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Explicit solutions to a free boundary model for the static/flowing transition in granular flows

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Abstract

This work is devoted to an analytical description of the dynamics of the static/flowing transition in thin-layer dry granular flows. Our study relies on a free boundary model that has been derived from an incompressible non-Newtonian dynamics with viscoplastic rheology characterized by a Drucker–Prager yield stress, in a non-averaged thin-layer asymptotics with space variable normal to the flow. Here a source term represents the effects of internal friction, gravity and nonhydrostatic pressure. We establish the validity of some explicit solutions to the model with time and space dependent source term, in the inviscid case. These solutions qualitatively reproduce features that are specific of granular flows. We show that the zero of the source term plays a monitoring role for the static/flowing interface. Depending on the time variation of this zero, we can represent different scenarios for the dynamics of the static/flowing interface, from the starting to the arrest of the flow, including a progressive starting, a progressive stopping and a sudden starting of a part of the granular mass.

KEYWORDS. Granular flows, static/flowing transition, Drucker–Prager yield stress, thin-layer flow, interface dynamics.

1 Introduction

Dense granular materials can behave as a solid or flow like a fluid. The static (solid) and flowing (fluid) zones are delimited by a static/flowing interface within the mass. This transition phenomenon occurs in most granular flows, and is relevant in both industrial and geophysical contexts. In particular it is known to play a key role in the erosion/deposition processes [26].

The behaviour of granular flows involving such a phase transition can be described by a viscoplastic rheology with yield stress. The yield stress formulation allows the granular material to flow if the stress is above a threshold value, and to remain at rest in the opposite case. One of the most simple viscoplastic laws with yield stress is the Drucker–Prager law [11, 20, 17, 22], for which the yield stress is proportional to the pressure. With this law, it is possible to obtain a flowing layer above a layer at rest, in contrast with the Bingham law involving a constant yield stress [4]. However, the static/flowing interface is not explicitly described by such viscoplastic laws. An explicit description of the evolution of this interface is a key issue for practical studies.

Our modelling uses the approach of [9], that initially describes the granular material by an incompressible viscoplastic model with Drucker-Prager yield stress. It assumes a thin-layer

geometry, i.e. the thickness is small with respect to the horizontal length. The material lies over an inclined topography. However, the flow is not depth-averaged as in usual shallow-water models. Indeed, in so-called shallow-water models, the conservation equations are averaged with respect to the direction normal to the topography, so that the dependency in this normal variable Z is lost. Therefore, in shallow-water models the static/flowing interface is generally neglected [7, 8, 24, 16], meaning that the material is fully static or fully flowing. Defining an equation for this interface in thin-layer models has been done based on phenomenological models or on strong assumptions such as a specified velocity profile or reducing the flow to a sliding block in [1, 5, 10, 15, 21, 2, 25, 3, 6, 18]. A review on static/flowing models can be found in [19].

As shown in [9, 23], the non-averaged thin-layer framework, on the basis of the viscoplastic model with Drucker-Prager yield stress, allows one to describe a static/flowing interface dynamics without any empirical input. This framework leads to a model driven by a source term that takes into account the effects of pressure, gravity, friction and space inhomogeneities. However, this model is still too elaborate to perform an analytical study. Having in mind to further simplify the model, we consider as in [23] that the flow-aligned variable X can be treated as a fixed parameter. Even if the variable X allows one to better take into account the inhomogeneities of the flow as described for example in [13], it can be omitted in a first approximation, since to each fixed X corresponds a specific source term. Therefore, to each fixed X corresponds a simplified model with source term that describes a specific dynamics. Then we are led to a simplified model, which is a one-dimensional simple shear flow over a flat inclined plane with constant slope angle where the only space variable is the normal space variable Z .

The simplified model we obtain takes the form of a free-boundary problem where the physical static/flowing interface is located by its position in the direction normal to the topography and the dynamical equations are set only in the flowing domain above this position. This free-boundary problem appears formally as an overdetermined boundary problem with one additional condition with respect to a problem with fixed boundary. This additional condition turns out to govern the time evolution of the static/flowing interface.

The present simplified model with source term has been studied in [23] for a constant and positive source term. More precisely, the dynamics is evaluated in [23] by both an analytical study in the inviscid case, and a numerical study in the viscous case, for geophysically relevant configurations. The relevance of the simplified model is confirmed by comparing the results to those obtained from the laboratory experiments of [14], with good agreement. As shown in [23], the simplified model with constant source term is able to reproduce the main features of experiments on granular flow on erodible beds such as the stopping time and the time evolution of the static/flowing interface. In this work, we investigate the more general case of a time and space dependent source term. We compute explicit solutions to the simplified model in the inviscid case and show that the time and space variations of the source term lead to a particularly rich structure that is able to capture several physically relevant effects from the starting to the arrest of the flow, including a progressive starting, a progressive stopping and a sudden starting of a part of the granular mass.

The outline of the paper is as follows. In Section 2 we recall the non-averaged thin-layer model of [9] and the simplified model of [23] involving only the normal space variable Z , with a given variable source term that represents the physical input. Doing so, we introduce the position b of the free static/flowing interface, which is an unknown of the problem. Several properties of this model are given. Section 3 is devoted to the inviscid case for the simplified model with source term. Throughout the section, we assume a variable source term which admits a zero in space denoted by b^* . We perform an analytical study of four different dynamical behaviours, which stem from four different behaviours of the given zero b^* and the initial data. In each case, we determine an explicit solution (u, b) , we provide some analytical properties and we illustrate our results with some particular data. Finally, in Section 4 we conclude the paper and provide an overview of all the dynamical behaviours exhibited in

Section 3. Our study proves the ability of the simplified model to describe physically relevant dynamics, from the starting to the arrest of the granular mass.

2 Simplified model with variable source term

2.1 Non-averaged thin-layer model

A non-averaged thin-layer model for viscoplastic flows with yield stress and static/flowing transition has been derived in [9]. It is formulated in the variables X in the direction tangent to the topography, and Z normal to it. The topography is described by its angle $\theta(X)$ with the horizontal (see Figure 1 in the case of a flat topography). The starting governing equations are the dynamics of an incompressible viscoplastic flow with Drucker–Prager yield stress, characterized by the rheological law

$$\boldsymbol{\sigma} = -p \text{Id} + 2\nu D\mathbf{u} + \mu_s p \frac{D\mathbf{u}}{\|D\mathbf{u}\|}, \quad (2.1)$$

where $\boldsymbol{\sigma}$ is the stress tensor (normalized by the density), p is the scalar pressure (also normalized by the density), $D\mathbf{u}$ is the strain rate tensor $D\mathbf{u} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^t)/2$ with \mathbf{u} the velocity vector. Here the norm of a matrix $A = (A_{ij})$ is $\|A\| = (\frac{1}{2} \sum_{ij} A_{ij}^2)^{1/2}$. The coefficients $\mu_s > 0$ and $\nu \geq 0$ are the internal friction and the kinematic viscosity, respectively. Under the assumptions that the width of the layer, the curvature of the topography, and the viscosity are small, the internal friction angle is close to the slope angle, the velocity is small, and the pressure is convex with respect to the normal variable Z , a formal expansion of the governing equations leads to the momentum balance equation

$$\partial_t u + S - \partial_Z(\nu \partial_Z u) = 0 \quad \text{for all } Z \in (b, h), \quad (2.2)$$

where $u(t, X, Z)$ is the velocity in the direction tangent to the topography. The kinematic viscosity ν is taken constant here. In (2.2), $h(t, X)$ is the thickness of the layer, and $b(t, X)$ is the position of the interface between the static part $Z < b(t, X)$ (where we set $u \equiv 0$) and the flowing part $Z > b(t, X)$. The thickness $h(t, X)$ evolves according to the mass conservation

$$\partial_t \left(h - \frac{h^2}{2} \frac{d\theta}{dX} \right) + \partial_X \left(\int_b^h u dZ \right) = 0, \quad (2.3)$$

and the evolution of the free interface b is implicitly governed by the boundary conditions for all $t > 0$,

$$u = 0 \quad \text{at } Z = b, \quad (2.4a)$$

$$\nu \partial_Z u = 0 \quad \text{at } Z = b, \quad (2.4b)$$

$$\nu \partial_Z u = 0 \quad \text{at } Z = h. \quad (2.4c)$$

Knowing that the material is at rest in the part below the interface, these boundary conditions mean that the velocity is continuously differentiable through the interface (in the viscous case), and that the viscous stress vanishes at the free surface $Z = h$.

The source S in (2.2) depends on t, X, Z and is expressed as

$$S = g(\sin\theta + \partial_X(h \cos\theta)) - \mu_s \partial_Z p \quad \text{for all } Z \in (b, h), \quad (2.5)$$

where $g > 0$ is the gravity acceleration, $\mu_s > 0$ is as above the friction coefficient related to the material, and $p(t, X, Z)$ is the pressure. The pressure is expressed as

$$p = g \left(\cos\theta + \sin\theta \partial_X h - 2|\sin\theta| \frac{\partial_X u}{|\partial_Z u|} \right) \times (h - Z), \quad \text{for all } Z \in (b, h). \quad (2.6)$$

It is nonlinear in terms of $\partial_X u$ and $\partial_Z u$, thus it is coupled to the unknown velocity u . The formula (2.5) is valid in the context of negative angle $\theta < 0$ and of an increasing velocity profile in the flowing layer, i.e. $\partial_Z u > 0$ for all $Z \in (b, h)$. An additional condition $S(t, X, b(t, X)) \geq 0$ is also necessary for the validity of the model. It states that the stress in the static part of the flow $Z < b$ is strong enough to maintain it static. If $S(t, X, Z)$ is negative for all Z this condition cannot be satisfied, meaning that there is no solution with static/flowing interface b ; instead the whole layer of material should flow down.

In the formulas (2.5), (2.6) that give S , we can identify a first part that is independent of Z , $g(\sin \theta + \mu_s \cos \theta)$, that takes into account the hydrostatic part $g \cos \theta (h - Z)$ of p in (2.6). The effect of this first part alone is described in [23], and the condition $S(t, X, b(t, X)) \geq 0$ reduces in this case to $|\tan \theta| \leq \mu_s$. Then there is a second part in S from (2.5), (2.6) that is also independent of Z , which is a correction to the first one due to flow-aligned variations of h , it is $g(\partial_X (h \cos \theta) + \mu_s \sin \theta \partial_X h)$. Finally there is a last term that involves nonlinearities in $\partial_X u$ and $\partial_Z u$, that could make S depend on Z . The aim of this paper is to investigate the possible effects on the flow of this dependence of S on Z due to this third term. Since the shape in Z of this third term is unknown, but should satisfy $\partial_Z S \leq 0$ (see [9]), we shall consider generic sources S satisfying this condition.

