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# Generalized Jones matrix method for homogeneous biaxial samples

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**Abstract:** The generalized Jones matrix (GJM) is a recently introduced tool to describe linear transformations of three-dimensional light fields. Based on this framework, a specific method for obtaining the GJM of uniaxial anisotropic media was recently presented. However, the GJM of biaxial media had not been tackled so far, as the previous method made use of a simplified rotation matrix that lacks a degree of freedom in the three-dimensional rotation, thus being not suitable for calculating the GJM of biaxial media. In this work we propose a general method to derive the GJM of arbitrarily-oriented homogeneous biaxial media. It is based on the differential generalized Jones matrix (dGJM), which is the three-dimensional counterpart of the conventional differential Jones matrix. We show that the dGJM provides a simple and elegant way to describe uniaxial and biaxial media, with the capacity to model multiple simultaneous optical effects. The practical usefulness of this method is illustrated by the GJM modeling of the polarimetric properties of a negative uniaxial KDP crystal and a biaxial KTP crystal for any three-dimensional sample orientation. The results show that this method constitutes an advantageous and straightforward way to model biaxial media, which show a growing relevance for many interesting applications.

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**OCIS codes:** (260.0260) Physical optics; (260.5430) Polarization; (260.2130) Ellipsometry and polarimetry.

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## 1. Introduction

The widely-used Jones calculus makes it possible to model light interactions with linear non-depolarizing samples, assuming normal incidence and paraxial propagation. Oblique propagation in layered media can be calculated by the Berreman method [1] and by the extended Jones matrix method [2, 3]. The former is used in generalized ellipsometry to measure optical birefringence of diverse samples like diffraction gratings, stretched plastic sheets, sculptured and magneto-optical films, and porous silicon Fabry-Perot resonators [4–6]. The latter has been extensively applied to model many type of samples, like liquid-crystal displays [7], laser resonators [8], and human cornea [9]. However, these methods assume transversal plane waves.

The three-dimensional (3D) Jones vector, also known as generalized Jones vector (GJV), enables a completely polarized beam to be described in the general case of 3D electric fields [10–12]. Azzam has recently presented a generalization of the Jones matrices calculus to model linear transformations of GJVs, by introducing the so-called  $3 \times 3$  generalized Jones matrix (GJM) [11]. This approach has the potential to model local transformations undergone by a three-dimensional light field. Moreover, this formalism also provides a convenient way to describe the evolution of propagating light beams in anisotropic media with arbitrary orientation.

The general parametric expression of the differential generalized Jones matrix (dGJM) was first obtained in [13]. It was derived by the extension of the differential Jones matrix [14] to the three-dimensional framework through the introduction of Gell-Mann matrices, which form the complete set of infinitesimal generators of the Lie group  $SU(3)$  [15]. Additionally, a method for obtaining the GJM of uniaxial samples was proposed, and it was subsequently applied to the study of turbid media with uniaxial birefringence [16]. However, the method for uniaxial media made use of a simplified rotation matrix that is not suitable for calculating the GJM of biaxial media, as it took advantage of the fact that only two degrees of freedom are needed to specify the optic axis direction in uniaxial media [11].

In this work, we propose a method to obtain the GJM of homogeneous biaxial samples with arbitrary orientation. Our development conveniently generalizes the previous method to cover biaxial samples, providing a straightforward and versatile way to calculate the polarimetric properties of anisotropic media based on the differential generalized Jones calculus. Within

the scope of this article, we assume transverse propagating light waves as a first and necessary step before this novel approach can be fully developed in forthcoming works to describe the interaction of three-dimensional fields with anisotropic media. The model includes co-aligned linear birefringence and linear dichroism, circular birefringence, and circular dichroism. Non-reciprocal effects are not considered in this work, although the model could tackle them in a similar way as the conventional Jones formalism does. The generalized Jones matrix method and its extension to biaxial media are described in Section 2. Subsequently, in Section 3 the presented approach is illustrated in the modeling of two widely-used uniaxial and biaxial crystals, namely KDP and KTP, which highlights the advantages and versatility of this method. Finally, the main conclusions of this work are included in Section 4.

