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## FIXED POINTS OF IMPLICIT CONTRACTION MAPPINGS

BY

MIHAI TURINICI

1. In this note, the notion of "implicit contraction mapping" of a complete metric space into itself is introduced and some fixed point theorems for such a class of mappings are presented.

Let  $R$  be the real axis,  $R_+$  the positive half-axis,  $D = R_+^0$  the strict positive half-axis (i.e.  $R_+ = [0, \infty[$ ,  $D = ]0, \infty[$ ),  $S \neq \emptyset$  an arbitrary set and  $\emptyset \neq P \subset S$  an arbitrary fixed subset.

**Definition 1.1.** A function  $f : D \times D \times R_+ \times R_+ \times R_+ \times R_+ \rightarrow S$  will be said to have property (P), if it satisfies

A. Global conditions:

$$a.1) \quad t, u > 0, v, w, p, q \geq 0 \implies \\ f(t, u, v, w, p, q) = f(t, u, w, v, p, q) = f(t, u, v, w, q, p)$$

$$a.2) \quad u, v > 0, 0 \leq p \leq u + v, f(u, v, v, u, p, 0) \in P \implies u \leq v$$

$$a.3) \quad w > 0 \implies f(w, w, 0, 0, w, w) \in S \setminus P$$

B. Local conditions:  $\forall r > 0, \exists a(r) \in ]0, r[$  such that

$$b.1) \quad u, v \in [r, r + a(r)[, u \leq v, 0 \leq p \leq u + v \implies f(u, v, v, u, p, 0) \in S \setminus P$$

$$b.2) \quad t, p, q \in ]r - a(r), r + a(r)[, u \in [r, r + a(r)[, v, w \in ]0, a(r)[ \implies \\ f(t, u, v, w, p, q) \in S \setminus P$$

$$b.3) \quad t, p \in ]r - a(r), r + a(r)[, u, v, q \in ]0, a(r)[ \implies f(t, u, v, r, p, q) \in S \setminus P.$$

**Definition 1.2.** Let  $f : D \times D \times R_+ \times R_+ \times R_+ \times R_+ \rightarrow S$  be a given function and let  $(X, d)$  be a complete metric space. A mapping  $T : X \rightarrow X$  will be called an implicit contraction mapping (abbreviated: i. c. m.) with respect to  $f$ , if

$$(1.1) \quad f(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \in P \\ \text{for all } x, y \in X, Tx \neq Ty.$$

Our main result can be stated as follows.

**Theorem 1.1.** *Suppose that  $T : X \rightarrow X$  satisfies*

(1.2)  *$T$  is an i. c. m. with respect to  $f$*

(1.3)  *$f$  has the property (P).*

*Then, the following conclusions hold:*

(1.4)  *$T$  has a unique fixed point  $z \in X$*

(1.5)  *$T^n x \rightarrow z$  as  $n \rightarrow \infty$ ,  $\forall x \in X$ .*

**Proof.** First, we prove the uniqueness of the fixed point of  $T$ . Let  $u, v \in X$  be such that  $u = Tu, v = Tv, u \neq v$ . From (1.1) and a.3) we obtain

$$(1.6) \quad P \ni f(d(u, v), d(u, v), 0, 0, d(u, v), d(u, v)) \in S \setminus P,$$

a contradiction. Therefore,  $u = v$ ; i.e., the fixed point of  $T$  is unique. Now we prove the existence. Take any  $x \in X$  and consider the sequence  $\{T^n x; n \geq 0\}$ . If  $T^n x = T^{n+1} x$  for some  $n$ , then the conclusion follows. Assume that  $T^n x \neq T^{n+1} x, \forall n \geq 0$ . From (1.1) we have,  $\forall n \geq 1$

$$(1.7) \quad f(d(T^n x, T^{n+1} x), d(T^{n-1} x, T^n x), d(T^{n-1} x, T^n x), d(T^n x, T^{n+1} x), d(T^{n-1} x, T^{n+1} x), 0) \in P;$$

and, on the other hand

$$(1.8) \quad d(T^{n-1} x, T^{n+1} x) \leq d(T^{n-1} x, T^n x) + d(T^n x, T^{n+1} x).$$

From (1.7), (1.8) and a.2) we obtain,  $\forall n \geq 1$ :

$$(1.9) \quad d(T^n x, T^{n+1} x) \leq d(T^{n-1} x, T^n x);$$

i.e., the sequence  $\{d(T^n x, T^{n+1} x); n \geq 0\}$  decreases.

