Dirichlet uniformly well-approximated numbers

Dong Han Kim, Lingmin Liao

DIRICHLET UNIFORMLY WELL-APPROXIMATED NUMBERS

DONG HAN KIM AND LINGMIN LIAO

Abstract. Fix an irrational number θ. For a real number $\tau > 0$, consider the numbers y satisfying that for all large number Q, there exists an integer $1 \le n \le Q$, such that $\|n\theta - y\| < Q^{-\tau}$, where $\|\cdot\|$ is the distance of a real number to its nearest integer. These numbers are called Dirichlet uniformly well-approximated numbers. For any $\tau > 0$, the Haussdorff dimension of the set of these numbers is obtained and is shown to depend on the Diophantine property of θ. It is also proved that with respect to τ, the only possible discontinuous point of the Hausdorff dimension is $\tau = 1$.

1. Introduction

In Diophantine approximation, we study the approximation of an irrational number by rationals. Denote by $\|t\| = \min_{n\in\mathbb{Z}} |t - n|$ the distance from a real t to the nearest integer. In 1842, Dirichlet [12] showed his celebrated theorem in Diophantine approximation:

Dirichlet theorem *Let θ, Q be real numbers with $Q \ge 1$. There exists an integer n with $1 \le n \le Q$, such that $\|n\theta\| < Q^{-1}$.*

Following Waldschmidt [42], we call Dirichlet theorem a *uniform approximation theorem*. A weak form of the theorem, called an *asymptotic approximation theorem*, was already known (e.g., Legendre's 1808 book [32, pp.18–19])[1] before Dirichlet: for any real θ, there exist infinitely many integers n such that $\|n\theta\| < n^{-1}$. In the literature, much more attention has been paid to the asymptotic approximation.

The first inhomogeneous asymptotic approximation result is due to Minkowski [35] in 1907. Let θ be an irrational number. Let y be a real number which is not equal to any $m\theta + \ell$ with $m, \ell \in \mathbb{N}$. Minkowski proved that there exist infinitely many integers n such that $\|n\theta - y\| < \frac{1}{4|n|}$.

Date: August 21, 2017.

2010 *Mathematics Subject Classification.* Primary 11J83; Secondary 11K50; 37M25.

Key words and phrases. Dirichlet theorem, inhomogeneous Diophantine approximation, irrational rotation, Hausdorff dimension.

[1]The authors thank Yann Bugeaud for telling us this history remark.

In 1924, Khinchine [24] proved that for a continuous function $\Psi : \mathbb{N} \to \mathbb{R}^+$, if $x \mapsto x\Psi(x)$ is non-increasing, then the set

$$\mathcal{L}_\Psi := \{\theta \in \mathbb{R} : \ \|n\theta\| < \Psi(n) \text{ for infinitely many } n\}$$

has Lebesgue measure zero if the series $\sum \Psi(n)$ converges and has full Lebesgue measure otherwise. The expected similar result by deleting the non-increasing condition on Ψ is the famous Duffin-Schaeffer conjecture [13]. One could find the recent progresses towards this conjecture in Haynes–Pollington–Velani–Sanju [18] and Beresnevich–Harman–Haynes–Velani [3].

For the inhomogeneous case, Khinchine's theorem was extended to the set

$$\mathcal{L}_\Psi(y) := \{\theta \in \mathbb{R} : \ \|n\theta - y\| < \Psi(n) \text{ for infinitely many } n\}$$

by Szüsz [41] and Schmidt [38]. On the other hand, it follows from the Borel-Cantelli Lemma that the Lebesgue measure of

$$\mathcal{L}_\Psi[\theta] := \{y \in \mathbb{R} : \ \|n\theta - y\| < \Psi(n) \text{ for infinitely many } n\}$$

is zero whenever the series $\sum \Psi(n)$ converges. However, it seems not easy to obtain a full Lebesgue measure result. In 1955, Kurzweil [30] showed that, if the irrational θ is of bounded type (the partial quotients of the continued fraction of θ is bounded), then for a monotone decreasing function $\Psi : \mathbb{N} \to \mathbb{R}^+$, with $\sum \Psi(n) = \infty$, the set $\mathcal{L}_\Psi[\theta]$ has full Lebesgue measure. In 1957, Cassels [8] proved that for almost all θ, the set $\mathcal{L}_\Psi[\theta]$ has full Lebesgue measure if $\sum \Psi(n) = \infty$. For new results in this direction, we refer to the recent works Laurent–Nogueira [31], Kim [27], and Fuchs–Kim [17].

At the end of twenties of last century, the concept of Hausdorff dimension had been introduced into the study of Diophantine approximation. We refer the reader to [14] for the definition and properties of the Hausdorff dimension. Using the notion of Hausdorff dimension, Jarník ([22], 1929) and independently Besicovitch ([2], 1934) studied the size of the set of asymptoticly well-approximated numbers. They proved that for any $\tau \geq 1$, the Hausdorff dimension of the set

$$\mathcal{L}_\tau(0) := \left\{\theta \in \mathbb{R} : \ \|n\theta\| < n^{-\tau} \text{ for infinitely many } n\right\}$$

is $2/(\tau + 1)$.

The corresponding inhomogeneous question was solved by Levesley [33] in 1998: for any $\tau \geq 1$, and any real number y, the Hausdorff dimension of the set

$$\mathcal{L}_\tau(y) := \left\{\theta \in \mathbb{R} : \ \|n\theta - y\| < n^{-\tau} \text{ for infinitely many } n\right\}$$

which is different from $\mathcal{L}_\tau(0)$, is also $2/(\tau + 1)$.

As in the Lebesgue measure problems, for the inhomogeneous case, one is also concerned with the Hausdorff dimension of the sets of inhomogeneous terms. For a fixed irrational θ, let us denote

$$\mathcal{L}_\tau[\theta] := \left\{ y \in \mathbb{R} : \ \|n\theta - y\| < n^{-\tau} \text{ for infinitely many } n \right\}.$$

In 2003, Bugeaud [4] and independently, Schmeling and Troubetzkoy [37] showed that for any $\tau \geq 1$ the Hausdorff dimension of the set $\mathcal{L}_\tau[\theta]$ is $1/\tau$.

In analogy to the asymptotic approximation, for $\tau > 0$, we consider the following uniform approximation sets:

$$\mathcal{U}_\tau(y) := \left\{ \theta \in \mathbb{R} : \ \text{for all large } Q, \ 1 \leq \exists n \leq Q \text{ such that } \|n\theta - y\| < Q^{-\tau} \right\},$$
$$\mathcal{U}_\tau[\theta] := \left\{ y \in \mathbb{R} : \ \text{for all large } Q, \ 1 \leq \exists n \leq Q \text{ such that } \|n\theta - y\| < Q^{-\tau} \right\}.$$

The points in $\mathcal{U}_\tau(y)$ and $\mathcal{U}_\tau[\theta]$ are called Dirichlet uniformly well-approximated numbers. The set $\mathcal{U}_\tau(0)$ is referred to as homogeneous uniform approximation. In general, except for a countable set, $\mathcal{U}_\tau(y)$ and $\mathcal{U}_\tau[\theta]$ are contained in $\mathcal{L}_\tau(y)$ and $\mathcal{L}_\tau[\theta]$ correspondingly, since the uniform approximation property is stronger than the asymptotic approximation property.

We see from Dirichlet Theorem that $\mathcal{U}_1(0) = \mathbb{R}$. However, Khintchine ([25], 1926) showed that for all $\tau > 1$, $\mathcal{U}_\tau(0)$ is $\mathbb{Q}$. Consult [29] for the uniform approximation by general error functions. In general, for $y \in \mathbb{R}$, the set $\mathcal{U}_1(y)$ does not always contain all irrationals. Thus, there is no inhomogeneous analogy of the Dirichlet Theorem. The question on the size of $\mathcal{U}_\tau(y)$ for general $y \in \mathbb{R}$ is largely open.

For higher dimensional analogy of $\mathcal{U}_\tau(0)$, Cheung [9] proved that the set of points (θ_1, θ_2) such that for any $\delta > 0$, for all large Q, there exists $n \leq Q$ such that

$$\max\left\{ \|n\theta_1\|, \|n\theta_2\| \right\} < \delta / Q^{\frac{1}{2}},$$

is of Hausdorff dimension $4/3$. This result was recently generalized to dimension larger than 2 by Cheung and Chevallier [10].

In this paper, we mainly study the set $\mathcal{U}_\tau[\theta]$. We will restrict ourselves to the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, for which the dimension results will be the same to those on $\mathbb{R}$. The points in $\mathbb{T}$ are considered as the same if their fractional parts are the same. The irrationality exponent of θ is given by

$$w(\theta) := \sup\{ s > 0 : \|n\theta\| < n^{-s} \text{ for infinitely many } n \}.$$

We remark that the usual irrationality exponent is defined as $1 + w(\theta)$. See for example [5, 42]. It was shown in Propositions 9 and 10 of [28] that $\mathcal{U}_\tau[\theta]$ is of Lebesgue measure 1 if $\tau < 1/w(\theta)$ or 0 if $\tau > 1/w(\theta)$ (see also [26] for a related result). Denote by $\dim_{\mathrm{H}}$ the Hausdorff dimension. Let $q_n = q_n(\theta)$ be the denominator of the

n-th convergent of the continued fraction of θ. The following main theorems show that $\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta])$ can be obtained using the sequence $q_n(\theta)$ and strongly depends on the irrationality exponent of θ:

Theorem 1. *Let θ be an irrational with $w(\theta) > 1$. Then $\mathcal{U}_\tau[\theta] = \mathbb{T}$ if $\tau < 1/w(\theta)$; $\mathcal{U}_\tau[\theta] = \{i\theta \in \mathbb{T} : i \geq 1, i \in \mathbb{Z}\}$ if $\tau > w(\theta)$; and*

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) = \begin{cases} \underset{k\to\infty}{\varliminf} \dfrac{\log\left(n_k^{1+1/\tau}\prod_{j=1}^{k-1} n_j^{1/\tau}\|n_j\theta\|\right)}{\log(n_k\|n_k\theta\|^{-1})}, & \dfrac{1}{w(\theta)} < \tau < 1, \\ \underset{k\to\infty}{\varliminf} \dfrac{-\log\left(\prod_{j=1}^{k-1} n_j\|n_j\theta\|^{1/\tau}\right)}{\log\left(n_k\|n_k\theta\|^{-1}\right)}, & 1 < \tau < w(\theta). \end{cases}$$

where n_k is the (maximal) subsequence of (q_k) such that

$$\begin{cases} n_k\|n_k\theta\|^\tau < 1, & \text{if } 1/w(\theta) < \tau < 1, \\ n_k^\tau\|n_k\theta\| < 2, & \text{if } 1 < \tau < w(\theta). \end{cases}$$

By a careful calculation for the case of $\tau = 1$ combined with Theorem 1, we have the following bounds of dimension in terms of $w(\theta)$.

Theorem 2. *For any irrational θ with $w(\theta) = 1$, we have $\mathcal{U}_\tau[\theta] = \mathbb{T}$ if $\tau < 1$; $\mathcal{U}_\tau[\theta] = \{i\theta : i \geq 1, i \in \mathbb{Z}\}$ if $\tau > 1$; and*

$$\frac{1}{2} \leq \dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \leq 1, \quad \text{if } \tau = 1.$$

Theorem 3. *For any irrational θ with $w(\theta) = w > 1$, we have*

$$\frac{w/\tau - 1}{w^2 - 1} \leq \dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \leq \frac{1/\tau + 1}{w + 1}, \qquad \frac{1}{w} \leq \tau \leq 1,$$

$$0 \leq \dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \leq \frac{w/\tau - 1}{w^2 - 1}, \qquad 1 < \tau \leq w.$$

Moreover, if $w(\theta) = \infty$, then $\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) = 0$ for all $\tau > 0$.

We will show in Examples 16, 17, 18 and 19, that the upper and lower bounds of Theorems 2 and 3 can be all reached.

Remark 4. Consider the case $\tau > 1$. By optimizing the upper bound in Theorem 3 with respect to w, we have for $\tau > 1$,

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \leq \frac{1}{2\tau(\tau + \sqrt{\tau^2 - 1})},$$

and the equality holds when $w = \tau + \sqrt{\tau^2 - 1}$. We then deduce that

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) < \frac{1}{2\tau^2} < \frac{1}{2\tau} \quad \text{for all } \tau > 1.$$

As we have mentioned, for all $\tau > 1$, $\mathcal{U}_\tau[\theta]$ is included in $\mathcal{L}_\tau[\theta]$ except for a countable set of points. In fact, excluding the countable set $\{n\theta : n \in \mathbb{N}\}$, the set $\mathcal{L}_\tau[\theta]$ is a

level set of the lower limit of the hitting time for the irrational rotation $x \mapsto x+\theta$ (see for example Lemma 4.2 of [16] and Lemma 3.2 of [34]), while the set $\mathcal{U}_\tau[\theta]$ is a level set of upper limit of the same hitting time. So the fact that $\mathcal{U}_\tau[\theta]$ is almost included in $\mathcal{L}_\tau[\theta]$ follows directly from the fact that lower limit is less than the upper limit. Recall that $\dim_{\mathrm{H}}(\mathcal{L}_\tau[\theta]) = 1/\tau$ for all $\tau > 1$. Our result then shows that the inclusion is strict in the sense of Hausdorff dimension: the former is strictly less than one-half of the latter one by Hausdorff dimension.

We will also prove the following theorem on the continuity of the Hausdorff dimension of the set $\mathcal{U}_\tau[\theta]$ with respect to the parameter τ.

Theorem 5. *For each irrational θ, $\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta])$ is a continuous function of τ on $(0,1)\cup(1,\infty)$.*

Finally, we note that our results give an answer for the case of dimension one of Problem 3 in Bugeaud and Laurent [6]. We also remark that the uniform approximation problem for the b-ary and β-expansion has been recently studied by Bugeaud and Liao [7]. The symbolic technique which is quite efficient in [7] falls in our context.

The paper is organized as follows. Some lemmas for the structure of uniform approximation set $\mathcal{U}_\tau[\theta]$ are stated in Section 2. The proof of Theorem 1 is given in Section 3. In Section 4 we discuss the set $\mathcal{U}_\tau[\theta]$ for $\tau = 1$ and prove Theorem 2. Section 5 is devoted to the proofs of Theorems 3 and 5. In the last section, we give the examples in which the bounds of Theorems 2 and 3 are attained.

