

# Supplementary material: Generation of atomic entanglement by cavity-feedback in the strong coupling regime

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## I. ABOUT THE PURITY CHANGE $PC$

### A. $PC$ as an entanglement criterion

Here we prove, that if the state is separable, then its purity change  $PC$  defined as

$$PC \equiv \text{Tr}_{1,2,\dots,N}[\rho^2] - \text{Tr}_{2,\dots,N}[(\text{Tr}_1 \rho)^2] \quad (1)$$

has to be or zero or negative. We remind that the mixed state  $\rho_{\text{sep}}$  is separable, if and only if it can be written in the form

$$\rho_{\text{sep}} = \sum_k c_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \dots \otimes \rho_k^{(N)}, \quad (2)$$

where  $\sum_k c_k = 1$  and  $\forall_i c_i \geq 0$ . Using the decomposition (2) it is easy to compute formally square of the density matrix

$$(\rho_{\text{sep}})^2 = \sum_{k,l} c_k c_l \left( \rho_k^{(1)} \rho_l^{(1)} \right) \otimes \left( \rho_k^{(2)} \rho_l^{(2)} \right) \otimes \dots \otimes \left( \rho_k^{(N)} \rho_l^{(N)} \right) \quad (3)$$

and the purity

$$\text{Tr} \left\{ (\rho_{\text{sep}})^2 \right\} = \sum_{k,l} c_k c_l \prod_{m=1}^N \text{Tr} \left\{ \rho_k^{(m)} \rho_l^{(m)} \right\}. \quad (4)$$

In order to compute the purity change  $PC$  we need to trace over a single atom. Without losing generality we trace out the Hilbert space of the "first" atom:

$$\text{Tr}_1 \{ \rho_{\text{sep}} \} = \sum_k c_k \rho_k^{(2)} \otimes \dots \otimes \rho_k^{(N)}. \quad (5)$$

Applying Eqs. (3) and (4), for the full and for the reduced density matrix, we compute PC:

$$PC = \sum_{k,l} c_k c_l \left( \text{Tr} \left\{ \rho_k^{(1)} \rho_l^{(1)} \right\} - 1 \right) \prod_{m=2}^N \text{Tr} \left\{ \rho_i^{(m)} \rho_j^{(m)} \right\} \quad (6)$$

For any matrices  $A$  and  $B$ , the matrix  $(A - B)^2$  is semi-positive, hence it has nonnegative trace. This leads to inequality  $\text{Tr} \{AB\} \leq (\text{Tr} \{A^2\} + \text{Tr} \{B^2\}) / 2$ . This inequality leads to the conclusion  $\text{Tr} \left\{ \rho_k^{(1)} \rho_l^{(1)} \right\} - 1 \leq 0$  for any  $\rho_k^{(1)}$  and  $\rho_l^{(1)}$ . On the other hand the product of two density matrices has to be positive-definite, thus  $\text{Tr} \left\{ \rho_k^{(m)} \rho_l^{(m)} \right\}$  is larger than zero. We conclude that each term under the sum (6) has to be non-positive, so is the whole sum.

Finally, if the state has positive purity change,  $PC > 0$ , this state has to be non-separable.

### B. PC in $\chi \hat{S}_z^2$ model

As an example, here we compute  $PC$  as a function of time for an initial coherent spin state (CSS):

$$|\psi(t=0)\rangle = \frac{1}{2^{N/2}} \sum_{n=0}^N \sqrt{\binom{N}{n}} |n, N-n\rangle_{\text{F}} \quad (7)$$

where  $|n, N-n\rangle_{\text{F}}$  is a bosonic Fock state, with  $n$  atoms in state  $|1\rangle$  and the rest of them occupying state  $|0\rangle$ , evolving according to the Hamiltonian  $H/\hbar = \chi \hat{S}_z^2$ . Notably, the unitary part of the evolution studied in the main part of the paper, in some regimes reduces to this model. We remind, that the initial state CSS, in the case without any dissipation, would evolve first to the squeezed states, then after time  $\pi/(2\chi\sqrt{N})$  to highly entangled states including macroscopic superpositions of few entangled states, eventually at time  $\pi/(2\chi)$  it will reach the cat state (see the same effects for photons [1])

At the time  $t$  the evolved state reads

$$|\psi(t)\rangle = \frac{1}{2^{N/2}} \sum_{n=0}^N \sqrt{\binom{N}{n}} e^{-i\chi(n-N/2)^2 t} |n, N-n\rangle_{\text{F}}. \quad (8)$$

As the state is pure, its purity is constant in time and equal to 1,  $\text{Tr} \{\rho^2\} = 1$ . In order to compute  $PC$  one has to find the reduced density matrix  $\text{Tr}_1 \{ |\psi(t)\rangle \langle \psi(t)| \}$ . This forces us to calculate  $\langle 0|n, N-n\rangle_{\text{F}}$  and  $\langle 1|n, N-n\rangle_{\text{F}}$  (where  $\langle 0|$  and  $\langle 1|$  represents the internal state of the

"first" atom). To accomplish this task, we come back from the Fock basis to the more general one:

$$\begin{aligned}
\langle 1|n, N-n\rangle_F &= \langle 1| \frac{1}{\sqrt{\binom{N}{n}}} \sum_{a_1+a_2+\dots+a_N=n} |a_1, a_2, \dots, a_N\rangle \\
&= \frac{1}{\sqrt{\binom{N}{n}}} \sum_{a_2+\dots+a_N=n-1} |a_2, \dots, a_N\rangle \\
&= \sqrt{\frac{n}{N}} |n-1, N-n\rangle_F.
\end{aligned} \tag{9}$$