Let  $r = \lim_{n \rightarrow \infty} d(T^n x, T^{n+1} x)$  and assume that  $r > 0$ . One can find some rank  $n(r) \geq 1$ , such that

$$(1.10) \quad n \geq n(r) \implies d(T^{n-1} x, T^n x) \in [r, r + a(r)].$$

Taking into account (1.8), (1.9), (1.10) and b.1) we have,  $\forall n \geq n(r)$

$$(1.11) \quad f(d(T^n x, T^{n+1} x), d(T^{n-1} x, T^n x), d(T^{n-1} x, T^n x), d(T^n x, T^{n+1} x), d(T^{n-1} x, T^{n+1} x), 0) \in S \setminus P,$$

which contradicts (1.7) for  $n \geq n(r)$ . Therefore  $r = 0$ .

Suppose that  $\{T^n x; n \geq 0\}$  is not a Cauchy sequence. Then there exist  $\varepsilon > 0$  and two sequences of natural numbers  $\{m(j); j \geq 1\}$  and  $\{n(j); j \geq 1\}$ ,  $m(j) < n(j), \forall j \geq 1, m(j) \rightarrow \infty$  as  $j \rightarrow \infty$  such that

$$d(T^{m(j)}x, T^{n(j)}x) \geq \varepsilon, \text{ while } d(T^{m(j)}x, T^{n(j)-1}x) < \varepsilon, \forall j \geq 1.$$

For the sake of simplicity, we shall write  $m, n$ , instead of  $m(j), n(j)$ , respectively.

Now, since  $d(T^kx, T^{k+1}x) \rightarrow 0$  as  $k \rightarrow \infty$ , we can find  $j(\varepsilon) \geq 1$ , such that

$$(1.12) \quad 0 < d(T^{n-1}x, T^n x) \leq d(T^m x, T^{m+1}x) < \\ (1/3)a(\varepsilon) < a(\varepsilon) < \varepsilon, \forall j \geq j(\varepsilon).$$

On the other hand, from the triangle inequality we have,  $\forall j \geq 1$

$$(1.13) \quad d(T^m x, T^n x) - d(T^m x, T^{m+1}x) - d(T^n x, T^{n+1}x) \leq \\ d(T^{m+1}x, T^{n+1}x) \leq \\ d(T^m x, T^n x) + d(T^m x, T^{m+1}x) + d(T^n x, T^{n+1}x),$$

$$(1.14) \quad d(T^m x, T^n x) \leq d(T^m x, T^{n-1}x) + d(T^{n-1}x, T^n x),$$

$$(1.15) \quad d(T^m x, T^n x) - d(T^n x, T^{n+1}x) \leq d(T^m x, T^{n+1}x) \leq \\ d(T^m x, T^n x) + d(T^n x, T^{n+1}x),$$

$$(1.16) \quad d(T^m x, T^n x) - d(T^m x, T^{m+1}x) \leq d(T^n x, T^{m+1}x) \leq \\ d(T^m x, T^n x) + d(T^m x, T^{m+1}x).$$

From (1.12)–(1.16) it easily follows,  $\forall j \geq j(\varepsilon)$

$$(1.17) \quad d(T^{m+1}x, T^{n+1}x), d(T^m x, T^{n+1}x), d(T^n x, T^{m+1}x) \in \\ ]\varepsilon - a(\varepsilon), \varepsilon + a(\varepsilon)[, \quad d(T^m x, T^n x) \in [\varepsilon, \varepsilon + a(\varepsilon)[.$$

Now, (1.1), (1.12), (1.17) and b.2) give us, for  $j \geq j(\varepsilon)$

$$(1.18) \quad P \ni f(d(T^{m+1}x, T^{n+1}x), d(T^m x, T^n x), d(T^m x, T^{m+1}x)), \\ d(T^n x, T^{n+1}x), d(T^m x, T^{n+1}x), d(T^n x, T^{m+1}x)) \in S \setminus P,$$

a contradiction. Therefore,  $\{T^n x; n \geq 0\}$  is a Cauchy sequence.

Since  $(X, d)$  is complete,  $T^n x \rightarrow z$ , for some  $z \in X$ . We have two possibilities:

**i)** There exists a sequence of natural numbers  $\{k(n); n \geq 0\}$ ,  $k(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , such that  $T^{k(n)}x = z$ . Then,  $T^{k(n)+1}x = Tz$ . Letting  $n$  tends to  $\infty$  and using the fact that  $\{T^{k(n)+1}x; n \geq 0\}$  is a subsequence of  $\{T^n x; n \geq 0\}$ , we get  $z = Tz$ .