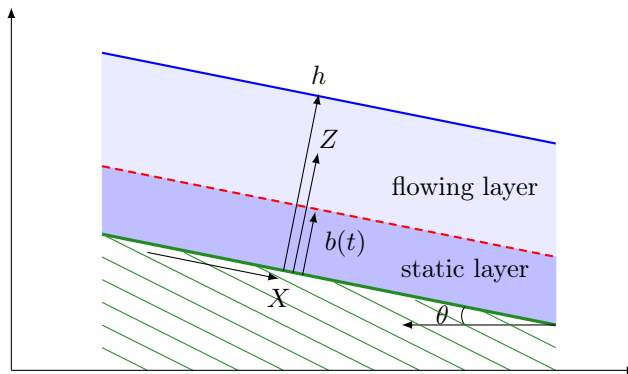


Figure 1: Simplified flow configuration consisting of a uniform flowing layer over a uniform static layer, both parallel to the rigid bed of slope angle θ . The static/flowing interface position is $b(t)$, and the total thickness h of the material is constant.

2.2 The simplified model and first properties

In this subsection we recall the simplified model introduced in [23] and its main properties. We propose full proofs of a few results, that were only sketched in [23].

The simplified model is obtained from the one of the previous subsection by eliminating the variable X in the flow-aligned direction. Thus there remains only the variable Z in the normal direction, together with the time variable t .

The procedure to simplify the thin-layer model in the variables (t, X, Z) is first to replace the coupled source term S by an empirical one that we assume given. Then, we consider X as a parameter. Indeed, to each fixed X corresponds a one-dimensional system involving only the space variable Z and the time variable t . Thus the variable X can be omitted, all the X dependencies being somehow hidden in the shape of the source $S(t, Z)$. The equation for the velocity u in the flow-aligned direction is then

$$\partial_t u(t, Z) + S(t, Z) - \nu \partial_{ZZ}^2 u(t, Z) = 0 \quad \text{for all } Z \in (b(t), h), \quad (2.7)$$

where $h > 0$ is the fixed width of the domain, $\nu \geq 0$ is the kinematic viscosity, and $S(t, Z)$ is a given source term, defined for all $t \in [0, T]$ and all $Z \in [0, h]$ and satisfying

$$S \text{ is continuous in } [0, T] \times [0, h]. \quad (2.8)$$

In (2.7), the position $b(t) \in (0, h)$ of the static/flowing interface is an unknown of the problem. We complete the equation (2.7) with the boundary conditions set for $t > 0$,

$$u = 0 \quad \text{at } Z = b(t), \quad (2.9a)$$

$$\nu \partial_Z u = 0 \quad \text{at } Z = b(t), \quad (2.9b)$$

$$\nu \partial_Z u = 0 \quad \text{at } Z = h. \quad (2.9c)$$

Typical expected velocity profiles are represented on Figure 2. The formulation (2.7), (2.9) is equivalent to a formulation on the entire domain $[0, h]$, see [9]. The initial condition is given by

$$u(0, Z) = u^0(Z) \quad \text{for all } Z \in (b^0, h), \quad (2.10)$$

where $b^0 \in [0, h]$ is the given initial position of the static/flowing interface. The initial velocity profile u^0 is assumed to satisfy

$$u^0 \in C^1([b^0, h]), \quad \partial_Z u^0 > 0 \quad \text{for all } Z \in (b^0, h), \quad (2.11a)$$

$$u^0(b^0) = 0. \quad (2.11b)$$

We extend u^0 by zero in $[0, b^0]$.

In what follows, we assume implicitly that $b(t)$ satisfies $0 < b(t) < h$. The case when $b(t)$ reaches one of the boundaries 0 or h is a special case that will be discussed every time that it is relevant.

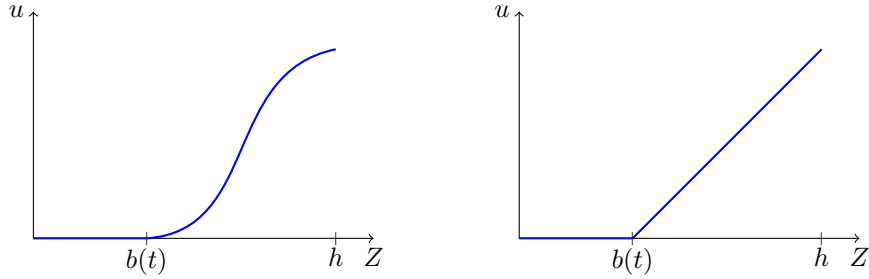


Figure 2: Typical velocity profile with respect to Z at fixed time $t > 0$, satisfying the boundary conditions (2.9). These profiles have been validated in [23]. The viscous case ($\nu > 0$) is represented on the left and the inviscid case ($\nu = 0$) is represented on the right. The velocity vanishes over $[0, b(t)]$ and increases on $(b(t), h)$.

The source term S is empirical here, thus we need to put some conditions in order to be consistent with the original model. In that one, we expect that for all $t \in [0, T]$, the velocity vanishes in the static part $[0, b(t)]$ and is increasing in the flowing part $(b(t), h)$. This will be the case under two assumptions. The first one requires that the initial velocity is increasing in the flowing part. The second one is the decrease in space (with Z increasing from 0 at the bed to h at the free surface) of the source term, that was mentioned in [9] as a stability condition for the simplified model with source term.

Lemma 2.1. *Assume that the velocity u solves (2.7), (2.9), (2.10) for some interface position $b(t)$. Moreover, assume that the viscosity ν is non zero, that the initial condition verifies (2.11a), and that the source term S verifies*

$$\partial_Z S \in C([0, T] \times [0, h]), \quad \partial_Z S \leq 0 \quad \text{for all } t \in [0, T], \quad \text{and all } Z \in [0, h]. \quad (2.12)$$

Then we have that

$$\partial_Z u > 0 \quad \text{for all } t \in [0, T], \quad \text{and all } Z \in (b(t), h). \quad (2.13)$$

Proof. By differentiating (2.7) with respect to Z , we obtain

$$\partial_t(\partial_Z u) + \partial_Z S - \nu \partial_{ZZ}^2(\partial_Z u) = 0 \quad \text{for all } Z \in (b(t), h). \quad (2.14)$$

Together with (2.12), this leads to

$$\partial_t(\partial_Z u) - \nu \partial_{ZZ}^2(\partial_Z u) \geq 0 \quad \text{for all } Z \in (b(t), h). \quad (2.15)$$

With the boundary conditions (2.9b), (2.9c) and (2.10), (2.11a), this leads to

$$\partial_t(\partial_Z u) - \nu \partial_{ZZ}^2(\partial_Z u) \geq 0 \quad \text{for all } Z \in (b(t), h), \quad (2.16a)$$

$$\nu \partial_Z u = 0 \quad \text{at } Z = b(t), \quad (2.16b)$$

$$\nu \partial_Z u = 0 \quad \text{at } Z = h, \quad (2.16c)$$

$$\partial_Z u > 0 \quad \text{at } t = 0. \quad (2.16d)$$

Knowing that $\nu > 0$, this formulation is suitable for applying the strong maximum principle to the function $\partial_Z u$. In order to deal with a time-independent interval, we perform the change of variable

$$\begin{aligned} \mathbb{R}^+ \times [b(t), h] &\rightarrow \mathbb{R}^+ \times [0, 1] \\ (t, Z) &\mapsto (\tau, Y), \end{aligned} \quad (2.17)$$

with $t = \tau$ and $Z = b(\tau) + Y(h - b(\tau))$. It leads to the formulas $\partial_Y = (h - b)\partial_Z$, $\partial_\tau = \partial_t + (1 - Y)\dot{b}\partial_Z$. Denoting $V = \partial_Z u$, (2.16a) yields

$$\partial_\tau V - \frac{(1 - Y)}{h - b} \dot{b} \partial_Y V - \frac{\nu}{(h - b)^2} \partial_{YY}^2 V \geq 0 \quad \text{for all } Y \in (0, 1), \quad (2.18a)$$

$$\nu V = 0 \quad \text{at } Y = 0, \quad (2.18b)$$

$$\nu V = 0 \quad \text{at } Y = 1, \quad (2.18c)$$

$$V > 0 \quad \text{at } \tau = 0. \quad (2.18d)$$

As $\nu > 0$, we deduce that $V > 0$, i.e. (2.13). \square

In [9], the thin-layer model was completed with a ‘‘static equilibrium condition’’, necessary for getting an approximate solution to the original viscoplastic model. This condition is

$$S(t, b(t)) \geq 0 \quad \text{for all } t \in (0, T]. \quad (2.19)$$

In the viscous case, this condition is automatically satisfied.

Lemma 2.2 ([23]). *Assume that the velocity u solves (2.7), (2.9), (2.10) for some interface position $b(t)$. Moreover, assume that the viscosity ν is non zero, and that (2.11a) and (2.8), (2.12) hold. Then (2.19) holds.*

Proof. Differentiating in time (2.9a) i.e. $u(t, b(t)) = 0$ yields that

$$\partial_t u(t, b(t)) + \partial_Z u(t, b(t)) \dot{b}(t) = 0. \quad (2.20)$$

Using (2.9b) and the assumption $\nu > 0$, we get $\partial_t u(t, b(t)) = 0$. Then evaluating (2.7) at $Z = b(t)$ gives

$$S(t, b(t)) = \nu \partial_{ZZ}^2 u(t, b(t)). \quad (2.21)$$

Now, according to Lemma 2.1, $\partial_Z u(t, Z) > 0$ for $Z > b(t)$. Since again by (2.9b) we have $\partial_Z u(t, b(t)) = 0$, we can write $\partial_{ZZ}^2 u(t, b(t)) = \lim_{Z \rightarrow b(t)^+} \partial_Z u(t, Z)/(Z - b(t))$ as $Z \rightarrow b(t)^+$, which yields that $\partial_{ZZ}^2 u(t, b(t)) \geq 0$. With (2.21) we get the result. \square

From the simplified model with source term (2.7), (2.9), we can get an explicit equation on the position $b(t)$ of the static/flowing interface, under regularity assumptions on the velocity and the source term.

