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Abstract

We suggest a new nonparametric estimate of the interaction function of pairwise interaction point process on $\mathbb{R}^d$ such that its Papangelou conditional intensity is translation invariant and satisfied finite-range interaction. We prove uniform strong consistency of the nonparametric estimate of the interaction function . Sufficient conditions of the rates of uniform strong consistency for the resulting kernel-type estimator are valid by the nonuniform mixing condition for a stationary point process, which can be viewed interchangeably as a lattice field in $\mathbb{Z}^d$.

Keywords: kernel-type estimator, mixing random fields, pairwise interaction spatial point process, Papangelou conditional intensity, rates of converge,uniform strong consistency

1 Introduction

Point processes are well studied objects in probability theory and statistics, which is of interest in such diverse disciplines as forestry, seismology, spatial epidemiology, materials science, astronomy, geography, ecology and other disciplines. In this work we will develop some recent results of nonparametric statistical spatial processes. Functional data analysis is a typical issue in modern statistics. During the last years, many papers have been devoted to theoretical results or applied studies on models involving functional data. The main goal of this paper is to present a nonparametric estimation of interaction function for stationary pairwise interaction point processes (they have been introduced in statistical literature by Ripley and Kelly [20], Daley and Vere-Jones [4] and Georgii [9]) defined in the space $\mathbb{R}^d$ for which its Papangelou conditional intensity (Papangelou [17]). In general it is not possible to deal with densities of infinite point processes, instead, the Papangelou conditional intensity becomes the appropriate starting point for modelling. A formal definition and interpretation of the Papangelou conditional intensity are given below. The estimate of the interaction function plays a very important role in the statistical analysis of spatial point pattern data. We explore the spatial dependence of random fields (mixing random fields) and we discuss the uniform strong consistent estimation. Many modelizations of the mixing random fields have been proposed through in literature, we refer to Bradley [2], Rosenblatt [21], I.A.Ibragimov [12]. Bosq [1], Delecroix [5], Heinrich and Liebscher [11], and Pelidrad [19]. The mixing hypothesis was introduced to extend the central limit theorem for more general classes dependent random variables (see Ibragimov and Linnik [13]). It also helps to generalize almost sure convergence obtained with independent data.

The paper is organized as follows. In Section 2, we set up the generic notation. In Section 3, we present the model that has been considered for stationary pairwise interaction point process and we suggest a new nonparametric estimator of interaction function. We present some conditions that will be helpful in the following paper and we present the main results, We suggest a new nonparametric estimate of the interaction function of pairwise interaction point process on $\mathbb{R}^d$ in the Papangelou conditional intensity and we prove uniform strong consistency of the resulting estimator. The Section 4 is devoted to the proofs. Our methods of proofs are based on new Kahane-Khintchine inequalities in Orlicz spaces induced by exponential Young functions for stationary uniformly strong mixing random fields, obtained by (El Machkouri [7].

2 Generic notation and Basic tools

We introduce some necessary definitions and notations and recall some basic facts from the theory of Gibbs point processes. Let $\mathscr{B}^d$ be the Borel σ-field on $\mathbb{R}^d$ of general dimension $d \geq 2$ and $\mathscr{B}^d_O = \{\Lambda \in \mathscr{B}^d : \Lambda \quad \text{bounded}\}$. For a set $\Lambda \in \mathscr{B}^d$, $|\Lambda|$ is the positive volume of Λ. We define the space of locally finite point configurations in $\mathbb{R}^d$ as $N_{lf} = \{\mathbf{x} \subseteq \mathbb{R}^d; n(\mathbf{x} \cap \Lambda) < \infty, \forall \Lambda \in \mathscr{B}^d_0\}$ equipped with the smallest σ-field $\mathscr{N}_{lf}$ containing all set of the form $\{\mathbf{x} \in N_{lf} : n(\mathbf{x} \cap \Lambda) = k\}$ for $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and any $\Lambda \in \mathscr{B}^d_0$. $\sum^{\neq}$ denotes summation over summands with index tuples having pairwise distinct components. The origin in $\mathbb{R}^d$ will be denoted by o.

