Big image of Galois representations associated with finite slope p-adic families of modular forms

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BIG IMAGE OF GALOIS REPRESENTATIONS ASSOCIATED WITH FINITE SLOPE p-ADIC FAMILIES OF MODULAR FORMS

ANDREA CONTI*, ADRIAN IOVITA, JACQUES TILOUINE*

1. Introduction

It is known by the works of Ribet [Ri2], [Ri3] and Momose [Mo], that the ℓ-adic Galois representation $\rho_{f,\ell}$ associated with a non-CM cuspidal eigenform f has large image for any prime ℓ, and that for almost all ℓ it satisfies

(cong_ℓ) $\mathrm{Im}\,\rho_{f,\ell}$ contains the conjugate of a principal congruence subgroup $\Gamma(\ell^m)$ of $\mathrm{SL}_2(\mathbb{Z}_\ell)$.

For instance, if $\mathrm{Im}\,\rho_{f,\ell}$ contains an element with eigenvalues in $\mathbb{Z}_\ell^\times$ distinct modulo ℓ, then (cong_ℓ) holds. In [Hi], Hida proved an analogous statement for p-adic families of non-CM ordinary cusp eigenforms. Let $\Lambda = \mathbb{Z}_p[[T]]$ be the Iwasawa algebra and $\mathfrak{m} = (p, T)$ be its maximal ideal. A special case of his first Theorem [Hi, Th.I] is the following

Theorem 1.1. *Let $\mathbf{f}$ be a non-CM Hida family of ordinary cusp eigenforms defined over a finite extension $\mathbb{I}$ of Λ, and let $\rho_{\mathbf{f}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I})$ be the associated Galois representation. Let us assume that it is residually irreducible and that there exists an element d in the image of Galois with eigenvalues $\alpha, \beta \in \mathbb{Z}_p^\times$ such that $\alpha^2 \not\equiv \beta^2 \pmod p$. Then there exists a non zero ideal $\mathfrak{l} \subset \Lambda$ and an element $g \in \mathrm{GL}_2(\mathbb{I})$ such that*

$$g\Gamma(\mathfrak{l})g^{-1} \subset \mathrm{Im}\,\rho_{\mathbf{f}}$$

where $\Gamma(\mathfrak{l})$ denotes the principal congruence subgroup of $\mathrm{SL}_2(\Lambda)$ of level $\mathfrak{l}$.

Moreover, under mild technical assumptions, he showed [Hi, Th. II] that if the residual representation contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_p)$, then $\mathfrak{l}$ is trivial or $\mathfrak{m}$-primary, and if the residual representation is dihedral "of CM type", the height one prime factors P of $\mathfrak{l}$ are exactly those of the g.c.d. of the adjoint p-adic L function of $\mathbf{f}$ and of anticyclotomic specializations of Katz p-adic L functions associated with certain Hecke characters of an imaginary quadratic field. This set of primes is precisely the set of congruence primes between the given non-CM family and CM families.

In her PhD dissertation (see her paper [La]), J. Lang improved upon Hida's Theorem I. Let $\mathbb{T}$ be Hida's big ordinary cuspidal Hecke algebra; it is finite and flat over Λ. Let $\mathrm{Spec}\,\mathbb{I}$ be an irreducible component of $\mathbb{T}$. It corresponds to a surjective Λ-algebra homomorphism $\theta\colon \mathbb{T} \to \mathbb{I}$ (a Λ-adic Hecke eigensystem). We also call θ a Hida family. Assume that it is not residually Eisenstein. It gives rise to a residually irreducible continuous Galois representation $\rho_\theta\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I})$ that is p-ordinary. We suppose for simplicity that $\mathbb{I}$ is normal. Consider the Λ-algebra automorphisms σ of $\mathbb{I}$ for which there exists a finite order character $\eta_\sigma\colon G_{\mathbb{Q}} \to \mathbb{I}^\times$ such that for any prime ℓ prime to the level, $\sigma \circ \theta(T_\ell) = \eta_\sigma(\ell)\theta(T_\ell)$ (see [Ri3] and [La]). These morphisms form a finite abelian 2-group Γ. Let $\mathbb{I}_0$ be the subring fixed by Γ. Let $H_0 = \bigcap_{\sigma\in\Gamma} \mathrm{Ker}\,\eta_\sigma$; it is a normal open subgroup of $G_{\mathbb{Q}}$. Then, by Carayol's theorem, one can assume, up to conjugation by an element of $\mathrm{GL}_2(\mathbb{I})$, that $\rho_\theta|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0)$.

Theorem 1.2. *(*[La] *Th.2.4) Let $\theta\colon \mathbb{T} \to \mathbb{I}$ be a non-CM Hida family such that $\overline{\rho}_\theta$ is absolutely irreducible. Assume after conjugation that $\rho_\theta(H_0) \subset \mathrm{GL}_2(\mathbb{I}_0)$ and that there exists $d \in \rho_\theta(D_p \cap H_0)$ with eigenvalues $\alpha, \beta \in \mathbb{Z}_p^\times$ such that $\alpha^2 \not\equiv \beta^2 \pmod p$. Then there exists a non zero ideal $\mathfrak{l} \subset \mathbb{I}_0$ and an element $g \in \mathrm{GL}_2(\mathbb{I})$ such that*

$$g\Gamma(\mathfrak{l})g^{-1} \subset \mathrm{Im}\,\rho_\theta$$

where $\Gamma(\mathfrak{l})$ denotes the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0)$ of level $\mathfrak{l}$.

For all these results, it is important to assume the ordinarity of the family as it implies the ordinarity of the Galois representation, and in particular that some element of the image of the inertia at p is conjugate to

$$A = \begin{pmatrix} u^{-1}(1+T) & * \\ 0 & 1 \end{pmatrix}$$

*Supported by the Programs ArShiFo ANR-10-BLAN-0114 and PerColaTor ANR-14-CE25-0002-01.

Hida and Lang use Pink's theory of Lie algebras of pro-p subgroups of $\mathrm{SL}_2(\mathbb{I})$. Conjugation by the element above defines a Λ-module structure on the Lie algebra of a pro-p subgroup of $\mathrm{Im}\,\rho_{\mathbf{f}}$.

In this paper, we propose a generalization of Hida's work to the finite slope case. We establish an analogue of Hida's Theorem I and Theorem II. These are Theorems 6.2, 7.1 and 7.4 in the text. Moreover, we put ourselves in the more general setting studied by Lang's [La] work. In the positive slope case, the existence of a conjugating matrix analoguous to A above is obtained by applying relative Sen theory ([Se1] and [Se2]) at the expense of extending scalars to the completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$.

More precisely, for any $h \in \mathbb{Q}^{\times +}$, we define an Iwasawa algebra $\Lambda_h = \mathcal{O}_h[[t]]$ (where $T = p^{s_h} \cdot t$ for some $s_h \in \mathbb{Q} \cap]0, \frac{1}{p-1}[$ and $\mathcal{O}_h$ is a finite extension of $\mathbb{Z}_p$ containing p^{s_h}) and a finite torsion free Λ_h-algebra $\mathbb{T}_h$ (see Section 3.1), called an adapted slope $\leq h$ Hecke algebra. Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be an irreducible component; it is finite torsion-free over Λ_h. The notation $\mathbb{I}^\circ$ is borrowed from the theory of Tate algebras. We write $\mathbb{I} = \mathbb{I}^\circ[p^{-1}]$. We assume again for simplicity that $\mathbb{I}^\circ$ is normal. The finite slope family θ gives rise to a continuous Galois representation $\rho_\theta\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}^\circ)$. We assume that the residual representation $\overline{\rho_\theta}$ is absolutely irreducible. We introduce the finite abelian 2-group Γ as above, together with its fixed ring $\mathbb{I}_0^\circ$ and the open normal subgroup $H_0 \subset G_{\mathbb{Q}}$. In Section 5.1 we define a ring $\mathbb{B}_r$ and a Lie algebra $\mathfrak{H}_r \subset \mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$ attached to the image of ρ_θ. In the positive slope case, CM families do not exist (see Section 3.3), hence no "non-CM" assumption is needed in the following.

Theorem 1.3. *(Theorem 6.2) Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a positive slope family such that $\overline{\rho}_\theta|_{H_0}$ is absolutely irreducible. Assume after a suitable conjugation that $\rho_\theta(H_0) \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$ and that there exists $d \in \rho_\theta(D_p \cap H_0)$ with eigenvalues $\alpha, \beta \in \mathbb{Z}_p^\times$ such that $\alpha^2 \not\equiv \beta^2 \pmod p$. Then there exists a non zero ideal $\mathfrak{l} \subset \mathbb{I}_0^\circ$ such that*

$$\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r) \subset \mathfrak{H}_r$$

The largest such ideal $\mathfrak{l}_\theta$ is called the Galois level of θ. We also introduce the notion of fortuitous CM congruence ideal for θ (Section 3.4). It is the ideal $\mathfrak{c}_\theta \subset \mathbb{I}$ given by the product of the primary ideals modulo which a congruence between θ and a slope $\leq h$ CM form occurs. In this context, following Hida's proof, we are also able to show (Section 7) that the set of primes of $\mathbb{I}_0 = \mathbb{I}_0^\circ[p^{-1}]$ containing $\mathfrak{l}_\theta$ coincides with the set of primes containing $\mathfrak{c}_\theta \cap \mathbb{I}_0$, except possibly for the primes of $\mathbb{I}_0$ above $P_1 = u^{-1}(1+T) - 1$ (the so-called weight 1 primes).

Several generalizations of the present work, including a generalization of [HT] (which treated the ordinary case for GSp_4, with a residual representation induced from the representation associated to a Hilbert form) to the finite slope case and to bigger groups and more cases of residual representations, are currently being studied by one of the authors[1].

Acknowledgments. This paper owes much to Hida's recent paper [Hi]. We also thank Jaclyn Lang for communicating us her dissertation [La], which proved very useful in writing Section 4, and for providing us with some helpful remarks.

2. The eigencurve

2.1. **The weight space.** Let $p > 2$ be a prime. The $\mathbb{Q}_p$-rigid space $\mathcal{W} = \underline{\mathrm{Hom}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ of continuous homomorphisms is called the weight space.

Remark 2.1. *Before Section 3 all the rigid analytic spaces we consider are defined over $\mathbb{Q}_p$. Indeed the weight space and the eigencurve can be admissibly covered by affinoid subdomains defined over $\mathbb{Q}_p$. Later we will need to consider rigid $\mathbb{Q}_p$-analytic spaces defined only over a finite extension $L/\mathbb{Q}_p$.*

For any $R > 0$, we denote by $B(0,R)$, respectively $B(0,R^-)$ the closed, respectively open, disc in $\mathbb{C}_p$ of centre 0 and radius R. The space $\mathcal{W}$ is isomorphic to a disjoint union of $p-1$ copies of the open unit disc $B(0,1^-)$ centered in 0 and indexed by the group $\mathbb{Z}/(p-1)\mathbb{Z} = \widehat{\mu}_{p-1}$. Let u be a topological generator of $1 + p\mathbb{Z}_p$, then an isomorphism is given by

$$\mathbb{Z}/(p-1)\mathbb{Z} \times B(0,1^-) \to \mathcal{W}, \quad (i,v) \mapsto \chi_{i,v}$$

where $\chi_{i,v}((\zeta, u^x)) = \zeta^i(1+v)^x$.

We use the following terminology.

Definition 2.2. *Let X be a $\mathbb{Q}_p$-rigid space and S a subset of X. We say that:*

- *S is Zariski-dense if for any global analytic function f on X the set $S \cap \{f \neq 0\}$ is nonempty;*

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- *S is an accumulation subset if for any affinoid domain* $\operatorname{Spm} R \subset X$ *the set* $S \cap \operatorname{Spm} R$ *is Zariski-dense in* $\operatorname{Spm} R$.

We say that a point $\chi \in \mathcal{W}$ is classical if there exist $k \in \mathbb{N}$ and a finite order character $\psi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ such that χ is the character $z \mapsto z^k \psi(z)$. The set of classical points is an accumulation subset of $\mathcal{W}$.

If $\operatorname{Spm} R \subset \mathcal{W}$ is an affinoid open subset, we denote by $\kappa = \kappa_R \colon \mathbb{Z}_p^\times \to R^\times$ its tautological character given by $\kappa(t)(\chi) = \chi(t)$ for any $\chi \in \operatorname{Spm} R$. Recall ([Bu, Prop. 8.3]) that κ_R is r-analytic for any sufficiently small radius $r > 0$ (by which we mean it extends to $\mathbb{Z}_p^\times B(1, r)$).

2.2. **Adapted pairs and the eigencurve.** Let N be a positive integer prime to p. Let us recall a definition of the spectral curve Z^N and of the cuspidal eigencurve C^N of tame level $\Gamma_1(N)$. We follow the presentation of [Bu, Part II]. Let $\operatorname{Spm} R \subset \mathcal{W}$ be an affinoid domain and let $r = p^{-s}$ for $s \in \mathbb{Q}$ be a radius smaller than the radius of analyticity of κ_R. We denote by $M_{R,r}$ the R-module of r-overconvergent modular forms of weight κ_R. It is endowed with a continous action of the Hecke operators T_ℓ, $l \nmid Np$, and U_p. The action of U_p on $M_{R,r}$ is completely continous, so we can consider its associated Fredholm series $F_{R,r}(T) = \det(1 - U_p T | M_{R,r}) \in R\{\{T\}\}$. These series are compatible when R and r vary, in the sense that there exists $F \in \mathcal{O}(\mathcal{W})\{\{T\}\}$ that restricts to $F_{R,r}(T)$ for any R and r.

The series $F_{R,r}(T)$ converges everywhere over the R-affine line $\operatorname{Spm} R \times \mathbb{A}^{1,an}$, so it defines a rigid curve $Z_{R,r}^{N,\pm} = \{F_{R,r}(T) = 0\}$ in $\operatorname{Spm} R \times \mathbb{A}^{1,an}$. When R and r vary, these curves glue into a rigid space Z^N endowed with a quasi-finite and flat morphism $w_Z : Z^N \to \mathcal{W}$. The curve Z^N is called the spectral curve. For any $h \geq 0$, let us consider

$$Z_R^{N,\leq h} = Z_R^N \cap \big(\operatorname{Spm} R \times B(0, p^h)\big)$$

By [Bu, Lemma 4.1] $Z_R^{N,\leq h}$ is quasi-finite and flat over $\operatorname{Spm} R$.

We recall how to construct an admissible covering of Z^N.

Definition 2.3. *We denote by $\mathcal{C}$ the set of affinoid subdomains $Y \subset Z$ such that:*

- *there exists an affinoid domain* $\operatorname{Spm} R \subset \mathcal{W}$ *such that Y is a union of connected components of* $w_Z^{-1}(\operatorname{Spm} R)$*;*
- *the map $w_Z|_Y : Y \to \operatorname{Spm} R$ is finite.*

Proposition 2.4. [Bu, Th. 4.6] *The covering $\mathcal{C}$ is admissible.*

Note in particular that an element $Y \in \mathcal{C}$ must be contained in $Z_R^{N,\leq h}$ for some h.

For any R and r as above and any $Y \in \mathcal{C}$ such that $w_Z(Y) = \operatorname{Spm} R$, we can associate to Y a direct factor M_Y of $M_{R,r}$ by the construction in [Bu, Sec. I.5]. The abstract Hecke algebra $\mathcal{H} = \mathbb{Z}[T_\ell]_{\ell \nmid Np}$ acts on $M_{R,r}$ and M_Y is stable with respect to this action. Let $\mathbb{T}_Y$ be the R-algebra generated by the image of $\mathcal{H}$ in $\operatorname{End}_R(M_Y)$ and let $C_Y^N = \operatorname{Spm} \mathbb{T}_Y$. Note that it is reduced as all Hecke operators are self-adjoint for a pairing and mutually commute.

For any Y the finite covering $C_Y^N \to \operatorname{Spm} R$ factors through $Y \to \operatorname{Spm} R$. The eigencurve C^N is defined by gluing the affinoids C_Y^N into a rigid curve, endowed with a finite morphism $C^N \to Z^N$. The curve C^N is reduced and flat over $\mathcal{W}$ since it is so locally.

We borrow the following terminology form Bellaïche.

Definition 2.5. [Be, Def. II.1.8] *Let* $\operatorname{Spm} R \subset \mathcal{W}$ *be an affinoid open subset and $h > 0$ be a rational number. The couple (R, h) is called adapted if $Z_R^{N,\leq h}$ is an element of $\mathcal{C}$.*

The sets of the form $Z_R^{N,\leq h}$ are actually sufficient to admissibly cover the spectral variety by [Be, Cor. II.1.13].

Now fix a finite slope h. We want to work with families of slope $\leq h$ which are finite over a wide open subset of the weight space. In order to do this it will be useful to know which pairs (R, h) in a connected component of $\mathcal{W}$ are adapted. If $\operatorname{Spm} R' \subset \operatorname{Spm} R$ are affinoid subdomains of $\mathcal{W}$ and (R, h) is adapted then (R', h) is also adapted by [Be, Prop. II.1.10]. A lower bound for the radius of an adapted disc in terms of h is given by a result of Wan. We denote by $F_k(T)$ the specialization of the Fredholm characteristic series $F(T)$ to a classical weight $k \colon t \mapsto t^k$.

Theorem 2.6. [Wa, Th. 2.5] *There exist constants $A, B \in \mathbb{Q}$ such that the following is true for $m(h) = Ah^2 + Bh$: if k_1 and k_2 are two non-negative integers such that $k_1 \equiv k_2 (\mathrm{mod}\ p^{m(h)}(p-1))$, then the slope $\leq h$ parts of the Newton polygons of $F_{k_1}(T)$ and $F_{k_2}(T)$ coincide.*

Remark 2.7. *In Wan's result about the dimension of spaces of classical forms a constant C appears in the quadratic polynomial for $m(h)$, but the proof of his Corollary 4.2 shows that we can take $C = 0$.*

We deduce from this theorem the estimate we need about the radius of a disc adapted to h:

Corollary 2.8. *If $h \in \mathbb{R}$ is a slope and $m(h) = Ah^2 + Bh$ is the polynomial defined by Wan's theorem, the disc $B(0, p^{-m(h)})$ is adapted to h.*

Proof. Let Z^N be the spectral variety defined as the locus $\{F = 0\}$ in $\mathcal{W} \times \mathbb{A}^1$ and $Z_h = Z^N \cap B(0, p^{-m(h)}) \times B(0, p^h)$. By definition $B(0, p^{-m(h)})$ is adapted to h if and only if the map $Z_h \to B(0, p^{-m(h)})$ is finite; this is equivalent by [Bu, Lemma 4.3] to the fact that all fibres of Z_h over points $\kappa \in B(0, p^{-m(h)})$ have the same degree. Since the map $Z_h \to B(0, p^{-m(h)})$ is locally finite (every point in $\mathcal{W}$ has a neighborhood adapted to h), the degree of its fibres is locally constant. We know that on the accumulation subset of $B(0, p^{-m(h)})$ consisting of classical weights this degree is constant by Wan's theorem. We conclude that the degree of the fibre of Z_h is constant everywhere on $B(0, p^{-m(h)})$, hence this disc is adapted to h. □

We work for simplicity with discs centered in 0, but the corollary above is clearly true for a disc $B(x, p^{-m(h)})$ of radius given by Wan's estimate and centered in any point $x \in \mathcal{W}$.

