New results for tails of probability distributions according to their asymptotic decay

Meitner Cadena, Marie Kratz

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Meitner Cadena* and Marie Kratz†

Abstract

This paper provides new properties for tails of probability distributions belonging to a class defined according to the asymptotic decay of the tails. This class contains the one of regularly varying tails of distributions. The main results concern the relation between this larger class and the maximum domains of attraction of Fréchet and Gumbel.

Keywords: asymptotic behavior; maximum domains of attraction; Fréchet; Gumbel; Pickands-Balkema-de Haan theorem regularly varying function

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1 Introduction

An extension of the class *RV* of regularly varying (RV) functions has been introduced and analyzed in details in a recent study ([3]). The characteristic properties of this new larger class allow one, not only to extend main RV properties, as described in [3], but also to deepen the understanding of some Tauberian theorems ([4]) and to build in a simple way an estimator of the tail index on this class, and consequently on the class *RV*, with a good rate of convergence ([5]).

In this paper, we focus on the probabilistic side of this large class, providing new results for tails of distributions belonging to it, according to the asymptotic decay of the tails.

Let us briefly introduce the definition of the sets that will be considered and recall their characteristic properties.

Let $\mathscr{M}$ be the class of measurable functions $U:\mathbb{R}^+\to\mathbb{R}^+$ satisfying

$$\exists\rho\in\mathbb{R},\ \forall\varepsilon>0,\quad \lim_{x\to\infty}\frac{U(x)}{x^{\rho+\epsilon}}=0\quad\text{and}\quad\lim_{x\to\infty}\frac{U(x)}{x^{\rho-\epsilon}}=\infty \tag{1}$$

where ϵ may be taken arbitrarily small. It can be proved that, for any $U\in\mathscr{M}$, ρ defined in (1) is unique; it is denoted by ρ_U and called the $\mathscr{M}$-index of U.

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Any function U of $\mathcal{M}$, bounded on finite intervals, satisfies the following ([3], Theorem 1.1, with a minor modification in the notation):

$$U \in \mathcal{M} \text{ with } \rho_U = \tau \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = \tau \tag{2}$$

where ρ_U is defined in (1).

Combining this characterization (2) with Theorem 1 in [13] provides ([3], Proposition 2.1) that

$$RV \subsetneq \mathcal{M}$$

and that for any RV function $U \in RV_\alpha$, its tail index α coincides with its $\mathcal{M}$-index ρ_U defined in (1):

$$U \in RV_\alpha \Rightarrow U \in \mathcal{M} \text{ with} \mathcal{M} - \text{index}\, \alpha\,. \tag{3}$$

Recall, for completeness, that a measurable function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is RV_α (see [12] and e.g. [2]) if, for some $\alpha \in \mathbb{R}$ called the tail index of U, and for any $t > 0$,

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^\alpha. \tag{4}$$

We also introduce a natural extension of $\mathcal{M}$ defined in [3] (with a small modification in the notation) by

$$\mathcal{M}_{-\infty} = \left\{ U : \mathbb{R}^+ \to \mathbb{R}^+ \ : \ \forall \rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = 0 \right\}.$$

As for $\mathcal{M}$, we have a characterization for $\mathcal{M}_{-\infty}$, namely ([3], Theorem 1.4)

$$U \in \mathcal{M}_{-\infty} \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{\log(U(x))}{\log(x)} = -\infty \tag{5}$$

for any positive measurable function U with support $\mathbb{R}^+$.

In view of (2) and (5), we see that $\mathcal{M}$ and $\mathcal{M}_{-\infty}$ allow one to sort the tails of distributions F by their behavior as $x \to \infty$, $\mathcal{M}$ including the tails of distributions with an asymptotic polynomial decay and $\mathcal{M}_{-\infty}$ the ones with an asymptotic exponential behavior.

For the tails of distributions which have neither a polynomial nor an exponential behavior, we introduce another class $\mathcal{O}$, namely ([3])

$$\mathcal{O} := \{ U : \mathbb{R}^+ \to \mathbb{R}^+ \ : \mu(U) < \nu(U) \} \tag{6}$$

where $\mu(U)$, $\nu(U)$ correspond to the *lower order* of U and *upper order* of U, respectively, defined by (see for instance [2], pp. 73)

$$\mu(U) = \underline{\lim}_{x\to\infty} \frac{\log(U(x))}{\log(x)} \quad \text{and} \quad \nu(U) = \overline{\lim}_{x\to\infty} \frac{\log(U(x))}{\log(x)}.$$

The class $\mathcal{O}$ is non empty, as shown in [3] where we provided explicit examples. Moreover $\mathcal{M}$, $\mathcal{M}_{-\infty}$, and $\mathcal{O}$ are disjoint sets.

