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Abstract

Specific kinematic assumptions are often adopted in structural analysis of civil engineering structures in order to simplify the global equilibrium equations and reduce the required number of degrees of freedom. The classical Timoshenko beam hypothesis, considering that plane sections remain plane after deformation but not necessary normal to the beam axis, is often chosen because it can (approximately) take into account the influence of shear strains. On the contrary, the Euler-Bernoulli assumption (sections remain plane and perpendicular the beam axis) neglects their influence and provides therefore accurate results only for the case of slender beam structures. This work is focused on the Timoshenko beam theory in the context of a multi-fiber approach: The section is considered as multi-fiber, it can have an arbitrary shape and each fiber has a local constitutive law representing a specific material. Various formulations of displacement based multi-fiber straight Timoshenko beam finite elements are re-visited. After a presentation of the shape functions leading to the stiffness matrices and the consistent nodal forces relative to each formulation, comparisons are made using elastic or elastic perfectly plastic constitutive laws. The advantages and disadvantages of each formulation are highlighted and general conclusions on the use of displacement based Timoshenko multi-fiber beams in engineering are drown.

Keywords: Timoshenko; Multi-fiber; beam.

Introduction

Different kinematic assumptions are used in structural analysis in order to simplify the global equilibrium equations and to reduce the required number of degree of freedom. The Timoshenko beam hypothesis, considers that plane sections remain plane after deformation but not necessary normal to the beam axis. The advantage of this theory is that it can take into account the influence of shear strains contrary to Euler-Bernoulli assumption

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which announces that sections remains plane and perpendicular to the beam axis after deformation and which neglects the influence of shear strains and provides accurate results only for the case of slender beam structures.

One of the first manuscripts introducing the idea of dividing a beam section in "fibers", where a specific stress/strain relation is defined, is the book of Owen and Hilton [1]. This type of finite element has been proven very efficient for various applications in civil engineering: the nonlinear analysis of beam type or bearing wall structures with non homogeneous sections (eg. reinforced concrete) [2],[3],[4],[5], arbitrarily geometrical plane or hollow shape sections [6],[7] submitted to flexion, shear or torsion [8], Soil Structure Interaction problems [6], vulnerability assessment cases [7] and Fiber-Reinforced Polymer retrofitting [9]...

In this article we review the performance of three (3) displacement based straight Timoshenko beam finite elements under the small rotations assumption (force based beam elements - see for example Spacone & al. [10] - are out of the scope of this article). The formulations differ on the shape functions interpolating the displacements and rotations fields inside each element. The first formulation, called hereafter **FLI** (Full-Linear-Independent), adopts linear polynomials while the transverse displacement and rotation fields are interpolated independently, see Pegon [11] and Guedes & al. [12]. Because of the low degree of the adopted polynomials, this formulation is prone to shear locking problems (see Stolarski and Belytschko [13], De Ville de Goyet [14], Crisfield [15], Ibrahimbegovic and Frey [16]) if specific measures are not considered. A way to avoid this is to calculate shear strains approximately, as proposed by Donea and Lamain [17] and adopted in [11] and [12] and commented hereafter. In the second formulation, referred hereafter as **FCQM** (Full-Cubic-Quadratic-Material) and proposed by Friedman and Kosmatka [18], the degree of the shape functions is of order three (3) for the transverse displacements and two (2) for the rotations. Displacements and rotations are now interdependent and the shear locking problem is avoided. The particularity of this formulation is that the shape functions depend on the properties of the materials, and thus its use for non linear problems can be problematic. The third formulation, called hereafter **FCQ** (Full-Cubic-Quadratic) and proposed by Caillerie & al. [19] uses shape functions of order three (3) for the transverse displacements and two (2) for the rotations and an additional internal node. This results to a finite element free of shear locking and to shape functions independent on the properties of the materials. Further bibliography on different Timoshenko beam finite elements can be found in [20],[21] and [22]. For more details about the comparison between these Finite Element formulations, we refer the reader to Bitar and al. [23].

The organization of the article is as follows: in the first section, the general form of the stiffness matrices within a multi-fiber beam is presented. In the second section, the specific equations for the **FLI**, **FCQM** and **FCQ** formulations are provided. The performance of each formulation is studied in the following section for the case of an elastic or elasto-plastic behavior under monotonic loading. The article ends with general conclusions and guidelines.

