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# Force-induced dispersion in heterogeneous media

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The effect of a constant applied external force, induced for instance by an electric or gravitational field, on the dispersion of Brownian particles in periodic media with spatially varying diffusivity, and thus mobility, is studied. We show that external forces can greatly enhance dispersion in the direction of the applied force and also modify, to a lesser extent and in some cases non-monotonically, dispersion perpendicular to the applied force. Our results thus open up the intriguing possibility of modulating the dispersive properties of heterogeneous media by using externally applied force fields. These results are obtained via a Kubo formula which can be applied to any periodic advection diffusion system in any spatial dimension.

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In diverse systems ranging from fluid mechanics, hydrology, soft matter to solid state physics, at mesoscopic length and time scales, the dynamics of tracer particles is described by stochastic differential equations and their associated Fokker-Planck equations [1–3]. In heterogeneous media, the local transport coefficients such as the diffusivity and the mobility can vary in space depending on the local material properties. In a locally isotropic material where a uniform force  $\mathbf{F}$  acts on a tracer particle, the probability density function (PDF)  $p(\mathbf{x}, t)$  for the tracer position at time  $t$  obeys

$$\partial_t p(\mathbf{x}, t) = \nabla \cdot [\kappa(\mathbf{x}) \nabla p - \beta \kappa(\mathbf{x}) \mathbf{F} p]. \quad (1)$$

The first term on the right hand side of Eq. (1) above corresponds to diffusion with a spatially varying diffusion constant. The second term represents the drift due to a constant applied external force and the term  $\beta \kappa(\mathbf{x}) = \mu(\mathbf{x})$  is the local mobility. The factor of the inverse temperature  $\beta$  results from the local Einstein relation between mobility and diffusivity. Specific physical examples include charge carriers in heterogeneous media, where  $\mu(\mathbf{x})$  is proportional to the local electrical conductivity, in the presence of an external electric field, as well as colloidal diffusion in porous media, with local diffusivity  $\kappa(\mathbf{x})$ , with an external field induced by gravitational or buoyancy forces. In this Letter, we will consider the effect that a constant external applied field has on the late time dispersion as characterized by the effective drift of a cloud of tracer particles

$$V_i = \lim_{t \rightarrow \infty} \frac{\langle X_i(t) - X_i(0) \rangle}{t}, \quad (2)$$

(where  $\mathbf{X}(t)$  denotes the position of a tracer particle and  $\langle \cdot \rangle$  denotes ensemble averaging) and the effective diffusivity

$$D_{ii} = \lim_{t \rightarrow \infty} \frac{\langle [X_i(t) - X_i(0)]^2 \rangle_c}{2t}, \quad (3)$$

( $c$  denotes the connected part, thus the variance of the displacement  $X_i(t) - X_i(0)$ ) characterising the dispersion

of a cloud of tracers about its mean position. These effective transport coefficients are important in several contexts, for example estimating the spread of pollutants and the determination of mixing and chemical reaction times [4].

In the case where  $\mathbf{F} = \mathbf{0}$ , the problem of determining  $D_{ii}$  and  $V_i$  has a history going back to Maxwell [5], where the equivalent problem of determining the dielectric constant of heterogeneous media was addressed. The Wiener bounds [6] state that  $(\overline{\kappa^{-1}})^{-1} \leq D \leq \overline{\kappa}$ , where  $\overline{\cdot}$  indicates spatial averaging. The upper bound shows that diffusion is slowed down with respect to the average value of the diffusion constant, while the lower bound is exact in one dimension and can be intuitively understood as the addition of conductors in series. In higher dimensions there are few exact results [7] but numerous approximations schemes exist [8–12]. However, the case where there is a finite external force appears not to have been studied and in this Letter we will address the force's effect on the dispersion of tracer particles.