## 2. Generalized Jones matrix method

We consider the propagation of a light beam through a linear anisotropic medium. The optical anisotropy of such a medium is characterized by its dielectric and conductivity tensors, whose principal axes are defined as the directions of its corresponding eigenvectors [17]. Throughout this work we will assume that the medium dichroism and birefringence are co-aligned, i.e. that the principal axes of the dielectric and conductivity tensors coincide. Such assumption is valid for the higher symmetry classes of crystals, including the tetragonal and orthorhombic lattices. Moreover, we consider weakly absorbing crystals, where dichroism is much smaller than birefringence. In this case, the electrical displacement field remains transverse to the wave normal, contrarily to the electric field [17]. Therefore, it will be assumed that the state of polarization of the beam within the sample is conveniently described by the Jones vector of the electrical displacement field [18]. Another important consequence of these premises is that the principal axes of the medium are orthogonal, so we will not discuss the so-called *non-orthogonal* optical elements, which have been analyzed elsewhere [19, 20] and which are outside the scope of this work.

The evolution of a  $3 \times 1$  GJV (denoted by  $\vec{\mathbf{D}}$ ) that completely characterizes a transverse light beam through an optical path length  $l_z$  along a propagation direction  $z$  in a uniform medium is given by:

$$\frac{d\vec{\mathbf{D}}}{dl_z} = \mathbf{g}\vec{\mathbf{D}}, \quad (1)$$

which is governed by  $\mathbf{g}$ , the  $3 \times 3$  differential generalized Jones matrix (dGJM) of the sample specified in the laboratory Cartesian coordinate system  $xyz$  (Fig. 1(a)). The GJM of a homogeneous sample is simply given by the matrix exponential of the dGJM weighted by the sample thickness:

$$\mathbf{G} = \exp(\mathbf{g}l_z), \quad (2)$$

such that the output GJV can be obtained by  $\vec{\mathbf{D}}_{\text{out}} = \mathbf{G}\vec{\mathbf{D}}_{\text{in}}$ . The upper-left  $2 \times 2$  block of the GJM corresponds to the conventional Jones matrix  $\mathbf{J}$ .

The general form of the dGJM can be written in compact notation as:

$$\mathbf{g} = \frac{1}{2} \begin{bmatrix} 2I + L_{xy} + \frac{L_z}{3} & L'_{xy} + C_{xy}^* & L'_{xz} + C_{xz}^* \\ L'_{xy} - C_{xy}^* & 2I - L_{xy} + \frac{L_z}{3} & L'_{yz} + C_{yz}^* \\ L'_{xz} - C_{xz}^* & L'_{yz} - C_{yz}^* & 2I - \frac{2L_z}{3} \end{bmatrix}, \quad (3)$$

where  $I = IA + iIR$ ,  $L = LD + iLB$ ,  $L' = LD' + iLB'$ ,  $C = CB + iCD$  (the subscript of  $L$ ,  $L'$  and  $C$  in Eq. (3) indicates the plane in which they are defined), and  $L_z = LD_z + iLB_z$ . The superscript \* denotes the complex conjugate operation. The definition of each of these symbols, as well as their physical meaning and their full correspondence with the differential parameters

defined in [13], is included in Table 1. It can be observed that  $\mathbf{g}$  presents a simple and clear form in which each matrix element is univocally linked to a specific anisotropic property of the medium. In the most general case, there are thus 18 differential parameters from which the number of independent ones depends on the symmetry properties of the material dielectric and conductivity tensors, being considerably less for the specific situations considered in this work as will be shown below. It shall also be noted that the differential formalism is able to model simultaneously occurring optical effects, which makes it an advantageous method in many applications like biological tissue polarimetry. We recall that the presented equations assume a complex propagation constant  $\gamma$  of the form  $\gamma = \eta + i\kappa$ , where  $\eta$  and  $\kappa$  are respectively linked to the refractive index  $n$  and to the extinction coefficient  $k$  of the medium.

Table 1. Differential generalized Jones matrix parameters.