1. Timoshenko multi-fiber beams

Consider a beam of length Ł discretized into n elements $e = [x_i; x_j]$ of length $L = x_j - x_i$ and external nodes i and j. The generalized displacement vector is approximated by an equation of the form $U = NU_e$, where U_e is a vector containing the external nodal displacements of the element e and N is the matrix of the shape functions depending on x.

The theoretical formulation, based on equilibrium equations an kinematic assumption and passing by the virtual work principle, provides the beam element stiffness matrix and the internal nodal forces expressions as the following

$$K_{element} = \int_0^L B^T K_S B dx \quad F_{int,element} = \int_0^L B^T F_S dx \tag{1}$$

where B is a matrix containing the derivatives with respect to x of the shape functions, K_S is the multi-fiber section stiffness matrix and F_S is the generalized force vector of the section.

$$\underbrace{\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix}}_{F_S} = \underbrace{\begin{bmatrix} \int_S E_f dS & 0 & -\int_S E_f y dS \\ 0 & \int_S kG_f dS & 0 \\ -\int_S E_f y dS & 0 & \int_S E_f y^2 dS \end{bmatrix}}_{K_S} \cdot \begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix}, \tag{2}$$

where $U_x(x)$ being the longitudinal displacement, $U_y(x)$ the transverse displacement and $\Theta_z(x)$ the rotation of the section. E_f and G_f the Young and Shear modulus respectively and k the shear correction factor.

2. FLI, FCQM and FCQ formulations

We present hereafter the shape functions for the **FLI**, **FCQM** and **FCQ** formulations.

*2.1. **FLI** formulation*

Linear independent shape functions are adopted for the generalized displacement fields [11],[12]:

The adopted generalized strain field has the following form:

$$\begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{2} & 0 & \frac{1}{L} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{L} & 0 & 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} U_{xi} \\ U_{yi} \\ \Theta_{zi} \\ U_{xj} \\ U_{yj} \\ \Theta_{zj} \end{bmatrix}, \tag{3}$$

where, following the proposal of Donea and Lamain [17], the expression of β_y is modified by eliminating the linear terms in the shape functions in order to avoid the shear locking problem.

2.2. ***FCQM*** *formulation*

Cubic and quadratic interdependent polynomials, functions of the material properties, are used for the transverse and rotational displacements and the formulation is free of shear locking [18]. The generalized strain field is:

$$\begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix} = \begin{bmatrix} N'_1 & 0 & 0 & N'_4 & 0 & 0 \\ 0 & N'_8 - N_{14} & N'_9 - N_{15} & 0 & N'_{11} - N_{17} & N'_{12} - N_{18} \\ 0 & N'_{14} & N'_{15} & 0 & N'_{17} & N'_{18} \end{bmatrix} \begin{bmatrix} U_{xi} \\ U_{yi} \\ \Theta_{zi} \\ U_{xj} \\ U_{yj} \\ \Theta_{zj} \end{bmatrix}. \tag{4}$$

and the shape functions:

$$\begin{cases} N_1 = 1 - \frac{x}{L} & N_4 = \frac{x}{L} \\ N_8 = \frac{1}{1+\phi}[2(\frac{x}{L})^3 - 3(\frac{x}{L})^2 - \phi(\frac{x}{L}) + 1 + \phi] & N_9 = \frac{L}{1+\phi}[(\frac{x}{L})^3 - (2 + \frac{\phi}{2})(\frac{x}{L})^2 + (1 + \frac{\phi}{2})(\frac{x}{L})] \\ N_{11} = -\frac{1}{1+\phi}[2(\frac{x}{L})^3 - 3(\frac{x}{L})^2 - \phi(\frac{x}{L})] & N_{12} = \frac{L}{1+\phi}[(\frac{x}{L})^3 - (1 - \frac{\phi}{2})(\frac{x}{L})^2 - \frac{\phi}{2}(\frac{x}{L})] \\ N_{14} = \frac{6}{(1+\phi)L}[(\frac{x}{L})^2 - (\frac{x}{L})] & N_{15} = \frac{1}{1+\phi}[3(\frac{x}{L})^2 - (4 + \phi)(\frac{x}{L}) + 1 + \phi] \\ N_{17} = -\frac{6}{(1+\phi)L}[(\frac{x}{L})^2 - (\frac{x}{L})] & N_{18} = \frac{1}{1+\phi}[3(\frac{x}{L})^2 - (2 - \phi)(\frac{x}{L})] \end{cases} \tag{5}$$

with ϕ the ratio of the beam bending to shear stiffness provided by (ν the Poisson's ratio):

$$\phi = \frac{12}{L^2}(\frac{EI}{kGA}) = \frac{24}{L^2}(\frac{I}{kA})(1 + \nu). \tag{6}$$