Lemma 2.3 ([23]). *Assume that the velocity u solves (2.7), (2.9). Then*
1) *If $\nu > 0$, if u and S are sufficiently differentiable and if $S(t, b(t)) \neq 0$, then*

$$\dot{b}(t) = \left(\frac{\partial_Z S(t, b(t)) - \nu \partial_{ZZ}^3 u(t, b(t))}{S(t, b(t))} \right) \nu. \quad (2.22)$$

2) *If $\nu = 0$, if u is of class C^1 in space and time and if $\partial_Z u(t, b(t)) \neq 0$, then*

$$\dot{b}(t) = \frac{S(t, b(t))}{\partial_Z u(t, b(t))}. \quad (2.23)$$

Proof. We again differentiate (2.9a) with respect to time, which provides (2.20).

1) If $\nu > 0$, as in the proof of Lemma 2.2 we get (2.21). By differentiating in time the boundary condition (2.9b), we obtain similarly

$$\partial_{tZ}^2 u(t, b(t)) + \partial_{ZZ}^2 u(t, b(t)) \dot{b}(t) = 0. \quad (2.24)$$

Then we differentiate (2.7) with respect to Z and evaluate it at $(t, b(t))$. We obtain

$$\partial_{tZ}^2 u(t, b(t)) + \partial_Z S(t, b(t)) - \nu \partial_{ZZ}^3 u(t, b(t)) = 0, \quad (2.25)$$

thus with (2.24),

$$\partial_{ZZ}^2 u(t, b(t)) \dot{b}(t) = \partial_Z S(t, b(t)) - \nu \partial_{ZZ}^3 u(t, b(t)). \quad (2.26)$$

Using (2.21) and according to the assumption $S(t, b(t)) \neq 0$, we obtain (2.22).

2) If $\nu = 0$, the equation (2.7) reduces to

$$\partial_t u(t, Z) + S(t, Z) = 0 \quad \text{for all } Z \in (b(t), h). \quad (2.27)$$

Thus, by continuity in space of $\partial_t u$ and S , we obtain $\partial_t u(t, b(t)) = -S(t, b(t))$. With (2.20) and the assumption $\partial_Z u(t, b(t)) \neq 0$ this give the result (2.23). \square

Lemma 2.3 provides a differential equation on $b(t)$, that could be used instead of the overdetermined boundary conditions. This way of thinking is however misleading, for several reasons. The first is that it requires too much regularity on the velocity. Numerically, this would lead to instability, in particular when ν is small, since (2.22) should converge to (2.23), which is a highly singular limit. Instead, numerical schemes developed in [23] use the boundary conditions (2.9), that yield stable solutions. The second reason is that the formulas (2.22), (2.23) are valid only in the case when the denominator does not vanish. If the denominator vanishes, the dynamics can be rather non trivial. This is the case for the inviscid case. We show in Section 3 that it is possible to have solutions with $\dot{b} < 0$, together with $\partial_Z u(t, b(t)) = 0$, while in the case $\partial_Z u(t, b(t)) > 0$ (u is increasing thus the denominator is never negative), (2.23) always give $\dot{b} \geq 0$ because of (2.19).

3 Analytical study of the inviscid case

Along this section, we perform an analytical study of the simplified model (2.7), (2.9) in the inviscid case, i.e. when $\nu = 0$. In the following, we suppose that the source term $S(t, Z)$ allows a zero $b^*(t)$ in space, that depends on time. Under physically relevant assumptions, this zero provides a monitoring of the position $b(t)$ of the static/flowing interface. We investigate several types of behaviour for $b^*(t)$ and the initial conditions, which in turn allow us to model different possible dynamics for our problem. In each case, we provide an explicit solution by first building the velocity u , then the position b of the static/flowing interface. Illustrations are given with a linear initial velocity profile and a source term linear in space.

In the inviscid case, the simplified model (2.7), (2.9) reduces to

$$\partial_t u(t, Z) + S(t, Z) = 0 \quad \text{for all } Z \in (b(t), h), \quad (3.1)$$

$$u(t, b(t)) = 0. \quad (3.2)$$

Moreover, as in Subsection 2.2 we assume that the source term S is continuous in time and space, and is nonincreasing in space, i.e. (2.8), (2.12). The initial condition is written

$$u(0, Z) = u^0(Z) \quad \text{for all } Z \in (b^0, h), \quad (3.3)$$

where u^0 satisfies (2.11). We seek for a solution (u, b) that satisfies the monotonicity condition

$$\partial_Z u > 0 \quad \text{for all } t \in (0, T], \text{ and all } Z \in (b(t), h), \quad (3.4)$$

and the “static equilibrium condition”

$$S(t, b(t)) \geq 0 \quad \text{for all } t \in (0, T]. \quad (3.5)$$

With these assumptions, we are going to provide an explicit solution (u, b) to (3.1), (3.2), (3.3), (3.4), (3.5), in several configurations.

We shall assume that the source $S(t, Z)$ admits a unique zero $b^*(t) \in [0, h]$ that depends on time t , hence satisfying $S(t, b^*(t)) = 0$. Together with the monotonicity in space of S written in (2.12), this assumption is equivalent to

$$S(t, Z) > 0 \quad \text{for all } Z < b^*(t), \quad (3.6a)$$

$$S(t, Z) < 0 \quad \text{for all } Z > b^*(t). \quad (3.6b)$$

Since S is continuous with respect to its two arguments, we deduce that $b^*(t)$ is continuous likewise. Furthermore, we get the following reformulation of the static equilibrium condition.

Lemma 3.1. *Assume that the source term S is continuous with respect to its two arguments (2.8), satisfies the monotonicity condition (2.12), and admits a unique zero $b^*(t)$ in space. Then, the condition (3.5) is equivalent to*

$$b(t) \leq b^*(t) \quad \text{for all } t \in (0, T]. \quad (3.7)$$

Proof. This is immediate with (3.6). \square

Let us give some comments.

a) The inequality (3.7) indicates that the source term vanishes at b^* which is in the flowing part of the flow $[b(t), h]$.

b) If the initial position of the static/flowing interface b^0 is set such that $b^0 > b^*(0)$, then the inequality (3.7) implies that $b(t)$ jumps instantaneously below $b^*(0)$. Therefore, this leads to a discontinuous static/flowing interface $b(t)$. This configuration allows to model a sudden starting of a part of the initially static material. Such configuration will be investigated in Subsection 3.4 below.

c) The complete stopping of the flow corresponds to the case when the position $b(t)$ of the static/flowing interface reaches the height h of the domain. Then it is natural to set $b(t) = h$ and $u(t, Z) = 0$ for all $t \geq t^{\text{stop}}$, where t^{stop} stands for the first stopping time. Therefore, when assuming

$$b^*(t) < h \quad \text{for all } t \in [0, T], \quad (3.8)$$

the inequality (3.7) provides a barrier for the static/flowing interface, that prevents the flow from fully stopping.

The outline of the present section is as follows. First, we investigate the case of a non-decreasing zero b^* , with the initial condition $b^0 < b^*(0)$, and we seek for a continuous and nondecreasing $b(t)$. Next, we consider a strictly decreasing zero b^* , with $b^0 = b^*(0)$, and we seek for a continuous and strictly decreasing $b(t)$. Then, we handle a more general case than the previous one by considering a strictly decreasing zero b^* , with $b^0 < b^*(0)$, and we seek for a continuous $b(t)$. Finally, we consider a nondecreasing zero b^* , with $b^0 > b^*(0)$. In this last case, the position of the static/flowing interface is discontinuous.

3.1 Source term with a nondecreasing zero b^* and $b^0 < b^*(0)$

In this subsection, we investigate the case of a nondecreasing zero $b^*(t)$ of the source term S , and we look for a position $b(t)$ of the static/flowing interface that is also nondecreasing. This configuration allows to model a progressive stopping of the flow.

We consider a source term S with a unique zero $b^*(t)$ (i.e. (3.6)) that satisfies

$$b^*(t) \text{ is nondecreasing,} \quad (3.9a)$$

$$b^0 < b^*(0). \quad (3.9b)$$

We seek for a continuous function $b(t)$, with $b(0) = b^0$.

We first build the velocity u solution to the evolution equation (3.1), under the assumption that the position $b(t)$ of the static/flowing interface is nondecreasing. Then, we manage to deduce explicitly the static/flowing interface position $b(t)$ by analyzing the boundary conditions. The following lemma provides the explicit expression of the velocity.

Lemma 3.2. *Assume (2.8), (2.12), (2.11), and that u solves (3.1), (3.3) for some continuous nondecreasing $b(t)$ satisfying $b(0) = b^0$. Then, for all $t \in [0, T]$ and for all $Z \in [b(t), h]$, one has*

$$u(t, Z) = u^0(Z) - \int_0^t S(\tau, Z) d\tau. \quad (3.10)$$

Proof. We would like to integrate (3.1) in time on $[0, t]$, for t fixed in $[0, T]$ and Z fixed in $[b(t), h]$. In order for (3.1) to be valid over the interval of integration (τ, Z) , $\tau \in [0, t]$, we need that $b(\tau) \leq Z$ for all $\tau \in [0, t]$. Since b is nondecreasing, this condition reduces to $b(t) \leq Z$. The integration path is represented on Figure 3. Integrating (3.1) yields (3.10). \square

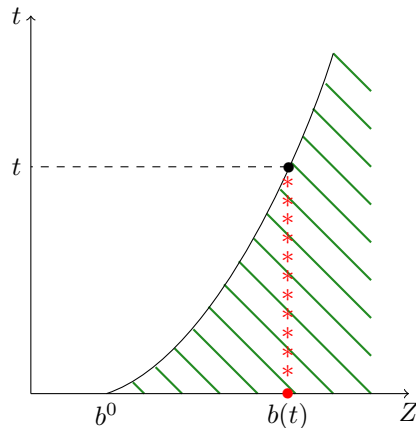


Figure 3: Position b of the static/flowing interface with respect to time t , starting from an initial position b^0 . The flowing domain where (3.1) holds is filled with oblique lines. The integration path for getting the velocity is represented with stars.

Next, in order to determine $b(t)$, we set an *extended velocity* defined on $[b^0, h]$ and denoted by \tilde{u} . The extended velocity is the right-hand side of (3.10), that is

$$\tilde{u}(t, Z) = u^0(Z) - \int_0^t S(\tau, Z) d\tau \quad \text{for all } Z \in [b^0, h]. \quad (3.11)$$

Lemma 3.2 says that u is equal to \tilde{u} for all $Z \in [b(t), h]$. Since $b(t)$ is nondecreasing, we are looking for $b(t) \in (b^0, h)$. In order to get the boundary condition (3.2), $b(t)$ must satisfy

$\tilde{u}(t, b(t)) = 0$. We note that the assumptions (2.12) and (2.11a) imply that $\partial_Z \tilde{u} > 0$ for all $Z \in (b^0, h)$, hence (3.4) is automatically satisfied. The equations (3.1) and (3.3) are also consequences of the formula (3.10). Therefore, it remains only to clarify if one can find a continuous nondecreasing function $b(t)$ such that $\tilde{u}(t, b(t)) = 0$, which satisfies $b(0) = b^0$, and (3.5). This is what gives the following lemma.

