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Abstract. We consider the following Hamiltonian equation on the L^2 Hardy space on the circle $\mathbb{S}^1$,

$$i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1)\,, \alpha \in \mathbb{R}\,, \tag{0.1}$$

where Π is the Szegő projector. The above equation with $\alpha = 0$ was introduced by Gérard and Grellier as an important mathematical model [5, 7, 3]. In this paper, we continue our studies started in [22], and prove our system is completely integrable in the Liouville sense. We study the motion of the singular values of the related Hankel operators and find a necessary condition of norm explosion. As a consequence, we prove that the trajectories of the solutions will stay in a compact subset, while more initial data will lead to norm explosion in the case $\alpha > 0$.

1. Introduction

The purpose of this paper is to study the following Hamiltonian system,

$$i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1)\,,\ x \in \mathbb{S}^1\,,\ t \in \mathbb{R}\,,\ \alpha \in \mathbb{R}\,. \tag{1.1}$$

where the operator Π is defined as a projector onto the non-negative frequencies, which is called the Szegő projector. When $\alpha = 0$, the equation above turns out to be the cubic Szegő equation,

$$i\partial_t u = \Pi(|u|^2 u)\,, \tag{1.2}$$

which was introduced by P. Gérard and S. Grellier as an important mathematical model of the completely integrable systems and non-dispersive dynamics [5, 7]. For $\alpha \neq 0$, by changing variables as $u = \sqrt{|\alpha|}\tilde{u}(|\alpha|t)$, then $\tilde{u}$ satisfies

$$i\partial_t \tilde{u} = \Pi(|\tilde{u}|^2 \tilde{u}) + \mathrm{sgn}(\alpha)(\tilde{u}|1)\,. \tag{1.3}$$

Thus our target equation with $\alpha \neq 0$ becomes

$$i\partial_t u = \Pi(|u|^2 u) \pm (u|1)\,. \tag{1.4}$$

1.1. Lax Pair structure.

Thanks to the Lax pairs for the cubic Szegő equation (1.2) [7], we are able to find a Lax pair for (1.1). To introduce the Lax pair structure, let us first define some useful operators and notation. For $X \subset \mathcal{D}'(\mathbb{S}^1)$, we denote

$$X_+(\mathbb{S}^1) := \left\{ u(e^{i\theta}) \in X,\ u(e^{i\theta}) = \sum_{k\geq 0} \hat{u}(k) e^{ik\theta} \right\}. \tag{1.5}$$

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For example, L^2_+ denotes the Hardy space of L^2 functions which extend to the unit disc $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ as holomorphic functions

$$u(z) = \sum_{k\geq 0} \hat{u}(k) z^k,\ \sum_{k\geq 0} |\hat{u}(k)|^2 < \infty\ . \tag{1.6}$$

Then the Szegő operator Π is an orthogonal projector $L^2(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$.

Now, we are to define a Hankel operator and a Toeplitz operator. By a Hankel operator we mean a bounded operator Γ on the sequence space ℓ^2 which has a Hankel matrix in the standard basis $\{e_j\}_{j\geq 0}$,

$$(\Gamma e_j,\ e_k\ ) = \gamma_{j+k},\ j,\ k \geq 0\ , \tag{1.7}$$

where $\{\gamma_j\}_{j\geq 0}$ is a sequence of complex numbers. More backgrounds on the Hankel operators can be found in [20].

Let S be the shift operator on ℓ^2,

$$S e_j = e_{j+1},\ j \geq 0\ .$$

It is easy to show that a bounded operator Γ on ℓ^2 is a Hankel operator if and only if

$$S^*\Gamma = \Gamma S\ . \tag{1.8}$$

Definition 1.1. *For any given $u \in H^{\frac{1}{2}}_+(\mathbb{S}^1)$, $b \in L^\infty(\mathbb{S}^1)$, we define two operators $H_u,\ T_b : L^2_+ \to L^2_+$ as follows. For any $h \in L^2_+$,*

$$H_u(h) = \Pi(u\bar{h})\ , \tag{1.9}$$

$$T_b(h) = \Pi(bh)\ . \tag{1.10}$$

Notice that H_u is $\mathbb{C}-$antilinear and symmetric with respect to the real scalar product $\mathrm{Re}(u|v)$. In fact, it satisfies

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1)\ .$$

T_b is $\mathbb{C}-$linear and is self-adjoint if and only if b is real-valued.

Moreover, H_u is a Hankel operator. Indeed, it is given in terms of Fourier coefficients by

$$\widehat{H_u(h)}(k) = \sum_{\ell\geq 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)}\ ,$$

then

$$\begin{aligned} S^*H_u(h) &= \sum_{k,\ell\geq 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)} S^* e_k = \sum_{k,\ell\geq 0} \hat{u}(k+\ell+1)\overline{\hat{h}(\ell)} e_k\ , \\ H_u S h &= \sum_{k\geq \ell,\ell\geq 0} \hat{u}(k) e_k \overline{\hat{h}(\ell) e_{\ell+1}} = \sum_{k,\ell\geq 0} \hat{u}(k+\ell+1)\overline{\hat{h}(\ell)} e_k\ , \end{aligned}$$

which means $S^*H_u = H_u S$, thus H_u is a Hankel operator. We may also represent T_b in terms of Fourier coefficients,

$$\widehat{T_b(h)}(k) = \sum_{\ell\geq 0} \hat{b}(k-\ell)\hat{h}(\ell)\ ,$$

then its matrix representation, in the basis $e_k, k \geq 0$, has constant diagonals, T_b is a Toeplitz operator.

We now define another operator $K_u := T_z^* H_u$. In fact T_z is exactly the shift operator S as above, we then call K_u the shifted Hankel operator, which satisfying the following identity

$$K_u^2 = H_u^2 - (\cdot \mid u)u \ . \tag{1.11}$$

Using the operators above, Gérard and Grellier found two Lax pairs for the Szegő equation (1.2).

Theorem 1.1. [5, Theorem 3.1] *Let $u \in C(R, H_+^s(\mathbb{S}^1))$ for some $s > 1/2$. The cubic Szegő equation (1.2) has two Lax pairs (H_u, B_u) and (K_u, C_u), namely, if u solves (1.2), then*

$$\frac{dH_u}{dt} = [B_u, H_u] \ , \ \frac{dK_u}{dt} = [C_u, K_u] \ , \tag{1.12}$$

where

$$B_u := \frac{i}{2} H_u^2 - i T_{|u|^2} \ , \ C_u = \frac{i}{2} K_u^2 - i T_{|u|^2} \ .$$

For $\alpha \neq 0$, the perturbed Szegő equation (1.1) is globally well-posed and by simple calculus, we find that (H_u, B_u) is no longer a Lax pair, in fact,

$$\frac{dH_u}{dt} = [B_u, H_u] - i\alpha(u|1)H_1 \ . \tag{1.13}$$

Fortunately, (K_u, C_u) is still a Lax pair.

Theorem 1.2. [22] *Given $u_0 \in H_+^{\frac{1}{2}}(\mathbb{S}^1)$, there exists a unique global solution $u \in C(\mathbb{R}; H_+^{\frac{1}{2}})$ of (1.1) with u_0 as the initial condition. Moreover, if $u_0 \in H_+^s(\mathbb{S}^1)$ for some $s > \frac{1}{2}$, then $u \in C^\infty(\mathbb{R}; H_+^s)$. Furthermore, the perturbed Szegő equation (1.1) has a Lax pair (K_u, C_u), namely, if u solves (1.1), then*

$$\frac{dK_u}{dt} = [C_u, K_u] \ . \tag{1.14}$$

An important consequence of this structure is that, if u is a solution of (1.1), then $K_{u(t)}$ is unitarily equivalent to K_{u_0}. In particular, the spectrum of the $\mathbb{C}$-linear positive self-adjoint trace class operator K_u^2 is conserved by the evolution.

Denote

$$\mathcal{L}(N) := \{u : \mathrm{rk}(K_u) = N, N \in \mathbb{N}^+\} \ . \tag{1.15}$$

Thanks to the Lax pair structure, the manifolds $\mathcal{L}(N)$ are invariant under the flow of (1.1). Moreover, they turn out to be spaces of rational functions as in the following Kronecker type theorem.

Theorem 1.3. [22] *$u \in \mathcal{L}(N)$ if and only if $u(z) = \frac{A(z)}{B(z)}$ is a rational function with*

$$A, B \in \mathbb{C}_N[z], A \wedge B = 1, \deg(A) = N \text{ or } \deg(B) = N, B^{-1}(\{0\}) \cap \overline{\mathbb{D}} = \emptyset \ ,$$

where $A \wedge B = 1$ means A and B have no common factors.

Our main objective of the study on this mathematical model (1.1) is on the large time unboundedness of the solution. This general question of existence of unbounded Sobolev trajectories comes back to [1], and was addressed by several authors for various Hamiltonian PDEs, see e.g. [2, 6, 12, 13, 14, 11, 15, 16, 17, 19, 21]. We have already considered the case with initial data $u_0 \in \mathcal{L}(1)$ and found that

Theorem 1.4. [22] *Let u be a solution to the α–Szegő equation,*

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1)\ ,\ \alpha = \mathbb{R}\ , \\ u(0,x) = u_0(x) \in \mathcal{L}(1)\ . \end{cases} \tag{1.16}$$

For $\alpha < 0$, the Sobolev norm of the solution will stay bounded, uniform if u_0 is in some compact subset of $\mathcal{L}(1)$,

$$\|u(t)\|_{H^s} \leq C\ ,\ C \text{ does not depend on time } t\ ,\ s \geq 0\ .$$

For $\alpha > 0$, the solution u of the α–Szegő equation has an exponential-on-time Sobolev norm growth,

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_s|t|}\ ,\ s > \frac{1}{2}\ ,\ C_s > 0\ ,\ |t| \to \infty\ , \tag{1.17}$$

if and only if

$$E_\alpha = \frac{1}{4}Q^2 + \frac{1}{2}Q, \tag{1.18}$$

with E_α and Q as the two conserved quantities, energy and mass.

1.2. **Main results.** We continue our studies on the cubic Szegő equation with a linear perturbation (1.1) on the circle $\mathbb{S}^1$ with more general initial data $u_0 \in \mathcal{L}(N)$ for any $N \in \mathbb{N}^+$.

Firstly, the system is integrable since there are a large amount of conservation laws which comes from the Lax pair structure(1.14).