To gain a flavor for the phenomenology of this problem we consider diffusion in a two dimensional medium, where  $\kappa(x, y)$  is shown in Fig. 1(a), with an applied force  $\mathbf{F}$  oriented in the  $x$  direction. We show in Fig. 1(c) the results of numerical simulations of the corresponding stochastic differential equation for the quantities  $D_{xx}$ ,  $D_{yy}$  and  $V_x/\beta F$ . At zero force, we see that all the quantities shown are equal, this is a result of the Stokes-Einstein relation  $D_{xx} = \beta \partial_F V_x$  which holds *only* [13] when  $F = 0$ . At small  $F$  upon increasing  $F$ , we see that both  $D_{xx}$  and  $V_x/F$  decrease while  $D_{yy}$  increases. As  $F$  increases further,  $V_x/F$  continues to decrease monotonically, however  $D_{xx}$  and  $D_{yy}$  attain minimal and maximal values respectively and eventually cross. This remarkable behavior shows that the fast and slow directions of dispersion can be inverted by an applied force and, even more surprisingly,  $D_{yy}(F)$  is a non-monotonic function. In Fig. 1(d), we see that  $D_{xx}$  continues to grow as  $F^2$  at large forces and can thus be made arbitrarily large (thus

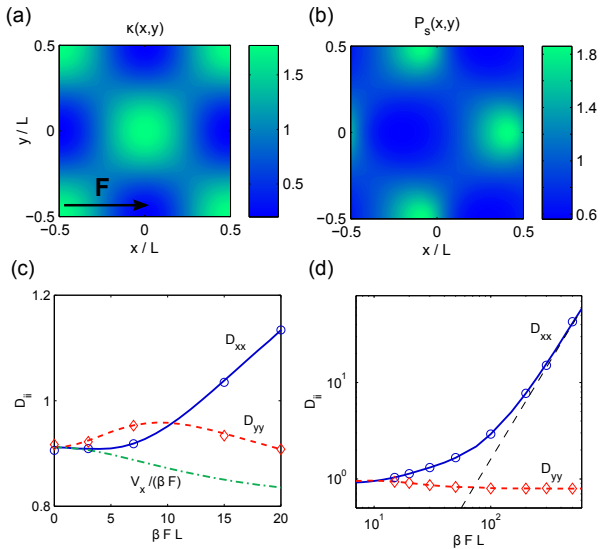


FIG. 1. (color online) (a) The 2D periodic diffusivity field  $\kappa(x, y) = \kappa_0[1 + 0.8 \cos(2\pi x/L) \cos(2\pi y/L)]$ , shown in units of  $\kappa_0$  on the fundamental rectangular unit cell. The arrow indicates the direction of the external force. (b) Stationary PDF in the diffusivity field shown in (a) with an external force of magnitude  $\beta FL = 10$ . (c) Components  $D_{xx}$  and  $D_{yy}$  of the effective diffusion tensor predicted by Eqs. (19,20) and the normalized effective drift  $V_x/\beta F$  from Eq. (7) (lines) along with simulation results for the stochastic differential equation (5) (symbols). (d) Same as (c) with different scales. The dashed line represents the behavior  $D_{xx} \approx cF^2$  with the coefficient  $c$  predicted by Eq. (25).

exceeding the upper Wiener bound for the forceless case), giving rise to force induced dispersion enhancement. The key difference between systems with and without an external force is that in the latter case the steady state probability distribution  $P_s(x, y)$  on the periodic unit cell of the system is constant, whereas in the presence of the field it becomes non-trivial as shown in Fig. 1(b).

To explain these results we will derive a Kubo-type formula for the transport coefficients for general Fokker-Planck equations with arbitrary periodic diffusion tensors and advection fields. This formula generalizes a number of existing results in fluid mechanics, notably for convection by incompressible velocity fields with constant molecular diffusivity as in the case of Taylor dispersion [16]. Examples include diffusion in Rayleigh-Bénard convection cells [17–19], diffusion in frozen turbulent flows [20] and transport by a fluid in porous media [21–24]. Our formula also encapsulates results from statistical physics for diffusion in periodic potentials [25–29]. In one dimension, results on diffusion in periodic potentials plus constant forces have been derived [30–34], as well as the more general case where the noise amplitude is a periodic function of position [35–37].