Similarly  $\langle 0|n, N-n\rangle_F = \sqrt{\frac{N-n}{N}} |n, N-n-1\rangle_F$ . The reduced density matrix reads:

$$\rho_1 = \frac{1}{2} |\psi_0\rangle \langle \psi_0| + \frac{1}{2} |\psi_1\rangle \langle \psi_1|, \tag{10}$$

where

$$|\psi_0(t)\rangle = \sqrt{2} \langle 0|\psi(t)\rangle = \sum_{n=0}^{N-1} \sqrt{\frac{\binom{N-1}{n}}{2^{N-1}}} e^{-i\chi(n-N/2)^2 t} |n, N-1-n\rangle_F \tag{11}$$

and

$$|\psi_1(t)\rangle = \sqrt{2} \langle 1|\psi(t)\rangle = \sum_{n=0}^{N-1} \sqrt{\frac{\binom{N-1}{n}}{2^{N-1}}} e^{-i\chi(n+1-N/2)^2 t} |n, N-1-n\rangle_F \tag{12}$$

The purity of the state (10) is

$$\text{Tr} \{\rho_1^2\} = \frac{1}{2} \left( 1 + |\langle \psi_0|\psi_1\rangle|^2 \right) = \frac{1}{2} \left( 1 + (\cos \chi t)^{2(N-1)} \right), \tag{13}$$

which leads us to the following  $PC$

$$PC = \frac{1}{2} [1 - (\cos \chi t)^{2(N-1)}] \tag{14}$$

The formula manifests different entangled states appearing in the evolution, mentioned before. Note that although  $PC$  reaches its maximal value  $1/2$  exactly at "cat" time, it stays close to  $1/2$  in a much wider range of times detecting entanglement [2].

### C. Properties and comparison with other criteria and entanglement measures

Essentially  $PC$  is used to qualify the bipartition entanglement, where one part is just a single atom (qbit), and the second part consists of the other  $N-1$  atoms. Up to prefactors,  $PC$  coincides in special cases with quantities sensitive to quantum correlations already discussed in the literature.

In case of two qubits, the concurrence is defined as

$$\text{Concurrence} \equiv \sqrt{2(1 - \text{Tr} \rho_{\text{red}}^2)}. \tag{15}$$

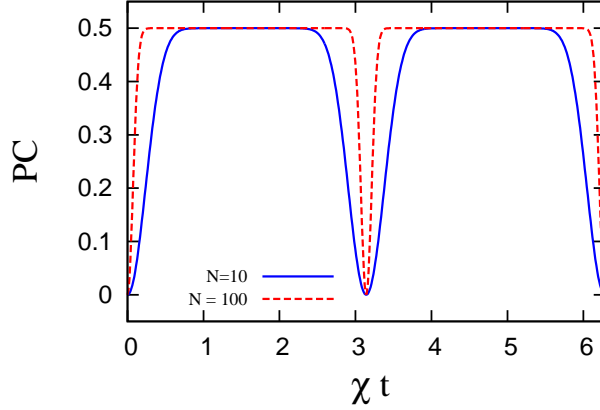


FIG. 1: Purity change  $PC$  as a function of time. The initial coherent spin state is evolving under the Hamiltonian  $H = \chi \hat{S}_z^2$ .

so that in this case

$$PC = \frac{(\text{Concurrence})^2}{2}. \quad (16)$$

$PC$  is also a particular form of the conditional entropy  $I$  that is the difference between the entropy of the total density matrix and of the reduced density matrix [3]:

$$I \equiv S(\rho_{AB}) - S(\text{Tr}_B \{\rho_{AB}\}). \quad (17)$$

When evaluated for the linear entropy  $S_{\text{lin}} = 1 - \text{Tr} \{\rho^2\}$  the conditional entropy  $I$  gives back  $PC$

$$I_{\text{lin}} = -PC \quad (18)$$

The conditional entropy with linearized entropy was already used, and benchmarked against different versions of conditional entropy in [4].

## II. ENTANGLED STATES IN THE STRONG COUPLING REGIME

The entangled states that we find in the strong coupling regime are due to the joint effect of cavity losses and atom-photon entanglement. To understand their nature, it is convenient to project the system density matrix onto the symmetric subspace spanned by Fock states  $|m\rangle \equiv |\frac{N}{2} + m, \frac{N}{2} - m\rangle_F$ , eigenstates of  $S_z$  with eigenvalue  $m \in [-N/2, N/2]$ .

$$\rho_{\text{sym}} = \sum_{m, m'} \langle m | \rho | m' \rangle |m\rangle \langle m'| \quad (19)$$

$$\langle m | \rho | m' \rangle = \sum_{\epsilon_1, \epsilon_2} \langle m | \epsilon_1 \rangle \langle \epsilon_1 | \rho | \epsilon_2 \rangle \langle \epsilon_2 | m' \rangle \quad (20)$$

