



# A review on various formulations of displacement based multi-fiber straight Timoshenko beam finite elements

Ibrahim Bitar, Stéphane Grange, Panagiotis Kotronis, Nathan Benkemoun

## ► To cite this version:

Ibrahim Bitar, Stéphane Grange, Panagiotis Kotronis, Nathan Benkemoun. A review on various formulations of displacement based multi-fiber straight Timoshenko beam finite elements. CIGOS 2015 - Conférence Internationale Géotechnique - Ouvrage - Structure, Innovations in Construction, ENS Cachan, May 2015, Paris, France. hal-01180335

**HAL Id: hal-01180335**

**<https://hal.science/hal-01180335>**

Submitted on 30 Oct 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A review on various formulations of displacement based multi-fiber straight Timoshenko beam finite elements

Ibrahim BITAR<sup>a,1,\*</sup>, Stéphane GRANGE<sup>b</sup>, Panagiotis KOTRONIS<sup>a</sup>, Nathan BENKEMOUN<sup>c</sup>

<sup>a</sup>LUNAM, École Centrale de Nantes, Université de Nantes, CNRS  
Institut de Recherche en Génie Civil et Mécanique (GeM), UMR 6183

1 rue de la Noë, BP 92101, 44321, Nantes, cedex 3, France

<sup>b</sup>Université Grenoble Alpes, 3SR, F-38000 Grenoble, France  
CNRS, 3SR, F-38000 Grenoble, France

<sup>c</sup>LUNAM, IUT Saint-Nazaire, Université de Nantes, CNRS  
Institut de Recherche en Génie Civil et Mécanique (GeM), UMR 6183  
58 rue Michel Ange, 44600 Saint-Nazaire, France

---

## Abstract

Specific kinematic assumptions are often adopted in structural analysis of civil engineering structures in order to simplify the global equilibrium equations and reduce the required number of degrees of freedom. The classical Timoshenko beam hypothesis, considering that plane sections remain plane after deformation but not necessary normal to the beam axis, is often chosen because it can (approximately) take into account the influence of shear strains. On the contrary, the Euler-Bernoulli assumption (sections remain plane and perpendicular the beam axis) neglects their influence and provides therefore accurate results only for the case of slender beam structures. This work is focused on the Timoshenko beam theory in the context of a multi-fiber approach: The section is considered as multi-fiber, it can have an arbitrary shape and each fiber has a local constitutive law representing a specific material. Various formulations of displacement based multi-fiber straight Timoshenko beam finite elements are re-visited. After a presentation of the shape functions leading to the stiffness matrices and the consistent nodal forces relative to each formulation, comparisons are made using elastic or elastic perfectly plastic constitutive laws. The advantages and disadvantages of each formulation are highlighted and general conclusions on the use of displacement based Timoshenko multi-fiber beams in engineering are drawn.

*Keywords:* Timoshenko; Multi-fiber; beam.

---

## Introduction

Different kinematic assumptions are used in structural analysis in order to simplify the global equilibrium equations and to reduce the required number of degree of freedom. The Timoshenko beam hypothesis, considers that plane sections remain plane after deformation but not necessary normal to the beam axis. The advantage of this theory is that it can take into account the influence of shear strains contrary to Euler-Bernoulli assumption

---

\*Corresponding author

Email address: [ibrahim.bitar@ec-nantes.fr](mailto:ibrahim.bitar@ec-nantes.fr) (Ibrahim BITAR)

<sup>1</sup>PhD student

---

which announces that sections remains plane and perpendicular to the beam axis after deformation and which neglects the influence of shear strains and provides accurate results only for the case of slender beam structures.

One of the first manuscripts introducing the idea of dividing a beam section in "fibers", where a specific stress/strain relation is defined, is the book of Owen and Hilton [1]. This type of finite element has been proven  
10 very efficient for various applications in civil engineering: the nonlinear analysis of beam type or bearing wall structures with non homogeneous sections (eg. reinforced concrete) [2],[3],[4],[5], arbitrarily geometrical plane or hollow shape sections [6],[7] submitted to flexion, shear or torsion [8], Soil Structure Interaction problems [6], vulnerability assessment cases [7] and Fiber-Reinforced Polymer retrofitting [9]...