In this study, our main object of interest is tails of distributions F satisfying $x^* = \sup\{x : F(x) < 1\} = \infty$. We denote $\overline{F} = 1 - F$ and will use abusively this notation when referring to tails of distributions.

The paper is organized as follows. Section 2 provides the main properties for tails of distributions belonging to $\mathcal{M}$ and $\mathcal{M}_{-\infty}$. The results given in Section 3 concern the maximum domains of attraction to $\mathcal{M}$ and $\mathcal{M}_{-\infty}$, followed by conclusions in the last section.

All along the paper, we will denote: $\min(a, b) = a \wedge b$, $\max(a, b) = a \vee b$, $\lfloor x \rfloor$ for the largest integer not greater than x, $\lceil x \rceil$ for the lowest integer greater or equal than x, and $\log(x)$ for the natural logarithm of x.

2 Properties for tails of distributions belonging to $\mathcal{M}$ and $\mathcal{M}_{-\infty}$

Let us summarize the main properties ([3], Properties 1.1 to 1.4, and Remark 1.1) when considering tails of distribution belonging to the two classes $\mathcal{M}$ or $\mathcal{M}_{-\infty}$. We refer also to [3] for the proofs.

Let F and G be two distributions.

1. For $(\overline{F}, \overline{G}) \in \mathcal{M} \times \mathcal{M}$ s.t. $\rho_{\overline{F}} > \rho_{\overline{G}}$, or $(\overline{F}, \overline{G}) \in \mathcal{M} \times \mathcal{M}_{-\infty}$, we have $\lim\limits_{x\to\infty} \dfrac{\overline{G}(x)}{\overline{F}(x)} = 0$.

2. Let $\overline{F} \in \mathcal{M}$. If $\rho_{\overline{F}} < -1$, then $\overline{F}$ is integrable on $\mathbb{R}^+$, whereas, if $\rho_{\overline{F}} > -1$, $\overline{F}$ is not integrable on $\mathbb{R}^+$. Note that in the case $\rho_{\overline{F}} = -1$, we can find examples of functions $\overline{F}$ which are integrable or not.

3. *Linear combination of tails of distribution in $\mathcal{M}$ or $\mathcal{M}_{-\infty}$.*
For any $a \geq 0$, $\overline{F} \in \mathcal{M}$, $\overline{G} \in \mathcal{M} \cup \mathcal{M}_{-\infty}$, we have $a\overline{F} + \overline{G} \in \mathcal{M}$ with $\rho_{a\overline{F}+\overline{G}} = \rho_{\overline{F}} \vee \rho_{\overline{G}}$, setting $\rho_{\overline{G}} = -\infty$ when $\overline{G} \in \mathcal{M}_{-\infty}$. If $\overline{F}, \overline{G} \in \mathcal{M}_{-\infty}$, then $\overline{F} + \overline{G} \in \mathcal{M}_{-\infty}$.

4. *Product of tails of distribution in $\mathcal{M}$ or $\mathcal{M}_{-\infty}$.*
If $\overline{F}, \overline{G} \in \mathcal{M}$, then $\overline{F}.\overline{G} \in \mathcal{M}$ with $\rho_{\overline{F}.\overline{G}} = \rho_{\overline{F}} + \rho_{\overline{G}}$. If $\overline{F} \in \mathcal{M}$ and $\overline{G} \in \mathcal{M} \cup \mathcal{M}_{-\infty}$, or $\overline{F}, \overline{G} \in \mathcal{M} \cup \mathcal{M}_{-\infty}$, then $\overline{F}.\overline{G} \in \mathcal{M}_{-\infty}$.