Symbol	Differential parameter	Physical property
$IR$	$\eta_i = 2\pi n_i / \lambda$	Isotropic retardance
$IA$	$\kappa_i = 2\pi k_i / \lambda$	Isotropic absorption
$LB$	$\eta_q^{xy} = 2\pi(n_y - n_x) / \lambda$	Linear birefringence ( $x - y$ )
$LB'$	$\eta_u^{xy,xz,yz} = 2\pi(n_{-45} - n_{+45}) / \lambda$	Linear birefringence ( $\pm 45^\circ$ )
$CB$	$\eta_v^{xy,xz,yz} = 2\pi(n_L - n_R) / \lambda$	Circular birefringence
$LD$	$\kappa_q^{xy} = 2\pi(k_y - k_x) / \lambda$	Linear dichroism ( $x - y$ )
$LD'$	$\kappa_u^{xy,xz,yz} = 2\pi(k_{-45} - k_{+45}) / \lambda$	Linear dichroism ( $\pm 45^\circ$ )
$CD$	$\kappa_v^{xy,xz,yz} = 2\pi(k_L - k_R) / \lambda$	Circular dichroism
$LB_z$	$\eta_q^z = \pi(2n_z - n_x - n_y) / \lambda$	Linear birefringence ( $xy - z$ )
$LD_z$	$\kappa_q^z = \pi(2k_z - k_x - k_y) / \lambda$	Linear dichroism ( $xy - z$ )

The conventional expressions of macroscopic Jones matrices for well-known uniaxial samples are usually specified in a reference system which coincides with the principal axes of the anisotropic medium. But when arbitrary 3D orientation of biaxial media is instead considered, the derivation of the conventional  $2 \times 2$  Jones matrix  $\mathbf{J}$  in the laboratory reference system is in general a cumbersome task. However, we will show that the differential generalized Jones matrix provides a straightforward and elegant way to calculate the GJM of biaxial media.

Let us first recall the previous results for uniaxial samples. We shall consider a local Cartesian reference system  $x'y'z'$  in which the  $z'$  axis coincides with the optic axis of the medium. The parameterized dGJM of a uniaxial sample with co-aligned linear birefringence and dichroism, and isotropic circular birefringence and dichroism (isotropic in the sense that it has the same value for the three principal planes), is given by:

$$\mathbf{g}_{\text{uniaxial}}^{x'y'z'} = \frac{1}{2} \begin{bmatrix} L_{z'}/3 & CB - iCD & CB - iCD \\ CB - iCD & L_{z'}/3 & CB - iCD \\ CB - iCD & CB - iCD & -2L_{z'}/3 \end{bmatrix}. \quad (4)$$

The previous matrix becomes diagonal if no circular birefringence is present, which is a direct consequence of the diagonal form of the dielectric tensor in the principal coordinate system of nonchiral media [21]. The GJM of such uniaxial sample in the local system, hereby denoted  $\mathbf{G}_{\text{uniaxial}}^{x'y'z'}$ , is readily calculated inserting  $\mathbf{g}_{\text{uniaxial}}^{x'y'z'}$  into Eq. (2). Given the GJM of a uniaxial medium in  $x'y'z'$ , the GJM in the laboratory reference system  $xyz$  is given by:

$$\mathbf{G}_{\text{uniaxial}}^{xyz} = \mathbf{C} \mathbf{G}_{\text{uniaxial}}^{x'y'z'} \mathbf{C}^{-1}, \quad (5)$$

with

$$\mathbf{C} = \begin{bmatrix} \sin \phi & \cos \theta \cos \phi & \sin \theta \cos \phi \\ -\cos \phi & \cos \theta \sin \phi & \sin \theta \sin \phi \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}, \quad (6)$$

where  $\theta$  and  $\phi$  are the polar and azimuthal angles of the optic axis specified in the  $xyz$  laboratory reference system [13]. We recall that the propagation direction is assumed to be parallel to  $z$ . Matrix  $\mathbf{C}$  is actually a simplified reference system change matrix that takes advantage of the fact that a uniaxial medium only requires two angles to fully determine its orientation. Note that  $\mathbf{C}$  is a unitary matrix, hence  $\mathbf{C}^{-1} = \mathbf{C}^T$  (i.e. its transpose matrix) and  $\mathbf{C}\exp(\mathbf{a})\mathbf{C}^T = \exp(\mathbf{C}\mathbf{a}\mathbf{C}^T)$ ,  $\forall \mathbf{a}$ . As a result, Eq. (5) can also be applied to the dGJM, yielding  $\mathbf{g}_{\text{uniaxial}}^{\text{xyz}} = \mathbf{C}\mathbf{g}_{\text{uniaxial}}^{\text{x'y'z'}}\mathbf{C}^{-1}$ .

