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# Directional Hölder Metric Regularity

Huynh Van Ngai · Nguyen Huu Tron · Michel Théra

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**Abstract** This paper sheds new light on regularity of multifunctions through various characterizations of directional Hölder/Lipschitz metric regularity, which are based on the concepts of slope and coderivative. By using these characterizations, we show that directional Hölder/Lipschitz metric regularity is stable, when the multifunction under consideration is perturbed suitably. Applications of directional Hölder/Lipschitz metric regularity to investigate the stability and the sensitivity analysis of parameterized optimization problems are also discussed.

**Keywords** Slope · Metric regularity · Hölder metric regularity · Generalized equation · Fréchet subdifferential · Asplund spaces · Ekeland variational principle, Hadamard directional differentiability.

**Mathematics Subject Classification (2000)** 49J52; 49J53; 58C06; 47H04; 54C60; 90C30

## 1 Introduction

Throughout the last three decades, metric regularity has been significantly developed and has become one of the central concepts of modern variational analysis. The terminology “metric regularity” was coined by Borwein [1], but the roots of this notion can be traced back to the classical *open mapping theorem* and its subsequent generalization to nonlinear mappings known as the *Lyusternik-Graves theorem*. The theory of metric regularity is extraordinary

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nary useful for investigating the behavior of solutions of a nonlinear equation under small perturbations of the data, or more generally the behavior of the solution set of generalized equations associated with a set-valued mapping. As a result, metric regularity plays an important role in many aspects of optimization, differential inclusions, control theory, numerical methods and in many problems of analysis. According to the long history of metric regularity there is an abundant literature on conditions ensuring this property. We refer the reader to the basic monographs [2, 3, 4, 5, 6], to the excellent survey of A. Ioffe [7] (in preparation) and to some (non exhaustives) references [1, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

Apart from the study of the usual (Lipschitz) metric regularity, Hölder metric regularity or more generally nonlinear metric regularity have been studied over the years 1980 – 1990s by several authors, including for example Borwein and Zhuang [1], Frankowska [16], Penot [25], and recently, for instance, Frankowska and Quincampoix [17], Ioffe [26], Li and Mordukhovich [27], Oyang and Mordukhovich [28].

Recently, several directional versions of metric regularity notions were considered. In [29, 30], Arutyunov et al have introduced and studied a notion of *directional metric regularity*. This notion is an extension of an earlier notion used by Bonnans and Shapiro [10] to study sensitivity analysis. Later, Ioffe [31] has introduced and investigated an extension called *relative metric regularity* which covers many notions of metric regularity in the literature. In particular, another version of directional metric regularity/subregularity has been introduced and extensively studied by Gfrerer in [32, 33] where some variational characterizations of this concept have been established and successfully applied to study optimality conditions for mathematical programs. In fact, this directional regularity property has been earlier used by Penot [34] to study second order optimality conditions. In the line of the directional version of metric subregularity considered by Gfrerer [32], Huynh, Nguyen and Tinh [35] have studied directional Hölder metric subregularity in order to investigate tangent cones to zero sets in degenerate cases.

It is our aim in the present article to study a *directional version* of Hölder metric regularity. The structure of the article is as follows. In the Section 2, we establish slope-based characterizations of directional Hölder/Lipschitz metric regularity. In Section 3, we present a stability property for directional Lipschitz metric regularity. In Section 4, a sufficient condition for directional Hölder/Lipschitz metric regularity based on the Fréchet coderivative is established in Asplund spaces. This condition becomes necessary when, either the multifunction under consideration is convex, or when considering directional Lipschitz metric regularity. It was simultaneously showed that under this condition, directional Hölder/Lipschitz metric regularity persists when the multifunction is perturbed by a Hadamard differentiable mapping. Applications to the study of the stability and the sensitivity analysis of parameterized optimization problems are discussed in Section 4. The last section contains concluding remarks.

## 2 Notations and Preliminaries

Throughout we let  $X$  and  $Y$  denote metric spaces endowed with metrics both denoted by  $d(\cdot, \cdot)$ . We denote the open and closed balls with center  $x$  and radius  $r > 0$  by  $B(x, r)$  and  $\bar{B}(x, r)$ , respectively. For a given set  $C$ , we write  $\text{int}C$  for its topological interior. A set-valued mapping (also called multifunction)  $F : X \rightrightarrows Y$  is a mapping assigning, to each point  $x \in X$ , a subset (possibly empty)  $F(x)$  of  $Y$ . We use the notations

$$\text{gph } F := \{(x, y) \in X \times Y : y \in F(x)\} \quad \text{and} \quad \text{Dom } F := \{x \in X : F(x) \neq \emptyset\}$$

for the *graph* of and the *domain* of  $F$ , respectively. For each set-valued mapping  $F : X \rightrightarrows Y$ , we define the *inverse* of  $F$ , as the mapping  $F^{-1} : Y \rightrightarrows X$  defined by  $F^{-1}(y) := \{x \in X : y \in F(x)\}$ ,  $y \in Y$  and satisfying

$$(x, y) \in \text{gph } F \iff (y, x) \in \text{gph } F^{-1}.$$

We use the standard notation  $d(x, C)$  to denote the distance from  $x$  to a set  $C$ ; it is defined by  $d(x, C) = \inf_{z \in C} d(x, z)$ , with the convention that  $d(x, S) = +\infty$  whenever  $S$  is the empty set. As pointed out in the introduction, main attention in this contribution is paid to the study of the concept of metric regularity. Recall that a mapping  $F$  is said to be *metrically regular* at  $(\bar{x}, \bar{y}) \in \text{gph } F$  with modulus  $\tau > 0$ , if there exists a neighborhood  $U \times V$  of  $(\bar{x}, \bar{y})$  such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \text{for all } (x, y) \in U \times V. \quad (1)$$

In other words, metric regularity allows to estimate the dependence of the distance of a trial point  $x \in X$  from the solution set  $F^{-1}(y)$  in terms of the residual quantity  $d(y, F(x))$  for all pairs  $(x, y)$  around the reference pair  $(\bar{x}, \bar{y}) \in \text{gph } F$ . The infimum of all moduli  $\tau$  is denoted by  $\text{reg } F(\bar{x}, \bar{y})$ .

If in the above definition we fix  $y = \bar{y}$  in (1), then we obtain a weaker notion called *metric subregularity*, see e.g. [4, 6, 36]. Observe that this latter property is equivalent to the existence of some neighborhood  $U$  of  $\bar{x}$  such that

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \quad \text{for all } x \in U.$$

It is also well known that metric subregularity can be treated in the framework of the theory of error bounds of extended-real-valued functions, see e.g. [37, 38].

An Hölder version of metric regularity is defined as follows (Frankowska and Quincampoix [17], Ioffe [26]). Let  $q \in (0, 1]$  be given. A mapping  $F$  is said to be *metrically  $q$ -regular* or Hölder metrically regular of order  $q$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  with modulus  $\tau > 0$  if there exists a neighborhood  $U \times V$  of  $(\bar{x}, \bar{y})$  such that

$$d(x, F^{-1}(y)) \leq \tau [d(y, F(x))]^q \quad \text{for all } (x, y) \in U \times V. \quad (2)$$

The infimum of all moduli  $\tau$  satisfying (2) is denoted by  $\text{reg}^q F(\bar{x}, \bar{y})$ , i.e.,

$$\text{reg}_q F(\bar{x}, \bar{y}) = \inf\{\tau > 0 : \exists \delta > 0 \text{ s.t. } d(x, F^{-1}(y)) \leq \tau (d(y, F(x)))^q \quad \text{for all } (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta)\}.$$

Fixing  $y = \bar{y}$  in the above definition, gives the concept of  $q$ -Hölder metric subregularity of the set-valued mapping  $F$  at  $(\bar{x}, \bar{y})$ .

In the present paper, we are interested in a *directional version* of Hölder metric regularity, defined as follows.

**Definition 2.1** Let  $X, Y$  be normed linear spaces. Let a real  $\gamma \in ]0, 1]$  and  $(u, v) \in X \times Y$  be given. A multifunction  $F$  is said to be (directionally) metrically  $\gamma$ -regular at  $(\bar{x}, \bar{y}) \in \text{gph } F$  in the direction  $(u, v)$  with a modulus  $\tau > 0$  iff there exists  $\delta, \varepsilon, \eta > 0$  such that

$$d(x, F^{-1}(y)) \leq \tau [d(y, F(x))]^\gamma \quad (3)$$

for all  $(x, y) \in B((\bar{x}, \bar{y}), \delta)$  with  $(x, y) \in (\bar{x}, \bar{y}) + \text{cone } B((u, v), \varepsilon)$ ;  $d(y, F(x)) \leq \eta \|(x, y) - (\bar{x}, \bar{y})\|^{1/\gamma}$ . Here  $\text{cone } B((u, v), \varepsilon)$  stands for the conic hull of  $B((u, v), \varepsilon)$ , i.e.,  $\text{cone } B((u, v), \varepsilon) = \cup_{\lambda \geq 0} \lambda B((u, v), \varepsilon)$ .

**Remark 2.1** In Definition 2.1, the triple  $\delta, \varepsilon, \eta > 0$  may be replaced by just a single positive number.

If (3) is required to be verified only at  $y = \bar{y}$ , and  $x \in B(\bar{x}, \delta)$  with  $x \in \bar{x} + \text{cone } B(u, \varepsilon)$ , we say that  $F$  is *directionally Hölder metrically subregular* at  $(\bar{x}, \bar{y})$  in the direction  $u$ . When  $\gamma = 1$ , one refers to the (*Lipschitz*) *directional metric regularity*, as equivalently introduced by Gfrerer [32].

Since in the definition above, the gauge condition  $d(y, F(x)) \leq \eta \|(x, y) - (\bar{x}, \bar{y})\|^{1/\gamma}$  is added, when  $(u, v) = (0, 0)$ , the version of Hölder/Lipschitz metric regularity in Definition 2.1 is even weaker than the usual ones defined by (2). For example, obviously, the function  $f(x) = x^2$ ,  $x \in \mathbb{R}$  is Hölder metrically regular of order  $1/2$  at  $(0, 0)$  in the direction  $(0, 0)$  in the sense of Definition 2.1, but is not in the usual sense of (2). The added gauge conditions in concepts of metric regularity are really needed when the usual regularity is not satisfied (see, e.g., Ioffe [26]). Let us mention that directional Hölder/Lipschitz metric regularity is obviously stronger than Hölder/Lipschitz metric subregularity. The main purpose of the present paper is to show that the tools of variational analysis such as the  $f$  and the concept of coderivative can be used to efficiently characterize directional Hölder/Lipschitz metric regularity. Our aim is to show that this directional version of Hölder/Lipschitz metric regularity, although weaker than the usual metric regularity, possesses the suitable stability properties which are lost in the case of metric subregularity.

### 3 Slope Characterizations of Directional Hölder Metric Regularity

Let  $X$  be a metric space. Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a given extended-real-valued function. As usual,  $\text{dom } f := \{x \in X : f(x) < +\infty\}$  denotes the domain of  $f$ .

Recall from [39], (see also [40] and the discussion in [38]) that the *local slope* and the *nonlocal slope* (see, e.g., [41]) of the function  $f$  at  $x \in \text{dom } f$ , are the quantities denoted respectively by  $|\nabla f|(x)$  and  $|\Gamma f|(x)$ . Using the notation  $[a]_+$  for  $\max\{a, 0\}$ , they are defined by  $|\nabla f|(x) = |\Gamma f|(x) = 0$  if  $x$  is a local minimum of  $f$  and otherwise by

$$|\nabla f|(x) = \limsup_{y \rightarrow x, y \neq x} \frac{f(x) - f(y)}{d(x, y)} \quad (4)$$

and

$$|\Gamma f|(x) = \sup_{y \neq x} \frac{[f(x) - f(y)]_+}{d(x, y)}. \quad (5)$$

For  $x \notin \text{dom} f$ , we set  $|\nabla f|(x) = |\Gamma f|(x) = +\infty$ . Obviously,  $|\nabla f|(x) \leq |\Gamma f|(x)$  for all  $x \in X$ .

Let  $X, Y$  be normed spaces. If not specified otherwise, we assume that the norm on  $X \times Y$  is defined by

$$\|(x, y)\| = \|x\| + \|y\|, (x, y) \in X \times Y.$$

For a *closed multifunction*  $F : X \rightrightarrows Y$ , (i.e., when the graph of  $F$  is closed in  $X \times Y$ ), the *lower semicontinuous envelope* of the distance function  $(x, y) \rightarrow d(y, F(x))$  is defined, for a given  $(x, y) \in X \times Y$ , by

$$\varphi(x, y) := \liminf_{(u, v) \rightarrow (x, y)} d(v, F(u)) = \liminf_{u \rightarrow x} d(y, F(u)).$$

In what follows, for  $u \in X$ , we shall use the notation  $x \xrightarrow{u} \bar{x}$  to mean

$$\begin{aligned} x \rightarrow \bar{x} & \quad \text{if } u = 0, \\ \left\| \frac{x - \bar{x}}{\|x - \bar{x}\|} - \frac{u}{\|u\|} \right\| & \rightarrow 0 \\ x \rightarrow \bar{x} & \quad \text{if } u \neq 0. \end{aligned}$$

Obviously, given a sequence  $\{x_n\} \subseteq X, u \in X$  the two facts are equivalent:

$$(C_1) : \{x_n\} \xrightarrow{u} \bar{x};$$

$$(C_2) : \{x_n\} \rightarrow \bar{x} \text{ and there is a sequence of nonnegative reals } \{\delta_n\} \rightarrow 0 \text{ such that}$$

$$x_n \in \bar{x} + \text{cone } B(u, \delta_n), \forall n \in \mathbb{N}.$$

We need the following series of useful lemmas whose proofs are straightforward.