2.3. **Pseudocharacters and Galois representations.** Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$. Let X be a rigid $\mathbb{Q}_p$-analytic variety defined over K. We denote by $\mathcal{O}(X)$ the ring of global analytic functions on X equipped with the coarsest locally convex topology making the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ continuous for any affinoid $U \subset X$. It is a Fréchet space isomorphic to the inverse limit over all affinoid domains U of the K-Banach spaces $\mathcal{O}(U)$. We denote by $\mathcal{O}(X)^\circ$ the $\mathcal{O}_K$-algebra of functions bounded by 1 on X, equipped with the topology induced by that on $\mathcal{O}(X)$. The question of the compactness of this ring is related to the following property of X:

Definition 2.9. [BC, Def. 7.2.10] *We say that a rigid $\mathbb{Q}_p$-analytic variety X defined over K is nested if there is an admissible covering $X = \bigcup X_i$ by open affinoids X_i defined over K such that the maps $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ induced by the inclusions are compact.*

As anticipated we have:

Lemma 2.10. [BC, Lemma 7.2.10] *We equip the ring $\mathcal{O}(X)^\circ$ with the topology induced by that on $\mathcal{O}(X) = \varprojlim_i \mathcal{O}(X_i)$. If X is reduced and nested, then $\mathcal{O}(X)^\circ$ is a compact (hence profinite) $\mathcal{O}_K$-algebra.*

Furthermore we have:

Proposition 2.11. [BC, Cor. 7.2.12] *The eigenvariety C^N is nested for $K = \mathbb{Q}_p$.*

Given any reduced nested subvariety X of C^N defined over a finite extension K of $\mathbb{Q}_p$, we have a pseudocharacter on X obtained by interpolating classical ones.

Proposition 2.12. [Be, Th. IV.4.1] *There exists a unique pseudocharacter*

$$\tau : G_{\mathbb{Q},Np} \to \mathcal{O}(X)^\circ$$

of dimension 2 such that for every l prime to Np $\tau(\mathrm{Frob}_\ell) = \psi_X(T_\ell)$, where ψ_X is the composition of $\psi : \mathcal{H} \to \mathcal{O}(C^N)^\circ$ with the restriction map $\mathcal{O}(C^N)^\circ \to \mathcal{O}(X)^\circ$.

Remark 2.13. *One can take as an example of X a union of irreducible components of C^N, in which case $K = \mathbb{Q}_p$. Later, we'll take other examples where $K \neq \mathbb{Q}_p$.*

3. The fortuitous congruence ideal

In this section we will define families with slope bounded by a finite constant and coefficients in a suitable profinite ring. We will show that any such family admits at most a finite number of classical specialization which are CM modular forms. Later we will define what it means for a point (not necessarily classical) to be CM and we will associate with a family a congruence ideal describing its CM points. Contrary to the ordinary case, the non-ordinary CM points do not come in families, so the points detected by the congruence ideal do not correspond to a crossing between a CM and a non-CM family. For this reason we call our ideal the "fortuitous congruence ideal".

3.1. **The adapted slope $\leq h$ Hecke algebra.** We fix a slope $h > 0$ throughout this section. Let $C^{N,\leq h}$ be the subvariety of C^N whose points have slope $\leq h$. Unlike the ordinary case treated in [Hi] the weight map $w^{\leq h} : C^{N,\leq h} \to \mathcal{W}$ is not finite, which means that a family of slope $\leq h$ is not in general defined by a finite map over the entire weight space. The best we can do in the finite slope situation is to place ourselves on the largest possible wide open subdomain U of $\mathcal{W}$ such that the restricted weight map $w^{\leq h}|_U : C^{N,\leq h} \times_{\mathcal{W}} U \to U$ is finite. Finiteness property will be used to apply going-up and going-down theorems.

Remark 3.1. *We work for simplicity with discs centred at* 0*, but all results hold for any other choice of a centre in* $\mathcal{W}$*.*

Let us fix a rational number s_h such that for $r_h = p^{-s_h}$ the closed disc $B(0, r_h)$ is adapted for h. By the results of Section 2.2, we can assume $s_h \leq m(h)$. We also assume that $s_h > \frac{1}{p-1}$. Let $\eta_h \in \overline{\mathbb{Q}}_p$ be an element of p-adic valuation s_h. Let $K_h = \mathbb{Q}_p(\eta_h)$ and let $\mathcal{O}_h$ be its valuation ring. Let $\Lambda_h = \mathcal{O}_h[[\eta_h^{-1}T]]$. This is the ring of analytic functions bounded by one on the wide open disc B_h of radius p^{-s_h}. For $i \geq 1$, let $s_i = s_h + 1/i$ and $B_i = B(0, p^{-s_i})$. The open disc B_h is the increasing union of the affinoid discs B_i. For each i, a model for B_i over K_h is given by Berthelot's construction of B_h as the rigid space associated with the $\mathcal{O}_h$-formal scheme $\operatorname{Spf} \Lambda_h$. We recall it briefly, following [dJ, Sec. 7]. Let $t = \eta_h^{-1}T$ and

$$A^\circ_{r_i} = \mathcal{O}_h\langle t, X_i\rangle/(pX_i - t^i)$$

We have $B_i = \operatorname{Spm} A^\circ_{r_i}[p^{-1}]$ as rigid space over K_h. For any i we have a morphism $A^\circ_{r_{i+1}} \to A^\circ_{r_i}$ given by

$$X_{i+1} \mapsto X_i t$$
$$t \mapsto t$$

We have induced morphisms $A^\circ_{r_{i+1}}[p^{-1}] \to A^\circ_{r_i}[p^{-1}]$, hence open immersions $B_i \to B_{i+1}$ defined over K_h. The wide open disc B_h is defined as the inductive limit of the affinoids B_i with these transition maps. We also have $\Lambda_h = \varprojlim_i A^\circ_{r_i}$.

Remark 3.2. *Since there is no constant term in Wan's radius, when the slope* h *is sufficiently close to* 0*, the wide open subdomain* B_h *can be taken to be the open disk* $B(0, p^{-\frac{1}{p-1}})$*.*

Since the s_i are strictly bigger than s_h, for each i, $B(0, p^{-s_i}) = \operatorname{Spm} A^\circ_{r_i}[p^{-1}]$ is adapted to h. Therefore, for any $r > 0$ sufficiently small and for any $i \geq 1$, the image of the abstract Hecke algebra acting on $M_{A^\circ_{r_i},r}$ provides a finite affinoid $A^\circ_{r_i}$-algebra $\mathbb{T}^{\leq h}_{A^\circ_{r_i},r}$. The morphism $w_i : \operatorname{Spm} \mathbb{T}^{\leq h}_{A^\circ_{r_i},r} \to \operatorname{Spm} A^\circ_{r_i}$ is finite. For $i < j$ we have natural inclusions $\operatorname{Spm} \mathbb{T}^{\leq h}_{A^\circ_{r_j},r} \to \operatorname{Spm}\mathbb{T}^{\leq h}_{A^\circ_{r_i},r}$ and corresponding restriction maps $\mathbb{T}^{\leq h}_{A^\circ_{r_i},r} \to \mathbb{T}^{\leq h}_{A^\circ_{r_j},r}$. We call C_h the increasing union $\bigcup_{i\in\mathbb{N},r>0} \operatorname{Spm} \mathbb{T}^{\leq h}_{A^\circ_{r_i},r}$; it is a wide open subvariety of C^N. We denote by $\mathbb{T}_h$ the ring of rigid analytic functions bounded by 1 on C_h. We have $\mathbb{T}_h := \mathcal{O}(C_h)^\circ = \varprojlim_{i,r} \mathbb{T}^{\leq h}_{A^\circ_{r_i},r}$. We have a natural weight map $w_h : C_h \to B_h$ that restricts to the maps $w_{A^\circ_{r_i},r} : \operatorname{Spm} \mathbb{T}^{\leq h}_{A^\circ_{r_i}} \to \operatorname{Spm} A^\circ_{r_i}$. It is finite since the closed ball of radius r_h is adapted to h.

3.2. **The Galois representation associated with a family of finite slope.** Since $\mathcal{O}(B_h)^\circ = \Lambda_h$, the ring $\mathbb{T}_h$ gets through w_h a structure of finite Λ_h-algebra; in particular it is profinite.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_h$. The residue field $k = \mathbb{T}_h/\mathfrak{m}$ is finite, say of order q. Let $\mathbb{T}_\mathfrak{m}$ be the localization of $\mathbb{T}_h$ at $\mathfrak{m}$. Since Λ_h is henselian, $\mathbb{T}_\mathfrak{m}$ is a direct factor or $\mathbb{T}_h$, hence it is finite over Λ_h; it is also local noetherian and profinite. Let $W = W(k)$ be the ring of Witt vectors of k. By the universal property of W, $\mathbb{T}_\mathfrak{m}$ is a W-algebra. By density of classical modular points, $\operatorname{Spm}\mathbb{T}_\mathfrak{m}$ contains points x corresponding to cusp eigenforms f_x, of weight $w(x) = k_x \geq 2$ and level Np. The Galois representations ρ_{f_x} associated with f_x give rise to a residual representation $\overline{\rho}\colon G_{\mathbb{Q},Np} \to \mathrm{GL}_2(k)$ which is independent of f_x. By Proposition 2.12, we have a pseudo-character

$$\tau_{\mathbb{T}_\mathfrak{m}} : G_{\mathbb{Q},Np} \to \mathbb{T}_\mathfrak{m}$$

such that for every classical point $x\colon \mathbb{T}_\mathfrak{m} \to L$, defined over some finite extension $L/\mathbb{Q}_p$, the specialization of $\tau_{\mathbb{T}_\mathfrak{m}}$ at x is the trace of the usual representation $G_{\mathbb{Q},Np} \to \mathrm{GL}_2(L)$ attached to x.

Proposition 3.3. *If* $\overline{\rho}$ *is irreducible, there exists a unique continuous irreducible Galois representation*

$$\rho_{\mathbb{T}_\mathfrak{m}} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{T}_\mathfrak{m})$$

lifting $\overline{\rho}$ *and whose trace is* $\tau_{\mathbb{T}_\mathfrak{m}}$*.*

This follows from a result by Nyssen and Rouquier ([Ro, Cor. 5.2]), since $\mathbb{T}_{\mathfrak{m}}$ is local henselian.

In the sequel, we assume that $\overline{\rho}$ is irreducible.

Let $\mathbb{I}^\circ$ be a finite torsion-free Λ_h-algebra. By a family, we mean an irreducible component of $\operatorname{Spec} \mathbb{T}_h$ defined by a surjective morphism $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$ of Λ_h-algebras. By Proposition 3.3 we obtain a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with θ.

Remark 3.4. *If $\eta_h \notin \mathbb{Q}_p$ then the open disc B_h is not defined over $\mathbb{Q}_p$. In particular Λ_h is not a power series ring over $\mathbb{Z}_p$.*

3.3. **Finite slope CM modular forms.** In this section we study non-ordinary finite slope CM modular forms. We say that a family is CM if all its classical points are CM. We prove that for any positive slope $h > 0$, there are no CM families with positive slope $\leq h$. However, contrary to the ordinary case, any family of finite positive slope may contain classical CM points of weight $k \geq 2$. Let F be an imaginary quadratic field, $\mathfrak{f}$ an integral ideal in F, $I_{\mathfrak{f}}$ the group of fractional ideals prime to $\mathfrak{f}$. Let σ_1, σ_2 be the embeddings of F into $\mathbb{C}$ (say that $\sigma_1 = \mathrm{Id}_F$) and let $(k_1, k_2) \in \mathbb{Z}^2$. A Grössencharacter ψ of infinity type (k_1, k_2) defined modulo $\mathfrak{f}$ is a homomorphism $\psi \colon I_{\mathfrak{f}} \to \mathbb{C}^*$ such that $\psi((\alpha)) = \sigma_1(\alpha)^{k_1}\sigma_2(\alpha)^{k_2}$ for all $\alpha \equiv 1 \pmod^{\times} \mathfrak{f}$. Consider the q-expansion

$$\sum_{\mathfrak{a} \subset \mathcal{O}_F(\mathfrak{a},\mathfrak{f})=1} \psi(\mathfrak{a}) q^{N(\mathfrak{a})}$$

where the sum is over ideals $\mathfrak{a} \subset \mathcal{O}_F$ and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. It is known that for an imaginary quadratic field $F/\mathbb{Q}$ of discriminant D and a Grössencharacter ψ of exact conductor $\mathfrak{f}$ and infinity type $(k-1, 0)$ the expansion above defines a cuspidal newform $f(F, \psi)$ of level $N(\mathfrak{f})D$.

Ribet proved in [Ri1, Th. 4.5] that if a newform g of weight $k \geq 2$ and level N has CM by a quadratic imaginary field F, one has $g = f(F, \psi)$ for some Grössencharacter ψ of F of infinity type $(k-1, 0)$.

Definition 3.5. *We say that a classical modular form g of level Np^k has CM by an imaginary quadratic field F if its Hecke eigenvalues for the operators T_ℓ, $l \nmid Np$ coincide with those of $f(F, \psi)$ for some Grössencharacter ψ of F of infinity type $(k-1, 0)$. We also say that g is CM without specifying the field.*

Remark 3.6. *For g as in the definition, the Galois representations $\rho_g, \rho_{f(F,\psi)} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ associated with g and $f(F, \psi)$ are isomorphic, hence the image of the representation ρ_g is contained in the normalizer of a torus in GL_2, if and only if the form g is CM.*

Proposition 3.7. *Let g be a CM modular form of level Np^m with N prime to p and $m \geq 0$. Then its p-slope is either 0, $\frac{k-1}{2}$, $k-1$ or infinite.*

Proof. We assume first that g is p-new.

We begin with the case $m = 0$. We determine the T_p-eigenvalue a_p. It is equal to 0 if p is inert in F. If p splits as $\mathfrak{p}\bar{\mathfrak{p}}$, then $a_p = \psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}})$. We can find an integer $h > 0$ such that $\mathfrak{p}^h$ is a principal ideal (α) with $\alpha \equiv 1 \pmod{\mathfrak{f}}$. Hence $\psi((\alpha)) = \alpha^{k-1}$. Since α is a generator of $\mathfrak{p}^h$ we have $\alpha \in \mathfrak{p}$ and $\alpha \notin \bar{\mathfrak{p}}$; moreover $\alpha^{k-1} = \psi((\alpha)) = \psi(\mathfrak{p})^h$, so we also have $\psi(\mathfrak{p}) \in \mathfrak{p} - \bar{\mathfrak{p}}$. In the same way we find $\psi(\bar{\mathfrak{p}}) \in \bar{\mathfrak{p}} - \mathfrak{p}$. We conclude that $\psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}})$ does not belong to $\mathfrak{p}$, so its p-adic valuation is 0.

Let $m = 1$. If p is split or ramified. If $p = \mathfrak{p}\bar{\mathfrak{p}}$, the U_p-eigenvalue a_p is either $\psi(\mathfrak{p})$ or $\psi(\bar{\mathfrak{p}})$ hence the slope is 0 or $k-1$. If p is ramified, $a_p = \psi(\mathfrak{p})$ and the slope is $\frac{k-1}{2}$.

If $m > 1$, then either $p = \mathfrak{p}^2$ divides D and $\mathfrak{f}$, or p is inert and divides $\mathfrak{f}$, or p splits. In the first two cases, the U_p-eigenvalue is 0. If p splits and $\mathfrak{p}^m$ or $\bar{\mathfrak{p}}^m$ divides $\mathfrak{f}$, then the U_p-eigenvalue has slope 0 or $k-1$.

If g is not p-new, it is the p-stabilization of a CM form $f(F, \psi)$ of level prime to p. If this form is p-ordinary, that is p splits in F and is prime to $\mathfrak{f}$, the U_p-eigenvalue of g is $\psi(\mathfrak{p})$ or $\psi(\bar{\mathfrak{p}})$, hence has slope 0 or $k-1$. If $f(F, \psi)$ is not ordinary, the Hecke polynomial at p is $X^2 \pm p^{k-1}$ and the stabilizations have slope $\frac{k-1}{2}$. □

As a corollary of interest to us we have:

Corollary 3.8. *There are no CM families of strictly positive slope.*

Proof. We show that the eigencurve C_h contains only a finite number of points corresponding to classical CM forms. It will follow that almost all classical points of a family in C_h are non-CM. Let f be a classical CM form of weight k and positive slope. By Proposition 3.7 its slope is at least $\frac{k-1}{2}$. If f corresponds to a point of C_h its slope must be $\leq h$, so we obtain an inequality $\frac{k-1}{2} \leq h$. The set of weights $\mathcal{K}$ satisfying this condition is finite. Since the weight map $C_h \to B_h$ is finite the set of points of C_h whose weight lies in $\mathcal{K}$ is finite, so the number of CM forms in C_h is also finite. □

So, in the finite positive slope case, classical CM forms can appear only as isolated points in an irreducible component of the eigencurve C_h. In the ordinary case, the CM congruence ideal of a non-CM irreducible component is defined as the intersection ideal of CM irreducible components with the given non-CM component. In the case of a positive slope $\leq h$ family $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$, we need to define differently its CM congruence ideal.

3.4. **Construction of the congruence ideal.** Let $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$ be a family.

Fix an imaginary quadratic field F where p is inert or ramified; let $-D$ be its discriminant. Let $\mathfrak{Q}$ be a primary ideal of $\mathbb{I}^\circ[p^{-1}]$; then $\mathfrak{q} = \mathfrak{Q} \cap \Lambda_h$ is a primary ideal of $\Lambda_h[p^{-1}]$. The projection $\Lambda_h \to \Lambda_h/\mathfrak{q}$ defines a point of B_h (possibly non-reduced) corresponding to a weight $\kappa_{\mathfrak{Q}} : \mathbb{Z}_p^* \to (\Lambda_h/\mathfrak{q})^*$. For $r > 0$ denote by $\mathbb{B}_r$ the ball of centre 1 and radius r in $\mathbb{C}_p$. By [Bu, Prop. 8.3] there exists $r > 0$ and a character $\kappa_{\mathfrak{Q},r} : \mathbb{Z}_p^\times \cdot \mathbb{B}_r \to (\Lambda_h/\mathfrak{q})^\times$ extending $\kappa_{\mathfrak{Q}}$.