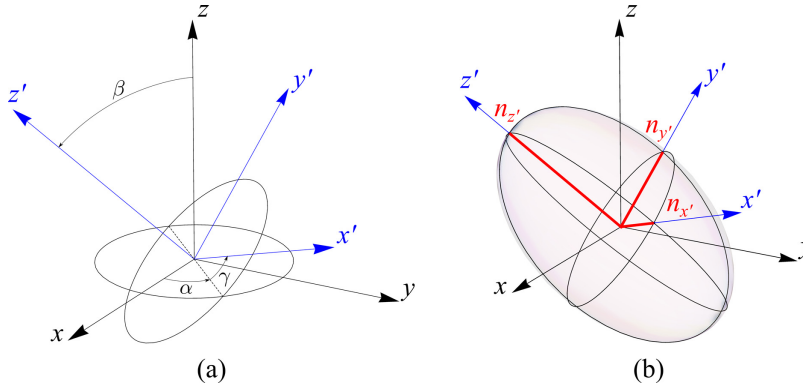


Fig. 1. (a) Euler angles  $\alpha$ ,  $\beta$ , and  $\gamma$  that specify the orientation of the local reference system  $x'y'z'$  relative to the laboratory reference system  $xyz$ . (b) Index ellipsoid of the biaxial medium, whose principal axes are aligned with the local reference system. The principal refractive indices are  $n_x'$ ,  $n_y'$ , and  $n_z'$ . The light beam propagates along  $z$ .

We now consider a biaxial medium whose principal axes are aligned with the local  $x'y'z'$  reference system, as shown in Fig. 1(b). If the sample shows parallel linear birefringence and linear dichroism, and isotropic circular birefringence and dichroism, the parameterization of the dGJM is:

$$\mathbf{g}_{\text{biaxial}}^{\text{x'y'z'}} = \frac{1}{2} \begin{bmatrix} L_{x'y'} + L_{z'}/3 & CB - iCD & CB - iCD \\ CB - iCD & -L_{x'y'} + L_{z'}/3 & CB - iCD \\ CB - iCD & CB - iCD & -2L_{z'}/3 \end{bmatrix}. \quad (7)$$

It can be observed that the only difference with  $\mathbf{g}_{\text{uniaxial}}^{\text{x'y'z'}}$  is the presence of the additional parameter  $L_{x'y'}$  in the first and second diagonal elements, which accounts for the non-equality of the corresponding permittivities in biaxial media. Introducing  $\mathbf{g}_{\text{biaxial}}^{\text{x'y'z'}}$  into Eq. (2) gives the GJM of such medium in the local reference system ( $\mathbf{G}_{\text{biaxial}}^{\text{x'y'z'}}$ ).

In biaxial media, three angles are obviously needed to completely specify the sample orientation in the laboratory reference system  $xyz$ . Such coordinate change is characterized by the Euler angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . In general, the Euler angles determine any rotation in a three-dimensional space [22]. In this work, those angles specify the orientation of the local reference system  $x'y'z'$  relative to the laboratory reference system  $xyz$ , as shown in Fig. 1(a). The GJM in the laboratory coordinate system is thus:

$$\mathbf{G}_{\text{biaxial}}^{\text{xyz}} = \mathbf{T}\mathbf{G}_{\text{biaxial}}^{\text{x'y'z'}}\mathbf{T}^{-1}, \quad (8)$$

where the coordinate transformation matrix is:

$$\mathbf{T} = \mathbf{R}_\alpha \mathbf{R}_\beta \mathbf{R}_\gamma, \quad (9)$$

and the three individual rotation matrices are:

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (10)$$

$$\mathbf{R}_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix}, \quad (11)$$

$$\mathbf{R}_\gamma = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (12)$$

Matrix  $\mathbf{T}$  is also unitary. It can be noticed that the simplified rotation matrix  $\mathbf{C}$  previously used for uniaxial media is the particular case of the general  $\mathbf{T}$  for Euler angles  $\alpha = \phi - \pi/2$ ,  $\beta = -\theta$ , and  $\gamma = 0$ , as a result of the specific rotations used in [11]. The previous equations allow the GJM of an arbitrarily-oriented biaxial sample to be calculated from the dGJM in its principal coordinate system, which can be readily obtained from Eq. (7). Finally, we should mention that the three-dimensional rotation relating the laboratory reference system and the local reference system could also be calculated from the directions of the optic axes of the crystal, as a geometric relationship holds between the principal axes and refractive indices of the crystal and its optic axes [17].