**Lemma 3.1** *Let  $X$  be a Banach space and  $Y$  be a normed space. Suppose a closed multifunction  $F : X \rightrightarrows Y$  and a point  $(\bar{x}, \bar{y}) \in \text{gph } F$  are given. Given  $(u, v) \in X \times Y$ , and  $\gamma \in ]0, 1]$ , then  $F$  is metrically  $\gamma$ -regular at  $(\bar{x}, \bar{y})$  in the direction  $(u, v)$  with modulus  $\tau > 0$ , if and only if, there exist real numbers  $\tau, \delta > 0$  such that*

$$d(x, F^{-1}(y)) \leq \tau \varphi^\gamma(x, y)$$

for all  $(x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta))$  with  $d(y, F(x)) \leq \delta \|(x, y) - (\bar{x}, \bar{y})\|^{\frac{1}{\gamma}}$ .

**Lemma 3.2** *Let  $u \in X$  and  $\bar{x} \in X$  be given as well as a sequence  $\{x_n\}$  such that  $x_n \xrightarrow{u} \bar{x}$ . Then, for any sequence  $\{\delta_n\} \downarrow 0$  of nonnegative reals and any sequence  $\{z_n\} \subseteq X$  with*

$$\|z_n - x_n\| \leq \delta_n \|x_n - \bar{x}\|, n \in \mathbb{N}, \quad (6)$$

one has  $z_n \xrightarrow{u} \bar{x}$ .

*Proof* It suffices to prove the result when  $\|u\| = 1$ . If  $x_n = \bar{x}$ , then from (6) we have  $z_n = \bar{x}$  and we are done. Otherwise, one has

$$(1 - \delta_n)\|x_n - \bar{x}\| \leq \|z_n - \bar{x}\|,$$

and since

$$\begin{aligned} \|z_n - \bar{x} - u\| \|z_n - \bar{x}\| &\leq 2\|z_n - x_n\| + \|x_n - \bar{x} - u\| \|x_n - \bar{x}\| \\ &\leq 2\delta_n \|x_n - \bar{x}\| + \|x_n - \bar{x} - u\| \|x_n - \bar{x}\|, \end{aligned}$$

it follows that

$$\left\| \frac{z_n - \bar{x}}{\|z_n - \bar{x}\|} - u \right\| = \frac{\|z_n - \bar{x} - u\| \|z_n - \bar{x}\|}{\|z_n - \bar{x}\|^2} \leq (1 - \delta_n)^{-1} \left( 2\delta_n + \left\| \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} - u \right\| \right).$$

As  $\left\| \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} - u \right\| \rightarrow 0$  and  $\{\delta_n\} \rightarrow 0$ , one obtains

$$\left\| \frac{z_n - \bar{x}}{\|z_n - \bar{x}\|} - u \right\| \rightarrow 0$$

as  $n$  tends to infinity. Moreover, it is easy to see that  $z_n \xrightarrow{u} \bar{x}$ .  $\square$

**Theorem 3.1** *Let  $X$  be a Banach space and  $Y$  be a normed space. Suppose a closed multifunction  $F : X \rightrightarrows Y$  and a point  $(\bar{x}, \bar{y}) \in \text{gph } F$  are given. Given  $(u, v) \in X \times Y$ , and  $\gamma \in ]0, 1]$ ,  $F$  is directional metrically  $\gamma$ -regular at  $(\bar{x}, \bar{y})$  in the direction  $(u, v)$ , if and only if,*

$$\liminf_{\substack{(x,y) \rightarrow (\bar{x}, \bar{y}), \varphi(x,y) > 0 \\ (u,v) \\ \frac{\varphi^\gamma(x,y)}{\|(x,y) - (\bar{x}, \bar{y})\|} \rightarrow 0}} |\Gamma \varphi^\gamma(\cdot, y)|(x) > 0. \quad (7)$$

*Proof* For the sufficiency, assume that (7) holds and assume on the contrary that  $F$  fails to be directional metrically  $\gamma$ -regular in the direction  $(u, v)$ . Then for every  $n \in \mathbb{N}$ , there exists  $(x_n, y_n)$  with

$$0 < \|x_n - \bar{x}\| + \|y_n - \bar{y}\| < \frac{1}{n}; \quad d(y_n, F(x_n))^\gamma \leq \frac{1}{n^2} \|(x_n, y_n) - (\bar{x}, \bar{y})\|;$$

$$(x_n, y_n) \xrightarrow{(u,v)} (\bar{x}, \bar{y})$$

and such that

$$d(x_n, F^{-1}(y_n)) > n^2 [d(y_n, F(x_n))]^\gamma (\geq n^2 \varphi^\gamma(x_n, y_n)).$$

From the relation  $d(y_n, F(x_n))^\gamma \leq \frac{1}{n^2} \|(x_n, y_n) - (\bar{x}, \bar{y})\|$ , it follows that

$$\varphi^\gamma(x_n, y_n) \leq \frac{1}{n^2} \|(x_n, y_n) - (\bar{x}, \bar{y})\|. \quad (8)$$

Applying the Ekeland variational principle to the lower semicontinuous function  $\varphi^\gamma(\cdot, y_n)$  on the Banach space  $X$ , one gets a point  $z_n$  satisfying the conditions:

$$\|z_n - x_n\| \leq \frac{1}{n} \|(x_n, y_n) - (\bar{x}, \bar{y})\|, \quad \varphi^\gamma(z_n, y_n) \leq \varphi^\gamma(x_n, y_n),$$

and

$$\varphi^\gamma(z_n, y_n) \leq \varphi^\gamma(x, y_n) + \frac{1}{n} \|x - z_n\|, \quad \forall x \in X.$$

Consequently,

$$|\Gamma \varphi^\gamma(\cdot, y_n)|(z_n) \leq \frac{1}{n}, \quad (9)$$

and by

$$\|(x_n, y_n) - (\bar{x}, \bar{y})\| \leq \|z_n - x_n\| + \|(z_n, y_n) - (\bar{x}, \bar{y})\| \leq \frac{1}{n} \|(x_n, y_n) - (\bar{x}, \bar{y})\| + \|(z_n, y_n) - (\bar{x}, \bar{y})\|,$$

one obtains

$$\|(x_n, y_n) - (\bar{x}, \bar{y})\| \leq \frac{n}{n-1} \|(z_n, y_n) - (\bar{x}, \bar{y})\|. \quad (10)$$

Hence, combining (8) and (10) we obtain:

$$\varphi^\gamma(z_n, y_n) \leq \varphi^\gamma(x_n, y_n) \leq \frac{1}{n^2} \|(x_n, y_n) - (\bar{x}, \bar{y})\| \leq \frac{1}{n(n-1)} \|(z_n, y_n) - (\bar{x}, \bar{y})\|.$$

The latter relation implies that

$$\lim_{n \rightarrow \infty} \frac{\varphi^\gamma(z_n, y_n)}{\|(z_n, y_n) - (\bar{x}, \bar{y})\|} = 0.$$

Moreover, invoking Lemma 3.2, relations  $(x_n, y_n) \xrightarrow{(u,v)} (\bar{x}, \bar{y})$  and  $\|z_n - x_n\| \leq 1/n \|(x_n, y_n) - (\bar{x}, \bar{y})\|$  imply that  $(z_n, y_n) \xrightarrow{(u,v)} (\bar{x}, \bar{y})$ . Hence, by (7),  $z_n \in F^{-1}(y_n)$  for  $n$  large, say,  $n \geq n_0$ .

For  $n \geq n_0$ , as

$$\varphi^\gamma(x_n, y_n) < \frac{1}{n^2} d(x_n, F^{-1}(y_n)),$$

applying again the Ekeland variational principle to the lower semicontinuous  $\varphi^\gamma(\cdot, y_n)$ , one gets a point  $w_n \in X$  satisfying the conditions:

$$\|w_n - x_n\| < \frac{1}{n} d(x_n, F^{-1}(y_n)), \quad \varphi^\gamma(w_n, y_n) \leq \varphi^\gamma(x_n, y_n)$$

and,

$$\varphi^\gamma(w_n, y_n) \leq \varphi^\gamma(x, y_n) + \frac{1}{n} \|x - w_n\|, \quad \forall x \in X.$$

We deduce that

$$|\Gamma \varphi^\gamma(\cdot, y_n)|(w_n) \leq \frac{1}{n}.$$

Moreover we claim that

$$\frac{\varphi^\gamma(w_n, y_n)}{\|(w_n, y_n) - (\bar{x}, \bar{y})\|} \leq \frac{1}{n^2} + \frac{1}{n^2(n^2 - 1)}. \quad (11)$$

Indeed, as

$$\begin{aligned} \|(x_n, y_n) - (\bar{x}, \bar{y})\| &\leq \\ &\leq \|(x_n, y_n) - (w_n, y_n)\| + \|(w_n, y_n) - (\bar{x}, \bar{y})\| \\ &= \|x_n - w_n\| + \|(w_n, y_n) - (\bar{x}, \bar{y})\| \\ &\leq \frac{1}{n} d(x_n, F^{-1}(y_n)) + \|(w_n, y_n) - (\bar{x}, \bar{y})\| \\ &\leq \frac{1}{n} \|x_n - z_n\| + \|(w_n, y_n) - (\bar{x}, \bar{y})\| \\ &\leq \frac{1}{n^2} \|(x_n, y_n) - (\bar{x}, \bar{y})\| + \|(w_n, y_n) - (\bar{x}, \bar{y})\|, \end{aligned}$$



we have

$$\|(x_n, y_n) - (\bar{x}, \bar{y})\| \leq \frac{n^2}{n^2 - 1} \|(w_n, y_n) - (\bar{x}, \bar{y})\|,$$

and

$$\begin{aligned} \varphi^\gamma(w_n, y_n) &\leq \varphi^\gamma(x_n, y_n) \leq \frac{1}{n^2} \|(x_n, y_n) - (\bar{x}, \bar{y})\| \\ &\leq \frac{1}{n^2} \|(w_n, y_n) - (\bar{x}, \bar{y})\| + \frac{1}{n^2} \|x_n - w_n\| \\ &\leq \frac{1}{n^2} \|(w_n, y_n) - (\bar{x}, \bar{y})\| + \frac{1}{n^3} d(x_n, F^{-1}(y_n)) \leq \frac{1}{n^2} \|(w_n, y_n) - (\bar{x}, \bar{y})\| + \frac{1}{n^3} \|x_n - z_n\| \\ &\leq \frac{1}{n^2} \|(w_n, y_n) - (\bar{x}, \bar{y})\| + \frac{1}{n^4} \|(x_n, y_n) - (\bar{x}, \bar{y})\|. \end{aligned}$$

Therefore,

$$\varphi^\gamma(w_n, y_n) \leq \left( \frac{1}{n^2} + \frac{1}{n^2(n^2 - 1)} \right) \|(w_n, y_n) - (\bar{x}, \bar{y})\|,$$

and (11) is established. By virtue of Lemma 3.2, since  $(x_n, y_n) \xrightarrow{(u,v)} (\bar{x}, \bar{y})$  and  $\|w_n - x_n\| \leq \frac{1}{n^2} \|(x_n, y_n) - (\bar{x}, \bar{y})\|$ , one has  $(w_n, y_n) \xrightarrow{(u,v)} (\bar{x}, \bar{y})$ .

In conclusion, we have obtained a sequence  $\{w_n\}$  which satisfies

$$w_n \notin F^{-1}(y_n); (w_n, y_n) \xrightarrow{(u,v)} (\bar{x}, \bar{y}); \frac{\varphi^\gamma(w_n, y_n)}{\|(w_n, y_n) - (\bar{x}, \bar{y})\|} \rightarrow 0 \quad \text{and} \quad |\Gamma \varphi^\gamma(\cdot, y_n)|(w_n) \leq \frac{1}{n}.$$

Hence, condition (7) is violated, and the sufficiency is proved.

For the necessary part, suppose that there exist reals  $\tau, \delta > 0$  such that

$$d(x, F^{-1}(y)) \leq \tau [d(y, F(x))]^\gamma$$

for all  $(x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta))$  with  $d(y, F(x)) \leq \delta \|(x, y) - (\bar{x}, \bar{y})\|^\frac{1}{\gamma}$ .