Lemma 3.3. *Assume (2.8), (2.12), (2.11), (3.6), (3.9). Then there exists a unique $b(t) \in [b^0, h]$ such that $\tilde{u}(t, b(t)) = 0$, with \tilde{u} defined by (3.11). Moreover,*
1) $b(t) < b^*(t)$ for all $t \in [0, T]$,
2) $b(t)$ is continuous, nondecreasing, and $b(0) = b^0$.

Proof. The uniqueness follows from the previously stated property $\partial_Z \tilde{u} > 0$. Then, we note that \tilde{u} is continuous in both variables, and that by (3.9) one has $b^*(t) > b^0$. Hereafter, we prove the existence of $b(t) \in [b^0, b^*(t)]$ such that $\tilde{u}(t, b(t)) = 0$. Indeed, we show that $\tilde{u}(t, b^0) < 0$ for $t > 0$, and $\tilde{u}(t, b^*(t)) > 0$. At first, by (2.11b) one has

$$\tilde{u}(t, b^0) = - \int_0^t S(\tau, b^0) d\tau. \quad (3.12)$$

Therefore, since $b^*(t) > b^0$ and using (3.6), we deduce that $S(\tau, b^0) > 0$ for all $\tau \in [0, T]$. Hence $\tilde{u}(t, b^0) < 0$ for all $t > 0$, while $\tilde{u}(0, b^0) = 0$, giving $b(0) = b^0$. Then,

$$\tilde{u}(t, b^*(t)) = u^0(b^*(t)) - \int_0^t S(\tau, b^*(t)) d\tau. \quad (3.13)$$

Using again $b^*(t) > b^0$ and (2.11), we get $u^0(b^*(t)) > 0$. Moreover, by (3.9a), $b^*(\tau) \leq b^*(t)$ for all $\tau \in [0, t]$, thus according to (3.6) we have $S(\tau, b^*(t)) \leq 0$. Therefore $\tilde{u}(t, b^*(t)) > 0$ as claimed. We deduce the existence of $b(t)$, and it only remains to prove that it is continuous and nondecreasing. One has $\tilde{u}(t, b(t)) = 0$ with \tilde{u} continuous, which proves the continuity of b . Then, differentiating with respect to t gives $\partial_t \tilde{u}(t, b) + \partial_Z \tilde{u}(t, b) \dot{b} = 0$. One has $\partial_t \tilde{u} = -S$, thus $\partial_t \tilde{u}(t, b(t)) = -S(t, b(t)) \leq 0$ by Lemma 3.1, and since $\partial_Z \tilde{u} > 0$ we get $\dot{b} \geq 0$. \square

The synthesis of Lemmas 3.2 and 3.3 is stated in the following main result of this subsection.

Proposition 3.4. *Assume (2.8), (2.12), (2.11), (3.6), (3.9). Then there exists a unique solution (u, b) to (3.1), (3.2), (3.3), (3.4), (3.5), such that the position $b(t)$ of the static/flowing interface is nondecreasing, continuous and $b(0) = b^0$. Moreover, the velocity u is given by (3.10), and the position $b(t)$ of the static/flowing interface is determined by the equation*

$$u^0(b(t)) = \int_0^t S(\tau, b(t)) d\tau, \quad (3.14)$$

and verifies $b(t) < b^*(t)$ for all $t \in [0, T]$.

Illustration

For the purpose of illustrating our results, we consider the most simple case of linear data. They are written in dimensionless units, meaning that appropriate rescaled variables have to be used. On the one hand we set a linear initial velocity

$$u^0(Z) = Z - b^0 \quad \text{for all } Z \in [b^0, h], \quad (3.15)$$

where b^0 is the initial position of the static/flowing interface. On the other hand, we set a linear and decreasing with respect to Z source term,

$$S(t, Z) = b^*(t) - Z \quad \text{for all } t \in [0, T], \text{ and all } Z \in [0, h], \quad (3.16)$$

where $b^*(t)$ is continuous and satisfies (3.9).

According to Proposition 3.4, we have a unique solution to our problem. The velocity is written, for all $t \in [0, T]$

$$u(t, Z) = (1+t)Z - b^0 - \int_0^t b^*(\tau) d\tau \quad \text{for all } Z \in [b(t), h]. \quad (3.17)$$

The expression of $b(t)$ is determined by (3.14), leading to

$$b(t) = \frac{b^0 + \int_0^t b^*(\tau) d\tau}{1+t} \quad \text{for all } t \in [0, T]. \quad (3.18)$$

In accordance with our result, this formula is nondecreasing, as can be checked directly with the assumption (3.9).

Let us consider a constant value $b^*(t) = b^* > b^0$ for all $t \in \mathbb{R}^+$. Then (3.17) and (3.18) give

$$u(t, Z) = (1+t)Z - (b^0 + tb^*) \quad \text{for all } Z \in [b(t), h], \quad (3.19)$$

and

$$b(t) = \frac{b^0 + tb^*}{1+t} \quad \text{for all } t \in \mathbb{R}^+. \quad (3.20)$$

The latter expression (3.20) satisfies $b^0 < b(t) < b^*$ for all $t > 0$ and $b(t) \rightarrow b^*$ as $t \rightarrow \infty$. We can remark that evaluating (3.19) at $Z = b^*$, we get

$$u(t, b^*) = u^0(b^*) \quad \text{for all } t \in \mathbb{R}^+, \quad (3.21)$$

meaning that the velocity profiles intersect at $Z = b^*$ for all times $t \in \mathbb{R}^+$. The solution (u, b) is plotted on Figure 4. Since $b(t) < b^* < h$, the static/flowing interface never reaches the top of the domain, modelling a flow that does never reach a complete stopping.

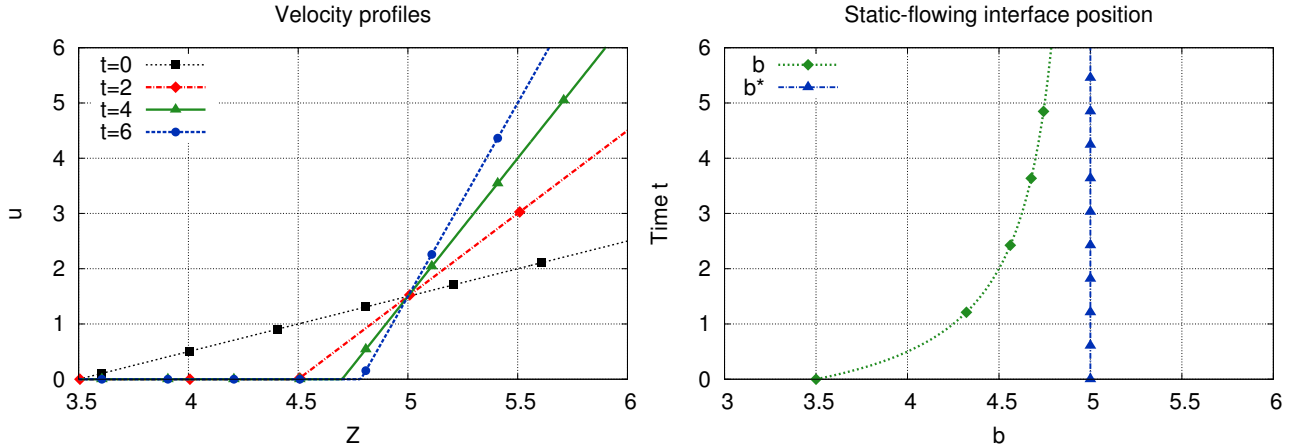


Figure 4: Illustration of the behaviour of the solution (u, b) in the case with nondecreasing zero b^* of the source and $b^0 < b^*(0)$. The source and initial velocity are linear, given by (3.16), (3.15), with $h = 6$, $b^0 = 3.5$, $b^* = 5$. Velocity profiles u are represented at times $t = 0$, $t = 2$, $t = 4$, $t = 6$ (left). The position $b(t)$ of the static/flowing interface with respect to time is shown (right).

3.2 Source term with a decreasing zero b^* and $b^0 = b^*(0)$

In this subsection we assume that the zero $b^*(t)$ of the source term S is strictly decreasing, starting from the initial position b^0 of the static/flowing interface. This configuration allows

to model a progressive start of the flow with a thickening fluid part.

We consider a source term S with a unique zero $b^*(t)$ (i.e. (3.6)) that satisfies

$$b^* \in C^1([0, T]), \quad \dot{b}^*(t) < 0 \text{ for } t > 0, \quad (3.22a)$$

$$b^0 = b^*(0). \quad (3.22b)$$

Here, the dot always denotes differentiation with respect to time. Let us first state a property on the position of the static/flowing interface that needs no hypothesis on b^* .

Lemma 3.5. *Assume (2.8), (2.12), (2.11), (3.6) and that (u, b) is a solution to (3.1), (3.2), (3.3), (3.4), (3.5) with b continuously differentiable on $(0, T]$. If there exists a time $t_0 > 0$ such that $\dot{b}(t_0) \leq 0$, then we have*

$$b(t_0) = b^*(t_0), \quad \partial_Z u(t_0, b(t_0))\dot{b}(t_0) = 0. \quad (3.23)$$

Proof. By differentiating (3.2) i.e. $u(t, b(t)) = 0$ with respect to t , we obtain

$$\partial_t u(t, b(t)) + \partial_Z u(t, b(t))\dot{b}(t) = 0. \quad (3.24)$$

According to (3.1) and (3.5) we have $\partial_t u(t, b(t)) = -S(t, b(t)) \leq 0$, and by (3.4) we have $\partial_Z u(t, b(t)) \geq 0$. Since $\dot{b}(t_0) \leq 0$, the equality (3.24) with the given signs leads to $S(t_0, b(t_0)) = 0$ and $\partial_Z u(t_0, b(t_0))\dot{b}(t_0) = 0$. Since S has a unique zero by (3.6), we obtain (3.23). \square

We can now identify the solution (u, b) under the hypothesis (3.22).