5. *Convolution on $\mathcal{M}$ and $\mathcal{M}_{-\infty}$.*
   (a) Let $\overline{F}, \overline{G} \in \mathcal{M}$. If $-1 < \rho_{\overline{F}} \leq \rho_{\overline{G}}$, then $\overline{F} * \overline{G} \in \mathcal{M}$ with $\rho_{\overline{F}*\overline{G}} = \rho_{\overline{F}} + \rho_{\overline{G}} + 1$.
   If $\rho_{\overline{F}} \leq \rho_{\overline{G}} < -1$ or $\rho_{\overline{F}} < -1 < \rho_{\overline{G}} = 0$, then $\overline{F} * \overline{G} \in \mathcal{M}$ with $\rho_{\overline{F}*\overline{G}} = \rho_{\overline{F}}$.
   (b) If $(\overline{F}, \overline{G}) \in \mathcal{M}_{-\infty} \times \mathcal{M}$ with $\rho_{\overline{G}} < -1$ or $\rho_{\overline{G}} = 0$, then $\overline{F} * \overline{G} \in \mathcal{M}$ with $\rho_{\overline{F}*\overline{G}} = \rho_{\overline{G}}$.
   (c) If $(\overline{F}, \overline{G}) \in \mathcal{M}_{-\infty} \times \mathcal{M}_{-\infty}$, then $\overline{F} * \overline{G} \in \mathcal{M}_{-\infty}$.

6. *Link to the notion of stochastic dominance.*
Let X and Y be rv's with distributions F_X and F_Y, respectively, having $\mathbb{R}^+$ support. X is said to be smaller than Y in the usual stochastic order (see e.g. [14], pp. 3) if
$$\overline{F}_X(x) \leq \overline{F}_Y(x) \quad \text{for all } x \in \mathbb{R}^+. \tag{7}$$

This relation is also interpreted as the first-order stochastic dominance of X over Y, as $F_X \geq F_Y$ (see e.g. [11], pp. 289).

We can deduce from property (i) that for X, Y such that $\overline{F}_X$ and $\overline{F}_Y$ belong to $\mathcal{M}$ with $\rho_{\overline{F}_X} > \rho_{\overline{F}_Y}$, (7) is satisfied at infinity, which means that X strictly dominates Y at infinity.

The properties (1)-(5) also hold when considering probability density functions instead of tails of distributions. Notice that if a probability density function belongs to $\mathcal{M}$, then its $\mathcal{M}$-index is less or equal to -1.

Property 5-(a) generalizes a result by Bingham, Goldie and Omey ([1]) proved for RV probability density functions f and g, where they first had to show that $\lim_{x\to\infty} \dfrac{f * g(x)}{f(x)+g(x)} = 1$ ([1], Theorem 1.1). To prove the property for general functions of $\mathcal{M}$, we proceed in a direct way, under the condition of integrability of the function having the lowest $\mathcal{M}$-index ([3]).

3 Maximum domains of attraction

Let $(X_n, n \in \mathbb{N})$ be a sequence of iid rv's with distribution F and $M_n := \max_{1\leq i\leq n} X_i$. If there exist numerical sequences $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$, with $a_n > 0$ and $b_n \in \mathbb{R}$, such that

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \underset{n\to\infty}{\to} G(x), \tag{8}$$

with G a non degenerate distribution function, then we say that F belongs to the maximum domain of attraction of G, which is denoted by $F \in MDA(G)$.

We will focus on $MDA(G)$ when G is one of the two distributions:

$$\begin{array}{rcl} \text{Gumbel} & : & \Lambda(x) := \exp\left\{e^{-x}\right\}, \quad \forall x \in \mathbb{R}, \\ \text{Fréchet} & : & \Phi_\alpha(x) := \exp\left\{-x^{-\alpha}\right\}, \quad \forall x \geq 0, \text{ for some } \alpha > 0. \end{array}$$

Recall the following characterization in the Fréchet case (see e.g. [7], Theorem 1.2.1):

$$F \in MDA(\Phi_\alpha) \quad \text{if and only if} \quad \overline{F} \in \text{RV}_{-\alpha} \text{ and } x^* = \sup\{x : F(x) < 1\} = \infty. \tag{9}$$

In the Gumbel case, we will consider distributions $F \in MDA(\Lambda)$ having infinite endpoint $x^* = \infty$ only, denoting by $MDA(\Lambda_\infty)$ the associated maximum domain of attraction, for which we have the following relation from Gnedenko (see [9], Theorem 7):