According to Lemma 3.1,

$$d(x, F^{-1}(y)) \leq \tau \varphi^\gamma(x, y) \quad \forall (x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta)) \quad \text{with} \quad 0 < \frac{\varphi^\gamma(x, y)}{\|(x, y) - (\bar{x}, \bar{y})\|} \leq \delta.$$

Let  $(x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta))$  with  $(x, y) \neq (\bar{x}, \bar{y})$  and  $0 < \frac{\varphi^\gamma(x, y)}{\|(x, y) - (\bar{x}, \bar{y})\|} \leq \delta$ . Then, for every  $\varepsilon > 0$ , there exists an element  $z \in F^{-1}(y)$  such that

$$\|x - z\| \leq (\tau + \varepsilon) \varphi^\gamma(x, y) = (\tau + \varepsilon) [\varphi^\gamma(x, y) - \varphi^\gamma(z, y)].$$

Consequently,

$$|\Gamma \varphi^\gamma(\cdot, y)|(x) \geq \frac{1}{(\tau + \varepsilon)}.$$

As  $\varepsilon > 0$  is arbitrary, one obtains

$$\liminf_{\substack{(x,y) \rightarrow (\bar{x}, \bar{y}), \\ (u,v) \\ \varphi^\gamma(x,y) > 0}} |\Gamma \varphi^\gamma(\cdot, y)|(x) \geq \frac{1}{\tau} > 0,$$

completing the proof.  $\square$

The theorem above yields the following local slope characterization of the directional metric  $\gamma$ -regularity .

**Theorem 3.2** *Let  $X$  be a Banach space and  $Y$  be a normed space. Suppose a closed multifunction  $F : X \rightrightarrows Y$  and a point  $(\bar{x}, \bar{y}) \in X \times Y$  are given such that  $\bar{y} \in F(\bar{x})$ . Let  $(u, v) \in X \times Y$ , and  $\gamma \in (0, 1]$  be fixed. If*

$$\liminf_{\substack{(x,y) \rightarrow (\bar{x}, \bar{y}), \varphi(x,y) > 0 \\ (u,v) \\ \frac{\varphi(x,y)}{\|(x,y) - (\bar{x}, \bar{y})\|} \rightarrow 0}} |\nabla \varphi^\gamma(\cdot, y)|(x) > 0, \quad (12)$$

then there exist reals  $\tau, \delta > 0$  such that

$$d(x, F^{-1}(y)) \leq \tau [d(y, F(x))]^\gamma$$

for all  $(x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta))$  with  $d(y, F(x)) \leq \delta \|(x, y) - (\bar{x}, \bar{y})\|^{1/\gamma}$ .

That is,  $F$  is directionally metrically  $\gamma$ -regular at  $(\bar{x}, \bar{y})$  in the direction  $(u, v)$  with modulus  $\tau$ .

**Remark 3.1** Condition (12) of Theorem 3.2 fails to be a necessary condition when  $\gamma \in ]0, 1[$ . To see this, let us consider the mapping  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$F(x) = (x_1 - x_2)^3, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Here,  $\mathbb{R}^2$  is equipped with the Euclidean norm. For  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $y \in \mathbb{R}$ ,  $F^{-1}(y) = \{(t + \sqrt[3]{y}, t) : t \in \mathbb{R}\}$ . Therefore,  $d(x, F^{-1}(y)) = |x_1 - x_2 - \sqrt[3]{y}|/\sqrt{2}$ . Noticing that from the inequality  $0 \leq 3(a+b)^2$ , we deduce that  $(a-b)^2 \leq 4(a^2 + ab + b^2)$  and therefore that  $|a-b|^3 \leq 4|a^3 - b^3|$ , for all  $a, b \in \mathbb{R}$ . Using the last inequality yields

$$d(x, F^{-1}(y)) = |x_1 - x_2 - \sqrt[3]{y}|/\sqrt{2} \leq 2^{1/6} |(x_1 - x_2)^3 - y|^{1/3} = 2^{1/6} |y - F(x)|^{1/3} \quad \text{for all } x \in \mathbb{R}^2, y \in \mathbb{R}.$$

Consequently,  $F$  is metrically  $1/3$ -regular at  $(0, 0)$  (in the direction  $(0, 0)$ ). However, for any  $x = (x_1, x_1) \in \mathbb{R}^2$  and  $y \in \mathbb{R}$  with  $y \neq 0$ , one has  $|\nabla \varphi^{1/3}(\cdot, y)|(x) = 0$ , where,  $\varphi(x, y) = |y - F(x)|$ .

As stated in the next theorem, when  $\gamma = 1$ , then condition (12) becomes a necessary condition.

**Theorem 3.3** *Let  $X$  be a Banach space and  $Y$  be a normed space. Suppose a closed multifunction  $F : X \rightrightarrows Y$  and a point  $(\bar{x}, \bar{y}) \in X \times Y$  are given such that  $\bar{y} \in F(\bar{x})$ . Let us fix  $(u, v) \in X \times Y$ . Then, the following are equivalent:*

(i)

$$\liminf_{\substack{(x,y) \rightarrow (\bar{x}, \bar{y}), \varphi(x,y) > 0 \\ (u,v) \\ \frac{\varphi(x,y)}{\|(x,y) - (\bar{x}, \bar{y})\|} \rightarrow 0}} |\nabla \varphi(\cdot, y)|(x) > 0, \quad (13)$$

(ii) *there exist reals  $\tau > 0$  and  $\delta > 0$  such that*

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x))$$

for all  $(x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta))$  with  $d(y, F(x)) \leq \delta \|(x, y) - (\bar{x}, \bar{y})\|$ .

That is  $F$  is directionally metrically regular at  $(\bar{x}, \bar{y})$  in the direction  $(u, v)$  with modulus  $\tau$ .

*Proof* According to Theorem 3.2, condition (13) is sufficient to obtain (ii). Conversely, suppose that there exist  $\tau > 0$  and  $\delta \in ]0, 1[$  such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad (14)$$

for all  $(x, y) \in B((\bar{x}, \bar{y}), 2\delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), 2\delta))$  with  $d(y, F(x)) \leq 2\delta \|(x, y) - (\bar{x}, \bar{y})\|$ .

Let  $(x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta))$  be such that

$$0 < d(y, F(x)) \leq \frac{\delta^2}{4} \|(x, y) - (\bar{x}, \bar{y})\|. \quad (15)$$

Since  $\varphi(x, y) = \sup_{\varepsilon > 0} \inf_{w \in B(x, \varepsilon)} d(y, F(w)) = \liminf_{w \rightarrow x} d(y, F(w))$ , we have

$$\frac{\varphi(x, y)}{\|(x, y) - (\bar{x}, \bar{y})\|} \leq \frac{\delta^2}{4}$$

and we can write  $\varphi(x, y) = \lim_{n \rightarrow +\infty} d(y, F(u_n))$  for some sequence  $\{u_n\}$  in  $X$  converging to  $x$ . Without loss of generality, we can suppose that  $d(y, F(u_n)) > (1 - \frac{1}{n^2})\varphi(x, y)$ , and

$$\|u_n - x\| < \frac{1}{n} \varphi(x, y) \leq \frac{\delta^2 \|(x, y) - (\bar{x}, \bar{y})\|}{4n} \leq \frac{\delta^3}{4n} < \frac{\delta}{n} \quad (16)$$

and

$$d(y, F(u_n)) < \left(1 + \frac{1}{n}\right) \varphi(x, y). \quad (17)$$

Note also that for every  $n \in \mathbb{N}$ , there exists  $y_n \in F(u_n)$  such that

$$d(y, F(u_n)) \leq \|y - y_n\| < \left(1 + \frac{1}{n}\right) d(y, F(u_n)). \quad (18)$$

Set  $z_n := \frac{1+n^{\frac{1}{2}}}{1+n}y + \frac{n(1-n^{-\frac{1}{2}})}{1+n}y_n$ .

*Claim 3.1*

$$\|y - z_n\| < \left(1 - \frac{1}{n}\right) \varphi(x, y) \text{ for large } n. \quad (19)$$

Indeed,

$$\|y_n - z_n\| \leq \frac{1+n^{1/2}}{1+n} \|y - y_n\|, \quad (20)$$

and

$$\begin{aligned} \|y - z_n\| &= \frac{n(1-n^{-\frac{1}{2}})}{1+n} \|y - y_n\| < \frac{n(1-n^{-\frac{1}{2}})}{1+n} \left(1 + \frac{1}{n}\right) d(y, F(u_n)) \\ &< \left(1 - n^{-\frac{1}{2}}\right) d(y, F(u_n)) < \left(1 - n^{-\frac{1}{2}}\right) \left(1 + \frac{1}{n}\right) \varphi(x, y) \\ &\leq \left(1 - n^{-\frac{1}{2}}\right) \left(1 + n^{-\frac{1}{2}}\right) \varphi(x, y). \end{aligned}$$

Hence,  $\|y - z_n\| < (1 - \frac{1}{n})\varphi(x, y)$ , as claimed.

*Claim 3.2*  $z_n \notin F(u_n)$  for large  $n$ .

Indeed, using (19), if we suppose that  $z_n \in F(u_n)$  (for  $n \geq n_0$ ), we deduce that

$$\left(1 - \frac{1}{n^2}\right)\varphi(x, y) < d(y, F(u_n)) \leq \|y - z_n\| < \left(1 - \frac{1}{n}\right)\varphi(x, y),$$

we have a contradiction. Hence, as claimed, for  $n$  large,  $z_n \notin F(u_n)$ .

*Claim 3.3*

$$(\star\star) \quad (u_n, z_n) \in (\bar{x}, \bar{y}) + \text{cone } B((u, v), 2\delta) \text{ for large } n.$$

As  $(x, y) \in (\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta)$ , there is some  $\lambda > 0$  such that  $\frac{(x, y) - (\bar{x}, \bar{y})}{\lambda} \in B((u, v), \delta)$ . Then,

$$\lambda \geq \frac{\|(x, y) - (\bar{x}, \bar{y})\|}{\|(u, v)\| + \delta}.$$

Let us observe that

$$\frac{\|z_n - y\|}{\lambda} \stackrel{\text{by (19)}}{<} \frac{(1 - 1/n)\varphi(x, y)}{\lambda} < \frac{(1 - 1/n)\delta^2}{4n\lambda} \|(x, y) - (\bar{x}, \bar{y})\| \leq \frac{(1 - 1/n)\delta^2}{4n} (\|(u, v)\| + \delta) \leq \frac{(1 - 1/n)\delta^2}{4} \leq \frac{\delta}{2}.$$

Therefore, one has

$$\begin{aligned} & \left\| \frac{(u_n, z_n) - (\bar{x}, \bar{y})}{\lambda} - (u, v) \right\| \\ & \leq \left\| \frac{(u_n, z_n) - (x, y)}{\lambda} \right\| + \left\| \frac{(x, y) - (\bar{x}, \bar{y})}{\lambda} - (u, v) \right\| \\ & \leq \frac{\|u_n - x\| + \|z_n - y\|}{\lambda} + \delta \\ & \leq \frac{\delta}{\lambda n} + \frac{\delta}{2} + \delta < 2\delta. \end{aligned}$$

This yields,  $(\star\star)$ .

*Claim 3.4*

$$(\star\star\star) \quad d(z_n, F(u_n)) < 2\delta^2 \|(u_n, z_n) - (\bar{x}, \bar{y})\| \text{ for large } n.$$

We know that

$$(\clubsuit) \quad d(z_n, F(u_n)) < \frac{1 + n^{1/2}}{1 + n} \left(1 + \frac{1}{n}\right)^2 \varphi(x, y) \leq \frac{1 + n^{1/2}}{1 + n} \left(1 + \frac{1}{n}\right)^2 \frac{\delta^2}{4} \|(x, y) - (\bar{x}, \bar{y})\|.$$

We have:

$$\|(x, y) - (\bar{x}, \bar{y})\| \leq \|(u_n, z_n) - (x, y)\| + \|(u_n, z_n) - (\bar{x}, \bar{y})\|,$$

and

$$\|(u_n, z_n) - (x, y)\| = \|u_n - x\| + \|z_n - y\| < \frac{\delta^2}{4n} \|(x, y) - (\bar{x}, \bar{y})\| + (1 - 1/n) \frac{\delta^2}{4n} \|(x, y) - (\bar{x}, \bar{y})\| = \frac{\delta^2}{4n} \|(x, y) - (\bar{x}, \bar{y})\|,$$

it follows

$$(\heartsuit) \quad \|(x, y) - (\bar{x}, \bar{y})\| < \frac{1}{(1 - \frac{\delta^2}{4n})} \|(u_n, z_n) - (\bar{x}, \bar{y})\|.$$

Hence combining  $(\clubsuit)$  and  $(\heartsuit)$  we derive

$$d(z_n, F(u_n)) < \frac{1 + n^{1/2}}{1 + n} \left(1 + \frac{1}{n}\right)^2 \frac{\delta^2}{4} \|(x, y) - (\bar{x}, \bar{y})\| < \frac{1 + n^{1/2}}{1 + n} \left(1 + \frac{1}{n}\right)^2 \frac{\delta^2}{4(1 - \frac{\delta^2}{4n})} \|(u_n, z_n) - (\bar{x}, \bar{y})\|.$$

Observing that the quantity  $\frac{1 + n^{1/2}}{1 + n} \frac{(1 + \frac{1}{n})^2}{(1 - \frac{\delta^2}{4n})}$  tends to 0 as  $n$  tends to  $+\infty$ , we obtain  $(\star\star\star)$ , as desired.