In this article we review the performance of three (3) displacement based straight Timoshenko beam finite elements under the small rotations assumption (force based beam elements - see for example Spacone & al. [10] - are out of the scope of this article). The formulations differ on the shape functions interpolating the displacements and rotations fields inside each element. The first formulation, called hereafter **FLI** (Full-Linear-Independent), adopts linear polynomials while the transverse displacement and rotation fields are interpolated independently, see Pegon [11] and Guedes & al. [12]. Because of the low degree of the adopted polynomials,  
20 this formulation is prone to shear locking problems (see Stolarski and Belytschko [13], De Ville de Goyet [14], Crisfield [15], Ibrahimbegovic and Frey [16]) if specific measures are not considered. A way to avoid this is to calculate shear strains approximately, as proposed by Donea and Lamain [17] and adopted in [11] and [12] and commented hereafter. In the second formulation, referred hereafter as **FCQM** (Full-Cubic-Quadratic-Material) and proposed by Friedman and Kosmatka [18], the degree of the shape functions is of order three (3) for the transverse displacements and two (2) for the rotations. Displacements and rotations are now interdependent and the shear locking problem is avoided. The particularity of this formulation is that the shape functions depend on the properties of the materials, and thus its use for non linear problems can be problematic. The third formulation, called hereafter **FCQ** (Full-Cubic-Quadratic) and proposed by Caillerie & al. [19] uses shape  
30 functions of order three (3) for the transverse displacements and two (2) for the rotations and an additional internal node. This results to a finite element free of shear locking and to shape functions independent on the properties of the materials. Further bibliography on different Timoshenko beam finite elements can be found in [20],[21] and [22]. For more details about the comparison between these Finite Element formulations, we refer the reader to Bitar and al. [23].

The organization of the article is as follows: in the first section, the general form of the stiffness matrices within a multi-fiber beam is presented. In the second section, the specific equations for the **FLI**, **FCQM** and **FCQ** formulations are provided. The performance of each formulation is studied in the following section for the case of an elastic or elasto-plastic behavior under monotonic loading. The article ends with general conclusions and guidelines.

40 **1. Timoshenko multi-fiber beams**

Consider a beam of length  $L$  discretized into  $n$  elements  $e = [x_i; x_j]$  of length  $L = x_j - x_i$  and external nodes  $i$  and  $j$ . The generalized displacement vector is approximated by an equation of the form  $U = NU_e$ , where  $U_e$  is a vector containing the external nodal displacements of the element  $e$  and  $N$  is the matrix of the shape functions depending on  $x$ .

The theoretical formulation, based on equilibrium equations and kinematic assumption and passing by the virtual work principle, provides the beam element stiffness matrix and the internal nodal forces expressions as the following

$$K_{element} = \int_0^L B^T K_S B dx \quad F_{int,element} = \int_0^L B^T F_S dx \quad (1)$$

where  $B$  is a matrix containing the derivatives with respect to  $x$  of the shape functions,  $K_S$  is the multi-fiber section stiffness matrix and  $F_S$  is the generalized force vector of the section.

$$\underbrace{\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix}}_{F_S} = \underbrace{\begin{bmatrix} \int_S E_f dS & 0 & -\int_S E_f y dS \\ 0 & \int_S k G_f dS & 0 \\ -\int_S E_f y dS & 0 & \int_S E_f y^2 dS \end{bmatrix}}_{K_S} \cdot \begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix}, \quad (2)$$

where  $U_x(x)$  being the longitudinal displacement,  $U_y(x)$  the transverse displacement and  $\Theta_z(x)$  the rotation of the section.  $E_f$  and  $G_f$  the Young and Shear modulus respectively and  $k$  the shear correction factor.

**2. FLI, FCQM and FCQ formulations**

We present hereafter the shape functions for the **FLI**, **FCQM** and **FCQ** formulations.

**2.1. FLI formulation**

Linear independent shape functions are adopted for the generalized displacement fields [11],[12]:

The adopted generalized strain field has the following form:

$$\begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{2} & 0 & \frac{1}{L} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{L} & 0 & 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} U_{xi} \\ U_{yi} \\ \Theta_{zi} \\ U_{xj} \\ U_{yj} \\ \Theta_{zj} \end{bmatrix}, \quad (3)$$

where, following the proposal of Donea and Lamain [17], the expression of  $\beta_y$  is modified by eliminating the linear terms in the shape functions in order to avoid the shear locking problem.