Theorem 1.5. *Let $u(t,x)$ be a solution of (1.1). For every Borel function f on $\mathbb{R}$, the following quantity*

$$L_f(u) := \Big(f(K_u^2)u|u\Big) - \alpha\Big(f(K_u^2)1|1\Big)$$

is conserved.

Let σ_k^2 be an eigenvalue of K_u^2, and f be the characteristic function of the singleton $\{\sigma_k^2\}$, then

$$\ell_k(u) := \|u_k'\|^2 - \alpha\|v_k'\|^2$$

is conserved, where u_k', v_k' are the projections of u and 1 onto $\ker(K_u^2 - \sigma_k^2)$, and $\|\cdot\|$ denotes the L^2–norm on the circle. Generically, on the $2N+1$–dimensional complex manifold $\mathcal{L}(N)$, we have $2N+1$ linearly independent and in involution conservation laws, which are $\sigma_k\ ,\ 1 \leq k \leq N$ and $\ell_m\ ,\ 0 \leq m \leq N$. Thus, the system (1.1) can be approximated by a sequence of systems of finite dimension which are completely integrable in the Liouville sense.

Secondly, we prove the existence of unbounded trajectories for data in $\mathcal{L}(N)$ for any arbitrary $N \in \mathbb{N}^+$. One way to capture the unbounded trajectories of solutions is via the motion of singular values of H_u^2 and K_u^2. In the case with $\alpha = 0$, all the eigenvalues of H_u^2 and K_u^2 are constants, but the eigenvalues of H_u^2 are no longer constants for $\alpha \neq 0$, which makes the system more complicated.

By studying the motion of singular values of H_u and K_u, we gain that the necessary condition and existence of crossing which means the two closest eigenvalues of H_u touch some eigenvalue of K_u at some finite time. A remarkable observation is that the Blaschke products of K_u never change their $\mathbb{S}^1$ orbits as time goes.

The main result on the large time behaviour of solutions is as below.

Theorem 1.6. *Let $u_0 \in \mathcal{L}(N)$ for any $N \in \mathbb{N}^+$.*

If $\alpha < 0$, the trajectory of the solution $u(t)$ of the α–Szegő (1.1) stays in a compact subset of $\mathcal{L}(N)$. In other words, the Sobolev norm of the solution $u(t)$ will stay bounded,

$$\|u(t)\|_{H^s} \leq C \ , \ C \text{ does not depend on time } t \ , \ s \geq 0 \ .$$

While for $\alpha > 0$, there exists $u_0 \in \mathcal{L}(N)$ which leads to a solution with norm explosion at infinity. More precisely,

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_\alpha(2s-1)|t|} \ , \ t \to \infty \ , \ \forall s \geq \frac{1}{2} \ .$$

Remark 1.1.

1. *In the case $\alpha = 0$, there are two Lax pairs, the conserved quantities are much simpler, which are the eigenvalues of H_u^2 and K_u^2. While in the case $\alpha \neq 0$, the eigenvalues of H_u^2 are no longer conserved, which makes our system more complicated.*

2. *For the cubic Szegő equation with $\alpha = 0$, Gérard and Grellier [4] have proved there exists a G_δ dense set $\mathfrak{g}$ of initial data in $C_+^\infty := \cap_s H^s$, such that for any $v_0 \in \mathfrak{g}$, there exist sequences of time $\overline{t_n}$ and $\underline{t_n}$, such that the corresponding solution v of the cubic Szegő equation*

$$i\partial_t v = \Pi_+(|v|^2 v) \ , \ v(0) = v_0 \ , \tag{1.19}$$

satisfies

$$\forall r > \frac{1}{2} \ , \ \forall M \geq 1 \ , \ \frac{\|v(\overline{t_n})\|_{H^r}}{|\overline{t_n}|^M} \to \infty \ , \ n \to \infty \ , \tag{1.20}$$

while

$$v(\underline{t_n}) \to v_0 \text{ in } C_+^\infty \ , \ n \to \infty \ . \tag{1.21}$$

Here, by considering the rational data in the case $\alpha \neq 0$, we proved the existence of solutions with exponential growth in time rather than lim sup.

There is another non dispersive example with norm growth by Oana Pocovnicu [21], who studied the cubic Szegő equation on the line $\mathbb{R}$, and found there exist solutions with Sobolev norms growing polynomially in time as $|t|^{2s-1}$ with $s \geq 1/2$.

3. *For the case $\alpha > 0$, we now have solutions of (1.1) with different growths, uniformly bounded, growing in fluctuations with a* lim sup *super-polynomial in time growth, and exponential in time growth. Indeed, it is easy to show that $zu(t, z^2)$ is a solution to the α–Szegő equation if $u(t, z)$ solves the cubic Szegő equation (1.19). Thus, for the cubic Szegő equation with a linear perturbation (1.1), there also exist solutions with such an energy cascade as in (1.20) and (1.21).*

4. *In this paper, we consider data in $\mathcal{L}(N)$ for any arbitrary $N \in \mathbb{N}^+$. The data we find which lead to a large time norm explosion are very special. An interesting observation is that the equations on u_k' and v_k' look similar to the original α–Szegő equation,*

$$\frac{\partial}{\partial t}\begin{pmatrix} u_k' \\ v_k' \end{pmatrix} = -i\begin{pmatrix} T_{|u|^2} & \alpha(u|1) \\ -(1|u) & T_{|u|^2} - \sigma_k^2 \end{pmatrix}\begin{pmatrix} u_k' \\ v_k' \end{pmatrix} , \tag{1.22}$$

which gives us some hope to extend our results to general rational data.

1.3. **Organization of this chapter.** In section 2, we recall the results about the singular values of H_u and K_u [9]. In section 3, we introduce the conservation laws and prove the integrability. In section 4, we study the motion of the singular values of the Hankel operators H_u and K_u, the eigenvalues of H_u move and may touch some eigenvalue of K_u at finite time while the eigenvalues of K_u stay fixed with the corresponding Blaschke products stay in the same orbits. In section 5, we present a necessary condition of the norm explosion, and as a direct consequence, we know that for $\alpha < 0$, the trajectories of the solutions stay in a compact subset. In section 6, we study the norm explosion with $\alpha > 0$ for data in $\mathcal{L}(N)$ with any $N \in \mathbb{N}^+$. We present some open problems in the last section.

2. Spectral analysis of the operators H_u and K_u

In this section, let us introduce some notation which will be used frequently and some useful results by Gérard and Grellier in their recent work [9]. We consider $u \in H^s_+(\mathbb{S}^1)$ with $s > \frac{1}{2}$. The Hankel operator H_u is compact by the theorem due to Hartman [18]. Let us introduce the spectral analysis of operators H_u^2 and K_u^2. For any $\tau \geq 0$, we set

$$E_u(\tau) := \ker(H_u^2 - \tau^2 \mathrm{I}),\ F_u(\tau) := \ker(K_u^2 - \tau^2 \mathrm{I})\ . \tag{2.1}$$

If $\tau > 0$, the $E_u(\tau)$ and $F_u(\tau)$ are finite dimensional with the following properties.

Proposition 2.1. [9] *Let $u \in H^s_+(\mathbb{S}^1) \setminus \{0\}$ with $s > 1/2$, and $\tau > 0$ such that*

$$E_u(\tau) \neq \{0\} \quad or \quad F_u(\tau) \neq \{0\}\ .$$

Then one of the following properties holds.

(1) $\dim E_u(\tau) = \dim F_u(\tau) + 1$, $u \not\perp E_u(\tau)$, *and* $F_u(\tau) = E_u(\tau) \cap u^\perp$.
(2) $\dim F_u(\tau) = \dim E_u(\tau) + 1$, $u \not\perp F_u(\tau)$, *and* $E_u(\tau) = F_u(\tau) \cap u^\perp$.

Moreover, if u_ρ and u'_σ denote respectively the orthogonal projections of u onto $E_u(\rho)$, $\rho \in \Sigma_H(u)$, and onto $F_u(\sigma)$, $\sigma \in \Sigma_K(u)$ with

$$\Sigma_H(u) := \{\tau > 0 :\ u \not\perp E_u(\tau)\},\quad \Sigma_K(u) := \{\tau \geq 0 :\ u \not\perp F_u(\tau)\}\ .$$

Then

(1) *$\Sigma_H(u)$ and $\Sigma_K(u)$ are disjoint, with the same cardinality;*
(2) *if $\rho \in \Sigma_H(u)$,*

$$u_\rho = \|u_\rho\|^2 \sum_{\sigma \in \Sigma_K(u)} \frac{u'_\sigma}{\rho^2 - \sigma^2}\ , \tag{2.2}$$

(3) *if $\sigma \in \Sigma_K(u)$,*

$$u'_\sigma = \|u'_\sigma\|^2 \sum_{\rho \in \Sigma_H(u)} \frac{u_\rho}{\rho^2 - \sigma^2}\ . \tag{2.3}$$

(4) *A non negative number σ belongs to $\Sigma_K(u)$ if and only if it does not belong to $\Sigma_H(u)$ and*

$$\sum_{\rho \in \Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2 - \sigma^2} = 1\ . \tag{2.4}$$

By the spectral theorem for H_u^2 and K_u^2, which are self-adjoint and compact, we have the following orthogonal decomposition

$$L_+^2 = \overline{\oplus_{\tau>0} E_u(\tau)} = \overline{\oplus_{\tau\geq 0} F_u(\tau)} \,. \tag{2.5}$$

Then we can write u as

$$u = \sum_{\rho\in\Sigma_H(u)} u_\rho = \sum_{\sigma\in\Sigma_K(u)} u'_\sigma \,. \tag{2.6}$$

In fact, we are able to describe these two sets $E_u(\tau)$ and $F_u(\tau)$ more explicitly. Recall that a finite Blaschke product of degree k is a rational function of the form

$$\Psi(z) = \mathrm{e}^{-i\psi}\frac{P(z)}{D(z)} \,,$$

where $\psi \in \mathbb{S}^1$ is called the angle of Ψ and P is a monic polynomial of degree k with all its roots in $\mathbb{D}$, $D(z) = z^k\overline{P}\left(\frac{1}{z}\right)$ as the normalized denominator of Ψ. Here a monic polynomial is a univariate polynomial in which the leading coefficient (the nonzero coefficient of highest degree) is equal to 1. We denote by $\mathcal{B}_k$ the set of all the Blaschke functions of degree k.