2.3. ***FCQ*** *formulation*

Cubic functions are used to interpolate the transverse displacements and quadratic for the rotations. The element is free of shear locking and uses an additional internal node [19]. Caillerie & al. [19] proved also that the **FCQM** formulation can be derived from the present formulation and that one **FCQ** element is able to predict the exact tip displacements for any complex loading (shear/flexion) submitted to an homogeneous elastic beam.

The nodal displacement field takes the following form:

$$U_e = \begin{bmatrix} U_{xi} & U_{yi} & \Theta_{zi} & \Delta U^1_{yi} & \Delta\Theta_i & \Delta U^2_{yi} & U_{xj} & U_{yj} & \Theta_{zj} \end{bmatrix}^T, \tag{7}$$

where ΔU^1_{yi}, $\Delta\Theta_i$ and ΔU^2_{yi} are the degrees of freedom of the internal node (with no specific physical meaning).

The generalized strain field is:

$$\begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix} = \begin{bmatrix} N'_1 & 0 & 0 & 0 & 0 & 0 & N'_7 & 0 & 0 \\ 0 & N'_{11} & -N_{21} & N'_{13} & -N_{23} & N'_{15} & 0 & N'_{17} & -N_{27} \\ 0 & 0 & N'_{21} & 0 & N'_{23} & 0 & 0 & 0 & N'_{27} \end{bmatrix} \begin{bmatrix} U_{xi} \\ U_{yi} \\ \Theta_{zi} \\ \Delta U^1_{yi} \\ \Delta\Theta_i \\ \Delta U^2_{yi} \\ U_{xj} \\ U_{yj} \\ \Theta_{zj} \end{bmatrix}. \tag{8}$$

$$\text{where} \begin{cases} N_1 = 1 - \frac{x}{L} & N_7 = \frac{x}{L} & N_{11} = (1-\frac{x}{L})^2(1+2\frac{x}{L}) \\ N_{13} = 2(1-\frac{x}{L})^2(\frac{x}{L}) & N_{15} = -2(\frac{x}{L})^2(1-\frac{x}{L}) & N_{17} = (\frac{x}{L})^2(3-2\frac{x}{L}) \\ N_{21} = (1-\frac{x}{L})(1-3\frac{x}{L}) & N_{23} = 1-(1-2\frac{x}{L})^2 & N_{27} = -(\frac{x}{L})(2-3\frac{x}{L}) \end{cases} \tag{9}$$

See [19] for more details and the analytical expressions of the condensed matrices and vectors).

3. Comparison of the FLI, FCQM and FCQ formulations

The different numerical examples chosen to illustrate the performance of the **FLI**, **FCQM** and **FCQ** beam formulations are 3.1 and 3.2. For more applications, we refer the reader to [23].

3.1. A cantilever elastic Timoshenko beam subjected to a transverse tip displacement

The geometrical characteristics of the beam and the material parameters are given in figure (1) and table (1).

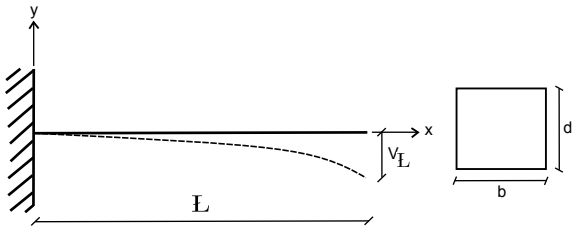


Figure 1: Cantilever beam

A displacement v_L is applied at $x = L$. The resulting analytical shear force and moment at $x = 0$ is given as [14]):

$$F_y^{ana} = \frac{v_L}{\frac{L^3}{3EI} + \frac{L}{kGS}} \quad , \quad M_z^{ana} = L \times F_y^{ana} \tag{10}$$