60 **2.2. FCQM formulation**

Cubic and quadratic interdependent polynomials, functions of the material properties, are used for the transverse and rotational displacements and the formulation is free of shear locking [18]. The generalized strain field is:

$$\begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix} = \begin{bmatrix} N'_1 & 0 & 0 & N'_4 & 0 & 0 \\ 0 & N'_8 - N_{14} & N'_9 - N_{15} & 0 & N'_{11} - N_{17} & N'_{12} - N_{18} \\ 0 & N'_{14} & N'_{15} & 0 & N'_{17} & N'_{18} \end{bmatrix} \begin{bmatrix} U_{xi} \\ U_{yi} \\ \Theta_{zi} \\ U_{xj} \\ U_{yj} \\ \Theta_{zj} \end{bmatrix}. \quad (4)$$

and the shape functions:

$$\left\{ \begin{array}{ll} N_1 = 1 - \frac{x}{L} & N_4 = \frac{x}{L} \\ N_8 = \frac{1}{1+\phi} [2(\frac{x}{L})^3 - 3(\frac{x}{L})^2 - \phi(\frac{x}{L}) + 1 + \phi] & N_9 = \frac{L}{1+\phi} [(\frac{x}{L})^3 - (2 + \frac{\phi}{2})(\frac{x}{L})^2 + (1 + \frac{\phi}{2})(\frac{x}{L})] \\ N_{11} = -\frac{1}{1+\phi} [2(\frac{x}{L})^3 - 3(\frac{x}{L})^2 - \phi(\frac{x}{L})] & N_{12} = \frac{L}{1+\phi} [(\frac{x}{L})^3 - (1 - \frac{\phi}{2})(\frac{x}{L})^2 - \frac{\phi}{2}(\frac{x}{L})] \\ N_{14} = \frac{6}{(1+\phi)L} [(\frac{x}{L})^2 - (\frac{x}{L})] & N_{15} = \frac{1}{1+\phi} [3(\frac{x}{L})^2 - (4 + \phi)(\frac{x}{L}) + 1 + \phi] \\ N_{17} = -\frac{6}{(1+\phi)L} [(\frac{x}{L})^2 - (\frac{x}{L})] & N_{18} = \frac{1}{1+\phi} [3(\frac{x}{L})^2 - (2 - \phi)(\frac{x}{L})] \end{array} \right. \quad (5)$$

with  $\phi$  the ratio of the beam bending to shear stiffness provided by ( $\nu$  the Poisson's ratio):

$$\phi = \frac{12}{L^2} \left( \frac{EI}{kGA} \right) = \frac{24}{L^2} \left( \frac{I}{kA} \right) (1 + \nu). \quad (6)$$

**2.3. FCQ formulation**

Cubic functions are used to interpolate the transverse displacements and quadratic for the rotations. The element is free of shear locking and uses an additional internal node [19]. Caillerie & al. [19] proved also that the **FCQM** formulation can be derived from the present formulation and that one **FCQ** element is able to predict the exact tip displacements for any complex loading (shear/flexion) submitted to an homogeneous elastic beam.

The nodal displacement field takes the following form:

$$U_e = \begin{bmatrix} U_{xi} & U_{yi} & \Theta_{zi} & \Delta U_{yi}^1 & \Delta \Theta_i & \Delta U_{yi}^2 & U_{xj} & U_{yj} & \Theta_{zj} \end{bmatrix}^T, \quad (7)$$

where  $\Delta U_{yi}^1$ ,  $\Delta \Theta_i$  and  $\Delta U_{yi}^2$  are the degrees of freedom of the internal node (with no specific physical meaning).

The generalized strain field is:

$$\begin{bmatrix} U'_x \\ \beta_y \\ \Theta'_z \end{bmatrix} = \begin{bmatrix} N'_1 & 0 & 0 & 0 & 0 & 0 & N'_7 & 0 & 0 \\ 0 & N'_{11} & -N_{21} & N'_{13} & -N_{23} & N'_{15} & 0 & N'_{17} & -N_{27} \\ 0 & 0 & N'_{21} & 0 & N'_{23} & 0 & 0 & 0 & N'_{27} \end{bmatrix} \begin{bmatrix} U_{xi} \\ U_{yi} \\ \Theta_{zi} \\ \Delta U_{yi}^1 \\ \Delta \Theta_i \\ \Delta U_{yi}^2 \\ U_{xj} \\ U_{yj} \\ \Theta_{zj} \end{bmatrix}. \quad (8)$$