Proposition 2.2. [9] *Let $\tau > 0$ and $u \in H_+^s(\mathbb{S}^1)$ with $s > \frac{1}{2}$.*

(1) *Assume $\tau \in \Sigma_H(u)$ and $\ell := \dim E_u(\tau) = \dim F_u(\tau) + 1$. Denote by u_τ the orthogonal projection of u onto $E_u(\tau)$. There exists a Blaschke function $\Psi_\tau \in \mathcal{B}_{\ell-1}$ such that*

$$\tau u_\tau = \Psi_\tau H_u(u_\tau) \,,$$

and if D denotes the normalized denominator of Ψ_τ,

$$E_u(\tau) = \left\{\frac{f}{D(z)}H_u(u_\tau) \,,\ f \in \mathbb{C}_{\ell-1}[z]\right\} \,, \tag{2.7}$$

$$F_u(\tau) = \left\{\frac{g}{D(z)}H_u(u_\tau) \,,\ g \in \mathbb{C}_{\ell-2}[z]\right\}, \tag{2.8}$$

and for $a = 0, \ldots, \ell-1$, $b = 0, \ldots, \ell-2$,

$$H_u\left(\frac{z^a}{D(z)}H_u(u_\tau)\right) = \tau\mathrm{e}^{-i\psi_\tau}\frac{z^{\ell-a-1}}{D(z)}H_u(u_\tau) \,, \tag{2.9}$$

$$K_u\left(\frac{z^b}{D(z)}H_u(u_\tau)\right) = \tau\mathrm{e}^{-i\psi_\tau}\frac{z^{\ell-b-2}}{D(z)}H_u(u_\tau) \,, \tag{2.10}$$

where ψ_τ denotes the angle of Ψ_τ.

(2) *Assume $\tau \in \Sigma_K(u)$ and $m := \dim F_u(\tau) = \dim E_u(\tau) + 1$. Denote by u'_τ the orthogonal projection of u onto $F_u(\tau)$. There exists an inner function $\Psi_\tau \in \mathcal{B}_{m-1}$ such that*

$$K_u(u'_\tau) = \tau\Psi_\tau u'_\tau \,,$$

and if D denotes the normalized denominator of Ψ_τ,

$$F_u(\tau) = \left\{\frac{f}{D(z)}u'_\tau \,,\ f \in \mathbb{C}_{m-1}[z]\right\} \,, \tag{2.11}$$

$$E_u(\tau) = \left\{\frac{zg}{D(z)}u'_\tau \,,\ g \in \mathbb{C}_{m-2}[z]\right\} \,, \tag{2.12}$$

and, for $a = 0, \dots, m-1$, $b = 0, \dots, m-2$,

$$K_u\left(\frac{z^a}{D(z)}u'_\tau\right) = \tau e^{-i\psi_\tau}\frac{z^{m-a-1}}{D(z)}u'_\tau \,, \tag{2.13}$$

$$H_u\left(\frac{z^{b+1}}{D(z)}u'_\tau\right) = \tau e^{-i\psi_\tau}\frac{z^{m-b-1}}{D(z)}u'_\tau \,, \tag{2.14}$$

where ψ_τ *denotes the angle of* Ψ_τ.

We call the elements $\rho_j \in \Sigma_H(u)$ and $\sigma_k \in \Sigma_K(u)$ as the dominant eigenvalues of H_u and K_u respectively. Due to the above achievements, they are in a finite or infinite sequence

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots \to 0 \,,$$

we denote by ℓ_j and m_k as the multiplicities of ρ_j and σ_k respectively. In other words,

$$\dim E_u(\rho_j) = \ell_j \,,$$
$$\dim F_u(\sigma_k) = m_k \,.$$

Therefore, we may define the dominant ranks of the operators as

$$\mathrm{rk_d}(H_u) := \sum_j \ell_j \,,$$
$$\mathrm{rk_d}(K_u) := \sum_k m_k \,,$$

while the ranks of the operators are

$$\mathrm{rk}(H_u) = \sum_j \ell_j + \sum_k (m_k - 1) \,,$$
$$\mathrm{rk}(K_u) = \sum_j (\ell_j - 1) + \sum_k m_k \,.$$

In this paper, u_j and u'_k denote the orthogonal projections of u onto $E_u(\rho_j)$ and $F_u(\sigma_k)$ respectively, while v_j and v'_k denote the orthogonal projections of 1 onto $E_u(\rho_j)$ and $F_u(\sigma_k)$. The L^2–norms of u_j and u'_k can be represented in terms of ρ_ℓ's and σ_ℓ's, which was already observed in [8].

Lemma 2.1. *Let* $u \in H^{\frac{1}{2}}(\mathbb{S}^1)$, $\Sigma_H(u) = \{\rho_j\}$ *and* $\Sigma_K(u) = \{\sigma_k\}$ *with*

$$\rho_1 > \sigma_1 > \rho_2 > \cdots \geq 0 \,.$$

Then

$$\|u_j\|^2 = \frac{\prod_\ell (\rho_j^2 - \sigma_\ell^2)}{\prod_{\ell \neq j} (\rho_j^2 - \rho_\ell^2)} \,, \quad \|u'_k\|^2 = \frac{\prod_\ell (\rho_\ell^2 - \sigma_k^2)}{\prod_{\ell \neq k} (\sigma_\ell^2 - \sigma_k^2)} \,.$$

Proof. First, we have

$$((I - xH_u^2)^{-1} 1 \mid 1) = \prod_\ell \frac{1 - x\sigma_\ell^2}{1 - x\rho_\ell^2} \,.$$

In fact, we can rewrite the left hand side as

$$((I - xH_u^2)^{-1}1 \mid 1) = \sum_\ell \frac{\|v_\ell\|^2}{1 - x\rho_\ell^2} + 1 - \sum_\ell \|v_\ell\|^2 .$$

From Proposition 2.2,

$$v_j = \Big(1, \frac{H_u(u_j)}{\|H_u(u_j)\|}\Big)\frac{H_u(u_j)}{\|H_u(u_j)\|} ,$$

combined with $\Psi_j H_u(u_j) = \rho_j u_j$, we get

$$\|v_j\|^2 = \frac{|(1, H_u(u_j))|^2}{\|H_u(u_j)\|^2} = \frac{|(H_u(1), u_j)|^2}{\rho_j^2\|u_j\|^2} = \frac{\|u_j\|^2}{\rho_j^2} .$$

Thus

$$\prod_\ell \frac{1 - x\sigma_\ell^2}{1 - x\rho_\ell^2} = \sum_\ell \frac{\|u_\ell\|^2}{\rho_\ell^2(1 - x\rho_\ell^2)} + 1 - \sum \frac{\|u_\ell\|^2}{\rho_\ell^2} .$$

We get, identifying the residues at $x = 1/\rho_j^2$,

$$\|u_j\|^2 = \frac{\prod_\ell(\rho_j^2 - \sigma_\ell^2)}{\prod_{\ell\neq j}(\rho_j^2 - \rho_\ell^2)} . \tag{2.15}$$

On the other hand, since

$$1 - x((I - xK_u^2)^{-1}u \mid u) = \frac{1}{((I - xH_u^2)^{-1}1 \mid 1)} ,$$

then

$$1 - x\Big(\sum_k \frac{\|u_k'\|^2}{1 - x\sigma_k^2} + \|u\|^2 - \sum_k \|u_k'\|^2\Big) = \prod_\ell \frac{1 - x\rho_\ell^2}{1 - x\sigma_\ell^2} ,$$

we get, identifying the residues at $x = 1/\sigma_k^2$,

$$\|u_k'\|^2 = \frac{\prod_\ell(\rho_\ell^2 - \sigma_k^2)}{\prod_{\ell\neq k}(\sigma_\ell^2 - \sigma_k^2)} . \tag{2.16}$$

□

3. Conservation laws and the α–Szegő hierarchy

We endow $L^2_+(\mathbb{S}^1)$ with the symplectic form

$$\omega(u, v) = 4\mathrm{Im}(u \mid v) .$$

Then (1.1) can be rewritten as

$$\partial_t u = X_{E_\alpha}(u) , \tag{3.1}$$

with X_{E_α} as the Hamiltonian vector field associated to the Hamiltonian function given by

$$E_\alpha(u) := \frac{1}{4}\int_{\mathbb{S}^1} |u|^4\frac{d\theta}{2\pi} + \frac{\alpha}{2}|(u|1)|^2 .$$

The invariance by translation and by multiplication by complex numbers of modulus 1 gives two other formal conservation laws

$$\text{mass:} \qquad Q(u) := \int_{\mathbb{S}^1} |u|^2 \frac{d\theta}{2\pi} = \|u\|_{L^2}^2 ,$$

$$\text{momentum:} \qquad M(u) := (Du|u),\ D := -i\partial_\theta = z\partial_z .$$

Moreover, the Lax pair structure leads to the conservation of the eigenvalues of K_u^2. So it is obvious the system is completely integrable for the data in the 3−dimensional complex manifold $\mathcal{L}(1)$. Then what about the general case, for example in $\mathcal{L}(N)$ with arbitrary $N \in \mathbb{N}^+$? Fortunately, we are able to find many more conservation laws by its Lax pair structure (1.14). We will then show our system is still completely integrable with data in $\mathcal{L}(N)$ in the Liouville sense.

3.1. **Conservation laws.** Thanks to the Lax pair structure, we are able to find an infinite sequence of conservation laws.

Theorem 3.1. *For every Borel function f on $\mathbb{R}$, the following quantity*

$$L_f(u) := \Big(f(K_u^2)u|u\Big) - \alpha\Big(f(K_u^2)1|1\Big)$$

is a conservation law.