Let σ be an embedding $F \hookrightarrow \mathbb{C}_p$. Let κ, r and $\kappa_{\mathfrak{Q},r}$ be as above. For m sufficiently large $\sigma(1+p^m\mathcal{O}_F)$ is contained in $\mathbb{Z}_p^\times \cdot \mathbb{B}_r$, the domain of definition of $\kappa_{\mathfrak{Q},r}$.

For an ideal $\mathfrak{f} \subset \mathcal{O}_F$ let $I_\mathfrak{f}$ be the group of fractional ideals prime to $\mathfrak{f}$. For any prime ℓ not dividing Np we denote $a_{\ell,\mathfrak{Q}}$ the image of the Hecke operator T_ℓ in $\mathbb{I}^\circ/\mathfrak{Q}$. We say what we mean by a non-classical CM point of C_h.

Definition 3.9. *Let F, σ, $\mathfrak{Q}$, κ, r, $\kappa_{\mathfrak{Q},r}$ be as above. We say that $\mathfrak{Q}$ defines a CM point of weight κ if there exist an integer $m > 0$, an ideal $\mathfrak{f} \subset \mathcal{O}_F$ with norm $N(\mathfrak{f})$ such that $DN(\mathfrak{f})$ divides N, a quadratic extension $(\mathbb{I}^\circ/\mathfrak{Q})'$ of $\mathbb{I}^\circ/\mathfrak{Q}$ and a homomorphism $\psi\colon I_{\mathfrak{f}p^m} \to (\mathbb{I}^\circ/\mathfrak{Q})'^\times$ such that:*

1. *$\sigma(1 + p^m\mathcal{O}_F) \subset \mathbb{Z}_p^\times \cdot \mathbb{B}_r$;*
2. *for any $\alpha \in \mathcal{O}_F$ with $\alpha \equiv 1 \pmod{\mathfrak{f}p^m}$, $\psi((\alpha)) = \kappa_{\mathfrak{Q},r}(\alpha)\alpha^{-1}$;*
3. *$a_{l,\mathfrak{Q}} = 0$ if l is a prime inert in F and not dividing Np;*
4. *$a_{l,\mathfrak{Q}} = \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}})$ if l is a prime splitting as $\mathfrak{l}\bar{\mathfrak{l}}$ in F and not dividing Np.*

Note that $\kappa_{\mathfrak{Q},r}(\alpha)$ is well defined thanks to condition 1.

Remark 3.10. *If $\mathfrak{P}$ is a prime of $\mathbb{I}^\circ$ corresponding to a classical form f then $\mathfrak{P}$ is a CM point if and only if f is a CM form in the sense of Section 3.3.*

Proposition 3.11. *The set of CM points in* $\operatorname{Spec}\mathbb{I}$ *is finite.*

Proof. By absurd, assume it is infinite. Then we have an injection $\mathbb{I} \hookrightarrow \prod_\mathfrak{P} \mathbb{I}/\mathfrak{P}$ where $\mathfrak{P}$ runs over the set of CM prime ideals of $\mathbb{I}$. One can assume that the imaginary quadratic field of CM is constant. We can also assume that the ramification of the associated Galois characters $\lambda_\mathfrak{P}\colon G_F \to (\mathbb{I}/\mathfrak{P})^\times$ is bounded (in support and in exponents). On the density one set of primes of F prime to $\mathfrak{f}p$ and of degree one, they take value in the image of $\mathbb{I}^\times$, hence they define a continuous Galois character $\lambda\colon G_F \to \mathbb{I}^\times$ such that $\rho_\theta = \operatorname{Ind}_{G_F}^{G_\mathbb{Q}}\lambda$, which is absurd (by specialization at non-CM classical points, which do exist). □

Definition 3.12. *The (fortuitous) congruence ideal $\mathfrak{c}_\theta$ associated with the family θ is defined as the intersection of all the primary ideals of $\mathbb{I}^\circ[p^{-1}]$ corresponding to CM points.*

Note that if we admit the Gouvêa-Mazur conjecture, for any elliptic curve with CM by F and for any prime p inert or ramified in F, the corresponding CM modular form is of slope $1/2$ and weight 2, hence fits in a p-adic family of slope $\leq 1/2$ over the disc $B(0, 1/p)$. This family is not CM, but has a CM specialization in weight 2.

Remark 3.13. *(Characterizations of the CM locus)*

1) Assume that $\overline{\rho}_\theta = \operatorname{Ind}_{G_K}^{G_\mathbb{Q}}\overline{\lambda}$ for a unique imaginary quadratic field K. Then, the closed subscheme $V(\mathfrak{c}_\theta) = \operatorname{Spec}\mathbb{I}/\mathfrak{c}_\theta \subset \operatorname{Spec}\mathbb{I}$ is the largest subscheme on which there is an isomorphism of Galois representations $\rho_\theta \cong \rho_\theta \otimes (\frac{K/\mathbb{Q}}{\bullet})$. Indeed, for any artinian $\mathbb{Q}_p$-algebra A, a CM point $x\colon \mathbb{I} \to A$ is characterized by the conditions $x(T_\ell) = x(T_\ell)(\frac{K/\mathbb{Q}}{\ell})$ for all primes ℓ prime to Np.

2) Note that N is divisible by the discriminant D of K. Assume that $\mathbb{I}$ is N-new and that D is prime to N/D. Let W_D be the Atkin-Lehner involution associated to D. Conjugation by W_D defines an automorphism ι_D of $\mathbb{T}_h$ and of $\mathbb{I}$. Then $V(\mathfrak{c}_\theta)$ coincides with the (schematic) invariant locus $(\operatorname{Spec}\mathbb{I})^{\iota_D=1}$.

4. The image of the representation associated with a finite slope family

It is shown by J. Lang in [La, Th. 2.4] that, under some technical hypotheses, the image of the Galois representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with a non-CM ordinary family $\theta : \mathbb{T} \to \mathbb{I}^\circ$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$, where $\mathbb{I}_0^\circ$ is the subring of $\mathbb{I}^\circ$ fixed by certain "symmetries" of the

representation ρ. We first recall some definitions and results of [La]. As we want to study the Galois representation associated with a non-ordinary family, we'll adapt some of the results in [La] to the non-ordinary case. Since the crucial step [La, Th. 4.3] requires the Galois ordinarity of the representation (as in [Hi, Lemma 2.9]), the results of this section will not imply the existence of a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ contained in the image of ρ. However, we will prove in later sections the existence of a "congruence Lie subalgebra" of $\mathfrak{sl}_2(\mathbb{I}_0^\circ)$ contained in a suitably defined Lie algebra of the image of ρ, by means of relative Sen theory.

4.1. **The group of self-twists of a family.** We follow [La, Sec. 2] in this subsection. Let $h \geq 0$ and $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$ be a family of slope $\leq h$ defined over a finite torsion free Λ_h-algebra $\mathbb{I}^\circ$.

Definition 4.1. *We say that $\sigma \in \mathrm{Aut}(Q(\mathbb{I}^\circ))$ is a conjugate self-twist for θ if there exists a Dirichlet character $\eta_\sigma : G_\mathbb{Q} \to \mathbb{I}^{\circ,\times}$ such that*

$$\sigma(\theta(T_\ell)) = \eta_\sigma(\ell)\theta(T_\ell)$$

for all but finitely many primes ℓ.

Any such σ acts on $\Lambda_h = \mathcal{O}_h[[t]]$ by acting trivially on t by an automorphism on $\mathcal{O}_h$. Moreover, given σ, the character η_σ is unique; from this it follows that the set Γ of elements σ in $\mathrm{Aut}(Q(\mathbb{I}^\circ))$ which are conjugate self-twists for θ is an abelian group. We recall the following result (which holds without assuming the ordinarity of θ):

Lemma 4.2. [La, Lemma 7.1] *The group Γ is a finite abelian 2-group.*

We suppose from now on that $\mathbb{I}^\circ$ is normal. The only reason for this hypothesis is that in this case $\mathbb{I}^\circ$ is stable under the action of Γ on $Q(\mathbb{I})$, which is not true in general. This makes it possible to define the subring $\mathbb{I}_0^\circ$ of elements of $\mathbb{I}^\circ$ fixed by Γ.

Remark 4.3. *The hypothesis of normality of $\mathbb{I}^\circ$ is just a simplifying one. We could work without it by introducing the Λ_h-order $\mathbb{I}^{\circ,\prime} = \Lambda_h[\theta(T_\ell), \ell \nmid Np] \subset \mathbb{I}^\circ$: it is an analogous of the Λ-order $\mathbb{I}'$ defined in* [La, Sec. 2] *and it is stable under the action of Γ. We would define $\mathbb{I}_0^\circ$ as the fixed subring of $\mathbb{I}^{\circ,\prime}$ and the arguments in the rest of the article could be adapted to this setting.*

We denote by $\mathcal{O}_{h,0}$ the largest integral extension of $\mathbb{Z}_p$ such that $\mathcal{O}_{h,0}[[t]] \subset \mathbb{I}_0^\circ$ and we put $\Lambda_{h,0} = \mathcal{O}_{h,0}[[t]]$. We also denote by $K_{h,0}$ the field of fractions of $\mathcal{O}_{h,0}$. Note that $\mathbb{I}_0^\circ$ is a finite extension of $\Lambda_{h,0}$ since $\mathbb{I}^\circ$ is a finite Λ_h-algebra. Moreover, we have $K_h^\Gamma = K_{h,0}$) (although the inclusion $\Lambda_h \cdot \mathbb{I}_0^\circ \subset \mathbb{I}^\circ$ may not be an equality).

We define two open normal subgroups of $G_\mathbb{Q}$ by:

- $H_0 = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma$;
- $H = H_0 \cap \ker(\det \overline{\rho})$.

Note that H_0 is an open normal subgroup of $G_\mathbb{Q}$ and that H is a pro-p open normal subgroup of H_0 and of $G_\mathbb{Q}$.

4.2. **The level of a general ordinary family.** We recall the main result of [La]. Denote by $\mathbb{T}$ the big ordinary Hecke algebra, which is finite over $\Lambda = \mathbb{Z}_p[[T]]$. Let $\theta : \mathbb{T} \to \mathbb{I}^\circ$ be a ordinary family with associated Galois representation $\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$. The representation ρ is p-ordinary, that is, its restriction $\rho|_{D_p}$ to a decomposition subgroup $D_p \subset G_\mathbb{Q}$ is reducible; there exist two characters $\epsilon, \delta : D_p \to \mathbb{I}^{\circ,\times}$, with δ unramified, such that $\rho|_{D_p}$ is an extension of ϵ by δ.

Denote by $\mathbb{F}$ the residue field of $\mathbb{I}^\circ$ and by $\overline{\rho}$ the representation $G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ obtained by reducing ρ.

Lang introduces the following technical condition:

Definition 4.4. *The p-ordinary representation $\overline{\rho}$ is called H_0-regular if $\overline{\epsilon}|_{D_p\cap H_0} \neq \overline{\delta}|_{D_p\cap H_0}$.*

The following result states the existence of a Galois level for ρ:

Theorem 4.5. [La, Th. 2.4] *Let $\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ be the representation associated with a ordinary family $\theta : \mathbb{T} \to \mathbb{I}^\circ$. Assume that $p > 2$, the cardinality of $\mathbb{F}$ is not 3 and the residual representation $\overline{\rho}$ is absolutely irreducible and H_0-regular.*

There exists $\gamma \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that $\gamma \cdot \mathrm{Im}\,\rho \cdot \gamma^{-1}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$.

The proof relies on the analogous result proved by Ribet [Ri2] and Momose [Mo] for the p-adic representation associated with a classical modular form.

4.3. **An approximation lemma.** In this subsection, we prove an analogue of [HT, Lemma 4.5]. It replaces in our approach the use of Pink's Lie algebra theory, which is relied on in the case of ordinary representations in [Hi] and [La]. Let $\mathbb{I}_0^\circ$ be a domain that is finite torsion free over Λ_h. It does not need to be related to a Hecke algebra for the moment.

Let N be an open normal subgroup of $G_{\mathbb{Q}}$ and let $\rho\colon N \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ be an arbitrary continuous representation. We denote by $\mathfrak{m}_{\mathbb{I}_0^\circ}$ the maximal ideal of $\mathbb{I}_0^\circ$ and by $\mathbb{F} = \mathbb{I}_0^\circ/\mathfrak{m}_{\mathbb{I}_0^\circ}$ its residue field, of cardinality q. In the lemma we do not suppose that ρ comes from a family of modular forms. We will only assume that it satisfies the following technical condition:

Definition 4.6. *Keep notations as above. We say that the representation $\rho : N \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ is $\mathbb{Z}_p$-regular if there exists $\delta \in N$ such that*

$$\rho(\delta) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

where the d_i are roots of unity of order dividing $p-1$ such that $d_1^2 \not\equiv d_2^2 \pmod{\mathfrak{m}_{\mathbb{I}_0^\circ}}$. We call $\rho(\delta)$ a $\mathbb{Z}_p$-regular diagonal element. If N' is an open normal subgroup of N then we say that ρ is $(N', \mathbb{Z}_p)$-regular if $\rho|_{N'}$ is $\mathbb{Z}_p$-regular.

Note that $\rho(\delta) \in \mathrm{Im}\,\rho$ is of finite order dividing $p-1$. Let $U^{\pm}$ be the unipotent radical of the Borel $B^{\pm}$ of upper respectively lower triangular matrices in SL_2.

Proposition 4.7. *Let $\mathbb{I}_0^\circ$ be a finite torsion free $\Lambda_{h,0}$-algebra, N an open normal subgroup of $G_{\mathbb{Q}}$ and ρ a continous representation $N \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ that is $(N, \mathbb{Z}_p)$-regular. Let $\mathbf{P}$ be an ideal of $\mathbb{I}_0^\circ$ and $\rho_{\mathbf{P}} : N \to \mathrm{GL}_2(\mathbb{I}_0^\circ/\mathbf{P})$ be the representation given by the reduction of ρ modulo $\mathbf{P}$. Let $U_\pm(\rho)$, respectively $U_\pm(\rho_{\mathbf{P}})$ be the upper and lower unipotent subgroups of the image $\mathrm{Im}\,\rho$, respectively $\mathrm{Im}\,\rho_{\mathbf{P}}$. Then the natural maps $U_+(\rho_\theta) \to U_+(\rho_{\mathbf{P}})$ and $U_-(\rho_\theta) \to U_-(\rho_{\mathbf{P}})$ are surjective.*

Remark 4.8. *The ideal $\mathbf{P}$ in the proposition is not necessarily prime. At a certain point we will need to take $\mathbf{P} = P\mathbb{I}_0^\circ$ for a prime ideal P of $\Lambda_{h,0}$.*

As in [HT, Lemma 4.5], we need two lemmas. Since the argument is the same for U_+ and U_-, we treat only the 'upper triangular' case $U = U_+$ and $B = B_+$.

For $* = U, B$, let us define for any $j \geq 1$ the groups

$$\Gamma_*(\mathbf{P}^j) = \{x \in \mathrm{SL}_2(\mathbb{I}_0^\circ) | x \mod \mathbf{P}^j \in *(\mathbb{I}_0^\circ/P^j)\}.$$

Let $\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^j)$ be the kernel of the reduction morphism $\pi_j\colon \mathrm{SL}_2(\mathbb{I}_0^\circ) \to \mathrm{SL}_2(\mathbb{I}_0^\circ/P^j)$. Note that $\Gamma_U(\mathbf{P}^j) = \Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^j)U(\mathbb{I}_0^\circ)$ consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, d \equiv 1 \pmod{\mathbf{P}^j}$, $c \equiv 0 \pmod{\mathbf{P}^j}$. Let $K = \mathrm{Im}\,\rho$ and

$$K_U(\mathbf{P}^j) = K \cap \Gamma_U(\mathbf{P}^j), \quad K_B(\mathbf{P}^j) = K \cap \Gamma_B(\mathbf{P}^j)$$

Since $U(\mathbb{I}_0^\circ)$ and $\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P})$ are p-profinite, the groups $\Gamma_U(\mathbf{P}^j)$ and $K_U(\mathbf{P}^j)$ for all $j \geq 1$ are also p-profinite. Note that

$$\left[\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ g & -e \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} bg-cf & 2(af-be) \\ 2(ce-ag) & cf-bg \end{smallmatrix}\right)$$

From this, we see that

Lemma 4.9. *If $X, Y \in \mathfrak{sl}_2(\mathbb{I}_0^\circ) \cap \begin{pmatrix} \mathbf{P}^j & \mathbf{P}^k \\ \mathbf{P}^i & \mathbf{P}^j \end{pmatrix}$ with $i \geq j \geq k$, $[X, Y] \in \begin{pmatrix} \mathbf{P}^{i+k} & \mathbf{P}^{j+k} \\ \mathbf{P}^{i+j} & \mathbf{P}^{i+k} \end{pmatrix}$.*

This tells us, for the topological commutator subgroup $D\Gamma_U(\mathbf{P}^j) := (\Gamma_U(\mathbf{P}^j), \Gamma_U(\mathbf{P}^j))$ that

$$D\Gamma_U(\mathbf{P}^j) \subset \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j) \tag{1}$$

Using now the $\mathbb{Z}_p$-regularity assumption, we consider a diagonal element $d \in K$, with distinct eigenvalues modulo $\mathfrak{m}_{\mathbb{I}_0^\circ}$ of prime-to-p finite order, say a. Note that d normalizes $K_U(\mathbf{P}^j)$ and $\Gamma_B(\mathbf{P}^j)$. In a p-profinite group H, any $x \in H$ has a unique a-th root. Recall that $\alpha(d) = d_1/d_2 \in \mathbb{F}_p^\times$; it can be viewed in $\mathbb{Z}_p^\times$ via the Teichmüller lift. We define for any p-profinite group normalized by d a map $\Delta\colon H \to H$ given by

$$\Delta(x) = [x \cdot ad(d)(x)^{\alpha(d)^{-1}} \cdot ad(d^2)(x)^{\alpha(d)^{-2}} \cdots ad(d^{a-1})(x)^{\alpha(d)^{1-a}}]^{1/a}$$

Lemma 4.10. *If $u \in \Gamma_U(\mathbf{P}^j)$ $(j \geq 1)$, then $\Delta^2(u) \in \Gamma_U(\mathbf{P}^{2j})$ and $\pi_j(\Delta(u)) = \pi_j(u)$.*

Proof. If $u \in \Gamma_U(\mathbf{P}^j)$, we have $\pi_j(\Delta(u)) = \pi_j(u)$ as Δ is the identity map on $U(\Lambda_h/P^j)$. Let $D\Gamma_U(\mathbf{P}^j)$ be the topological commutator subgroup of $\Gamma_U(\mathbf{P}^j)$. Since Δ induces the projection of the $\mathbb{Z}_p$-module $\Gamma_U(\mathbf{P}^j)/D\Gamma_U(\mathbf{P}^j)$ onto its α-eigenspace for $ad(d)$, it is a projection onto $U(\mathbb{I}_0^\circ)D\Gamma_U(\mathbf{P}^j)/D\Gamma_U(\mathbf{P}^j)$. The fact that this is exactly the α-eigenspace comes from the Iwahori decomposition of $\Gamma_U(\mathbf{P}^j)$, hence a similar direct sum decomposition holds in the abelianization $\Gamma_U(\mathbf{P}^j)/D\Gamma_U(\mathbf{P}^j)$.