Claim 3.5

$$\|(u_n, z_n) - (\bar{x}, \bar{y})\| \leq 2\delta \text{ for large } n. \quad (21)$$

$$\begin{aligned} \|(u_n, z_n) - (\bar{x}, \bar{y})\| &\leq \|(u_n, z_n) - (x, y)\| + \|(x, y) - (\bar{x}, \bar{y})\| \\ &\leq \frac{\delta}{n} + \left(1 - \frac{1}{n}\right)\varphi(x, y) + \|(x, y) - (\bar{x}, \bar{y})\| \\ &\leq \frac{\delta}{n} + \left(2 - \frac{1}{n}\right)\frac{\delta^2}{4}\|(x, y) - (\bar{x}, \bar{y})\| \\ &\leq 2\delta. \end{aligned}$$

Combining relations (21), (\*\*\*) and (\*\*\*) we see that the point  $(u_n, z_n)$  verifies (14). Hence by assumption we have

$$d(u_n, F^{-1}(z_n)) \leq \tau d(z_n, F(u_n)).$$

Next, select  $\tilde{x}_n \in F^{-1}(z_n)$  such that

$$\begin{aligned} \|\tilde{x}_n - u_n\| &\leq (1 + n^{-\frac{1}{2}})d(u_n, F^{-1}(z_n)) \\ &\leq \tau(1 + n^{-\frac{1}{2}})d(z_n, F(u_n)) \\ &\leq \tau(1 + n^{-\frac{1}{2}})\|z_n - y_n\| \\ &= \tau(1 + n^{-\frac{1}{2}})\frac{(1 + n^{\frac{1}{2}})}{1 + n}\|y - y_n\| \\ &= \tau\frac{2 + n^{\frac{1}{2}} + n^{-\frac{1}{2}}}{1 + n}\|y - y_n\|. \end{aligned} \quad (22)$$

Consequently,  $\|\tilde{x}_n - u_n\| < \tau\frac{2 + n^{\frac{1}{2}} + n^{-\frac{1}{2}}}{1 + n}\left(1 + \frac{1}{n}\right)^2\varphi(x, y)$  and therefore,  $\lim_{n \rightarrow \infty} \|\tilde{x}_n - x\| = 0$ . Next, for large  $n$ , we have the following estimate:

$$\begin{aligned} \varphi(x, y) - \varphi(\tilde{x}_n, y) &> \frac{n}{n+1} \left( d(y, F(u_n)) - d(y, F(\tilde{x}_n)) \right) \\ &> \frac{n^2}{(n+1)^2} \|y - y_n\| - \|y - z_n\|, \end{aligned}$$

and by the definition of  $z_n$ , we derive

$$\varphi(x, y) - \varphi(\tilde{x}_n, y) > \frac{n^{3/2} - n + n^{1/2}}{(n+1)^2} \|y - y_n\|.$$

Thus, using also (16), we obtain

$$\begin{aligned} \frac{\varphi(x, y) - \varphi(\tilde{x}_n, y)}{\|\tilde{x}_n - x\|} &\geq \frac{\varphi(x, y) - \varphi(\tilde{x}_n, y)}{\|u_n - x\| + \|\tilde{x}_n - u_n\|} \\ &> \frac{(n^{3/2} - n + n^{1/2})(n+1)^{-2}\|y - y_n\|}{\delta n^{-1} + \tau(2 + n^{\frac{1}{2}} + n^{-\frac{1}{2}})(n+1)^{-1}\|y - y_n\|} \\ &= \frac{n^{3/2} - n + n^{1/2}}{(2 + n^{\frac{1}{2}} + n^{-\frac{1}{2}})(n+1)} \frac{1}{\tau + \frac{\delta(n+1)}{n(2 + n^{1/2} + n^{-\frac{1}{2}})}\|y - y_n\|^{-1}}. \end{aligned}$$

Since  $\frac{n^{3/2}-n+n^{1/2}}{(2+n^{1/2}+n^{-1/2})(n+1)} \rightarrow 1$ ,  $\lim_{n \rightarrow \infty} \|y - y_n\| = \varphi(x, y) > 0$  and

$$\frac{\delta(n+1)}{n(2+n^{1/2}+n^{-1/2})} \cdot \frac{1}{\|y - y_n\|} \rightarrow 0,$$

we deduce that

$$|\nabla \varphi(\cdot, y)|(x) = \limsup_{u \rightarrow x} \frac{\varphi(x, y) - \varphi(u, y)}{\|x - u\|} \geq \limsup_{n \rightarrow \infty} \frac{\varphi(x, y) - \varphi(\tilde{x}_n, y)}{\|\tilde{x}_n - x\|} \geq \frac{1}{\tau},$$

establishing the proof.  $\square$

#### 4 Directional Metric Regularity under Perturbations

In [42, 43, 19, 44], it was established that metric regularity is stable under Lipschitz (single-valued or set-valued) perturbations with a suitable Lipschitz modulus. We shall show that directional metric regularity (with  $\gamma = 1$ ) is also stable under suitable Lipschitz perturbations. Recall that a mapping  $g : X \rightarrow Y$  between normed spaces is Hadamard differentiable at  $\bar{x} \in X$  with respect to the direction  $u \in X$ , if the following limit exists:

$$\lim_{\substack{t \downarrow 0 \\ w \rightarrow u}} \frac{g(\bar{x} + tw) - g(\bar{x})}{t} = Dg(\bar{x})(u).$$

Obviously, if  $g$  is locally Lipschitz around  $\bar{x}$ , then  $g$  is Hadamard differentiable at  $\bar{x}$  with respect to the direction 0 and  $Dg(\bar{x})(0) = 0$ .

**Theorem 4.1** *Let  $X$  be a Banach space,  $Y$  be a normed space. For a given  $(u, v) \in X \times Y$ , suppose  $F : X \rightrightarrows Y$  is a set-valued mapping which is directionally metrically regular at  $(\bar{x}, \bar{y}) \in \text{gph } F$  in direction  $(u, v)$  with modulus  $\tau > 0$ . If  $g : X \rightarrow Y$  is locally Lipschitz around  $\bar{x}$  with a constant  $\lambda > 0$  satisfying  $\lambda\tau < 1$  and if  $g$  is Hadamard differentiable at  $\bar{x}$  with respect to the direction  $u$ , then the set-valued mapping  $F + g$  is directionally metrically regular at  $(\bar{x}, \bar{y} + g(\bar{x}))$  in the direction  $(u, v + Dg(\bar{x})(u))$ .*

*Proof* Since  $g$  is locally Lipschitz around  $\bar{x}$  with constant  $\lambda$  and  $F$  is directionally metrically regular at  $(\bar{x}, \bar{y}) \in \text{gph } F$  in direction  $(u, v)$  with modulus  $\tau > 0$ , take  $\delta \in (0, \lambda)$  such that

$$\|g(x) - g(z)\| \leq \lambda \|x - z\| \quad \forall x, z \in B(\bar{x}, \delta),$$

and

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \text{for all } (x, y) \in B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta)), \quad (23)$$

with  $d(y, F(x)) \leq 2\delta \|(x, y) - (\bar{x}, \bar{y})\|$ .

Taking into account the Hadamard differentiability of  $g$  at  $\bar{x}$  with respect to the direction  $u$ , pick some  $\varepsilon \in ]0, \frac{\delta}{2}[$  and then some  $\eta \in (0, 1)$  such that

$$\left\| \frac{g(\bar{x} + tw) - g(\bar{x})}{t} - Dg(\bar{x})(u) \right\| < \varepsilon \quad \text{for all } t \in ]0, \eta[, w \in B(u, \eta). \quad (24)$$

Set

$$\varphi_{F+g}(x, y) = \liminf_{z \rightarrow x} d(y, F(z) + g(z)), \quad (x, y) \in X \times Y.$$

Take  $\rho > 0$  such that

$$\rho < \begin{cases} \min \left\{ \frac{\delta}{\lambda+3}, \delta - \varepsilon, \frac{\eta \|u\|}{1+\eta}, \frac{\delta}{2\tau+1} \right\}, & \text{if } u \neq 0 \\ \rho < \min \left\{ \frac{\delta}{\lambda+3}, \delta - \varepsilon, \frac{\delta}{2\tau+1} \right\}, & \text{if } u = 0 \end{cases}$$

and fix

$$(x, y) \in B((\bar{x}, \bar{y} + g(\bar{x})), \rho) \cap ((\bar{x}, \bar{y} + g(\bar{x})) + \text{cone } B((u, v + Dg(\bar{x})(u)), \rho))$$

with

$$0 < d(y, F(x) + g(x)) < \rho \|(x, y) - (\bar{x}, \bar{y} + g(\bar{x}))\|, \quad (25)$$

and select a sequence  $\{x_n\}$  converging to  $x$  such that  $\varphi_{F+g}(x, y) = \lim_{n \rightarrow \infty} d(y, F(x_n) + g(x_n))$ .

With the aim of making the proof clearer, we will establish some claims.

*Claim 4.1*

$$\|(x_n, y) - (\bar{x}, \bar{y} + g(\bar{x}))\| < \delta, \quad \text{for } n \text{ large.} \quad (26)$$

*Claim 4.2*

$$(x_n, y) \in (\bar{x}, \bar{y} + g(\bar{x})) + \text{cone } B((u, v + Dg(\bar{x})(u)), \rho) \quad \text{for } n \text{ large.} \quad (27)$$

Indeed, since  $(x, y) - (\bar{x}, \bar{y} + g(\bar{x})) \notin \text{cone } B((u, v + Dg(\bar{x})(u)), \rho) \setminus \{(0, 0)\}$ , for large  $n$ , one has

$$(x, y) - (\bar{x}, \bar{y} + g(\bar{x})) \in \text{int}(\text{cone } B((u, v + Dg(\bar{x})(u)), \rho))$$

It follows that  $(x_n, y) - (\bar{x}, \bar{y} + g(\bar{x})) \in \text{cone } B((u, v + Dg(\bar{x})(u)), \rho)$ .

*Claim 4.3*

$$0 < d(y, F(x_n) + g(x_n)) < \rho \|(x_n, y) - (\bar{x}, \bar{y} + g(\bar{x}))\| \quad \text{for } n \text{ large.} \quad (28)$$

Indeed, reasoning ad absurdum, suppose the existence of a subsequence still denoted by  $\{x_n\}$  such that

$$d(y, F(x_n) + g(x_n)) \geq \rho \|(x_n, y) - (\bar{x}, \bar{y} + g(\bar{x}))\|.$$

Since  $F$  has closed graph then

$$0 < \varphi_{F+g}(x, y) = \lim_{n \rightarrow +\infty} d(y, F(x_n) + g(x_n)) \leq d(y, F(x) + g(x)),$$

due to the fact that  $\{x_n\}$  converges to  $x$  and thanks to (25), we obtain a contradiction.