Proof. From the Lax pair identity

$$\frac{dK_u}{dt} = [C_u, K_u] ,\ C_u = -iT_{|u|^2} + \frac{i}{2}K_u^2 ,$$

we infer

$$\frac{d}{dt}K_u^2 = [-iT_{|u|^2}, K_u^2] ,$$

and consequently, for every Borel function f on $\mathbb{R}$,

$$\frac{d}{dt}f(K_u^2) = [-iT_{|u|^2}, f(K_u^2)] .$$

On the other hand, the equation reads

$$\frac{d}{dt}u = -iT_{|u|^2}u - i\alpha(u|1) .$$

Therefore we obtain

$$\begin{aligned}\frac{d}{dt}\Big(f(K_u^2)u|u\Big) &= \Big([-iT_{|u|^2}, f(K_u^2)]u|u\Big) - i\Big(f(K_u^2)T_{|u|^2}u|u\Big) + i\Big(u|f(K_u^2)T_{|u|^2}u\Big)\\ &\quad -i\alpha(u|1)\Big(f(K_u^2)(1)|u\Big) + i\alpha(1|u)\Big(f(K_u^2)(u)|1\Big)\\ &= -i\alpha\Big[\big(f(K_u^2)(1)|(1|u)u\big) - \big((1|u)u|f(K_u^2)(1)\big)\Big] .\end{aligned}$$

Now observe that

$$(1|u)u = H_u^2(1) - K_u^2(1) = T_{|u|^2}(1) - K_u^2(1) .$$

We obtain

$$\begin{aligned}\frac{d}{dt}\Big(f(K_u^2)u|u\Big) &= -i\alpha\Big[(f(K_u^2)(1)|T_{|u|^2}(1)) - (T_{|u|^2}(1)|f(K_u^2)(1))\Big]\\ &= \alpha\Big([-iT_{|u|^2}, f(K_u^2)](1)|1\Big)\\ &= \alpha\frac{d}{dt}\Big(f(K_u^2)(1)|1\Big)\,.\end{aligned}$$

□

3.2. **The α–Szegő hierarchy.** By the theorem above, for any $n \in \mathbb{N}$,

$$L_n(u) := (K_u^{2n}(u) \mid u) - \alpha(K_u^{2n}(1) \mid 1)$$

is conserved. Then the manifold $\mathcal{L}(N)$ is of $2N+1-$ complex dimension and admits $2N+1$ conservation laws, which are

$$\sigma_k,\ k = 1, \cdots, N \text{ and } L_n(u),\ n = 0, 1, \cdots, N\,.$$

We are to show that all these conservation laws are in involve. Since the σ_k's are constants, it is sufficient to show that all these L_n satisfy the Poisson commutation relations

$$\{L_n, L_m\} = 0\,. \tag{3.2}$$

Let us begin with the following lemma which helps us better understand the conserved quantities.

Lemma 3.1. *Let $u \in H^{\frac{1}{2}}(\mathbb{S}^1)$, $\Sigma_H(u) = \{\rho_j\}$ and $\Sigma_K(u) = \{\sigma_k\}$ with*

$$\rho_1 > \sigma_1 > \rho_2 > \cdots \geq 0\,.$$

Denote

$$\begin{aligned}J_x(u) &:= ((1 - xH_u^2)^{-1}(1) \mid 1)\,,\\ Z_x(u) &:= (1 \mid (1 - xH_u^2)^{-1}(u))\,,\\ F_x(u) &:= ((1 - xK_u^2)^{-1}(u) \mid u)\,,\\ E_x(u) &:= ((1 - xK_u^2)^{-1}(1) \mid 1)\,.\end{aligned}$$

Then

$$F_x(u) = \frac{J_x(u) - 1}{xJ_x(u)}\,, \tag{3.3}$$

$$E_x(u) = J_x(u) - x\frac{|Z_x(u)|^2}{J_x(u)}\,. \tag{3.4}$$

Proof. Recall (1.11), for any $f \in H^{\frac{1}{2}}$, we have

$$K_u^2 f = H_u^2 f - (f \mid u)u\,.$$

Denote

$$w(f) = (1 - xH_u^2)^{-1}(f) - (1 - xK_u^2)^{-1}(f)\,, \tag{3.5}$$

then

$$w(f) = x\Big(f \mid (1-xK_u^2)^{-1}(u)\Big)(1-xH_u^2)^{-1}(u)$$
$$= x\Big(f \mid (1-xH_u^2)^{-1}(u)\Big)(1-xK_u^2)^{-1}(u)\,.$$

We may observe the two vectors $(1-xH_u^2)^{-1}(u)$ and $(1-xK_u^2)^{-1}(u)$ are co-linear,

$$(1-xK_u^2)^{-1}(u) = A(1-xH_u^2)^{-1}(u),\ A \in \mathbb{R}\,. \tag{3.6}$$

Let us choose $f = u$, then

$$\Big(w(u) \mid u\Big) = (1-A)\Big((1-xH_u^2)^{-1}(u) \mid u\Big) = Ax\Big(u \mid (1-xH_u^2)^{-1}(u)\Big)^2\,. \tag{3.7}$$

We are to calculate the factor A. Since

$$x\Big(u \mid (1-xH_u^2)^{-1}(u)\Big) = x\Big(1 \mid (1-xH_u^2)^{-1}H_u^2(1)\Big)$$
$$= \sum_{n\geq 0} x^{n+1}\Big(H_u^{2(n+1)}(1) \mid 1\Big) = \sum_{n\geq 0} x^n\Big(H_u^{2n}(1) \mid 1\Big) - 1 = J_x - 1\,,$$

thus (3.7) yields

$$1 - A = (J_x - 1)A\,,$$

which means

$$A = \frac{1}{J_x}\,.$$

So (3.6) turns out to be

$$(1-xK_u^2)^{-1}(u) = \frac{1}{J_x}(1-xH_u^2)^{-1}(u)\,, \tag{3.8}$$

then combined with the definition of $w(f)$, we have

$$(1-xH_u^2)^{-1}(f) - (1-xK_u^2)^{-1}(f) = \frac{x}{J_x}\Big(f \mid (1-xH_u^2)^{-1}(u)\Big)(1-xH_u^2)^{-1}(u)\,. \tag{3.9}$$

Using the equality (3.8),

$$F_x = \Big((1-xK_u^2)^{-1}(u) \mid u\Big) = \frac{1}{J(x)}\Big((1-xH_u^2)^{-1}(u) \mid u\Big)$$
$$= \frac{1}{J(x)}\Big((1-xH_u^2)^{-1}H_u^2(1) \mid 1\Big) = \frac{J_x - 1}{xJ_x}\,.$$

Now, we turn to prove (3.4). Use again (3.5) with $f = 1$,

$$\begin{aligned}\Big(w(1)|1\Big) &= \Big((1-xH_u^2)^{-1}(1) - (1-xK_u^2)^{-1}(1)|1\Big) = J_x - E_x\\ &= x\Big(1|1-xH_u^2)^{-1}(1)\Big)\Big((1-xK_u^2)^{-1}(1)|1\Big) = x\overline{Z_x}\Big((1-xK_u^2)^{-1}(u)|1\Big)\,,\end{aligned}$$

plugging (3.6),

$$\Big((1-xK_u^2)^{-1}(u)|1\Big) = \frac{1}{J_x}\Big((1-xH_u^2)^{-1}(u)|1\Big) = \frac{Z_x}{J_x}\,,$$

then

$$J_x - E_x = x\overline{Z_x}\frac{Z_x}{J_x} = x\frac{|Z_x|^2}{J_x}\,, \tag{3.10}$$

which leads to (3.4). □

Now, we are ready to show the following cancellation for the Poisson brackets of the conservation laws.

Theorem 3.2. *For any $x \in \mathbb{R}$, we set*

$$L_x(u) = ((1 - xK_u^2)^{-1}(u) \mid u) - \alpha((1 - xK_u^2)^{-1}(1) \mid 1) ,$$

Then $L_x(u(t))$ is conserved, and for every x, y,

$$\{L_x,\ L_y\} = 0 . \tag{3.11}$$

Proof. Using the previous Lemma, we may rewrite

$$L_x = \frac{1}{x}(1 - \frac{1}{J_x}) - \alpha E_x , \tag{3.12}$$

with

$$\begin{aligned} J_x(u) &:= ((1 - xH_u^2)^{-1}(1) \mid 1) = 1 + x((1 - xH_u^2)^{-1}(u) \mid u) , \\ E_x(u) &:= ((1 - xK_u^2)^{-1}(1) \mid 1) = J_x(u) - x\frac{|Z_x(u)|^2}{J_x(u)} , \\ Z_x(u) &:= (1 \mid (1 - xH_u^2)^{-1}(u)) . \end{aligned}$$

Recall that the identity

$$\{J_x,\ J_y\} = 0 \tag{3.13}$$

which was obtained in [5, section 8]. We then have

$$\{L_x,\ L_y\} = \alpha\Big(\frac{y}{xJ_x^2J_y}\{J_x,\ |Z_y|^2\} - \frac{x}{yJ_y^2J_x}\{J_y,\ |Z_x|^2\}\Big) + \alpha^2\{E_x,\ E_y\} . \tag{3.14}$$

Let us first prove that $\{E_x,\ E_y\} = 0$. Notice that

$$E_x(u) = J_x(S^*u) , \tag{3.15}$$

therefore

$$dE_x(u) \cdot h = dJ_x(S^*u) \cdot (S^*h) = \omega(S^*h, X_{J_x}(S^*u)) = \omega(h, SX_{J_x}(S^*u)) .$$

We conclude

$$X_{E_x}(u) = SX_{J_x}(s^*U) ,$$

thus

$$\begin{aligned} \{E_x,\ E_y\}(u) &= dE_y(u) \cdot X_{E_x}(u) = dJ_y(S^*u) \cdot S^*SX_{J_x}(S^*u) \\ &= dJ_y(S^*u) \cdot X_{J_x}(S^*u) = \{J_x, J_y\}(S^*u) = 0 . \end{aligned}$$

We now show that the coefficient of α in (3.14) vanishes identically. It is enough to work on the generic states of $\mathcal{L}(N)$, so we can use the coordinates

$$(\rho_1, \cdots, \rho_{N+1}, \sigma_1, \cdots, \sigma_N, \varphi_1, \cdots, \varphi_{N+1}, \theta_1, \cdots, \theta_N)$$

for which we recall that

$$\omega = \sum_{j=1}^{N+1} d(\frac{\rho_j^2}{2}) \wedge d\varphi_j + \sum_{k=1}^{N} d(\frac{\sigma_k^2}{2}) \wedge d\theta_k .$$

Moreover, we have

$$\rho_j u_j = \mathrm{e}^{-i\varphi_j} H_u(u_j) ,$$

therefore,

$$Z_x(u) = \sum_{j=1}^{N+1} \frac{\|u_j\|^2}{\rho_j(1-x\rho_j^2)} \mathrm{e}^{i\varphi_j} \ .$$

Since

$$J_x(u) = \frac{\prod_{k=1}^{N}(1-x\sigma_k^2)}{\prod_{j=1}^{N+1}(1-x\rho_j^2)} \ ,$$

we know that

$$\{J_x,\ \varphi_j\} = \frac{2xJ_x}{1-x\rho_j^2} \ ,$$

and we infer

$$\{J_x,\ Z_y\} = 2ixJ_x \sum_{j=1}^{N+1} \frac{\|u_j\|^2}{\rho_j(1-x\rho_j^2)(1-y\rho_j^2)} \mathrm{e}^{i\varphi_j} = \frac{2ixJ_x}{x-y}(xZ_x - yZ_y) \ . \tag{3.16}$$