The Papangelou conditional intensity of Gibbs point processes $\mathbf{X}$ in $\mathbb{R}^d$ (Møller and Waagepetersen [14]) is a function $\lambda : \mathbb{R}^d \times N_{lf} \to \mathbb{R}_+$ and characterized by the Georgii-Nguyen-Zessin (GNZ) formula (see Papangelou [18] and Zessin [22] for historical comments and Georgii [8] or Nguyen and Zessin [16] for a general presentation). The GNZ formula states that for any nonnegative measurable function h on $\mathbb{R}^d \times N_{lf}$

$$\mathrm{E}\sum_{\xi\in\mathbf{X}} h(\xi,\mathbf{X}\setminus\xi) = \mathrm{E}\int_{\mathbb{R}^d} h(\xi,\mathbf{X})\lambda(\xi,\mathbf{X})\mathrm{d}\xi. \tag{2.1}$$

Heuristically, $\lambda(\xi,\mathbf{x})\mathrm{d}\xi$ can be interpreted as the conditional probability of $\mathbf{X}$ having a point in an infinitesimal small region containing ξ and of size $\mathrm{d}\xi$ given the rest of $\mathbf{X}$ is $\mathbf{x}$. Using induction we obtain the second-order GNZ-formula

$$\mathrm{E}\sum_{\xi,\eta,\in\mathbf{X}}^{\neq} h(\xi,\eta,\mathbf{X}\setminus\{\xi,\eta\}) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{E}h(\xi,\eta,\mathbf{X})\lambda(\xi,\eta,\mathbf{X})\mathrm{d}\xi\mathrm{d}\eta \tag{2.2}$$

where $\lambda(\xi,\eta,\mathbf{x}) = \lambda(\xi,\mathbf{x})\lambda(\eta,\mathbf{x}\cup\{\xi\})$, for $\xi,\eta\in\mathbb{R}^d$, $\mathbf{x}\in N_{lf}$ with a nonnegative measurable function $h(\xi,\eta,\mathbf{x})$ on $\mathbb{R}^d\times\mathbb{R}^d\times N_{lf}$.

3 Main results

3.1 The model and its estimator

Throughout this paper, we shall focus on stationary pairwise interaction point processes (Daley and Vere-Jones [4], Georgii [9]). We specify that the Papangelou conditional intensity is translation invariant for a configuration $\mathbf{x}\in N_{lf}$ and a location $\xi\in\mathbb{R}^d$ should be of the form

$$\lambda(\xi,\mathbf{x}) = \beta^\star \exp\left(-\sum_{\eta\in\mathbf{x}\setminus\xi} V_2(\eta-\xi)\right) \tag{3.3}$$

where $\beta^\star$ is the true value of the intensity parameter. The function V_2 represents the pair potential defined on $\mathbb{R}^d$. To do this, we will assume the following condition on the potential V_2:

$$0 < V_2(\xi) < \infty. \tag{3.4}$$

Condition 3.4 says that the interaction is purely repulsive. In physics applications, V_2 is interpreted as the energy of the configuration $\mathbf{x}$. $V = \exp(-V_2)$ represents the pairwise interaction function. The Papangelou conditional intensity is said to have finite interaction range R with $0 < R < \infty$, if

$$\lambda(\xi,\mathbf{x}) = \lambda(\xi,\mathbf{x}\cap B(\xi,R)), \tag{3.5}$$

for any $\xi \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$, where $B(\xi, R)$ is the closed ball in $\mathbb{R}^d$ with center ξ and radius R.

Our objective is to study problems of estimation of the interaction function V in the model (3.3). For this, let $\Lambda_n \in \mathscr{B}_O^d$, for $n \in \mathbb{N}$, be a sequence, denotes a bounded observation window in such a way that $|\Lambda_n| \to \infty$, as $n \to \infty$. In most application examples Λ_n cubes growing up to $\mathbb{R}^d$. Throughout in this paper, f is a nonnegative measurable function defined for all $w, w' \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$, by

$$f(w, w', \mathbf{x}) = 1\!\!1(\mathbf{x} \cap B(w, R) = \emptyset, \mathbf{x} \cap B(w', R) = \emptyset)$$

and we also introduce the following function

$$\bar{C}(o, t) = \mathrm{E}[f(o, t, \mathbf{X})] = \mathbb{P}(\mathbf{X} \cap B(o, R) = \emptyset, \mathbf{X} \cap B(t, R) = \emptyset). \tag{3.6}$$