$$\text{where } \begin{cases} N_1 = 1 - \frac{x}{L} & N_7 = \frac{x}{L} & N_{11} = (1 - \frac{x}{L})^2(1 + 2\frac{x}{L}) \\ N_{13} = 2(1 - \frac{x}{L})^2(\frac{x}{L}) & N_{15} = -2(\frac{x}{L})^2(1 - \frac{x}{L}) & N_{17} = (\frac{x}{L})^2(3 - 2\frac{x}{L}) \\ N_{21} = (1 - \frac{x}{L})(1 - 3\frac{x}{L}) & N_{23} = 1 - (1 - 2\frac{x}{L})^2 & N_{27} = -(\frac{x}{L})(2 - 3\frac{x}{L}) \end{cases} \quad (9)$$

See [19] for more details and the analytical expressions of the condensed matrices and vectors).

### 3. Comparison of the **FLI**, **FCQM** and **FCQ** formulations

The different numerical examples chosen to illustrate the performance of the **FLI**, **FCQM** and **FCQ** beam formulations are 3.1 and 3.2. For more applications, we refer the reader to [23].

80

#### 3.1. A cantilever elastic Timoshenko beam subjected to a transverse tip displacement

The geometrical characteristics of the beam and the material parameters are given in figure (1) and table (1).

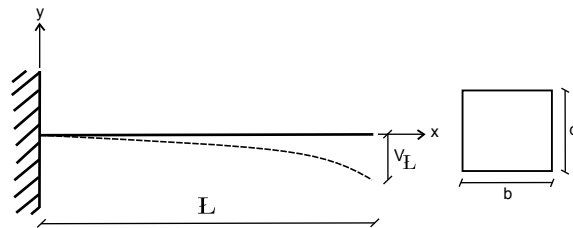


Figure 1: Cantilever beam

A displacement  $v_L$  is applied at  $x = L$ . The resulting analytical shear force and moment at  $x = 0$  is given as [14]):

$$F_y^{ana} = \frac{v_L}{\frac{L^3}{3EI} + \frac{L}{kGS}}, \quad M_z^{ana} = L \times F_y^{ana} \quad (10)$$

### 3 COMPARISON OF THE **FLI**, **FCQM** AND **FCQ** FORMULATIONS

Geometry		Material properties			
$L$	$1.53m$	Young modulus	$E$	$210GPa$	
$b$	$0.25m$	Poisson's ratio	$\nu$	$0.3$	
$d$	$0.25m$	Shear correction factor	$k$	$\frac{5}{6}$	

Table 1: Cantilever beam - Geometrical characteristics and material properties

Comparison of the performances of the three formulations is provided in tables (2) and (3) where  $n$  is the number of the multi-fiber Timoshenko beam elements used for the spatial discretization and  $R_E(F_y)$  the relative error on shear forces defined as:

$$R_E(F_y) = \left| \frac{F_y^{ana} - F_y^{num}}{F_y^{ana}} \right| \quad (11)$$

with  $F_y^{num}$  the result of the numerical calculations.

	$n$	$v_L(m)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$
Analytical	1	0.1	5609.1	8581.9	—
<b>FLI</b>	1	0.1	7428.3	11365.3	32%
<b>FCQM</b>	1	0.1	5609.1	8581.9	0%
<b>FCQ</b>	1	0.1	5609.1	8581.9	0%

Table 2: A cantilever elastic Timoshenko beam subjected to a transverse tip displacement - Results of the 3 formulations considering one (1) multi-fiber Timoshenko beam element

$n$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$	$n$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$
4	5696.3	8715.3	1.5%	20	5612.2	8587.2	0.053%
8	5630.6	8614.9	0.37%	31	5610.5	8584.1	0.025%
16	5614.5	8590.1	0.087%	51	5609.6	8582.73	0.009%

Table 3: A cantilever elastic Timoshenko beam subjected to a transverse tip displacement - Results of the **FLI** formulation considering  $n$  multi-fiber Timoshenko beam elements

One **FCQM** or **FCQ** element provides the exact solution in terms of forces and moments whereas the **FLI** formulation presents a relative error  $R_E(F_y)$  of 32%, see table (2). This is due to the fact that its shape functions corresponding to the transverse displacements and rotations are linear whereas the analytical solutions of  $\Theta_z$  and  $U_y$  are a second and a third order polynomial respectively. By increasing the number of elements, the accuracy of the **FLI** formulation however quickly improves (see table (3)).