2. Cantor structures

In this section, we first give some basic notations and properties on the continued fraction expansion of irrational numbers which will be useful later. Then we describe in detail the Cantor structure of the sets $\mathcal{U}_\tau[\theta]$.

Let $\theta \in [0,1]$ be an irrational and $\{a_k\}_{k\geq 1}$ be the partial quotients of θ in its continued fraction expansion. The denominator q_k and the numerator p_k of the k-th convergent ($q_0 = 1, p_0 = 0$), satisfy the following relations

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad \forall n \geq 1. \tag{1}$$

A corresponding useful recurrence property is

$$\|q_{n-1}\theta\| = a_{n+1}\|q_n\theta\| + \|q_{n+1}\theta\|. \tag{2}$$

We also have the equality

$$q_{n+1}\|q_n\theta\| + q_n\|q_{n+1}\theta\| = 1, \tag{3}$$

and the estimation

$$\frac{1}{2q_{n+1}} < \frac{1}{q_{n+1}+q_n} < \|q_n\theta\| \le \frac{1}{q_{n+1}}. \tag{4}$$

Recall that the irrationality exponent of θ is defined by $w(\theta) := \sup\{s > 0 : \liminf_{j\to\infty} j^s\|j\theta\| = 0\}$. By the theorem of best approximation (e.g. [36]), we can show that

$$w(\theta) = \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}. \tag{5}$$

Since (q_n) is increasing, we have $w(\theta) \ge 1$ for every irrational number θ. The set of irrational numbers with $w(\theta) = 1$ has measure 1 and includes the set of irrational numbers with bounded partial quotients, which is of measure 0 and of Hausdorff dimension 1. There exist numbers with $w(\theta) = \infty$, called the Liouville numbers. For more details on continued fractions, we refer to Khinchine's book [23].

In the following, we will investigate the Cantor structure of our main object $\mathcal{U}_\tau[\theta]$. Denote by $B(x, r)$ the open ball of center x and radius r in $\mathbb{T}$. Fix $\tau > 0$. Let

$$G_n = \bigcup_{i=1}^{n} B\left(i\theta, \frac{1}{n^\tau}\right) \quad \text{and} \quad F_k = \bigcap_{n=q_k}^{q_{k+1}} G_n.$$

Then we have

$$\mathcal{U}_\tau[\theta] = \bigcup_{\ell=1}^{\infty}\bigcap_{n=\ell}^{\infty} G_n = \bigcup_{\ell=1}^{\infty}\bigcap_{k=\ell}^{\infty} F_k.$$

We will calculate the Hausdorff dimension of $\bigcap_{k=1}^{\infty} F_k$. From the construction, we will see that for all ℓ, the Hausdorff dimensions of $\bigcap_{k=\ell}^{\infty} F_k$ are the same to that of $\bigcap_{k=1}^{\infty} F_k$. Thus by countable stability of the Hausdorff dimension,

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) = \dim_{\mathrm{H}}\left(\bigcap_{k=1}^{\infty} F_k\right).$$

For $m \ge 1$, set

$$E_m := \bigcap_{k=1}^{m} F_k.$$

Then for each m, E_m is a union of intervals, and we have

$$\forall m \ge 1,\ E_{m+1} \subset E_m, \quad \text{and} \quad \bigcap_{m=1}^{\infty} E_m = \bigcap_{k=1}^{\infty} F_k.$$

We are thus led to the calculation of the Hausdorff dimension of the nested Cantor set $\bigcap_{m=1}^{\infty} E_m$. To this end, let us first investigate the structure of F_k.

We note that $q_k\theta - p_k > 0$ if and only if k is even. In the following lemmas, we will only consider formulae of F_k for even k's since for the odd k's we will have symmetric formulae.

The well-known Three Step Theorem (e.g. [39]) shows that by the points $\{i\theta\}_{i=1}^{q_k}$, the unit circle $\mathbb{T}$ is partitioned into q_k intervals of length $\|q_{k-1}\theta\|$ or $\|q_{k-1}\theta\|+\|q_k\theta\|$. Furthermore, for even k, we have

$$\begin{aligned}
&\mathbb{T}\setminus\{-i\theta : 0\le i<q_k\}\\
&=\bigcup_{i=1}^{q_{k-1}}((i-q_k)\theta,(i-q_{k-1})\theta)\cup\bigcup_{i=q_{k-1}+1}^{q_k}((i-q_k)\theta,(i-q_k-q_{k-1})\theta)\\
&=\bigcup_{i=1}^{q_{k-1}}(i\theta-\|q_k\theta\|,i\theta+\|q_{k-1}\theta\|)\cup\bigcup_{i=q_{k-1}+1}^{q_k}(i\theta-\|q_k\theta\|,i\theta+\|q_k\theta\|-\|q_{k-1}\theta\|).
\end{aligned}\tag{6}$$

We remind that here and further throughout the paper, we will always consider $i\theta$ as a point in $\mathbb{T}$, but not in $\mathbb{R}$. So the absolute values of these points are always less than 1. In particular, $q_k\theta=\|q_k\theta\|$ if k is even.

Lemma 6. *(i) If*

$$2\left(\frac{1}{q_{k+1}}\right)^{\tau}>\|q_{k-1}\theta\|+\|q_k\theta\|,\tag{7}$$

then we have $F_k=\mathbb{T}$.

(ii) For the case of $\tau=1$ and $a_{k+1}=1$, we have $F_k=\mathbb{T}$.

Proof. (i) For each $q_k\le n\le q_{k+1}$ we have

$$2\left(\frac{1}{n}\right)^{\tau}\ge 2\left(\frac{1}{q_{k+1}}\right)^{\tau}>\|q_{k-1}\theta\|+\|q_k\theta\|.$$

Since any two neighboring points in $\{i\theta : 1\le i\le q_k\}$ are distanced by $\|q_{k-1}\theta\|$ or $\|q_{k-1}\theta\|+\|q_k\theta\|$, all intervals overlap. Hence,

$$G_n=\bigcup_{i=1}^{n}B\left(i\theta,n^{-\tau}\right)=\mathbb{T}.$$

The result then follows.

(ii) If $a_{k+1}=1$, then by (2) and (3) we have

$$\begin{aligned}
q_{k+1}\left(\|q_{k-1}\theta\|+\|q_k\theta\|\right)&=q_{k+1}\left(2\|q_k\theta\|+\|q_{k+1}\theta\|\right)\\
&=2q_{k+1}\|q_k\theta\|+\left(q_k+q_{k-1}\right)\|q_{k+1}\theta\|\\
&=2-q_k\|q_{k+1}\theta\|+q_{k-1}\|q_{k+1}\theta\|<2.
\end{aligned}$$

Hence, by (i), if $\tau=1$, $F_k=\mathbb{T}$. □

Lemma 7. *For any $\tau\le 1$, we have*

(i)

$$F_k\supset\bigcup_{i=1}^{q_k}\left(i\theta-\left(\frac{1}{q_{k+1}}\right)^{\tau},\ i\theta+\min_{1\le c\le a_{k+1}+1}\left((c-1)\|q_k\theta\|+\frac{1}{(cq_k+i-1)^{\tau}}\right)\right).$$

(ii)

$$F_k \supset \bigcup_{i=1}^{q_k} \left(i\theta - \|q_k\theta\|,\ i\theta + \left(C_\tau \left(\frac{1}{q_k^\tau \|q_k\theta\|} \right)^{\frac{1}{\tau+1}} - 2 \right) \|q_k\theta\| \right),$$

where $C_\tau = \tau^{\frac{1}{\tau+1}} + \tau^{-\frac{\tau}{\tau+1}}$. *Note that* $1 < C_\tau \leq 2$.

(iii)

$$F_k \subset \bigcup_{i=1}^{q_k} \left(i\theta - \tau^{-\frac{\tau}{\tau+1}} \left(\frac{\|q_k\theta\|}{q_k} \right)^{\frac{\tau}{\tau+1}},\ i\theta + C_\tau \left(\frac{\|q_k\theta\|}{q_k} \right)^{\frac{\tau}{\tau+1}} \right).$$

Proof. (i) Let n be an integer such that $q_k \leq n \leq q_{k+1}$ for some $k \in \mathbb{N}$. Then if k is even (the case when k is odd is the same up to symmetry), for each i with $1 \leq i \leq q_k$ we have

$$\begin{aligned} G_n &= \bigcup_{j=1}^{n} B\left(j\theta, n^{-\tau}\right) \\ &\supset B\left(i\theta, \frac{1}{n^\tau}\right) \cup B\left((q_k+i)\theta, \frac{1}{n^\tau}\right) \cup \cdots \cup B\left(\left(\left\lfloor \frac{n-i}{q_k} \right\rfloor q_k + i\right)\theta, \frac{1}{n^\tau}\right). \end{aligned}$$

Notice that for $q_k \leq n \leq q_{k+1}$

$$\frac{1}{n^\tau} \geq \frac{1}{n} \geq \frac{1}{q_{k+1}} > \|q_k\theta\|. \tag{8}$$

Thus, the above $\left\lfloor \frac{n-i}{q_k} \right\rfloor$ intervals overlap and for each $1 \leq i \leq q_k$

$$G_n \supset \left(i\theta - \frac{1}{n^\tau},\ i\theta + \left\lfloor \frac{n-i}{q_k} \right\rfloor q_k\theta + \frac{1}{n^\tau} \right).$$

For each $1 \leq i \leq q_k$, if $(c-1)q_k + i \leq n \leq cq_k + i - 1$, then

$$\left\lfloor \frac{n-i}{q_k} \right\rfloor \|q_k\theta\| + \frac{1}{n^\tau} \geq (c-1)\|q_k\theta\| + \frac{1}{(cq_k+i-1)^\tau}.$$

Therefore, we have for each $1 \leq i \leq q_k$

$$\begin{aligned} F_k &= \bigcap_{n=q_k}^{q_{k+1}} G_n \supset \bigcap_{n=q_k}^{q_{k+1}} \left(i\theta - \frac{1}{n^\tau},\ i\theta + \left\lfloor \frac{n-i}{q_k} \right\rfloor q_k\theta + \frac{1}{n^\tau} \right) \\ &\supset \left(i\theta - \frac{1}{q_{k+1}^\tau},\ i\theta + \min_{1 \leq c \leq a_{k+1}+1} \left((c-1)\|q_k\theta\| + \frac{1}{(cq_k+i-1)^\tau} \right) \right). \end{aligned}$$

(ii) By elementary calculus,

$$\inf_{x \geq 0} \left(x\|q_k\theta\| + \frac{1}{(xq_k)^\tau} \right) = \left(\tau^{\frac{1}{\tau+1}} + \tau^{-\frac{\tau}{\tau+1}} \right) \left(\frac{\|q_k\theta\|}{q_k} \right)^{\frac{\tau}{\tau+1}}.$$

Thus, we have

$$\begin{aligned} &\min_{1 \leq c \leq a_{k+1}+1} \left((c-1)\|q_k\theta\| + \frac{1}{((c+1)q_k)^\tau} \right) \\ &\qquad \geq \left(\tau^{\frac{1}{\tau+1}} + \tau^{-\frac{\tau}{\tau+1}} \right) \left(\frac{\|q_k\theta\|}{q_k} \right)^{\frac{\tau}{\tau+1}} - 2\|q_k\theta\|. \end{aligned}$$

Hence, by (i) and (8), we have

$$F_k \supset \bigcup_{i=1}^{q_k} \left(i\theta - \|q_k\theta\|,\ i\theta + \left(\tau^{\frac{1}{\tau+1}} + \tau^{-\frac{\tau}{\tau+1}}\right)\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} - 2\|q_k\theta\| \right).$$

(iii) Let

$$c := \left\lceil \left(\frac{\tau}{q_k^{\tau}\|q_k\theta\|}\right)^{\frac{1}{\tau+1}} \right\rceil.$$

We will distinguish two cases. If $c \le a_{k+1}$, then we have

$$\begin{aligned} F_k &= \bigcap_{n=q_k}^{q_{k+1}} G_n \subset G_{cq_k} = \bigcup_{i=1}^{cq_k} B(i\theta, (cq_k)^{-\tau}) \\ &= \bigcup_{i=1}^{q_k} B(i\theta, (cq_k)^{-\tau}) \cup B((q_k+i)\theta, (cq_k)^{-\tau}) \cup \cdots \cup B\Big(((c-1)q_k+i)\theta, (cq_k)^{-\tau}\Big). \end{aligned}$$

Since

$$\left(\frac{1}{cq_k}\right)^{\tau} \ge \frac{1}{cq_k} \ge \frac{1}{a_{k+1}q_k} \ge \frac{1}{q_{k+1}} > \|q_k\theta\|,$$

the above c intervals in the union overlap and we have

$$F_k \subset \bigcup_{i=1}^{q_k} \left(i\theta - (cq_k)^{-\tau}, i\theta + (c-1)q_k\theta + (cq_k)^{-\tau}\right).$$

By the definition of c, we have

$$\begin{aligned} F_k &\subset \bigcup_{i=1}^{q_k} \left(i\theta - \left(\frac{\tau q_k}{\|q_k\theta\|}\right)^{-\frac{\tau}{\tau+1}}, i\theta + \frac{1}{q_k}\left(\frac{\tau q_k}{\|q_k\theta\|}\right)^{\frac{1}{\tau+1}}\|q_k\theta\| + \left(\frac{\tau q_k}{\|q_k\theta\|}\right)^{-\frac{\tau}{\tau+1}} \right) \\ &= \bigcup_{i=1}^{q_k} \left(i\theta - \tau^{-\frac{\tau}{\tau+1}}\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}}, i\theta + \left(\tau^{\frac{1}{\tau+1}} + \tau^{-\frac{\tau}{\tau+1}}\right)\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} \right). \end{aligned}$$

Then the assertion follows.