The Kubo formula we derive here is valid in any dimension. The terms in the Kubo formula can be analytically

evaluated in the case where the diffusivity varies only in one direction, and we give analytical results for such stratified systems. We also solve the generic problem analytically in the limit of large forces, proving that the coefficient of  $D_{ii}$ , where  $i$  is the direction of the force, is generically proportional to  $F^2$  in the limit of large forces. Finally, the Kubo formula can be evaluated numerically by solving a set of associated partial differential equations and the simulations is shown in Figs. 1(c,d), and we see that the agreement is excellent.

*Kubo formula for the dispersion.* Consider the Fokker-Planck equation in its most general form

$$\partial_t p = \sum_{i,j=1}^d \partial_{x_i} \{ -u_i(\mathbf{x})p + \partial_{x_j} [\kappa_{ij}(\mathbf{x})p] \} \equiv \mathcal{L}_{\mathbf{x}} p, \quad (4)$$

where  $\kappa_{ij}(\mathbf{x})$  is a local (symmetric) diffusion tensor,  $\mathbf{u}(\mathbf{x})$  represents a drift field, and where we have called  $\mathcal{L}_{\mathbf{x}}$  the transport operator acting on the variable  $\mathbf{x}$ . Our only assumption in the derivation of our formula for the dispersion is that all fields  $u_i(\mathbf{x})$  and  $\kappa_{ij}(\mathbf{x})$  are periodic functions in space. Let  $\Omega$  denote the fundamental unit cell of the periodic structure. We call  $p(\mathbf{x}, t|\mathbf{y})$  the propagator of the stochastic process in infinite space, defined as the solution of Eq. (4) in infinite space with initial condition  $p(\mathbf{x}, 0|\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ , and representing the probability of observing a particle at position  $\mathbf{x}$  at time  $t$  given that the starting position is  $\mathbf{y}$ . We distinguish this infinite space propagator  $p(\mathbf{x}, t|\mathbf{y})$  from the propagator calculated with periodic boundary conditions on the boundaries of  $\Omega$ , denoted  $P(\mathbf{x}, t|\mathbf{y})$ , and representing the probability density to observe a particle at time  $t$  at a position  $\mathbf{x}$  modulo an integer number of translations along the lattice vectors of the periodic structure. Finally, we define  $P_s(\mathbf{x}) = \lim_{t \rightarrow \infty} P(\mathbf{x}, t|\mathbf{y})$  the stationary PDF of the particles with periodic boundary conditions. Our first goal is to express the macroscopic quantities  $V_i, D_{ii}$  defined in Eqs. (2,3), which characterize the motion at the scales of many periods, as a function of local quantities such as  $P$ , which only depends on the properties of the system at the scale of one period.

In the Ito prescription, the stochastic differential equation corresponding to the Fokker-Planck equation (4) in the direction  $i$  [2, 3] is

$$dX_i = u_i(\mathbf{X}(t)) dt + \sum_{j=1}^d (\kappa^{1/2}(\mathbf{X}(t)))_{ij} dW_j, \quad (5)$$

where  $\kappa^{1/2}$  represents the square-root matrix of the positive symmetric matrix  $\kappa$ . The noise increments  $dW_i$  are Gaussian, independent, of zero mean and are only correlated at equal times as

$$\langle dW_i dW_j \rangle = 2\delta_{ij} dt. \quad (6)$$

Averaging Eq. (5) over different realizations, it is immediate that the effective drift is given by

$$V_i = \int_{\Omega} d\mathbf{x} P_s(\mathbf{x}) u_i(\mathbf{x}), \quad (7)$$

as shown by Stratonovich [39]. To calculate the effective diffusivity we first subtract  $u_i dt$  from both sides of Eq. (5), integrate over time, square both sides of the resulting equation and then average to find

$$\begin{aligned} & \langle [X_i(t) - X_i(0)]^2 \rangle + \int_0^t dt_1 \int_0^{t_1} dt_2 \langle u_i(\mathbf{X}(t_1)) u_i(\mathbf{X}(t_2)) \rangle \\ & - 2 \int_0^t dt' \langle \{X_i(t) - X_i(t') + X_i(t') - X_i(0)\} u_i(\mathbf{X}(t')) \rangle \\ & = 2t \int_{\Omega} d\mathbf{x} P_s(\mathbf{x}) \kappa_{ii}(\mathbf{x}). \end{aligned} \quad (8)$$