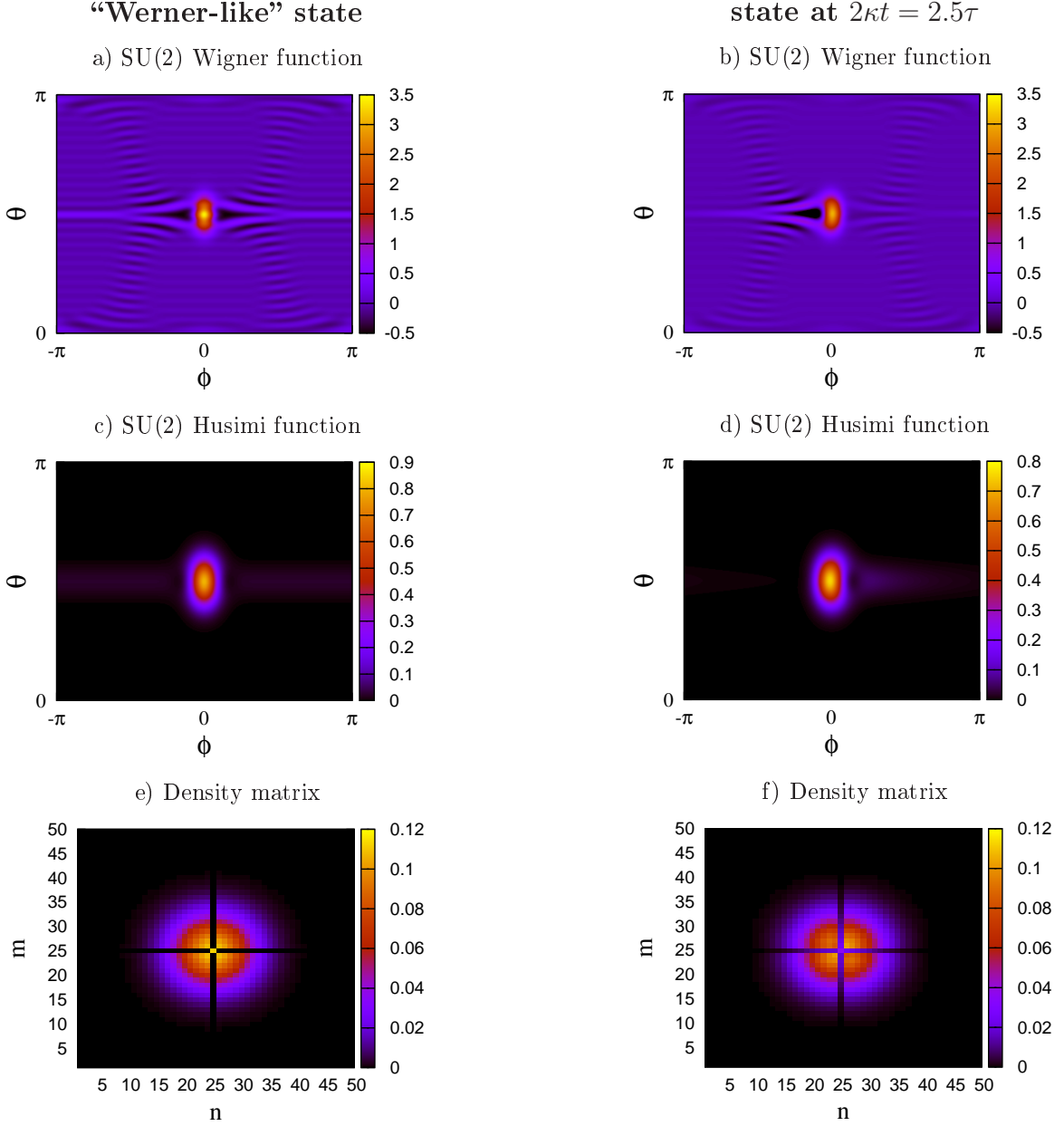


FIG. 2: Left column: “Werner-like” state (23), Right column: state obtained in our system at  $2\kappa t = 2.5t_0$ , where we restricted to symmetrized subspace. The weight of the symmetrized subspace given by the trace of  $\rho_{\text{sym}}$  (19) is 0.91 at this time. From top to bottom, we show the Wigner function, the Husimi function, and the modulus of the density matrix elements in the Fock basis

The trace of  $\rho_{\text{sym}}$  tells us about the importance of the symmetric subspace in our state.

Let us consider for simplicity the case without spontaneous emission:  $\Gamma = 0$ . From combinatory

arguments and using Eq (5) of the main text, one derive

$$\begin{aligned} \langle m|\rho|m'\rangle_{\Gamma=0} &= p_m p_{m'} \langle \alpha(m)|\alpha(m')\rangle^{1+2\kappa t} \\ &\times e^{i\kappa t[|\alpha(m)|^2(\delta/\kappa+\phi_0 m)-|\alpha(m')|^2(\delta/\kappa+\phi_0 m')]} \end{aligned} \quad (21)$$

where  $p_m = \sqrt{\frac{1}{2^N} \binom{N}{m+\frac{N}{2}}}$  is the probability of finding the Fock state  $|m\rangle$  in the initial superposition and  $\text{Tr}[\rho_{\text{sym}}] = 1$ , as the system remains in the symmetric state. We now evaluate the overlap  $\langle \alpha(m)|\alpha(m')\rangle_{m \neq m'}^{1+2\kappa t}$  (first line in (21)) for large  $\phi_0$ . Using the formula  $|\langle \alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$ , and the dependence of the cavity coherent state  $\alpha(m)$  upon the atomic state with  $\delta = \kappa \simeq \kappa_{\text{eff}}$

$$\alpha(m) = \frac{\eta/\kappa}{1 + i(1 + \phi_0 m)} \quad (22)$$

that we expand for  $\phi_0 m \gg 1$  if  $m \neq 0$ , one sees that  $\langle \alpha(m)|\alpha(m')\rangle_{m \neq m'}^{1+2\kappa t}$  tends to zero if  $mm' = 0$  on a timescale  $t_0 = 2\kappa/\eta^2$  while if  $mm' \neq 0$  it stays close to one for a long time scale  $(\kappa/\eta^2)\phi_0^2(mm'/(m'-m))^2 \geq t_1 = (\kappa/\eta^2)\phi_0^2/4$ .