Now, we suggest a new edge-corrected nonparametric estimator for the function $\beta^\star V(t)$ by

$$\widehat{V}_n(t) = \frac{\widehat{J}_n(t)}{\widehat{C}_n(t)}, \quad \text{for} \quad t \in \mathbb{R}^d \tag{3.7}$$

where

$$\widehat{J}_n(t) = \sum_{\xi, \eta \in \mathbf{X} \cap \Lambda_n}^{\neq} \frac{1\!\!1(\eta - \xi \in B(o, R)) f(\xi, \eta, \mathbf{X} \setminus \{\xi, \eta\}) K\left(\frac{\eta - \xi - t}{b_n}\right)}{b_n^d |\Lambda_n \cap (\Lambda_n + (\eta - \xi))|} \tag{3.8}$$

is a kernel-type estimator estimating $\beta^{\star 2} V(t) \bar{C}(o, t)$, where $|\Lambda_n \cap (\Lambda_n + (\eta - \xi))|$ is an edge correction factor and assume this has the positive volume. Recall that $\Lambda_n + \xi = \{\eta + \xi : \eta \in \Lambda_n\}$ denotes Λ_n translated by ξ. A kernel function K will be any measurable function associated with a sequence $(b_n)_{n \geq 1}$ of bandwidths. To estimate the function $\beta^\star \bar{C}(o, t)$, we propose an empirical estimator $\widehat{C}_n(t)$ defined by

$$\widehat{C}_n(t) = \sum_{\xi \in \mathbf{X} \cap \Lambda_n} \frac{f(\xi, t + \xi, \mathbf{X} \setminus \{\xi\})}{|\Lambda_n \cap (\Lambda_n + \xi)|}. \tag{3.9}$$

In this case, the denominator equals 0, we choose (by convention) an usual way to define our estimator : $\widehat{V}_n(t) = 0$.

3.2 Uniform strong consistency

The proof of uniform strong consistency of the nonparametric estimator $\widehat{V}_n(t)$ defined by (3.7), requires more delicate conditions. We have to impose certain restriction on the kernel function K and the sequence of bandwidths $(b_n)_{n \geq 1}$. The following assumptions are required.

(A1). The kernel function $K:\mathbb{R}^d \longrightarrow [0,\infty)$ is bounded with bounded support, and satisfies: Let $\xi = (\xi_1,\ldots,\xi_d)'$, $\xi_i \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \xi_1^{i_1}\ldots\xi_d^{i_d} K({}_x i_1,\ldots,\xi_d)\mathrm{d}\xi_1\ldots\mathrm{d}\xi_d = \begin{cases} 1 & \text{if} \quad i_1 = i_2 = \ldots i_d = 0, \\ 0 & \text{if} \quad 0 < i_1 + i_2 + \ldots + i_d < s. \end{cases}$$

$$\int_{\mathbb{R}^d} ||\xi||^i |K(\xi)|\mathrm{d}\xi < \infty \quad \text{for} \quad i = 0 \quad \text{and} \quad i = s, \quad \int_{\mathbb{R}^d} K(\xi)\mathrm{d}\xi = 1.$$

(A2). The kernel function K satisfies a Lipschitz condition, i.e. there exists a constant $L > 0$ such that

$$\big|K(\xi) - K(\eta)\big| \leq L||\xi - \eta|| \quad \text{for any} \quad \xi,\eta \in \mathbb{R}^d.$$

(A3). The sequence of bandwidths $(b_n)_{n\geq 1}$ is a decreasing sequence of positive real numbers satisfying:

$$\lim_{n\to\infty} b_n^d = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n^d |\Lambda_n| = \infty.$$

These assumptions are more than enough to guarantee rates of convergence of mean and a uniform strong consistency of the kernel-type estimator $\widehat{J}_n(t)$ (defined by (3.8)). Now, we establish thereafter interesting results for establishing the uniform strong consistency of the estimator $\widehat{V}_n(t)$.

Theorem 1. *Let* $\mathbf{X}$ *be a stationary pairwise interaction point process in* $\mathbb{R}^d$ *with Papangelou conditional intensity* (3.3) *satisfying conditions* (3.4) *and* (3.5)*. Let the kernel function* K *satisfy* $(A1)$ *and* $(A2)$*. Let the bandwidths* $(b_n)_{n\geq 1}$ *satisfy* $(A3)$*. Furthermore, we assume that* $V(t)\bar{C}(o,t)$ *has bounded and continuous partial derivatives of order* s *on for any fixed compact set* $K_0 \subset \mathbb{R}^d$*. Then,*

$$\sup_{t\in K_0} \big|\widehat{V}_n(t) - \beta^\star V(t)\big| \xrightarrow[n\to\infty]{a.s} 0.$$

The proof of uniform strong consistency of the nonparametric estimator $\widehat{V}_n(t)$ (Theorem 1) is based on some asymptotic properties of the estimators $\widehat{C}_n(t)$ and $\widehat{J}_n(t)$. For the estimator $\widehat{J}_n(t)$, we are exploring the spatial dependence of point processes, i.e. ϕ-mixing conditions.