Consequently,

$$\{J_x,\ |Z_y|^2\} = 2\mathrm{Re}(\overline{Z}_y\{J_x,\ Z_y\}) = -\frac{4x^2J_x}{x-y}\mathrm{Im}(\overline{Z}_yZ_x) \ . \tag{3.17}$$

We conclude that

$$\frac{y}{xJ_x^2J_y}\{J_x,\ |Z_y|^2\} - \frac{x}{yJ_y^2J_x}\{J_y,\ |Z_x|^2\} = -\frac{4xy}{(x-y)J_xJ_y}(\mathrm{Im}(\overline{Z}_yZ_x) + \mathrm{Im}(\overline{Z}_xZ_y)) = 0 \ . \tag{3.18}$$

This completes the proof. □

The last part of this section is devoted to proving that functions $(L_n(u))_{0\le n\le N}$ are generically independent on $\mathcal{L}(N)$. Actually, it is sufficient to discuss the case $|\alpha| << 1$. For α small enough, we may consider the term $\alpha(K_u^{2n}(1)|1)$ as a perturbation, then we only need to study the independence of $F_n := (K_u^{2n}(u)|u)$. Using the formula (3.12), for any $0 \le n \le N$,

$$F_n = J_{n+1} - \sum_{\substack{k+j=n \\ j\ge 1, k\ge 0}} F_kJ_j \ ,$$

with $J_n = (H_u^{2n}1|1)$. Assume there exists a sequence c_n such that

$$\sum_{n\ge 0} c_nF_n = 0 \ ,$$

we are to prove that $c_n \equiv 0$. Indeed,

$$\sum_{n\ge 0} c_nJ_{n+1} - \sum_{n\ge 0}\sum_{\substack{k+j=n \\ j\ge 1, k\ge 0}} c_nF_kJ_j = \sum_n (c_n - \sum_{0\le k\le N-(n+1)} c_{n+k+1}F_k)J_{n+1} = 0 \ ,$$

since all the J_{n+1} are independent in the complement of a closed subset of measure 0 of $\mathcal{L}(N)$ [5], then for every n,

$$c_n - \sum_{0\le k\le N-(n+1)} c_{n+k+1}F_k = 0 \ .$$

Thus $c_N = c_{N-1} = \cdots = c_0 = 0$.

Finally, we now have $2N+1$ linearly independent and in involution conservation laws on a dense open subset of $2N+1$ dimensional complex manifold $\mathcal{L}(N)$, thus our system is completely integrable in the Liouville sense.

4. Multiplicity and Blaschke product

Recall the notation in section 2, there are two kinds of eigenvalues of K_u, some are the dominant eigenvalues of K_u, which are denoted as $\sigma_k \in \Sigma_K(u)$, while the others are the dominant eigenvalues of H_u with multiplicities larger than 1. Let us denote $u(t)$ as the solution of the α–Szegő equation with $\alpha \neq 0$. Fortunately, we are able to show that for almost all $t \in \mathbb{R}$, the Hankel operator $H_{u(t)}$ has single dominant eigenvalues with multiplicities equal to 1. In other words, for almost every time $t \in \mathbb{R}$,

$$\mathrm{rk_d} K_{u(t)} = \mathrm{rk} K_{u(t)} = \mathrm{rk} K_{u_0} \ .$$

We call the phenomenon that $H_{u(t_0)}$ has some eigenvalue σ with multiplicity $m \geq 2$ as *crossing at σ at t_0*.

4.1. **The motion of singular values.** Let us first introduce the following Kato-type lemma.

Lemma 4.1 (Kato)**.** *Let $P(t)$ be a projector on a Hilbert space $\mathcal{H}$ which is smooth in $t \in I$, then there exists a smooth unitary operator $U(t)$, such that*

$$P(t) = U(t)P(0)U^*(t) \ ,$$

and

$$\frac{d}{dt}U(t) = Q(t)U(t) \ , \ U(0) = \mathrm{Id} \ , \tag{4.1}$$

with $Q(t) = [P'(t), P(t)]$.

Proof. By simple calculus, we can prove $Q^* = -Q$. Since $P(t)$ is smooth in time, then by the Cauchy theorem for linear ordinary equations, $U(t)$ is well defined. The unitary property of $U(t)$ for every t is a consequence of the anti self-adjointness of Q.

$$\frac{d}{dt}(U(t)^*U(t)) = \frac{d}{dt}U^*U + U^*\frac{d}{dt}U = U^*Q^*U + U^*QU = 0 \ ,$$

thus $U(t)^*U(t) = \mathrm{Id}$. On the other hand,

$$\frac{d}{dt}(U(t)U(t)^*) = \frac{d}{dt}UU^* + U\frac{d}{dt}U^* = QUU^* - UU^*Q \ .$$

It is obvious that Id is a solution to the linear equation $\frac{d}{dt}A = QA - AQ$ with $A(0) = \mathrm{Id}$, using the uniqueness of solutions, we have $U(t)U^*(t) = \mathrm{Id}$. We now prove that $U^*(t)P(t)U(t)$ does not depend on t.

$$\begin{aligned}
\frac{d}{dt}(U^*(t)P(t)U(t)) &= \frac{d}{dt}U^*(t)P(t)U(t) + U^*(t)\frac{d}{dt}P(t)U(t) + U^*(t)P(t)\frac{d}{dt}U(t) \\
&= U^*Q^*PU + U^*P'U + U^*PQU \\
&= U^*(P' + [P,Q])U \\
&= U^*(P' - PP' - P'P)U = 0
\end{aligned}$$

where we have used $P^2 = P$. This completes the proof. □

If $u_0 \in H^s_+$ with $s > 1$, then the solution $u(t)$ of the α–Szegő equation (1.1) is real analytic in t valued in H^s_+. By the Lax pair for K_u, we know that the singular values of K_u are fixed, with constant multiplicities.

Proposition 4.1. *Given any initial data $u_0 \in H^s_+$ with $s > 1$, let u be the corresponding solution to the α–Szegő equation. Let $\sigma > 0$ be a singular eigenvalue of K_u with multiplicity m, and write*

$$\sigma_+ > \sigma > \sigma_-$$

where σ_+, σ_- are the closest singular values of K_u, possibly, $\sigma_+ = +\infty$ or $\sigma_- = 0$. Then one of the following two possibilities occurs.

(1) *σ is a singular value of $H_{u(t)}$ with multiplicity $m + 1$ for every time t, and u is a solution of the cubic Szegő equation (1.2).*

(2) *There exists a discrete subset T_c of times outside of which the singular values of $H_{u(t)}$ in the interval (σ_-, σ_+) are ρ_1, ρ_2 of multiplicity 1, and σ of multiplicity $m - 1$ if $m \geq 2$, with*

$$\rho_1 > \sigma > \rho_2 \, ,$$

and ρ_1, ρ_2 are analytic on every interval contained into the complement of T_c.

Proof. Let us assume that σ is a singular value of multiplicity $m + 1$ of $H_{u(t_0)}$ for some time t_0. Then we may select $\delta > 0$ and $\epsilon > 0$ such that

$$\sigma_+ > \sigma + \epsilon > \sigma > \sigma - \epsilon > \sigma_-$$

such that, for every $t \in [t_0 - \delta, t_0 + \delta]$, $\sigma^2 - \epsilon$ and $\sigma^2 + \epsilon$ are not eigenvalues of $H^2_{u(t)}$. Then we know that $H^2_{u(t)}$ has either σ^2 as an eigenvalue of multiplicity $m+1$, or admits in $(\sigma^2-\epsilon, \sigma^2+\epsilon)$ two eigenvalues of multiplicity 1, ρ_1, ρ_2 on both sides of σ. Set

$$P(t) := (2i\pi)^{-1} \int_{C(\sigma^2,\epsilon)} (z\mathrm{Id} - H^2_{u(t)})^{-1} dz \, . \tag{4.2}$$

We know that $P(t)$ is an orthogonal projector, depending analytically of $t \in (t_0 - \delta, t_0 + \delta)$, and that $P(t_0)$ is just the projector onto

$$E(t_0) := \ker(H^2_{u(t_0)} - \sigma^2 \mathrm{Id}) \, .$$

Consider the selfadjoint operator

$$A(t) := H^2_{u(t)} P(t)$$

acting on the $(m + 1)$-dimensional space $E(t) = \mathrm{Ran} P(t)$. Then its characteristic polynomial is

$$\mathcal{P}(\lambda, t) = (\sigma^2 - \lambda)^{m-1}(\lambda^2 + a(t)\lambda + b(t)) \, ,$$

where a, b are real analytic, real valued functions, such that

$$a^2 - 4b \geq 0 \, .$$

Notice that the condition $a(t)^2 - 4b(t) = 0$ is precisely equivalent to the fact that $H^2_{u(t)}$ has σ^2 as an eigenvalue of multiplicity $m + 1$. Since this function is analytic, it is either identically 0, or different from 0 for $0 < |t - t_0| < \delta$ and $\delta > 0$ small enough. Moreover, by the following perturbation analysis, the first condition only occurs if

$$(1|u(t)) = 0$$

for every $t \in (t_0 - \delta, t_0 + \delta)$. Since $(1|u)$ is a real analytic function of t, this would imply that it is identically 0, whence u is a solution of the cubic Szegő equation. We now come back to the perturbation analysis, let $U(t)$ be a unitary operator given as in the Kato-type lemma above, denote

$$B(t) = U^*(t)A(t)U(t) ,$$

then

$$B(t_0) = \sigma^2 \mathrm{Id} P(t_0) .$$

Let us calculate the derivative of B, we find

$$\frac{d}{dt}B(t) = \frac{d}{dt}\left(U^*(t)H^2_{u(t)}U(t)U^*(t)P(t)U(t)\right) = \frac{d}{dt}\left(U^*(t)H^2_{u(t)}U(t)P(t_0)\right) .$$

Since $\frac{d}{dt}U(t) = Q(t)U(t)$ with $Q(t) = [P'(t), P(t)]$, then

$$\frac{d}{dt}B(t) = U^*\Big(\frac{d}{dt}H^2_{u(t)} + [H^2_{u(t)}, Q(t)]\Big)UP(t_0) ,$$

using (1.13),

$$\frac{d}{dt}H^2_{u(t)} = [B_u, H^2_u] - i\alpha(u|1)H_1H_u + i\alpha(1|u)H_uH_1 .$$