The average of the right hand side of Eq. (8) follows from Eq. (6) and the independence of the  $dW_i$  at different time steps. Exploiting the periodicity of the field  $\mathbf{u}(\mathbf{x})$ , we can evaluate the second term of Eq. (8) for  $t_1 < t_2$  as

$$\begin{aligned} & \langle u_i(\mathbf{X}(t_1)) u_i(\mathbf{X}(t_2)) \rangle = \\ & \iint_{\Omega} d\mathbf{x}_1 d\mathbf{x}_2 u_i(\mathbf{x}_2) u_i(\mathbf{x}_1) P(\mathbf{x}_2, t_2 - t_1 | \mathbf{x}_1) P_s(\mathbf{x}_1). \end{aligned} \quad (9)$$

The second line of Eq. (8) contains the term [40]

$$\begin{aligned} & \langle [X_i(\tau) - X_i(0)] u_i(\mathbf{X}(0)) \rangle = \\ & \int_{\mathbb{R}^d} d\mathbf{x} \int_{\Omega} d\mathbf{y} p(\mathbf{x}, \tau | \mathbf{y}) P_s(\mathbf{y}) (x_i - y_i) u_i(\mathbf{y}). \end{aligned} \quad (10)$$

Differentiating with respect to  $\tau$ , using Eq. (4) and integrating by parts over  $\mathbf{x}$ , we obtain

$$\begin{aligned} & \partial_{\tau} \langle [X_i(\tau) - X_i(0)] u_i(\mathbf{X}(0)) \rangle = \int_{\Omega} d\mathbf{y} P_s(\mathbf{y}) u_i(\mathbf{y}) \times \\ & \int_{\mathbb{R}^d} d\mathbf{x} \left[ u_i(\mathbf{x}) p(\mathbf{x}, \tau | \mathbf{y}) - \sum_{j=1}^d \partial_{x_j} \kappa_{ij}(\mathbf{x}) p(\mathbf{x}, \tau | \mathbf{y}) \right]. \end{aligned} \quad (11)$$

Finally, exploiting the periodicity of the field  $\mathbf{u}$ , we can replace the integral over  $\mathbf{x}$  over the infinite space by an integral over the unit cell  $\Omega$  if one replaces the infinite space propagator  $p$  by the propagator with periodic boundary conditions  $P$ , yielding for any  $t > t'$  [41]

$$\begin{aligned} & \partial_t \langle [X_i(t) - X_i(t')] u_i(\mathbf{X}(t')) \rangle = \\ & \int_{\Omega} d\mathbf{x} \int_{\Omega} d\mathbf{y} u_i(\mathbf{y}) u_i(\mathbf{x}) P(\mathbf{x}, t - t' | \mathbf{y}) P_s(\mathbf{y}). \end{aligned} \quad (12)$$

Now, the last term to be computed in Eq. (8) is

$$\begin{aligned} & \langle [X_i(t) - X_i(0)] u_i(\mathbf{X}(t)) \rangle = \\ & \int_{\mathbb{R}^d} d\mathbf{x} \int_{\Omega} d\mathbf{y} p(\mathbf{x}, t | \mathbf{y}) P_s(\mathbf{y}) (x_i - y_i) u_i(\mathbf{x}). \end{aligned} \quad (13)$$