*Claim 4.4* For  $n$  sufficiently large,

$$(x_n, y - g(x_n)) \in B((\bar{x}, \bar{y}), \delta), \quad (29)$$

$$d(y - g(x_n), F(x_n)) \leq 2\delta \|(x_n, y - g(x_n)) - (\bar{x}, \bar{y})\|. \quad (30)$$

and

$$(x_n, y - g(x_n)) \in (\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta). \quad (31)$$

We know that

$$\|(x - \bar{x}, y - \bar{y} - g(\bar{x}))\| < \rho.$$

For  $n$  sufficiently large, one has  $\|x_n - x\| < \delta$  and

$$\begin{aligned} \|(x_n - \bar{x}, y - g(x_n) - \bar{y})\| &= \|x_n - \bar{x}\| + \|y - \bar{y} - g(x_n)\| \\ &\leq \|x_n - x\| + \|x - \bar{x}\| + \|y - \bar{y} - g(\bar{x})\| + \|g(\bar{x}) - g(x_n)\| \\ &\leq \|x_n - x\| + \|(x - \bar{x}, y - \bar{y} - g(\bar{x}))\| + \lambda \|x_n - x\| \\ &= (\lambda + 1)\|x_n - x\| + 2\rho \\ &< (\lambda + 3)\rho < \delta \quad (\text{since } \rho < \frac{\delta}{\lambda + 3}). \end{aligned}$$

Thus, one obtains (29). Next, we have

$$\begin{aligned} d(y - g(x_n), F(x_n)) &< \rho(\|(x_n, y - g(x_n)) - (\bar{x}, \bar{y})\| + \|g(x_n) - g(\bar{x})\|) \\ &\leq \rho(\|(x_n, y - g(x_n)) - (\bar{x}, \bar{y})\| + \lambda \|x_n - \bar{x}\|) \\ &\leq \rho(\lambda + 1)\|(x_n, y - g(x_n)) - (\bar{x}, \bar{y})\| \\ &< 2\delta\|(x_n, y - g(x_n)) - (\bar{x}, \bar{y})\| \quad (\text{since } \rho < \frac{\delta}{\lambda + 2}). \end{aligned} \tag{32}$$

So, we receive (30). By relation (27), for  $n$  sufficiently large, say,  $n \geq n_0$ , as  $(x_n, y_n) \neq (\bar{x}, \bar{y} + g(\bar{x}))$  we may find  $t_n > 0$  and

$(u_n, w_n) \in B((u, v + Dg(\bar{x})(u)), \rho)$  such that

$$x_n = \bar{x} + t_n u_n, \quad y = \bar{y} + g(\bar{x}) + t_n w_n.$$

Set

$$y - g(x_n) = \bar{y} + t_n v_n; \quad v_n = w_n - \frac{g(\bar{x} + t_n u_n) - g(\bar{x})}{t_n}.$$

If  $u = 0$  then

$$\begin{aligned} \|(u_n, v_n) - (0, v)\| &= \|(u_n, (v_n - w_n + Dg(\bar{x})) + (w_n - v - Dg(\bar{x}))\| \\ &\leq \|(u_n w_n - v - Dg(\bar{x}))\| + \|w_n - v_n - Dg(\bar{x})\| \\ &\leq \rho + \left\| \frac{g(\bar{x} + t_n u_n) - g(\bar{x})}{t_n} - Dg(\bar{x}) \right\| \\ &< \rho + \varepsilon < \delta; \end{aligned}$$

otherwise,  $u \neq 0$ , since  $u_n \in B(u, \rho)$ , one has

$$t_n = \frac{\|x_n - \bar{x}\|}{\|u_n\|} \leq \frac{\rho}{\|u\| - \rho} < \eta,$$

and therefore, by (24),

$$\begin{aligned} \|(u_n, v_n) - (u, v)\| &\leq \|(u_n, w_n) - (u, v + Dg(\bar{x})(u))\| + \|w_n - v_n - Dg(\bar{x})(u)\| \\ &\leq \rho + \left\| \frac{g(\bar{x} + t_n u_n) - g(\bar{x})}{t_n} - Dg(\bar{x})(u) \right\| \leq \rho + \varepsilon < \delta. \end{aligned}$$



Now, pick  $z_n \in F^{-1}(y - g(x_n))$  such that

$$\|x_n - z_n\| \leq (1 + 1/n)d(x_n, F^{-1}(y - g(x_n))).$$

Hence, according to (23), (29), (30), (31), we obtain,

$$\|x_n - z_n\| \leq (1 + 1/n)\tau d(y - g(x_n), F(x_n)). \quad (33)$$

Consequently,

$$\begin{aligned} \|z_n - \bar{x}\| &\leq \|x_n - z_n\| + \|x_n - \bar{x}\| \\ &\leq (1 + 1/n)\tau\rho\|(x_n, y) - (\bar{x}, \bar{y} + g(\bar{x}))\| + \|x_n - \bar{x}\| \\ &< (1 + 1/n)\tau\rho^2 + \rho < (1 + 1/n)\tau\rho + \rho \\ &< 2\tau\rho + \rho = \rho(1 + 2\tau) \\ &< \delta. \end{aligned}$$

Using the local Lipschitz continuity around  $\bar{x}$  of  $g$  gives,  $\|g(z_n) - g(x_n)\| \leq \lambda\|z_n - x_n\|$ . As  $y - g(x) \notin F(x)$  and  $\lim_{n \rightarrow \infty} x_n = x$ , then  $\liminf_{n \rightarrow \infty} \|x_n - z_n\| > 0$ . Hence, by relation (33), one has

$$\begin{aligned} |\Gamma\varphi_{F+g}(\cdot, y)|(x) &\geq \limsup_{n \rightarrow \infty} \frac{\varphi_{F+g}(x, y) - \varphi_{F+g}(z_n, y)}{\|x - z_n\|} \\ &\geq \limsup_{n \rightarrow \infty} \frac{d(y - g(x_n), F(x_n)) - d(y - g(z_n), F(z_n))}{\|x_n - z_n\|} \\ &\geq \limsup_{n \rightarrow \infty} \frac{d(y - g(x_n), F(x_n)) - d(y - g(x_n), F(z_n)) - \|g(x_n) - g(z_n)\|}{\|x_n - z_n\|} \\ &\geq \limsup_{n \rightarrow \infty} \frac{d(y - g(x_n), F(x_n))}{\|x_n - z_n\|} - \lambda \\ &\geq \limsup_{n \rightarrow \infty} \tau^{-1} \frac{n}{n+1} - \lambda = \tau^{-1} - \lambda. \end{aligned}$$

Therefore,

$$\begin{aligned} \liminf_{(x, y) \xrightarrow{(u, v + Dg(\bar{x})(u))} (\bar{x}, \bar{y} + g(\bar{x}))} |\Gamma\varphi_{F+g}(\cdot, y)|(x) &\geq \tau^{-1} - \lambda. \\ x \notin (F + g)^{-1}(y), \frac{\varphi_{F+g}(x, y)}{\|(x, y) - (\bar{x}, \bar{y})\|} &\rightarrow 0 \end{aligned}$$

Thanks to Theorem 3.1, the proof is complete.  $\square$

## 5 Stability of Directional Hölder Metric Regularity under Mixed Coderivative-Tangency Conditions

For the usual metric regularity, sufficient conditions in terms of coderivatives have been given by various authors, for instance, in [9, 21, 4, 24]. In this section, we establish a characterization of directional Hölder metric regularity using the Fréchet subdifferential in Asplund spaces, i.e., in Banach spaces in which every convex continuous function is generically Fréchet differentiable. There are many equivalent descriptions of Asplund spaces, see, e.g., Mordukhovich's book [4] and its bibliography. In particular, any reflexive space is Asplund, as well as each Banach space such that each of its separable subspaces has a separable dual. We shall show that the proposed characterization also ensures the

stability of directional Hölder metric regularity under suitable differentiable perturbations. In this section, in order to formulate some coderivative characterizations of directional Hölder metric regularity, some additional definitions are required. Let  $X$  be a Banach space. Consider now an extended-real-valued function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ . The *Fréchet subdifferential* of  $f$  at  $\bar{x} \in \text{Dom } f$  is given as

$$\partial f(\bar{x}) = \left\{ x^* \in X^* : \liminf_{\substack{x \rightarrow \bar{x}, \\ x \neq \bar{x}}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

For the convenience of the reader, we would like to mention that the terminology *regular subdifferential* instead of Fréchet subdifferential is also popular due to its use in Rockafellar and Wets [36]. Every element of the Fréchet subdifferential is termed as a Fréchet (regular) subgradient. If  $\bar{x}$  is a point where  $f(\bar{x}) = \infty$ , then we set  $\partial f(\bar{x}) = \emptyset$ . In fact one can show that an element  $x^*$  is a Fréchet subgradient of  $f$  at  $\bar{x}$  iff

$$f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \quad \text{where} \quad \lim_{x \rightarrow \bar{x}} \frac{o(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = 0.$$

It is well-known that the Fréchet subdifferential satisfies a fuzzy sum rule in Asplund spaces (see [4], Theorem 2.33). More precisely, if  $X$  is an Asplund space and  $f_1, f_2 : X \rightarrow \mathbb{R} \cup \{\infty\}$  are such that  $f_1$  is Lipschitz continuous around  $\bar{x} \in \text{Dom } f_1 \cap \text{Dom } f_2$  and  $f_2$  is lower semicontinuous around  $\bar{x}$ , then for any  $\gamma > 0$  one has

$$\partial(f_1 + f_2)(\bar{x}) \subset \bigcup \{ \partial f_1(x_1) + \partial f_2(x_2) \mid x_i \in \bar{x} + \gamma \overline{B}_X, |f_i(x_i) - f_i(\bar{x})| \leq \gamma, i = 1, 2 \} + \gamma B_{X^*}. \quad (34)$$

For a nonempty closed set  $C \subseteq X$ , denote by  $\delta_C$  the *indicator function* associated with  $C$  (i.e.  $\delta_C(x) = 0$ , when  $x \in C$  and  $\delta_C(x) = \infty$  otherwise). The *Fréchet normal cone* to  $C$  at  $\bar{x}$  is denoted by  $N(C, \bar{x})$ . It is a closed and convex object in  $X^*$  which is defined as  $\partial \delta_C(\bar{x})$ . Equivalently a vector  $x^* \in X^*$  is a Fréchet normal to  $C$  at  $\bar{x}$  if

$$\langle x^*, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|), \quad \forall x \in C,$$

where  $\lim_{x \rightarrow \bar{x}} \frac{o(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = 0$ . Let  $F : X \rightrightarrows Y$  be a set-valued map and  $(x, y) \in \text{gph } F$ . Then the *Fréchet coderivative* at  $(x, y)$  is the set-valued map  $D^*F(x, y) : Y^* \rightrightarrows X^*$  given by

$$D^*F(x, y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\text{gph } F, (x, y))\}.$$

In the spirit of Gfrerer [45], see also Kruger [38], Ngai-Tinh [46] we introduce the following *limit set for directional Hölder metric regularity of order  $\gamma$*  ( $\gamma \in (0, 1]$ ).

**Definition 5.1** Let  $F : X \rightrightarrows Y$  be a closed multifunction and  $(\bar{x}, \bar{y}) \in \text{gph } F$  be fixed. For a given  $(u, v) \in X \times Y$  and each  $\gamma \in (0, 1]$ , the *critical limit set* for metric regularity of order  $\gamma$  of  $F$  in the direction  $(u, v)$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  is denoted by  $\text{Cr}^\gamma F((\bar{x}, \bar{y}), (u, v))$  and is defined as the set of all  $(w, x^*) \in Y \times X^*$  such that there exist sequences  $\{t_n\} \downarrow 0$ ,  $\{\varepsilon_n\} \downarrow 0$ ,  $(u_n, v_n) \in \text{cone } B((u, v), \varepsilon_n)$  with  $\|(u_n, v_n)\| = 1$ ,  $w_n \in Y$ ,  $y_n^* \in \mathcal{S}_{Y^*}$  (the unit sphere of  $Y^*$ ),  $x_n^* \in D^*F(\bar{x} + t_n u_n, \bar{y} + t_n v_n + t_n^{1/\gamma} w_n)(y_n^*)$  with

$$\frac{\langle y_n^*, w_n \rangle}{\|w_n\|} \rightarrow 1 \quad \text{and} \quad (w_n, t_n^{(\gamma-1)/\gamma} \|w_n\|^{\gamma-1} x_n^*) \rightarrow (w, x^*).$$

For a closed multifunction  $F : X \rightrightarrows Y$ , we define the *coderivative slope* of  $F$  at  $(x, y) \in \text{gph } F$  as the following quantity:

$$m_F(x, y) = \inf \{ \|x^*\| : x^* \in D^*F(x, y)(y^*), y^* \in \mathcal{S}_Y^* \}.$$

By applying Theorem 4.1, we receive the following result.