### 3 COMPARISON OF THE **FLI**, **FCQM** AND **FCQ** FORMULATIONS

#### 3.2. A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement.

We consider hereafter an elastic perfectly plastic material [24] assuming that only the normal component of the axial stress  $\sigma_x$  can enter to plasticity. No interaction between the shear stresses and the normal stresses at the material constitutive law level is considered. The stress elastic limit  $f_y$  is taken equal to  $450MPa$  while the other parameters are provided in table 1. The plastic moment and the plastic shear force are (Eurocode 2 [25]):

$$M_{pl} = W_{pl}f_y = \frac{bd^2}{4}f_y \quad , \quad F_{yp} = \frac{M_{pl}}{L} \quad (12)$$

These two values are obtained considering that all the fibers in the section are plastified. This is an asymptotic section behavior state that cannot be reached with the adopted kinematic assumption of plane sections, since the axial strain at the neutral axis equals zero and thus the corresponding fibers cannot be plastified.

The cantilever beam is submitted to an imposed displacement  $v_L$  at its free end ( $x = L$ ). Comparison of the performances of the three formulations for different discretizations are shown in tables 4 and 5. As before, one **FLI** element does not provide good results. However, increasing the number of elements greatly improves its performance. Results are better for the **FCQ** formulation that always provides the smallest error among the three formulations.

	$n$	$v_L(m)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$
Analytical	1	0.1	1148.9	1757.8	—
<b>FLI</b>	1	0.1	2263	3462	97%
<b>FCQM</b>	1	0.1	1609	2462	40%
<b>FCQ</b>	1	0.1	1581	2417	37.6 %

Table 4: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Results of the 3 formulations considering one (1) multi-fiber Timoshenko beam

	<b>FLI</b>			<b>FCQM</b>			<b>FCQ</b>		
$n$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$	$F_y(KN)$	$M_z(KNm)$	$R_E(F_y)$
4	1312	2006	14%	1311	2005	17.4%	1236	1889	7.7%
8	1225	1872	7%	1225	1872	6.6 %	1191	1819	4.1%
16	1186	1811	3.2%	1186	1811	3.2%	1169	1785	1.7%

Table 5: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Results of the 3 formulations considering  $n$  multi-fiber Timoshenko beam elements

Figure (2) presents the moment diagrams along the beam for the three formulations and for different number of elements. The moment nodal values are plotted and linked with straight lines. By increasing the number of



## 4 CONCLUSION

multi-fiber beam elements, the moment diagrams of the three formulations coincide. Conclusions are similar in figure (3) showing the evolution of the shear force  $F_y$  with respect to the vertical tip displacement  $v_L$ .

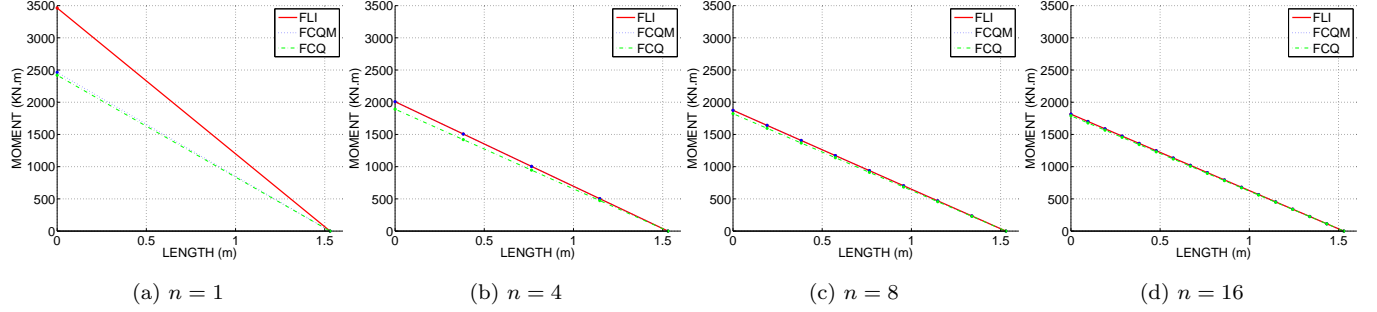


Figure 2: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Moment diagrams along the beams for the 3 formulations considering  $n$  multi-fiber Timoshenko beam elements