If $c > a_{k+1}$, i.e.,

$$\left(\frac{\tau}{q_k^{\tau}\|q_k\theta\|}\right)^{\frac{1}{\tau+1}} > a_{k+1},$$

then we have

$$\begin{aligned} \left(\tau^{\frac{1}{\tau+1}} + 2\tau^{-\frac{\tau}{\tau+1}}\right)\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} &> \left(1+\frac{2}{\tau}\right) a_{k+1}\|q_k\theta\| \ge 3a_{k+1}\|q_k\theta\| \\ &> \|q_{k-1}\theta\| + \|q_k\theta\|. \end{aligned}$$

Thus,

$$\bigcup_{i=1}^{q_k} \left(i\theta - \tau^{-\frac{\tau}{\tau+1}}\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}}, i\theta + \left(\tau^{\frac{1}{\tau+1}} + \tau^{-\frac{\tau}{\tau+1}}\right)\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} \right) = \mathbb{T},$$

and the assertion trivially holds. □

Lemma 8. *Suppose $\tau > 1$.*

(i) We have

$$\bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right) \subset F_k \subset \bigcup_{i=1}^{q_{k+1}} B\left(i\theta, q_{k+1}^{-\tau}\right)$$

and for large q_k the balls $B\left(i\theta, q_{k+1}^{-\tau}\right)$, $1 \le i \le q_{k+1}$, are disjoint.

(ii) If $q_{k+1}^{-\tau} + q_k^{-\tau} \le \|q_k\theta\|$, then

$$F_k = \bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right).$$

(iii) For large q_k

$$\bigcup_{i=1}^{\max(c_k,1)\cdot q_k} B\left(i\theta, q_{k+1}^{-\tau}\right) \subset F_k \subset \bigcup_{i=1}^{(2c_k+3)q_k} B\left(i\theta, q_{k+1}^{-\tau}\right).$$

where

$$c_k := \left\lfloor \left(\frac{1}{q_k^{\tau}\|q_k\theta\|}\right)^{\frac{1}{\tau+1}} \right\rfloor.$$

Proof. (i) For each $1 \le i \le q_k$ and $q_k \le n \le q_{k+1}$,

$$B\left(i\theta, q_{k+1}^{-\tau}\right) \subset \bigcup_{j=1}^{n} B\left(j\theta, n^{-\tau}\right).$$

Thus,

$$\bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right) \subset \bigcap_{n=q_k}^{q_{k+1}} \left(\bigcup_{j=1}^{n} B\left(j\theta, n^{-\tau}\right)\right) = F_k.$$

On the other hand,

$$F_k = \bigcap_{n=q_k}^{q_{k+1}} G_n \subset G_{q_{k+1}} = \bigcup_{i=1}^{q_{k+1}} B\left(i\theta, q_{k+1}^{-\tau}\right). \tag{9}$$

Since $\tau > 1$, for large q_k, (hence for lager q_{k+1}),

$$2q_{k+1}^{-\tau} < \frac{1}{2q_{k+1}} < \|q_k\theta\|. \tag{10}$$

Thus, the balls $B\left(i\theta, q_{k+1}^{-\tau}\right)$, $1 \le i \le q_{k+1}$, are disjoint.

(ii) Suppose that there exists $x \in F_k \setminus \bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right)$. By (i), we have $x \in B\left(j\theta, q_{k+1}^{-\tau}\right)$ for some $q_k + 1 \le j \le q_{k+1}$. Since $x \in F_k \subset G_{q_k}$, there exists $1 \le i \le q_k$ such that $x \in B\left(i\theta, q_k^{-\tau}\right)$. Since $|i\theta - j\theta| \ge \|q_k\theta\|$ and $x \in B\left(j\theta, q_{k+1}^{-\tau}\right) \cap B\left(i\theta, q_k^{-\tau}\right) \ne \emptyset$, we have $\|q_k\theta\| < {q_k}^{-\tau} + {q_{k+1}}^{-\tau}$, which is a contradiction.

(iii) Suppose $c_k \ge 2$. Then for $1 \le m \le c_k - 1$, and for large q_k,

$$\begin{aligned} m\|q_k\theta\| &\le \left(\left(\frac{1}{q_k^{\tau}\|q_k\theta\|}\right)^{\frac{1}{\tau+1}} - 1\right)\|q_k\theta\| = \left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} - \|q_k\theta\| \\ &\le \frac{1}{(c_k q_k)^{\tau}} - \frac{1}{(q_{k+1})^{\tau}} \le \frac{1}{((c_k - m + 1)q_k)^{\tau}} - \frac{1}{(q_{k+1})^{\tau}}, \end{aligned} \tag{11}$$

where for the second inequality we use (10).

Let i be an integer satisfying $q_k < i \le c_k q_k$. For each n with $q_k \le n < i$, choose m as $i - mq_k \le n < i - (m-1)q_k$. Then $1 \le m \le c_k - 1$ and $n \le (c_k - m + 1)q_k$. By (11) we have

$$B\left(i\theta, q_{k+1}^{-\tau}\right) \subset B\left((i-mq_k)\theta, ((c_k-m+1)q_k)^{-\tau}\right) \subset B\left((i-mq_k)\theta, n^{-\tau}\right) \subset G_n.$$

We also have for $i \le n \le q_{k+1}$,

$$B\left(i\theta, q_{k+1}^{-\tau}\right) \subset G_n$$

Therefore, for $q_k < i \le c_k q_k$,

$$B(i\theta, q_{k+1}^{-\tau}) \subset \bigcap_{n=q_k}^{q_{k+1}} G_n = F_k.$$

Hence, if $c_k \ge 2$, we have

$$\bigcup_{i=q_k+1}^{c_k q_k} B\left(i\theta, q_{k+1}^{-\tau}\right) \subset F_k.$$

On the other hand, we have already proved in (i) that

$$\bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right) \subset F_k.$$

Therefore, the first inclusion for the case $c_k \le 1$ in (iii) follows.

For large q_k,

$$\begin{aligned}(c_k+2)\|q_k\theta\| &> \left(\left(\frac{1}{q_k^{\tau}\|q_k\theta\|}\right)^{\frac{1}{\tau+1}} + 1\right)\|q_k\theta\| = \left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} + \|q_k\theta\| \\ &> \frac{1}{((c_k+1)q_k)^{\tau}} + \frac{1}{(q_{k+1})^{\tau}}.\end{aligned}$$

Suppose $(2c_k+3)q_k < i \le q_{k+1}$. Then for any j with $1 \le j \le (c_k+1)q_k$ we have $|i\theta - j\theta| \ge (c_k+2)\|q_k\theta\|$, thus

$$B\left(i\theta, q_{k+1}^{-\tau}\right) \cap B\left(j\theta, ((c_k+1)q_k)^{-\tau}\right) = \emptyset,$$

which implies

$$B\left(i\theta, q_{k+1}^{-\tau}\right) \cap G_{(c_k+1)q_k} = \emptyset.$$

Hence,

$$B\left(i\theta, q_{k+1}^{-\tau}\right) \cap F_k = \emptyset.$$

Therefore, by (9) we have

$$F_k \subset \bigcup_{i=1}^{(2c_k+3)q_k} B\left(i\theta, q_{k+1}^{-\tau}\right), \tag{12}$$

which is the second inclusion in (iii). □

3. Proof of Theorem 1

We will use the following known facts in fractal geometry to calculate the Hausdorff dimensions. Let $E_0 \supset E_1 \supset E_2 \supset \dots$ be a decreasing sequence of sets, with each E_n a union of finite number of disjoint intervals. Set

$$F = \bigcap_{n=0}^{\infty} E_n.$$

Fact 9 ([14], p.64)**.** *Suppose each interval of E_{i-1} contains at least m_i intervals of E_i $(i = 1, 2, \dots)$ which are separated by gaps of at least ε_i, where $0 < \varepsilon_{i+1} < \varepsilon_i$ for each i. Then*

$$\dim_{\mathrm{H}}(F) \geq \varliminf_{i\to\infty} \frac{\log(m_1 \cdots m_{i-1})}{-\log(m_i \varepsilon_i)}.$$

Fact 10 ([14], p.59)**.** *Suppose F can be covered by ℓ_i sets of diameter at most δ_i with $\delta_i \to 0$ as $i \to \infty$. Then*

$$\dim_{\mathrm{H}}(F) \leq \varliminf_{i\to\infty} \frac{\log \ell_i}{-\log \delta_i}.$$

Now we are ready to prove Theorem 1. Recall that

$$F_k = \bigcap_{n=q_k}^{q_{k+1}} \left(\bigcup_{i=1}^{n} B\left(i\theta, n^{-\tau}\right) \right).$$

By the discussion at the beginning of Section 2, we need to calculate the Hausdorff dimension of the set

$$F = \bigcap_{n=1}^{\infty} E_n, \quad \text{with} \quad E_n = \bigcap_{k=1}^{n} F_k.$$

The dimension of F is the same to that of $\mathcal{U}_\tau[\theta]$.

Proof of Theorem 1. (i) If $\tau < 1/w(\theta)$, by (5) we have for all large k,

$$2\left(\frac{1}{q_{k+1}}\right)^{\tau} > \frac{2}{q_k} > \frac{1}{q_k} + \frac{1}{q_{k+1}} \geq \|q_{k-1}\theta\| + \|q_k\theta\|.$$

Thus by Lemma 6 (i), for all large k, F_k is the whole circle $\mathbb{T}$. Hence,

$$\mathcal{U}_\tau[\theta] = \bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} F_k = \mathbb{T}.$$

(ii) If $\tau > w(\theta)$, then we have $q_k^{\tau}\|q_k\theta\| > 2$ for all large k, thus

$$q_{k+1}^{-\tau} + q_k^{-\tau} < 2q_k^{-\tau} \leq \|q_k\theta\| \quad \text{for large } k. \tag{13}$$

By Lemma 8 (ii), for large k

$$F_k = \bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right).$$

Thus,

$$F_k \cap F_{k+1} = \left(\bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right) \right) \cap \left(\bigcup_{j=1}^{q_{k+1}} B\left(i\theta, q_{k+2}^{-\tau}\right) \right).$$

By (13), for $1 \le i \ne j \le q_{k+1}$ we have $|i\theta - j\theta| \ge \|q_{k+1}\theta\| > q_{k+2}^{-\tau} + q_{k+1}^{-\tau}$, thus

$$F_k \cap F_{k+1} = \bigcup_{j=1}^{q_k} B\left(i\theta, q_{k+2}^{-\tau}\right).$$

Inductively, for each $\ell \ge 0$ we get

$$F_k \cap F_{k+1} \cap \cdots \cap F_{k+\ell} = \bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+\ell+1}^{-\tau}\right). \tag{14}$$

Hence, we conclude

$$\mathcal{U}_\tau[\theta] = \bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} F_k = \{i\theta : i \ge 1\}.$$

(iii) Assume that $1/w(\theta) < \tau < 1$. If $q_k \|q_k\theta\|^\tau \ge 1$ then we have

$$\begin{aligned} 2\left(\frac{1}{q_{k+1}}\right)^\tau &> 2\|q_k\theta\|^\tau > \|q_k\theta\|^\tau + \|q_k\theta\| \\ &\ge \frac{1}{q_k} + \|q_k\theta\| > \|q_{k-1}\theta\| + \|q_k\theta\|. \end{aligned}$$

By Lemma 6 (i), we have $F_k = \mathbb{T}$. Since removing such sets F_k from the intersection $F = \bigcap_{k=1}^\infty F_k$ does not change F, we only consider F_k such that $q_k\|q_k\theta\|^\tau < 1$.

Suppose for some k

$$q_k\|q_k\theta\|^\tau < 1. \tag{15}$$

Then

$$\frac{1}{4}\Big(\frac{\|q_k\theta\|}{q_k}\Big)^{\frac{\tau}{\tau+1}} < \frac{1}{4q_k} < \frac{1}{2}\|q_{k-1}\theta\|. \tag{16}$$

For $1 \le i \le q_k$, put

$$\tilde{F}_k(i) := \left(i\theta - \|q_k\theta\|,\ i\theta + \frac{1}{4}\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} - \|q_k\theta\| \right).$$

By (15), for any constant $c > 0$ for large k

$$c\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} > c\|q_k\theta\|^\tau > \|q_k\theta\|. \tag{17}$$

Since $C_\tau > 1$, by (17) and Lemma 7 (ii)

$$\tilde{F}_k := \bigcup_{i=1}^{q_k} \tilde{F}_k(i) \subset F_k. \tag{18}$$

By (16), the intervals in $\tilde{F}_k(i)$'s are disjoint and distanced by more than $\frac{1}{2}\|q_{k-1}\theta\|$.

We estimate the number of subintervals of $\tilde{F}_{k+\ell}$ in each $\tilde{F}_k(i)$ by the Denjoy-Koksma inequality (see, e.g., [19]): let T be an irrational rotation by θ and f be

a real valued function of bounded variation on the unit interval. Denote by $\mathrm{var}(f)$ the total variation of f on the unit interval. Then for any x

$$\left|\sum_{n=0}^{q_k-1} f(T^n x) - q_k \int f \,\mathrm{d}x\right| \le \mathrm{var}(f). \tag{19}$$

For a given interval I, by the Denjoy-Koksma inequality (19), we have

$$\#\{1 \le n \le q_k : n\theta \in I\} = \sum_{n=0}^{q_k-1} 1_I(T^n x) \ge q_k|I| - 2.$$

Since $\tilde{F}_{k+\ell}$ consists of the disjoint intervals at $q_{k+\ell}$ orbital points, we have for each $1 \le i \le q_k$

$$\#\left\{1 \le n \le q_{k+\ell} : \tilde{F}_{k+\ell}(n) \cap \tilde{F}_k(i) \ne \emptyset\right\} \ge q_{k+\ell} \cdot \frac{1}{4}\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} - 2$$

and

$$\#\left\{1 \le n \le q_{k+\ell} : \tilde{F}_{k+\ell}(n) \subset \tilde{F}_k(i)\right\} \ge \frac{q_{k+\ell}}{4}\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}} - 4.$$

By applying (17), we deduce

$$\#\left\{1 \le n \le q_{k+\ell} : \tilde{F}_{k+\ell}(n) \subset \tilde{F}_k(i)\right\} \ge \frac{q_{k+\ell}}{5}\left(\frac{\|q_k\theta\|}{q_k}\right)^{\frac{\tau}{\tau+1}}.$$

Let $\{n_i\}$ be the sequence of all integers satisfying

$$n_i\|n_i\theta\|^\tau < 1.$$

We remark that since $1/w(\theta) < \tau$, by the definition of $w(\theta)$, there are infinitely many such n_i's. Further, by the Legendre's theorem ([32], pp. 27–29), we have $n_i = q_{k_i}$ for some k_i.