In a timescale  $t_0$ , the initial coherent state  $|\psi(t=0)\rangle = \sqrt{p_0}|m=0\rangle + \sqrt{1-p_0}|\psi^\perp\rangle$  is thus mapped to the mixture of a twin Fock state  $|m=0\rangle$  and the state  $|\psi^\perp\rangle$ :

$$|\psi(t=0)\rangle \mapsto p_0 |m=0\rangle \langle m=0| + (1-p_0) |\psi^\perp\rangle \langle \psi^\perp| \quad (23)$$

that survives for a time  $t_1 \gg t_0$ . We will refer here to the state (23) as to the "Werner-like" state. A comparison between the "Werner-like" state and the state that we actually produce in our system, with spontaneous emission is shown in Fig.2 and we note that the main features are indeed the same. Furthermore, the purity of the state obtained in the simulation (with spontaneous emission) nicely matches the analytical value  $p_0^2 + (1-p_0)^2$  obtained for the state (23). It is also straightforward to trace out a single atom from the "Werner-like" state and to compute purity of the resulting reduced density matrix

$$\text{Tr}_{2,\dots,N}[(\text{Tr}_1 \rho_{\text{Werner-like}})^2] = \frac{1}{2} p_0^2 + p_0 q_0 + \frac{1}{2} (1-q_0)^2 + \frac{1}{2} (1-2q_0)^2, \quad (24)$$

where  $q_0 = \frac{1}{2^{N-1}} \binom{N-1}{N/2}$ . Finally the purity change of "Werner-like" state is:

$$PC = -\frac{1}{2} p_0^2 + p_0 q_0 + \frac{1}{2} (1-q_0)^2 + \frac{1}{2} (1-2q_0)^2 - (1-p_0)^2. \quad (25)$$

We checked for  $N = 50$  that the state appearing in evolution at the time  $t$  has approximately this purity change. An example of such comparison is given also for  $N = 10$  in Fig. 3 of the main text.

### A. Fisher Information of "Werner-like" state

The Fisher Information of any mixed state is defined via the covariance matrix:

$$\gamma_{ij}[\rho] = \frac{1}{2} \sum_{l \neq m} C_{ij} [|m\rangle, |l\rangle], \quad (26)$$

where

$$C_{ij} [|m\rangle, |l\rangle] = \frac{(\lambda_l - \lambda_m)^2}{\lambda_l + \lambda_m} \Re \{ \langle l | S_i | m \rangle \langle m | S_j | l \rangle \} \quad (27)$$

and  $|l\rangle, |m\rangle$  are eigenvectors of the density matrix  $\rho$  with eigenvalues  $\lambda_l$  and  $\lambda_m$ , respectively.  $S_i$  with  $i = x, y, z$  are angular momentum operators. The optimal Fisher information  $I_F$  of a mixed state is equal to the largest eigenvalue of the matrix  $\gamma$ , multiplied by 4:

$$I_F = 4 \max \{ \lambda \in \mathcal{R}_+ : \gamma |\lambda\rangle = \lambda |\lambda\rangle \}. \quad (28)$$

To find the explicit form of the matrix (26) one has to find the eigenvectors and eigenvalues of the "Werner-like" state. Two eigenvectors can be read directly from Eq. (23):

- the twin Fock state  $|m = 0\rangle$  with eigenvalue  $p_0$
- $|\psi^\perp\rangle$  with eigenvalue  $(1 - p_0)$ .

The other eigenstates span the  $N - 1$  dimensional space, orthogonal to both  $|\psi^\perp\rangle$  and  $|N/2\rangle$ . Each of these eigenvectors have the eigenvalue equal to 0.

We checked that the matrix  $\gamma$  is diagonal. These diagonal terms are equal to

$$\gamma_{xx} = N^2 p_0 \left( \frac{3}{4} - p_0 - \frac{1}{4(1 - p_0)} \right) + \frac{1}{4} p_0 (N + 1), \quad (29)$$

$$\gamma_{yy} = \frac{1}{8} N^2 p_0 - \frac{3}{8} N p_0 + \frac{1}{2} N, \quad (30)$$

$$\gamma_{zz} = \frac{N}{4}. \quad (31)$$

In the main text we compute the Fisher information of the Werner-like state directly from (28) and from the formulas giving the diagonal terms of the covariance matrix (29)-(31).