By (1) $D\Gamma_U(\mathbf{P}^j) \subset \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j)$. Since the α-eigenspace of $\Gamma_U(\mathbf{P}^j)/D\Gamma_U(\mathbf{P}^j)$ is inside $\Gamma_B(\mathbf{P}^{2j})$, Δ projects $u\Gamma_U(\mathbf{P}^j)$ to

$$\overline{\Delta}(u) \in (\Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j))/D\Gamma_U(\mathbf{P}^j)$$

In particular, $\Delta(u) \in \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j)$. Again apply Δ. Since $\Gamma_B(\mathbf{P}^{2j})/\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^{2j})$ is sent to $\Gamma_U(\mathbf{P}^{2j})/\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^{2j})$ by Δ, we get $\Delta^2(u) \in \Gamma_U(\mathbf{P}^{2j})$ as desired. □

Proof. We can now prove Proposition 4.7. Let $\overline{u} \in U(\mathbb{I}_0^\circ/\mathbf{P}) \cap \mathrm{Im}(\rho_{\mathbf{P}})$. Since the reduction map $\mathrm{Im}(\rho) \to \mathrm{Im}(\rho_{\mathbf{P}})$ induced by π_1 is surjective, there exists $v \in \mathrm{Im}(\rho)$ such that $\pi_1(v) = \overline{u}$. Take $u_1 \in U(\mathbb{I}_0^\circ)$ such that $\pi_1(u_1) = \overline{u}$ (as $\pi_1 \colon U(\Lambda_h) \to U(\Lambda_h/P)$ is onto). Then $vu_1^{-1} \in \Gamma_{\mathbb{I}_0^\circ}(\mathbf{P})$, so that $v \in K_U(\mathbf{P})$.

By compactness of $K_U(\mathbf{P})$ and by 4.10, starting with v as above, we see that $\lim_{m\to\infty} \Delta^m(v)$ converges $\mathbf{P}$-adically to $\Delta^\infty(v) \in U(\mathbb{I}_0^\circ) \cap K$ with $\pi_1(\Delta^\infty(v)) = \overline{u}$. □

Remark 4.11. *Proposition 4.7 is true with the same proof if we replace $\Lambda_{h,0}$ by Λ_h and $\mathbb{I}_0^\circ$ by a finite torsion free Λ_h-algebra.*

As a first application of Proposition 4.7 we give a result that we will need in the next subsection. Given a representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ and any ideal $\mathbf{P}$ of $\mathbb{I}^\circ$ we define $\rho_{\mathbf{P}}$, $U_\pm(\rho)$ and $U_\pm(\rho_{\mathbf{P}})$ as above, by replacing $\mathbb{I}_0^\circ$ by $\mathbb{I}^\circ$.

Proposition 4.12. *Let $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$ be a Hecke family of slope $\leq h$ and $\rho_\theta \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ be the representation associated with θ. Suppose that there exists $g \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that $g \cdot \rho_\theta \cdot g^{-1}$ is $(H_0, \mathbb{Z}_p)$-regular. Let $\rho = g \cdot \rho_\theta \cdot g^{-1}|_{H_0}$. Then $U_+(\rho)$ and $U_-(\rho)$ are both non-trivial.*

Recall that for a $\mathbb{Z}_p$-regular representation ρ we always work with an element of its $\mathrm{GL}_2(\mathbb{I}^\circ)$ conjugacy class for which the image of an element giving regularity is diagonal.

Proof. By density of classical points in $\mathbb{T}_h$ we can choose a prime ideal $\mathbf{P} \subset \mathbb{I}^\circ$ corresponding to a classical modular form f. The m*od* $\mathbf{P}$ representation $\rho_{\mathbf{P}}$ is then the p-adic representation classically associated with f. By the results of [Ri2] and [Mo], there exists an ideal $\mathfrak{l}_{\mathbf{P}}$ of $\mathbb{Z}_p$ such that $\mathrm{Im}\,\rho_{\mathbf{P}}$ contains the congruence subgroup $\Gamma_{\mathbb{Z}_p}(\mathfrak{l}_{\mathbf{P}})$. In particular $U_+(\rho_{\mathbf{P}})$ and $U_-(\rho_{\mathbf{P}})$ are nontrivial. Since the maps $U_+(\rho) \to U_+(\rho_{\mathbf{P}})$ and $U_-(\rho) \to U_-(\rho_{\mathbf{P}})$ are surjective we find nontrivial elements in $U_+(\rho)$ and $U_-(\rho)$. □

We now show the following:

Proposition 4.13. *Suppose that the representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ is $(H_0, \mathbb{Z}_p)$-regular. Assume moreover that $\overline{\rho}|_{H_0}$ is absolutely irreducible. Then there exists $g \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that the conjugate representation $g\rho g^{-1}$ satisfies the following two properties:*

1. *the image of $g\rho g^{-1}|_{H_0}$ is contained in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$;*
2. *the image of $g\rho g^{-1}|_{H_0}$ contains a diagonal $\mathbb{Z}_p$-regular element.*

Remark 4.14. *The hypothesis of irreducibility of $\overline{\rho}|_{H_0}$ is not necessary in* [La, Th. 4.1] *under the hypothesis of ordinarity of ρ. In a subsequent work by one of the authors, it will be removed without assuming ordinarity.*

The existence of a conjugate of ρ satisfying only (i) is an easy consequence of [Ca]. Indeed, by definition of H_0, the character $\eta_\sigma : G_{\mathbb{Q}} \to \mathbb{I}^{\circ,\times}$ is trivial on H_0, and hence

$$\rho|_{H_0} \cong \rho^\sigma|_{H_0}$$

for any $\sigma \in \Gamma$. By taking traces of both sides we deduce that the pseudocharacter $\mathrm{Tr}(\rho|_{H_0}) \colon H_0 \to \mathbb{I}^\circ$ satisfies $\mathrm{Tr}(\rho|_{H_0}) = \mathrm{Tr}(\rho|_{H_0})^\sigma$ for any $\sigma \in \Gamma$, so $\mathrm{Tr}(\rho|_{H_0})$ takes values in $\mathbb{I}_0^\circ$. Since $\overline{\rho}|_{H_0}$ is absolutely irreducible and $\mathbb{I}_0^\circ$ is a local ring, the theorem of Carayol tells us that there is a representation $\rho' \colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ such that $\mathrm{Tr}(\rho')$ is the pseudocharacter $\mathrm{Tr}(\rho|_{H_0})$; furthermore ρ' is unique up to isomorphism. Since $\rho|_{H_0}$ is a representation $H_0 \to \mathrm{GL}_2(\mathbb{I}^\circ)$ with the same property we deduce that $\rho|_{H_0}$ and ρ' are isomorphic, so $\rho' = g\rho|_{H_0}g^{-1}$ for some $g \in \mathrm{GL}_2(\mathbb{I}^\circ)$ and $g\rho g^{-1}$ satisfies (i).

In order to have (ii) at the same time we will need the following lemma:

Lemma 4.15. *If ρ is $(H_0, \mathbb{Z}_p)$-regular then there exists a $\mathbb{Z}_p$-regular element in $\mathrm{Im}\,\rho|_{H_0} \cap \mathrm{GL}_2(\mathbb{I}_0^\circ)$.*

Proof. By hypothesis there exists a $\mathbb{Z}_p$-regular element $\rho(\delta) \in \operatorname{Im}\rho|_{H_0}$. By definition $\rho(\delta)$ is a diagonal matrix $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, with d_1 and d_2 roots of unity of order dividing $p-1$ and $d_1^2 \not\equiv d_2^2 \pmod{\mathfrak{m}_{\mathbb{I}^\circ}}$. Since $\rho|_{H_0} \cong \rho^\sigma|_H$ for any $\sigma \in \Gamma$, there exist matrices $T_\sigma \in \mathrm{GL}_2(\mathbb{I}^\circ)$ satisfying

$$\rho^\sigma = T_\sigma \rho T_\sigma^{-1} \tag{2}$$

We prove that the matrices T_σ are diagonal. Let $\rho(t)$ be any non-scalar diagonal element in $\operatorname{Im}\rho$ (for example $\rho(\delta)$). Evaluating (2) at $g = t$ we find that T_σ must be either a diagonal or an antidiagonal matrix. Now by Proposition 4.12 there exists a nontrivial element $\rho(u_+) \in \operatorname{Im}\rho \cap U_+(\mathbb{I}^\circ)$. Evaluating (2) at $g = u_+$ we find that T_σ cannot be antidiagonal.

In particular any diagonal element commutes with T_σ, so by evaluating 2 on δ we obtain

$$\begin{pmatrix} d_1^\sigma & 0 \\ 0 & d_2^\sigma \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

for any $\sigma \in \Gamma$. We deduce, for $i = 1, 2$, that $d_i = d_i^\sigma$ for any $\sigma \in \Gamma$, so $d_i \in \mathbb{I}_0^\circ$. □

We complete now the proof of the proposition.

Proof. Choose a conjugate of ρ such that $\rho|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$, as given by Carayol's theorem. We still denote it by ρ. Since ρ is $(H_0, \mathbb{Z}_p)$-regular, $\operatorname{Im}\rho|_{H_0}$ contains a $\mathbb{Z}_p$-regular element $\rho(\delta)$. By the lemma above, the eigenvalues d_1, d_2 of $\rho(\delta)$ belong to $\mathbb{I}_0^\circ$. Since they are distinct there exists $h \in \mathrm{GL}_2(\mathbb{I}_0^\circ)$ such that $h\rho(\delta)h^{-1}$ is diagonal. The representation $h\rho h^{-1}$ satisfies the conditions (i) and (ii) of the proposition. □

4.4. **Fullness of the unipotent subgroups.** From now on we write ρ for the element in its $\mathrm{GL}_2(\mathbb{I}^\circ)$ conjugacy class such that $\rho|_{H_0} \in \mathrm{GL}_2(\mathbb{I}_0^\circ)$. Recall that H is the open subgroup of H_0 defined by the condition $\det\overline{\rho}(h) = 1$ for any $h \in H$. As in [La, Sec. 4] we define a representation $H \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$:

$$\rho_0 = \rho|_H \otimes \sqrt{\det\rho|_H}^{-1}$$

We can take the square root of the determinant thanks to the definition of H.

We will use the results of [La] to deduce that the $\Lambda_{h,0}$-module generated by the unipotent subgroups of the image of ρ_0 is big. We will later deduce the same for ρ.

We fix from now on a height one prime $P \subset \Lambda_{h,0}$ with the following properties:

(1) there is an arithmetic prime $P_k \subset \Lambda_h$ associated with an integer $k > h+1$ such that $P = P_k \cap \Lambda_{h,0}$;
(2) every prime $\mathfrak{P} \subset \mathbb{I}^\circ$ lying above P_k corresponds to a non-CM point.

Such a point exists since non-critical classical points form a Zariski dense set in $\operatorname{Spm}\mathbb{T}_h$ (hence there are infinitely many of them on every irreducible component) and by the results of Section 3.3 there are only finitely many classical CM points in $\operatorname{Spm}\mathbb{T}_h$.

Remark 4.16. *Since $k > h+1$ every point of $\operatorname{Spm}\mathbb{T}_h$ above P_k is classical and non critical. It is known that the weight map is étale at every such point. In particular since $P = P_k \cap \Lambda_{h,0}$ and $P\mathbb{I}_0^\circ = P_k\mathbb{I}^\circ \cap \mathbb{I}_0^\circ$ the prime $P\mathbb{I}_0^\circ$ splits as a product of distinct prime ideals of $\mathbb{I}_0^\circ$.*

Make the technical assumption that the order of the residue field $\mathbb{F}$ of $\mathbb{I}^\circ$ is not 3. For any ideal $\mathbf{P}$ of $\mathbb{I}_0^\circ$ over P we let $\pi_{\mathbf{P}}$ be the projection $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathbf{P}) \to \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathbf{P})$. We still denote by $\pi_{\mathbf{P}}$ the restricted maps $U_\pm(\mathbb{I}_0^\circ/\mathbf{P}) \to U_\pm(\mathbb{I}_0^\circ/\mathbf{P})$.

Let $G = \operatorname{Im}\rho_0$. For any ideal $\mathbf{P}$ of $\mathbb{I}_0^\circ$ we denote by $\rho_{0,\mathbf{P}}$ the representation $\pi_{\mathbf{P}}(\rho_0)$ and by $G_{\mathbf{P}}$ the image of $\rho_{\mathbf{P}}$. Clearly $G_{\mathbf{P}} = \pi_{\mathbf{P}}(G))$. We state two results from Lang's work that come over unchanged to the non-ordinary setting.

Proposition 4.17. [La, Cor. 6.3] *Let $\mathfrak{P}$ be a prime of $\mathbb{I}_0^\circ$ over P. Then $G_{\mathfrak{P}}$ contains a congruence subgroup $\Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}) \subset \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$; in particular it is open.*

Proposition 4.18. [La, Prop. 5.1] *Assume that for any prime $\mathfrak{P} \subset \mathbb{I}_0^\circ$ over P the subgroup $G_{\mathfrak{P}}$ is open in $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$. Then the image of G in $\prod_{\mathfrak{P}|P}\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$ through the map $\prod_{\mathfrak{P}|P}\pi_{\mathfrak{P}}$ contains a product of congruence subgroups $\prod_{\mathfrak{P}|P}\Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}_{\mathfrak{P}})$.*

Remark 4.19. *The proofs of Propositions 4.17 and 4.18 rely on the fact that the big ordinary Hecke algebra is étale over Λ at any arithmetic point. In order for these proofs to adapt to the non-ordinary setting it is essential that the prime P satisfies the properties above Remark 4.16.*

We let $U_\pm(\rho_0) = G \cap U_\pm(\mathbb{I}_0^\circ)$ and $U_\pm(\rho_{\mathbf{P}}) = G_{\mathbf{P}} \cap U_\pm(\mathbb{I}_0^\circ/\mathbf{P})$ We denote by $U(\rho_{\mathbf{P}})$ either the upper or lower unipotent subgroups of $G_{\mathbf{P}}$ (the choice will be fixed throughout the proof). By projecting to the upper right element we identify $U_+(\rho_0)$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}_0^\circ$ and $U_+(\rho_{0,\mathbf{P}})$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}_0^\circ/\mathbf{P}$. We make analogous identifications for the lower unipotent subgroups. We will use Proposition 4.18 and Proposition 4.7 to show that for both signs $U_\pm(\rho)$ spans $\mathbb{I}_0^\circ$ over $\Lambda_{h,0}$.

First we state a version of [La, Lemma 4.10], with the same proof. Let A and B be Noetherian rings with B integral over A. We call *A-lattice* any A-submodule of B generated by the elements of a basis for the total ring of fractions $Q(B)$ over $Q(A)$.

Lemma 4.20. *Any A-lattice in B contains a non-zero ideal of B. Conversely, any non-zero ideal of B contains an A-lattice.*

We prove the following proposition by means of Proposition 4.7. We could also use Pink theory as in [La, Sec. 4].

Proposition 4.21. *Consider $U_\pm(\rho_0)$ as subsets of $Q(\mathbb{I}_0^\circ)$. For each choice of sign the $Q(\Lambda_{h,0})$-span of $U_\pm(\rho_0)$ is $Q(\mathbb{I}_0^\circ)$. Equivalently the $\Lambda_{h,0}$-span of $U_\pm(\rho_0)$ contains a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Proof. Keep notations as above. We omit the sign when writing unipotent subgroups and we refer to either the upper or lower ones (the choice is fixed throughout the proof). Let P be the prime of $\Lambda_{h,0}$ we chose above. By Remark 4.16 $P\mathbb{I}_0^\circ$ splits as a product of distinct prime ideals in $\mathbb{I}_0^\circ$. When $\mathfrak{P}$ varies among these primes, the map $\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}}$ gives embeddings of $\Lambda_{h,0}/P$-modules $\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ \hookrightarrow \bigoplus_{\mathfrak{P}|P} \mathbb{I}_0^\circ/\mathfrak{P}$ and $U(\rho_{P\mathbb{I}_0^\circ}) \hookrightarrow \bigoplus_{\mathfrak{P}|P} U(\rho_{\mathfrak{P}})$. The following diagram commutes:

$$
\begin{array}{ccc}
U(\rho_{P\mathbb{I}_0^\circ}) & \xrightarrow{\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}}} & \bigoplus_{\mathfrak{P}|P} U(\rho_{\mathfrak{P}}) \\
\downarrow & & \downarrow \\
\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ & \xrightarrow{\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}}} & \bigoplus_{\mathfrak{P}|P} \mathbb{I}_0^\circ/\mathfrak{P}
\end{array}
\tag{3}
$$

By Proposition 4.18 there exist ideals $\mathfrak{a}_{\mathfrak{P}} \subset \mathbb{I}_0^\circ/\mathfrak{P}$ such that $(\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}})(G_{P\mathbb{I}_0^\circ}) \supset \bigoplus_{\mathfrak{P}|P} \Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}_{\mathfrak{P}})$. In particular $(\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}})(U(\rho_{P\mathbb{I}_0^\circ})) \supset \bigoplus_{\mathfrak{P}|P}(\mathfrak{a}_{\mathfrak{P}})$. By Lemma 4.20 each ideal $\mathfrak{a}_{\mathfrak{P}}$ contains a basis for $Q(\mathbb{I}_0^\circ/\mathfrak{P})$ over $Q(\Lambda_{h,0}/P)$, so that the $Q(\Lambda_{h,0}/P)$-span of $\bigoplus_{\mathfrak{P}|P} \mathfrak{a}_{\mathfrak{P}}$ is the whole $\bigoplus_{\mathfrak{P}|P} Q(\mathbb{I}_0^\circ/\mathfrak{P})$. Then the $Q(\Lambda_{h,0}/P)$-span of $(\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}})(G_{\mathfrak{P}} \cap U(\rho_{\mathfrak{P}}))$ is also $\bigoplus_{\mathfrak{P}|P} Q(\mathbb{I}_0^\circ/\mathfrak{P})$. By commutativity of diagram (3) we deduce that the $Q(\Lambda_{h,0}/P)$-span of $G_P \cap U(\rho_{P\mathbb{I}_0^\circ})$ is $Q(\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ)$. In particular $G_{P\mathbb{I}_0^\circ} \cap U(\rho_{P\mathbb{I}_0^\circ})$ contains a $\Lambda_{h,0}/P$-lattice, hence by Lemma 4.20 a non-zero ideal $\mathfrak{a}_P$ of $\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ$.