Since $F_k = \mathbb{T}$ if $k \ne k_i$, the Cantor set F is

$$F = \bigcap_{k=1}^{\infty} F_k = \bigcap_{i=1}^{\infty} F_{k_i}.$$

Now we will apply Fact 9. Let

$$\tilde{E}_i := \bigcap_{j=1}^{i} \tilde{F}_{k_j} \subset \bigcap_{j=1}^{i} F_{k_j}.$$

Then $\bigcap_{i=1}^{\infty} \tilde{E}_i \subset F$. Keeping the notations m_i, ε_i as in Fact 9, we have for i large enough,

$$m_i \ge \frac{q_{k_i}}{5}\left(\frac{\|q_{k_{i-1}}\theta\|}{q_{k_{i-1}}}\right)^{\frac{\tau}{\tau+1}}, \quad \varepsilon_i \ge \frac{1}{2}\|q_{k_i-1}\theta\|. \tag{20}$$

Since the lower limit will not be changed if we modify finite number of m_i and ε_i's, we can suppose that the estimates (20) hold for all i. Hence, by Fact 9

$$\begin{aligned}\dim_{\mathrm{H}}(F) &\geq \varliminf_i \frac{\log(m_1\cdots m_i)}{-\log(m_{i+1}\varepsilon_{i+1})}\\ &\geq \varliminf_i \frac{\frac{\tau}{\tau+1}\log(\frac{\|n_1\theta\|\cdots\|n_{i-1}\theta\|}{n_1\cdots n_{i-1}})+\log(n_1\cdots n_i)-i\log 5}{\frac{\tau}{\tau+1}\log(n_i/\|n_i\theta\|)}\\ &= \varliminf_i \frac{\log(\|n_1\theta\|\cdots\|n_{i-1}\theta\|)+\frac{1}{\tau}\log(n_1\cdots n_{i-1})+(1+\frac{1}{\tau})\log n_i}{\log(n_i/\|n_i\theta\|)}.\end{aligned}$$

The last equality follows from the fact that n_k increases super-exponentially when $w(\theta)>1$.

For the the upper bound of $\dim_{\mathrm{H}}(F)$, by Lemma 7 (iii), we have

$$F_k \subset \bigcup_{i=1}^{q_k}\left(i\theta - C_\tau\Big(\frac{\|q_k\theta\|}{q_k}\Big)^{\frac{\tau}{\tau+1}},\ i\theta + C_\tau\Big(\frac{\|q_k\theta\|}{q_k}\Big)^{\frac{\tau}{\tau+1}}\right) := \bar{F}_k.$$

By the Denjoy-Koksma inequality (19), the number of subintervals of $\bar{F}_{k_i}$ contained in each interval of $\bar{F}_{k_{i-1}}$ is at most

$$2C_\tau q_{k_i}\Big(\frac{\|q_{k_{i-1}}\theta\|}{q_{k_{i-1}}}\Big)^{\frac{\tau}{\tau+1}}+4.$$

Therefore, $\bar{E}_i := \bigcap_{j=1}^{i}\bar{F}_{k_j}$ can be covered by ℓ_i sets of diameter at most δ_i, with

$$\begin{aligned}&\ell_i \leq q_{k_1}\left(2C_\tau q_{k_2}\Big(\frac{\|q_{k_1}\theta\|}{q_{k_1}}\Big)^{\frac{\tau}{\tau+1}}+4\right)\cdots\left(2C_\tau q_{k_i}\Big(\frac{\|q_{k_{i-1}}\theta\|}{q_{k_{i-1}}}\Big)^{\frac{\tau}{\tau+1}}+4\right),\\ &\delta_i \leq 2C_\tau\left(\frac{\|q_{k_i}\theta\|}{q_{k_i}}\right)^{\frac{\tau}{\tau+1}}.\end{aligned} \tag{21}$$

By (17)

$$2C_\tau q_{k_i}\Big(\frac{\|q_{k_{i-1}}\theta\|}{q_{k_{i-1}}}\Big)^{\frac{\tau}{\tau+1}} > 2C_\tau q_{k_i}\|q_{k_{i-1}}\theta\|^\tau > 2C_\tau q_{k_i}\|q_{k_{i-1}}\theta\| > C_\tau > 1.$$

Thus by the fact that $x+4\leq 5x$ for $x\geq 1$, we have

$$\ell_i \leq (10C_\tau)^{i-1}(n_1\cdots n_{i-1})^{\frac{1}{\tau+1}}n_i\left(\|n_1\theta\|\cdots\|n_{i-1}\theta\|\right)^{\frac{\tau}{\tau+1}},\ \delta_i \leq 2C_\tau\left(\frac{\|n_i\theta\|}{n_i}\right)^{\frac{\tau}{\tau+1}}.$$

Hence, by Fact 10, we have

$$\begin{aligned}\dim_{\mathrm{H}}(F) &\leq \varliminf_i \frac{\log \ell_i}{-\log\delta_i}\\ &\leq \varliminf_i \frac{\log(\|n_1\theta\|\cdots\|n_{i-1}\theta\|)+\frac{1}{\tau}\log(n_1\cdots n_{i-1})+(1+\frac{1}{\tau})\log n_i}{\log(n_i/\|n_i\theta\|)}.\end{aligned}$$

(iv) Suppose $1<\tau<w(\theta)$. Let $\{n_i\}$ be the sequence of all integers satisfying

$$n_i^\tau\|n_i\theta\|<2.$$

Remark that by the definition of $w(\theta)$, there are infinitely many such n_i's. Applying the Legendre's theorem ([32], pp. 27–29), we have $n_i = q_{k_i}$ for some k_i.

If $k \neq k_i$, then $q_{k+1}^{-\tau} + q_k^{-\tau} \leq 2q_k^{-\tau} \leq \|q_k\theta\|$. Thus by Lemma 8 (ii)

$$F_k = \bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right).$$

Therefore, by (14)

$$\bigcap_{\ell=k_i+1}^{k_{i+1}-1} F_\ell = \bigcup_{j=1}^{q_{k_i+1}} B\left(j\theta, q_{k_i+1}^{-\tau}\right). \tag{22}$$

Also since $|i\theta - j\theta| \geq \|q_{k_i}\theta\| > q_{k_i+1}^{-\tau}$ for $1 \leq i \neq j \leq q_{k_i+1}$, we deduce that

$$\bigcup_{j=1}^{\max(c_{k_i},1)q_{k_i}} B\left(j\theta, q_{k_{i+1}}^{-\tau}\right) = \left(\bigcup_{j=1}^{\max(c_{k_i},1)q_{k_i}} B\left(j\theta, q_{k_i+1}^{-\tau}\right)\right) \cap \left(\bigcup_{j=1}^{q_{k_i+1}} B\left(j\theta, q_{k_{i+1}}^{-\tau}\right)\right).$$

Thus, by Lemma 8 (iii) and (22)

$$\bigcup_{j=1}^{\max(c_{k_i},1)q_{k_i}} B\left(j\theta, q_{k_{i+1}}^{-\tau}\right) \subset F_{k_i} \cap \left(\bigcap_{\ell=k_i+1}^{k_{i+1}-1} F_\ell\right) = \bigcap_{\ell=k_i}^{k_{i+1}-1} F_\ell.$$

Take

$$\tilde{F}_i := \bigcup_{j=1}^{\max(c_{k_i},1)q_{k_i}} B\left(j\theta, q_{k_{i+1}}^{-\tau}\right) \qquad \text{and} \qquad \tilde{E}_i := \bigcap_{j=1}^{i} \tilde{F}_j.$$

Then

$$\bigcap_{i=1}^{\infty} \tilde{E}_i \subset F.$$

By the definition of c_k, if $c_{k_i} \geq 1$, then $q_{k_i}^\tau \|q_{k_i}\theta\| \leq 1$. Using $\tau > 1$, we have for large k

$$\begin{aligned}(c_{k_i} - 1)\|q_{k_i}\theta\| + \frac{1}{q_{k_{i+1}}^\tau} &\leq \left(\frac{1}{q_{k_i}^\tau \|q_{k_i}\theta\|}\right)^{\frac{1}{\tau+1}} \|q_{k_i}\theta\| - \|q_{k_i}\theta\| + \frac{1}{q_{k_{i+1}}^\tau} \\ &< \frac{1}{q_{k_i}^\tau \|q_{k_i}\theta\|}\|q_{k_i}\theta\| - \frac{1}{2q_{k_i+1}} + \frac{1}{q_{k_{i+1}}^\tau} < \frac{1}{q_{k_i}^\tau}.\end{aligned}$$

Therefore, for each $1 \leq j \leq q_{k_i}$

$$B\left(j\theta, q_{k_i}^{-\tau}\right) \cap \tilde{F}_i = \bigcup_{h=0}^{\max(c_{k_i},1)-1} B\left((hq_{k_i} + j)\theta, q_{k_{i+1}}^{-\tau}\right).$$

The number of intervals of $\tilde{F}_i$ in each interval $B\left(j\theta, q_{k_i}^{-\tau}\right)$ of $\tilde{F}_{i-1}$ is

$$m_i = \max(c_{k_i}, 1) = \max\left(\left\lfloor \left(\frac{1}{q_{k_i}^\tau \|q_{k_i}\theta\|}\right)^{\frac{1}{\tau+1}} \right\rfloor, 1\right) \tag{23}$$

and the gaps between intervals in $\tilde{F}_i$ is at least

$$\epsilon_i \geq \|q_{k_i}\theta\| - \frac{2}{(q_{k_{i+1}})^\tau}.$$

Since $\max(\lfloor x \rfloor, 1) \geq \frac{x}{2}$ for any real $x \geq 0$, we have

$$m_i \geq \frac{1}{2}\left(\frac{1}{q_{k_i}^\tau \|q_{k_i}\theta\|}\right)^{\frac{1}{\tau+1}}.$$

For large i, from $\tau > 1$, we deduce

$$\epsilon_i \geq \|q_{k_i}\theta\| - \frac{2}{(q_{k_{i+1}})^\tau} \geq \frac{\|q_{k_i}\theta\|}{2}.$$

Therefore, by Fact 9

$$\begin{aligned}
\dim_{\mathrm{H}}(F) &\geq \varliminf_i \frac{\log(m_1 \cdots m_{i-1})}{-\log(m_i \varepsilon_i)} \\
&\geq \varliminf_k \frac{-\frac{\tau}{\tau+1}\log(n_1\|n_1\theta\|^{1/\tau} n_2\|n_2\theta\|^{1/\tau} \cdots n_{i-1}\|n_{i-1}\theta\|^{1/\tau}) - (i-1)\log 2}{\frac{\tau}{\tau+1}\log(n_i/\|n_i\theta\|) + \log 4} \\
&= \varliminf_k \frac{-\log(n_1\|n_1\theta\|^{1/\tau} n_2\|n_2\theta\|^{1/\tau} \cdots n_{i-1}\|n_{i-1}\theta\|^{1/\tau})}{\log(n_i/\|n_i\theta\|)}.
\end{aligned}$$

For the upper bound, by (22) and Lemma 8 (i), (iii),

$$\bar{F}_i := \bigcup_{j=1}^{\min((2c_{k_i}+3)q_{k_i}, q_{k_i+1})} B\left(j\theta, q_{k_{i+1}}^{-\tau}\right) \supset \bigcap_{\ell=k_i}^{k_{i+1}-1} F_\ell.$$

Then

$$F \subset \bigcap_{i=1}^{\infty} \bar{F}_i.$$

By a similar calculation of (23), we deduce that each $\bar{E}_i := \bigcap_{j=1}^{i} \bar{F}_j$ can be covered by ℓ_i sets of diameter at most δ_i, with

$$\begin{aligned}
\ell_i &\leq (2c_{k_1}+3)\cdots(2c_{k_{i-1}}+3) \\
&\leq \left(2\left(\frac{1}{n_1^\tau\|n_1\theta\|}\right)^{\frac{1}{\tau+1}} + 5\right)\cdots\left(2\left(\frac{1}{n_{i-1}^\tau\|n_{i-1}\theta\|}\right)^{\frac{1}{\tau+1}} + 5\right), \\
\delta_i &\leq (2c_{k_i}+3)\|q_{k_i}\theta\| + \frac{2}{q_{k_{i+1}}^\tau} \leq \left(2\left(\frac{1}{n_i^\tau\|n_i\theta\|}\right)^{\frac{1}{\tau+1}} + 5\right)\cdot\|n_i\theta\|.
\end{aligned}$$

Note that

$$(q_{k_i}^\tau\|q_{k_i}\theta\|)^{-1/(\tau+1)} > 2^{-1/(\tau+1)} > 2^{-1/2},$$

and $2x + 5 < 10x$ for $x > 2^{-1/2}$. Then we have

$$\ell_i \leq 10^{i-1}\left(\frac{1}{n_1^\tau\|n_1\theta\|}\cdots\frac{1}{n_{i-1}^\tau\|n_{i-1}\theta\|}\right)^{\frac{1}{\tau+1}}, \qquad \delta_i \leq 10\left(\frac{\|n_i\theta\|}{n_i}\right)^{\frac{\tau}{\tau+1}}.$$

Thus by Fact 10,

$$\begin{aligned}\dim_{\mathrm{H}}(F) &\le \varliminf_i \frac{\log \ell_i}{-\log \delta_i} \\ &\le \varliminf_i \frac{-\log(n_1\|n_1\theta\|^{1/\tau} n_2\|n_2\theta\|^{1/\tau}\cdots n_{i-1}\|n_{i-1}\theta\|^{1/\tau}) + (i-1)\log 10}{\log(n_i/\|n_i\theta\|) - \log 10} \\ &= \varliminf_i \frac{-\log(n_1\|n_1\theta\|^{1/\tau} n_2\|n_2\theta\|^{1/\tau}\cdots n_{i-1}\|n_{i-1}\theta\|^{1/\tau})}{\log(n_i/\|n_i\theta\|)}.\end{aligned}$$

The last equality is from the super-exponentially increasing of n_k when $w(\theta) > 1$. □

4. The case of $\tau = 1$ and proof of Theorem 2

For the case of $\tau = 1$, we need more accurate estimation on the size of intervals of F_k. We first prove the following two lemmas which describe the subintervals contained in F_k.