### 3. Application to model uniaxial and biaxial crystals

The applicability of the method is now illustrated to calculate the polarimetric properties of a slab of an arbitrarily-oriented anisotropic crystal. We consider a homogeneous sample whose plane-parallel surfaces are normal to the  $z$  axis of the Cartesian coordinate system, which is the assumed propagation direction of the light beam, as shown in Fig. 2. The orientation of the local reference system  $x'y'z'$  with respect to the laboratory system is specified by the three Euler angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . We assume that the crystal principal axes are aligned with the local reference system, as stated in the previous section.

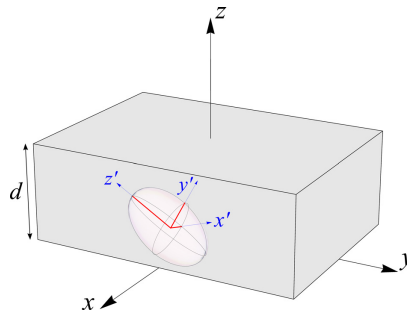


Fig. 2. Anisotropic crystal sample of thickness  $d$  with its principal axes aligned with the local  $x'y'z'$  reference system. The beam propagates along  $z$  in the laboratory system  $xyz$ .

We first model a uniaxial crystal to validate the presented approach. In particular, we consider a 3 mm thick KDP sample, a negative uniaxial crystal which has a tetragonal lattice. KDP



ordinary and extraordinary refractive indices are respectively  $n_o = 1.4942$  and  $n_e = 1.4603$  at  $\lambda = 1064$  nm [23]. It is transparent at such wavelength ( $\kappa \simeq 0$ ) and does not have circular birefringence nor circular dichroism. We assume that the optic axis is aligned with  $z'$ , as discussed in the previous section, so  $n_{x'} = n_{y'} = n_o$  and  $n_{z'} = n_e$ . Under these conditions, the general parametric form of the dGJM of a uniaxial sample given by Eq. (6) simplifies to

$$\mathbf{g}_1^{x'y'z'} = \frac{1}{2} \begin{bmatrix} L_{z'}/3 & 0 & 0 \\ 0 & L_{z'}/3 & 0 \\ 0 & 0 & -2L_{z'}/3 \end{bmatrix}, \quad (13)$$

as the only non-null differential parameter of this crystal is  $L_{z'}$ . Moreover,  $L_{z'} = iLB_{z'}$  due to the absence of dichroism (i.e.  $LD_{z'} = 0$ ), with  $LB_{z'} = \pi(2n_{z'} - n_{x'} - n_{y'})/\lambda = 2\pi(n_e - n_o)/\lambda$ . Consequently, the differential generalized Jones matrix of this sample in the local system is:

$$\mathbf{g}_1^{x'y'z'} = 10^5 \begin{bmatrix} -1.3346i & 0 & 0 \\ 0 & -1.3346i & 0 \\ 0 & 0 & 2.6692i \end{bmatrix}. \quad (14)$$

The GJM of the modeled sample in the laboratory reference system  $xyz$  for any crystal orientation can be straightforwardly calculated by inserting  $\mathbf{g}_1^{x'y'z'}$  in Eq. (2) and introducing the latter into Eq. (8), yielding, for sample thickness  $d$ :

$$\mathbf{G}_1^{xyz}(\alpha, \beta, \gamma) = \mathbf{T} \exp\left(\mathbf{g}_1^{x'y'z'} d\right) \mathbf{T}^{-1}. \quad (15)$$

In the specific situation where the local and the laboratory coordinate systems coincide, the coordinate transformation matrix  $\mathbf{T}$  is the  $3 \times 3$  identity matrix, so the GJM is simply the matrix exponential of  $\mathbf{g}_1^{x'y'z'}$  weighted by the sample thickness, which gives:

$$\mathbf{G}_1^{xyz}(0, 0, 0) = \mathbf{G}_1^{x'y'z'} = \begin{bmatrix} -0.176 + 0.984i & 0 & 0 \\ 0 & -0.176 + 0.984i & 0 \\ 0 & 0 & -0.938 + 0.347i \end{bmatrix}. \quad (16)$$