Lemma 3.6. *Assume (2.8), (2.12), (2.11), (3.6), (3.22). If (u, b) solves (3.1), (3.2), (3.3), (3.4), (3.5) with $b \in C^1([0, T])$ with $b(0) = b^0$, then*

$$b(t) = b^*(t) \quad \text{for all } t \in [0, T], \quad (3.25)$$

and for all $t \in [0, T]$ the velocity is given by

$$u(t, Z) = - \int_{(b^*)^{-1}(Z)}^t S(\tau, Z) d\tau \quad \text{for all } Z \in [b^*(t), b^*(0)], \quad (3.26a)$$

$$u(t, Z) = u^0(Z) - \int_0^t S(\tau, Z) d\tau \quad \text{for all } Z \in [b^*(0), h]. \quad (3.26b)$$

Proof. Setting $\Delta(t) = b(t) - b^*(t)$, by Lemma 3.1 we have $\Delta \leq 0$, while by (3.22b) $\Delta(0) = 0$. For proving (3.25), assume by contradiction that there exists a time $t_1 > 0$ such that $\Delta(t_1) < 0$. By the mean value theorem, there exists a time $t_0 \in (0, t_1)$ such that $\Delta(t_0) < 0$ and $\dot{\Delta}(t_0) < 0$. Thus $\dot{b}(t_0) < \dot{b}^*(t_0) \leq 0$ by (3.22a), and according to Lemma 3.5 this implies that $\Delta(t_0) = 0$, hence a contradiction. This proves (3.25).

Next, by (3.22a) we get for all $t > 0$ that $b^*(t) < b^*(0)$. The value of Z being fixed in $[b^*(t), b^*(0)]$, we integrate (3.1) in time between t' and t , where $t' \in [0, t]$ is such that $Z = b^*(t')$, i.e. $t' = (b^*)^{-1}(Z)$. This is possible because for all $\tau \in [t', t]$, $b(\tau) \leq b(t') = Z$. The path of integration is represented on Figure 5. We obtain

$$u(t, Z) = u(t', Z) - \int_{t'}^t S(\tau, Z) d\tau. \quad (3.27)$$

But since $b(t') = b^*(t') = Z$, we infer that $u(t', Z) = u(t', b(t')) = 0$ by (3.2), giving (3.26a). Similarly, for a fixed Z in $[b^*(0), h]$, we are allowed to integrate (3.1) between 0 and t , therefore with (3.3) we obtain (3.26b). \square

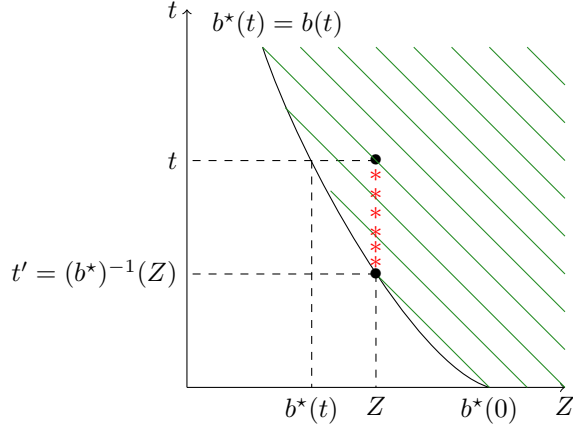


Figure 5: Position $b = b^*$ of the static/flowing interface with respect to time t , starting from $b^0 = b^*(0)$. The flowing domain where (3.1) holds is filled with oblique lines. The path of integration is represented with stars.

In the following proposition, we check that the velocity and position of the static/flowing interface given by (3.25), (3.26) provide a solution to (3.1), (3.2), (3.3), (3.4), (3.5), under a strictly decreasing hypothesis on the source term. We also establish an additional boundary condition at $Z = b(t)$,

$$\partial_Z u(t, b(t)) = 0 \quad \text{for all } t > 0, \quad (3.28)$$

hence recovering in the context of (3.22) the boundary condition (2.9b) of the viscous case.

Proposition 3.7. *Assume (2.8), (2.12), (2.11), (3.6), (3.22) and that $\partial_Z S < 0$. Then there is a unique solution (u, b) to (3.1), (3.2), (3.3), (3.4), (3.5) such that $b \in C^1([0, T])$ with $b(0) = b^0$. Moreover, (u, b) is given by (3.25), (3.26), and verifies the boundary condition (3.28).*

Proof. The uniqueness is given by Lemma 3.6. For existence, define (u, b) by (3.25), (3.26). Then since $b = b^*$ and by (3.22b), one has $b(0) = b^0$. The assumption (3.22) enables to define u by (3.26a), (3.26b), indeed the two formulas match at $Z = b^*(0)$ because of (2.11b). Thus u is continuous. It obviously satisfies (3.3) according to (3.26b). Then, (3.1) is obtained easily by differentiating (3.26) with respect to t . Next, (3.2) is immediate by (3.26a). The condition (3.5) is obviously satisfied since $S(t, b(t)) = S(t, b^*(t)) = 0$. Let us show the condition (3.4). For $Z \in (b^*(0), h]$ it is immediate by (2.11a) and (2.12). For $t > 0$ and $Z \in [b^*(t), b^*(0)]$, since $b^* \neq 0$ by (3.22a), we compute

$$\partial_Z u(t, Z) = - \int_{(b^*)^{-1}(Z)}^t \partial_Z S(\tau, Z) d\tau + S((b^*)^{-1}(Z), Z) \partial_Z ((b^*)^{-1}(Z)). \quad (3.29)$$

Denoting $t' = (b^*)^{-1}(Z)$, one has $S((b^*)^{-1}(Z), Z) = S(t', b^*(t')) = 0$. Therefore,

$$\partial_Z u(t, Z) = - \int_{t'}^t \partial_Z S(\tau, Z) d\tau \quad \text{for all } t > 0, Z \in [b^*(t), b^*(0)]. \quad (3.30)$$

Since we assumed $\partial_Z S < 0$, we get $\partial_Z u > 0$ if $t' < t$, i.e. $Z > b^*(t)$, and (3.4) holds. For $Z = b(t)$ one has $t' = t$, thus the latter equality provides the additional boundary condition (3.28) (it is also possible to use (3.23)). We can remark additionally that for $t > 0$, the limits of $\partial_Z u$ on both sides of $b^*(0) = b^0$ differ,

$$\partial_Z u(t, b^0+) - \partial_Z u(t, b^0-) = \partial_Z u^0(b^0) \geq 0. \quad (3.31)$$

Therefore, as long as $\partial_Z u^0(b^0) > 0$, the velocity u is not continuously differentiable at b^0 . \square

Illustration

In order to illustrate the configuration of Proposition 3.7, we consider linear initial data and source term in the form (3.15) and (3.16), for some b^* satisfying (3.22). We take for b^*

$$b^*(t) = \frac{b^0 + t^2}{1 + t^2}, \quad (3.32)$$

where $1 < b^0 < h$. Then $b = b^*(t)$, and the velocity is determined for all $t \in [0, T]$, by

$$u(t, Z) = (Z - 1) \left(t - \sqrt{\frac{b^0 - Z}{Z - 1}} \right) + (1 - b^0) \left(\arctan t - \arctan \sqrt{\frac{b^0 - Z}{Z - 1}} \right) \quad (3.33)$$

for all $Z \in [b(t), b^0]$,

$$u(t, Z) = (1 + t)Z - (b^0 + t) + (1 - b^0) \arctan t \quad \text{for all } Z \in [b^0, h]. \quad (3.34)$$

At a fixed time t , these expressions lead to nonlinear profiles in $[b(t), b^0]$ and linear profiles in $[b^0, h]$, as illustrated on Figure 6.

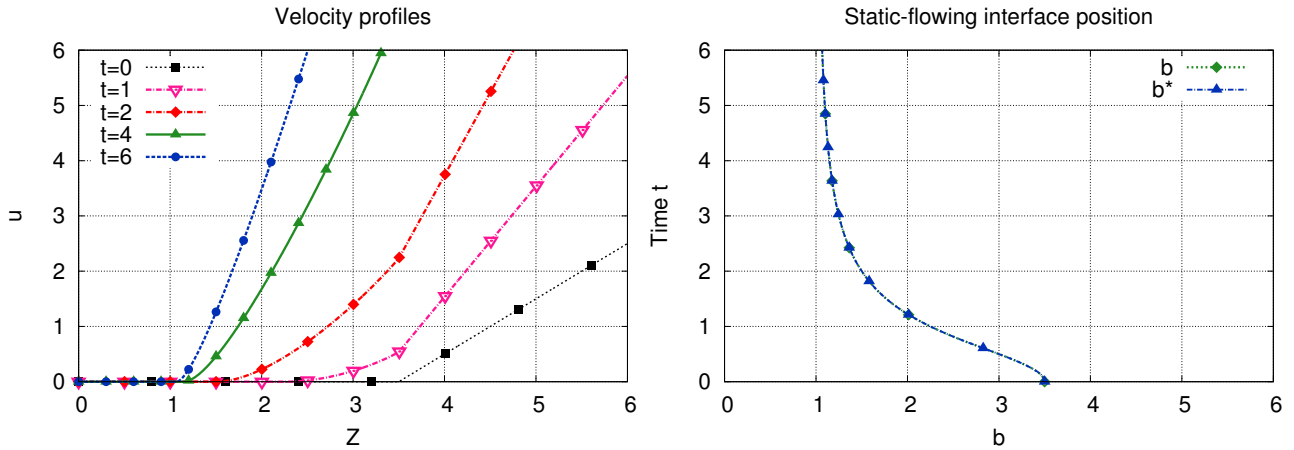


Figure 6: Illustration of the behaviour of the solution (u, b) in the case with decreasing zero b^* of the source and $b^0 = b^*(0)$. The source and initial velocity are linear, given by (3.16), (3.15), with $h = 6$, $b^0 = 3.5$, b^* given by (3.32). Velocity profiles u are represented at times $t = 0$, $t = 1$, $t = 2$, $t = 4$, $t = 6$ (left). The position $b(t)$ of the static/flowing interface with respect to time is shown (right). It coincides with the zero $b^*(t)$ of the source.

3.3 Source term with a decreasing zero b^* and $b^0 < b^*(0)$

In this subsection, we handle a more general case than previously in Subsection 3.2. We consider still a decreasing zero b^* of the source term, but starting from an initial value $b^*(0)$ larger than the initial position b^0 of the static/flowing interface. This configuration allows to model a change of variation in the static/flowing interface position, from the initial thickening of the static part of the flow to the progressive thinning of that part.

