vertical components of  $\varphi_*(\sigma^{-1}(H))$  is equal to  $(\tilde{f}')^{-1}(\infty)$  (see § 2.1.1) and on the other hand, it follows from the proof of Theorem 7 and § 3.1 that the restrictions of the two horizontal irreducible components  $E_1$  and  $E_2$  of  $\varphi_*(\sigma^{-1}(H))$  to the generic fiber  $S'_\eta$  of  $\tilde{f}'$  are either a pair of 0-curves  $C_1$  and  $C_2$  defined over  $\mathbb{C}(\lambda)$  with intersecting each other twice at  $\mathbb{C}(\lambda)$ -rational points if  $H \cap S$  consist of three irreducible components, or the union of a  $(-1)$ -curve  $C_1$  and a 0-curve  $C_2$  defined over  $\mathbb{C}(\lambda)$  with  $(C_1 \cdot C_2) = 2$  in the case where  $H \cap S$  consists of two irreducible components. In the first case, one of the curves  $C_i$  is a 2-section of the induced conic bundle  $\xi_\eta : S'_\eta \rightarrow W_\eta \simeq \mathbb{P}_{\mathbb{C}(\lambda)}^1$  while the other one is a full fiber of it, and in the second case,  $C_1$  is a 2-section of  $\xi_\eta$  while  $C_2$  is a full fiber. So up to a permutation, we may assume that in both cases,  $E_1$  is a birational 2-section of  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  while  $E_2$  is mapped by  $\xi$  onto a section  $D$  of  $q : W \rightarrow \mathbb{P}^1$ . The open subset  $U = W \setminus \xi(E_2 \cup (\tilde{f}')^{-1}(\infty)) = W \setminus (D \cup \eta^{-1}(\infty))$  of  $W$  is thus isomorphic to  $\mathbb{A}^2$ , and by construction, the composition  $\xi_0 = \xi \circ \varphi \circ \sigma^{-1} : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \rightarrow W$  factors through  $U$ . Since  $E_1$  is an irreducible birational 2-section of the conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$ , the generic fiber of  $\xi_0$  is a nontrivial form of the punctured affine line over the function field of  $W$ , so  $\xi_0 : \mathbb{A}^3 \rightarrow U$  is a twisted  $\mathbb{A}_*^1$ -fibration.  $\square$

4.1. The twisted  $\mathbb{A}_*^1$ -fibrations  $\xi_0 : \mathbb{A}^3 \rightarrow \mathbb{A}^2$  obtained in Theorem 10 above can be described in terms of the initial data consisting of the smooth cubic surface  $S \subset \mathbb{P}^3$  and the hyperplane  $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$  as follows.

a) In the case where  $H \cap S$  consists of the union of three lines  $\ell_1, \ell_2$  and  $\ell_3$ , then given a good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  of  $\bar{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ , the fiber of  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$  over the generic point  $\eta$  of  $\mathbb{P}^1$  is a smooth cubic surface  $S_\eta \subset \mathbb{P}_{\mathbb{C}(\lambda)}^3$  defined over  $\mathbb{C}(\lambda)$  and the horizontal irreducible components  $E_1, E_2$  and  $E_3$  of  $\sigma^{-1}(H)$ , corresponding respectively to  $\ell_1, \ell_2$  and  $\ell_3$  intersect  $S_\eta$  along three  $(-1)$ -curves defined over  $\mathbb{C}(\lambda)$ . For a relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$ , it follows from the description given in the proof of Theorem 7 that exactly one horizontal irreducible component of  $\sigma^{-1}(H)$  is contracted by  $\varphi$ , say  $E_3$  up to a permutation. The intersection of the proper transforms  $\varphi_*(E_1)$  and  $\varphi_*(E_2)$  of  $E_1$  and  $E_2$  with the generic fiber  $S'_\eta$  of  $\tilde{f}' : \tilde{\mathbb{P}}' \rightarrow \mathbb{P}^1$  are 0-curves defined over  $\mathbb{C}(\lambda)$  intersecting each other twice at  $\mathbb{C}(\lambda)$ -rational points. Furthermore, one of them, say  $\varphi_*(E_2)|_{S'_\eta}$  is a fiber of the induced Mori conic bundle structure  $\xi_\eta : S'_\eta \rightarrow W_\eta \simeq \mathbb{P}_{\mathbb{C}(\lambda)}^1$ , the other one  $\varphi_*(E_1)|_{S'_\eta}$  being a 2-section of  $\xi_\eta$ . Therefore  $\xi_\eta$  coincides with the proper transform by the restriction  $\varphi_\eta$  of  $\varphi$  of the conic bundle  $\theta : S_\eta \rightarrow \mathbb{P}_{\mathbb{C}(\lambda)}^1$  defined by the mobile part of the restriction to  $S_\eta$  of the pencil of hyperplanes in  $\mathbb{P}_{\mathbb{C}(\lambda)}^3$  containing  $E_1|_{S_\eta}$ . So letting  $\Theta_{\ell_1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  be the projection from the line  $\ell_1 \subset H \cap S$ , we conclude that  $\xi_0 : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \rightarrow \mathbb{A}^2$  coincides with the restriction to  $\mathbb{P}^3 \setminus H$  of the rational map  $\bar{f} \times \Theta_{\ell_1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ .

b) In the case where  $H \cap S$  consists of the union of a line  $\ell$  and a smooth conic, the description given in § 3.1 implies by a similar argument that  $\xi_0 : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \rightarrow \mathbb{A}^2$  coincides with the restriction to  $\mathbb{P}^3 \setminus H$  of the rational map  $\bar{f} \times \Theta_\ell : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  where  $\Theta_\ell : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  denotes the projection from the line  $\ell$ .