The real and imaginary part of  $\mathbf{G}_1^{xyz}$  are depicted in Fig. 3 for Euler angles  $0 \leq \alpha \leq 2\pi$  and  $0 \leq \beta \leq \pi$ . In this first example, it is observed that the results are identical for whatever angle  $\gamma$  as a consequence of the sample uniaxiality. It can be checked that the results obtained using the general method coincide with those calculated with the simplified approach for uniaxial media,

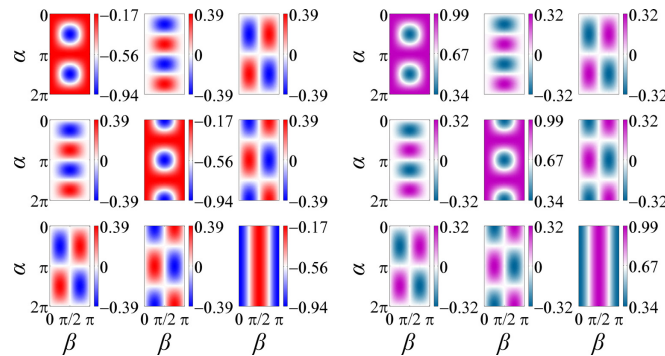


Fig. 3. Real part (left) and imaginary part (right) of the nine generalized Jones matrix (GJM) elements for a 3 mm section of KDP at  $\lambda = 1064$  nm.



considering the sign and shift between angles stated after Eq. (12). This expected feature arises from the fact that in uniaxial media the index ellipsoid degenerates to an ellipsoid of revolution, so the third Euler angle  $\gamma$  (which determines the final orientation of the  $x'$  and  $y'$  axes, see Fig. 1(a)) is obviously unnecessary,  $\mathbf{C}$  and  $\mathbf{T}$  both providing valid means of calculating the polarimetric properties of the sample in this case. This verification constitutes a first validation of the presented method for the degenerate situation of uniaxial media.

From Fig. 3, two aspects can be highlighted. Firstly, the symmetric pattern of all the elements of uniaxial samples clearly shows up, which is especially noticeable in the radial structure of the first and second diagonal elements. And secondly, it can be seen that the fact that the matrix becomes diagonal for  $\beta = 0$  and  $\beta = \pi$  with any  $\alpha$  (as the optic axis keeps parallel to  $z$ ), and for  $\beta = \pi/2$  with  $\alpha = n\pi/2$ , i.e. when the optic axis lies perpendicular to the propagation direction and it is parallel to either the  $x$  or  $y$  axis. It is also worth recalling that the upper-left  $2 \times 2$  block of the GJM corresponds to the conventional Jones matrix, which effectively shows such feature for uniaxial anisotropic samples (linear retarders and polarizers), confirming these results.

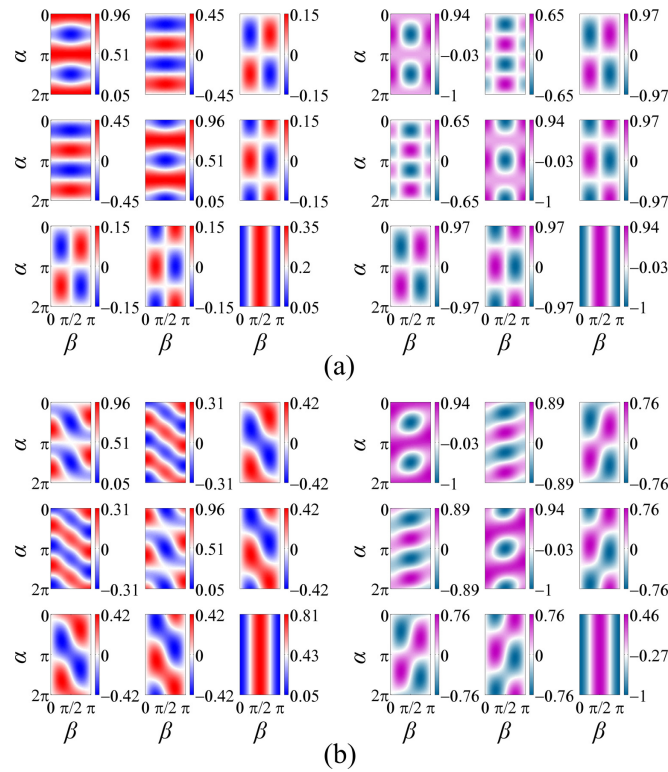


Fig. 4. Real part (left) and imaginary part (right) of the GJM elements of the KTP sample at  $\lambda = 1064$  nm for: (a)  $\gamma = 0$ , and (b)  $\gamma = \pi/3$ .