We consider a source term S with a unique zero $b^*(t)$ (i.e. (3.6)) that satisfies

$$b^* \in C^1([0, T]), \quad \dot{b}^*(t) < 0 \text{ for } t > 0, \quad (3.35a)$$

$$b^0 < b^*(0). \quad (3.35b)$$

We seek for a continuous function $b(t)$ with $b(0) = b^0$. Let us first state a general result following Lemma 3.5.

Lemma 3.8. *Assume (2.8), (2.12), (2.11), (3.6) and that (u, b) is a solution to (3.1), (3.2), (3.3), (3.4), (3.5). If $b(t) < b^*(t)$ for all $t \in I$ where I is a time interval not containing 0 where b is C^1 , then $\dot{b}(t) > 0$ for all $t \in I$.*

Proof. According to Lemma 3.1, one has $b(t) \leq b^*(t)$ for all $t > 0$. Thus the assumption $b(t) < b^*(t)$ for all $t \in I$ means that $b(t) \neq b^*(t)$ for $t \in I$. Arguing by contradiction, the result is immediate with Lemma 3.5. \square

In the context of (3.35) and for a continuous b with $b(0) = b^0 < b^*(0)$, one has necessarily $b(t) < b^*(t)$ for t small enough. Then there are two possible situations:

- (i) $b(t) < b^*(t)$ for all $t \in [0, T]$,
- (ii) there exists a time $t^* > 0$ such that $b(t^*) = b^*(t^*)$, and for all $t \in [0, t^*)$, $b(t) < b^*(t)$.

Case (i) means that there is no intersection between the graphs of b and b^* . In case (ii), there is an intersection at time t^* . Then, from $t \geq t^*$ we encounter the situation of Subsection 3.2. Herein, we focus on case (ii), that can be formalized as

$$\begin{cases} b(t) < b^*(t) & \text{for all } t \in [0, t^*), \\ b(t) = b^*(t) & \text{for all } t \geq t^*. \end{cases} \quad (3.36)$$

Therefore, the position b of the static/flowing interface is increasing over $[0, t^*]$, and is decreasing over $[t^*, T]$.

In what follows, we first build the velocity u under the assumption (3.36), then we determine the position $b(t)$ of the static/flowing interface.

Lemma 3.9. *Assume (2.8), (2.12), (2.11), (3.6), (3.35), and that (u, b) solves (3.1), (3.2), (3.3), (3.4), (3.5) with b continuous in $[0, T]$, piecewise C^1 with $b(0) = b^0$. We suppose that there exists $t^* \in (0, T)$ such that (3.36) holds. Then, for all $t \geq t^*$, the velocity is given by*

$$u(t, Z) = - \int_{(b^*)^{-1}(Z)}^t S(\tau, Z) d\tau \quad \text{for all } Z \in [b^*(t), b^*(t^*)], \quad (3.37a)$$

$$u(t, Z) = u^0(Z) - \int_0^t S(\tau, Z) d\tau \quad \text{for all } Z \in [b^*(t^*), h]. \quad (3.37b)$$

For all $t \in [0, t^*]$, the velocity is given by

$$u(t, Z) = u^0(Z) - \int_0^t S(\tau, Z) d\tau \quad \text{for all } Z \in [b(t), h]. \quad (3.38)$$

Proof. The geometrical setting is represented on Figure 7. The domain $Z \geq b(t)$ where the velocity u is defined is filled with oblique lines. It is split in three sets A, B, C corresponding to (3.37a), (3.37b), (3.38) respectively. If $t \leq t^*$ then $b(t)$ is increasing, therefore the velocity is given by Lemma 3.2, which gives (3.38). If $t \geq t^*$, we are in the configuration of Subsection 3.2. Applying Lemma 3.6 we deduce the formula (3.37a), and

$$u(t, Z) = u(t^*, Z) - \int_{t^*}^t S(\tau, Z) d\tau \quad \text{for all } Z \in [b^*(t^*), h]. \quad (3.39)$$

But according to (3.38) applied for $t = t^*$, we have

$$u(t^*, Z) = u^0(Z) - \int_0^{t^*} S(\tau, Z) d\tau \quad \text{for all } Z \in [b^*(t^*), h]. \quad (3.40)$$

Replacing the value of $u(t^*, Z)$ given by (3.40) in (3.39) gives (3.37b). \square

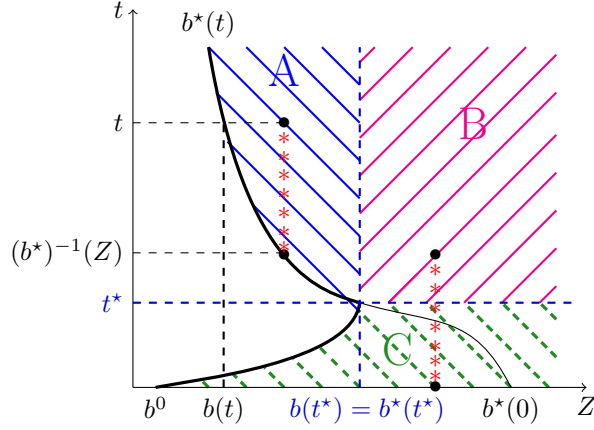


Figure 7: Position b (thick dark) of the static/flowing interface with respect to time t , starting from an initial position $b^0 < b^*(0)$. The flowing domain where (3.1) holds is filled with oblique lines. It is split in three sets $A : t \geq t^*, b^*(t) \leq Z \leq b^*(t^*)$, $B : t \geq t^*, b^*(t^*) \leq Z \leq h$, $C : t \leq t^*, b(t) \leq Z \leq h$. The path of integration is represented with stars.

The determination of the solution (u, b) is completed through the following result giving the position $b(t)$ of the static/flowing interface for $t \leq t^*$, and the crossing time t^* . The position b of the static/flowing interface is determined by the boundary condition $u(t, b(t)) = 0$, and as in Lemma 3.3, we extend the formula (3.38) by defining \tilde{u} , for $Z \in [b^0, h]$ and $t \in [0, T]$ by (3.11).

Lemma 3.10. *Assume (2.8), (2.12), (2.11), (3.6), (3.35). Then there exists a maximal time $t^* > 0$ such that for all $t \in (0, t^*)$, the extended velocity \tilde{u} defined by (3.11) has a unique zero $b(t)$ in Z , such that $b^0 < b(t) < b^*(t)$. Moreover, b is continuous and increasing in $[0, t^*)$ with $b(0) = b^0$. Finally, if $t^* < T$ then $b(t^*) = b^*(t^*)$.*

Proof. We denote by $Prop(t)$ the following property: there exists a unique zero $b(t)$ of $Z \mapsto \tilde{u}(t, Z)$, such that $b^0 < b(t) < b^*(t)$. We consider the set

$$I = \{t^+ > 0 \text{ such that } \forall t \in (0, t^+), Prop(t) \text{ holds}\}. \quad (3.41)$$

The set I is convex since for any $t_1^+ \in I$, $t_2^+ \in (0, t_1^+) \Rightarrow t_2^+ \in I$. Thus I is an interval, and one of the three situations must occur:

- (a) $I = \emptyset$,
- (b) $I = (0, t^*]$ for some $t^* \in (0, T)$,
- (c) $I = (0, T]$.

In order to prove the first statement of the lemma, we have to establish that (a) cannot happen, i.e. that $I \neq \emptyset$. In other words we need to prove that for $t > 0$ small enough, $\tilde{u}(t, \cdot)$ admits a unique zero $b(t)$ in $(b^0, b^*(t))$. The uniqueness follows from the property $\partial_Z \tilde{u} > 0$, which is obtained with (2.11a) and (2.12). In order to get existence, we first notice that according to (3.35), for t small enough one has $b^*(t) > b^0$. Then, let us prove that for t small enough $\tilde{u}(t, b^0) < 0$, and $\tilde{u}(t, b^*(t)) > 0$. We have by the definition (3.11) and by (2.11b) that $\tilde{u}(t, b^0) = -\int_0^t S(\tau, b^0) d\tau$. Thus, by (3.6) we deduce that $S(\tau, b^0) > 0$, and that $\tilde{u}(t, b^0) < 0$. Next, since S is continuous by (2.8), there exists a constant $C_1 \geq 0$ such that $S(\tau, Z) \leq C_1$ for all $\tau \in [0, T]$ and all $Z \in [0, h]$. This leads to $\tilde{u}(t, b^*(t)) \geq u^0(b^*(t)) - C_1 t$. This expression tends to $u^0(b^*(0))$ as $t \rightarrow 0$. Since by (3.35b) and (2.11) this limit is positive, we deduce that for t small enough we have $\tilde{u}(t, b^*(t)) > 0$, concluding the existence of a zero $b(t)$ in the interval $(b^0, b^*(t))$ and the proof that $I \neq \emptyset$.

Then, for all $t \in (0, t^*)$ one has $\tilde{u}(t, b(t)) = 0$ with \tilde{u} continuous, thus b is continuous in $(0, t^*)$. Moreover, defining $b(0) = b^0$, with the same argument b is continuous in $[0, t^*)$. Since \tilde{u} is continuously differentiable with respect to (t, Z) with $\partial_Z \tilde{u} > 0$, one gets that b is differentiable with $\dot{b}(t) = -\partial_t \tilde{u}(t, b(t)) / \partial_Z \tilde{u}(t, b(t))$. Since $\partial_t \tilde{u} = -S$, we get $\dot{b}(t) > 0$ according to (3.6) and the property $b(t) < b^*(t)$.

It remains to prove the last statement concerning the case $t^* < T$. We are going to prove that in this case we have $\tilde{u}(t^*, b^*(t^*)) = 0$, giving the claim. Indeed, the assumption $t^* < T$ yields by definition of t^* that there exists a sequence $t_n \in (t^*, T)$ such that $t_n \rightarrow t^*$ and not satisfying $Prop(t_n)$, which means that $\tilde{u}(t_n, \cdot)$ has no zero in $(b^0, b^*(t_n))$ (recall that uniqueness always holds because $\partial_Z \tilde{u} > 0$).