Note that the representation $\rho_0 : H \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ satisfies the hypotheses of Proposition 4.7. Indeed we assumed that $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I})$ is $(H_0, \mathbb{Z}_p)$-regular, so the image of $\rho|_{H_0}$ contains a diagonal $\mathbb{Z}_p$-regular element d. Since H is a normal subgroup of H_0, $\rho(H)$ is a normal subgroup of $\rho(H_0)$ and it is normalized by d. By a trivial computation we see that the image of $\rho_0 = \rho \otimes (\det \rho)^{-1/2} : H_0 \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is also normalized by d.

Let $\mathfrak{a}$ be an ideal of $\mathbb{I}_0^\circ$ projecting to $\mathfrak{a}_P \subset U(\rho_{0,P\mathbb{I}_0^\circ})$. By Proposition 4.7 applied to ρ_0 we obtain that the map $U(\rho_0) \to U(\rho_{0,P\mathbb{I}_0^\circ})$ is surjective, so the $\mathbb{Z}_p$-module $\mathfrak{a} \cap U(\rho_0)$ also surjects to $\mathfrak{a}_P$. Since $\Lambda_{h,0}$ is local we can apply Nakayama's lemma to the $\Lambda_{h,0}$-module $\Lambda_{h,0}(\mathfrak{a} \cap U(\rho_0)$ to conclude that it coincides with $\mathfrak{a}$. Hence $\mathfrak{a} \subset \Lambda_{h,0} \cdot U(\rho_0)$, so the $\Lambda_{h,0}$-span of $U(\rho_0)$ contains a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$. □

We show that Proposition 4.21 is true if we replace ρ_0 by $\rho|_H$. This is done in [La, Prop. 4.2] for an ordinary representation by using the description of subnormal sugroups of $\mathrm{GL}_2(\mathbb{I}^\circ)$ presented in [Taz]. We will also follow this approach, but since we cannot induce a $\Lambda_{h,0}$-module structure on the unipotent subgroups of G we need a preliminary step. For a subgroup $\mathcal{G} \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$ define $\widetilde{\mathcal{G}} = \mathcal{G}^p \cap (1 + pM_2(\mathbb{I}_0^\circ))$. Let $\widetilde{\mathcal{G}}^{\Lambda_{h,0}}$ be the subgroup of $\mathrm{GL}_2(\mathbb{I}^\circ)$ generated by the set $\{g^\lambda \colon g \in \widetilde{\mathcal{G}}, \lambda \in \Lambda_{h,0}\}$ where $g^\lambda = \exp(\lambda \log g)$. We have the following.

Lemma 4.22. *The group $\widetilde{\mathcal{G}}^{\Lambda_{h,0}}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if both the unipotent subgroups $\mathcal{G} \cap U_+(\mathbb{I}_0^\circ)$ and $\mathcal{G} \cap U_-(\mathbb{I}_0^\circ)$ contain a basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Proof. It is easy to see that $\mathcal{G} \cap U_+(\mathbb{I}_0^\circ)$ contains the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$ if an only if the same is true for $\widetilde{\mathcal{G}} \cap U_+(\mathbb{I}_0^\circ)$. The same is true for U_-. By a standard argument, used in the proofs of [Hi, Lemma 2.9] and [La, Prop. 4.3], $\mathcal{G}^{\Lambda_{h,0}} \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if both its upper and lower unipotent subgroup contain an ideal of $\mathbb{I}_0^\circ$. We have $U_+(\mathbb{I}_0^\circ) \cap \mathcal{G}^{\Lambda_{h,0}} = \Lambda_{h,0}(\mathcal{G} \cap U_+(\mathbb{I}_0^\circ))$,

so by Lemma 4.20 $U_+(\mathbb{I}_0^\circ)\cap\mathcal{G}^{\Lambda_{h,0}}$ contains an ideal of $\mathbb{I}_0^\circ$ if and only if $\mathcal{G}\cap U_+(\mathbb{I}_0^\circ)$ contains a basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$. We proceed in the same way for U_-. □

Now let $G_0=\operatorname{Im}\rho|_H$, $G=\operatorname{Im}\rho_0$. Note that $G_0\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of G. Let $f:\mathrm{GL}_2(\mathbb{I}_0^\circ)\to\mathrm{SL}_2(\mathbb{I}_0^\circ)$ be the homomorphism sending g to $\det(g)^{-1/2}g$. We have $G=f(G_0)$ by definition of ρ_0. We show the following.

Proposition 4.23. *The subgroups $G_0\cap U_\pm(\mathbb{I}_0^\circ)$ both contain the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$ if and only if $G\cap U_\pm(\mathbb{I}_0^\circ)$ both contain the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Proof. Since $G=f(G_0)$ we have $\widetilde{G}=f(\widetilde{G_0})$. This implies $\widetilde{G}^{\Lambda_{h,0}}=f(\widetilde{G_0}^{\Lambda_{h,0}})$. We remark that $\widetilde{G_0}^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of $\widetilde{G}^{\Lambda_{h,0}}$. Indeed $\widetilde{G_0}^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ is normal in $\widetilde{G_0}^{\Lambda_{h,0}}$, so its image $f(G_0^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ))=G_0^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ is normal in $f(G_0^{\Lambda_{h,0}})=\widetilde{G}^{\Lambda_{h,0}}$.

By [Taz, Cor. 1] a subgroup of $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if it is subnormal in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ and it is not contained in the subgroup of scalar matrices. We note that $\widetilde{G_0}^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)=(\widetilde{G_0}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ))^{\Lambda_{h,0}}$ is not contained in the subgroup $\{\pm1\}$. Otherwise also $\widetilde{G_0}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ would be contained in $\{\pm1\}$ and $\operatorname{Im}\rho\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ would be finite, since $\widetilde{G_0}$ is of finite index in G_0^p. This gives a contradiction: indeed if $\mathfrak{P}$ is an arithmetic prime of $\mathbb{I}^\circ$ of weight greater than 1 and $\mathfrak{P}'=\mathfrak{P}\cap\mathbb{I}_0^\circ$, the image of ρ modulo $\mathfrak{P}'$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P}')$ by the result of [Ri2].

Now since $\widetilde{G_0}^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of $\widetilde{G}^{\Lambda_{h,0}}$, we deduce by Tazhetdinov's result that $\widetilde{G_0}^{\Lambda_{h,0}}\cap\mathrm{SL}_2(\mathbb{I}_0^\circ)$ (hence $\widetilde{G_0}^{\Lambda_{h,0}}$) contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if $\widetilde{G}^{\Lambda_{h,0}}$ does. By applying Lemma 4.22 to $\mathcal{G}=G_0$ and $\mathcal{G}=G$ we obtain the desired equivalence. □

By combining the above proposition with Proposition 4.21 we obtain the following.

Corollary 4.24. *The $\Lambda_{h,0}$-span of each unipotent subgroups $\operatorname{Im}\rho\cap U_\pm$ contains a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0^\circ$.*

Unlike in the ordinary case we cannot deduce from the corollary that $\operatorname{Im}\rho$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$, since we are working over $\Lambda_h\neq\Lambda$ and we cannot induce a Λ_h-module structure (not even a Λ-module structure) on $\operatorname{Im}\rho\cap U_\pm$. The proofs of [Hi, Lemma 2.9] and [La, Prop. 4.3] rely on the existence in the image of Galois of an element inducing by conjugation a Λ-module structure on $\operatorname{Im}\rho\cap U_\pm$; in their situation this is predicted by the condition of Galois ordinarity of ρ. In the non-ordinary case we will find an element with a similar property via relative Sen theory. In order to do this we will need to work with a suitably defined Lie algebra rather than with the group itself.

5. Relative Sen theory

We recall the notations of Section 3.1. In particular $r_h=p^{-s_h}$, with $s_h\in\mathbb{Q}$, is the h-adapted radius (which we also take smaller than $p^{-\frac{1}{p-1}}$), η_h is an element in $\mathbb{C}_p$ of norm r_h, $K_h=\mathbb{Q}_p(\eta_h)$ and $\mathcal{O}_h$ is the ring of integers in K_h. The ring Λ_h of analytic functions on the open disc $B_h=B(0,r_h^-)$ is identified to $\mathcal{O}_h[[t]]$. We take a sequence of radii $r_i=p^{-s_h-1/i}$ converging to r_h and denote by $A_{r_i}=K_h\langle t,X_i\rangle/(pX_i-t^i)$ the K_h-algebra defined in Section 3.1 which is a form over K_h of the $\mathbb{C}_p$-algebra of analytic functions on the closed ball $B(0,r_i)$ (its Berthelot model). We denote by $A_{r_i}^\circ$ the $\mathcal{O}_h$-subalgebra of functions bounded by 1. Then $\Lambda_h=\varprojlim_i A_{r_i}^\circ$ where $A_{r_j}^\circ\to A_{r_i}^\circ$ for $i<j$ is the restriction of analytic functions.

We defined in Section 4.1 a subring $\mathbb{I}_0^\circ\subset\mathbb{I}^\circ$, finite over $\Lambda_{h,0}\subset\Lambda_h$. For r_i as above, we write $A_{0,r_i}^\circ=K_{h,0}\langle t,X_i\rangle/(pX_i-t^i)$ with maps $A_{0,r_j}^\circ\to A_{0,r_i}^\circ$ for $i<j$, so that $\Lambda_{h,0}=\varprojlim_i A_{0,r_i}^\circ$. Let $\mathbb{I}_r^\circ=\mathbb{I}^\circ\widehat{\otimes}_{\Lambda_h}A_r^\circ$ and $\mathbb{I}_{0,r}^\circ=\mathbb{I}_0^\circ\widehat{\otimes}_{\Lambda_{h,0}}A_{0,r}^\circ$, both endowed with their p-adic topology. Note that $(\mathbb{I}_r^\circ)^\Gamma=\mathbb{I}_{r,0}^\circ$.

Consider the representation $\rho:G_\mathbb{Q}\to\mathrm{GL}_2(\mathbb{I}^\circ)$ associated with a family $\theta:\mathbb{T}_h\to\mathbb{I}^\circ$. We observe that ρ is continuous with respect to the profinite topology of $\mathbb{I}^\circ$ but not with respect to the p-adic topology. For this reason we fix an arbitrary radius r among the r_i defined above and consider the representation $\rho_r:G_\mathbb{Q}\to\mathrm{GL}_2(\mathbb{I}_r^\circ)$ obtained by composing ρ with the inclusion $\mathrm{GL}_2(\mathbb{I}^\circ)\hookrightarrow\mathrm{GL}_2(\mathbb{I}_r^\circ)$. This inclusion is continuous, hence the representation ρ_r is continous with respect to the p-adic topology on $\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$.

Recall from Proposition 4.13 that, after replacing ρ by a conjugate, there is an open normal subgroup $H_0\subset G_\mathbb{Q}$ such that the restriction $\rho|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ and is $(H_0,\mathbb{Z}_p)$-regular. The restriction $\rho_r|_{H_0}$ then gives a representation $H_0\to\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$ which is continuous with respect to the p-adic topology on $\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$.

5.1. **Big Lie algebras.** Fix a decomposition group $G_p \subset G_{\mathbb{Q}}$ at p. Let G_r and G_r^{loc} be the images respectively of H_0 and $G_p \cap H_0$ under the representation $\rho_r|_{H_0}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)$. Note that they are actually independent of r, since they coincide with the images of H_0 and $G_p \cap H_0$ under ρ.

For any ring R and ideal $I \subset R$ we denote by $\Gamma_{\mathrm{GL}_2(R)}(I)$ the GL_2-congruence subgroup consisting of elements $g \in \mathrm{GL}_2(R)$ such that $g \equiv \mathrm{Id}_2 \pmod I$. Let $G'_r = G_r \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)}(p)$ and $G'^{,\mathrm{loc}}_r = G_r^{\mathrm{loc}} \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ)}(p)$, so that G'_r and $G'^{,\mathrm{loc}}_r$ are pro-p-groups. Note that the congruence subgroups $\Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r})}(p^m)$ are open in $\mathrm{GL}_2(\mathbb{I}_{0,r})$ for the p-adic topology. In particular G'_r and $G'^{,\mathrm{loc}}_r$ can be identified with the image under ρ of the absolute Galois group of a finite extension respectively of $\mathbb{Q}$ and $\mathbb{Q}_p$.

Remark 5.1. *We remark that we can choose an arbitrary r_0 and set, for any r, $G'_r = G_r \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r_0}^\circ)}(p)$, which is independent of r since G_r is. In particular it is a pro-p and finite index subgroup of G_r for any r. We can identify it to the image under ρ_r (and ρ) of an open subgroup of $G_{\mathbb{Q}}$ which is independent of r. This will be important in Section 7.1, where we will take projective limits over r of various objects.*

We set $A_{0,r} = A_{0,r}^\circ[p^{-1}]$ and $\mathbb{I}_{0,r} = \mathbb{I}_{0,r}^\circ[p^{-1}]$. We consider from now on G'_r and $G'^{,\mathrm{loc}}_r$ as subgroups of $\mathrm{GL}_2(\mathbb{I}_{0,r})$ through the inclusion $\mathrm{GL}_2(\mathbb{I}_{0,r}^\circ) \hookrightarrow \mathrm{GL}_2(\mathbb{I}_{0,r})$.

We want to define big Lie algebras associated with the groups G'_r and $G'^{,\mathrm{loc}}_r$. For any non zero ideal $\mathfrak{a}$ of the principal ideal domain $A_{0,r}$, we denote by $G'_{r,\mathfrak{a}}$ and $G_{r,\mathfrak{a}}^{\mathrm{loc},\prime}$ the images respectively of G'_r and $G_r^{\mathrm{loc},\prime}$ under the natural projection $\mathrm{GL}_2(\mathbb{I}_{0,r}) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$. The pro-$p$ groups $G'_{r,\mathfrak{a}}$ and $G_{r,\mathfrak{a}}^{\mathrm{loc},\prime}$ are topologically of finite type, so we can define the corresponding $\mathbb{Q}_p$-Lie algebras $\mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}}$ using the p-adic logarithm map: $\mathfrak{H}_{r,\mathfrak{a}} = \mathbb{Q}_p \cdot \mathrm{Log}\, G'_{r,\mathfrak{a}}$ and $\mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}} = \mathbb{Q}_p \cdot \mathrm{Log}\, G_{r,\mathfrak{a}}^{\mathrm{loc},\prime}$. They are closed Lie subalgebras of the finite dimensional $\mathbb{Q}_p$-Lie algebra $M_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$.

Let $B_r = \varprojlim_{(\mathfrak{a},P_1)=1} A_{0,r}/\mathfrak{a} \cdot A_{0,r}$ where the inverse limit is taken over non zero ideals ideals $\mathfrak{a} \subset A_{0,r}$ prime to $P_1 = (u^{-1}(1+T)-1)$ (the reason for excluding P_1 will become clear later). We endow B_r with the projective limit topology coming from the p-adic topology on each quotient. We have a topological isomorphism of $K_{h,0}$-algebras

$$B_r \cong \prod_{P \neq P_1} \widehat{(A_{0,r})}_P$$

where the product is over primes P and $\widehat{(A_{0,r})}_P = \varprojlim_{m \geq 1} A_{0,r}/P^m A_{0,r}$ denotes the $K_{h,0}$-Fréchet space inverse limit of the finite dimensional $K_{h,0}$-vector spaces $A_{0,r}/P^m A_{0,r}$. Similarly, let $\mathbb{B}_r = \varprojlim_{(\mathfrak{a},P_1)=1} \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$, where as before $\mathfrak{a}$ varies over all non-zero ideals of $A_{0,r}$ prime to P_1. We have

$$\mathbb{B}_r \cong \prod_{P \neq P_1} \widehat{(\mathbb{I}_{0,r})}_{P\mathbb{I}_{0,r}} \cong \prod_{\mathfrak{P} \nmid P_1} \widehat{(\mathbb{I}_{0,r})}_{\mathfrak{P}}$$

where the second product is over primes $\mathfrak{P}$ of $\mathbb{I}_{0,r}$ and $\widehat{(\mathbb{I}_{0,r})}_{\mathfrak{P}}$ denotes the projective limit of finite dimensional $K_{h,0}$-algebras (endowed with the p-adic topology). The last isomorphism follows from the fact that $\mathbb{I}_{0,r}$ is finite over $A_{0,r}$, so that there is an isomorphism $\mathbb{I}_{0,r} \otimes \widehat{(A_{0,r})}_P = \prod_{\mathfrak{P}} \widehat{(\mathbb{I}_{0,r})}_{\mathfrak{P}}$ where P is any prime of $A_{0,r}$ and $\mathfrak{P}$ varies among the primes of $\mathbb{I}_{0,r}$ above P. We have natural continous inclusions $A_{0,r} \hookrightarrow B_r$ and $\mathbb{I}_{0,r} \hookrightarrow \mathbb{B}_r$, both with dense image. The map $A_{0,r} \hookrightarrow \mathbb{I}_{0,r}$ induces an inclusion $B_r \hookrightarrow \mathbb{B}_r$ with closed image. Note however that $\mathbb{B}_r$ is not finite over B_r. We will work with $\mathbb{B}_r$ for the rest of this section, but we will need B_r later.

For any $\mathfrak{a}$ we have defined Lie algebras $\mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}}$ associated with the finite type Lie groups $G'_{r,\mathfrak{a}}$ and $G_{r,\mathfrak{a}}^{\mathrm{loc},\prime}$. We take the projective limit of these algebras to obtain Lie subalgebras of $M_2(\mathbb{B}_r)$.

Definition 5.2. *The Lie algebras associated with G'_r and $G_r^{\mathrm{loc},\prime}$ are the closed Lie subalgebras of $M_2(\mathbb{B}_r)$ given respectively by*

$$\mathfrak{H}_r = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}_{r,\mathfrak{a}}$$

and

$$\mathfrak{H}_r^{\mathrm{loc}} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}}$$

where as usual the products are taken over non zero ideals $\mathfrak{a} \subset A_{0,r}$ prime to P_1.

For any ideal $\mathfrak{a}$ prime to P_1, we have continuous homomorphisms $\mathfrak{H}_r \to \mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{H}_r^{\mathrm{loc}} \to \mathfrak{H}_{r,\mathfrak{a}}^{\mathrm{loc}}$. Since the transition maps are surjective, these homomorphisms are surjective.

