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# A nonparametric estimator for stationary pairwise interaction point processes under mixing conditions

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## Abstract

We consider a nonparametric estimator of the interaction function of pairwise interaction point process such that its Papangelou conditional intensity is translation invariant and satisfied finite-range interaction. This method of nonparametric inference is based on a single observation in a window which is assumed its volume grows unboundedly in all directions as  $n$  tends infinity. We prove uniform strong consistency of multivariate nonparametric estimates. The main problem consists in deriving appropriate for uniform strong mixing and strong mixing random fields in  $\mathbb{Z}^d$  under Dobrushin's uniqueness condition.

**Keywords:** kernel-type estimator, edge effects, pairwise interaction point process, mixing random fields, rates of converge, Dobrushin uniqueness conditions.

# 1 Introduction

Statistical problems related to modeling and data study spatial know lately great interest in statistics. Found in many areas such as statistical physics, ecology (Diggle [8]), forestry (Matérn [19]), spatial epidemiology (Lawson [17]) and astrophysics (Neyman and Scott [23]). A useful tool for modeling and analysis of Gibbs processes is the Papangelou conditional intensity (Papangelou, [25]). In this work we will develop some recent results of non-parametric statistical processes. The main goal of this paper is to present a nonparametric estimation of interaction function for stationary pairwise interaction point processes (have been introduced in statistical literature by Ripley and Kelly [29], Daley and Vere-Jones [5] and Georgii [11]) defined in the plane  $\mathbb{R}^d$  for which its Papangelou conditional intensity is translation invariant and satisfied finite-range interaction, a formal definition and interpretation are given below. The estimate of the interaction function plays a very important role in the statistical analysis of spatial point pattern data. This method of nonparametric inference is based on a single observation in a window which is assumed its volume grows unboundedly in all directions as  $n$  tends infinity. We exploring the spatial dependence of random fields (mixing random fields) and discuss some of the connections with the uniform strong consistent estimation. Many modelizations of weak dependence have been proposed through in literature, we refer to Bradley [3], Rosenblatt [30], Ibragimov [14], Bosq [2], Delecroix [7], Heinrich et al. [13] and Peligrad [27]. Other hand, Doukhan and Louhichi [10] introduced a new concept of dependence, which can be checked for more general process. The mixing hypothesis was introduced to extend the central limit theorem for more general classes dependent random variables (see Linnik and Ibragimov [15]). It also helps to generalize almost sure convergence obtained with independent data. The paper is organized as follows. In Section 2, we set up the generic notation. In Section 3, we present the model that has been considered for stationary pairwise interaction point process and we introduce the nonparametric estimator for interaction function. In Section 4, we present some conditions that will be helpful in the following paper and we present a result for uniformly strongly consistent of the nonparametric estimator for interaction function. The proof of this result requires more delicate conditions of uniform strong consistency of the kernel-type estimator (3.5) and we exploring the spatial dependence of point processes. In Section 4.1, sufficient conditions to uniform strong consistency are obtained of kernel-type estimator (3.5) in the setting of  $\Phi$ -mixing random fields and also give uniform rate of convergence. Our method of proof of is based on Kahane-Khintchine inequalities in Orlicz spaces induced by exponential Young functions for stationary  $\Phi$ -mixing

random fields which are bounded or satisfy some finite exponential moment. Section 4.2 deals with the uniform strong consistency of the kernel-type estimator (3.5) and uniform rate of convergence are also given. We focus more precisely on estimation issues in the case of a strong mixing ( $\alpha$ -mixing) random fields. Our proof is based on an exponential inequality of Hoeffding type derived from a Marcinkiewicz- Zygmund type inequality (see Serfling [31] or McLeish [20]) for  $p$ -integrable real random fields. The Section 5 is devoted to the proofs. In Appendix, we collect some preliminary corollary and proposition, which are used in the proofs of our main results.

## 2 Generic notation and Basic tools

We introduce some necessary definitions and notations and recall some basic facts from the theory of Gibbs point processes. Let  $N_{lf}$  be the set of all locally-finite counting measures on  $\mathbb{R}^d$  of general dimension  $d \geq 2$ . Let  $\mathcal{B}^d$  be the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$  and  $\mathcal{B}_O^d = \{\Lambda \in \mathcal{B}^d : \Lambda \text{ bounded}\}$ . For a set  $\Lambda \in \mathcal{B}^d$ ,  $|\Lambda|$  is the volume of  $\Lambda$ .  $\sum^{\neq}$  denotes summation over summands with index tuples having pairwise distinct components. The origin in  $\mathbb{R}^d$  will be denoted by  $o$ . A Young function  $\psi$  is a real convex nondecreasing function defined on  $\mathbb{R}^+$  which satisfies  $\lim_{t \rightarrow \infty} \psi(t) = +\infty$  and  $\psi(0) = 0$ . We define the Orlicz space  $L_\psi$  as the space of real random variables  $Z$  defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $E[\psi(|Z|/c)] < +\infty$  for some  $c > 0$ . The Orlicz space  $L_\psi$  equipped with the so-called Luxemburg norm  $\|\cdot\|_\psi$  defined for any real random variable  $Z$  by

$$\|Z\|_\psi = \inf\{c > 0; E[\psi(|Z|/c)] \leq 1\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [16]. Let  $\theta > 0$ . We denote by  $\psi_\theta$  the Young function defined for any  $x \in \mathbb{R}^+$  by

$$\psi_\theta(x) = \exp((x + l_\theta)^\theta) - \exp(l_\theta^\theta) \quad \text{where} \quad l_\theta = ((1 - \theta)/\theta)^{1/\theta} \mathbb{1}_{\{0 < \theta < 1\}}.$$