Since we have proved that b is increasing in $(0, t^*)$, we have $b(t) > b^0$ for all $t \in (0, t^*)$, and $b(t) \rightarrow \ell$ as $t \rightarrow t^*$, for some $\ell > b^0$. Moreover, as $b(t) < b^*(t)$ for all $t \in (0, t^*)$, we obtain $\ell \leq b^*(t^*)$. Thus we obtain $b^0 < b^*(t^*)$. The continuity of b^* at t^* implies that there exists a time $\tilde{t} \in (t^*, T)$ such that

$$b^*(t) \geq \frac{b^*(t^*) + b^0}{2} \quad \text{for all } t \in [t^*, \tilde{t}]. \quad (3.42)$$

Since $t_n \rightarrow t^*$, there exists \tilde{n} such that $t_n \leq \tilde{t}$ for all $n \geq \tilde{n}$. Therefore, by (3.42) this gives

$$b^*(t_n) \geq \frac{b^*(t^*) + b^0}{2} > b^0 \quad \text{for all } n \geq \tilde{n}. \quad (3.43)$$

By the decrease of b^* (3.35a), we deduce that $b^*(\tau) > b^*(t_n) > b^0$ for all $\tau \in (0, t_n)$. By (3.6) we deduce that $S(\tau, b^0) > 0$, and that $\tilde{u}(t_n, b^0) = -\int_0^{t_n} S(\tau, b^0) d\tau < 0$. As $Prop(t_n)$ is false, there is no zero of $\tilde{u}(t_n, \cdot)$ in $(b^0, b^*(t_n))$, hence we conclude that $\tilde{u}(t_n, b^*(t_n)) \leq 0$. By passing to the limit $n \rightarrow \infty$, we get $\tilde{u}(t^*, b^*(t^*)) \leq 0$. Moreover, taking a sequence $t'_n < t^*$ such that $t'_n \rightarrow t^*$, we have that $Prop(t'_n)$ is true, and we infer that $\tilde{u}(t'_n, b^*(t'_n)) > 0$. Passing to the limit $n \rightarrow \infty$, this leads to $\tilde{u}(t^*, b^*(t^*)) \geq 0$. Therefore, we conclude that $\tilde{u}(t^*, b^*(t^*)) = 0$, and this ends the proof of the lemma. \square

Proposition 3.11. *Assume (2.8), (2.12), (2.11), (3.6), (3.35) and that $\partial_Z S < 0$. Then there is a unique solution (u, b) to (3.1), (3.2), (3.3), (3.4), (3.5) with b continuous in $[0, T]$, piecewise C^1 with $b(0) = b^0$. Moreover, for all $t \in [0, t^*]$, $u(t, Z)$ is given by (3.38), and $b(t)$ is the unique value in $[b^0, b^*(t)]$ such that this formula (3.38) gives $u(t, b(t)) = 0$. The interface position b is increasing in $[0, t^*]$. For larger times, one of the two following cases occurs*

- (i) $b(t) < b^*(t)$ for all $t \in [0, T]$, and $t^* = T$,
- (ii) $b(t)$ reaches $b^*(t)$ at the time t^* . Then $\dot{b}(t^* - 0) = 0$, and for all $t \in [t^*, T]$ one has $b(t) = b^*(t)$ (thus b is decreasing in $[t^*, T]$) with u given by (3.37).

Proof. The result follows from Lemmas 3.9, 3.10 and Proposition 3.7. Only the property $\dot{b}(t^* - 0) = 0$ has not been stated. For $t < t^*$ one has $\dot{b}(t) = S(t, b(t)) / \partial_Z \tilde{u}(t, b(t))$. When $t \rightarrow t^*$ we have $S(t, b(t)) \rightarrow S(t^*, b^*(t^*)) = 0$, which yields the result. \square

Illustration

In order to illustrate the proposition, we consider a linear initial velocity in the form (3.15) and a linear source term in the form (3.16). We set

$$b^*(t) = \frac{h + b^0 t^2}{1 + t^2}, \quad (3.44)$$

which is strictly decreasing from the height h of the domain to the initial position b^0 of the static/flowing interface. Thus it satisfies (3.35). By Proposition 3.11, the velocity profiles

are determined as follows. For all $t \in [0, t^*]$,

$$\begin{aligned} u(t, Z) &= (1+t)(Z - b^0) - (h - b^0) \arctan t \quad \text{for all } Z \in [b(t), h], \\ b(t) &= b^0 + (h - b^0) \frac{\arctan t}{1+t}, \end{aligned} \quad (3.45)$$

and for all $t \in [t^*, T]$, $b(t) = b^*(t)$ and

$$\begin{aligned} u(t, Z) &= (Z - b^0) \left(t - \sqrt{\frac{h-Z}{Z-b^0}} \right) - (h - b^0) \left(\arctan t - \arctan \sqrt{\frac{h-Z}{Z-b^0}} \right) \\ &\quad \text{for all } Z \in [b^*(t), b^*(t^*)], \\ u(t, Z) &= (1+t)(Z - b^0) - (h - b^0) \arctan t \quad \text{for all } Z \in [b^*(t^*), h]. \end{aligned} \quad (3.46)$$

As in Subsection 3.2, for $t \geq t^*$ we encounter nonlinear velocity profiles in $[b(t), b^*(t^*)]$, and linear velocity profiles in $[b^*(t^*), h]$. The configuration is illustrated on Figure 8.

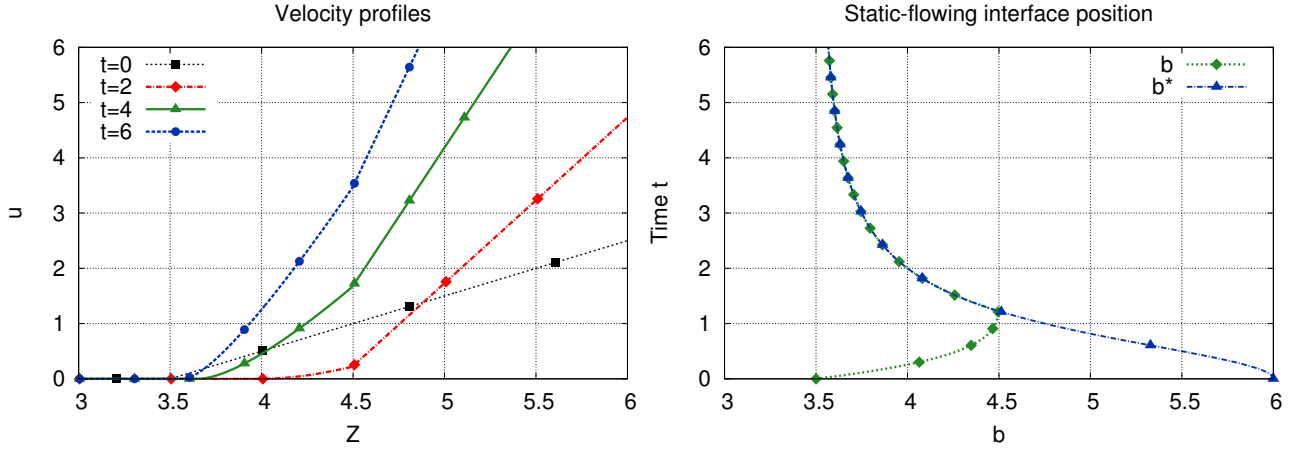


Figure 8: Illustration of the behaviour of the solution (u, b) in the case with decreasing zero b^* of the source and $b^0 < b^*(0)$. The source and initial velocity are linear, given by (3.16), (3.15), with $h = 6$, $b^0 = 3.5$, b^* given by (3.44) (thus $b^*(0) = h$). We have $t^* \simeq 1.23$ and $b^*(t^*) \simeq 4.5$. Velocity profiles u are represented at times $t = 0, t = 2, t = 4, t = 6$ (left). The position $b(t)$ of the static/flowing interface with respect to time is shown, together with the zero $b^*(t)$ of the source (right).

3.4 Source term with an increasing zero b^* and $b^0 > b^*(0)$

In this subsection we show how the static/flowing interface position $b(t)$ can be discontinuous with respect to time. We assume that the zero $b^*(t)$ of the source term is increasing, starting from an initial value lower than the initial position of the static/flowing interface. This configuration allows to model a sudden starting of a part of the initially static material, as well as a progressive stopping of the flow.

Thereafter, we consider a source term S with a unique zero $b^*(t)$ (i.e. (3.6)) that satisfies

$$b^*(t) \text{ is increasing,} \quad (3.47a)$$

$$b^0 > b^*(0). \quad (3.47b)$$

We recall that according to Lemma 3.1, we are looking for an interface position b that satisfies

$$b(t) \leq b^*(t) \quad \text{for all } t \in (0, T]. \quad (3.48)$$

Together with (3.47b) and since b^* is continuous, this condition implies that $b(t)$ cannot tend to b^0 as $t \rightarrow 0$, we have necessarily $b(0^+) \equiv \lim_{t \rightarrow 0^+} b(t) \leq b^*(0) < b^0$. We are going to take the minimal possible jump. Thus we consider an interface position b that satisfies

$$b(0^+) = b^*(0). \quad (3.49)$$

For consistency with this configuration, the initial condition (3.3) has to be modified to

$$u(t, Z) \xrightarrow[t \rightarrow 0^+]{} u^0(Z) \quad \text{for all } Z \in (b(0^+), h), \quad (3.50)$$

where the initial velocity u^0 , that was defined for $Z \in [b^0, h]$, is extended on $[0, b^0]$ by setting

$$u^0(Z) = 0 \quad \text{for all } Z \in [0, b^0]. \quad (3.51)$$

Lemma 3.12. *Assume (2.8), (2.12), (2.11), and that u solves (3.1), (3.50), (3.51) for some continuous nondecreasing $b(t)$ satisfying $b(0^+) < b^0$. Then, for all $t \in [0, T]$ and for all $Z \in [b(t), h]$, one has*

$$u(t, Z) = u^0(Z) - \int_0^t S(\tau, Z) d\tau. \quad (3.52)$$

Proof. It is identical as that of Lemma 3.2. The domain of integration is represented on Figure 9. \square

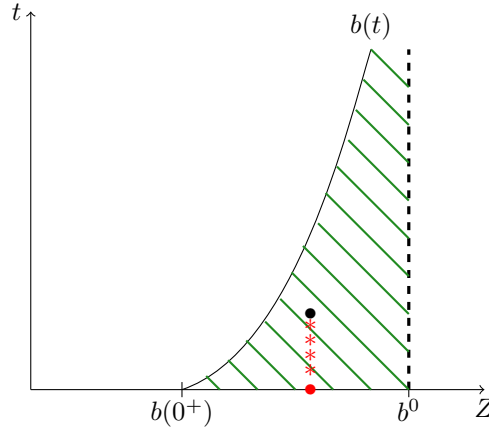


Figure 9: Position b of the static/flowing interface with respect to time t , starting from $b(0^+) < b^0$. The domain where (3.1) holds is filled with oblique lines. The integration path for getting the velocity is represented with stars.