We note that, due to the periodicity, we can exchange the integration domains of  $\mathbf{y}$  and  $\mathbf{x}$  in this equation. We now use the backward Fokker-Planck equation [3]  $\partial_t p(\mathbf{x}, t | \mathbf{y}) = \mathcal{L}_{\mathbf{y}}^{\dagger} p$ , (where  $\mathcal{L}^{\dagger}$  is the adjoint of the transport operator  $\mathcal{L}$ ) to find

$$\begin{aligned} & \partial_t \langle [X_i(t) - X_i(0)] u_i(\mathbf{X}(t)) \rangle = \\ & \int_{\Omega} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{y} [\mathcal{L}_{\mathbf{y}}^{\dagger} p(\mathbf{x}, t | \mathbf{y})] P_s(\mathbf{y}) (x_i - y_i) u_i(\mathbf{x}). \end{aligned} \quad (14)$$

Using the definition of the adjoint operator, we write

$$\begin{aligned} & \partial_t \langle [X_i(t) - X_i(0)] u_i(\mathbf{X}(t)) \rangle = \\ & \int_{\mathbb{R}^d} d\mathbf{y} \int_{\Omega} d\mathbf{x} u_i(\mathbf{x}) p(\mathbf{x}, t | \mathbf{y}) \mathcal{L}_{\mathbf{y}} \{ P_s(\mathbf{y}) (x_i - y_i) \}. \end{aligned} \quad (15)$$

The periodicity of  $\mathbf{u}$  means that we can again replace the infinite space propagator  $p$  by the propagator in the unit domain if one changes the integration domain of  $\mathbf{x}$  to  $\Omega$ . Explicit calculation of  $\mathcal{L}_{\mathbf{y}} \{ P_s(\mathbf{y}) (x_i - y_i) \}$  gives

$$\begin{aligned} & \partial_{t'} \langle [X_i(t') - X_i(0)] u_i(\mathbf{X}(t')) \rangle = \int_{\Omega} d\mathbf{x} u_i(\mathbf{x}) \times \\ & \int_{\Omega} d\mathbf{y} P(\mathbf{x}, t | \mathbf{y}) \left\{ J_{s,i}(\mathbf{y}) - \sum_{j=1}^d \partial_{y_j} [\kappa_{ij}(\mathbf{y}) P_s(\mathbf{y})] \right\}, \end{aligned} \quad (16)$$

where we have introduced  $\mathbf{J}_s(\mathbf{y})$  the local current in the stationary state at position  $\mathbf{y}$ , given by

$$J_{s,i}(\mathbf{y}) = u_i(\mathbf{y}) P_s(\mathbf{y}) - \sum_{j=1}^d \partial_{y_j} [\kappa_{ij}(\mathbf{y}) P_s(\mathbf{y})]. \quad (17)$$

Finally, all the terms appearing in Eq. (8) can be evaluated by using Eqs. (10,12,16). Taking the large time limit, we obtain our final Kubo formula for the effective diffusion tensor

$$\begin{aligned} D_{ii} & = \int_{\Omega} d\mathbf{y} P_s(\mathbf{y}) \kappa_{ii}(\mathbf{y}) + \\ & \iint_{\Omega} d\mathbf{x} d\mathbf{y} u_i(\mathbf{x}) G(\mathbf{x} | \mathbf{y}) [2J_{s,i}(\mathbf{y}) - u_i(\mathbf{y}) P_s(\mathbf{y})], \end{aligned} \quad (18)$$

where we have introduced  $G$  the integral of the propagator, defined as  $G(\mathbf{x} | \mathbf{y}) = \int_0^{\infty} dt \{ P(\mathbf{x}, t | \mathbf{y}) - P_s(\mathbf{x}) \}$ , which is the pseudo-Green function [42] of  $\mathcal{L}$  on  $\Omega$ . The equation (18) gives in an explicit way the dispersion properties in terms of quantities that are defined at the level of an individual cell  $\Omega$ , with periodic boundary conditions. For practical uses, it is useful to re-express  $D_{ii}$  by introducing a function  $\mathbf{f}(\mathbf{x})$  defined as the solution of