Geometry		Material properties		
Ł	$1.53m$	Young modulus	E	$210GPa$
b	$0.25m$	Poisson's ratio	ν	0.3
d	$0.25m$	Shear correction factor	k	$\frac{5}{6}$

Table 1: Cantilever beam - Geometrical characteristics and material properties

Comparison of the performances of the three formulations is provided in tables (2) and (3) where n is the number of the multi-fiber Timoshenko beam elements used for the spatial discretization and $R_E(F_y)$ the relative error on shear forces defined as:

$$R_E(F_y) = \left| \frac{F_y^{ana} - F_y^{num}}{F_y^{ana}} \right| \tag{11}$$

with F_y^{num} the result of the numerical calculations.

	n	$v_{\text{L}}(m)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$
Analytical	1	0.1	5609.1	8581.9	–
FLI	1	0.1	7428.3	11365.3	32%
FCQM	1	0.1	5609.1	8581.9	0%
FCQ	1	0.1	5609.1	8581.9	0%

Table 2: A cantilever elastic Timoshenko beam subjected to a transverse tip displacement - Results of the 3 formulations considering one (1) multi-fiber Timoshenko beam element

n	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$	n	$F_y(KN)$	$M_z(KNm)$	$R_E\ (F_y)$
4	5696.3	8715.3	1.5%	20	5612.2	8587.2	0.053%
8	5630.6	8614.9	0.37%	31	5610.5	8584.1	0.025%
16	5614.5	8590.1	0.087%	51	5609.6	8582.73	0.009%

Table 3: A cantilever elastic Timoshenko beam subjected to a transverse tip displacement - Results of the **FLI** formulation considering n multi-fiber Timoshenko beam elements

One **FCQM** or **FCQ** element provides the exact solution in terms of forces and moments whereas the **FLI** formulation presents a relative error $R_E(F_y)$ of 32%, see table (2). This is due to the fact that its shape functions corresponding to the transverse displacements and rotations are linear whereas the analytical solutions of Θ_z and U_y are a second and a third order polynomial respectively. By increasing the number of elements, the accuracy of the **FLI** formulation however quickly improves (see table (3)).

3.2. A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement.

We consider hereafter an elastic perfectly plastic material [24] assuming that only the normal component of the axial stress σ_x can enter to plasticity. No interaction between the shear stresses and the normal stresses at the material constitutive law level is considered. The stress elastic limit f_y is taken equal to $450MPa$ while the other parameters are provided in table 1. The plastic moment and the plastic shear force are (Eurocode 2 [25]):

$$M_{pl} = W_{pl} f_y = \frac{bd^2}{4} f_y \quad , \quad F_{yp} = \frac{M_{pl}}{\text{Ł}} \tag{12}$$

These two values are obtained considering that all the fibers in the section are plastified. This is an asymptotic section behavior state that cannot be reached with the adopted kinematic assumption of plane sections, since the axial strain at the neutral axis equals zero and thus the corresponding fibers cannot plastified.

The cantilever beam is submitted to an imposed displacement $v_{\text{Ł}}$ at its free end ($x = \text{Ł}$). Comparison of the performances of the three formulations for different discretizations are shown in tables 4 and 5. As before, one **FLI** element does not provide good results. However, increasing the number of elements greatly improves its performance. Results are better for the **FCQ** formulation that always provides the smallest error among the three formulations.

	n	$v_{\text{Ł}}(m)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$
Analytical	1	0.1	1148.9	1757.8	–
FLI	1	0.1	2263	3462	97%
FCQM	1	0.1	1609	2462	40%
FCQ	1	0.1	1581	2417	37.6 %