**Theorem 5.1** *Let  $X$  and  $Y$  be Asplund spaces and suppose given a closed multifunction  $F : X \rightrightarrows Y$ , as well as  $(\bar{x}, \bar{y}) \in \text{gph } F$ ,  $(u, v) \in X \times Y$  and  $\gamma \in ]0, 1[$ . If  $(0, 0) \notin \text{Cr}^\gamma F((\bar{x}, \bar{y}), (u, v))$ , then for any mapping  $g : X \rightarrow Y$ , differentiable in a neighborhood of  $\bar{x}$ , and verifying for some  $c \in ]0, 1[$ ,  $\delta > 0$*

$$\|Dg(x)\| \leq cm_F(x, y) \quad \forall (x, y) \in \text{gph } F \cap B((\bar{x}, \bar{y}), \delta) \cap ((\bar{x}, \bar{y}) + \text{cone } B((u, v), \delta)), \quad (35)$$

*the multifunction  $F + g$  is directionally Hölder metrically regular of order  $\gamma$  at  $(\bar{x}, \bar{y} + g(\bar{x}))$  in the direction  $(u, v + Dg(\bar{x})(u))$ . In particular,  $F$  is directionally Hölder metrically regular of order  $\gamma$  at  $(\bar{x}, \bar{y})$  in the direction  $(u, v)$ .*

*Proof* gAssume on the contrary that there is a differentiable function  $g : X \rightarrow Y$ , verifying (35) for some  $c \in (0, 1)$  and  $\delta > 0$ , such that Hölder metric regularity of order  $\gamma$  at  $(\bar{x}, \bar{y} + g(\bar{x}))$  in the direction  $(u, v + Dg(\bar{x})(u))$  fails for  $F + g$ . According to Theorem 3.2, setting

$$\varphi(x, y) = \liminf_{u \rightarrow x} d(y, F(u) + g(u)) = \liminf_{u \rightarrow x} d(y - g(x), F(u)), \quad (x, y) \in X \times Y,$$

we can find a sequence  $\{(x_n, y_n)\} \subseteq X \times Y$  and a sequence of nonnegative reals  $\{\delta_n > 0\}$  (we can set  $\delta_n = \frac{1}{n}$ ) such that

$$\varphi(x_n, y_n) > 0, \quad \delta_n \downarrow 0, \quad \|x_n - \bar{x}\| < \delta_n, \quad \|y_n - \bar{y} - g(\bar{x})\| < \delta_n;$$

$$(x_n, y_n) \in (\bar{x}, \bar{y} + g(\bar{x})) + \text{cone } B((u, v + Dg(\bar{x})(u)), \delta_n); \quad (36)$$

$$\lim_{n \rightarrow \infty} \frac{\varphi^\gamma(x_n, y_n)}{\|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\|} = 0,$$

and

$$|\nabla \varphi^\gamma(\cdot, y_n)|(x_n) < \delta_n, \quad \forall n \in \mathbb{N}.$$

We can assume that  $\delta_n \in (0, \delta/4)$  for all  $n \in \mathbb{N}$ . Then, for each  $n \in \mathbb{N}$ , there is  $\eta_n \in (0, \delta_n)$  with

$$\eta_n / \varphi(x_n, y_n) \rightarrow 0; \quad 2\eta_n < \varphi(x_n, y_n)(1 - \delta_n); \quad \eta_n^2/4 + 5\eta_n/4 < \|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\| \quad \forall n \in \mathbb{N} \quad (37)$$

such that

$$d(y_n - g(z), F(z)) \geq \varphi(x_n, y_n)(1 - \delta_n), \quad \forall z \in B(x_n, 4\eta_n),$$

and

$$\delta_n \geq \frac{\varphi^\gamma(x_n, y_n) - \varphi^\gamma(z, y_n)}{\|x_n - z\|} \quad \text{for all } z \in \bar{B}(x_n, \eta_n).$$

Equivalently,

$$\varphi^\gamma(x_n, y_n) \leq \varphi^\gamma(z, y_n) + \delta_n \|z - x_n\| \quad \text{for all } z \in \bar{B}(x_n, \eta_n).$$

Since

$$\eta_n / \varphi(x_n, y_n) \rightarrow 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\varphi^\gamma(x_n, y_n)}{\|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\|} = 0,$$

one has

$$\lim_{n \rightarrow \infty} \frac{\eta_n^\gamma}{\|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\|} = 0. \quad (38)$$

Take  $z_n \in B(x_n, \eta_n^2/4)$ ,  $w_n \in F(z_n) + g(z_n)$  such that

$$\|y_n - w_n\|^\gamma \leq \varphi(x_n, y_n)^\gamma + \eta_n^2/4.$$

Then,

$$\|y_n - w_n\|^\gamma \leq \varphi^\gamma(z, y_n) + \delta_n \|z - x_n\| + \eta_n^2/4, \quad \forall z \in \bar{B}(x_n, \eta_n).$$

By

$$\varphi^\gamma(z, y_n) \leq [d(y_n, (F + g)(z))]^\gamma \leq \|y_n - w\|^\gamma + \delta_{\text{gph}(F+g)}(z, w), \quad \forall w \in Y.$$

one has

$$\|y_n - w_n\|^\gamma \leq \|y_n - w\|^\gamma + \delta_{\text{gph}(F+g)}(z, w) + \delta_n \|z - z_n\| + (\delta_n + 1)\eta_n^2/4$$

for all  $(z, w) \in \bar{B}(x_n, \eta_n) \times Y$ .

Applying the Ekeland variational principle to the function

$$(z, w) \mapsto \|y_n - w\|^\gamma + \delta_{\text{gph}(F+g)}(z, w) + \delta_n \|z - z_n\|$$

on  $\bar{B}(x_n, \eta_n) \times Y$ , we can select  $(z_n^1, w_n^1) \in (z_n, w_n) + \frac{\eta_n}{4} B_{X \times Y}$  with  $(z_n^1, w_n^1) \in \text{gph}(F + g)$  such that

$$\|y_n - w_n^1\|^\gamma + \delta_n \|z_n^1 - z_n\| \leq \|y_n - w_n\|^\gamma (\leq \varphi^\gamma(x_n, y_n) + \eta_n^2/4); \quad (39)$$

and such that the function

$$(z, w) \mapsto \|y_n - w\|^\gamma + \delta_{\text{gph}(F+g)}(z, w) + \delta_n \|z - z_n\| + (\delta_n + 1)\eta_n \|(z, w) - (z_n^1, w_n^1)\|$$

attains a minimum on  $\bar{B}(x_n, \eta_n) \times Y$  at  $(z_n^1, w_n^1)$ . Observing that the functions

$$(z, w) \rightarrow \|y_n - w\|^\gamma, \quad (z, w) \rightarrow \|z - z_n\| \quad \text{and} \quad (z, w) \rightarrow \|(z, w) - (z_n^1, w_n^1)\|$$

are locally Lipschitz around  $(z_n^1, w_n^1)$ , thanks to the fuzzy sum rule, we can select

$$w_n^2 \in B_Y(w_n^1, \eta_n); \quad (z_n^3, w_n^3) \in B_{X \times Y}((z_n^1, w_n^1), \eta_n) \cap \text{gph}(F + g);$$

$$w_n^{2*} \in \partial(\|y_n - \cdot\|^\gamma)(w_n^2); \quad (z_n^{3*}, -w_n^{3*}) \in N(\text{gph}(F + g), (z_n^3, w_n^3))$$

satisfying

$$w_n^{3*} = w_n^{2*} + (\delta_n + 2)\eta_n w_n^{4*},$$

$$\|w_n^{2*} - w_n^{3*}\| < (\delta_n + 2)\eta_n, \quad (40)$$

and,

$$\|z_n^{3*}\| \leq \delta_n + (\delta_n + 2)\eta_n.$$

Since  $\|y_n - w_n^2\| \geq \|y_n - w_n\| - \|w_n^2 - w_n\| \geq \varphi(x_n, y_n)(1 - \delta_n) - 2\eta_n > 0$ , then

$$w_n^{2*} \in \partial(\|y_n - \cdot\|^\gamma)(w_n^2) = \gamma\|y_n - w_n^2\|^{\gamma-1} \partial(\|y_n - \cdot\|)(w_n^2),$$

and therefore,

$$w_n^{2*} = \gamma\|y_n - w_n^2\|^{\gamma-1} t_n^{2*}$$

with  $\|t_n^{2*}\| = 1$  and  $\langle t_n^{2*}, w_n^2 - y_n \rangle = \|y_n - w_n^2\|$ . Thus, from (40), it follows that

$$\begin{aligned} \|w_n^{3*}\| &\geq \|w_n^{2*}\| - (\delta_n + 2)\eta_n \\ &= \gamma\|y_n - w_n^2\|^{\gamma-1} \|t_n^{2*}\| - (\delta_n + 2)\eta_n \\ &= \gamma\|y_n - w_n^2\|^{\gamma-1} - (\delta_n + 2)\eta_n > 0, \end{aligned}$$

$$\begin{aligned} \|w_n^{3*}\| &\leq \|w_n^{2*}\| + (\delta_n + 2)\eta_n \\ &= \gamma\|y_n - w_n^2\|^{\gamma-1} \|t_n^{2*}\| + (\delta_n + 2)\eta_n \\ &= \gamma\|y_n - w_n^2\|^{\gamma-1} + (\delta_n + 2)\eta_n. \end{aligned}$$

Since

$$\begin{aligned} t_n &\geq \|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\| - \|z_n^3 - x_n\| \\ &\geq \|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\| - \eta_n^2/4 - 5\eta_n/4 > 0, \end{aligned}$$

it makes sense to set

$$t_n = \|(z_n^3, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\|; \quad (u_n, v_n) = (z_n^3 - \bar{x}, y_n - \bar{y} - g(\bar{x}))/t_n; \quad \zeta_n = (w_n^3 - y_n)/t_n^{\frac{1}{\gamma}}, \quad (41)$$

and

$$y_n^* = w_n^{3*}/\|w_n^{3*}\|; \quad x_n^* = z_n^{3*}/\|z_n^{3*}\|.$$

As  $(z_n^3, y_n) \rightarrow (\bar{x}, \bar{y} + g(\bar{x}))$  and  $(z_n^3, y_n) \neq (\bar{x}, \bar{y} + g(\bar{x}))$  for  $n$  sufficiently large, then  $(t_n) \downarrow 0$  as  $n \rightarrow \infty$ . Since

$$\begin{aligned} \varphi(x_n, y_n)(1 - \delta_n) &\leq d(y_n, (F + g)(\bar{x} + t_n u_n)) \leq t_n^{\frac{1}{\gamma}} \|\zeta_n\| \\ &\leq \|y_n - w_n^1\| + \eta_n \\ &\leq \varphi(x_n, y_n) + \eta_n^2/4 + \eta_n, \end{aligned}$$

one has

$$\|\zeta_n\| \leq \frac{\varphi(x_n, y_n) + \eta_n^2/4 + \eta_n}{(\|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\| - \eta_n^2/4 - 5\eta_n/4)^{1/\gamma}}.$$

As  $\varphi^\gamma(x_n, y_n)/\|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\| \rightarrow 0$  as well as  $\eta_n^\gamma/\|(x_n, y_n) - (\bar{x}, \bar{y} + g(\bar{x}))\| \rightarrow 0$ , one obtains

$$\lim_{n \rightarrow \infty} \zeta_n = 0. \quad (42)$$

Next, by using the standard formula for the Fréchet coderivative of  $F + g$ , one has

$$\begin{aligned} x_n^* &\in D^*(F + g)(\bar{x} + t_n u_n, \bar{y} + g(\bar{x}) + t_n v_n + t_n^{\frac{1}{\gamma}} \zeta_n)(y_n^*) \\ &= D^*F(\bar{x} + t_n u_n, \bar{y} + g(\bar{x}) - g(\bar{x} + t_n u_n) + t_n v_n + t_n^{\frac{1}{\gamma}} \zeta_n)(y_n^*) + Dg^*(\bar{x} + t_n u_n)y_n^*. \end{aligned}$$

In other words,

$$z_n^* := x_n^* - Dg^*(\bar{x} + t_n u_n) y_n^* \in D^*F(\bar{x} + t_n u_n, \bar{y} + g(\bar{x}) - g(\bar{x} + t_n u_n) + t_n v_n + t_n^{\frac{1}{\gamma}} \zeta_n)(y_n^*). \quad (43)$$

From (41) and relations

$$(36), \lim_{n \rightarrow \infty} \zeta_n = 0; g(\bar{x} + t_n u_n) - g(\bar{x}) = t_n Dg(\bar{x})(u_n) + o(t_n),$$

by checking directly, we derive that

$$\left( u_n, \frac{g(\bar{x}) - g(\bar{x} + t_n u_n)}{t_n} + v_n \right) \in \text{cone } B((u, v), \varepsilon_n), \quad (44)$$

and

$$(t_n u_n, g(\bar{x}) - g(\bar{x} + t_n u_n) + t_n v_n + t_n^{\frac{1}{\gamma}} \zeta_n) \in \text{cone } B((u, v), \varepsilon_n),$$

for some sequence  $\{\varepsilon_n\} \downarrow 0$ .