**Example 11.** Let  $S \subset \mathbb{P}^3 = \text{Proj}_{\mathbb{C}}(\mathbb{C}[x, y, z, w])$  be the smooth cubic surface defined by the vanishing of the polynomial  $F = w^2z + y^2x + wx^2 + z^3$ , let  $\bar{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  be the pencil generated by  $S$  and  $3H$ , where  $H = \{x = 0\}$  and let

$$f : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \simeq \text{Spec}(\mathbb{C}[y, z, w]) \rightarrow \mathbb{A}^1, (y, z, w) \mapsto w^2z + y^2 + w + z^3$$

be the induced morphism. The intersection  $H \cap S$  consists of three lines  $\ell_1 = \{z = t = 0\}$ ,  $\ell_2 = \{w + iz = t = 0\}$  and  $\ell_3 = \{w - iz = t = 0\}$  meeting in the Eckardt point  $[0 : 1 : 0 : 0]$  of  $S$ , and the morphism  $\xi_0 = (f, \text{pr}_z) : \mathbb{A}^3 \rightarrow \mathbb{A}^2$  is a surjective twisted  $\mathbb{A}_*^1$ -fibration induced by the restriction of  $\bar{f} \times \Theta_{\ell_1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . The fact that  $\xi_0$  is twisted can be seen directly as follows: its generic fiber is isomorphic to the curve  $C \subset \mathbb{A}_{\mathbb{C}(\lambda, z)}^2 = \text{Spec}(\mathbb{C}(\lambda, z)[y, w])$  defined by the equation  $w^2z + y^2 + w + z^3 - \lambda = 0$ . Extending the scalars to the quadratic extension  $K = \mathbb{C}(\lambda, z)[v]/(v^2 - z)$ ,

we have

$$\begin{aligned}
 C_K &\simeq \operatorname{Spec}(K[x, y]/(w^2v^2 + y^2 + w + v^6 - \lambda)) \\
 &\simeq \operatorname{Spec}(K[x, y]/((wv + \frac{1}{2v})^2 + y^2 - (\frac{1}{4v^2} - v^6 + \lambda))) \\
 &\simeq \operatorname{Spec}(K[U, V]/(UV - (\frac{1}{4v^2} - v^6 + \lambda))) \\
 &\simeq \operatorname{Spec}(K[U^{\pm 1}])
 \end{aligned}$$

where  $U = wv + \frac{1}{2v} + iy$  and  $V = wv + \frac{1}{2v} - iy$ , on which the Galois group  $\operatorname{Gal}(K/\mathbb{C}(\lambda, z)) \simeq \mathbb{Z}_2$  acts by  $U \mapsto -U^{-1}$ . So  $C$  is a nontrivial  $\mathbb{C}(\lambda, z)$ -form of the punctured affine line over  $\mathbb{C}(\lambda, z)$ .

*Remark 12.* In the case where  $d = 3$  and  $H \cap S$  consists of a line  $\ell$  and smooth conic, the fact that the projection  $\Theta_\ell : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  gives rise to a twisted  $\mathbb{A}_*^1$ -fibration  $\xi_0 = (\bar{f}, \Theta_\ell)|_{\mathbb{P}^3 \setminus H} : \mathbb{A}^3 = \mathbb{P}^3 \setminus H \rightarrow \mathbb{A}^2$  does not necessarily imply that a relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  ran from a good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  of  $\bar{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$  terminates with a Mori conic bundle  $\xi : \tilde{\mathbb{P}}' \rightarrow W$  inducing  $\xi_0$  (see Proposition 9). Note that since the base locus of  $\Theta_\ell$  is contained in that of  $\bar{f}$ , we can choose a good resolution  $(\tilde{\mathbb{P}}, \sigma, \tilde{f})$  of  $\bar{f}$  which simultaneously resolves the indeterminacies of  $\Theta_\ell$ . Every MMP  $\psi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}_1$  relative to the morphism  $(\tilde{f}, \Theta_\ell \circ \sigma) : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  being also a part of a MMP relative to  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$ , it preserves the open subset  $\mathbb{A}^3 = \tilde{\mathbb{P}} \setminus \sigma^{-1}(H)$  by virtue of Proposition 6. Such a MMP process  $\psi$  does not contract any horizontal irreducible component of  $\sigma^{-1}(H)$  and terminates with a Mori conic bundle  $\xi_1 : \tilde{\mathbb{P}}_1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ , whose restriction to  $\mathbb{A}^3$  coincides with  $\xi_0$  by construction. But there is no guarantee in general that  $\tilde{f}_1 = \operatorname{pr}_1 \circ \xi_1 : \tilde{\mathbb{P}}_1 \rightarrow \mathbb{P}^1$  coincides with the final output of a MMP relative to  $\tilde{f} : \tilde{\mathbb{P}} \rightarrow \mathbb{P}^1$ : there could exist a relative MMP  $\varphi : \tilde{\mathbb{P}} \dashrightarrow \tilde{\mathbb{P}}'$  which factorizes through  $\psi$  and for which the induced rational map  $\psi' = \varphi \circ \psi^{-1} : \tilde{\mathbb{P}}_1 \dashrightarrow \tilde{\mathbb{P}}'$  contracts an irreducible component of  $\psi_*(\sigma^{-1}(H))$  that is horizontal for  $\tilde{f}_1$ .

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