3.3 Rates of uniform strong consistency of the kernel-type estimator $\widehat{J}_n(t)$

In this section we discuss sufficient conditions which ensure uniform strong consistency (including a convergence rate) of the kernel-type estimator $\widehat{J}_n(t)$ under uniformly strong mixing (i.e. ϕ-mixing) random fields. We use Kahane-Khintchine inequalities in Orlicz spaces induced by exponential Young functions for stationary ϕ-mixing real random fields which satisfy some finite exponential moment condition.

Now let us the nonuniform ϕ-mixing condition. Given two sub-σ-algebras $\mathscr{U}$ and $\mathscr{V}$ of $\mathscr{F}$. The ϕ-mixing coefficients had been introduced by I.A.Ibragimov [12] and can be defined by:

$$\phi(\mathscr{U},\mathscr{V}) = \sup\{\|\mathbb{P}(V|\mathscr{U}) - \mathbb{P}(V)\|_\infty, V \in \mathscr{V}\}.$$

In the sequel, we consider the nonuniform $\phi_{1,\infty}(n)$-mixing coefficients $\phi_{1,\infty}(n)$ defined for each positive integer n by

$$\phi_{1,\infty}(n) = \sup\{\phi(\sigma(X_k),\mathscr{F}_\Gamma), k \in \mathbb{Z}^d, \Gamma \subset \mathbb{Z}^d, d(\Gamma,\{k\}) \geq n\},$$

where $\mathscr{F}_\Gamma = \sigma(X_i, i \in \Gamma)$ is the σ-algebra genereted by X_i, $i \in \Gamma$, $|\Gamma|$ is the cardinal of Γ and $d(\Gamma_1,\Gamma_2) = \min\{|i-k| : i \in \Gamma_1, k \in \Gamma_2\}$. We say that the random field $(X_i)_{i\in\mathbb{Z}^d}$ is ϕ-mixing if $\lim_{n\to\infty}\phi_{1,\infty}(n) = 0$. As shown by (Dobrushin [6]; Georgii [9]; Guyon [10]), the weak dependence conditions based on the above mixing coefficients are satisfied by large classes of random fields including Gibbs fields.

Now, we assume that the domain Λ_n is divided into a fixed number of sub-domains as follows $\Lambda_n = \cup_{i\in\Gamma_n}\Lambda_i$, where Λ_i be the unit cube centered at $i \in \mathbb{Z}^d$ and assume that $\Gamma_n \subset \mathbb{Z}^d$, such that $|\Gamma_n| \to \infty$, as $n \to \infty$. We will consider estimation of $\beta^{\star 2}V(t)\bar{C}(o,t)$ from $\widehat{J_n}(t)$, where the process is observed in $\cup_{i\in\Gamma_n}\Lambda_i$ as

$$\widehat{J_n}(t) = \sum_{i\in\Gamma_n}\sum_{\xi,\eta\in\mathbf{X}\cap\Lambda_i}^{\neq} \frac{1\!\!1(\eta-\xi\in B(o,R))f(\xi,\eta,\mathbf{X}\setminus\{\xi,\eta\})K\left(\frac{\eta-\xi-t}{b_n}\right)}{b_n^d|\Lambda_n\cap(\Lambda_n+(\eta-\xi))|}.$$

Put

$$J_{n,i} = \sum_{\xi,\eta\in\mathbf{X}\cap\Lambda_i}^{\neq} 1\!\!1(\eta-\xi\in B(o,R))f(\xi,\eta,\mathbf{X}\setminus\{\xi,\eta\})K\left(\frac{\eta-\xi-t}{b_n}\right)$$

and we note that: $\bar{J}_{n,i} = J_{n,i} - \mathrm{E}(J_{n,i})$.

Recall that a Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to\infty}\psi(t) = +\infty$ and $\psi(0) = 0$. We define the Orlicz space L_ψ as the space of real random variables Z defined on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ such that $E[\psi(|Z|/c)] < +\infty$ for some $c > 0$. The Orlicz space L_ψ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi = \inf\{c > 0; E[\psi(|Z|/c)] \leq 1\}$$

is a Banach space. Let $\theta > 0$. We denote by ψ_θ the exponential Young function defined for any $x \in \mathbb{R}^+$ by

$$\psi_\theta(x) = \exp((x+\xi_\theta)^\theta) - \exp(\xi_\theta^\theta) \quad \text{where} \quad \xi_\theta = ((1-\theta)/\theta)^{1/\theta} 1\!\!1\{0 < \theta < 1\}. \tag{3.10}$$

On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1, \ldots, i_d)$ and $j = (j_1, \ldots, j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{lex} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, \ldots, d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \leq q < p$. Let the sets V_i^1 ; $i \in \mathbb{Z}^d$ be defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d\,;\, j <_{lex} i\}.$$

To establish the proof of Theorem 2, we need the following result.