Therefore, by (35), one has (for  $n$  sufficiently large)

$$\|x_n^*\| = \|z_n^* + Dg^*(\bar{x} + t_n u_n) y_n^*\| \geq \|z_n^*\| - \|Dg^*(\bar{x} + t_n u_n)\| \geq (1 - c) \|z_n^*\|.$$

By (36), one derives that

$$\begin{aligned} t_n^{(\gamma-1)/\gamma} \|\zeta_n\|^{\gamma-1} \|x_n^*\| &= t_n^{(\gamma-1)/\gamma} \|\zeta\|^{\gamma-1} \|z_n^*\| / \|w_n^{3*}\| \\ &\leq \frac{t_n^{(\gamma-1)/\gamma} \|\zeta_n\|^{\gamma-1} (\delta_n + (\delta_n + 2)\eta_n)}{\gamma \|y_n - w_n^2\|^{\gamma-1} - (\delta_n + 2)\eta_n} \\ &\leq \frac{\|y_n - w_n^3\|^{\gamma-1} (\delta_n + (\delta_n + 2)\eta_n)}{\gamma (\|y_n - w_n^2\| - 2\eta_n)^{\gamma-1} - (\delta_n + 2)\eta_n}. \end{aligned} \quad (45)$$

Observing that

$$\|y_n - w_n^2\| \geq \|y_n - w_n\| - \|w_n - w_n^2\| \geq \varphi(x_n, y_n) - \delta_n - 2\eta_n > 0,$$

then,

$$\lim_{n \rightarrow \infty} \frac{\eta_n}{\|y_n - w_n^2\|} \leq \lim_{n \rightarrow \infty} \frac{\eta_n}{\varphi(x_n, y_n) - \delta_n - 2\eta_n} = 0.$$

Therefore, (45) yields

$$\lim_{n \rightarrow \infty} t_n^{(\gamma-1)/\gamma} \|\zeta_n\|^{\gamma-1} \|x_n^*\| \leq \lim_{n \rightarrow \infty} (1 - c)^{-1} t_n^{(\gamma-1)/\gamma} \|\zeta\|^{\gamma-1} \|x_n^*\| = 0. \quad (46)$$

Next, one has

$$\begin{aligned} \frac{\langle y_n^*, \zeta_n \rangle}{\|\zeta_n\|} &= \frac{\langle y_n^*, w_n^3 - y_n \rangle}{\|w_n^3 - y_n\|} \\ &\geq \frac{\langle w_n^{3*}, w_n^3 - y_n \rangle}{\|w_n^{3*}\| \|w_n^3 - y_n\|} \\ &\geq \frac{\langle w_n^{2*}, w_n^2 - y_n \rangle - \|w_n^{2*} - w_n^{3*}\| \|w_n^2 - y_n\| - \|w_n^{2*}\| \|w_n^2 - w_n^3\|}{\|w_n^{2*}\| \|w_n^2 - y_n\| + \|w_n^{2*}\| \|w_n^3 - w_n^2\| + \|w_n^3 - w_n^2\| \|w_n^{2*} - w_n^{3*}\| + \|w_n^2 - w_n^3\| \|w_n^{2*}\| \|w_n^2 - y_n\|} \\ &\geq \frac{\gamma \|w_n^2 - y_n\|^\gamma - (\delta_n + 2)\eta_n (\varphi(x_n, y_n) + \delta_n + 2\eta_n) - 2\eta_n \gamma \|y_n - w_n^2\|^{\gamma-1}}{\gamma \|w_n^2 - y_n\|^\gamma + 2\eta_n \gamma \|y_n - w_n^2\|^{\gamma-1} + 2\eta_n^2 (\delta_n + 2) + (\delta_n + 2)\eta_n (\varphi(x_n, y_n) + \delta_n + 2\eta_n)}. \end{aligned}$$

Here, we use the following relations

$$\langle w_n^{2*}, w_n^2 - y_n \rangle = \gamma \|w_n^2 - y_n\|^\gamma; \|w_n^{2*}\| = \gamma \|w_n^2 - y_n\|^{(\gamma-1)}$$

$$\|y_n - w_n^2\| \geq \|y_n - w_n\| - \|w_n - w_n^2\| \geq \varphi(x_n, y_n)(1 - \delta_n) - 2\eta_n > 0.$$

Hence, since

$$\frac{\eta_n}{\|y_n - w_n^2\|} \leq \frac{\eta_n}{\varphi(x_n, y_n)(1 - \delta_n) - 2\eta_n} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

one obtains

$$\lim_{n \rightarrow \infty} \frac{\langle y_n^*, \zeta_n \rangle}{\|\zeta_n\|} = 1. \quad (47)$$

Relations (42), (43), (44), (46), (47) show that  $(0, 0) \in \text{Cr}^\gamma F((\bar{x}, \bar{y}), (u, v))$ , completing the proof.  $\square$

An open question is whether the inverse implication is true in general? In the particular case when  $F$  is a closed multifunction and either  $F$  is a closed convex multifunction (i.e.,  $\text{gph } F$  is closed convex) or  $\gamma = 1$ , the inverse holds as shown in the following proposition.

**Proposition 5.1** *Let  $X, Y$  be Banach spaces and let  $(u, v) \in X \times Y$ ,  $\gamma \in (0, 1]$  be given.*

- (i) *If  $F : X \rightrightarrows Y$  is a closed convex multifunction, which is directionally Hölder metrically regular of order  $\gamma$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  in the direction  $(u, v)$ , then  $(0, 0) \notin \text{Cr}^\gamma F((x, y), (u, v))$ ;*
- (ii) *If a closed multifunction  $F : X \rightrightarrows Y$  is directionally metrically regular at  $(\bar{x}, \bar{y}) \in \text{gph } F$  in the direction  $(u, v)$ , then  $(0, 0) \notin \text{Cr} F((x, y), (u, v))$ .*

*Proof (i).* Let us consider sequences  $\{t_n\} \downarrow 0$ ;  $\{\varepsilon_n\} \downarrow 0$ ;  $(u_n, v_n) \in \text{cone } B((u, v), \varepsilon_n)$  with  $\|(u_n, v_n)\| = 1$ ;  $\zeta_n \in Y$ ;  $y_n^* \in \mathcal{S}_{Y^*}$  and  $x_n^* \in X^*$  such that

$$\|\zeta_n\| \rightarrow 0; \quad \frac{\langle y_n^*, \zeta_n \rangle}{\|\zeta_n\|} \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

$$x_n^* \in D^* F(\bar{x} + t_n u_n, \bar{y} + t_n v_n + t_n^{1/\gamma} \zeta_n)(y_n^*).$$

Thanks to the convexity of  $F$ ,

$$\langle x_n^*, x - \bar{x} - t_n u_n \rangle - \langle y_n^*, y - \bar{y} - t_n v_n - t_n^{1/\gamma} \zeta_n \rangle \leq 0 \quad \forall (x, y) \in \text{gph } F. \quad (48)$$

Since  $F$  is directionally Hölder metrically regular of order  $\gamma$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  in the direction  $(u, v)$ , with some constant  $\tau > 0$ , and,

$$\frac{d(\bar{y} + t_n v_n, F(\bar{x} + t_n u_n))}{(t_n \|(u_n, v_n)\|)^{1/\gamma}} \leq \|\zeta_n\| \rightarrow 0,$$

for  $n$  sufficiently large, one has

$$\begin{aligned} d(\bar{x} + t_n u_n, F^{-1}(\bar{y} + t_n v_n)) &\leq \tau d(\bar{y} + t_n v_n, F(\bar{x} + t_n u_n))^\gamma \\ &\leq \tau \|\bar{y} + t_n v_n - \bar{y} - t_n v_n - t_n^{1/\gamma} \zeta_n\|^\gamma = \tau t_n \|\zeta_n\|^\gamma. \end{aligned}$$

Thus, we can find  $x_n \in F^{-1}(\bar{y} + t_n v_n)$  such that

$$\|x_n - \bar{x} - t_n u_n\| \leq \tau(1 + t_n)t_n \|\zeta_n\|^\gamma.$$

Taking into account (48), one obtains

$$\|x_n^*\| \tau(1+t_n)t_n \|\zeta_n\|^\gamma \geq \|x_n^*\| \|x_n - \bar{x} - t_n u_n\| \geq \langle x_n^*, \bar{x} + t_n u_n - x_n \rangle \geq t_n^{1/\gamma} \langle y_n^*, \zeta_n \rangle.$$

Therefore,

$$\liminf_{n \rightarrow \infty} \|x_n^*\| t_n^{(\gamma-1)/\gamma} \|\zeta_n\|^{\gamma-1} \geq \lim_{n \rightarrow \infty} \tau^{-1} (1+t_n)^{-1} \frac{\langle y_n^*, \zeta_n \rangle}{\|\zeta_n\|} = \tau^{-1},$$

which shows that  $(0, 0) \notin \text{Cr}^\gamma F((\bar{x}, \bar{y}), (u, v))$ .

(ii). Suppose that  $F$  is directionally metrically regular at  $(\bar{x}, \bar{y}) \in \text{gph } F$  in the direction  $(u, v)$ . Let be given sequences  $\{t_n\} \downarrow 0$ ;  $\{\varepsilon_n\} \downarrow 0$ ;  $(u_n, v_n) \in \text{cone } B((u, v), \varepsilon_n)$  with  $\|(u_n, v_n)\| = 1$ ;  $\zeta_n \in Y$ ;  $y_n^* \in \mathcal{S}_{Y^*}$  and  $x_n^* \in X^*$  such that

$$\|\zeta_n\| \rightarrow 0; \quad \frac{\langle y_n^*, \zeta_n \rangle}{\|\zeta_n\|} \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

$$x_n^* \in D^* F(\bar{x} + t_n u_n, \bar{y} + t_n v_n + t_n \zeta_n)(y_n^*).$$

Then, there is a sequence of nonnegative reals  $\{\delta_n\}$  with  $\delta_n \in ]0, t_n[$  such that

$$\begin{aligned} \langle x_n^*, x - \bar{x} - t_n u_n \rangle - \langle y_n^*, y - \bar{y} - t_n v_n - t_n \zeta_n \rangle &\leq \varepsilon_n \|((x, y) - (\bar{x} + t_n u_n, \bar{y} + t_n v_n + t_n \zeta_n))\| \\ &\text{for all } (x, y) \in \text{gph } F \text{ with } \|(x, y) - (\bar{x} + t_n u_n, \bar{y} + t_n v_n + t_n \zeta_n)\| < \delta_n. \end{aligned} \quad (49)$$

By setting  $y_n = \bar{y} + t_n v_n + (t_n - \delta_n)\zeta_n$ , then  $(\bar{x} + t_n u_n, y_n) \xrightarrow{(u, v)} (\bar{x}, \bar{y})$ . Hence, by the directional metric regularity of  $F$  at  $(\bar{x}, \bar{y})$  in the direction  $(u, v)$  with some constant  $\tau > 0$ , for  $n$  large enough, there is  $x_n \in F^{-1}(y_n)$  such that

$$\begin{aligned} \|x_n - \bar{x} - t_n u_n\| &\leq \tau(1+t_n)d(y_n, F(\bar{x} + t_n u_n)) \\ &\leq \tau(1+t_n)\|y_n - \bar{y} - t_n v_n - t_n \zeta_n\| = \tau(1+t_n)\delta_n \|\zeta_n\|. \end{aligned}$$

Thus,  $\|(x_n, y_n) - (\bar{x} + t_n u_n, \bar{y} + t_n v_n + t_n \zeta_n)\| < \delta_n$  for  $n$  sufficiently large, and therefore, taking into account of (49), one derives that

$$\|x_n^*\| \tau(1+t_n)\delta_n \|\zeta_n\| \geq \delta_n \langle y_n^*, \zeta_n \rangle,$$

which implies  $\liminf_{n \rightarrow \infty} \|x_n^*\| \geq \tau^{-1}$ , and the proof is complete.  $\square$

## 6 Applications to the Directional Differentiability of the Optimal Value Function

Let  $X, Y$  be Banach spaces,  $f : X \times Y \rightarrow \mathbb{R}$  and  $g : X \rightarrow Y$  be continuous functions. We suppose that  $K$  is a closed convex subset of  $Y$ . In this section, we consider a parameterized optimization problem of the form

$$(\mathcal{P}_y) \quad \min_{x \in X} f(x, y) \quad \text{s.t.} \quad y \in g(x) - K,$$

depending on a parameter  $y \in Y$ . The *feasible set* of  $(\mathcal{P}_y)$  is denoted by

$$\Phi(y) := \{x \in X : y \in g(x) - K\}.$$



For  $y = 0$ , the corresponding problem  $(\mathcal{P}_0)$  denoted by  $(\mathcal{P})$  is viewed as an unperturbed problem. We set  $f(x) := f(x, 0)$ ,  $\Phi_0 := \Phi(0)$  and we denote by  $v(y)$  the *optimal value function* of  $(\mathcal{P}_y)$  and by  $\mathcal{S}(y)$  the associated set of optimal solutions:

$$v(y) := \inf_{x \in \Phi(y)} f(x, y);$$

$$\mathcal{S}(y) := \operatorname{argmin}_{x \in \Phi(y)} f(x, y).$$

Recall that  $x_\varepsilon$  is said to be an  $\varepsilon$ -*optimal solution* of  $(\mathcal{P}_y)$  if  $x_\varepsilon \in \Phi(y)$  and  $f(x_\varepsilon, y) \leq v(y) + \varepsilon$ . In this section, we apply the concept of directional metric regularity to discuss the directional differentiability of the optimal value function  $v(y)$ . We now assume that  $f(\cdot, \cdot)$  and  $g$  are mappings of class  $\mathcal{C}^1$ . Associated to a given direction  $d \in Y$ , we consider a path  $y(t)$  of the form  $y(t) = td + o(t)$ , with  $t \in \mathbb{R}_+$ . Let us recall the notion of *feasible direction* [10]:

**Definition 6.1** Let  $x_0 \in \Phi(0)$  be given. A direction  $h \in X$  is said to be a feasible direction at  $x_0$ , relative to the direction  $d \in Y$ , iff for any path  $y(t) = td + o(t)$  with  $t \geq 0$  in  $Y$ , there exists  $r(t) = o(t)$  in  $X$  such that  $x_0 + th + r(t) \in \Phi(y(t))$ .