$F \in MDA(\Lambda_\infty)$ if and only if there exists a continuous function A such that $A(z) \underset{z\to\infty}{\to} 0$, and

$$\forall x \in \mathbb{R}, \quad \lim_{z\to\infty} \frac{1 - F(z(1 + A(z)\,x))}{1 - F(z)} = e^{-x}, \tag{10}$$

and the sufficient condition from de Haan (see [6], Corollary 2.5.3)

$$F \in MDA(\Lambda_\infty) \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\log\left(\overline{F}(x)\right)}{\log(x)} = -\infty. \tag{11}$$

Our main goal is to study the relation between $\mathcal{M}$ or $\mathcal{M}_{-\infty}$ and the maximum domains of attraction $MDA(\Phi_\alpha)$ and $MDA(\Lambda_\infty)$. We can prove the following:

Proposition 3.1.

(i) $\forall\alpha>0$, *if* $F\in MDA(\Phi_\alpha)$, *then* $\overline{F}\in\mathcal{M}$ *with* $\mathcal{M}$*-index* $(-\alpha)$*, but the converse does not hold:*

$$\left\{F\in MDA(\Phi_\alpha),\,\alpha>0\right\}\subsetneq\left\{F:\ \overline{F}\in\mathcal{M}\right\}.$$

(ii) $MDA(\Lambda_\infty)\subsetneq\left\{F:\ \overline{F}\in\mathcal{M}_{-\infty}\right\}$.

Proof of Proposition 3.1.

(i) Let $F\in MDA(\Phi_\alpha)$, $\alpha>0$. Applying (9) then (3) gives immediately that $\overline{F}\in\mathcal{M}$ with $\mathcal{M}$-index $(-\alpha)$.
To prove that the converse does not hold, it is enough to consider a typical example of a non RV function (see [10], or e.g. [8]), the Peter and Paul distribution, defined for $x>0$ by $F_P(x)=1-\sum_{k\geq1:\,2^k>x}2^{-k}$. It is straightforward to check ([3]) that $\overline{F}_P$ satisfies $\lim_{x\to\infty}\frac{\log\left(\overline{F}_P(x)\right)}{\log(x)}=-1$, hence, using (2), that $\overline{F}_P\in\mathcal{M}$ with $\mathcal{M}$-index -1.

(ii) Assume $F\in MDA(\Lambda_\infty)$. Applying (11) then (5) implies that $\overline{F}\in M_{-\infty}$.
To show that it is a strict subset inclusion, we consider the distribution F defined in a left neighborhood of ∞ by

$$F(x):=1-\exp\left\{-\lfloor x\rfloor\,\log(x)\right\}\tag{12}$$

and prove that $\overline{F}\in\mathcal{M}_{-\infty}$ but $F\notin MDA(\Lambda_\infty)$.
To verify that $\overline{F}\in\mathcal{M}_{-\infty}$ is immediate using (5), since F satisfies $\lim_{x\to\infty}\frac{\lfloor x\rfloor\,\log(x)}{\log(x)}=\infty$.
It remains to check that $F\notin MDA(\Lambda_\infty)$. We proceed by contradiction and so assume that F belongs to $MDA(\Lambda_\infty)$. It comes back to say that there exists a function A such that $A(x)\to0$ as $x\to\infty$ and (10) holds, which gives, $\forall x\in\mathbb{R}$,

$$\begin{aligned}&\lim_{z\to\infty}\Big(\lfloor z\,(1+A(z)\,x)\rfloor\,\log\big(z\,(1+A(z)\,x)\big)-\lfloor z\rfloor\,\log(z)\Big)=\\&\lim_{z\to\infty}\Big(\big(\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor\big)\log(z)+\lfloor z\,(1+A(z)\,x)\rfloor\,\log\big(1+A(z)\,x\big)\Big)=x\,.\end{aligned}\tag{13}$$