For any $h_1, h_2 \in E(t_0)$,

$$([B_u, H^2_u]h_1, h_2) + ([H^2_u, Q]h_1, h_2) = 0 ,$$

then

$$(\frac{d}{dt}B(t_0)h_1, h_2) = -i\alpha[(u(t_0)|1)(h_1|u(t_0))(1|h_2) - (1|u(t_0))(u(t_0)|h_2)(h_1|1)] .$$

Denote by v, w as the projections onto $E(t_0)$ of 1 and u respectively. If $(u(t_0)|1) \neq 0$, then the corresponding matrix under the base (v, w) turns out to be

$$\begin{pmatrix} -i\alpha(u|1)(v|w) & i\alpha(1|u)\|w\|^2 \\ -i\alpha(u|1)\|v\|^2 & i\alpha(1|u)(w|v) \end{pmatrix}$$

which has a negative determinant if $(u(t_0)|1) \neq 0$. For the case $(u(t_0)|1) = 0$ with $\frac{d^n}{dt^n}(u|1)(t_0) \neq 0$ for some $n \in \mathbb{N}$, we only need to consider $\frac{d^{n+1}}{dt^{n+1}}(B(t))(t_0)$,

$$\left(\frac{d^{n+1}}{dt^{n+1}}B(t_0)h_1, h_2\right) = -i\alpha\left[\Big(\frac{d^n}{dt^n}(u|1)\Big)(t_0)(h_1|u(t_0))(1|h_2) - \Big(\frac{d^n}{dt^n}(1|u)\Big)(t_0)(u(t_0)|h_2)(h_1|1)\right] ,$$

with any $h_1, h_2 \in E(t_0)$. It is similar as the case $n = 0$. This completes the proof. □

Since $u(t)$ satisfying $(1|u(t)) \equiv 0$ would be a solution of the cubic Szegő equation, which is well studied by Gérard and Grellier [5, 7, 6, 10]. We assume $(1|u)$ is not identically zero in the rest of this article. From the discussion above, we have

Corollary 4.1. *The dominant eigenvalues of $H_{u(t)}$ are of multiplicity 1 for almost all $t \in \mathbb{R}$.*

Recall the notation in section 2, by rewriting the conservation laws in Theorem 3.1 as

$$L_n := \Big(K^{2n}_u(u) \mid u\Big) - \alpha\Big(K^{2n}_u(1) \mid 1\Big) = \sum_k \sigma_k^{2n}\left(\|u_k'\|^2 - \alpha\|v_k'\|^2\right) , \tag{4.3}$$

we get the following conserved quantities

$$\ell_k := \|u_k'\|^2 - \alpha\|v_k'\|^2 \,. \tag{4.4}$$

Lemma 4.2. *Let $\alpha > 0$. If there exists a crossing at σ_k at time $t = t_0$, then $\ell_k < 0$.*

Proof. Since there is a crossing at σ_k, then $\sigma_k \in \Sigma_H(u(t_0))$ with multiplicity $m \geq 2$. Then

$$F_u(\sigma_k) = E_u(\sigma_k) \cap u^\perp = \left\{ \frac{g}{D} H_u(u_k) : \ g \in \mathbb{C}_{m-2}[z] \right\} \,.$$

Hence, $u_k' = 0$ while $v_k' \neq 0$, since

$$\|v_k'\| = \frac{(1, H_u(u_k))}{\|H_u(u_k)\|} = \frac{\|u_k\|}{\sigma_k} \neq 0 \,. \tag{4.5}$$

Thus $\ell_k = \|u_k'\|^2 - \alpha\|v_k'\|^2 < 0$ for $\alpha > 0$. □

Here, we present an example to show the existence of crossing.

Example 4.1 (Existence of crossing)**.** *Let $u_0(z) = \frac{z-p}{1-pz}$ with $p \neq 0$ and $|p| < 1$, and u be the corresponding solution to the equation*

$$i\partial_t u = \Pi(|u|^2 u) + (u|1) \,. \tag{4.6}$$

It is obvious that $u_0 \in \mathcal{L}(1)$ and $1 \in \Sigma_H(u_0)$ with multiplicity 2, and

$$L_1(u) = \left(K_u^2(u) \mid u\right) - \left(K_u^2(1) \mid 1\right) = -(1 - |p|^2) < 0 \,.$$

Let us represent the Hamiltonian function $E = \frac{1}{4}\|u\|_{L^4}^4 + \frac{1}{2}|(u|1)|^2$ under the coordinates

$$\rho_1, \rho_2, \sigma, \varphi_1, \varphi_2, \theta \,,$$

$$\begin{aligned} E &= \frac{1}{4}(\rho_1^4 + \rho_2^4 - \sigma^4) \\ &\quad + \frac{1}{2}\frac{\rho_1^2(\rho_1^2 - \sigma^2)^2 + \rho_2^2(\sigma^2 - \rho_2^2)^2 + 2\rho_1\rho_2(\rho_1^2 - \sigma^2)(\sigma^2 - \rho_2^2)\cos(\varphi_1 - \varphi_2)}{(\rho_1^2 - \rho_2^2)^2} \\ &= \frac{1}{4} + \frac{1}{2}|p|^2 \,. \end{aligned}$$

Notice that $\sigma = 1$ and $\rho_1^2 + \rho_2^2 - \sigma^2 = \|u\|_{L^2}^2 = 1$, then $\rho_1^2 + \rho_2^2 = 2$. Set $I = \frac{\rho_1^2 - \rho_2^2}{2}$, $\varphi = \varphi_1 - \varphi_2$, then $\rho_1^2 = 1 + I$ and $\rho_2^2 = 1 - I$, thus we can rewrite E as

$$E = \frac{1}{4}(1 + 2I^2) + \frac{1}{4}(1 + \sqrt{1 - I^2}\cos(\varphi)) \,.$$

Thus

$$\begin{aligned} \frac{dI}{dt} &= -2\frac{\partial E}{\partial \varphi} = \frac{1}{2}\sqrt{1 - I^2}\sin(\varphi) \\ &= \pm\frac{1}{2}\sqrt{-4I^2 + (8|p|^2 - 5)I^2 + 4|p|^2(1 - |p|^2)} \\ &= \pm\sqrt{(a - I^2)(b + I^2)} \,, \end{aligned}$$

with a, b satisfy

$$\begin{cases} a > 0, b > 0 \,, \\ ab = |p|^2(1-|p|^2) \,, \\ a - b = 2|p|^2 - 5/4 \,. \end{cases}$$

Recall the definition of Jacobi elliptic functions. The incomplete elliptic integral of the first kind F is defined as

$$F(\varphi, k) \equiv \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \,,$$

then the Jacobi elliptic function sn and cn are defined as follows,

$$sn(F(\varphi, k), k) = \sin \varphi \,,$$
$$cn(F(\varphi, k), k) = \cos \varphi \,.$$

Then we may solve the above equation,

$$I(t) = \sqrt{a}\mathrm{cn}\left(\sqrt{a+b}(t - t_0) + F\Big(\frac{\pi}{2}, \sqrt{\frac{a}{a+b}}\Big), \sqrt{\frac{a}{a+b}} \right) .$$

Therefore, there exists a discrete set of time $0 \in T_c$, such that $I(t) = 0$ for every $t \in T_c$. In other words, crossing happens at $t \in T_c$.

4.2. **Blaschke product.** We aim to show that the Blaschke products $\Psi(t)$ of $K_{u(t)}$ do not change their $\mathbb{S}^1$–orbits as times grows even before or after crossings.

Proposition 4.2. *For any open interval Ω contained into the complement of T_c, for any $\sigma_k \in \Sigma_K(u(t))$ with $t \in \Omega$,*

$$K_{u(t)} u_k'(t) = \sigma_k \Psi_k(t) u_k'(t) \,. \tag{4.7}$$

Then there exists a function $\psi_k(t) : \Omega \to \mathbb{S}^1$, such that

$$\Psi_k(t) = \mathrm{e}^{i\psi_k(t)} \Psi_k(0) \,, \ t \in \Omega \,. \tag{4.8}$$

Proof. Differentiating the above equation (4.7) and using the Lax pair structure (1.14), one obtains

$$[C_u, K_u](u_k') + K_u\left(\frac{du_k'}{dt}\right) = \sigma_k \dot{\Psi}_k u_k' + \sigma_k \Psi_k \frac{du_k'}{dt} \,. \tag{4.9}$$

Recall $u_k' = P_k(u)$, where P_k as (4.2) by replacing H_u with K_u, then

$$\frac{d}{dt} P_k(t) = [C_u, P_k] \,.$$

Rewriting $\Pi(|u|^2 u) = T_{|u|^2}(u) = (iC_u + \frac{1}{2}K_u^2)u$, then the α–Szegő equation (1.1) turns out to be

$$\frac{du}{dt} = (C_u - \frac{i}{2}K_u^2)u - i\alpha(u|1) \,,$$

then

$$\begin{aligned} \frac{du_k'}{dt} &= (\frac{d}{dt}P_k)(u) + P_k(\frac{du}{dt}) \\ &= [C_u, P_k]u + P_k C_u u - \frac{i}{2}K_u^2 P_k(u) - i\alpha(u|1)P_k(1) \end{aligned}$$

thus

$$\frac{du'_k}{dt} = -iT_{|u|^2}u'_k - i\alpha(u \mid 1)\frac{(1 \mid u'_k)}{(u'_k \mid u'_k)}u'_k \,. \tag{4.10}$$

Then (4.9) and (4.10) obtained above lead to

$$\left(\dot{\Psi}_k - i\left(\sigma_k^2 + 2\alpha \mathrm{Re}[\frac{(u \mid 1)(1 \mid u'_k)}{(u'_k \mid u'_k)}]\right)\Psi_k\right)u'_k = -i[T_{|u|^2}, \Psi_k](u'_k) \,.$$

We claim that

$$[T_{|u|^2}, \Psi_k](u'_k) = 0 \,.$$

therefore

$$\Psi_k(t) = \mathrm{e}^{i(\sigma_k^2 t + \gamma_k(t))}\Psi_k(0) \,,$$

where

$$\gamma_k(t) = 2\alpha \int_0^t \frac{\mathrm{Re}[(u(t') \mid 1)(1 \mid u'_k(t')]}{|u'_k(t')|^2}\, dt' \,.$$