$$\begin{aligned} \mathcal{L}_{\mathbf{x}} \mathbf{f}_i(\mathbf{x}) & = -2J_{s,i}(\mathbf{x}) + u_i(\mathbf{x}) P_s(\mathbf{x}) \\ & + P_s(\mathbf{x}) \int_{\Omega} d\mathbf{y} [2J_{s,i}(\mathbf{y}) - u_i(\mathbf{y}) P_s(\mathbf{y})], \end{aligned} \quad (19)$$

again with periodic boundary conditions on  $\Omega$ , and with the integral condition  $\int_{\Omega} d\mathbf{x} \mathbf{f}(\mathbf{x}) = \mathbf{0}$ . The diffusion tensor is then given in terms of  $\mathbf{f}$  by

$$D_{ii} = \int_{\Omega} d\mathbf{x} \{P_s(\mathbf{x})\kappa_{ii}(\mathbf{x}) + u_i(\mathbf{x})f_i(\mathbf{x})\}. \quad (20)$$

The non-equilibrium component of the problem is manifested in Eq. (18) by the presence of the local currents of the stationary state, which are absent in the case of equilibrium problems. Our formula is a generalization of similar Kubo formulas derived for equilibrium problems. In the case of transport by incompressible fluid flows, the stationary PDF  $P_s(\mathbf{x})$  is uniform, the current  $\mathbf{J}_s$  is equal to the flow  $\mathbf{u}$  and one recovers the equations describing dispersion in incompressible hydrodynamic flows (compare for example Eqs. (19,20) to Eqs. (35,48) of Ref. [24]).

*Periodic diffusivity with an external uniform force.*

We now come back to the original problem of determining the dispersion properties of particles submitted to a force in media of heterogeneous diffusivity. Here the corresponding Fokker-Planck equation (1) falls in the class of the general equation (4) with

$$\kappa_{ij}(\mathbf{x}) = \delta_{ij}\kappa(\mathbf{x}), \quad \mathbf{u}(\mathbf{x}) = \kappa(\mathbf{x})\beta\mathbf{F} + \nabla\kappa(\mathbf{x}). \quad (21)$$

The effective dispersion tensor  $D_{ii}$  can be obtained by solving numerically the partial differential equations (19,20), leading to the results on Fig. 1, which compare very well to numerical simulations of the stochastic equation (5), thereby validating our approach.

*Stratified media.* Now consider systems where the local diffusivity varies only in one dimension,  $\kappa(x, y) = \kappa(x)$  as illustrated in Fig. 2(a); consequently the functions  $\mathbf{f}$  in this case depend only on  $x$  and can be calculated analytically [38]. At low forces,  $|\mathbf{F}| \rightarrow 0$ , the diffusivity tensor reads

$$D_{xx} = 1/\overline{\kappa^{-1}}, \quad D_{yy} = \overline{\kappa}, \quad D_{xy} = 0. \quad (22)$$

The anisotropy of the dispersion is in this case imposed by the anisotropy of the field  $\kappa$ ; from Jensen's inequality we see that  $D_{xx} \leq D_{yy}$ , indicating that dispersion is faster in the direction parallel to the strata of the medium [Fig. 2(b)]. For large forces however, we find that the dispersion tensor becomes

$$D_{ij} = (\overline{\kappa^{-1}})^{-1} \left\{ \delta_{ij} + \frac{F_i F_j}{|\mathbf{F} \cdot \mathbf{e}_x|^2} \left[ \frac{\overline{\kappa^{-2}}}{(\overline{\kappa^{-1}})^2} - 1 \right] \right\}, \quad (23)$$

so the dispersion becomes larger in the direction parallel to the force than in the perpendicular direction [43]. The dispersion is highly sensitive to the projection of the force normal to the strata [Fig. 2(c)], and the diffusion coefficients in the planes of the strata diverge when  $\mathbf{F}$  is in the plane of the strata (in fact they grow as  $|\mathbf{F}|^2$ ).