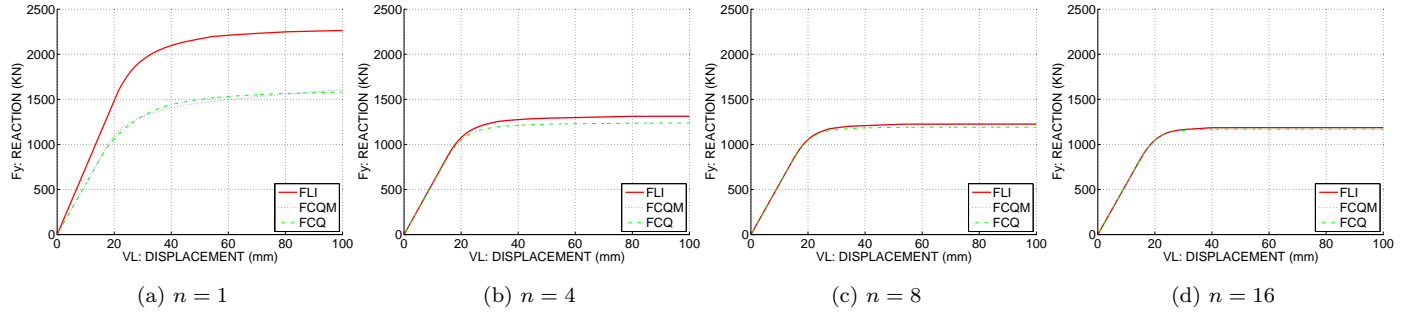


Figure 3: A cantilever elastic perfectly plastic Timoshenko beam subjected to a transverse tip displacement - Shear forces Vs. vertical displacements considering  $n$  multi-fiber Timoshenko beam elements

*Remark:* In the **FCQM** formulation the shape functions depend on the material properties (5),(6). These shape functions were considered constant in this example (using the elastic material parameters) even when in the plasticity regime. Although this approach provides sometimes good results (see tables 4 and 5), the performance of the multi-fiber Timoshenko beam for non linear calculations can be unpredictable [19].

## 4. Conclusion

Three different Timoshenko multi-fiber beam formulations have been presented in this paper: the **FLI** formulation with linear shape functions, the **FCQM** formulation with higher order shape functions dependent on the material properties and the **FCQ** formulation with higher order shape functions and additional internal degrees of freedom. A comparison between the three formulations through numerical examples showed that the **FLI** formulation does not provide accurate results when only one element is adopted. Results are however improved using a finer discretization. The **FCQM** formulation is not suitable for non linear calculations as its performance can be unpredictable. This is because of the dependency on the material properties. Finally,

the **FCQ** formulation provides the best results with the smaller number of finite elements. Nevertheless, an increased number of finite elements is needed in order to reproduce correctly the axial force bending moment interaction.

### Acknowledgments

130 This work done within the *SINAPS@* project benefited from state funding managed by the National Research Agency under program *RNSR* Future Investments bearing reference *No. ANR-11-RSNR-0022-04*.