It remains to prove the claim (one can also refer to [9, Theorem 8] for the proof). We first prove that, for any $\chi_p(z) = \frac{z-p}{1-pz}$ with $|p| < 1$,

$$[T_{|u|^2}, \chi_p]f = 0$$

for any $f \in F_u(\sigma_k)$ such that $\chi_p f \in F_u(\sigma_k)$. For any L^2 function g,

$$[\Pi, \chi_p]g = (1 - |p|^2)H_{1/(1-\overline{p}z)}(h) \,,$$

where $\overline{(\mathrm{Id} - \Pi)g} = Sh$. Consequently, the range of $[\Pi, \chi_p]$ is one dimensional, directed by $\frac{1}{1-\overline{p}z}$. In particular, $[T_{|u|^2}, \chi_p]f$ is proportional to $\frac{1}{1-\overline{p}z}$. Since

$$\begin{aligned}
([T_{|u|^2}, \chi_p]f|1) &= (T_{|u|^2}\chi_p f - \chi_p T_{|u|^2} f|1) \\
&= (\chi_p f|H_u^2(1)) - (\chi_p|1)(H_u^2 f|1) \\
&= (H_u^2(\chi_p f)|1) - (\chi_p|1)(H_u^2 f|1) \\
&= (\chi_p f - (\chi_p|1)f|u)(u|1) \,,
\end{aligned}$$

We used (3.6) to gain the last equality. Since $\chi_p f - (\chi_p|1)f \in F_u(\sigma_k)$ is orthogonal to 1, by Proposition 2.2, $\chi_p f - (\chi_p|1)f \in E_u(\sigma_k)$, hence $\chi_p f - (\chi_p|1)f \in F_u(\sigma_k)$ is orthogonal to u. This proves that $[T_{|u|^2}, \chi_p]f = 0$. □

Therefore, we have

Corollary 4.2.

$$\mathrm{rk}K_{u(t)} = \mathrm{rk}_{\mathrm{d}}K_{u(t)} = \mathrm{rk}K_{u_0}, \ a.e.\ t < \infty.$$

We know that $\Psi_k(t)$ is defined for every t in an open subset Ω of $\mathbb{R}$ consisting of the complement of a discrete closed subset, corresponding to crossings at σ_k^2. Furthermore, by Proposition 4.2, on each connected component of Ω, the zeroes of $\Psi_k(t)$ are constant. Together with the following property, $\Psi_k(t)$ never changes it orbit even after the crossings.

Proposition 4.3. *For every time t such that $\Psi_k(t)$ is defined, the zeroes of $\Psi_k(t)$ are the same.*

Proof. The proposition is a consequence of the following lemma. □

Lemma 4.3. *There exists an analytic function $\Psi_k^\sharp$ defined in a neighborhood Ω' of Ω^c and valued into rational functions, and, for every $t \in \Omega \cap \Omega'$, there exists $\beta(t) \in \mathbb{T}$ such that*

$$\Psi_k(t,z) = \mathrm{e}^{i\beta_k(t)}\Psi_k^\sharp(t,z)\,.$$

Proof. Since σ_k^2 is an eigenvalue of constant multiplicity m of $K_{u(t)}^2$, the orthogonal projector $P_k(t)$ onto $F_{u(t)}(\sigma_k)$ is an analytic function of $t \in \mathbb{R}$. Consequently, the vector

$$v_k'(t) := P_k(t)(1)$$

depends analytically on t. Furthermore, $v_k'(t)$ is not 0 if $t \notin \Omega$. Indeed, from the description of $F_u(\tau)$ provided by Proposition 2.2 when τ is a singular value associated to the pair (H_u, K_u), we observe that, if τ is H dominant, the space $F_u(\tau)$ is not orthogonal to 1. Consequently, we can define, for t in a neighborhood Ω' of Ω^c,

$$\Psi_k^\sharp(t,z) := \frac{K_{u(t)}(v_k'(t))(z)}{\sigma_k v_k'(t,z)}$$

as an analytic function of t valued into rational functions of z. On the other hand, if $t \in \Omega$, Proposition 2.2 shows that

$$F_{u(t)}(\sigma_k) \cap u(t)^\perp = E_{u(t)}(\sigma_k) = F_{u(t)}(\sigma_k) \cap 1^\perp\,,$$

therefore $v_k'(t)$ is collinear to $u_k'(t)$,

$$v_k'(t) = (1|u_k'(t))\frac{u_k'(t)}{\|u_k'(t)\|^2}\,.$$

Since, from the definition of $\Psi_k(t)$,

$$K_{u(t)}(u_k'(t)) = \sigma_k\Psi_k(t)u_k'(t)\,,$$

we infer that there exists an analytic β_k on $\Omega \cap \Omega'$ valued into $\mathbb{T}$ such that

$$K_{u(t)}(v_k'(t)) = \sigma_k\mathrm{e}^{-i\beta_k(t)}\Psi_k(t)v_k'(t)\,.$$

This completes the proof. □

5. Necessary condition of norm explosion

In this section, let $u(t)$ be the solution of α–Szegő equation (1.1) with initial data $u_0 \in \mathcal{L}(N)$, $N \in \mathbb{N}^+$, $u^\infty = \lim u(t_n)$ for the weak * topology of $H^{1/2}$, for some sequence t_n going to infinity. To study the large time behavior of solutions, it is equivalent to study the rank of the shifted Hankel operator K_u.

Lemma 5.1. *The solution $u(t)$ to the α–Szegő equation will stay in a compact subset of $\mathcal{L}(N)$ if and only if for all the adherent values u^∞ of $u(t)$ at infinity,*

$$\mathrm{rk}K_{u^\infty} = \mathrm{rk}K_{u_0}\,. \tag{5.1}$$

Proof. By the explicit formula of functions in $\mathcal{L}(N) \subset H^s$ for every s in Theorem 1.3, $\mathrm{rk}u(t) = N$ if and only if

$$u(z) = \frac{A(z)}{B(z)}$$

with $A, B \in \mathbb{C}_N[z], A \wedge B = 1, \deg(A) = N$ or $\deg(B) = N, B^{-1}(\{0\}) \cap \overline{\mathbb{D}} = \emptyset$.

Then a sequence of $(u_n)_n$ is in a relatively compact subset of $\mathcal{L}(N)$ unless one of the poles of u_n approaches the unit disk $\mathbb{D}$, then the corresponding limit $u(z)$ will be in some $\mathcal{L}(N')$ with $N' < N$. □

We first present a necessary condition of the norm explosion for any $\alpha \in \mathbb{R} \setminus \{0\}$.

Theorem 5.1. *If* $\mathrm{rk}K_{u^\infty} < \mathrm{rk}K_{u_0}$, *then there exists some* k *such that* $\ell_k(u_0) = 0$.

Corollary 5.1. *If* $\alpha < 0$, *for any* $N \in \mathbb{N}^+$, *given initial data* $u_0 \in \mathcal{L}(N)$, *then the solution to the* α*–Szegő equation stays in a compact subset of* $\mathcal{L}(N)$.

Proof of Corollary 5.1. Since $\alpha < 0$, then $\ell_k := \|u_k'\|^2 - \alpha\|v_k'\|^2 > 0$, due to Theorem 5.1, $\mathrm{rk}K_{u^\infty} \equiv \mathrm{rk}K_{u_0}$. □

Proof of Theorem 5.1. Assume $\mathrm{rk}K_{u^\infty} < \mathrm{rk}K_{u_0}$, then there exists some k such that $\dim F_{u^\infty}(\sigma_k) < \dim F_{u_0}(\sigma_k) = m$. We are to prove $\|u_k'^\infty\|^2 = 0$ and $\|v_k'^\infty\|^2 = 0$.

- $\|u_k'^\infty\|^2 = 0$.
  There exists a time dependent Blaschke product Ψ_k of degree $m-1$ such that
  $$K^2_{u(t_n)}(u_k'(t_n)) = \sigma_k^2 u_k'(t_n)\ ,\ K_{u(t_n)}(u_k'(t_n)) = \sigma_k \Psi_k(t_n) u_k'(t_n)\ , \tag{5.2}$$
  By Proposition 4.3, any limit point of $\Psi_k(t)$ as t goes to ∞ is of degree $m-1$ as well. Since $u_k'(t_n)$ is bounded in L^2_+, up to a subsequence it converges weakly to some $u_k'^\infty \in L^2_+$. Passing to the limit in the identities (5.2), we get
  $$K^2_{u^\infty}(u_k'^\infty) = \sigma_k^2 u_k'^\infty\ ,\ K_{u^\infty}(u_k'^\infty) = \sigma_k \Psi_k^\infty u_k'^\infty\ , \tag{5.3}$$
  where Ψ_k^∞ is a Blaschke product of degree $m-1$. The latter identities (5.3) show that $u_k'^\infty$ and $\Psi_k^\infty u_k'^\infty$ belong to $F_{u^\infty}(\sigma_k)$, hence, if $u_k'^\infty$ is not zero, the dimension of $F_{u^\infty}(\sigma_k)$ is at least m. Indeed, if we write $\Psi_k^\infty = \mathrm{e}^{-i\psi}\frac{P(z)}{D(z)}$, then
  $$F_{u^\infty}(\sigma_k) = \left\{ \frac{f}{D(z)} u_k'^\infty,\ f \in \mathbb{C}_{m-1}[z] \right\}\ . \tag{5.4}$$
- $\|v_k'^\infty\|^2 = 0$.
  Recall the structure of $F_u(\sigma_k)$ with $\sigma_k \in \Sigma_K(u)$ in Proposition 2.2, the orthogonal projection of 1 onto the space $F_u(\sigma_k)$, v_k' can be represented as
  $$v_k' = \Big(1 \mid \frac{u_k'}{\|u_k'\|}\Big)\frac{u_k'}{\|u_k'\|}\ .$$
  If $v_k'^\infty \neq 0$, since $\|v_k'\| = \left|(1 \mid \frac{u_k'}{\|u_k'\|})\right|$, thus $\frac{u_k'}{\|u_k'\|} \rightharpoonup v$ in L^2 with $v \neq 0$. Using the strategy in the first step above by replacing u_k' by $\frac{u_k'}{\|u_k'\|}$, we have $\dim F_{u^\infty}(\sigma_k) = m$.