5.2. **The Sen operator associated with a Galois representation.** Recall that there is a finite extension $K/\mathbb{Q}_p$ such that $G_r'^{,\mathrm{loc}}$ is the image of $\rho|_{\mathrm{Gal}(\overline{K}/K)}$ and, for any ideal $P \subset A_{0,r}$ and any $m \geq 1$, $G_{r,P^m}^{\mathrm{loc},\prime}$ is the image of $\rho_{r,P^m}|_{\mathrm{Gal}(\overline{K}/K)}$. Following [Se1] and [Se2] we can define a Sen operator associated with $\rho_r|_{\mathrm{Gal}(\overline{K}/K)}$ and $\rho_{r,P^m}|_{\mathrm{Gal}(\overline{K}/K)}$ for any ideal $P \subset A_{0,r}$ and any $m \geq 1$. We will see that these operators satisfy a compatibility property. We write for the rest of the section ρ_r and ρ_{r,P^m} while implicitly taking the domain to be $\mathrm{Gal}(\overline{K}/K)$.

We begin by recalling the definition of the Sen operator associated with a representation $\tau : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathcal{R})$ where $\mathcal{R}$ is a Banach algebra over a p-adic field L. We follow [Se2]. We can suppose $L \subset K$; if not we just restrict τ to the open subgroup $\mathrm{Gal}(\overline{K}/KL) \subset \mathrm{Gal}(\overline{K}/K)$.

Let L_∞ be a ramified $\mathbb{Z}_p$-extension of L. Let γ be a topological generator of $\Gamma = \mathrm{Gal}(L_\infty/L)$, $\Gamma_n \subset \Gamma$ the subgroup generated by γ^{p^n} and $L_n = L_\infty^{\gamma^{p^n}}$, so that $L_\infty = \cup_n L_n$. Let $L_n' = L_n K$ and $G_n' = \mathrm{Gal}(\overline{L}/L_n')$. If $\mathcal{R}^m$ is the $\mathcal{R}$-module over which $\mathrm{Gal}(\overline{K}/K)$ acts via τ, define an action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathcal{R}\widehat{\otimes}_L\mathbb{C}_p$ by letting $\sigma \in \mathrm{Gal}(\overline{K}/K)$ map $x \otimes y$ to $\tau(\sigma)(x) \otimes \sigma(y)$. Then by the results in [Se1] and [Se2] there is a matrix $M \in \mathrm{GL}_m\left(\mathcal{R}\widehat{\otimes}_L\mathbb{C}_p\right)$, $n \geq 0$ and a representation $\delta : \Gamma_n \to \mathrm{GL}_m(\mathcal{R} \otimes_L L_n')$ such that for all $\sigma \in G_n'$ we have

$$M^{-1}\tau(\sigma)\sigma(M) = \delta(\sigma)$$

Definition 5.3. *The Sen operator associated with τ is*

$$\phi := \lim_{\sigma \to 1} \frac{\log(\delta(\sigma))}{\log(\chi(\sigma))} \in M_2(\mathcal{R}\widehat{\otimes}_L\mathbb{C}_p)$$

The limit exists as for σ close to 1 the map $\sigma \mapsto \dfrac{\log(\delta(\sigma))}{\log(\chi(\sigma))}$ is constant. It is proved in [Se2, Sec. 2.4] that ϕ does not depend on the choice of δ and M.

If $L = \mathcal{R} = \mathbb{Q}_p$ we define the Lie algebra $\mathfrak{g}$ associated with $\tau(\mathrm{Gal}(\overline{K}/K))$ as the $\mathbb{Q}_p$-vector space generated by the image of the Log map in $M_m(\mathbb{Q}_p)$. In this situation the Sen operator ϕ associated with τ has the following property, which will be essential in the following.

Theorem 5.4. [Se1, Th. 1] *For any continuous representation $\tau\colon G_K \to \mathrm{GL}_m(\mathbb{Q}_p)$, the Lie algebra $\mathfrak{g}$ of the group $\tau(\mathrm{Gal}(\overline{K}/K))$ is the smallest $\mathbb{Q}_p$-subspace of $M_m(\mathbb{Q}_p)$ such that $\mathfrak{g}\otimes\mathbb{C}_p$ contains ϕ.*

This theorem is valid in the absolute case above, but relies heavily on the fact that the image of Galois is a finite dimensional Lie group. In the relative case, it is doubtful that its proof can be generalized.

5.3. **The Sen operator associated with ρ_r.** Set $\mathbb{I}_{0,r,\mathbb{C}_p} = \mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p$. It is a Banach space for the natural norm. Let $\mathbb{B}_{r,\mathbb{C}_p} = \mathbb{B}_r\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p$; it is the topological $\mathbb{C}_p$-algebra completion of $\mathbb{B}_r \otimes_{K_{h,0}} \mathbb{C}_p$ for the (uncountable) set of nuclear seminorms $p_{\mathfrak{a}}$ given by the norms on $\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}$ via the specialization morphisms $\pi_{\mathfrak{a}}\colon \mathbb{B}_r \otimes_{K_{h,0}} \mathbb{C}_p \to \mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}$. Let $\mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p} = \mathfrak{H}_{r,\mathfrak{a}} \otimes_{K_{h,0}} \mathbb{C}_p$ and $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p} = \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a}}, \otimes_{K_{h,0}} \mathbb{C}_p$. Then, we define $\mathfrak{H}_{r,\mathbb{C}_p} = \mathfrak{H}_r\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p$ as the topological $\mathbb{C}_p$-Lie algebra completion of $\mathfrak{H}_r \otimes_{K_{0,h}} \mathbb{C}_p$ for the (uncountable) set of seminorms $p_{\mathfrak{a}}$ given by the norms on $\mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p}$ and similar specialization morphisms $\pi_{\mathfrak{a}}\colon \mathfrak{H}_{r,} \otimes_{K_{h,0}} \mathbb{C}_p \to \mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p}$. We define in the same way $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$ in terms of the norms on $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$. Note that by definition, we have

$$\mathfrak{H}_{r,\mathbb{C}_p} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p}, \quad \text{and } \mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}.$$

We apply the construction of the previous subsection to $L = K_{h,0}$, $\mathcal{R} = \mathbb{I}_{0,r}$, which is a Banach L-algebra with the p-adic topology, and $\tau = \rho_r$. We obtain an operator $\phi_r \in M_2(\mathbb{I}_{0,r,\mathbb{C}_p})$. Recall that we have a natural continous inclusion $\mathbb{I}_{0,r} \hookrightarrow \mathbb{B}_r$, inducing inclusions $\mathbb{I}_{0,r,\mathbb{C}_p} \hookrightarrow \mathbb{B}_{r,\mathbb{C}_p}$ and $M_2(\mathbb{I}_{0,r,\mathbb{C}_p}) \hookrightarrow M_2(\mathbb{B}_{r,\mathbb{C}_p})$. We denote all these inclusions by $\iota_{\mathbb{B}_r}$ since it will be clear each time to which we are referring. We'll prove in this section that $\iota_{\mathbb{B}_r}(\phi_r)$ is an element of $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$.

Let $\mathfrak{a}$ be a non zero ideal of $A_{0,r}$. Let us apply Sen's construction to $L = K_{h,0}$, $\mathcal{R} = \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$ and $\tau = \rho_{r,\mathfrak{a}} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$; we obtain an operator $\phi_{r,\mathfrak{a}} \in M_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p)$.

Let

$$\pi_{\mathfrak{a}} : M_2(\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p) \to M_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p)$$

and

$$\pi_{\mathfrak{a}}^{\times} : \mathrm{GL}_2(\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}\mathbb{C}_p)$$

be the natural projections.

Proposition 5.5. *We have $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$ for all $\mathfrak{a}$.*

Proof. Recall from the construction of ϕ_r that there exist $M \in \mathrm{GL}_2\left(\mathbb{I}_{0,r,\mathbb{C}_p}\right)$, $n \geq 0$ and $\delta : \Gamma_n \to \mathrm{GL}_2(\mathbb{I}_{0,r}\widehat{\otimes}_{K_{h,0}}K'_{h,0,n})$ such that for all $\sigma \in G'_n$ we have

$$M^{-1}\rho_r(\sigma)\sigma(M) = \delta(\sigma) \tag{4}$$

and

$$\phi_r := \lim_{\sigma\to 1}\frac{\log(\delta(\sigma))}{\log(\chi(\sigma))}. \tag{5}$$

Let $M_{\mathfrak{a}} = \pi_{\mathfrak{a}}^{\times}(M) \in \mathrm{GL}_2(\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p})$ and $\delta_{\mathfrak{a}} = \pi_{\mathfrak{a}}^{\times}\circ\delta : \Gamma_n \to \mathrm{GL}_2((\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\widehat{\otimes}_{K_{h,0}}K'_{h,0,n})$ Denote by $\phi_{r,\mathfrak{a}} \in M_2((\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\widehat{\otimes}_{K_{h,0}}K'_{h,0,n})$ the Sen operator associated with $\rho_{r,\mathfrak{a}}$. Now (4) gives

$$M_{\mathfrak{a}}^{-1}\rho_{r,\mathfrak{a}}(\sigma)\sigma(M_{\mathfrak{a}}) = \delta_{\mathfrak{a}}(\sigma) \tag{6}$$

so we can calculate $\phi_{r,\mathfrak{a}}$ as

$$\phi_{r,\mathfrak{a}} = \lim_{\sigma\to 1}\frac{\log(\delta_{\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))} \in M_2(\mathcal{R}\widehat{\otimes}_L\mathbb{C}_p) \tag{7}$$

By comparing with (5) we see that $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$. □

Let $\phi_{r,\mathbb{B}_r} = \iota_{\mathbb{B}_r}(\phi_r)$. For any non zero ideal $\mathfrak{a}$ of $A_{0,r}$ let $\pi_{\mathbb{B}_r,\mathfrak{a}}$ be the natural projection $\mathbb{B}_r \to \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$. Clearly $\pi_{\mathbb{B}_r,\mathfrak{a}}(\phi_{r,\mathbb{B}_r}) = \pi_{\mathfrak{a}}(\phi_r)$ and $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$ by Proposition 5.5, so we have $\phi_{r,\mathbb{B}_r} = \varprojlim_{(\mathfrak{a},P_1)=1}\phi_{r,\mathfrak{a}}$.

We apply Theorem 5.4 to show the following.

Proposition 5.6. *Let $\mathfrak{a}$ be any non zero ideal of $A_{0,r}$ prime to P_1. The operator $\phi_{r,\mathfrak{a}}$ belongs to the Lie algebra $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$.*

Proof. Let n be the dimension over $\mathbb{Q}_p$ of $\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$; by choosing a $\mathbb{Q}_p$-basis $(\omega_1,\dots,\omega_n)$ of this algebra, we can define an injective ring morphism $\alpha : M_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}) \hookrightarrow M_{2n}(\mathbb{Q}_p)$ and an injective group morphism $\alpha^{\times} : \mathrm{GL}_2(\mathbb{I}_{0,r}/Q\mathbb{I}_{0,r}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{Q}_p)$. In fact, an endomorphism f of the $(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$-module $(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})^2 = (\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\cdot e_1 \oplus (\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\cdot e_2$ is $\mathbb{Q}_p$-linear, so it induces an endomorphism $\alpha(f)$ of the $\mathbb{Q}_p$-vector space $(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})^2 = \bigoplus_{i,j}\mathbb{Q}_p\cdot\omega_i e_j$; furthermore if α is an automorphism then $\alpha(f)$ is one too. In particular $\rho_{r,\mathfrak{a}}$ induces a representation $\rho^{\alpha}_{r,\mathfrak{a}} = \alpha^{\times}\circ\rho_{r,\mathfrak{a}} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2n}(\mathbb{Q}_p)$. The image of $\rho^{\alpha}_{r,\mathfrak{a}}$ is the group $G^{\mathrm{loc},\alpha}_{r,\mathfrak{a}} = \alpha^{\times}(G^{\mathrm{loc}}_{r,\mathfrak{a}})$. We consider its Lie algebra $\mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a}} = \mathbb{Q}_p\cdot\mathrm{Log}\,(G^{\mathrm{loc},\alpha}_{r,\mathfrak{a}}) \subset M_{2n}(\mathbb{Q}_p)$. The p-adic logarithm commutes with α in the sense that $\alpha(\mathrm{Log}\,x) = \mathrm{Log}\,(\alpha^{\times}(x))$ for every $x \in \Gamma_{\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}}(p)$, so we have $\mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a}} = \alpha(\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a}})$ (recall that $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a}} = \mathbb{Q}_p\cdot\mathrm{Log}\,G^{\mathrm{loc}}_{r,\mathfrak{a}}$).

Let $\phi^{\alpha}_{r,\mathfrak{a}}$ be the Sen operator associated with $\rho^{\alpha}_{r,\mathfrak{a}} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2n}(\mathbb{Q}_p)$. By Theorem 5.4 we have $\phi^{\alpha}_{r,\mathfrak{a}} \in \mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a},\mathbb{C}_p} = \mathfrak{H}^{\mathrm{loc},\alpha}_{r,\mathfrak{a}}\widehat{\otimes}\mathbb{C}_p$. Denote by $\alpha_{\mathbb{C}_p}$ the map $\alpha\widehat{\otimes}1$: $M_2(\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}) \hookrightarrow M_{2n}(\mathbb{C}_p)$. We show that $\phi^{\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a}} = \alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$, from which it follows that $\phi_{r,\mathfrak{a}} \in \mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$ since $\mathfrak{H}^{\mathrm{loc},\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a},\mathbb{C}_p} = \alpha_{\mathbb{C}_p}(\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p})$ and $\alpha_{\mathbb{C}_p}$ is injective. Now let $M_{\mathfrak{a}}$, $\delta_{\mathfrak{a}}$ be as in (6) and $M^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}} = \alpha_{\mathbb{C}_p}(M_{\mathfrak{a}})$, $\delta^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}} = \alpha_{\mathbb{C}_p}\circ\delta_{\mathfrak{a}}$. By applying α_C to (4) we obtain $(M^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}})^{-1}\rho^{\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a}}(\sigma)\sigma(M^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}}) = \delta^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}}(\sigma)$ for every $\sigma \in G'_n$, so we can calculate

$$\phi^{\alpha_{\mathbb{C}_p}}_{r,\mathfrak{a}} = \lim_{\sigma\to 1}\frac{\log(\delta^{\alpha_{\mathbb{C}_p}}_{\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))}$$

which coincides with $\alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$. □

Proposition 5.7. *The element $\phi_{r,\mathbb{B}_r}$ belongs to $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$.*

Proof. By definition of the space $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$ as completion of the space $\mathfrak{H}^{\mathrm{loc}}_{r}\otimes_{K_{h,0}}\mathbb{C}_p$, for the seminorms $p_{\mathfrak{a}}$ given by the norms on $\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$, we have $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p} = \varprojlim_{(\mathfrak{a},P_1)=1}\mathfrak{H}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$. By Proposition 5.5, we have $\phi_{r,\mathbb{B}_r} = \varprojlim_{\mathfrak{a}}\phi_{r,\mathfrak{a}}$ and by Proposition 5.6 we have for any $\mathfrak{a}$, $\phi_{r,\mathfrak{a}} \in \mathfrak{H}_{r,\mathfrak{a},\mathbb{C}_p}$. We conclude that $\phi_{r,\mathbb{B}_r} \in \mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p}$. □

Since $\mathfrak{H}^{\mathrm{loc}}_{r,\mathbb{C}_p} \subset \mathfrak{H}_{r,\mathbb{C}_p}$ we also have $\phi_{r,\mathbb{B}_r} \in \mathfrak{H}_{r,\mathbb{C}_p}$.

From now on we identify $\mathbb{I}_{0,r,\mathbb{C}_p}$ with a subring of $\mathbb{B}_{r,\mathbb{C}_p}$ via $\iota_{\mathbb{B}_r}$, so we also identify $M_2(\mathbb{I}_{0,r})$ with a subring of $M_2(\mathbb{B}_r)$ and $\mathrm{GL}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ with a subgroup of $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$. In particular we identify ϕ_r with $\phi_{r,\mathbb{B}_r}$ and we say that $\phi_r \in \mathfrak{H}_{r,\mathbb{C}_p} \cap M_2(\mathbb{I}_{0,r,\mathbb{C}_p})$. Now consider $\Phi_r = \exp(\phi_r) \in \mathrm{GL}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$. Since $\phi_r \in \mathfrak{H}_{r,\mathbb{C}_p}$ the matrix Φ_r normalizes $\mathfrak{H}_{r,\mathbb{C}_p}$.

5.4. **The characteristic polynomial of the Sen operator.** Sen proved the following result.

Theorem 5.8. *Let L_1 and L_2 be two p-adic fields, and let $\mathbb{C}_p = \widehat{\overline{L_1}}$. Let us assume for simplicity that L_2 contains the normal closure of L_1. Let $\tau : \mathrm{Gal}(\overline{L}_1/L_1) \to \mathrm{GL}_m(L_2)$ be a continuous representation. For each embedding $\sigma\colon L_1 \to L_2$, we have a Sen operator $\phi_{\tau,\sigma} \in M_m(\mathbb{C}_p \otimes_{L_1,\sigma} L_2)$ associated to τ and σ. If τ is Hodge-Tate with σ-Hodge-Tate weights $h_{1,\sigma}, \dots, h_{m,\sigma}$ (with multiplicities, if any), then the characteristic polynomial of $\phi_{\tau,\sigma}$ is $\prod_{i=1}^m (X - h_{i,\sigma})$.*

Now let $k \in \mathbb{N}$ and $P_k = u^{-k}(1+T) - 1$ be the corresponding arithmetic prime of $A_{0,r}$. Let $\mathfrak{P}_f$ a prime of $\mathbb{I}_r$ above P, associated with the system of Hecke eigenvalues of a classical modular form f. Let $\rho : \mathbb{G}_Q \to \mathrm{GL}_2(\mathbb{I}_r)$ be as usual. The specialization of ρ modulo $\mathfrak{P}$ is the representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}_r/\mathfrak{P})$ classically associated with f, defined over the field $K_f = \mathbb{I}_r/\mathfrak{P}\mathbb{I}_r$. By a theorem of Faltings [Fa], when the weight of the form f is k, the representation ρ_f is Hodge-Tate of Hodge-Tate weights 0 and $k-1$; hence, by Th. 5.8 the Sen operator ϕ_f associated with ρ_f has characteristic polynomial $X(X-(k-1))$. Let $\mathfrak{P}_{f,0} = \mathfrak{P}_f \cap \mathbb{I}_{0,r}$. With the notations of the previous subsection, the specialization of ρ_r modulo $\mathfrak{P}_{0,f}$ gives a representation $\rho_{r,\mathfrak{P}_{f,0}} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{P}_{f,0})$, which coincides with $\rho_f|_{\mathrm{Gal}(\overline{K}/K)}$. In particular the Sen operator $\phi_{r,\mathfrak{P}_{f,0}}$ associated with $\rho_{r,\mathfrak{P}_{f,0}}$ coincides with ϕ_f.