Corollary 1. *(El Machkouri [7]). Let $(X_i)_{i\in\mathbb{Z}^d}$ be a zero mean stationary real random field which satisfies the assumption:*

$$\exists q \in]0,2[\quad \exists c > 0 \quad \mathrm{E}[\exp(c|X_0|^{\theta(q)})] < +\infty,$$

where $\theta(q) = 2q/(2-q)$. There exists a positive universal constant $M_1(q) > 0$ depending only on q such that for any family $(a_i)_{i\in\mathbb{Z}^d}$ of real numbers and any finite subset Γ in $\mathbb{Z}^d$,

$$\Big\| \sum_{i\in\Gamma} a_i X_i \Big\|_{\psi_q} \leq M_1(q) \|X_0\|_{\psi_{\theta(q)}} \Big(\sum_{i\in\Gamma} |a_i| \widetilde{b}_{i,q}(X) \Big)^{1/2}$$

where

$$\widetilde{b}_{i,q}(X) = |a_i| + C(q) \sum_{k\in V_0^1} |a_{k+i}| \sqrt{\phi_{\infty,1}(|k|)}.$$

$C(q)$ is the positive constant depending only on q.

Theorem 2. *Let the kernel function K satisfy $(A2)$. Then, for any fixed compact set $K_0 \subset \mathbb{R}^d$, we have:*

If there exists $0 < q < 2$ such that $\bar{J}_{n,0} \in \mathbb{L}_{\psi_{\theta(q)}}$ and

$$\sum_{k\in\mathbb{Z}^d} \sqrt{\phi_{1,\infty}(|k|)} < \infty. \tag{3.11}$$

Then

$$\sup_{t\in K_0} \big| \widehat{J}_n(t) - \mathrm{E}(\widehat{J}_n(t)) \big| = \mathscr{O}_{a.s.}\left(\frac{(\log n)^{1/q}}{(b_n\sqrt{n})^d} \right) \quad as \quad n \to \infty.$$

Corollary 2. *If in addition the function $V(t)\bar{C}(o,t)$ has bounded and continuous partial derivatives of order s on $K_0 \subset \mathbb{R}^d$ and we suppose that $(A1)$ and $(A3)$ are fulfilled. If there exists $0 < q < 2$ such that $\bar{J}_{n,0} \in \mathbb{L}_{\psi_{\theta(q)}}$ and*

$$\sum_{k\in\mathbb{Z}^d} \sqrt{\phi_{1,\infty}(|k|)} < \infty.$$

Then

$$\sup_{t\in K_0} \left|\widehat{J}_n(t) - \beta^{\star 2} V(t)\bar{C}(o,t)\right| = \mathcal{O}_{a.s.}\left(\frac{(\log n)^{1/q}}{(b_n\sqrt{n})^d}\right) + \mathcal{O}(b_n^s) \quad as \quad n\to\infty.$$

4 Proofs

Proof of Theorem 2. Let $(a_n)_{n\geq 1}$ be sequence of positive numbers going to zero. Following Carbon et al. [3], the compact set K_0 can be covered by v_n cubes T_k having sides of length $L_n = a_n b_n^{d+1}$ and center at c_k. Clearly there exists $c > 0$, such that $v_n \leq c/L_n^d$. We use the following classical decomposition

$$\begin{aligned}
\sup_{t\in K_0} |\widehat{J}_n(t) - \mathrm{E}\widehat{J}_n(t)| \leq & \max_{1\leq k\leq v_n} \sup_{t\in T_k} \left|\widehat{J}_n(t) - \widehat{J}_n(c_k)\right| \\
& + \max_{1\leq k\leq v_n} \sup_{t\in T_k} \left|\mathrm{E}\widehat{J}_n(t) - \mathrm{E}\widehat{J}_n(c_k)\right| \\
& + \max_{1\leq k\leq v_n} \left|\widehat{J}_n(c_k) - \mathrm{E}\widehat{J}_n(c_k)\right| \\
& = B_1 + B_2 + B_3.
\end{aligned}$$

For any $t \in T_k$, by assumption $(A2)$, we derive that there exists constant $L > 0$ such that n sufficiently large

$$\left|\widehat{J}_n(t) - \widehat{J}_n(c_k)\right| \leq L a_n \frac{1}{cn^d} \sum_{\xi,\eta\in\mathbf{X}\cap\Lambda_n}^{\neq} 1\!\!1(\eta - \xi \in B(o,R)) f(\xi,\eta,\mathbf{X}\setminus\{\xi,\eta\}).$$