Lemma 11. *If $\frac{1}{(b+1)(b+2)} \le q_k\|q_k\theta\| < \frac{1}{b(b+1)}$, for some $b \ge 1$, then*

$$\bigcup_{1\le i\le q_k} \left(i\theta - \frac{1}{q_{k+1}},\ i\theta + (b-1)q_k\theta + \frac{1}{(b+1)q_k}\right) \subset F_k.$$

Proof. Since $\frac{1}{(b+1)(b+2)q_k} \le \|q_k\theta\| < \frac{1}{b(b+1)q_k}$, for any integer $c \ge 1$

$$(b-c)\|q_k\theta\| + \frac{1}{(b+1)q_k} - \frac{1}{(c+1)q_k} = (b-c)\left(\|q_k\theta\| - \frac{1}{(b+1)(c+1)q_k}\right) \le 0.$$

Therefore, for all $c \ge 1$ and $1 \le i \le q_k$

$$(b-1)\|q_k\theta\| + \frac{1}{(b+1)q_k} \le (c-1)\|q_k\theta\| + \frac{1}{(c+1)q_k} \le (c-1)\|q_k\theta\| + \frac{1}{cq_k+i}.$$

Applying Lemma 7 (i), we complete the proof. □

For each $k \ge 0$, denote

$$r_{k+1} := \begin{cases} \left\lfloor \sqrt{4a_{k+1}+5} \right\rfloor - 3, & a_{k+1} \ne 2, \\ 1, & a_{k+1} = 2. \end{cases}$$

We remark that $0 \le r_{k+1} < a_{k+1}$ and the first values of r_{k+1} are

$$r_{k+1} = \begin{cases} 0, & a_{k+1} = 1, \\ 1, & a_{k+1} = 2,3,4, \\ 2, & a_{k+1} = 5,6,7. \end{cases}$$

Define inductively

$$\tilde{r}_{k+1} := \begin{cases} r_{k+1} + 1 = 2, & \text{if } a_{k+1} = 4 \text{ and } a_{k+2} \ge 2, \\ r_{k+1}, & \text{otherwise.} \end{cases}$$

The first values of $\tilde{r}_{k+1}$ can be easily calculated:

$$\tilde{r}_{k+1} = \begin{cases} 0, & a_{k+1} = 1, \\ 1, & a_{k+1} = 2, 3, \\ 1, & a_{k+1} = 4, a_{k+2} = 1, \\ 2, & a_{k+1} = 4, a_{k+2} \geq 2, \\ 2, & a_{k+1} = 5, 6, 7. \end{cases}$$

Note that for $a_{k+1} \neq 4$ or $a_{k+2} \geq 2$ (i.e., for all cases except $a_{k+1} = 4, a_{k+2} = 1$)

$$\tilde{r}_{k+1} + 1 \geq \sqrt{a_{k+1} + 1} \geq \sqrt{\frac{q_{k+1}}{q_k}}. \tag{24}$$

We can also check

$$\tilde{r}_{k+1} \leq \frac{a_{k+1}}{2} \quad \text{for} \quad a_{k+1} \geq 1, \tag{25}$$

$$\tilde{r}_{k+1} + 1 \leq \frac{1}{5}(4a_{k+1} - 1) \quad \text{for} \quad a_{k+1} \geq 3. \tag{26}$$

Lemma 12. *For each $k \geq 1$ with $a_{k+1} \geq 2$, we have*

$$\bigcup_{i=1}^{q_k} (i\theta - \|q_k\theta\|, i\theta + r_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|) \subset F_k.$$

Moreover, if $a_{k+1} = 4$ and $a_{k+2} \geq 2$, then

$$\begin{cases} \displaystyle\bigcup_{i=1}^{q_k} \big(i\theta - \|q_k\theta\|, i\theta + \tilde{r}_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|\big) \subset F_k, & \text{if } a_k = 1, \\ \displaystyle\bigcup_{i=1}^{(\tilde{r}_k+1)q_{k-1}} \big(i\theta - \|q_k\theta\|, i\theta + \tilde{r}_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|\big) \subset F_k, & \text{if } a_k \geq 2. \end{cases}$$

Proof. For the first part of the proof, We distinguish two cases.

(i) Suppose $a_{k+1} = 2$. Then $r_{k+1} = 1$ and $\frac{1}{4} < q_k\|q_k\theta\| < \frac{1}{2}$. Thus, by applying Lemma 11 for $b = 1$, we have

$$\bigcup_{1 \leq i \leq q_k} \left(i\theta - \frac{1}{q_{k+1}},\ i\theta + \frac{1}{2q_k}\right) \subset F_k. \tag{27}$$

Using the equality (3) for $n = k - 1$, and observing $q_{k+1} = a_{k+1}q_k + q_{k-1} = 2q_k + q_{k-1}$, we have

$$\begin{aligned} \frac{1}{2q_k} + \frac{1}{q_{k+1}} &= \frac{1}{q_k} - \left(\frac{1}{2q_k} - \frac{1}{q_{k+1}}\right) \\ &= \frac{q_k\|q_{k-1}\theta\| + q_{k-1}\|q_k\theta\|}{q_k} - \frac{q_{k+1} - 2q_k}{2q_kq_{k+1}} \\ &= \|q_{k-1}\theta\| + \frac{q_{k-1}}{q_k}\left(\|q_k\theta\| - \frac{1}{2q_{k+1}}\right) > \|q_{k-1}\theta\|. \end{aligned}$$

Then by (27), for $q_{k-1} < i \le q_k$

$$\begin{aligned}&(i\theta - \|q_k\theta\|,\ i\theta + \|q_k\theta\| + \|q_{k+1}\theta\|)\\ &\subset \left(i\theta - \frac{1}{q_{k+1}},\ i\theta + \frac{1}{2q_k}\right) \cup \left((i-q_{k-1})\theta - \frac{1}{q_{k+1}},\ (i-q_{k-1})\theta + \frac{1}{2q_k}\right) \subset F_k.\end{aligned} \tag{28}$$

On the other hand, by (3), and the assumption $a_{k+1} = 2$, we can check

$$\|q_k\theta\| + \|q_{k+1}\theta\| < \frac{1}{q_k + q_{k-1}}, \qquad \|q_{k+1}\theta\| < \frac{1}{2q_k + q_{k-1}}.$$

Thus, for $1 \le i \le q_{k-1}$

$$\begin{aligned}&(i\theta - \|q_k\theta\|,\ i\theta + \|q_k\theta\| + \|q_{k+1}\theta\|)\\ &\qquad \subset \left(i\theta - \frac{1}{q_{k+1}},\ i\theta + \min\left(\frac{1}{q_k + q_{k-1}}, \|q_k\theta\| + \frac{1}{2q_k + q_{k-1}}\right)\right) \subset F_k,\end{aligned} \tag{29}$$

where the second inclusion is from Lemma 7 (i).

Combining (28) and (29), we conclude that for $a_{k+1} = 2$

$$\bigcup_{1\le i\le q_k} (i\theta - \|q_k\theta\|,\ i\theta + \|q_k\theta\| + \|q_{k+1}\theta\|) \subset F_k.$$

(ii) Assume $a_{k+1} \ge 3$. There exists an integer $b \ge 1$ satisfying

$$b(b+1) < \frac{1}{q_k\|q_k\theta\|} \le (b+1)(b+2).$$

Thus, we have $b(b+1) - 1 \le a_{k+1} \le (b+1)(b+2) - 1$.

By the fact

$$\frac{1}{q_k} > \|q_{k-1}\theta\| = a_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|,$$

we have

$$\begin{aligned}\frac{1}{(b+1)q_k} &> \frac{a_{k+1}-b}{b+1}\|q_k\theta\| + \frac{b\|q_k\theta\| + \|q_{k+1}\theta\|}{b+1}\\ &> \frac{a_{k+1}-b}{b+1}\|q_k\theta\| + \|q_{k+1}\theta\|.\end{aligned} \tag{30}$$

We will apply Lemma 11 and we will distinguish three parts according to the value of a_{k+1}.

If $b^2 + b - 1 \le a_{k+1} \le b^2 + 2b - 1$, then $\lfloor\sqrt{4a_{k+1}+5}\rfloor = 2b+1$ and by (30)

$$(2b-2)\|q_k\theta\| + \|q_{k+1}\theta\| < (b-1)\|q_k\theta\| + \frac{1}{(b+1)q_k}.$$

If $b^2 + 2b \le a_{k+1} \le b^2 + 3b$, then $\lfloor\sqrt{4a_{k+1}+5}\rfloor = 2b+2$ and by (30)

$$(2b-1)\|q_k\theta\| + \|q_{k+1}\theta\| < (b-1)\|q_k\theta\| + \frac{1}{(b+1)q_k}.$$

Finally if $a_{k+1} = b^2 + 3b + 1$ we have $\lfloor \sqrt{4a_{k+1}+5} \rfloor = 2b+3$ and by (30)

$$2b\|q_k\theta\| + \|q_{k+1}\theta\| < (b-1)\|q_k\theta\| + \frac{1}{(b+1)q_k}.$$

Therefore, in all cases, we have

$$\left(\left\lfloor \sqrt{4a_{k+1}+5} \right\rfloor - 3 \right) \|q_k\theta\| + \|q_{k+1}\theta\| \le (b-1)\|q_k\theta\| + \frac{1}{(b+1)q_k}.$$

By Lemma 11, we have

$$\bigcup_{1 \le i \le q_k} (i\theta - \|q_k\theta\|,\ i\theta + r_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|) \subset F_k.$$

Now we prove the second assertion of the lemma. We will apply Lemma 7 (i). To this end, we will obtain in the following many estimates of the form:

$$(b-1)\|q_k\theta\| + \frac{1}{bq_k + i} \qquad (1 \le i \le q_k).$$

(a) If $a_k = 1$, then we have

$$\begin{aligned} q_k\|q_{k+1}\theta\| &= q_{k-1}\|q_{k+1}\theta\| + q_{k-2}\|q_{k+1}\theta\| \\ &\le (a_{k+2}-1)q_{k-1}\|q_{k+1}\theta\| + q_{k-2}\|q_{k+1}\theta\| \\ &< a_{k+2}q_{k-1}\|q_{k+1}\theta\| < q_{k-1}\|q_k\theta\|. \end{aligned}$$

Hence, for all $b \ge 1$

$$\begin{aligned} (b+1)q_k\left((3-b)\|q_k\theta\| + \|q_{k+1}\theta\|\right) &\le 4q_k\|q_k\theta\| + 2q_k\|q_{k+1}\theta\| \\ &< 4q_k\|q_k\theta\| + q_{k-1}\|q_k\theta\| + q_k\|q_{k+1}\theta\| \\ &= q_{k+1}\|q_k\theta\| + q_k\|q_{k+1}\theta\| = 1, \end{aligned}$$

which yields that for all $b \ge 1$

$$2\|q_k\theta\| + \|q_{k+1}\theta\| < (b-1)\|q_k\theta\| + \frac{1}{(b+1)q_k}.$$

Therefore, by Lemma 7 (i), we have

$$\bigcup_{1 \le i \le q_k} (i\theta - \|q_k\theta\|,\ i\theta + 2\|q_k\theta\| + \|q_{k+1}\theta\|) \subset F_k.$$

(b) Suppose $a_k \ge 2$. We will prove for all $b \ge 1$

$$2\|q_k\theta\| + \|q_{k+1}\theta\| < (b-1)\|q_k\theta\| + \frac{1}{bq_k + (\tilde{r}_k+1)q_{k-1}},$$

which is equivalent to

$$(bq_k + (\tilde{r}_k+1)q_{k-1})\left((3-b)\|q_k\theta\| + \|q_{k+1}\theta\|\right) < 1.$$

In fact, for $1 \le b \le 3$, by (25), we have

$$\begin{aligned}
&(bq_k + (\tilde{r}_k+1)q_{k-1})\left((3-b)\|q_k\theta\| + \|q_{k+1}\theta\|\right)\\
&\le \left(bq_k + \left(\frac{a_k}{2}+1\right)q_{k-1}\right)\left((3-b)\|q_k\theta\| + \|q_{k+1}\theta\|\right)\\
&= (3b-b^2)q_k\|q_k\theta\| + (b-1)q_k\|q_{k+1}\theta\| + \left(\frac{(3-b)a_k}{2} + 2 - b\right)q_{k-1}\|q_k\theta\|\\
&\quad + \left(\frac{a_k}{2}+1\right)q_{k-1}\|q_{k+1}\theta\| + q_k\|q_{k+1}\theta\| + q_{k-1}\|q_k\theta\|.
\end{aligned} \tag{31}$$

By (2) and (1) respectively, we have the estimations:

$$\|q_{k+1}\theta\| \le \frac{1}{a_{k+2}}\|q_k\theta\| \quad \text{and} \quad q_{k-1} < \frac{q_k}{a_k}. \tag{32}$$

Thus, for $1 \le b \le 2$

$$\begin{aligned}
&(3b-b^2)q_k\|q_k\theta\| + (b-1)q_k\|q_{k+1}\theta\|\\
&\qquad + \left(\frac{(3-b)a_k}{2} + 2 - b\right)q_{k-1}\|q_k\theta\| + \left(\frac{a_k}{2}+1\right)q_{k-1}\|q_{k+1}\theta\|\\
&\qquad\quad < \left(3b - b^2 + \frac{b-1}{a_{k+2}} + \frac{3-b}{2} + \frac{2-b}{a_k} + \left(\frac{1}{2} + \frac{1}{a_k}\right)\frac{1}{a_{k+2}}\right)q_k\|q_k\theta\|.
\end{aligned}$$

By using the assumption $a_{k+2} \ge 2$ and $1 \le b \le 2$, we then deduce

$$\begin{aligned}
&(bq_k + (\tilde{r}_k+1)q_{k-1})\left((3-b)\|q_k\theta\| + \|q_{k+1}\theta\|\right)\\
&\le \left(3b - b^2 + \frac{b-1}{2} + \frac{3-b}{2} + \frac{2-b}{2} + \frac{1}{2}\right)q_k\|q_k\theta\| + q_k\|q_{k+1}\theta\| + q_{k-1}\|q_k\theta\|\\
&= \left(\frac{5}{2} + \frac{5b}{2} - b^2\right)q_k\|q_k\theta\| + q_k\|q_{k+1}\theta\| + q_{k-1}\|q_k\theta\|\\
&\le 4q_k\|q_k\theta\| + q_{k-1}\|q_k\theta\| + q_k\|q_{k+1}\theta\| = 1.
\end{aligned}$$

For the last equality, we have used the assumption $a_{k+1} = 4$ and the fact (3).