By Proposition 5.5 the Sen operator $\phi_r \in M_2(\mathbb{I}_{0,r})$ specializes modulo $Q\mathbb{I}_{0,r}$ to the operator $\phi_{r,Q}$ associated with $\rho_{r,Q}$ for any primary ideal Q of $A_{0,r}$. In the exact same way we can show that it specializes to $\rho_{r,\mathfrak{P}_{f,0}}$ for a prime $\mathfrak{P}_{f,0} \subset \mathbb{I}_{r,0}$ as above. The characteristic polynomial of its Sen operator $\phi_{r,\mathfrak{P}_{f,0}}$ is $X(X-(k-1))$. Since the primes of the form $\mathfrak{P}_{f,0}$ are dense in $\mathbb{I}_{r,0}$ the eigenvalues of $\rho_{r,\mathfrak{P}_{f,0}}$ can be interpolated in a unique way. The two elements obtained via this interpolation are the eigenvalues of $\phi_{r,Q}$. In particular we deduce the following.

Proposition 5.9. *The eigenvalues of the matrix $\Phi_r = \exp \phi_r$ are $u^{-1}(1+T)$ and 1. In particular Φ_r is a conjugate in $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ of the diagonal matrix C_T with entries $u^{-1}(1+T)$ and 1.*

Remark 5.10. *The second assertion in the proposition is true because in the definition*

$$\mathbb{B}_r = \varprojlim_{(\mathfrak{a},P_1)=1} \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$$

we have asked that $\mathfrak{a}$ is prime to P_1. Indeed, the difference of the eigenvalues of Φ_r is the generator $u^{-1}(1+T)-1$ of P_1, which is invertible in $\mathbb{B}_r$.

6. Existence of the Galois level for a family with finite positive slope

Let $r_h \in p^{\mathbb{Q}} \cap]0, p^{-\frac{1}{p-1}}]$ be the radius chosen in Section 3. Recall that $\mathfrak{H}_r \subset M_2(\mathbb{B}_r)$ is the Lie algebra attached to the image of ρ_r (see Definition 5.2) and $\mathfrak{H}_{0,r,\mathbb{C}_p} = \mathfrak{H}_r \widehat{\otimes} \mathbb{C}_p$. Let $\mathfrak{u}^{\pm}$, resp. $\mathfrak{u}^{\pm}_{\mathbb{C}_p}$, be the upper and lower nilpotent subalgebras of $\mathfrak{H}_r$, resp. of $\mathfrak{H}_{r,\mathbb{C}_p}$.

Remark 6.1. *The commutative Lie algebra $\mathfrak{u}^{\pm}$ is independent of r because it is equal to $\mathbb{Q}_p \cdot \mathrm{Log}(U(\mathbb{I}_0^\circ) \cap G'_r)$ which is independent of r, provided $r_0 \le r < r_h$.*

We fix $r_0 \in p^{\mathbb{Q}} \cap]0, r_h[$ and we work from now on with $r_0 \le r < r_h$. All subsequent rings in this section are indexed by r, with $r \ge r_0$. As in Remark 5.1, this fixes a finite extension of $\mathbb{Q}$ corresponding to the inclusion $G'_r \subset G_r$. For $r < r'$ we have a natural inclusion $\mathbb{I}_{0,r'} \hookrightarrow \mathbb{I}_{0,r}$. Since $\mathbb{B}_r = \varprojlim_{(\mathfrak{a}P_1)=1} \mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r}$ this induces an inclusion $\mathbb{B}_{r'} \hookrightarrow \mathbb{B}_r$. We will consider from now on $\mathbb{B}_{r'}$ as a subring of $\mathbb{B}_r$ for any $r < r'$. We will also consider $M_2(\mathbb{I}_{r',\mathbb{C}_p})$ and $M_2(\mathbb{B}_{r'})$ as subsets respectively of $M_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ and $M_2(\mathbb{B}_r)$. These inclusions still hold after taking completed tensors with $\mathbb{C}_p$.

For $r < r'$ the maps $A_{r'} \hookrightarrow A_{0,r}$ and $\mathbb{I}_{0,r'} \hookrightarrow \mathbb{I}_{0,r}$ induce injections $\mathbb{B}_{r'} \hookrightarrow \mathbb{B}_r$. We identify $\mathbb{B}_{r'}$ with a subring of $\mathbb{B}_r$ (so every $\mathbb{B}_r$ is a subring of $\mathbb{B}_{r_0}$).

Recall that $\phi_r \in M_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ is the Sen operator associated with ρ_r and $\Phi_r \in \mathrm{GL}_2(\mathbb{I}_r)$ is its exponential. The operator ϕ_r is independent of r in the following sense: if $r < r' < r_h$ and $\mathbb{I}_{r',\mathbb{C}_p} \to \mathbb{I}_{r,\mathbb{C}_p}$ is the natural inclusion then the image of $\phi_{r'}$ under the induced map $M_2(\mathbb{I}_{r',\mathbb{C}_p}) \to M_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ is ϕ_r.

We deduce that the exponential Φ_r is also independent of r (in the same sense).

By Proposition 5.9, for any $r < r_h$ we see that there exists an element $\beta_r \in \mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ such that $\beta_r \Phi_r \beta_r^{-1} = C_T$. Since Φ_r normalizes $\mathfrak{H}_{r,\mathbb{C}_p}$, $C_T = \beta_r \Phi \beta_r^{-1}$ normalizes $\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1}$.

We denote by $\mathfrak{U}^{\pm}$ the upper and lower unipotent subalgebras of $\mathfrak{sl}_2$. The action of C_T on $\mathfrak{H}_{r,\mathbb{C}_p}$ by conjugation is semisimple and we can decompose $\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1}$ as a sum of eigenspaces for C_T:

$$\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1} = \left(\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1}\right)[1] \oplus \left(\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1}\right)[u^{-1}(T+1)] \oplus \left(\beta_r \mathfrak{H}_{r,\mathbb{C}_p} \beta_r^{-1}\right)[u(T+1)^{-1}]$$

with

$$\left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u^{-1}(T+1)]\subset\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})\quad\text{and}\quad\left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u(T+1)^{-1}]\subset\mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$$

Note that $\left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u^{-1}(T+1)]\neq 0$ and $\left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u(T+1)^{-1}]\neq 0$ because otherwise $\mathfrak{H}_{r,\mathbb{C}_p}$ would be solvable, and this is not the case as $\mathfrak{H}_r\cap\mathfrak{U}^{\pm}\neq 0$ by the approximation lemma.

Moreover, the formula

$$\begin{pmatrix} u^{-1}(1+T) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & \lambda\\ 0 & 1\end{pmatrix}\begin{pmatrix} u^{-1}(1+T) & 0\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix} 1 & u^{-1}(1+T)\lambda\\ 0 & 1\end{pmatrix}$$

shows that the action of C_T by conjugation coincides with multiplication by $u^{-1}(1+T)$. By linearity this gives an action of the polynomial ring $\mathbb{C}_p[T]$ on $\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\cap\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$, compatible with the action of $\mathbb{C}_p[T]$ on $\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$ given by the inclusions $C[T]\subset\Lambda_{h,0,\mathbb{C}_p}\subset B_{r,\mathbb{C}_p}\subset\mathbb{B}_{r,\mathbb{C}_p}$. Since $\mathbb{C}_p[T]$ is dense in $A_{h,0,\mathbb{C}_p}$ for the p-adic topology, it is also dense in $B_{r,\mathbb{C}_p}$. Since $\mathfrak{H}_{r,\mathbb{C}_p}$ is closed in $M_2(B_{r,\mathbb{C}_p})$, we can define by continuity a structure of $B_{r,\mathbb{C}_p}$-module on $\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\cap\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$, compatible with that on $\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$. Similarly we have

$$\begin{pmatrix} u^{-1}(1+T) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \mu & 1\end{pmatrix}\begin{pmatrix} u^{-1}(1+T) & 0\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix} 1 & 0\\ u(1+T)^{-1}\mu & 1\end{pmatrix}.$$

We note that $1+T$ is invertible for the p-adic topology in $A_{0,r}$ since $T=p^{s_h}t$ where $r_h=p^{-s_h}$. Therefore C_T is invertible and by twisting by $(1+T)\mapsto(1+T)^{-1}$ we can also give $\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\cap\mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$ a structure of $B_{r,\mathbb{C}_p}$-module compatible with that on $\mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$.

By combining the previous remarks with Proposition 4.24, we prove the following 'fullness' result for the big Lie algebras $\mathfrak{H}_r$.

Theorem 6.2. *There exists a non-zero ideal $\mathfrak{l}$ of $\mathbb{I}_0$ independent of $r<r_h$ such that for any such r the Lie algebra $\mathfrak{H}_r$ contains $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_r)$.*

Proof. Since $U^{\pm}(\mathbb{B}_r)\cong\mathbb{B}_r$, we can and shall identify $\mathfrak{u}^+=\mathbb{Q}_p\cdot\mathrm{Log}G'_r\cap\mathfrak{U}^+(\mathbb{B}_r)$ with a $\mathbb{Q}_p$-vector subspace of $\mathbb{B}_r$ (actually of $\mathbb{I}_0$), resp. $\mathfrak{u}^+_{\mathbb{C}_p}$ with a $\mathbb{C}_p$-vector subspace of $\mathbb{B}_{r,\mathbb{C}_p}$. We repeat that these spaces are independent of r since G'_r is, provided $r_0\leq r<r_h$ (see Remark 5.1). By Proposition 4.24, $\mathfrak{u}^{\pm}\cap\mathbb{I}_0$ contains a basis $\{e_{i,\pm}\}_{i\in I}$ for $Q(\mathbb{I}_0)$ over $Q(\Lambda_{h,0})$. The set $\{e_{i,+}\}_{i\in I}\subset\mathfrak{u}^+$ is a basis for $Q(\mathbb{I}_0)$ over $Q(\Lambda_{h,0})$, so $\mathfrak{u}^+$ contains the basis of a $\Lambda_{h,0}$-lattice in $\mathbb{I}_0$. By Lemma 4.20 we deduce that the span $\Lambda_{h,0}\mathfrak{u}^+$ contains an ideal $\mathfrak{a}^+$ of $\mathbb{I}_0$. Hence we also have $B_{r,\mathbb{C}_p}\mathfrak{u}^+_{\mathbb{C}_p}\supset B_{r,\mathbb{C}_p}\mathfrak{a}^+$. Now $\mathfrak{a}^+$ is an ideal of $\mathbb{I}_0$ and $B_{r,\mathbb{C}_p}\mathbb{I}_{0,\mathbb{C}_p}=\mathbb{B}_{r,\mathbb{C}_p}$, so $B_{r,\mathbb{C}_p}\mathfrak{a}^+=\mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}$ is an ideal in $\mathbb{B}_{r,\mathbb{C}_p}$. We conclude that $B_{r,\mathbb{C}_p}\cdot\mathfrak{u}^+\supset\mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}$ for a non zero ideal $\mathfrak{a}^+$ of $\mathbb{I}_0$. We proceed in the same way for the lower unipotent subalgebra, obtaining $B_{r,\mathbb{C}_p}\cdot\mathfrak{u}^-\supset\mathfrak{a}^-\mathbb{B}_{r,\mathbb{C}_p}$ for some non zero ideal $\mathfrak{a}^-$ of $\mathbb{I}_0$.

Consider now the Lie algebra $B_{r,\mathbb{C}_p}\mathfrak{H}_{\mathbb{C}_p}\subset M_2(\mathbb{B}_{r,\mathbb{C}_p})$. Its unipotent subalgebras are $B_{r,\mathbb{C}_p}\mathfrak{u}^+$ and $B_{r,\mathbb{C}_p}\mathfrak{u}^-$, and we showed $B_{r,\mathbb{C}_p}\mathfrak{u}^+\supset\mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}$ and $B_{r,\mathbb{C}_p}\mathfrak{u}^-\supset\mathfrak{a}^-\mathbb{B}_{r,\mathbb{C}_p}$. Denote by $\mathfrak{t}\subset\mathfrak{sl}_2$ the subalgebra of diagonal matrices over $\mathbb{Z}$. By taking the Lie bracket, we see that $[\mathfrak{U}^+(\mathfrak{a}^+\mathbb{B}_{r,\mathbb{C}_p}),\mathfrak{U}^-(\mathfrak{a}^-\mathbb{B}_{r,\mathbb{C}_p})]$ spans $\mathfrak{a}^+\cdot\mathfrak{a}^-\mathfrak{t}(\mathbb{B}_{r,\mathbb{C}_p})$ over $B_{r,\mathbb{C}_p}$. We deduce that $B_{r,\mathbb{C}_p}\mathfrak{H}_{\mathbb{C}_p}\supset\mathfrak{a}^+\cdot\mathfrak{a}^-\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Let $\mathfrak{a}=\mathfrak{a}^+\cdot\mathfrak{a}^-$. Now $\mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$ is a $\mathbb{B}_{r,\mathbb{C}_p}$-Lie subalgebra of $\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Recall that $\beta_r\in\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$; hence by stability by conjugation we have $\beta_r\left(\mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})\right)\beta_r^{-1}=\mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Thus, we constructed $\mathfrak{a}$ such that $B_{r,\mathbb{C}_p}\left(\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\right)\supset\mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$, In particular, $B_{r,\mathbb{C}_p}\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}\supset\mathfrak{a}\mathbb{B}_{r,\mathbb{C}_p}$ for both signs. By the discussion preceding the proposition the subalgebras $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$ have a structure of $B_{r,\mathbb{C}_p}$-modules, which means that $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}=B_{r,\mathbb{C}_p}\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$. We conclude that $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}\supset\beta_r\left(\mathfrak{a}\cdot\mathfrak{U}^{\pm}(\mathbb{B}_{r,\mathbb{C}_p})\right)\beta_r^{-1}$ for both signs, so by the usual argument of taking the bracket, we have $\beta_r\mathfrak{H}_{r,\mathbb{C}_p}\beta_r^{-1}\supset\mathfrak{a}^2\mathfrak{sl}_2(B_{r,\mathbb{C}_p})$. We can then untwist by the invertible matrix β_r to conclude that for $\mathfrak{l}=\mathfrak{a}^2$, we have $\mathfrak{H}_{r,\mathbb{C}_p}\supset\mathfrak{l}\cdot\mathfrak{sl}_2(B_{r,\mathbb{C}_p})$.

Now let us get rid of the completed extension of scalars to $\mathbb{C}_p$. For any ideal $\mathfrak{a}\subset\mathbb{I}_0$ not dividing P_1, let $\mathfrak{H}_{r,\mathfrak{a}}$ be the image of $\mathfrak{H}_r$ in $M_2(\mathbb{I}_{0,r}/\mathfrak{a})\mathbb{I}_{0,r}$; consider the two finite dimensional $\mathbb{Q}_p$-vector subspaces $\mathfrak{H}_{r,\mathfrak{a}}$ and $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})$ of $\mathfrak{gl}_2(\mathbb{I}_0/\mathfrak{a})$. After extending scalars to $\mathbb{C}_p$, we have

$$\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\otimes\mathbb{C}_p\subset\mathfrak{H}_{r,\mathfrak{a}}\otimes\mathbb{C}_p$$

Therefore, we have

$$\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{a}\mathbb{I}_{0,r})\subset\mathfrak{H}_{r,\mathfrak{a}}$$

By taking projective limit over $\mathfrak{a}$, we conclude

$$\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_r)\subset\mathfrak{H}_r$$

□

Definition 6.3. *The Galois level of the family $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$ is the largest ideal $\mathfrak{l}_\theta$ of $\mathbb{I}_0[P_1^{-1}]$ such that $\mathfrak{H}_r \supset \mathfrak{l}_\theta \cdot \mathfrak{sl}_2(\mathbb{B}_r)$.*

It follows by the previous remarks that $\mathfrak{l}_\theta$ is non-zero.

7. Comparison between the Galois level and the fortuitous congruence ideal

Let $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$ be a family. We keep all the notations from the previous sections. In particular $\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ is the Galois representation associated with θ. We suppose throughout this section that $\overline{\rho}|_{H_0}$ is irreducible and that the restriction of ρ to H_0 takes values in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$. As noted in Remark 5.1 we can replace H_0 by a finite index subgroup H such that the image of $\rho|_H$ is a pro-p-group. Recall that $\mathbb{I} = \mathbb{I}^\circ[p^{-1}]$ and $\mathbb{I}_{\mathbb{C}_p} = \mathbb{I}\widehat{\otimes}_{K_h}\mathbb{C}_p$. Recall that P_1 is the prime of $\Lambda_{h,0}$ generated by $(u^{-1}(1+T)-1)$. Let $\mathfrak{c} \subset \mathbb{I}$ be the congruence ideal associated with θ. Set $\mathfrak{c}_0 = \mathfrak{c} \cap \mathbb{I}_0$ and $\mathfrak{c}_1 = \mathfrak{c}_0\mathbb{I}_0[P_1^{-1}]$. Let $\mathfrak{l} = \mathfrak{l}_\theta \subset \mathbb{I}_0$ be the Galois level associated with θ. Recall that this ideal is prime to P_1. For an ideal $\mathfrak{a}$ of $\mathbb{I}_0[P_1^{-1}]$ we denote by $V(\mathfrak{a})$ the set of prime ideals of $\mathbb{I}_0[P_1^{-1}]$ containing $\mathfrak{a}$. We show the following.

Theorem 7.1. *Suppose that that there exists no pair (F, ψ) where F is a real quadratic field and $\psi : \mathrm{Gal}(\overline{F}/F) \to \mathbb{F}^\times$ is a character, such that $\overline{\rho} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F}) \equiv \mathrm{Ind}_F^\mathbb{Q}\psi$. Then in $\operatorname{Spec}\mathbb{I}_0[P_1^{-1}]$ we have $V(\mathfrak{l}) = V(\mathfrak{c}_1)$.*

Before giving the proof we make some remarks. Let P be a prime of $\mathbb{I}_0$ and Q a prime factor of $P\mathbb{I}$. We have a representation $\rho_{\mathbb{I},Q} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}/Q)$ obtained by reducing $\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I})$ modulo Q. Its restriction $\rho_{\mathbb{I},Q}|_H$ takes values in $\mathrm{GL}_2(\mathbb{I}_0/(Q \cap \mathbb{I}_0)) = \mathrm{GL}_2(\mathbb{I}_0/P)$ and coincides with the reduction ρ_P of $\rho|_H : H \to \mathrm{GL}_2(\mathbb{I}_0)$ modulo P. In particular $\rho_{\mathbb{I},Q}|_H$ is independent of the chosen prime factor Q of $P\mathbb{I}$.