Using the spatial ergodic theorem of Nguyen and Zessin [15], and by the second-order Georgii-Nguyen-Zessin formula (2.2) with

$$h(\xi,\eta,\mathbf{X}) = 1\!\!1(\eta - \xi \in B(o,R)) f(\xi,\eta,\mathbf{X})$$

it easily follows

$$B_1 = \mathcal{O}_{a.s.}(a_n).$$

From L^1-version of the ergodic theorem of Nguyen and Zessin [15], it follows

$$B_2 = \mathcal{O}_{a.s.}(a_n).$$

Now, we focus on the term B_3. In the sequel, the letter C denotes any generic positive constant. Let $\varepsilon > 0$ and $t \in K_0$ be fixed. We use the Tchebychev-Markov inequality, we have

$$\mathbb{P}\left(|\widehat{J}_n(t) - \mathrm{E}\widehat{J}_n(t)| > \varepsilon a_n\right) = \mathbb{P}\left(|\sum_{i\in\Gamma_n} \bar{J}_{n,i}| > \varepsilon a_n (nb_n)^d\right)$$
$$\leq \exp\left[-\left(\frac{\varepsilon\, a_n (nb_n)^d}{||\sum_{i\in\Gamma_n} \bar{J}_{n,i}||_{\psi_q}} + l_q\right)^q\right] \mathrm{E}\exp\left[\left(\frac{|\sum_{i\in\Gamma_n} \bar{J}_{n,i}|}{||\sum_{i\in\Gamma_n} \bar{J}_{n,i}||_{\psi_q}} + l_q\right)^q\right]. \quad (4.12)$$

Therefore, we assume that there exists a real $0 < q < 2$, such that $\bar{J}_{n,0} \in \mathbb{L}_{\psi_{\theta(q)}}$. As a direct application of Kahane-Khintchine inequality of Corollary 1 to the zero mean sequence $\bar{J}_{n,i}$, for $i \in \Gamma_n$, we obtain that:

$$\mathbb{P}\left(|\sum_{i\in\Gamma_n} \bar{J}_{n,i}| > \varepsilon a_n (b_n n)^d\right) \leq (1 + e^{l_q^q}) \exp\left[-\left(\frac{\varepsilon\, a_n (b_n n)^d}{M_1(q) \|\bar{J}_{n,0}\|^2_{\psi_{\theta(q)}} (\sum_{i\in\Gamma_n} \widetilde{b}_{i,q}(\bar{J}))^{1/2}} + l_q\right)^q\right]$$

where

$$\widetilde{b}_{i,q}(\bar{J}) = 1 + C(q) \sum_{k\in V_0^1} \sqrt{\phi_{1,\infty}(|k|)},$$

$M_1(q)$ and $C(q)$ are positive universal constants depending only on q. Thus, under condition (3.11) and from the stationary of $\mathbf{X}$ and by definition of the norm $||\cdot||_{\psi_{\theta(q)}}$, we infer that there exists a constant $C > 0$ such that

$$\mathbb{P}(|\widehat{J}_n(t) - \mathrm{E}\widehat{J}_n(t)| > \varepsilon a_n) \leq (1 + e^{l_q^q}) \exp\left[-\frac{\varepsilon^q a_n^q (b_n\sqrt{n})^{dq}}{C^q}\right].$$

Inserting $a_n = (\log n)^{1/q}/(b_n\sqrt{n})^d$, we see after a short calculation that

$$\mathbb{P}(|\widehat{J}_n(t) - \mathrm{E}\widehat{J}_n(t)| > \varepsilon a_n) \leq (1 + e^{l_q^q}) \exp\left[-\frac{\varepsilon^q \log n}{C^q}\right].$$

From the last equality, we find that

$$\sup_{t\in K_0} \mathbb{P}(|\widehat{J}_n(t) - \mathrm{E}\widehat{J}_n(t)| > \varepsilon a_n) \leq (1 + e^{l_q^q}) \exp\left[-\frac{\varepsilon^q \log n}{C^q}\right]. \quad (4.13)$$