In the limit  $N \gg 1$ , we approximate  $p_0$  with  $\frac{1}{\sqrt{2\pi N}}$ . It turns out that in this limit the largest eigenvalue of the covariance matrix  $C_{ij}$  is  $\gamma_{xx}$ . This leads to the following scaling of the Fisher Information

$$I_F \stackrel{N \gg 1}{\approx} 2 \sqrt{\frac{2}{\pi}} N^{3/2}. \quad (32)$$

### B. Role of spontaneous emission

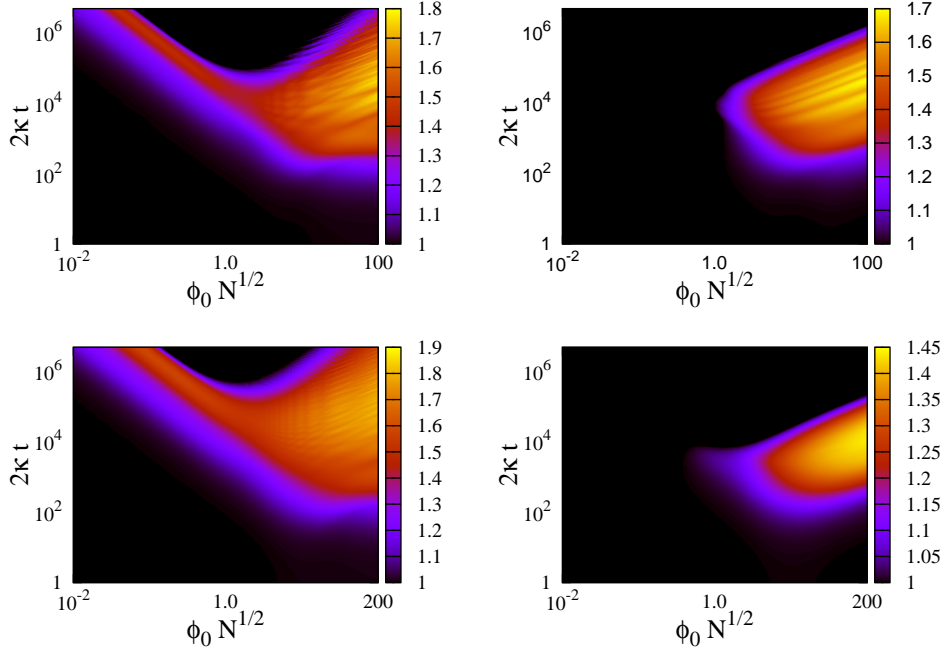


FIG. 3: Fisher information  $I_F^{sym}$  of the density matrix projected onto symmetric subspace  $\rho_{sym}$ . The colored Z axis is  $\log(I_F)/\log N$  (which is equal to 2 for a cat state and equal to 1 for a coherent state). All values smaller than 1 are projected to 1.

Top: 50 atoms, bottom: 200 atoms. The first column : no spontaneous emission emission ( $\Gamma = 0$ ), the second column : Rubidium like spontaneous emission ( $\Delta/\Gamma = 500$ ).

In the presence of spontaneous emission, the state leaves the space spanned by the Fock basis (so the "symmetric subspace"), where the "Werner-like" states lives. It is not easy to verify to which extent the state produced during evolution matches the "Werner-like" state in case of spontaneous emission and large number of atoms.

To understand better the role of spontaneous emission at least for mesoscopic systems, we compared the Fisher Information of the symmetric part of the density matrix, denoted as  $I_F^{sym}$  in two cases: evolution with and without spontaneous emission. In case without spontaneous emission the Fisher Information of the symmetric subspace,  $I_F^{sym}$  is equal to the total Fisher Information  $I_F$ . However if the spontaneous emission is present, then we have  $I_F^{sym} < I_F$ . In Figure 3 we show landscape of the Fisher Information (a counterpart of Fig. (2) from the main text) for these two cases and for two number of atoms:  $N = 50$  and  $N = 200$ . The Fisher Information in the strong coupling regime corresponding to large values of  $\phi_0\sqrt{N}$  is affected, but not completely changed.



This suggests that in this regime the state will remain in the symmetric subspace and it will remain entangled. We see also the typical feature of entangled states: the more atoms in the sample, the more sensitive is the state to decoherence. On the other hand, the Fisher Information in the region  $\phi_0\sqrt{N} \ll 1$  is completely damped by the spontaneous emission. The reason is that the state leaves the symmetric subspace, and in this region one has to analyze the whole density matrix, as it is done in the main part of the paper.

### III. QUANTUM AVERAGES FOR A GENERALIZED $\hat{S}_z^2$ MODEL WITH DECOHERENCE

We consider an effective Hamiltonian of the form

$$H_{\text{eff}}/\hbar = \chi \hat{S}_z^2 - \frac{i}{2} \sum_{\alpha \in \text{jumps}} d_\alpha^\dagger d_\alpha \quad (33)$$

with the jump operators

$$d_c = \sqrt{\Lambda_C} \hat{S}_z \quad (34)$$

$$d_{i\text{el}} = \sqrt{\Lambda_{\text{Ray}}/2} |\sigma\rangle\langle\sigma|_i; \quad \sigma = 0, 1 \quad (35)$$

$$d_{i\text{Ram}} = \sqrt{\Lambda_{\text{Ram}}} |\sigma\rangle\langle\sigma'|_i; \quad \sigma \neq \sigma'; \quad \sigma, \sigma' = 0, 1 \quad (36)$$

$$d_{iX} = \sqrt{\Lambda_X} |X\rangle\langle\sigma|_i; \quad \sigma = 0, 1; \quad X \neq 0, 1 \quad (37)$$