□

6. Large time behavior of the solution for the case $\alpha > 0$

In this section, we prove for any N, there exist solutions in $\mathcal{L}(N)$ which admit an exponential on time norm explosion.

Theorem 6.1. *For $\alpha > 0$, $u_0 \in H^s_+$ such that $\Sigma_K(u_0) = \{\sigma\}$ with multiplicity $k = \mathrm{rk}K_{u_0}$. Then $\|u(t)\|_{H^s}$ grows exponentially on time,*

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_\alpha(2s-1)|t|} ,$$

if and only if

$$L_1(u) := (K_u^2(u)|u) - \alpha(K_u^2(1)|1) = 0 . \tag{6.1}$$

Let u_0 as in the theorem above. If u_0 is not a Blaschke product, we have

$$\Sigma_H(u_0) = \{\rho_1, \rho_2\} , \ \rho_1 > \sigma > \rho_2 .$$

Using the results by Gérard and Grellier [9], we have the explicit formula for the solution u as

$$u(t,z) = \frac{\triangle_{11} - \triangle_{21}}{\det(C(z))}\mathrm{e}^{-i\varphi_1} + \frac{\triangle_{22} - \triangle_{12}}{\det(C(z))}\mathrm{e}^{-i\varphi_2} , \tag{6.2}$$

with $\triangle_{jk}$ as the minor determinant of $C(z)$ corresponding to line k and column j, and

$$C(z) = \begin{pmatrix} \frac{\rho_1 - \sigma z\Psi\mathrm{e}^{-i\varphi_1}}{\rho_1^2-\sigma^2} & \frac{\rho_2 - \sigma z\Psi\mathrm{e}^{-i\varphi_2}}{\rho_2^2-\sigma^2} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} \end{pmatrix}$$

Then

$$u(t,z) = \frac{(\frac{1}{\rho_2} - \frac{\rho_2 - \sigma z\Psi\mathrm{e}^{-i\varphi_2}}{\rho_2^2-\sigma^2})\mathrm{e}^{-i\varphi_1} + (\frac{\rho_1 - \sigma z\Psi\mathrm{e}^{-i\varphi_1}}{\rho_1^2-\sigma^2} - \frac{1}{\rho_1})\mathrm{e}^{-i\varphi_2}}{\frac{1}{\rho_2}(\frac{\rho_1 - \sigma z\Psi\mathrm{e}^{-i\varphi_1}}{\rho_1^2-\sigma^2}) - \frac{1}{\rho_1}(\frac{\rho_2 - \sigma z\Psi\mathrm{e}^{-i\varphi_2}}{\rho_2^2-\sigma^2})} .$$

An interesting fact is that u is under the form

$$u(t,z) = b(t) + \frac{c'(t)z\Psi(t,z)}{1 - p'(t)z\Psi(t,z)} ,$$

where b , p', $c' \in \mathbb{C}$. Since $\Psi(t,z) = \mathrm{e}^{i\psi(t)}\chi(z)$ with χ as a time independent Blaschke product, we then rewrite

$$u(t,z) = b(t) + \frac{c(t)z\chi(z)}{1 - p(t)z\chi(z)} . \tag{6.3}$$

Lemma 6.1. *Let χ be a time-independent Blaschke product. A function $u \in C^\infty(\mathbb{R}, H^s_+)$ with $s > \frac{1}{2}$ is a solution of the α–Szegő equation,*

$$i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1) ,$$

if and only if

$$\widetilde{u}(t,z) := u(t, z\chi(z))$$

satisfies the α–Szegő equation.

Proof. First of all, $z\chi(z) \in C^\infty_+(\mathbb{S}^1)$, then $(z\chi(z))^n \in C^\infty_+(\mathbb{S}^1)$ for any n, so that $u \in H^s_+$ implies $\widetilde{u} \in H^s_+$. Assume u is a solution of the α–Szegő equation, it is equivalent to

$$i\partial_t\hat{u}(t,n) = \sum_{p-q+r=n} \hat{u}(t,p)\overline{\hat{u}}(t,q)\hat{u}(t,r) + \alpha\hat{u}(t,0)\delta_{n0} , \ \forall n \geq 0 . \tag{6.4}$$

Since

$$\Pi(|u(z\chi(z))|^2 u(z\chi(z))) = \sum_{p-q+r\geq 0} \hat{u}(p)\overline{\hat{u}}(q)\hat{u}(r)(z\chi(z))^{p-q+r} ,$$

we obtain that $\widetilde{u}$ satisfies the α–Szegő equation.

Conversely, assume $\widetilde{u}$ satisfies the α–Szegő equation, then we have

$$i\partial_t \hat{u}(n)(z\chi(z))^n = \sum_{p-q+r\geq 0} \hat{u}(p)\overline{\hat{u}}(q)\hat{u}(r)(z\chi(z))^{p-q+r} + \hat{u}(0)\,. \tag{6.5}$$

Identifying the Fourier coefficients of 0 mode of both sides, we get equation (6.4) with $n = 0$. Then withdraw this quantity from both sides of (6.5) and simplify by $z\chi(z)$. Continuing this process, we get all the equations (6.4) for every n. □

Lemma 6.2. *Let Ψ be a Blaschke product of finite degree d and $s \in [0,1)$. There exists $C_{\Psi,s} > 0$ such that, for every $p \in \mathbb{D}$,*

$$\left\| \frac{1}{1-p\Psi} \right\|_{H^s(\mathbb{S}^1)} \geq \frac{C_{\Psi,s}}{(1-|p|)^{s+\frac{1}{2}}}\,.$$

Proof. It is a classical fact that, for every $u \in H^s_+(\mathbb{S}^1)$, for every $s \in [0,1)$,

$$\|u\|^2_{H^s(\mathbb{S}^1)} \simeq \int_{\mathbb{D}} |u'(z)|^2 (1-|z|^2)^{1-2s}\, dL(z)\,,$$

where L denotes the bi-dimensional Lebesgue measure.

Let $p \in \mathbb{D}$ close to the unit circle and

$$\omega := \frac{p}{|p|}\,.$$

Since Ψ is a Blaschke product of finite degree d, the equation

$$\omega\Psi(z) = 1$$

admits d solutions on the circle. Moreover, these solutions are simple. Indeed, writing

$$\Psi(z) = \mathrm{e}^{-i\psi} \prod_{j=1}^{d} \frac{z-p_j}{1-\overline{p}_j z}\,,\ |p_j| < 1\,,$$

we have, for every $z \in \mathbb{S}^1$,

$$\frac{\Psi'(z)}{\Psi(z)} = \frac{1}{z}\sum_{j=1} \frac{1-|p_j|^2}{|z-p_j|^2} \neq 0\,.$$

Let α be such a solution. For every z such that

$$|z-\alpha| \leq (1-|p|),$$

we have, if $1-|p|$ is small enough,

$$|1-p\Psi(z)| = |1-p\Psi(\alpha) - p\Psi'(\alpha)(z-\alpha) + O(|z-\alpha|^2)| \leq C(1-|p|).$$

Therefore

$$\begin{aligned}\left\|\frac{1}{1-p\Psi}\right\|^2_{H^s(\mathbb{S}^1)} &\geq A_s \int\limits_{\mathbb{D}\cap\{|z-\alpha|\leq(1-|p|)\}} \left|\frac{\Psi'(z)}{(1-p\Psi(z))^2}\right|^2 (1-|z|^2)^{1-2s}\, dL(z)\\ &\geq B_{\Psi,s}(1-|p|)^{-4} \int\limits_{\mathbb{D}\cap\{|z-\alpha|\leq(1-|p|)\}} (1-|z|^2)^{1-2s}\, dL(z)\\ &\geq \frac{C^2_{\Psi,s}}{(1-|p|)^{2s+1}}\,.\end{aligned}$$

□

Let us turn back to prove the theorem.

Proof of Theorem 6.1. Recall that

$$\begin{aligned}L_1(u) &= (K^2_u(u)\mid u) - \alpha(K^2_u(1)\mid 1)\\ &= \frac{1}{2}(\|u\|^4_{L^4} - \|u\|^4_{L^2}) - \alpha(\|u\|^2_{L^2} - |(u\mid 1)|^2)\,.\end{aligned}$$

Since $\chi(z)$ is an inner function, we have

$$(\widetilde{u}\mid\widetilde{v}) = (u|v)\,,\forall\ u,v\,,$$

thus

$$(\widetilde{u}|1) = (u|1)\,,\|\widetilde{u}\|_{L^2} = \|u\|_{L^2}\,,$$

and since

$$\widetilde{u^2} = (\widetilde{u})^2\,,$$

then

$$\|\widetilde{u}\|_{L^4} = \|u\|_{L^4}\,.$$

As a consequence, $L_1(u) = L_1(\widetilde{u}) = 0$.

The solution $\widetilde{u}$ is under the form (6.3),

$$u(t,z) = b(t) + \frac{c(t)z\chi(z)}{1-p(t)z\chi(z)} = b - \frac{c}{p} + \frac{c}{p}\frac{1}{1-pz\chi(z)}\,,$$

thus

$$\begin{aligned}\|u\|_{H^s} &\simeq |c|\|\frac{1}{1-pz\chi(z)}\|_{H^s}\\ &\geq C_{\chi,s}\frac{|c|}{(1-|p|)^{s+1/2}}\,,\end{aligned}$$

where we used Lemma 6.2. Using the result in [22, Theorem 3.1] and its proof, we have

$$\frac{|c|}{(1-|p|)^{s+1/2}} \simeq (1-|p|)^{-s+1/2} \simeq \mathrm{e}^{C_\alpha(2s-1)|t|}\,.$$

Therefore, $\widetilde{u}$ admit an exponential on time growth of the Sobolev norm H^s with $s > \frac{1}{2}$. The proof is complete. □

7. Perspectives

The main purpose of this work is to study the dynamics of the general solutions of the α–Szegő equation (1.1). We have already observed the weak turbulence by considering some special rational data. We proved the existence of data with exponential in time growth, a natural question is about the genericity of data with such a high growth. Besides, an important open problem is to gain new informations on the solutions with infinite rank.

Another interesting question is about the cubic Szegő equation with other perturbations, for example, consider a Hamiltonian function

$$E(u) = \frac{1}{4}\|u\|_{L^4}^4 + \frac{1}{2}F(|(u|1)|^2)\ ,$$

with a non linear function F. In this case, we still have one Lax pair (K_u, C_u) while the conservation laws we found no longer exist. The question is to study the integrability and also the existence of turbulent solutions of this new Hamiltonian system.

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