If $b = 3$, then from (31) and (32) we have

$$\begin{aligned}
(bq_k + (\tilde{r}_k+1)q_{k-1})\left((3-b)\|q_k\theta\| + \|q_{k+1}\theta\|\right) &\le \left(3q_k + \left(\frac{a_k}{2}+1\right)q_{k-1}\right)\|q_{k+1}\theta\|\\
&< \left(\frac{7}{2} + \frac{1}{a_k}\right)q_k\|q_{k+1}\theta\|\\
&\le 4q_k\|q_{k+1}\theta\| < 1.
\end{aligned}$$

For $b \ge 4$, it is easy to see that

$$(bq_k + (\tilde{r}_k+1)q_{k-1})\left((3-b)\|q_k\theta\| + \|q_{k+1}\theta\|\right) < 0 < 1.$$

Thus, for each $1 \le i \le (\tilde{r}_k+1)q_{k-1}$, we have for any $b \ge 1$

$$2\|q_k\theta\| + \|q_{k+1}\theta\| < (b-1)\|q_k\theta\| + \frac{1}{bq_k + (\tilde{r}_k+1)q_{k-1}} \le (b-1)\|q_k\theta\| + \frac{1}{bq_k + i}.$$

By Lemma 7 (i), we have

$$\bigcup_{i=1}^{(\tilde{r}_k+1)q_{k-1}} \left(i\theta - \|q_k\theta\|, i\theta + \tilde{r}_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|\right) \subset F_k. \qquad \square$$

The proof of Lemma 12 is completed.

Now we are ready to give a new nested Cantor subset of $\mathcal{U}_\tau[\theta]$. Remind that we assume k is even. We denote

$$D_k := \begin{cases} \mathbb{T}, & a_{k+1} = 1, \\ \bigcup_{i=1}^{q_k} \left(i\theta - \|q_k\theta\|, i\theta + \tilde{r}_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|\right). & a_{k+1} \geq 2. \end{cases} \tag{33}$$

For the case k is odd, we have the symmetric formula:

$$D_k := \bigcup_{i=1}^{q_k} \left(i\theta - \tilde{r}_{k+1}\|q_k\theta\| - \|q_{k+1}\theta\|, i\theta + \|q_k\theta\|\right). \tag{34}$$

Then, by Lemma 12, we have $D_k \subset F_k$, thus

$$D := \bigcap_{k=1}^{\infty} D_k \subset \bigcap_{k=1}^{\infty} F_k.$$

Now we will investigate the numbers of subintervals of $D_{k+\ell}$ in each interval of D_k. Let (u_m) be the Fibonacci sequence defined by $u_0 = 0, u_1 = 1$ and $u_{m+1} = u_m + u_{m-1}$.

Lemma 13. *Suppose that $a_{k+1} \geq 2$, $a_{k+\ell+1} \geq 2$ and $a_{k+m} = 1$ for all $2 \leq m \leq \ell$. Then the number of points of $j\theta$, $1 \leq j \leq q_{k+\ell}$ in each interval of D_k is*

$$u_\ell \tilde{r}_{k+1} + u_{\ell+1} \geq \frac{q_{k+\ell}}{\sqrt{q_k q_{k+1}}}.$$

Proof. For each integer $n \geq 0$ we have a unique representation (called Ostrowski's expansion, see [36]):

$$n = \sum_{m=0}^{\infty} c_{m+1} q_m,$$

where $0 \leq c_1 < a_1$, $0 \leq c_{m+1} \leq a_{m+1}$, and $c_m = 0$ if $c_{m+1} = a_{m+1}$.

If

$$j = \sum_{m=k}^{k+\ell-1} c_{m+1} q_m$$

is an integer with its representation coefficients:

$$0 \leq c_{k+1} \leq \tilde{r}_{k+1} < a_{k+1}, \quad 0 \leq c_{m+1} \leq a_{m+1} = 1 \ (k < m \leq k+\ell), \tag{35}$$

then, by the fact that $q_k\theta - p_k > 0$ if and only if k is even, we have

$$\begin{aligned} j\theta &= c_{k+1}q_k\theta + c_{k+2}q_{k+1}\theta + \cdots + c_{k+\ell}q_{k+\ell-1}\theta \\ &\le \tilde{r}_{k+1}q_k\theta + a_{k+3}q_{k+2}\theta + a_{k+5}q_{k+4}\theta + \cdots < \tilde{r}_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|, \\ j\theta &\ge a_{k+2}q_{k+1}\theta + a_{k+4}q_{k+3}\theta + \cdots > -\|q_k\theta\|. \end{aligned}$$

Thus, for each i with $1 \le i \le q_k$

$$i\theta - \|q_k\theta\| < (i+j)\theta < i\theta + \tilde{r}_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|.$$

The number of the above integer j's of which expansion satisfying (35) is the number of ℓ-tuples of $(c_{k+1}, c_{k+2}, \ldots, c_{k+\ell})$ such that

$$0 \le c_{k+1} \le \tilde{r}_{k+1} < a_{k+1}, \quad 0 \le c_{m+1} \le 1 = a_{m+1} \ \text{ for } \ k+1 \le m \le k+\ell-1$$

and

$$c_m c_{m+1} = 0 \ \text{ for } k+1 \le m \le k+\ell-1,$$

which is $u_\ell\tilde{r}_{k+1} + u_{\ell+1}$. Note that if $\ell = 1$, then the number of j's satisfying (35) is $\tilde{r}_{k+1} + 1 = u_1\tilde{r}_{k+1} + u_2$. Hence, for each $1 \le i \le q_k$

$$\#\{1 \le j \le q_{k+\ell} : j\theta \in (i\theta - \|q_k\theta\|,\ i\theta + \tilde{r}_{k+1}\|q_k\theta\| + \|q_{k+1}\theta\|)\} = u_\ell\tilde{r}_{k+1} + u_{\ell+1}.$$

If $a_{k+1} \ne 4$ or $\ell = 1$, then using (24) and the fact $q_{k+\ell} = u_\ell q_{k+1} + u_{\ell-1}q_k$, the number of points satisfies

$$\begin{aligned} u_\ell\tilde{r}_{k+1} + u_{\ell+1} = u_\ell(\tilde{r}_{k+1}+1) + u_{\ell-1} &\ge u_\ell\sqrt{\frac{q_{k+1}}{q_k}} + u_{\ell-1}\sqrt{\frac{q_k}{q_{k+1}}} \\ &= \frac{u_\ell q_{k+1} + u_{\ell-1}q_k}{\sqrt{q_k q_{k+1}}} = \frac{q_{k+\ell}}{\sqrt{q_k q_{k+1}}}. \end{aligned}$$

If $a_{k+1} = 4$ and $\ell \ge 2$, then $\frac{q_{k+1}}{q_k} < 5$, thus

$$\frac{\sqrt{\frac{q_{k+1}}{q_k}} - 2}{1 - \sqrt{\frac{q_k}{q_{k+1}}}} < \frac{\sqrt{5}-2}{1-\frac{1}{\sqrt{5}}} < \frac{1}{2} \le \frac{u_{\ell-1}}{u_\ell},$$

which is equivalent to

$$2u_\ell + u_{\ell-1} > u_\ell\sqrt{\frac{q_{k+1}}{q_k}} + u_{\ell-1}\sqrt{\frac{q_k}{q_{k+1}}}.$$

Therefore, we have

$$\begin{aligned} u_\ell\tilde{r}_{k+1} + u_{\ell+1} = u_\ell + u_{\ell+1} &= 2u_\ell + u_{\ell-1} \\ &> u_\ell\sqrt{\frac{q_{k+1}}{q_k}} + u_{\ell-1}\sqrt{\frac{q_k}{q_{k+1}}} = \frac{u_\ell q_{k+1} + u_{\ell-1}q_k}{\sqrt{q_k q_{k+1}}} = \frac{q_{k+\ell}}{\sqrt{q_k q_{k+1}}}. \end{aligned}$$

□

We use the mass distribution principle (e.g. [15]):

Fact 14 (Mass Distribution Principle)**.** *Let $E \subset \mathbb{R}^n$ and let μ be a finite Borel measure with $\mu(E) > 0$. Suppose that there are numbers $s \geq 0$, $c > 0$ and $\delta_0 > 0$ such that*

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| \leq \delta_0$, where $|\cdot|$ stands for the Euclidean diameter. Then

$$\dim_{\mathrm{H}}(E) \geq s.$$

Now we are ready to estimate the Hausdorff dimension of $\mathcal{U}_1[\theta]$.

Theorem 15. *For $\tau = 1$ and for any irrational θ*

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right) \geq \frac{1}{w(\theta)+1}.$$

Proof. We may assume $w(\theta) < \infty$. If $a_k = 1$ for all large k, then Lemma 6 (2) implies that $\mathcal{U}_\tau[\theta] = \mathbb{T}$. Thus we assume that $a_k \geq 2$ for infinitely many k's. Let (k_i) be the increasing sequence of integers such that $k_0 = 0$ and

$$\{k_1, k_2, \dots\} = \{k \in \mathbb{N} : a_{k+1} \geq 2\}.$$

Denote by m_i the number of intervals of D_{k_i} contained in each interval of $D_{k_{i-1}}$. Then by Lemma 13 we have

$$m_i \geq \frac{q_{k_i}}{\sqrt{q_{k_{i-1}} q_{k_{i-1}+1}}}. \tag{36}$$

Define μ on D given by

$$\mu(I) = \prod_{n=1}^{i} \frac{1}{m_n}$$

for each interval I of the form $(j\theta - \|q_{k_i}\theta\|, j\theta + \tilde{r}_{k_i+1}\|q_{k_i}\theta\| + \|q_{k_i+1}\theta\|)$ with $1 \leq j \leq q_{k_i}$ in D_{k_i}. Note that

$$|j_1 - j_2| \geq \|q_{k_i-1}\theta\| \quad \text{for } \ 1 \leq j_1, j_2 \leq q_{k_i} \text{ and } j_1 \neq j_2. \tag{37}$$

Let U be an interval with

$$\|q_{k_{i+1}-1}\theta\| \leq |U| < \|q_{k_i-1}\theta\|$$

for some $i \geq 1$. Then by (37), U intersects at most $(|U|/\|q_{k_{i+1}-1}\theta\| + 2)$ interval of $D_{k_{i+1}}$. Thus, we have

$$\mu(U) \leq \frac{1}{m_1 m_2 \cdots m_{i+1}} \left(\frac{|U|}{\|q_{k_{i+1}-1}\theta\|} + 2 \right) \leq \frac{3|U|}{m_1 m_2 \cdots m_{i+1} \|q_{k_{i+1}-1}\theta\|}. \tag{38}$$

If $a_{k_i+1} \geq 3$, then by (37) the smallest gap between two intervals in D_{k_i} is at least

$$\begin{aligned}\|q_{k_i-1}\theta\| - (\tilde{r}_{k_i+1}+1)\,\|q_{k_i}\theta\| - \|q_{k_i+1}\theta\| &= (a_{k_i+1}-1-\tilde{r}_{k_i+1})\,\|q_{k_i}\theta\| \\ &> \frac{a_{k_i+1}-1-\tilde{r}_{k_i+1}}{a_{k_i+1}+1}\|q_{k_i-1}\theta\| \\ &\geq \frac{\|q_{k_i-1}\theta\|}{5} > \frac{|U|}{5},\end{aligned}$$

where we use (2) and (26) for the first and the second inequalities. Thus U intersects at most 6 intervals of D_{k_i} and

$$\mu(U) \leq \frac{6}{m_1 m_2 \cdots m_i}. \tag{39}$$

If $a_{k_i+1} = 2$, then each interval in D_{k_i} is of length

$$(\tilde{r}_{k_i+1}+1)\|q_{k_i}\theta\| + \|q_{k_i+1}\theta\| = 2\|q_{k_i}\theta\| + \|q_{k_i+1}\theta\| = \|q_{k_i-1}\theta\| > |U|$$

Therefore, U intersects at most 2 intervals of D_{k_i}. Thus

$$\mu(U) \leq \frac{2}{m_1 m_2 \cdots m_i}. \tag{40}$$

Hence, (38), (39) and (40) imply that

$$\mu(U) \leq \frac{6}{m_1 m_2 \cdots m_{i+1}} \min\left(\frac{|U|}{\|q_{k_{i+1}-1}\theta\|}, m_{i+1}\right).$$

For any $0 < s < 1$, since $\min(x,y) \leq x^s y^{1-s}$ for $x, y \geq 1$, we have

$$\mu(U) \leq \frac{6}{m_1 m_2 \cdots m_i}\left(\frac{|U|}{m_{i+1}\|q_{k_{i+1}-1}\theta\|}\right)^s.$$

By (36), we have

$$\begin{aligned}\mu(U) &\leq 6\frac{\sqrt{q_{k_0}q_{k_0+1}}}{q_{k_1}}\frac{\sqrt{q_{k_1}q_{k_1+1}}}{q_{k_2}}\cdots\frac{\sqrt{q_{k_{i-1}}q_{k_{i-1}+1}}}{q_{k_i}}\left(\frac{\sqrt{q_{k_i}q_{k_i+1}}|U|}{q_{k_{i+1}}\|q_{k_{i+1}-1}\theta\|}\right)^s \\ &\leq 6\sqrt{\frac{q_{k_0}}{q_{k_1}}}\sqrt{\frac{q_{k_1}}{q_{k_2}}}\cdots\sqrt{\frac{q_{k_{i-1}}}{q_{k_i}}}\left(2\sqrt{q_{k_i}q_{k_i+1}}|U|\right)^s \leq 12\frac{\left(\sqrt{q_{k_i}q_{k_i+1}}|U|\right)^s}{\sqrt{q_{k_i}}}.\end{aligned} \tag{41}$$

Let s be any real number satisfying

$$s < \frac{1}{w+1} = \varliminf_{i\to\infty}\frac{\log q_{k_i}}{\log q_{k_i} + \log q_{k_i+1}}.$$

Then by (41) for sufficiently small $|U|$

$$\mu(U) \leq 12|U|^s.$$

Therefore, by Fact 14, we have

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right) \geq \dim_{\mathrm{H}}\left(\bigcap_{i=1}^{\infty} D_i\right) \geq s. \qquad \square$$

Proof of Theorem 2. When $\tau < 1$ or $\tau > 1$, the proof is the same as that of Theorem 1. The case of $\tau = 1$ follows from Theorem 15. □

5. Proofs of Theorems 3 and 5

Using Theorem 15, we can prove Theorem 3.

Proof of Theorem 3. Let us use the same notation $(q_{k_j})_{j\geq 1}$ for the subsequences selected in Theorem 1 for the two cases $1/w(\theta) < \tau < 1$ and $1 < \tau < w(\theta)$. Then by the fact that $n_j = q_{k_j}$ increases super-exponentially, we can replace $\|n_j\theta\|$ by $q_{k_j+1}^{-1}$ and rewrite the formula in Theorem 1 as follows.