The quantum averages can be obtained analytically :

$$\begin{aligned} \langle S_x \rangle &= \frac{N}{2} e^{-\Lambda_C t/2} e^{-(\Lambda_X + \Lambda_{\text{Ram}} + \Lambda_{\text{Ray}})t} (H_1(t))^{N-1} \\ \langle S_z^2 \rangle &= \frac{N}{4} e^{-\Lambda_X t} \\ \langle \hat{n}_0 \hat{n}_1 \rangle &= \frac{N(N-1)}{4} e^{-2\Lambda_X t} \\ \langle S_y^2 \rangle &= \frac{N}{4} e^{-\Lambda_X t} + \frac{(N-1)N}{8} e^{-2(\Lambda_X + \Lambda_{\text{Ray}} + \Lambda_{\text{Ram}})t} \left(1 - e^{-2\Lambda_C t} (H_2(t))^{N-2}\right) \\ 2\text{Re} \langle S_z S_y \rangle &= \frac{N(N-1)}{2} e^{-\Lambda_C t/2} e^{-(2\Lambda_X + \Lambda_{\text{Ray}} + 2\Lambda_{\text{Ram}})t} G(t) (H_1(t))^{N-2} \end{aligned}$$

where, by introducing  $\beta = 1, 2$  we have:

$$H_\beta(t) = \frac{\beta^2 \chi^2 e^{-(\Lambda_{\text{Ram}} + \Lambda_X)t} \left( \cosh(u_\beta t) + \frac{\Lambda_{\text{Ram}} + \Lambda_X}{u_\beta} \sinh(u_\beta t) \right) + \Lambda_X (\Lambda_X + 2\Lambda_{\text{Ram}})}{\beta^2 \chi^2 + \Lambda_X (\Lambda_X + 2\Lambda_{\text{Ram}})} \quad (38)$$

$$G(t) = \frac{\chi \sinh(u_1 t)}{u_1} \quad (39)$$

$$u_\beta = \sqrt{\Lambda_{\text{Ram}}^2 - \beta^2 \chi^2} \quad (40)$$

Starting from these exact formulas for quantum averages, in the limit of large number of atoms and short evolution times and in the case of Rayleigh and Raman scatterings only (namely if  $\Lambda_X = 0$ ), we recover the approximate results given in Eqns. (20) - (24) in the paper [6].

### A. Recovering an effective $\hat{S}_z^2$

**Adiabatic elimination of the cavity field.** The field within the cavity falls onto a stationary state on a timescale  $1/\kappa_{\text{eff}}$ . As a consequence, for longer times, at which squeezing arises, we can treat the cavity field as a constant, but with the value depending on the state of atoms. Hence, it is reasonable to eliminate the cavity field from the description by performing adiabatic elimination. In practice it amounts to replace in the initial master equation the operators  $\hat{c}$  and  $\hat{c}^\dagger$  by  $\alpha(\hat{S}_z)$  and  $\alpha^*(\hat{S}_z)$  where  $\alpha(\hat{S}_z)$  is the amplitude of the stationary coherent state in the cavity for a given  $\hat{S}_z$

$$\alpha(\hat{S}_z) = \frac{\eta}{\kappa_{\text{eff}} + i(\delta + \kappa\phi_0\hat{S}_z)} \quad \text{with} \quad \kappa_{\text{eff}} = \kappa \left( 1 + \frac{(N_0 + N_1)\phi_0\Gamma}{4\Delta} \sum_{L=\text{Ray,Ram,X}} a_L \right)^{N \ll \frac{\Delta}{\Gamma\phi_0}} \simeq \kappa, \quad (41)$$

where we have introduced  $a_{\text{Ray}} = 2a_{\sigma,\sigma}$ ,  $a_{\text{Ram}} = |a_{\sigma',\sigma}|^2/a_{\sigma,\sigma}$  and  $a_X = \sum_{X \neq 0,1} |a_{X\sigma}|^2/a_{\sigma\sigma}$  and  $a_{\sigma\sigma'}$  are defined in the last section of this supplementary material. Here  $N_\sigma$  is the number of atoms in the state  $|\sigma\rangle$ . Then the Hamiltonian, Eq. (1) of the main text, takes the form:

$$H/\hbar = - \left| \alpha(\hat{S}_z) \right|^2 \left( \delta + \kappa\phi_0\hat{S}_z \right) \quad (42)$$

whereas the jumps, Eq. (2) of the main text, are transformed into

$$d_{i,\text{el}} = \sqrt{\frac{\Gamma_{\text{Ray}}}{2}} (|1\rangle\langle 1| - |0\rangle\langle 0|)_i \alpha(\hat{S}_z) \quad (43)$$

$$d_{i,\sigma\sigma'} = \sqrt{\Gamma_{\text{Ram}}} |\sigma\rangle\langle\sigma'| \alpha(\hat{S}_z) \quad (44)$$

$$d_{i,X\sigma} = \sqrt{\Gamma_X} |X\rangle\langle\sigma'| \alpha(\hat{S}_z) \quad (45)$$

$$d_c = \sqrt{2\kappa} \alpha(\hat{S}_z) \quad (46)$$

where rates  $\Gamma_{\text{Ray}}, \Gamma_{\text{Ram}}, \Gamma_X$  are defined in the main text.

This elimination is valid as long as the entanglement between internal degrees of freedom of atoms and photons is relatively small. More precisely, in the calculation of atomic observables for squeezing, that involve only one or two body coherences, the adiabatic elimination amounts to replacing by one the overlaps  $\langle\alpha(m)|\alpha(m \pm 1)\rangle$  and  $\langle\alpha(m)|\alpha(m \pm 2)\rangle$ , which is correct as long as  $\phi_0 \ll 1$ .