We are going to see that the assumption of the existence of such function A leads to a contradiction for some x, when considering all possible values of $\lim_{z\to\infty}z\,A(z)$.
- Suppose $\lim_{z\to\infty}z\,A(z)=c\neq0$, and choose $x\geq2/|c|$.
On one hand, if $c>0$, for z large enough such that $z\,A(z)\geq c/2$, we have $\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor>0$ (since $z\,(1+A(z)\,x)\geq z+cx/2\geq z+1$), which implies that

$$\lim_{z\to\infty}\big(\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor\big)\log(z)=\infty,$$

whereas, if $c<0$, for z large enough such that $z\,A(z)\leq c/2$, we have $\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor<0$ (since $z\,(1+A(z)\,x)\leq z-1$), which implies that

$$\lim_{z\to\infty}\left(\lfloor z(1+A(z)\,x)\rfloor - \lfloor z\rfloor\right)\log(z) = -\infty.$$

On the other hand, for z large enough so that $\log\left(1+A(z)\,x\right)\approx A(z)x$ and $z\,A(z)\le 2|c|$, we have $\lfloor z(1+A(z)\,x)\rfloor\,\log\left(1+A(z)\,x\right) \;\le\; z(1+A(z)\,x)\,\log\left(1+A(z)\,x\right)\approx$

$$z(1+A(z)\,x)\,A(z)\,x \le 2|c|(1+A(z)\,x)\,x < \infty.$$

Combining both results and taking $z\to\infty$ contradict (13).

- Suppose $\lim_{z\to\infty} z\,A(z) = 0$ and take $x>0$.

On one hand, $\lim_{z\to\infty}\left(\lfloor z(1+A(z)\,x)\rfloor - \lfloor z\rfloor\right)\log(z)$ could be 0 or ∞, depending on the asymptotic behavior of $z\,A(z)$ as $z\to\infty$.

On the other hand, we have for z large enough such that $\log\left(1+A(z)\,x\right)\approx A(z)\,x$,

$$\lfloor z(1+A(z)\,x)\rfloor\,\log\left(1+A(z)\,x\right)\le z(1+A(z)\,x)\,\log\left(1+A(z)\,x\right)\approx z(1+A(z)\,x)\,A(z)\,x \underset{z\to\infty}{\longrightarrow} 0.$$

Combining these results contradicts (13).

- Suppose $\lim_{z\to\infty} z\,A(z) = \infty$ and take $x>0$. Note that the function $\lfloor x\rfloor\log(x)$ is increasing for $x>1$ and $\lfloor x\rfloor\log(x) \underset{x\to\infty}{\longrightarrow} \infty$. Then taking $z>1$ such that $u = z(1+A(z)\,x)>1$ and $\lfloor u\rfloor\log(u)>x$, shows that (13) does not hold for any x.

- Suppose $\lim_{z\to\infty} z\,A(z) = -\infty$. Then $\exists z_0>0$ s.t. $z_0\,A(z_0)<0$. Taking $x > -z_0\big/(z_0\,A(z_0))$ leads also to an immediate contradiction of (13). □

Remark 3.1.

De Haan ([6], Lemma 2.4.3) proved that if $F\in MDA(\Lambda_\infty)$, then there exists a continuous and increasing distribution function G satisfying

$$\lim_{x\to\infty}\frac{\overline{F}(x)}{\overline{G}(x)} = 1. \tag{14}$$

Is it possible to extend this result to $\mathcal{M}$? The answer is no. To see that, it is enough to consider the distribution F defined in (12) *which satisfies $F\in\mathcal{M}\setminus MDA(\Lambda_\infty)$.*

Indeed, suppose that, for F defined in (12)*, there exists a continuous and increasing distribution function G satisfying* (14)*, which comes back to suppose that there exits a positive and continuous function h such that $G(x) = 1-\exp\left(-h(x)\log(x)\right)$ ($x>0$), in particular in a neighborhood of ∞. So* (14) *may be rewritten as*

$$\lim_{x\to\infty}\frac{\overline{F}(x)}{\overline{G}(x)} = \lim_{x\to\infty}\exp\left(-(\lfloor x\rfloor - h(x))\log(x)\right) = \lim_{x\to\infty} x^{h(x)-\lfloor x\rfloor} = 1.$$

However, since $\lfloor x\rfloor$ cannot be approximated for any continuous function, the previous limit does not hold.

Now let us turn to the tails of distributions which belong to $\mathcal{O}$ defined in (6).

A natural question is whether the Pickands-Balkema-de Haan theorem (see e.g. [8], Theorem 3.4.5) applies when restricting $\mathcal{O}$ to tails of distributions. The answer follows.

Theorem 3.1.