Since

$$\mathbb{P}(|B_3| > \varepsilon a_n) \leq v_n \sup_{t\in K_0} \mathbb{P}(|\widehat{J}_n(t) - \mathrm{E}\widehat{J}_n(t)| > \varepsilon a_n),$$

using (4.13), it follows with Borel-Cantelli's lemma

$$\mathbb{P}(\limsup_{n\to\infty} |B_3| > \varepsilon a_n) = 0$$

We conclude the proof of Theorem 2. □

Proof of Corollary 2. Using the second-order Georgii-Nguyen-Zessin formula (2.2) with

$$h(\xi,\eta,\mathbf{X}) = \frac{1\!\!1(\eta-\xi\in B(o,R))}{b_n^d|\Lambda_n\cap(\Lambda_n+(\eta-\xi)|} f(\xi,\eta,\mathbf{X}) K\left(\frac{\eta-\xi-t}{b_n}\right),$$

we have

$$\mathrm{E}\widehat{J}_n(t) = \mathrm{E}\int_{\Lambda_n^2} \frac{1\!\!1(\eta-\xi\in B(o,R))}{b_n^d|\Lambda_n\cap(\Lambda_n+(\eta-\xi)|} f(\xi,\eta,\mathbf{X}) K\left(\frac{\eta-\xi-t}{b_n}\right)\lambda(\xi,\eta,\mathbf{X})\mathrm{d}\xi\mathrm{d}\eta.$$

We remember the second order Papangelou conditional intensity by:

$$\lambda(\xi,\eta,\mathbf{X}) = \lambda(\xi,\mathbf{X})\lambda(\eta,\mathbf{X}\cup\{\xi\}) \quad \text{for any } \xi,\eta\in\mathbb{R}^d.$$

Using the finite range property (3.5) for each function $\lambda(\xi,\mathbf{X})$ and $\lambda(\eta,\mathbf{X}\cup\{\xi\})$, we have

$$\begin{aligned}\lambda(\xi,\mathbf{X}) &= \lambda(\xi,\mathbf{X}\cap B(\xi,R))\\ &= \beta^\star \quad \text{when} \quad \mathbf{X}\cap B(\xi,R)=\emptyset\end{aligned}$$

and

$$\begin{aligned}\lambda(\eta,\mathbf{X}\cup\{\xi\}) &= \lambda(\eta,(\mathbf{X}\cap B(\eta,R))\cup\{\xi\})\\ &= \beta^\star V(\eta-\xi) \quad \text{when} \quad \mathbf{X}\cap B(\eta,R)=\emptyset \quad \text{and} \quad \xi\in B(\eta,R).\end{aligned}$$

We obtain by stationarity of $\mathbf{X}$ and from the definition of f giving by (3.1) and $\bar{C}(o,t)$ is defined through (3.6):

$$\begin{aligned}\mathrm{E}\widehat{J}_n(t) &= \beta^{\star 2}\int_{\mathbb{R}^{2d}} \frac{1\!\!1(\xi\in\Lambda_n,\eta\in\Lambda_n,\eta-\xi\in B(o,R))}{b_n^d|\Lambda_n\cap(\Lambda_n+(\eta-\xi)|}\bar{C}(o,\eta-\xi)K\left(\frac{\eta-\xi-t}{b_n}\right)V(\eta-\xi)\mathrm{d}\xi\mathrm{d}\eta\\ &= \beta^{\star 2}\int_{\mathbb{R}^{2d}} \frac{1\!\!1(\xi\in\Lambda_n,\eta+\xi\in\Lambda_n,\eta\in B(o,R))}{b_n^d|\Lambda_n\cap(\Lambda_n+\eta|}\bar{C}(o,\eta)K\left(\frac{\eta-t}{b_n}\right)V(\eta)\mathrm{d}\xi\mathrm{d}\eta\\ &= \beta^{\star 2}\int_{\mathbb{R}^d} \frac{1\!\!1(\eta\in B(o,R))}{b_n^d|\Lambda_n\cap(\Lambda_n+\eta)|}\bar{C}(o,\eta)K\left(\frac{\eta-t}{b_n}\right)V(\eta)\int_{\mathbb{R}^d}1\!\!1(\xi\in\Lambda_n\cap\Lambda_n-\eta)\mathrm{d}\xi\mathrm{d}\eta\\ &= \beta^{\star 2}\int_{\mathbb{R}^d} \frac{|\Lambda_n\cap(\Lambda_n-\eta)|}{b_n^d|\Lambda_n\cap(\Lambda_n+\eta)|}1\!\!1(\eta\in B(o,R)\bar{C}(o,\eta)K\left(\frac{\eta-t}{b_n}\right)V(\eta)\mathrm{d}\eta.\end{aligned}$$