**Expansions with respect to small parameter.** The main idea for further simplifying the master equation relies on the Taylor expansion of  $\alpha$  in powers of  $\epsilon \hat{S}_z$ , once we identify a small parameter  $\epsilon$ . To this aim we introduce the critical atom number, Eq. (12) in the main text

$$N_c \equiv \left( \frac{4\Delta/\Gamma}{a_{\text{Ray}} + a_{\text{Ram}} + a_X} \right)^2. \quad (47)$$

$$1. \text{ Case : } N \ll N_c \text{ and } \phi_0 \sqrt{N} \ll 1$$

Both conditions  $N \ll N_c$  and  $\phi_0 \sqrt{N} \ll 1$  together guaranties that the broadening of the cavity line width due to spontaneous emission is negligible, namely

$$\frac{\kappa_{\text{eff}}}{\kappa} - 1 \approx N \frac{\phi_0}{4} \frac{\Gamma}{\Delta} < \frac{1}{4} \sqrt{\frac{N}{N_c}} \ll 1 \quad (48)$$

The dispersion of  $\hat{S}_z$  operator is equal to  $\sqrt{N}/2$ , hence if  $\phi_0 \sqrt{N} \ll 1$  ( which is the second condition characterizing this regime) then the quantity  $\phi_0 \hat{S}_z$  is small and can be used as the small parameter in the expansion.

The Taylor expansion of the master equation lead us to

$$H_{\text{eff}}/\hbar = v \hat{S}_z + \chi \hat{S}_z^2 - \frac{i}{2} \sum_{\alpha \in \text{jumps}} d_\alpha^\dagger d_\alpha \quad (49)$$

which has the form of the general Hamiltonian (33) of section III, plus a drift term, with the jump operators (37) where now the rates are given by

$$v = \frac{|\eta|^2 \phi_0 \kappa}{\delta^2 + \kappa^2} \quad \chi = -\frac{|\eta|^2 \phi_0^2 \delta \kappa^2}{(\delta^2 + \kappa^2)^2} \quad (50)$$

$$\Lambda_C = \frac{2\kappa}{\delta} |\chi| \quad \Lambda_{\text{Ram}}/a_{\text{Ram}} = \Lambda_{\text{Ray}}/a_{\text{Ray}} = \Lambda_X/a_X = \frac{\kappa \phi_0}{4} \frac{\Gamma}{\Delta} \frac{|\eta|^2}{\delta^2 + \kappa^2} \propto \frac{\chi}{\phi_0} \quad (51)$$

For  $\delta = \kappa$  we finally obtain

$$\chi = -\frac{1}{4} \eta^2 \phi_0^2; \quad \Lambda_C = 2|\chi|; \quad (52)$$

$$\Lambda_{\text{Ram}}/a_{\text{Ram}} = \Lambda_{\text{Ray}}/a_{\text{Ray}} = \Lambda_X/a_X = \frac{|\chi|}{2\phi_0} \frac{\Gamma}{\Delta} \equiv \frac{|\chi|}{4C}, \quad (53)$$

where we introduced the cooperativity  $C = g^2/\kappa\Gamma = \phi_0\Delta/(2\Gamma) \approx \phi_0\sqrt{N}\sqrt{N_c/N}$ . From (53) the ratio  $\Gamma/\chi$  between the spontaneous emission and squeezing rates scales as  $1/C$ . This explains the fact that spontaneous emission deteriorate the squeezing in the weak coupling regime as one can see in Fig. 4 of the main text. On the other hand one can show that if  $N^{-1/10} \ll \phi_0 \sqrt{N} \ll 1$  then

the squeezing is limited mostly by cavity losses, which plays here larger role than scattering into free space. The best squeezing and the time at which this squeezing is reached are given by:

$$\xi_{\min}^2 = \frac{5}{6} (3)^{4/5} N^{-2/5}; \quad (54)$$

$$t_{\min} = \frac{2}{\eta^2 \phi_0^2} (3)^{1/5} N^{-3/5}. \quad (55)$$

## 2. Case $N \gg N_c$ , $\phi_0 \sqrt{N} \gg 1$

Due to the assumption  $N \gg N_c$  the rate of photon losses in the empty cavity,  $\kappa$ , is much smaller than the rate of scattering a single photons on some of atoms.

$$\frac{\kappa_{\text{eff}}}{\kappa} = 1 + N \frac{\phi_0}{4} \frac{\Gamma}{\Delta} > \sqrt{\frac{N}{N_c}} > 1 \quad (56)$$

We again expand the coherent state amplitude in a Taylor expansion,  $\frac{\Delta}{N\Gamma} \sqrt{N} = \sqrt{\frac{N_c}{N}}$  being the small parameter. When repeating the steps from the previous part we fall back again into the Hamiltonian (33) of section III, plus a drift term. In case of Rayleigh jumps, the resulting quantum averages are well described by the model (33) with rates:

$$\chi = \frac{\eta^2}{2} \left( \frac{\kappa \phi_0}{\kappa_{\text{eff}}} \right)^3; \quad \Lambda_C = 2\chi; \quad \Lambda_{\text{Ray}} = \frac{\chi}{4} \left( \frac{\Gamma}{\Delta} a_{\text{Ray}} \right)^2 N \quad (57)$$