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right) = \begin{cases} \underline{\lim}\limits_{i\to\infty} \dfrac{\log\left(\prod_{j=1}^{i-1}(q_{k_j}^{1/\tau}q_{k_j+1}^{-1})\cdot q_{k_i}^{1+1/\tau}\right)}{\log(q_{k_i}q_{k_i+1})}, & \text{if } \dfrac{1}{w(\theta)} < \tau < 1, \\ \underline{\lim}\limits_{i\to\infty} \dfrac{-\log\left(\prod_{j=1}^{i-1} q_{k_j}q_{k_j+1}^{-1/\tau}\right)}{\log\left(q_{k_i}q_{k_i+1}\right)}, & \text{if } 1 < \tau < w(\theta). \end{cases}$$

Further, let w_j be the real numbers defined by $2q_{k_j+1} = q_{k_j}^{w_j}$ for the case $1/w(\theta) < \tau < 1$ and $4q_{k_j+1} = q_{k_j}^{w_j}$ for the case $1 < \tau < w(\theta)$. Then by (4), $w_j \geq 1/\tau$ if $1/w(\theta) < \tau < 1$ and $w_j \geq \tau$ if $1 < \tau < w(\theta)$. By (5), we have

$$\overline{\lim_{j\to\infty}}\, w_j = w(\theta), \tag{42}$$

and the dimension $\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right)$ is equal to

$$\begin{cases} \underline{\lim}\limits_{i\to\infty}\left(\dfrac{1+\frac{1}{\tau}}{w_i+1} - \sum\limits_{j=1}^{i-1}\dfrac{w_j-\frac{1}{\tau}}{w_i+1}\cdot\dfrac{\log q_{k_j}}{\log q_{k_i}}\right), & \text{if } \dfrac{1}{w(\theta)} < \tau < 1, \\ \underline{\lim}\limits_{i\to\infty}\sum\limits_{j=1}^{i-1}\dfrac{\frac{w_j}{\tau}-1}{w_i+1}\cdot\dfrac{\log q_{k_j}}{\log q_{k_i}}, & \text{if } 1 < \tau < w(\theta). \end{cases} \tag{43}$$

Now fix $w(\theta) = w \in (1, +\infty]$. For all $j < i$, we have

$$0 < \frac{\log q_{k_j}}{\log q_{k_i}} = \frac{\log q_{k_j}}{\log q_{k_{j+1}}}\cdots\frac{\log q_{k_{i-1}}}{\log q_{k_i}} \leq \frac{\log q_{k_j}}{\log q_{k_j+1}}\cdots\frac{\log q_{k_{i-1}}}{\log q_{k_{i-1}+1}} = \frac{1}{w_j\cdots w_{i-1}}.$$

Hence, if $\frac{1}{w} < \tau < 1$,

$$0 \leq (w_j - \frac{1}{\tau})\cdot\frac{\log q_{k_j}}{\log q_{k_i}} \leq \frac{1}{w_{j+1}\cdots w_{i-1}} - \frac{1}{\tau w_j\cdots w_{i-1}},$$

and if $1 < \tau < w$,

$$0 \leq \left(\frac{w_j}{\tau} - 1\right)\cdot\frac{\log q_{k_j}}{\log q_{k_i}} \leq \frac{1}{\tau w_{j+1}\cdots w_{i-1}} - \frac{1}{w_j\cdots w_{i-1}}.$$

Let

$$S_{i-1} = \frac{1}{w_1\cdots w_{i-1}} + \frac{1}{w_2\cdots w_{i-1}} + \cdots + \frac{1}{w_{i-1}}.$$

Then for $1/w < \tau < 1$

$$\underline{\lim}_{i\to\infty} \frac{1}{w_i+1}\left(\frac{1}{\tau}+\Big(\frac{1}{\tau}-1\Big)S_{i-1}+\frac{1}{w_1\cdots w_{i-1}}\right) \le \dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \le \underline{\lim}_{i\to\infty}\frac{1+\frac{1}{\tau}}{w_i+1}$$

and for $1 < \tau < w$

$$0 \le \dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \le \underline{\lim}_{i\to\infty}\frac{1}{w_i+1}\left(\frac{1}{\tau}-\Big(1-\frac{1}{\tau}\Big)S_{i-1}-\frac{1}{\tau w_1\cdots w_{i-1}}\right).$$

If $w=\infty$, then $\overline{\lim}\, w_i = \infty$ for both two cases $0 = 1/w(\theta) < \tau < 1$ and $1 < \tau < w(\theta) = \infty$. Thus by (43), we have

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \le \begin{cases} \underline{\lim}_{i\to\infty}\dfrac{1+\frac{1}{\tau}}{w_i+1} = 0, & 0<\tau<1,\\[2ex] \underline{\lim}_{i\to\infty}\dfrac{1}{w_i+1}\cdot\dfrac{1}{\tau} = 0, & 1<\tau<\infty. \end{cases}$$

Therefore, $\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) = 0$ for all $\tau > 0$.

If $w < \infty$, then by (42), for any $\varepsilon > 0$ there is N such that if $i \ge N$ then

$$S_{i-1} = \frac{1}{w_1\cdots w_{i-1}}+\cdots+\frac{1}{w_{i-1}} > \frac{1}{(w+\varepsilon)^{i-N}}+\cdots+\frac{1}{w+\epsilon} = \frac{1-(w+\epsilon)^{-i+N}}{w+\epsilon-1}.$$

Thus, for $1/w < \tau < 1$

$$\frac{1}{w+1}\left(\frac{1}{\tau}+\Big(\frac{1}{\tau}-1\Big)\frac{1}{w-1}\right) \le \dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \le \frac{1+\frac{1}{\tau}}{w+1},$$

and for $1 < \tau < w$

$$0 \le \dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) \le \frac{1}{w+1}\left(\frac{1}{\tau}-\Big(1-\frac{1}{\tau}\Big)\frac{1}{w-1}\right).$$

For the case of $\tau = 1$, we complete the proof by Theorem 15. □

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Let $1/w < \tau' < \tau < 1$ and (k_i) and (k_i') be the maximal sequences of

$$q_{k_i}\|q_{k_i}\theta\|^{\tau} < 1, \qquad q_{k_i'}\|q_{k_i'}\theta\|^{\tau'} < 1.$$

Note that (k_i') is a subsequence of (k_i).

Let w_j, w_j' be the real numbers defined by $2q_{k_j+1} = q_{k_j}^{w_j}$, $2q_{k_j'+1} = q_{k_j'}^{w_j'}$ as in the proof of Theorem 3. Recall that for all j, we have $w_j\tau \ge 1$ and $w_j'\tau' \ge 1$. Thus, by

noting the fact $q_{k_{j+1}} \geq q_{k_j+1} > {q_{k_j}}^{1/\tau}$, we have

$$\begin{aligned}
&\frac{1+1/\tau}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} \\
=&\frac{1+1/\tau'}{w_i+1} - \frac{1/\tau' - 1/\tau}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} - \sum_{j=1}^{i-1} \frac{1/\tau' - 1/\tau}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} \\
\geq&\frac{1+1/\tau'}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} - \sum_{j=1}^{i} \frac{1/\tau' - 1/\tau}{1/\tau^{i-j}(1+1/\tau)} \\
\geq&\frac{1+1/\tau'}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} - \frac{\tau - \tau'}{\tau'(1-\tau^2)}. \qquad (44)
\end{aligned}$$

Let s be the index such that $k'_s < k_i < k'_{s+1}$. Noting that $w_j - 1/\tau' \leq 0$ if k_j is not in the subsequence (k'_i), we have

$$\frac{1+1/\tau'}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} \geq \frac{1+1/\tau'}{w_i+1} - \sum_{j=1}^{s} \frac{w'_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k'_j}}{\log q_{k_i}}. \qquad (45)$$

By the choice of s, we know $q_{k_i} \geq q_{k'_s+1} = q_{k'_s}^{w'_s}$. Hence, the right hand side of (45) is bigger than

$$\frac{1+1/\tau'}{w_i+1} - \frac{w'_s - 1/\tau'}{(w_i+1)w'_s} - \sum_{j=1}^{s-1} \frac{w'_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k'_j}}{w'_s \log q_{k'_s}},$$

which is equal to

$$\frac{1+1/w'_s}{\tau'(w_i+1)} - \sum_{j=1}^{s-1} \frac{w'_j - 1/\tau'}{(w_i+1)w'_s} \cdot \frac{\log q_{k'_j}}{\log q_{k'_s}}.$$

Reminding the fact $1/\tau \leq w_i \leq 1/\tau'$, we then deduce that

$$\frac{1+1/\tau'}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} \geq \frac{1+1/w'_s}{\tau'+1} - \sum_{j=1}^{s-1} \frac{w'_j - 1/\tau'}{(1+1/\tau)w'_s} \cdot \frac{\log q_{k'_j}}{\log q_{k'_s}}.$$

By verifying $(1+1/\tau)w'_s > w'_s + 1$ and

$$\frac{1+1/w'_s}{\tau'+1} \geq \frac{1+1/\tau'}{w'_s+1},$$

we obtain

$$\frac{1+1/\tau'}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau'}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} \geq \frac{1+1/\tau'}{w'_s+1} - \sum_{j=1}^{s-1} \frac{w'_j - 1/\tau'}{w'_s+1} \cdot \frac{\log q_{k'_j}}{\log q_{k'_s}}. \qquad (46)$$

Therefore, combining (44) and (46), we have for $k'_s \leq k_i < k'_{s+1}$,

$$\frac{1+1/\tau}{w_i+1} - \sum_{j=1}^{i-1} \frac{w_j - 1/\tau}{w_i+1} \cdot \frac{\log q_{k_j}}{\log q_{k_i}} \geq \frac{1+1/\tau'}{w'_s+1} - \sum_{j=1}^{s-1} \frac{w'_j - 1/\tau'}{w'_s+1} \cdot \frac{\log q_{k'_j}}{\log q_{k'_s}} - \frac{\tau - \tau'}{\tau'(1-\tau^2)}.$$

Hence, by (43), we have

$$\begin{aligned}\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) &= \varliminf_{i\to\infty}\left(\frac{1+1/\tau}{w_i+1}-\sum_{j=1}^{i-1}\frac{w_j-1/\tau}{w_i+1}\cdot\frac{\log q_{k_j}}{\log q_{k_i}}\right)\\ &\geq \varliminf_{s\to\infty}\left(\frac{1/\tau'+1}{w'_s+1}-\sum_{j=1}^{s-1}\frac{w'_j-1/\tau'}{w'_s+1}\cdot\frac{\log q_{k'_j}}{\log q_{k'_s}}\right)-\frac{\tau-\tau'}{\tau'(1-\tau^2)}\\ &= \dim_{\mathrm{H}}(\mathcal{U}_{\tau'}[\theta])-\frac{\tau-\tau'}{\tau'(1-\tau^2)}.\end{aligned}$$

Let $1<\tau<\tau'<w$. Let (k_i) and (k'_i) be the sequence of

$$q_{k_i}^{\tau}\|q_{k_i}\theta\|<2, \qquad q_{k'_i}^{\tau'}\|q_{k'_i}\theta\|<2.$$

Clearly, (k'_i) is a subsequence of (k_i).

Let w_i be the real numbers defined by $4q_{k_i+1}=q_{k_i}^{w_i}$ as in the proof of Theorem 3. Recall that for all j, we have $w_j\geq\tau$. Then, by (43), we have

$$\begin{aligned}\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) &= \varliminf_{i\to\infty}\sum_{j=1}^{i-1}\frac{w_j/\tau-1}{w_i+1}\cdot\frac{\log q_{k_j}}{\log q_{k_i}}\\ &= \varliminf_{i\to\infty}\left(\sum_{j=1}^{i-1}\frac{w_j/\tau'-1}{w_i+1}\cdot\frac{\log q_{k_j}}{\log q_{k_i}}+\sum_{j=1}^{i-1}\frac{w_j(\tau'-\tau)}{\tau\tau'(w_i+1)}\cdot\frac{\log q_{k_j}}{\log q_{k_i}}\right)\\ &\leq \varliminf_{i\to\infty}\left(\sum_{j=1}^{i-1}\frac{w_j/\tau'-1}{w_i+1}\cdot\frac{\log q_{k_j}}{\log q_{k_i}}\right)+\varlimsup_{i\to\infty}\left(\sum_{j=1}^{i-1}\frac{w_j(\tau'-\tau)}{\tau\tau'(\tau+1)\tau^{i-j}}\right)\\ &\leq \varliminf_{i\to\infty}\left(\sum_{j=1}^{i-1}\frac{w_j/\tau'-1}{w_i+1}\cdot\frac{\log q_{k'_j}}{\log q_{k'_i}}\right)+\frac{w(\tau'-\tau)}{\tau\tau'(\tau^2-1)}.\end{aligned}$$

Hence,

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta])-\dim_{\mathrm{H}}(\mathcal{U}_{\tau'}[\theta])\leq\frac{(\tau'-\tau)w}{\tau\tau'(\tau^2-1)}.$$

Since $\mathcal{U}_\tau[\theta]\supset\mathcal{U}_{\tau'}[\theta]$,

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta])-\dim_{\mathrm{H}}(\mathcal{U}_{\tau'}[\theta])\geq 0.$$

Therefore, the claim holds. □

6. Examples

The following examples show that the upper and lower bounds in Theorems 2 and 3 can not be replaced by smaller or larger numbers.

Example 16. Let θ be of irrational exponent $w(\theta) = w > 1$ with $q_{k+1} > q_k^w$ for all k. Then the subsequence k_i in the proof of Theorem 1 is given by $k_i = i$.

Put $q_{k_i+1} = q_{k_i}^{w_i}$. Then $\lim_{i\to\infty} w_i = w$.