We now consider a sample of KTP, an orthorhombic biaxial crystal whose principal refractive indices at  $\lambda = 1064$  nm are  $n_{x'} = 1.7404$ ,  $n_{y'} = 1.7479$  and  $n_{z'} = 1.8296$  [24]. It should be noted that for other biaxial structures, like the monoclinic or the triclinic lattice, one cannot readily assign the principal refractive indices of the crystal to each of the local axes  $x'y'z'$ . In that case, it would be necessary to perform an additional projection to determine  $n_{x'}$ ,  $n_{y'}$  and  $n_{z'}$  from the principal indices [6]. For this sample, the non-null differential parameters of the general dGJM

shown in Eq. (7) are  $L_{z'}$  and  $L_{x'y'}$ :

$$\mathbf{g}_2^{x'y'z'} = \frac{1}{2} \begin{bmatrix} L_{x'y'} + L_{z'}/3 & 0 & 0 \\ 0 & -L_{x'y'} + L_{z'}/3 & 0 \\ 0 & 0 & -2L_{z'}/3 \end{bmatrix}, \quad (17)$$

with  $L_{z'} = iLB_{z'} = i\pi(2n_{z'} - n_{x'} - n_{y'})/\lambda$  and  $L_{x'y'} = iLB_{x'y'} = i2\pi(n_{y'} - n_{x'})/\lambda$ . Introducing the principal refractive indices of KTP into these expressions yields the following dGJM in the local reference system:

$$\mathbf{g}_2^{x'y'z'} = 10^5 \begin{bmatrix} 3.1426i & 0 & 0 \\ 0 & 3.5855i & 0 \\ 0 & 0 & -6.7280i \end{bmatrix}. \quad (18)$$

Taking into account the same thickness as above ( $d = 3$  mm), the GJM of the KTP sample when the crystal principal axes are aligned with the laboratory coordinate system is:

$$\mathbf{G}_2^{\text{xyz}}(0,0,0) = \mathbf{G}_2^{x'y'z'} = \begin{bmatrix} 0.957 + 0.291i & 0 & 0 \\ 0 & 0.347 + 0.938i & 0 \\ 0 & 0 & 0.059 - 0.998i \end{bmatrix}. \quad (19)$$

The real and imaginary part of the sample GJM are shown in Fig. 4(a) for local coordinate system orientations defined by Euler angles  $0 \leq \alpha \leq 2\pi$  and  $0 \leq \beta \leq \pi$ , with  $\gamma = 0$ . The

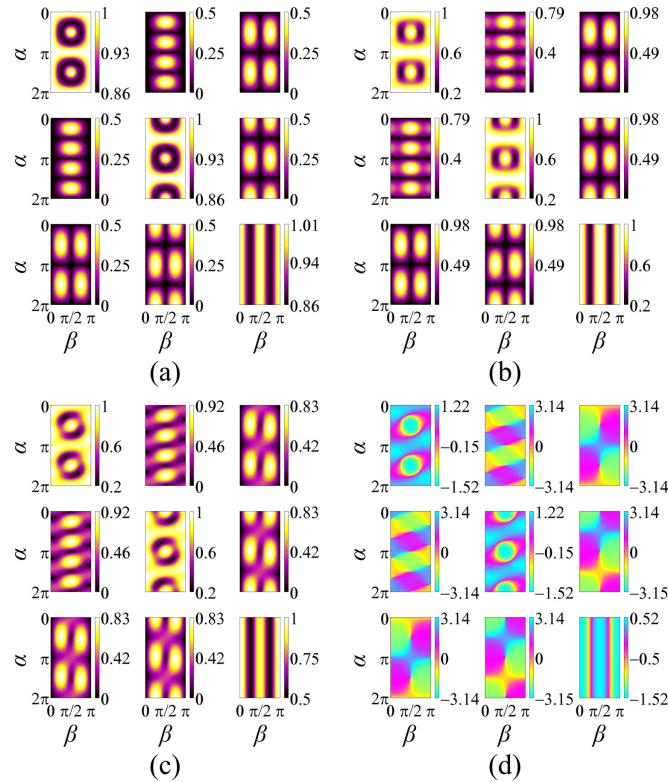


Fig. 5. Absolute value of the GJM elements of a 3 mm sample of: (a) KDP, (b) KTP for  $\gamma = 0$ , (c) KTP for  $\gamma = \pi/3$ , and (d) phase of the latter.