Note from [10] that if  $h$  is a feasible direction relative to  $d \in Y$  at  $x_0$ , then

$$Dg(x_0)h - d \in T_K(g(x_0)), \quad (50)$$

where,  $T_K(g(x_0)) = \{d \in Y : d(g(x_0) + td, K) = o(t), \quad t \geq 0\}$  stands for the *contingent cone* to the convex set  $K$  at  $g(x_0)$ .

Conversely, one has the following:

**Lemma 6.1** Let  $x_0 \in \Phi(0)$  be given. If relation (50) holds, and if in addition,  $G(x) := g(x) - K$  is directionally metrically regular in the direction  $(h, d)$  at  $(x_0, 0)$ , then  $h$  is a feasible direction relative to  $d$ .

*Proof* By the assumption, there exist  $\tau, \varepsilon > 0$  such that

$$d(x, \Phi(y)) \leq \tau d(y, g(x) - K) \quad (51)$$

$$\forall (x, y) \in B((x_0, 0), \varepsilon) \cap [(x_0, 0) + \operatorname{cone} B((h, d), \varepsilon)], \quad d(y, g(x) - K) \leq \varepsilon \|(x, y) - (x_0, 0)\|.$$

Let  $y(t) = td + o(t)$  be given and set  $x(t) = x_0 + th$ , one has

$$g(x_0) + tDg(x_0)h - td + o(t) \in K, \text{ as } t \downarrow 0.$$

As  $g(x(t)) = g(x_0) + tDg(x_0)h + o(t)$ , then we have  $g(x(t)) - td + o(t) \in K$  from which we obtain

$$d(y(t), g(x(t)) - K) = o(t).$$

Hence, when  $t > 0$  is sufficiently small,

$$(x(t), y(t)) \in B((x_0, 0), \varepsilon) \cap [(x_0, 0) + \operatorname{cone} B((h, d), \varepsilon)], \quad d(y(t), g(x(t)) - K) \leq \varepsilon \|(x(t), y(t)) - (x_0, 0)\|.$$

Thus, there is  $\bar{x}(t) := x_0 + th + o(t) \in \Phi(y(t))$ , as  $t > 0$ . □

**Lemma 6.2** Assume that  $Y$  is finite dimensional. Let  $x_0 \in \Phi(0)$  and  $d \in Y \setminus \{0\}$ ,  $h \in X$  such that (50) holds. If  $G$  is directionally metrically regular at  $(x_0, 0)$  in the direction  $(h, d)$ , then one has

$$d \in \text{int}\{Dg(x_0)X - T_K(g(x_0))\}. \quad (52)$$

*Proof* Let  $\tau > 0$ ,  $\varepsilon \in (0, 1)$  be such that (51) happens. Let  $0 < \delta < \varepsilon\|d\|/2$  and fix  $\tilde{d} \in B(d, \delta)$ . Since (50) holds, one has

$$g(x_0) + tDg(x_0)h - td + o(t) \in K, \quad \text{ast } t > 0.$$

Hence,

$$g(x_0 + th) - td + o(t) \in K, \quad \text{ast } t > 0.$$

Moreover, one has  $\delta \leq \frac{\varepsilon\|\tilde{d}\|}{2(1-\delta/2)} < \varepsilon\|\tilde{d}\|$  since  $\delta < \varepsilon\|d\|/2 < \frac{\varepsilon(\delta+\|\tilde{d}\|)}{2}$ . Consequently, when  $t$  is sufficiently small,

$$d(t\tilde{d}, g(x_0 + th) - K) \leq t\delta + o(t) < \varepsilon t(\|h\| + \|\tilde{d}\|).$$

According to (51), we now select  $x(t) \in \Phi(t\tilde{d})$  such that

$$\|x_0 + th - x(t)\| \leq \tau t\delta + o(t).$$

Setting  $h(t) = \frac{x(t) - x_0}{t}$ , one has

$$\|h - h(t)\| \leq \tau\delta + \frac{o(t)}{t}.$$

As  $x(t) \in \Phi(t\tilde{d})$ , then

$$t\tilde{d} \in g(x_0 + h(t)) - K,$$

and therefore,

$$\tilde{d} \in Dg(x_0)(h(t)) + \frac{o(t)}{t} - \frac{K - g(x_0)}{t}.$$

Since  $Y$  is finite dimensional, we can take a sequence  $(t_n)_{n \in \mathbb{N}} \downarrow 0$  such that the sequence  $(Dg(x_0)h(t_n))_{n \in \mathbb{N}}$  converges to some  $w \in Dg(x_0)X$ . Then, thanks to the preceding relation we obtain

$$\tilde{d} \in Dg(x_0)X - T_K(g(x_0)),$$

which completes the proof of the lemma.  $\square$

Denote by  $L(x, \lambda, y)$  and  $\Lambda(x_0)$  the *Lagrangian* of  $(\mathcal{P}_y)$  and the set of *Lagrange multipliers* of the problem  $(\mathcal{P}_0)$  for  $x_0 \in S(0)$ , respectively. More precisely, if  $N_K(g(x_0))$  stands for the *normal cone* to the convex set  $K$  at  $g(x_0)$ , we have:

$$L(x, \lambda, y) = f(x, y) + \langle \lambda, g(x) - y \rangle, \quad (x, \lambda) \in X \times Y^*;$$

$$\Lambda(x_0) = \{\lambda \in N_K(g(x_0)) : D_x L(x_0, \lambda, 0) = 0\}.$$

For a given  $d \in Y$ , we now consider the following *linearization* of  $(\mathcal{P}_Y)$ :

$$(\mathcal{P}\mathcal{L}_d) \quad \min_{h \in X} Df(x_0, 0)(h, d) \quad \text{s.t.} \quad Dg(x_0)h - d \in T_K(g(x_0)).$$

From [10, p. 278], we observe that relation (52) is exactly Robinson's constraint qualification for the problem  $(\mathcal{P}\mathcal{L}_d)$ , and the dual of  $(\mathcal{P}\mathcal{L}_d)$  is

$$(\mathcal{D}\mathcal{L}_d) \quad \max_{\lambda \in \Lambda(x_0)} D_y L(x_0, \lambda, 0)d.$$

According to the standard duality result in extended linear programming (see, e.g., [10, Theorem 2.165]), from Lemma 6.2, one has the following dual result for the linearization problem  $(\mathcal{D}\mathcal{L}_d)$ :

**Lemma 6.3** *Let  $x_0 \in S(0)$  and  $d \in Dg(x_0)X - T_K(g(x_0))$  be given. Assume that  $Y$  is finite dimensional and  $G = g - K$  is directionally metrically regular at  $(x_0, 0)$  in the direction  $(h, d)$  for some  $h \in X$  with  $Dg(x_0)h - d \in T_K(g(x_0))$ . Then there is no duality gap between problems  $(\mathcal{P}\mathcal{L}_d)$  and  $(\mathcal{D}\mathcal{L}_d)$ . Moreover, the common optimal value is finite, if and only if, the set  $\Lambda(x_0)$  is nonempty; and in this case, the set of optimal solutions of  $(\mathcal{D}\mathcal{L}_d)$  is a nonempty compact set.*

The following theorem offers a result related to the Hadamard directional differentiability of the optimal value function  $v(y)$ . In order to establish this result, we use the concept of directional metric regularity which is weaker than the Robinson constraint qualification in general used in the literature (see, e.g., [10, Theorem 4.26]).

**Theorem 6.1** *Let  $Y$  be finite dimensional and let  $d \in Y$  with*

$$\{d, -d\} \subseteq Dg(x)X - T_K(g(x)) \quad \text{for all } x \in S(x_0).$$

*Assume that*

- (i) *the multifunction  $G = g - K$  is directionally metrically regular in the directions  $(0, d)$  and  $(0, -d)$  at all  $(x, 0)$  with  $x \in S(0)$ ;*
- (ii) *for any sequence  $y_n = t_n d + o(t_n)$  with  $t_n \downarrow 0$ , there exists a sequence of  $o(t_n)$ -optimal solutions  $(x_n)$  of  $(\mathcal{P}_{y_n})$ , converging to some  $x_0 \in S(0)$ .*

*Then denoting by  $v'_-(0, d)$  and  $v'_+(0, d)$ , the lower and upper Hadamard directional derivatives of  $v$  at 0 in the direction  $d$ , one has*

$$\begin{aligned} v'_-(0, d) &\geq \inf_{x \in S(0)} \inf_{\lambda \in \Lambda(x)} D_y L(x, \lambda, 0)d; \\ v'_+(0, d) &\leq \inf_{x \in S(0)} \sup_{\lambda \in \Lambda(x)} D_y L(x, \lambda, 0)d. \end{aligned} \tag{53}$$

*As a result, if  $\Lambda(x)$  is a singleton  $\{\lambda(x)\}$  for all  $x \in S(0)$ , then the Hadamard directional derivative in the direction  $d$  of  $v(y)$  at 0 exists and*

$$v'(0, d) = \inf_{x \in S(0)} D_y L(x, \lambda(x), 0)d.$$

*Proof* Let  $x \in S(0)$  and let  $h \in X$  be such that  $Dg(x)h - d \in T_K(g(x))$ . As  $G$  is directionally metrically regular at  $(x, 0)$  in the direction  $(0, d)$ , then it is also directionally metrically regular at  $(x, 0)$  in the direction  $(h, d)$ . By Lemma 6.1,  $h$  is a feasible direction relative to  $d$ , i.e.,  $x + th + o(t) \in \Phi(td)$ ,  $t \downarrow 0$ . Therefore,

$$v(td) \leq f(x + th + o(t), td) = f(x, 0) + tDf(x, 0)(h, d) + o(t),$$

and consequently,

$$\limsup_{t \downarrow 0} \frac{v(td) - v(0)}{t} \leq Df(x, 0)(h, d).$$

Since  $x$  is arbitrary in  $S(0)$  and  $h$  is an arbitrary feasible point of  $(\mathcal{P}\mathcal{L}_d)$ , the second inequality in (53) is proved.

For the first inequality, let  $t_n \downarrow 0$ ;  $y_n = t_n d + o(t_n)$  and  $(x_n)$  be a sequence of  $o(t_n)$ -solutions of  $(\mathcal{P}_{y_n})$  as in (ii), which converges to  $x_0 \in S(0)$ . Pick  $h \in X$  such that

$$Dg(x_0)h + d \in T_K(g(x_0));$$

equivalently,

$$g(x_0) + tDg(x_0)h + td + o(t) \in K, \quad t \downarrow 0.$$

Since  $g(x_n) - y_n \in K$ , and as  $K$  is convex, for any  $t > 0$ , when  $n$  is sufficiently large and such that  $t_n/t < 1$ , one has

$$\begin{aligned} & (1 - t_n/t)[g(x_n) - y_n] + t_n/t[g(x_0) + tDg(x_0)h + td + o(t)] \\ &= g(x_n) - y_n + t_n/t[g(x_0) - g(x_n)] + t_nDg(x_0)h + t_nd + t_no(t)/t \in K. \end{aligned}$$

Therefore, for  $\varepsilon > 0$ , when  $n$  is sufficiently large, one has  $d(g(x_n) + t_nDg(x_0)h, K) \leq t_n\varepsilon$ . Since

$$g(x_n + t_nh) = g(x_n) + t_nDg(x_0)h + o(t_n),$$

one obtains

$$d(g(x_n + t_nh), K) \leq t_n\varepsilon + o(t_n).$$

By the directional metric regularity of  $G$ , we get some  $z_n \in \Phi_0 = G^{-1}(0)$  such that  $x_n + t_nh - z_n = o(t_n\varepsilon)$ . Thus,

$$\begin{aligned} \frac{v(y_n) - v(0)}{t_n} &\geq \frac{f(x_n, y_n) - f(z_n, 0) + o(t_n)}{t_n} \\ &= \frac{f(x_n, y_n) - f(x_n + t_nh, 0) - o(t_n\varepsilon)}{t_n} \\ &= -Df(x_0, 0)(h, -d) - o(t_n\varepsilon)/t_n. \end{aligned}$$

Finally, by Lemma 6.3, there is no duality gap between  $(\mathcal{P}\mathcal{L}_{-d})$  and  $(\mathcal{D}\mathcal{L}_{-d})$ ; as  $\varepsilon > 0$  is arbitrary, one derives that

$$\liminf_{n \rightarrow \infty} \frac{v(y_n) - v(0)}{t_n} \geq \inf_{\lambda \in \Lambda(x_0)} D_y L(x_0, \lambda, 0)d,$$

from which follows the first inequality of (53). □

## 7 Conclusions

In this contribution we have tried to demonstrate how Hölder directional metric regularity of set-valued mappings is an useful concept for studying the stability and the sensitivity analysis of parameterized optimization problems. This has been achieved in the last section, where we have investigated the Hadamard directional differentiability of the optimal value function of a general parametrized optimization problem.

## 8 Conflict of Interest

The author declares that he has no conflict of interest.

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