In the limit  $N \rightarrow \infty$  the squeezing converges to a small constant

$$\xi_{\text{opt}}^2 = e \left( \frac{\Gamma a_{\text{Ray}}}{\Delta} \right)^2 \quad \chi t_{\text{best}} = 2 \left( \frac{\Delta}{\Gamma a_{\text{Ray}}} \right)^2 \frac{1}{N} \quad (58)$$

In the more general case, assuming laser-cavity detuning equal to  $\delta = \lambda \kappa_{\text{eff}}$  one obtains:

$$\xi^2 = \frac{1 + \lambda^2}{2\lambda^2} \left( \frac{\Gamma}{\Delta} a_{\text{Ray}} \right)^2 e \quad \chi t_{\text{best}} = \frac{4\lambda}{1 + \lambda^2} \left( \frac{\Delta}{\Gamma a_{\text{Ray}}} \right)^2 \frac{1}{N} \quad (59)$$

## IV. CALCULATION OF THE EFFECTIVE JUMPS RATES

The  $a_{\sigma'\sigma}$  coefficients entering the expressions of the jump operators modeling the effect of spontaneous emission are calculated for the  $^{87}\text{Rb}$  D2 line as follows [5]:

$$a_{\sigma'\sigma} = \Delta \left| \sum_{e,q'} \frac{\langle \sigma' | d_{q'} | e \rangle \langle e | d_q | \sigma \rangle}{\omega_p - \omega_{\sigma,e}} \right|. \quad (60)$$

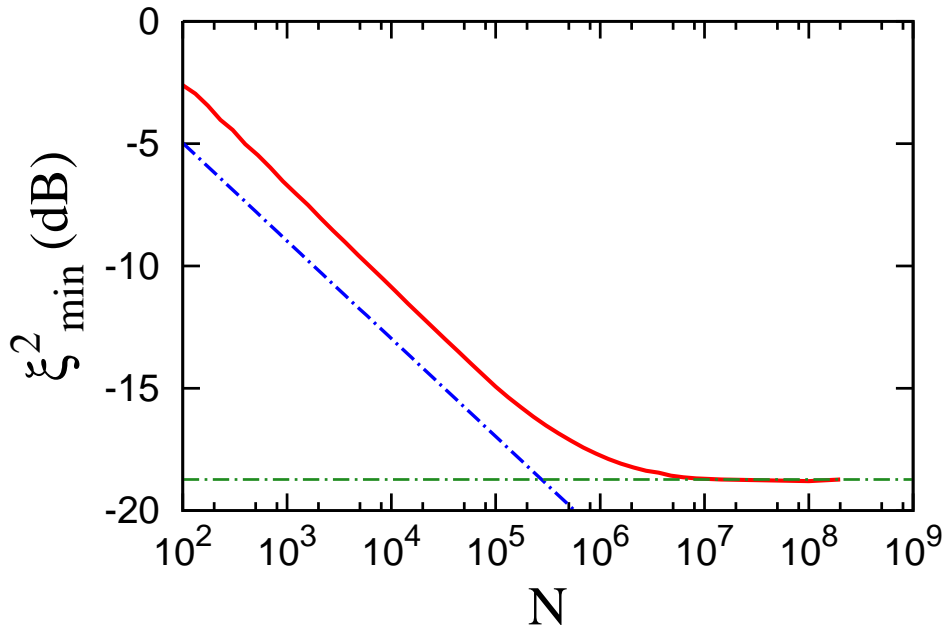


FIG. 4: Spin squeezing optimized over time and  $\phi_0$  as a function of total number of atoms, for  $\Delta/\Gamma = 10$  and  $(\eta/\kappa)^2 = 10^{-2}$ . Solid line: full model, Eq. (5) of the main part of the paper, with cavity losses and Rayleigh scattering with  $a_{\text{Ray}} = 0.702$ . Dot-dashed blue line: analytical results (55) in the regime of small number of atoms  $N \ll N_c$  and small  $\phi_0\sqrt{N} \ll 1$  described in the subsection III A 1. Horizontal dot-dashed green line: analytical results (58) in the regime with large number of atoms  $N \gg N_c$  and large  $\phi_0\sqrt{N} \gg 1$  described in the subsection III A 2.

The electric dipole moment matrix element  $\langle \sigma | d_q | e \rangle$  is normalized to one for the closed transition  $|F = 2, m_F = 2\rangle \rightarrow |F_e = 3, m_e = 3\rangle$ ,  $q$  is the laser polarization,  $\omega_p$  the laser frequency and  $\omega_{\sigma,e}$  the atomic transition frequency from  $|\sigma\rangle$  to  $|e\rangle$ . The sum goes over all excited states  $|e\rangle$  of the  $5P_{3/2}$  manifold and all polarizations  $q' = -1, 0, 1$ .

The lightshifts for the  $|\sigma\rangle$  and  $|\sigma'\rangle$  states are equal and opposite when

$$\left| \sum_e \frac{|\langle e | d_q | \sigma \rangle|^2}{\omega_p - \omega_{\sigma,e}} \right| = \left| \sum_e \frac{|\langle e | d_q | \sigma' \rangle|^2}{\omega_p - \omega_{\sigma',e}} \right|. \quad (61)$$

We find the laser frequency that fulfills this condition for  $|\sigma\rangle = |F = 1, m_F = 0\rangle$  and  $|\sigma'\rangle = |F = 2, m_F = 0\rangle$  and  $q = 0$ . We then calculate the  $a_{\sigma'\sigma}$  from (60).

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