In this way we get:

$$\begin{aligned}&\mathrm{E}\widehat{J}_n(t)-\beta^{\star 2}V(t)\bar{C}(o,t)\\ &= \beta^{\star 2}\int_{\mathbb{R}^d}K(z)\Big(1\!\!1(b_nz+t\in B(o,R))\bar{C}(o,b_nz+t)V(b_nz+t)-V(t)\bar{C}(o,t)\Big)\mathrm{d}z.\end{aligned}$$

By Taylor expansion of the integrand in neighborhood of t and making use of $(A1)$, $(A3)$ and the function $V(t)\bar{C}(o,t)$ has bounded and continuous partial derivatives of order s of in $B^o(t,\delta)$ (for some $\delta > 0$), we get the following rate of convergence

$$\mathrm{E}\widehat{J}_n(t) - \beta^{\star 2}V(t)\bar{C}(o,t) = \mathscr{O}(b_n^s) \quad \text{as} \quad n \to \infty.$$

By the Theorem 2, we complete the proof. □

Proof of Theorem 1. We consider the decomposition

$$\widehat{V}_n(t) - \beta^\star V(t) = \frac{\widehat{J}_n(t) - \beta^{\star 2}V(t)\bar{C}(o,t)}{\widehat{C}_n(t)} + \beta^\star V(t)\frac{\widehat{C}_n(t) - \beta^\star \bar{C}(o,t)}{\widehat{C}_n(t)}. \tag{4.14}$$

Now, we prove by the ergodic theorem (Nguyen and Zessin [15]) that

$$\left|\widehat{C}_n(t) - \beta^\star \bar{C}(o,t)\right| \xrightarrow{a.s.} 0, \quad \text{as} \quad n \to \infty. \tag{4.15}$$

Let $(\tau_y)_{y\in\mathbb{R}^d}$ be the shift group, where τ_y is the translation by the vector $-y \in \mathbb{R}^d$. Let consider the following quantity for each Borel Λ

$$L_\Lambda = \sum_{\xi\in\mathbf{X}\cap\Lambda} f(\xi, t+\xi, \mathbf{X}\setminus\{\xi\}).$$

It is seen that L_Λ is covariante:

$$L_{\tau_y\Lambda}(\tau_y\mathbf{x}) = L_\Lambda(\mathbf{x})$$

and additive: if $\Lambda_1 \cap \Lambda_2 = \emptyset$,

$$L_{\Lambda_1\cap\Lambda_2} = L_{\Lambda_1} + L_{\Lambda_2}.$$

Hence, from the Nguyen and Zessin [15] ergodic theorem, and additionally assume that $\mathbb{P}$ is ergodic, $\widehat{C}_n(t)$ converges almost surely to

$$L = \frac{1}{|\Lambda_0|}\sum_{\xi\in\mathbf{X}\cap\Lambda_0} f(\xi, t+\xi, \mathbf{X}\setminus\{\xi\})$$

where Λ_0 is a bounded Borel subset of $\mathbb{R}^d$. It follows immediately from the GNZ formula (2.1)

$$\begin{aligned} L &= \frac{1}{|\Lambda_0|}\mathrm{E}\sum_{\xi\in\mathbf{X}\cap\Lambda_0} f(\xi, t+\xi, \mathbf{X}\setminus\{\xi\}) \\ &= \frac{1}{|\Lambda_0|}\mathrm{E}\int_{\mathbb{R}^d} 1\!\!1_{\Lambda_0}(\xi) f(\xi, t+\xi, \mathbf{X})\lambda(\xi, \mathbf{X})\mathrm{d}\xi. \end{aligned}$$

Since $\mathbf{X}$ is stationary with translation invariant interaction function and using the finite range property (3.5), i.e.

$$\begin{aligned}\lambda(\xi,\mathbf{X}) &= \lambda(\xi,\mathbf{X}\cap B(\xi,R)) \\ &= \beta^{\star} \quad \text{when} \quad \mathbf{X}\cap B(\xi,R)=\emptyset\end{aligned}$$

then, it follows that $L=\beta^{\star}\bar{C}(o,t)$. We conclude that $\widehat{C}_n(t)$ defined by (4) turns out to be unbiased estimator of $\beta^{\star}\bar{C}(o,t)$ and strongly consistent as n tends infinity.

Using the monotony of functions $\bar{C}$ and $\widehat{C}_n$, we can approach the functions $\bar{C}$ and $\widehat{C}_n$ by their values in a finite number of points, furthermore by the result (4.15), we have as $n\to\infty$,

$$\sup_{t\in K_0}\left|\widehat{C}_n(t)-\beta^{\star}\bar{C}(o,t)\right| \longrightarrow 0 \tag{4.16}$$

almost surely.

By Corollary 2, the result (4.16) and the expression (4.14), we complete the proof. □

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