**ii)** There exists  $n_0 \in N$  such that  $n \geq n_0 \implies T^n x \neq z$ . Suppose that  $z \neq Tz$ ; then  $r := d(z, Tz) > 0$ . We can find  $n(r) \in N$  such that,  $\forall n \geq n(r)$

$$(1.19) \quad 0 < d(T^n x, T^{n+1}x), d(T^n x, z) < (1/3)a(r) < a(r) < r.$$

On the other hand, from the triangle inequality we have,  $\forall n \geq 1$

$$(1.20) \quad r - d(T^n x, z) \leq d(T^n x, Tz) \leq r + d(T^n x, z).$$

From (1.19), (1.20) we obtain,  $\forall n \geq n(r)$

$$(1.21) \quad d(T^n x, Tz) \in ]r - a(r), r + a(r)[.$$

Now, (1.1), (1.19), (1.21) and b.3) give us, for  $n \geq n(r)$

$$(1.22) \quad P \ni f(d(T^{n+1}x, Tz), d(T^n x, z), d(T^n x, T^{n+1}x), \\ r, d(T^n x, Tz), d(z, T^{n+1}x)) \in S \setminus P,$$

a contradiction. Therefore,  $z = Tz$ , which completes the proof. **Q. E. D.**

**2.** In this paragraph, we give some corollaries of the preceding main result. Let  $R_+$ ,  $D$ ,  $S$ ,  $P$  be defined as before.

**Definition 2.1.** A function  $f : D \times D \times R_+ \times R_+ \rightarrow S$  will be said to have property (P), if it satisfies

**A. Global conditions:**

$$a.1) \quad t, u > 0, v, w \geq 0 \implies f(t, u, v, w) = f(t, u, w, v),$$

$$a.2) \quad u, v > 0, f(u, v, v, u) \in P \implies u \leq v,$$

$$a.3) \quad w > 0 \implies f(w, w, 0, 0) \in S \setminus P$$

**B. Local conditions:**  $\forall r > 0, \exists a(r) \in ]0, r[$  such that

$$b.1) \quad u, v \in [r, r + a(r)[, u \leq v \implies f(u, v, v, u) \in S \setminus P,$$

$$b.2) \quad t \in ]r - a(r), r + a(r)[, u \in [r, r + a(r)[, v, w \in ]0, a(r)[ \implies \\ f(t, u, v, w) \in S \setminus P,$$

$$b.3) \quad t \in ]r - a(r), r + a(r)[, u, v \in ]0, a(r)[ \implies f(t, u, v, r) \in S \setminus P.$$

**Definition 2.2.** Let  $f : D \times D \times R_+ \times R_+$  be a given function. A mapping  $T : X \rightarrow X$  will be called an i. c. m. with respect to  $f$ , if

$$(2.1) \quad f(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty)) \in P, \\ \forall x, y \in X, Tx \neq Ty.$$

**Theorem 2.1.** Suppose that  $T : X \rightarrow X$  satisfies

$$(2.2) \quad T \text{ is an i. c. m. with respect to } f$$

$$(2.3) \quad f \text{ has the property (P)}.$$

Then, the following conclusions hold:

$$(2.4) \quad T \text{ has a unique fixed point } z \in X$$

$$(2.5) \quad T^n x \rightarrow z \text{ as } n \rightarrow \infty, \forall x \in X.$$

**Proof.** Clearly, the preceding theorem is applicable.

Q. E. D.

**Definition 2.3.** A function  $f : D \times D \rightarrow S$  will be said to have property (P), if it satisfies

**A. Global conditions**

$$a.1) \quad u, v > 0, f(u, v) \in P \implies u \leq v,$$

$$a.2) \quad w > 0 \implies f(w, w) \in S \setminus P$$

**B. Local conditions:**  $\forall r > 0, \exists a(r) \in ]0, r[$  such that

$$b.1) \quad u, v \in [r, r + a(r)[, u \leq v \implies f(u, v) \in S \setminus P,$$

$$b.2) \quad t \in ]r - a(r), r + a(r)[, u \in [r, r + a(r)[ \implies f(t, u) \in S \setminus P,$$

$$b.3) \quad t \in ]r - a(r), r + a(r)[, u \in ]0, a(r)[ \implies f(t, u) \in S \setminus P.$$

**Definition 2.4.** Let  $f : D \times D \rightarrow S$  be a given function. A mapping  $T : X \rightarrow X$  will be called an i. c. m. with respect to  $f$ , if

$$(2.6) \quad f(d(Tx, Ty), d(x, y)) \in P, \forall x, y \in X, Tx \neq Ty.$$

**Theorem 2.2.** Suppose that  $T : X \rightarrow X$  satisfies

$$(2.7) \quad T \text{ is an i. c. m. with respect to } f$$

$$(2.8) \quad f \text{ has the property (P)}.$$

Then, the following conclusions hold:

$$(2.9) \quad T \text{ has a unique fixed point } z \in X,$$

$$(2.10) \quad T^n x \rightarrow z \text{ as } n \rightarrow \infty, \forall x \in X.$$

**Proof.** See the preceding result.

**Remark.** The main result is a partial extension of a result due to Hardy and Rogers [1]. Theorem 2.1 extends some results of Reich [3, 4], Rus [5, 6] and of the author [7, 8]; and Theorem 2.2 represents an extension of a result due to Boyd and Wong [2].

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