For $1/w < \tau < 1$, we have

$$\begin{aligned}
\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) &= \varliminf_{i\to\infty} \frac{\log(q_1^{1/\tau}\|q_1\theta\|q_2^{1/\tau}\|q_2\theta\|\cdots q_{i-1}^{1/\tau}\|q_{i-1}\theta\|\cdot q_i^{1+\frac{1}{\tau}})}{\log(q_i/\|q_i\theta\|)} \\
&= \varliminf_{i\to\infty} \frac{\log(q_1^{1/\tau}q_2^{1/\tau-1}\cdots q_{i-1}^{1/\tau-1}\cdot q_i^{1/\tau})}{\log(q_iq_{i+1})} \\
&= \varliminf_{i\to\infty} \frac{1}{1+w_i}\left(\frac{\frac{1}{\tau}}{w_1\cdots w_{i-1}} + \frac{\frac{1}{\tau}-1}{w_2\cdots w_{i-1}} + \cdots + \frac{\frac{1}{\tau}-1}{w_{i-1}} + \frac{1}{\tau}\right) \\
&= \frac{1}{1+w}\left(\frac{\frac{1}{\tau}-1}{w-1} + \frac{1}{\tau}\right).
\end{aligned}$$

For $1 < \tau < w$, we have

$$\begin{aligned}
\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) &= \varliminf_{i\to\infty} \frac{-\log(q_1\|q_1\theta\|^{1/\tau}q_2\|q_2\theta\|^{1/\tau}\cdots q_{i-1}\|q_{i-1}\theta\|^{1/\tau})}{\log(q_i/\|q_i\theta\|)} \\
&= \varliminf_{i\to\infty} \frac{-\log\left(q_1q_2^{1-1/\tau}\cdots q_{i-1}^{1-1/\tau}\cdot q_i^{-1/\tau}\right)}{\log(q_iq_{i+1})} \\
&= \varliminf_{i\to\infty} \frac{1}{1+w_i}\left(\frac{\frac{1}{\tau}}{w_1\cdots w_{i-1}} + \frac{\frac{1}{\tau}-1}{w_2\cdots w_{i-1}} + \cdots + \frac{\frac{1}{\tau}-1}{w_{i-1}} + \frac{1}{\tau}\right) \\
&= \frac{1}{1+w}\left(\frac{\frac{1}{\tau}-1}{w-1} + \frac{1}{\tau}\right).
\end{aligned}$$

Therefore, for each $1/w < \tau < w$ we have

$$\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) = \frac{\frac{w}{\tau}-1}{w^2-1}.$$

Example 17. Assume that θ is an irrational of $w(\theta) = w > 1$ with the subsequence $\{k_i\}$ of $q_{k_i+1} > q_{k_i}^w$ satisfying that $a_{n+1} = 1$ for $n \neq k_i$ and $q_{k_i} > \left(q_{k_{i-1}+1}\right)^{2^i}$. Then we have

$$\lim_{i\to\infty}\left(\frac{\log q_{k_1}}{\log q_{k_i}} + \frac{\log q_{k_2}}{\log q_{k_i}} + \cdots + \frac{\log q_{k_{i-1}}}{\log q_{k_i}}\right) = 0.$$

Since w_i converges to w, by (43), the Hausdorff dimension of $\mathcal{U}_\tau[\theta]$ is $\frac{1/\tau+1}{w+1}$ and 0, respectively for $1/w < \tau < 1$ and $\tau > 1$.

If $\tau = 1$, then, by the proof of (4),

$$\begin{aligned}\dim_{\mathrm{H}}(\mathcal{U}_\tau[\theta]) &\geq \varliminf_{i\to\infty} \frac{\log q_{k_i} + \sum\limits_{\substack{1\leq k<k_i\\ a_{k+1}=1}} \log \frac{q_{k+1}}{q_k}}{\log q_{k_i} + \log q_{k_{i+1}} - \sum\limits_{\substack{k_i\leq k<k_{i+1}\\ a_{k+1}=1}} \log \frac{q_{k+1}}{q_k}} \\ &\geq \varliminf_{i\to\infty} \frac{\log q_{k_i} + \big(\log q_{k_i} - \log q_{k_{i-1}+1}\big)}{\log q_{k_i} + \log q_{k_i+1}} \geq \varliminf_{i\to\infty} \frac{2 - \log q_{k_{i-1}+1}/\log q_{k_i}}{1 + \log q_{k_i+1}/\log q_{k_i}} \\ &\geq \varliminf_{i\to\infty} \frac{2 - 2^{-i}}{1 + \log q_{k_i+1}/\log q_{k_i}} = \frac{2}{w+1}.\end{aligned}$$

Hence, we have

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right) = \begin{cases} \dfrac{1+\frac{1}{\tau}}{w+1}, & \text{for } 1/w < \tau \leq 1, \\ 0, & \text{for } \tau > 1. \end{cases}$$

Example 18. Let $\theta = \frac{\sqrt{5}-1}{2}$, of which partial quotients $a_k = 1$ for all k. Note that $w(\theta) = 1$. By Lemma 6, $\mathcal{U}_\tau[\theta] = \mathbb{T}$ for $\tau = 1$. Thus, we have

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right) = \begin{cases} 1, & \tau \leq 1, \\ 0 & \tau > 1. \end{cases}$$

Example 19. Let θ be the irrational with partial quotient $a_k = k$ for all k. Then $w(\theta) = 1$. Consider the case of $\tau = 1$. By Lemma 7 (iii), we have

$$F_k \subset \bigcup_{i=1}^{q_k} \left(i\theta - 2\Big(\frac{\|q_k\theta\|}{q_k}\Big)^{\frac{1}{2}},\ i\theta + 2\Big(\frac{\|q_k\theta\|}{q_k}\Big)^{\frac{1}{2}} \right).$$

Thus, by (21), F_k can be covered by ℓ_k sets of diameter at most δ_k, with

$$\begin{aligned}\ell_k &\leq q_1 \left(4q_2\Big(\frac{\|q_1\theta\|}{q_1}\Big)^{\frac{1}{2}} + 4\right) \cdots \left(4q_k\Big(\frac{\|q_{k-1}\theta\|}{q_{k-1}}\Big)^{\frac{1}{2}} + 4\right) \\ &\leq 8^{k-1} q_1 \left(\frac{q_2}{q_1}\right)^{\frac{1}{2}} \cdots \left(\frac{q_k}{q_{k-1}}\right)^{\frac{1}{2}} = 8^{k-1} \left(q_1 q_k\right)^{\frac{1}{2}}, \\ \delta_k &\leq 4\Big(\frac{\|q_k\theta\|}{q_k}\Big)^{\frac{1}{2}} < 4\Big(\frac{1}{q_k q_{k+1}}\Big)^{\frac{1}{2}}.\end{aligned}$$

Here we use the fact $x + 1 \leq 2x$ for $x \geq 1$ for the second inequality for ℓ_k. Thus,

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right) \leq \varliminf_{k} \frac{\log \ell_k}{-\log \delta_k} = \varliminf_{k} \frac{(k-1)\log 8 + \frac{1}{2}(\log q_k + \log q_1)}{-\log 4 + \frac{1}{2}(\log q_k + \log q_{k+1})}.$$

Since

$$\log q_{k+1} \geq \sum_{i=1}^{k+1} \log a_i = \sum_{i=2}^{k+1} \log i \geq \int_1^{k+1} (\log x)\mathrm{d}x = (k+1)\log(k+1) - k,$$

one has

$$\lim_{k\to\infty}\frac{k}{\log q_k}=0,\qquad 1\le\lim_{k\to\infty}\frac{\log q_{k+1}}{\log q_k}\le\lim_{k\to\infty}\frac{\log(a_{k+1}+1)+\log q_k}{\log q_k}=1.$$

Therefore,

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right)\le\underline{\lim}_{k}\frac{\log q_k}{\log q_k+\log q_{k+1}}=\frac{1}{2}.$$

Hence, by Theorem 15, we have

$$\dim_{\mathrm{H}}\left(\mathcal{U}_\tau[\theta]\right)=\begin{cases}1, & \tau<1,\\ \frac{1}{2}, & \tau=1,\\ 0 & \tau>1.\end{cases}$$

Acknowledgement

The authors thank Yann Bugeaud and Michel Laurent for their valuable remarks. Dong Han Kim was partially supported by NRF-2012R1A1A2004473 and NRF-2015R1A2A2A01007090 (Korea).

References

[1] P. Alessandri and V. Berthé, Three distance theorems and combinatorics on words, *Enseign. Math.* **44** (1998), no. 1-2, 103–132.

[2] A. S. Besicovitch, Sets of fractional dimensions (IV): on rational approximation to real numbers, *J. London Math. Soc.* **9** (1934), 126–131.

[3] V. Beresnevich, G. Harman, A. Haynes and S. Velani, The Duffin-Schaeffer conjecture with extra divergence II, *Math. Z.* **275** (2013), no. 1-2, 127–133.

[4] Y. Bugeaud, A note on inhomogeneous diophantine approximation, *Glasgow Math. J.* **45** (2003), 105–110.

[5] Y. Bugeaud, Distribution Modulo One and Diophantine Approximation, Cambridge University Press, 2012.

[6] Y. Bugeaud and M. Laurent, On exponents of homogeneous and inhomogeneous Diophantine approximation, *Moscow Math. J.* **5** (2005), 747–766.

[7] Y. Bugeaud and L. Liao, Uniform Diophantine approximation related to b-ary and β-expansions, *Ergod. Th. Dynam. Sys.*, **36**, no. 1, (2016), 1–22.

[8] J. W. S. Cassels. *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. **45**. Cambridge University Press, New York, 1957.

[9] Y. Cheung, Hausdorff dimension of the set of singular pairs, *Ann. Math.* **173** (2011), 127–167.

[10] Y. Cheung and N. Chevallier, Hausdorff dimension of singular vectors, *Duke Math. J.*, **65** (2016), 2273-2329.

[11] Z. Coelho and E. de Faria, Limit laws of entrance times for homeomorphisms of the circle, *Israel J. Math.* **93** (1996), 93–112.

[12] L. G. P. Dirichlet, Verallgemeinerung eines Satzes aus der Lehre von den Kettenbrüchen nebst einige Anwendungen auf die Theorie der Zahlen, *S.-B. Preuss. Akad. Wiss.* (1842), 93–95.

[13] R. J. Duffin and A. C. Schaeffer. Khintchine's problem in metric Diophantine approximation, *Duke Math. J.* **8** (1941), 243–255.

[14] K. J. Falconer, *Fractal geometry. Mathematical foundations and applications*, Second edition, John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[15] K. J. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, Inc., Hoboken, NJ, 1997.

[16] A.-H. Fan, J. Schmeling, and S. Troubetzkoy, A multifractal mass transference principle for Gibbs measures with applications to dynamical Diophantine approximation, *Proc. Lond. Math. Soc.* **107** (2013), 1173–1219.

[17] M. Fuchs and D. H. Kim, On Kurzweil's 0-1 law in inhomogeneous Diophantine approximation, *Acta. Arith.* **173** (2016), 41–57.

[18] A. Haynes, A. Pollington and S. Velani, The Duffin-Schaeffer conjecture with extra divergence, *Math. Ann.* **353** (2012), no. 2, 259–273.

[19] M. R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Inst. Hautes Études Sci. Publ. Math.*, **49** (1979), 5–233.

[20] R. Hill and S. Velani, Ergodic theory of shrinking targets, *Invent. math.* **119** (1995), 175–198.

[21] R. Hill and S. Velani, The shrinking target problem for matrix transformations of tori, *J. London Math. Soc. (2)* **60** no. 2 (1999), 381–398.

[22] V. Jarník, Zur metrischen Theorie der diophantischen Approximation, *Prace Mat.-Fiz.* **36** (1928/1929), 91–106.

[23] A. Ya. Khinchine, *Continued Fractions*, Univ. Chicago Press, Chicago, 1964.

[24] A. Ya. Khinchine, Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen, *Math. Ann.* **92** (1924), no. 1-2, 115–125.

[25] A. Ya. Khinchine, Über eine Klasse linearer diophantischer Approximationen, *Rendiconti Circ. Mat. Palermo* **50** (1926), 170–195.

[26] D.H. Kim, The recurrence time of irrational rotations, *Osaka J. Math.* **43** (2006) 351–364.

[27] D.H. Kim, Refined shrinking target property of rotations, *Nonlinearity* **27** (2014), 1985–1997.

[28] D.H. Kim and B.K. Seo, The waiting time for irrational rotations, *Nonlinearity* **16** (2003), 1861–1868.

[29] D. Kleinbock and N. Wadleigh, A zero-one Law for improvements to Dirichlet's Theorem, Proc. Amer. Math. Soc., to appear.

[30] J. Kurzweil, On the metric theory of inhomogeneous Diophantine approximations, *Studia Math.* **15** (1955), 84–112.

[31] M. Laurent and A. Nogueira, Inhomogeneous approximation with coprime integers and lattice orbits, *Acta Arith.* **154** (2012), 413–427.

[32] A. M. Legendre, *Essai sur la théorie des nombres*, Chez Courcier, Imprimeur-Libaraire pour les Mathématiques, Paris, 1808.

[33] J. Levesley, A general inhomogeneous Jarník-Besicovitch theorem, *J. Number Theory* **71** (1998), 65–80.

[34] L. Liao and S. Seuret, Diophantine approximation by orbits of expanding Markov maps, *Ergodic Theory Dynam. Systems.* **33** (2013), 585–608.

[35] H. Minkowski, *Diophantische Approximationen*, Leipzig, Berlin 1907.

[36] A. Rockett and P. Szüsz, *Continued Fractions*. World Scientific, 1992.

[37] J. Schmeling and S. Troubetzkoy, Inhomogeneous Diophantine approximation and angular recurrence properties of the billiard flow in certain polygons, *Math. Sbornik* **194** (2003), 295–309.

[38] W. M. Schmidt, Metrical theorems on fractional parts of sequences, *Trans. Amer. Math. Soc.* **110** (1964), 493–518.

[39] N. B. Slater, Gaps and steps for the sequence $n\theta$ (mod 1), *Proc. Camb. Phil. Soc.* **63** (1967), 1115–1123.

[40] V. Sprindžuk, *Metric Theory of Diophantine Approximations*, V. H. Winston & Sons, Washington, D.C., 1979.

[41] P. Szüsz, Über die metrische Theorie der diophantischen Approximation, *Acta Math. Acad. Sci. Hung.* **9** (1958), 177–193.

[42] M. Waldschmidt, Recent advances in Diophantine approximation, *Number theory, analysis and geometry*. 659–704, Springer, New York, 2012.

Department of Mathematics Education, Dongguk University - Seoul, 30 Pil-dongro 1-gil, Jung-gu, Seoul 04620, Korea

E-mail address: kim2010@dongguk.edu

LAMA UMR 8050, CNRS Université Paris-Est Créteil, 61 Avenue du Général de Gaulle, 94010 Créteil, Cedex, France

E-mail address: lingmin.liao@u-pec.fr