We say that a subgroup of $\mathrm{GL}_2(A)$ for some algebra A finite over a p-adic field K is *small* if it admits a finite index abelian subgroup. Let P, Q be as above, G_P be the image of $\rho_P : H \to \mathrm{GL}_2(\mathbb{I}_0/P)$ and $G_{\mathbb{I},Q}$ the image of $\rho_{\mathbb{I},Q} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}/Q)$. By our previous remark ρ_P coincides with the restriction $\rho_{\mathbb{I}_0,Q}|_H$, so G_P is a finite index subgroup of $G_{\mathbb{I},Q}$ for any Q. In particular G_P is small if and only if $G_{\mathbb{I},Q}$ is small for all prime factors Q of $P\mathbb{I}$.

Now if Q is a CM point the representation $\rho_{\mathbb{I},Q}$ is induced by a character of $\mathrm{Gal}(F/\mathbb{Q})$ for an imaginary quadratic field F. Hence $G_{\mathbb{I},Q}$ admits an abelian subgroup of index 2 and G_P is also small.

Conversely, if G_P is small, $G_{\mathbb{I},Q'}$ is small for any prime Q' above P. Choose any such Q'; by the argument in [Ri1, Prop. 4.4] $G_{\mathbb{I},Q'}$ has an abelian subgroup of index 2. It follows that $\rho_{\mathbb{I},Q'}$ is induced by a character of $\mathrm{Gal}(\overline{F}_{Q'}/F_{Q'})$ for a quadratic field $F_{Q'}$. If $F_{Q'}$ is imaginary then Q' is a CM point. In particular, if we suppose that the residual representation $\bar{\rho} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ is not induced by a character of $\mathrm{Gal}(\overline{F}/F)$ for a real quadratic field $F/\mathbb{Q}$, then $F_{Q'}$ must be imaginary, hence Q' is CM. The above argument proves that G_P is small if and only if all points $Q' \subset \mathbb{I}$ above P are CM.

Proof. We prove (i). By absurd, suppose that a prime $P \nmid P_1$ of $\mathbb{I}_0$ contains $\mathfrak{c}_0$ but P does not contain $\mathfrak{l}$. Then there exists a prime factor Q of $P\mathbb{I}$ such that $\mathfrak{c} \subset Q$. By definition of $\mathfrak{c}$ we have that Q is a CM point in the sense of Section 3.4, hence the representation $\rho_{\mathbb{I},Q}$ has image contained in the normalizer of a torus in $\mathrm{GL}_2(\mathbb{I}/Q)$. Then its restriction $\rho_{\mathbb{I},Q}|_H = \rho_P$ has image contained in the normalizer of a torus in $\mathrm{GL}_2(I_0/P)$. We deduce that there is no non-zero ideal $\mathfrak{I}$ of $\mathbb{I}_0$ such that the Lie algebra $\mathfrak{H}_{r,P}$ contains $\mathfrak{I} \cdot \mathfrak{sl}_2(\mathbb{B}_r)$.

Now by definition of $\mathfrak{l}$ we have $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r) \subset \mathfrak{H}_r$. Since reduction modulo P gives a surjection $\mathfrak{H} \to \mathfrak{H}_P$, by looking at the previous inclusion modulo P we find $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}/P\mathbb{I}_{0,r}) \subset \mathfrak{H}_{r,P}$. If $\mathfrak{l} \not\subset P$ we have $\mathfrak{l}/P \neq 0$, which contradicts our earlier statement. We deduce that $\mathfrak{l} \subset P$.

We prove (ii). Let $P \subset \mathbb{I}_0$ be a prime containing $\mathfrak{l}$. Recall that $\mathbb{I}_0$ has Krull dimension one, so $\kappa_P = \mathbb{I}_0/P$ is a field. Let $\mathfrak{Q} \subset \mathbb{I}_0$ be the P-primary component of $\mathfrak{l}$ and let $\mathfrak{a}$ be an ideal of $\mathbb{I}_0$ containing $\mathfrak{Q}$ such that the localization at P of $\mathfrak{a}/\mathfrak{Q}$ is one dimensional over κ_P. Let $\tilde{\mathfrak{a}}$ and $\widetilde{\mathfrak{H}}_r$ be the images of $\mathfrak{a}$ and $\mathfrak{H}_r$ under reduction modulo $\mathfrak{Q}$. Let $\mathfrak{s} = \tilde{\mathfrak{a}} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{Q}\mathbb{I}_{0,r}) \cap \widetilde{\mathfrak{H}}_r \subset \tilde{\mathfrak{a}} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r})/\mathfrak{Q}\mathbb{I}_{0,r})$. Note that $\mathfrak{s}$ is stable under the adjoint action $\mathrm{Ad}(\rho)$ of H, so we consider it as a Galois representation via $\mathrm{Ad}(\rho)$. By the proof of Theorem 6.2 we can assume, by considering possibly a sub-Galois representation, that $\mathfrak{H}_r$ is a $\mathbb{B}_r$-submodule of $\mathfrak{sl}_2(\mathbb{B}_r)$ containing $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r)$ but not $\mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{B}_r)$ for any $\mathfrak{a}$ strictly bigger than $\mathfrak{l}$. This allows to speak of the localization $\mathfrak{s}_P$ of $\mathfrak{s}$ at P. The localization at P of $\tilde{\mathfrak{a}} \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r}/\mathfrak{Q}\mathbb{I}_{0,r})$ is $\mathfrak{sl}_2(\kappa_P)$, so $\mathfrak{s}_P$ is contained in $\mathfrak{sl}_2(\kappa_P)$. It is a κ_P-submodule of rank at most 3 stable for the action of H via $\mathrm{Ad}(\rho_P)$, where $\rho_P : H \to \mathrm{GL}_2(\mathbb{I}_0/P)$ denotes the reduction of $\rho|_H$ modulo P.

Note that, since $\mathfrak{Q}$ is the P-primary component of $\mathfrak{l}$ and $\mathfrak{a}_P/\mathfrak{Q}_P \cong \kappa_P$, when P-localizing we find $\mathfrak{H}_{r,P} \supset \mathfrak{Q}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P})$ and $\mathfrak{H}_{r,P} \not\supset \mathfrak{a}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P})$.

In order to get a contradiction, we study the κ_P-vector space $\mathfrak{s}_P$.

We cannot have $\mathfrak{s}_P = 0$. In this case we would obtain $(\mathfrak{a}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P}) \cap \mathfrak{H}_{r,P})/\mathfrak{Q}_P = 0$. By Nakayama's lemma $\mathfrak{a}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P}) \cap \mathfrak{H}_{r,P} = 0$ which is absurd since $\mathfrak{a}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P}) \cap \mathfrak{H}_{r,P} \supset \mathfrak{Q}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P}) \neq 0$.

We also exclude the 3-dimensional case: indeed if $\mathfrak{s}_P = \mathfrak{sl}_2(\kappa_P)$, we would have by exchanging the quotient with the localization $(\mathfrak{a}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P}) \cap \mathfrak{H}_{r,P})/\mathfrak{Q}_P = (\mathfrak{a}_P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,P}))/\mathfrak{Q}_P\mathbb{I}_{0,r,P}$, because $\mathfrak{a}_P\mathbb{I}_{0,r,P}/\mathfrak{Q}_P\mathbb{I}_{0,r,P} = (\mathbb{I}_{0,r,P}/\mathfrak{Q}_P\mathbb{I}_{0,r,P})$ and this is isomorphic to κ_P. By Nakayama's lemma we would conclude that $\mathfrak{H}_{r,P} \supset \mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P})$, which is absurd.

We are left with the 1- and 2-dimensional cases. If $\mathfrak{s}_P$ is two-dimensional we can always replace it by its orthogonal in $\mathfrak{sl}_2(\kappa_P)$ which is one dimensional; indeed the action of $G_\mathbb{Q}$ via $\mathrm{Ad}(\rho_P)$ is isometric with respect to the scalar product $\mathrm{Tr}(XY)$ on $\mathfrak{sl}_2(\kappa_P)$.

Suppose then that $\mathfrak{sl}_2(\kappa_P)$ contains a 1-dimensional stable subspace. Let ϕ be a generator of this subspace over κ_P. For any $g \in G_\mathbb{Q}$ there must be a character $\chi : H \to \kappa_P$ such that $\rho_P(g)\phi\rho_P(g)^{-1} = \chi(g)\phi$. Let F_0 be the finite extension of $\mathbb{Q}$ fixed by χ and G_{F_0} be its absolute Galois group. For $g \in G_{F_0}$ we have $\rho_P(g)\phi\rho_P(g)^{-1} = \phi$, so by Schur's lemma we deduce that $\rho_P|_{G_{F_0}}$ acts by scalars, so ρ_P is induced by a character $G_{F_0} \to (\mathbb{I}_0/P\mathbb{I}_0)^\times$. In particular the image of ρ_P is small. By the remarks preceding this proof, if $Q \subset \mathbb{I}$ is any primary component of $P\mathbb{I}$ then $\rho_{\mathbb{I},Q}$ is induced by the character of a quadratic field $F_1/\mathbb{Q}$ (which will be a subfield of F_0). Since we supposed that the residual representation $\rho_{\mathfrak{m}_\mathbb{I}} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ is not of the form $\mathrm{Ind}_F^\mathbb{Q}\psi$ for a real quadratic field F and a character $\psi : \mathrm{Gal}(\overline{F}/F) \to \mathbb{F}^\times$, we deduce that F_1 must be imaginary. We conclude that Q is a CM point. By construction of the congruence ideal we conclude that $\mathfrak{c} \subset Q$, hence $\mathfrak{c}_0 \subset Q \cap \mathbb{I}_0 = P$. □

We prove two corollaries.

Corollary 7.2. *If the residual representation $\overline{\rho} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ is not dihedral then $\mathfrak{l} = 1$.*

Proof. Since $\overline{\rho}$ is not dihedral there cannot be any CM point on the family $\theta : \mathbb{T}_h \to \mathbb{I}$. By Theorem 7.1 we deduce that $\mathfrak{l}$ has no nontrivial prime factor, hence is trivial (recall that we are not considering the prime P_1 since $\mathfrak{l} \subset \mathbb{I}_0[P_1^{-1}]$). □

Remark 7.3. *Theorem 7.1 gives another proof of Proposition 3.11. Indeed the CM points of a family $\theta : \mathbb{T}_h \to \mathbb{I}^\circ$ correspond to the prime factors of its Galois level, which are finite in number.*

We also give a partial result about the comparison of the exponents of each prime factor in $\mathfrak{c}_1$ and $\mathfrak{l}$. This is an analogous of what is proved in [Hi, Th. 8.6] for the ordinary case; our proof also relies on the strategy there. Denote by P any prime of $\mathbb{I}_0$ with $P \nmid P_1$. We denote by $\mathfrak{c}^P$ and $\mathfrak{l}^P$ the P-primary components respectively of $\mathfrak{c}_1$ and $\mathfrak{l}$.

Theorem 7.4. *Suppose that $\overline{\rho}$ is not induced by a character of G_F for a real quadratic field $F/\mathbb{Q}$. We have $(\mathfrak{c}^P)^2 \subset \mathfrak{l}^P \subset \mathfrak{c}^P$.*

Proof. The inclusion $\mathfrak{l}^P \subset \mathfrak{c}^P$ is proved in the same way as part (i) of Theorem 7.1.

We show that the inclusion $(\mathfrak{c}^P)^2 \subset \mathfrak{l}^P$ holds. If $\mathfrak{c}^P$ is trivial this reduces to Theorem 7.1, so we can suppose $\mathfrak{c}^P \neq 1$. In particular P is a factor of $\mathfrak{c}_1$, hence if $P = \mathfrak{P} \cap \mathbb{I}_0[P_1^{-1}]$ then P is a factor of $\mathfrak{c}_1$. This means that P is a CM point, so we have an isomorphism $\rho_P \cong \mathrm{Ind}_F^\mathbb{Q}\psi$ for an imaginary quadratic field $F/\mathbb{Q}$ and a character $\psi : G_F \to \mathbb{C}_p^\times$. We keep the notations of the proof of Theorem 7.1. Consider the κ_P-vector space $\mathfrak{s}_{\mathfrak{c}^P} = \mathfrak{H}_r \cap \mathfrak{c}^P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r})/\mathfrak{H}_r \cap \mathfrak{c}^P P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r})$. We see it as a subspace of $\mathfrak{sl}_2(\kappa_P)$ and it is stable under the adjoint representation $\mathrm{Ad}(\rho_P) : G_\mathbb{Q} \to \mathrm{Aut}(\mathfrak{sl}_2(\kappa_P))$.

Let $\chi_{F/\mathbb{Q}} : G_\mathbb{Q} \to \mathbb{C}_p^\times$ is the quadratic character defined by the extension $F/\mathbb{Q}$. Let $\epsilon \in G_\mathbb{Q}$ be an element projecting to the generator of $\mathrm{Gal}(F/\mathbb{Q})$, $\psi^\epsilon : G_F \to \mathbb{C}_p^\times$ the character given by $\psi^\epsilon(\tau) = \psi(\epsilon\tau\epsilon^{-1})$ and $\psi^- = \psi/\psi^\epsilon$. Since $\rho_P \cong \mathrm{Ind}_F^\mathbb{Q}\psi$ we have a decomposition $\mathrm{Ad}(\rho_P) \cong \chi_{F/\mathbb{Q}} \oplus \mathrm{Ind}_F^\mathbb{Q}\psi^-$ where the two factors are irreducible. Now we have three possibilities for the Galois isomorphism class of $\mathfrak{s}_{\mathfrak{c}^P}$: either it is $\mathrm{Ad}(\rho_P)$ or it is isomorphic to one of the two irreducible factors.

If $\mathfrak{s}_{\mathfrak{c}^P} \cong \mathrm{Ad}(\rho_P)$, then $\mathfrak{s}_{\mathfrak{c}^P} = \mathfrak{sl}_2(\kappa_P)$, and we proceed as in the proof of Theorem 7.1 to obtain $\mathfrak{s}_{\mathfrak{c}^P} = \mathfrak{sl}_2(\kappa_P)$. By Nakayama's lemma $\mathfrak{H}_r \supset \mathfrak{c}^P \cdot \mathfrak{sl}_2(\mathbb{B}_r)$. This implies $\mathfrak{c}^P \subset \mathfrak{l}^P$, hence $\mathfrak{c}^P = \mathfrak{l}^P$ in this case.

If $\mathfrak{s}_{\mathfrak{c}^P}$ is one-dimensional then we proceed as in the proof of Theorem 7.1 to show that $\rho_{\mathfrak{c}^P P} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{c}^P P)$ is dihedral. We deduce that there exists a character $\psi_{\mathfrak{c}^P P} : G_F \to \mathbb{C}_p^\times$ such that $\rho_{\mathfrak{c}^P P} \cong \mathrm{Ind}_F^\mathbb{Q}\psi_{\mathfrak{c}^P P}$. This is a contradiction, since $\mathfrak{c}^P$ is the P-primary component of $\mathfrak{c}_1$, hence it is the smallest P-primary ideal $\mathfrak{a}$ such that $\rho_\mathfrak{a} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}_{0,r}/\mathfrak{a})$ is induced by a character of G_F.

We are left with the two-dimensional case. Suppose then that $\mathfrak{s}_{\mathfrak{c}^P} \cong \mathrm{Ind}_F^{\mathbb{Q}}\psi^-$. Let $d = \mathrm{diag}(d_1, d_2) \in \rho(G_{\mathbb{Q}})$ be the image of a $\mathbb{Z}_p$-regular element. The image of d modulo P is a nontrivial diagonal element $d_P = \mathrm{diag}(d_{1,P}, d_{2,P}) \in \rho_P(G_{\mathbb{Q}})$, since d_1 and d_2 are nontrivial modulo the maximal ideal of $\mathbb{I}_0^\circ$. We decompose $\mathfrak{s}_{\mathfrak{c}^P}$ in eigenspaces for the adjoint action of d_P as $\mathfrak{s}_{\mathfrak{c}^P}[a] \oplus \mathfrak{s}_{\mathfrak{c}^P}[1] \oplus \mathfrak{s}_{\mathfrak{c}^P}[a^{-1}]$, where $a = d_{1,P}/d_{2,P}$. Now $\mathfrak{s}_{\mathfrak{c}^P}[1]$ is contained in the diagonal torus, on which the adjoint action of $G_{\mathbb{Q}}$ is given by the character $\chi_{F/\mathbb{Q}}$. Since $\chi_{F/\mathbb{Q}}$ does not appear as a factor of $\mathfrak{s}_{\mathfrak{c}^P}$ we must have $\mathfrak{s}_{\mathfrak{c}^P}[1] = 0$. This implies that $\mathfrak{s}_{\mathfrak{c}^P}[a] \neq 0$ and $\mathfrak{s}_{\mathfrak{c}^P}[a^{-1}] \neq 0$. Since $\mathfrak{s}_{\mathfrak{c}^P}[a] = \mathfrak{s}_{\mathfrak{c}^P} \cap \mathfrak{u}^+(\kappa_P)$ and $\mathfrak{s}_{\mathfrak{c}^P}[a^{-1}] = \mathfrak{s}_{\mathfrak{c}^P} \cap \mathfrak{u}^-(\kappa_P)$ we deduce that $\mathfrak{s}_{\mathfrak{c}^P}$ contains nontrivial upper and lower unipotent elements $\overline{u}^+$ and $\overline{u}^-$. Then $\overline{u}^+$ and $\overline{u}^-$ are the images of some elements u^+ and u^- of $\mathfrak{H}_r \cap \mathfrak{c}^P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r})$ which are nontrivial modulo $\mathfrak{c}^P P$. The Lie bracket $t = [u^+, u^-]$ gives an element in $\mathfrak{H}_r \cap \mathfrak{t}(\mathbb{I}_{0,r})$ (where $\mathfrak{t}$ denotes the diagonal torus) and t is nontrivial modulo $(\mathfrak{c}^P)^2 P$. Hence the κ_P-vector space $\mathfrak{s}_{(\mathfrak{c}^P)^2} = \mathfrak{H}_r \cap (\mathfrak{c}^P)^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,\mathbb{C}_p})/\mathfrak{H}_r \cap (\mathfrak{c}^P)^2 P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ contains nontrivial diagonal, upper and lower unipotent elements, so it is three-dimensional. As before we can conclude with Nakayama's lemma that $\mathfrak{H}_r \supset (\mathfrak{c}^P)^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r})$, so $(\mathfrak{c}^P)^2 \subset \mathfrak{l}^P$. □

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