### References

- [1] D. R. Owen, E. Hinton, Finite elements in plasticity, Vol. 271, Pineridge Press Swansea, 1980.
- [2] P. Kotronis, J. Mazars, Simplified modelling strategies to simulate the dynamic behaviour of R/C walls, Journal of Earthquake Engineering 9 (2) (2005) 285–306. doi:[10.1142/S1363246905001979](https://doi.org/10.1142/S1363246905001979).
- [3] P. Kotronis, F. Ragueneau, J. Mazars, A simplified modelling strategy for R/C walls satisfying PS92 and EC8 design, Engineering Structures 27 (8) (2005) 1197–1208. doi:<http://dx.doi.org/10.1016/j.engstruct.2005.03.003>.
- [4] N. Ile, X.-H. Nguyen, P. Kotronis, J. Mazars, J. M. Reynouard, Shaking table tests of lightly RC  
140 walls: Numerical simulations, Journal of Earthquake Engineering 12 (6) (2008) 849–878. doi:[10.1080/13632460801890430](https://doi.org/10.1080/13632460801890430).
- [5] S. Grange, P. Kotronis, J. Mazars, Numerical modelling of the seismic behaviour of a 7-story building: NEES benchmark, Materials and Structures 42 (10) (2009) 1433–1442. doi:[10.1617/s11527-008-9462-y](https://doi.org/10.1617/s11527-008-9462-y).
- [6] S. Grange, L. Botrugno, P. Kotronis, C. Tamagnini, The effects of soil - structure interaction on a reinforced concrete viaduct, Earthquake Engineering & Structural Dynamics 40 (1) (2011) 93–105. doi:[10.1002/eqe.1034](https://doi.org/10.1002/eqe.1034).
- [7] C. Desprez, P. Kotronis, J. Mazars, Seismic vulnerability assessment of a RC structure before and after FRP retrofitting, Bulletin of Earthquake Engineering (2014) 1–26doi:[10.1007/s10518-014-9621-1](https://doi.org/10.1007/s10518-014-9621-1).
- [8] J. Mazars, P. Kotronis, F. Ragueneau, G. Casaux, Using multifiber beams to account for shear and torsion:  
150 Applications to concrete structural elements, Computer Methods in Applied Mechanics and Engineering 195 (52) (2006) 7264–7281. doi:[10.1016/j.cma.2005.05.053](https://doi.org/10.1016/j.cma.2005.05.053).
- [9] C. Desprez, J. Mazars, P. Kotronis, P. Paultre, Damage model for FRP-confined concrete columns under cyclic loading, Engineering Structures 48 (2013) 519 – 531. doi:[10.1016/j.engstruct.2012.09.019](https://doi.org/10.1016/j.engstruct.2012.09.019).
- [10] E. Spacone, F. C. Filippou, F. F. Taucer, Fibre beam-column model for non-linear analysis of R/C frames: Part I. formulation, Earthquake Engineering and Structural Dynamics 25 (7) (1996) 711–726.

- [11] P. Pegon, A Timoshenko simple beam element in Castem 2000, Special publication Nr. I.94.04, Applied Mechanics Unit, Institute for Safety Technology, Joint Research Centre, Commission of the European Communities, I-21020 ISPRA (VA), Italy, (1994).
- [12] J. Guedes, P. Pegon, A. Pinto, A fibre/Timoshenko beam element in Castem 2000, Special publication  
160 Nr. I.94.31, Applied Mechanics Unit, Institute for Safety Technology, Joint Research Centre, Commission of the European Communities, I-21020 ISPRA (VA), Italy, (1994).
- [13] H. Stolarski, T. Belytschko, Membrane locking and reduced integration for curved elements, *Journal of Applied Mechanics* 49 (1) (1982) 172–176.
- [14] V. De Ville de Goyet, L’analyse statique non linéaire par la méthode des éléments finis des structures spatiales formées de poutres à section non symétrique, Ph.D. thesis, Université de Liège (1989).
- [15] M. A. Crisfield, *Nonlinear finite element analysis of solids and structures. Volume 1: Essentials*, Wiley, New York, NY, United States, 1991.
- [16] A. Ibrahimbegović, F. Frey, Finite element analysis of linear and non-linear planar deformations of elastic initially curved beams, *International Journal for Numerical Methods in Engineering* 36 (19) (1993) 3239–  
170 3258.
- [17] J. Donea, L. Lamain, A modified representation of transverse shear in  $C^0$  quadrilateral plate elements, *Computer Methods in Applied Mechanics and Engineering* 63 (2) (1987) 183–207.
- [18] Z. Friedman, J. B. Kosmatka, An improved two-node Timoshenko beam finite element, *Computers & Structures* 47 (3) (1993) 473–481.
- [19] D. Caillerie, P. Kotronis, R. Cyburski, A new Timoshenko finite element beam with internal degrees of freedom, *International Journal of Numerical and Analytical Methods in Geomechanics* (in press) 2015.
- [20] R. Nickel, G. Secor, Convergence of consistently derived Timoshenko beam finite elements, *International Journal for Numerical Methods in Engineering* 5 (2) (1972) 243–252.
- [21] J. Thomas, B. Abbas, Finite element model for dynamic analysis of Timoshenko beam, *Journal of Sound and Vibration* 41 (3) (1975) 291–299.  
180
- [22] A. Tessler, S. Dong, On a hierarchy of conforming Timoshenko beam elements, *Computers & structures* 14 (3) (1981) 335–344.
- [23] I. Bitar, S. Grange, P. Kotronis, N. Benkemoun, A review on Timoshenko multi-fiber beams Finite Element formulations and elasto-plastic applications, *Engineering Structures* (submitted) 2015.
- [24] J. Simo, T. Hughes, *Computational inelasticity*, 1998.
- [25] C. E. De Normalisation, Eurocode 2: design of concrete structures.