Any distribution of a rv having a tail in $\mathcal{O}$ does not satisfy Pickands-Balkema-de Haan theorem.

Proof of Theorem 3.1.

Let us prove this theorem by contradiction, assuming that F satisfies $\mu(\overline{F}) < \nu(\overline{F})$ and the Pickands-Balkema-de Haan theorem that we recall here for completeness.

Pickands-Balkema-de Haan theorem :
For $\xi \in \mathbb{R}$, G_ξ denoting the Generalized Pareto Distribution, the following assertions are equivalent.

(i) $F \in MDA(\exp(-G_\xi))$,

(ii) There exists a positive function $a > 0$ such that for $1 + \xi x > 0$,

$$\lim_{u\to\infty} \frac{\overline{F}(u + x\,a(u))}{\overline{F}(u)} = G_\xi(x).$$

We consider the two possibilities (i) and (ii) given in the Pickands-Balkema-de Haan theorem. Recall that $x^* = \infty$.

• *Assume that F satisfies (i) in the Pickands-Balkema-de Haan theorem with $\xi \geq 0$ (since $x^* = \infty$).* Let $\epsilon > 0$. By (ii) in Pickands-Balkema-de Haan theorem , there exists $u_0 > 0$ such that, for $u \geq u_0$ and $x \geq 0$,

$$\frac{\overline{F}(u+x)}{\overline{F}(u)\,G_\xi(x/a(u))} \leq 1 + \epsilon. \tag{15}$$

By the definition of upper order, we have that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n \to \infty$ as $n \to \infty$ such that

$$\begin{aligned}
\nu(\overline{F}) &= \lim_{x_n\to\infty} \frac{\log\left(\overline{F}(u+x_n)\right)}{\log(u+x_n)} \leq \lim_{x_n\to\infty} \frac{\log\left((1+\epsilon)\,\overline{F}(u)\,G_\xi(x_n/a(u))\right)}{\log(x_n)} \quad \text{by (15)} \\
&= \lim_{x_n\to\infty} \frac{\log\left(\overline{F}(u)\right)}{\log(x_n)} + \lim_{x_n\to\infty} \frac{\log\left(G_\xi(x_n/a(u))\right)}{\log(x_n)} = \begin{cases} -\dfrac{1}{\xi} \displaystyle\lim_{x_n\to\infty} \frac{\log(1+\xi\,x_n/a(u))}{\log(x_n)} & \text{if } \xi > 0 \\ -\displaystyle\lim_{x_n\to\infty} \frac{x_n/a(u)}{\log(x_n)} & \text{if } \xi = 0 \end{cases} \\
&= \begin{cases} -\frac{1}{\xi} & \text{if } \xi > 0 \\ -\infty & \text{if } \xi = 0. \end{cases}
\end{aligned}$$

If $\xi > 0$, we conclude that $\nu(\overline{F}) \leq -1/\xi$. A similar procedure provides $\mu(\overline{F}) \geq -1/\xi$. Therefore we obtain $\mu(\overline{F}) = \nu(\overline{F})$, which contradicts $\mu(\overline{F}) < \nu(\overline{F})$.

If $\xi = 0$, we conclude that $-\infty \leq \mu(\overline{F}) \leq \nu(\overline{F}) \leq -\infty$. Hence $\mu(\overline{F}) = \nu(\overline{F}) = -\infty$, which contradicts $\mu(\overline{F}) < \nu(\overline{F})$.

• *Assuming that F satisfies the Pickands-Balkema-de Haan theorem , (ii),* and following the previous proof (done when assuming (i)), we deduce that $\mu(\overline{F}) = \nu(\overline{F})$ which contradicts $\mu(\overline{F}) < \nu(\overline{F})$. $\square$

4 Conclusion

This paper is a contribution to the study of tails of distributions according to their asymptotic decay. First it extends some properties known for regularly varying tails, to tails belonging to the larger class $\mathcal{M} \cup \mathcal{M}_{-\infty}$. Then it shows that the maximum domains of attraction of Fréchet and Gumbel (with infinite endpoint) are properly included in $\mathcal{M}$ and $\mathcal{M}_{-\infty}$, respectively. Finally tails of distribution having neither an exponential nor a polynomial asymptotic behavior are proved not to satisfy the Pickands-Balkema-de Haan theorem, which is worth knowing in view of the importance of this theorem.

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