Then we define again the extended velocity \tilde{u} for $Z \in [0, h]$ by the formula (3.52).

Lemma 3.13. *Assume (2.8), (2.12), (2.11), (3.6), (3.47), and $\partial_Z S < 0$. Then for all $t \in (0, T]$ there exists a unique $b(t) \in [b^*(0), h]$ such that $\tilde{u}(t, b(t)) = 0$. Moreover,*

- 1) $b(t) < b^*(t)$ for all $t \in (0, T]$,
- 2) $b(t)$ is continuous, increasing, and $b(0^+) = b^*(0)$.

We skip the proof since it is almost identical to that of Lemma 3.3. Using Lemmas 3.12 and 3.13 we conclude the following proposition.

Proposition 3.14. *Assume (2.8), (2.12), (2.11), (3.6), (3.47), and $\partial_Z S < 0$. Then there exists a unique solution (u, b) to (3.1), (3.2), (3.50), (3.51), (3.4), (3.5), such that the position $b(t)$ of the static/flowing interface is nondecreasing, continuous and $b(0^+) = b^*(0)$.*

Moreover, the velocity u is given by (3.52), and the position $b(t)$ of the static/flowing interface is determined by the equation

$$u^0(b(t)) = \int_0^t S(\tau, b(t)) d\tau, \quad (3.53)$$

and verifies $b(t) < b^*(t)$ for all $t \in (0, T]$.

Illustration

We consider linear data of the form (3.15) and (3.16), with b^* continuous satisfying (3.47). The initial velocity is extended over $[0, h]$ by (3.51). We take

$$b^*(t) = \frac{b^*(0) + ht^2}{1 + t^2}, \quad (3.54)$$

with $0 < b^*(0) < b^0$. Then according to Proposition 3.14, there is a unique solution (u, b) to our problem.

We first determine the velocity for all $t \in [0, T]$ such that $b(t) \geq b^0$. By (3.52), in this situation we have

$$u(t, Z) = -b^0 + (1 + t)Z - ht + (h - b^*(0)) \arctan t \quad \text{for all } Z \in [b(t), h]. \quad (3.55)$$

By writing the zero of this expression, we get

$$b(t) = \frac{b^0 + ht - (h - b^*(0)) \arctan t}{1 + t}. \quad (3.56)$$

This formula is valid if it satisfies $b(t) \geq b^0$, which can be rewritten

$$\frac{\arctan t}{t} \leq \frac{h - b^0}{h - b^*(0)}. \quad (3.57)$$

Then, for all $t \in (0, T]$ such that (3.57) is not satisfied (which corresponds to small times), we have to look for $b(t)$ such that $b^*(0) < b(t) < b^0$. By Lemma 3.12, the velocity is then given by

$$u(t, Z) = -b^0 + (1 + t)Z - ht + (h - b^*(0)) \arctan t \quad \text{for all } Z \in [b^0, h], \quad (3.58a)$$

$$u(t, Z) = Zt - ht + (h - b^*(0)) \arctan t \quad \text{for all } Z \in [b(t), b^0]. \quad (3.58b)$$

We deduce the value of $b(t)$ by getting the zero of (3.58b),

$$b(t) = h - (h - b^*(0)) \frac{\arctan t}{t}. \quad (3.59)$$

The velocity profiles are linear for all $t \in [0, T]$ such that $b(t) \geq b^0$. They are piecewise linear for all t such that $b^*(0) < b(t) < b^0$. The velocity and interface position are represented on Figure 10. The position $b(t)$ of the static/flowing interface jumps from b^0 to $b^*(0)$ instantaneously. We also notice that $b(t)$ increases to $b^*(t)$. This illustrates the monitoring effect of b^* on b .

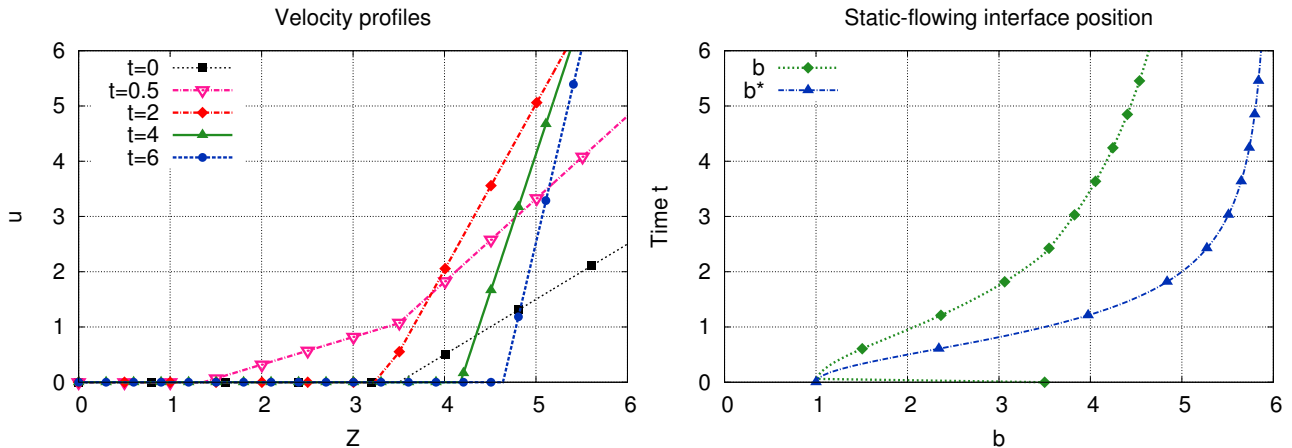


Figure 10: Illustration of the behaviour of the solution (u, b) in the case with increasing zero b^* of the source and $b^0 > b^*(0)$. The source and initial velocity are linear, given by (3.16), (3.15), with $h = 6$, $b^0 = 3.5$, $b^*(0) = 1$, b^* given by (3.54). The switching time for which $b(t) = b^0$ is $t \simeq 2.33$. Velocity profiles u are represented at times $t = 0$, $t = 0.5$, $t = 2$, $t = 4$, $t = 6$ (left). The position $b(t)$ of the static/flowing interface with respect to time is shown, together with the zero $b^*(t)$ of the source (right).

4 Conclusion and overview

We have studied a model for non-averaged thin-layer flows that has been derived in [9] from an incompressible viscoplastic model with Drucker–Prager yield stress describing granular materials with static/flowing transition. Following [23], this model has been simplified in a one-dimensional setting, keeping the normal to the flow variable. In [23] we considered the case of constant source and we provided a comparison between analytical results and laboratory experiments in the inviscid case and between numerical and experimental results in the viscous case, showing that the simplified model was able to predict physically relevant dynamics. Here we have performed an analytical study of the inviscid case for a non-constant source term. In this inviscid context, we provided the solution explicitly, namely the velocity u and the position b of the static/flowing interface. We identified some important properties of the solution, proving the ability of the simplified model to represent different types of interface dynamics, exhibiting a richer structure than for constant source term.

The model involves a source term that can represent gravity, nonlinear friction and non-hydrostatic effects. In our study, the source term is variable in time and space. We have shown that the shape of the source term, and especially its zero in space and its time dependence, is a key factor in order to determine the evolution of the interface. We have managed to exhibit different dynamics of the flow, from the starting to the arrest, including a progressive starting, a progressive stopping and a sudden starting. These behaviours represent relevant situations for real flows, although further studies are needed to evaluate the results in comparison to real data.

An overview of these features is as follows. At first, the zero $b^*(t)$ of the source is supposed to be nondecreasing with respect to time, with an additional assumption on the initial data that $b^0 < b^*(0)$. In this configuration, the position $b(t)$ of the static/flowing interface increases until reaching the zero $b^*(t)$. Physically, this means that the static part of the flow thickens until reaching $b^*(t)$. If $b^*(t)$ remains smaller than the height h of the domain, this prevents the flow from complete stopping. On the contrary, if $b^*(t)$ reaches the height of the domain h , then there is a progressive and complete stopping.

In a second configuration, we have assumed that $b^*(t)$ is decreasing, with the additional assumption on the initial data that $b^0 = b^*(0)$. In this case the position of the static/flowing

interface follows the zero, i.e. $b(t) = b^*(t)$ for all times, and the velocity profiles have a nonlinear part and a linear one. As b^* decreases, b decreases likewise. Therefore, the fluid part of the flow thickens. If the zero $b^*(t)$ of the source term tends to 0, then the position $b(t)$ of the static/flowing interface tends to 0 as well, meaning that the whole material flows. Therefore, if $b^*(0) = h$, this models a flow that progressively starts until becoming fully flowing.

Next, we have considered a decreasing zero $b^*(t)$ as previously, but with initial data satisfying $b^0 < b^*(0)$. In this configuration, we obtain that the position $b(t)$ of the static/flowing interface starts to increase (with $b(t) < b^*(t)$), until it reaches the zero $b^*(t)$ at some time t^* . Then we are in the situation of the previous case, and for larger times $b(t)$ remains equal to $b^*(t)$. This models a progressive stopping, followed by a progressing starting of the granular material. Therefore, in this configuration we have a change in the sense of variation of the position of the static/flowing interface.

Finally, we have assumed that the zero $b^*(t)$ of the source term is increasing, with the initial data satisfying $b^0 > b^*(0)$. In this case, $b(t)$ jumps instantaneously to the value $b^*(0)$, and then increases with $b(t) < b^*(t)$. This particular case involves a static/flowing interface position which is discontinuous with respect to time. This models a sudden starting of a part of the mass.

All these configurations show the following behaviour of the static/flowing interface position $b(t)$. First, the zero $b^*(t)$ of the source plays the role of a barrier, since as stated in Lemma 3.1, we must have $b(t) \leq b^*(t)$. Thus it can prevent the flow from a complete stopping. Second, the interface position $b(t)$ always “tries to follow” the zero $b^*(t)$. If $b^*(t)$ increases then $b(t)$ increases also, while remaining lower than $b^*(t)$, and reaching $b^*(t)$ in large time. If $b^*(t)$ decreases, then at some time t^* one has $b(t^*) = b^*(t^*)$ and then $b(t) = b^*(t)$ for all $t \geq t^*$.

These dynamical properties of the static/flowing interface should be interpreted in the context of the coupling of the source term with the velocity dependency in the flow-aligned variable, as derived in the full thin-layer model of [9] recalled in Subsection 2.1. This will be the object of forthcoming studies.

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