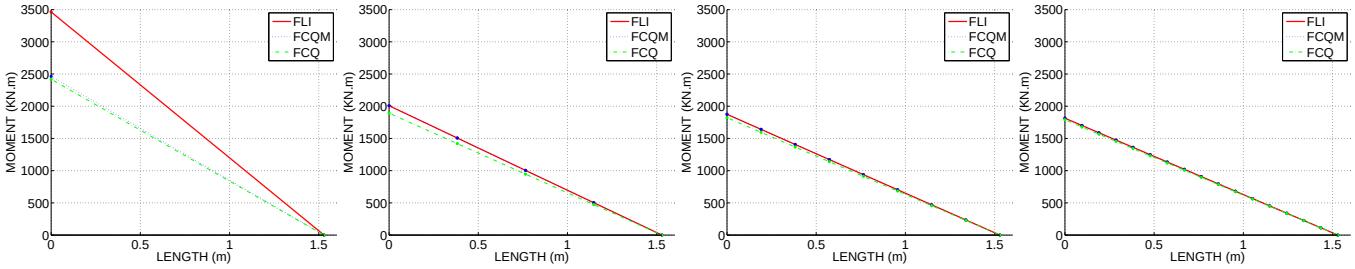
Table 4: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Results of the 3 formulations considering one (1) multi-fiber Timoshenko beam

	FLI			**FCQM**			**FCQ**		
n	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$
4	1312	2006	14%	1311	2005	17.4%	1236	1889	7.7%
8	1225	1872	7%	1225	1872	6.6 %	1191	1819	4.1%
16	1186	1811	3.2%	1186	1811	3.2%	1169	1785	1.7%

Table 5: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Results of the 3 formulations considering n multi-fiber Timoshenko beam elements

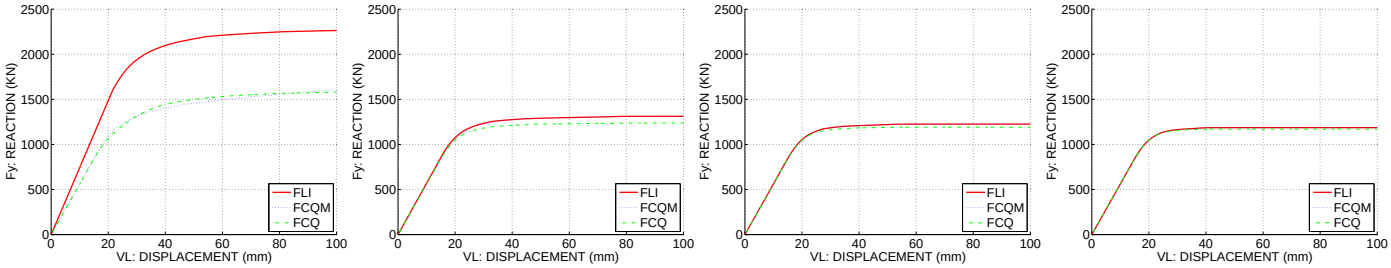
Figure (2) presents the moment diagrams along the beam for the three formulations and for different number of elements. The moment nodal values are plotted and linked with straight lines. By increasing the number of

multi-fiber beam elements, the moment diagrams of the three formulations coincide. Conclusions are similar in figure (3) showing the evolution of the shear force F_y with respect to the vertical tip displacement v_{L}.



(a) $n = 1$ (b) $n = 4$ (c) $n = 8$ (d) $n = 16$

Figure 2: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Moment diagrams along the beams for the 3 formulations considering n multi-fiber Timoshenko beam elements



(a) $n = 1$ (b) $n = 4$ (c) $n = 8$ (d) $n = 16$

Figure 3: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Shear forces Vs. vertical displacements considering n multi-fiber Timoshenko beam elements

Remark: In the **FCQM** formulation the shape functions depend on the material properties (5),(6). These shape functions were considered constant in this example (using the elastic material parameters) even when in the plasticity regime. Although this approach provides sometimes good results (see tables 4 and 5), the performance of the multi-fiber Timoshenko beam for non linear calculations can be undpredictable [19].

4. Conclusion

Three different Timoshenko multi-fiber beam formulations have been presented in this paper: the **FLI** formulation with linear shape functions, the **FCQM** formulation with higher order shape functions dependent on the material properties and the **FCQ** formulation with higher order shape functions and additional internal degrees of freedom. A comparison between the three formulations through numerical examples showed that the **FLI** formulation does not provide accurate results when only one element is adopted. Results are however improved using a finer discretization. The **FCQM** formulation is not suitable for non linear calculations as its performance can be unpredictable. This is because of the dependency on the material properties. Finally,

the **FCQ** formulation provides the best results with the smaller number of finite elements. Nevertheless, an increased number of finite elements is needed in order to reproduce correctly the axial force bending moment interaction.

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