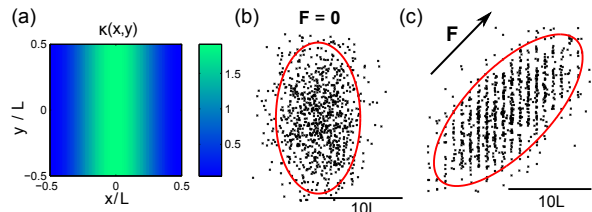


FIG. 2. (color online) (a) The 2D periodic diffusivity field for our example of stratified medium,  $\kappa(x, y) = \kappa_0[1 + 0.95 \cos(2\pi x/L)]$ , shown in units of  $\kappa_0$  on the fundamental rectangular unit cell. (b) and (c): Cloud of particles diffusing in the local diffusivity field shown in (a) in the presence of external force at a time  $t = 10L^2/\kappa_0$ . In (b) no external force and in (c) the force has magnitude given by  $\beta FL = 100$ , and acts in the direction indicated by the arrow. The ellipses represent the region in which 95% of the points should fall and are determined from Eqs. (22,23).

*Force induced dispersion enhancement in 2D.* We now return to the general 2D problem, where we derive a formula describing the physics for large forces (again oriented in the direction  $x$ ). For large forces, it is natural to suppose that the equilibration time in the direction of the force is much shorter than in the other direction. We thus make a quasi-static approximation for the propagator,

$$P(x, y, t) \simeq \pi(y, t)P_s(x|y), \quad (24)$$

where  $P_s(x|y) \sim \kappa^{-1}(x, y)$  is the stationary probability to observe  $x$  given the value of  $y$ . An effective Fokker-Planck equation can then be derived for the PDF  $\pi(y, t)$  by integrating over  $x$ , and using Eqs. (19,20), to obtain the following compact formula for the dispersion in the direction of the force [38]

$$D_{xx} = \frac{[\beta FR(L)]^2}{W(L)} \int_0^L dy \left[ \frac{W(y)}{W(L)} - \frac{R(y)}{R(L)} \right]^2 e^{-\overline{\ln \kappa}(y)}, \quad (25)$$

where  $L$  is the length of the period in the direction  $y$ , the notation  $\overline{g}(y)$  representing uniform spatial averaging over  $x$  for any function  $g(x, y)$ , and the following quantities have been defined:

$$R(y) = \int_0^y du e^{\overline{\ln \kappa}(u)}; \quad W(y) = \int_0^y du \overline{\kappa^{-1}}(u) e^{\overline{\ln \kappa}(u)}. \quad (26)$$

Equation (25) shows that local heterogeneities generically give rise to diffusion coefficients scaling as the square of the force for large forces, implying that *the force-induced diffusivity can be much larger than the microscopic diffusion coefficients*. Simple quadrature of the integrals appearing in Eq. (25) give a coefficient of  $F^2$  which is in very good agreement with the simulations, as seen in Fig. 1(d).

*Conclusion.* Taylor dispersion [16] represents a textbook example of a phenomenon where spatial variations of a time-independent compressible velocity field along with locally constant molecular diffusivity, lead to enhanced dispersion. Here, external uniform forces lead to increased dispersion in the direction of the force. The mechanism is similar to that behind Taylor dispersion in that particles with different trajectories experience very different advection by the applied force due to its coupling to the local mobility which is proportional to the diffusivity. We have also seen that the presence of an external force modifies in a non-monotonic way the dispersion in the direction perpendicular to the applied force, this is a somewhat surprising effect and is due to the fact that an applied force yields a non-uniform stationary distribution over the fundamental periodic cell of the problem. It is possible that one may construct experimental systems where the effects predicted here could be observed. Lasers have been used to generate periodic optical potentials in which colloidal particles can be tracked [44, 45]. It would be interesting to see if experimental realizations of media with spatially modulated diffusivities could be similarly produced, in order to observe the effects predicted in this Letter. Finally, we stress that the results here can be applied to any periodic advection-diffusion system and thus has a wide range of applicability. For instance, one can use our formulas to study the dispersion in periodic potentials in *any* dimension, in the presence of an external force [30, 31] and even with varying local mobility. The condition of local detailed balance is not necessary for the application of the formulas derived here and our results can thus be extended to study dispersion in active particle systems.

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