main feature to be highlighted is the loss of radial symmetry in the upper-left block elements as a result of the biaxial nature of the sample. Additionally, it can be observed that the GJM becomes diagonal for  $\beta = 0$ ,  $\beta = \pi/2$ , and  $\beta = \pi$ , with  $\alpha = n\pi/2$  (which is of course a more restrictive condition than in the uniaxial situation). The results for the same sample with a fixed  $\gamma = \pi/3$  are presented in Fig. 4(b), which consequently shows an oblique orientation, with similar characteristics to the previous one but with a tilted asymmetric pattern as a result of the additional rotation arising from the non-zero  $\gamma$  angle. We shall stress that the results straightforwardly obtained by the presented method would be more tedious and lengthy to derive by other approaches like the extended Jones matrix method. The absolute value of the three considered examples is included in Fig. 5 for the sake of completeness, as well as the phase of the latter.

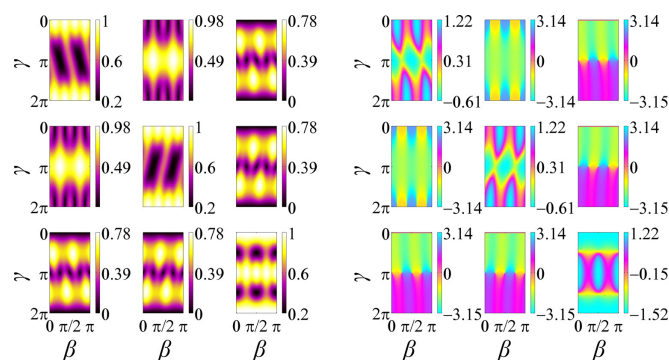


Fig. 6. Absolute value (a) and phase (b) of the GJM elements of the KTP sample for  $\alpha = \pi/4$ .

Finally, we should mention that the pattern observed in element  $\mathbf{G}(3,3)$  is due to the symmetry properties of rotations about  $\alpha$  and  $\beta$  with a fixed  $\gamma$  in the orthorhombic system. Different patterns are actually observed in such element for other rotations. As a particular example, Fig. 6 shows the absolute value and the phase of the KTP sample for Euler angles  $0 \leq \beta \leq \pi$  and  $0 \leq \gamma \leq 2\pi$  with a fixed  $\alpha = \pi/4$ . It can be observed that the pattern of  $\mathbf{G}(3,3)$  is no longer symmetric, which prevents any misleading interpretation of the presented results.

#### 4. Conclusions

In conclusion, we have presented a general method to calculate the GJM of arbitrarily-oriented homogeneous biaxial samples. This method relies on the use of the differential generalized Jones matrix, which constitutes a quite straightforward way to model uniaxial and biaxial media. Moreover, it enables coupled anisotropic effects to be easily described, which is an interesting advantage of this method that makes it possible to perform realistic models of a wide range of anisotropic media. We have considered parallel linear birefringence and dichroism, and both circular birefringence and dichroism, although the method shall be extended to cover additional anisotropic effects. Weakly-absorbing crystals and transverse propagating light waves have been assumed, the interesting task of extending this method to three-dimensional fields being out of the scope of this paper. The applicability of this versatile method has been illustrated in the modeling of two widely-used uniaxial and biaxial crystals, namely KDP and KTP. These results highlight the capacity of this calculation to significantly simplify the analytical expression and numerical calculation of the polarization transfer properties of biaxial samples. Thus, this method has a considerable potential to model optical systems involving biaxial crystals, including relevant phenomena produced therein, like conical refraction. Inter-

esting aspects like multilayered media and effects at the interfaces should be further studied to fully develop the potential of the generalized Jones calculus for applications like light propagation in turbid media, high numerical aperture microscopy, near-field optics and light interaction with nanoparticles.