## 3 The model and its estimator

Throughout this paper, we shall focus on stationary pairwise interaction point processes (Daley and Vere-Jones [5], Georgii [11]). We specify that the Papangelou conditional intensity translation is invariant for a configuration  $\mathbf{x} \in N_{lf}$  and location  $\xi \in \mathbb{R}^d$  should be of the form

$$\lambda(\xi, \mathbf{x}) = \beta^* \exp\left(-\sum_{\eta \in \mathbf{x} \setminus \xi} V_2(\eta - \xi)\right) \quad (3.1)$$

where  $\beta^*$  is the true value of the intensity parameter,  $V_2$  represents the pair potential defined on  $\mathbb{R}^d$  and is assumed a nonnegative function (says that the interaction is purely repulsive) and  $V = \exp(-V_2)$  represents the pairwise interaction function. Heuristically,  $\lambda(\xi, \mathbf{x})d\xi$  can be interpreted as the conditional probability of  $\mathbf{X}$  having a point in an infinitesimal small region containing  $\xi$  and of size  $d\xi$  given the rest of  $\mathbf{X}$  is  $\mathbf{x}$ . The Papangelou conditional intensity is said to have finite interaction range  $R$  with  $0 < R < \infty$  if

$$\lambda(\xi, \mathbf{x}) = \lambda(\xi, \mathbf{x} \cap B(\xi, R)), \quad (3.2)$$

for any  $\xi \in \mathbb{R}^d$ ,  $\mathbf{x} \in N_{lf}$ , where  $B(\xi, R)$  is the closed ball in  $\mathbb{R}^d$  with center  $\xi$  and radius  $R$ . An equivalent approach is to assume that

$$V_2(\eta - \xi) = 0 \quad \text{whenever} \quad \|\eta - \xi\| \geq R.$$

Examples include most pairwise interaction point processes, such as the Strauss process, and more complicated interaction point processes, such as the area-interaction point process (see Baddeley et al [1], Møller and Waagepetersen ([21], [22])). Our proofs are mainly based on the following relation (known as the iterated Georgii-Nguyen-Zessin formula, see Papangelou [26])

$$\mathbb{E} \sum_{\xi, \eta \in \mathbf{X}}^{\neq} h(\xi, \eta, \mathbf{X} \setminus \{\xi, \eta\}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E} h(\xi, \eta, \mathbf{X}) \lambda(\xi, \eta, \mathbf{X}) d\xi d\eta \quad (3.3)$$

where  $\lambda(\xi, \eta, \mathbf{x}) = \lambda(\xi, \mathbf{x})\lambda(\eta, \mathbf{x} \cup \{\xi\})$ , for  $\xi, \eta \in \mathbb{R}^d$ ,  $\mathbf{x} \in N_{lf}$  with a nonnegative measurable function  $h(\xi, \eta, \mathbf{x})$  on  $\mathbb{R}^d \times \mathbb{R}^d \times N_{lf}$ .

New, we let a sequence  $\Lambda_n \in \mathcal{B}_O^d$ , for  $n \in \mathbb{N}$ , denote a bounded observation window which is assumed to expand undoundly in all directions as  $n \rightarrow \infty$ . In most application examples  $\Lambda_n$  cubes growing up to  $\mathbb{R}^d$ . Now, we define nonparametric estimator for the function  $\beta^*V(t)$  by

$$\widehat{V}_n(t) = \frac{\widehat{J}_n(t)}{\widehat{C}_n(t)}, \quad \text{for } t \in \mathbb{R}^d \quad (3.4)$$

where

$$\widehat{J}_n(t) = \sum_{\xi, \eta \in \mathbf{X} \cap \Lambda_n}^{\neq} \frac{\mathbb{1}(\eta - \xi \in B(o, R)) f(\xi, \eta, \mathbf{X} \setminus \{\xi, \eta\}) K\left(\frac{\eta - \xi - t}{b_n}\right)}{b_n^d |\Lambda_n \cap (\Lambda_n + (\eta - \xi))|} \quad (3.5)$$

is a kernel-type estimator estimating  $\beta^{*2}V(t)\bar{C}(o, t)$ , where  $\Lambda_n + u = \{\eta + u : \eta \in \Lambda_n\}$  denotes  $\Lambda_n$  translated by  $u$  and  $|\Lambda_n \cap (\Lambda_n + (\eta - \xi))|$  is an edge correction factor and assume this has volume equal to  $cn^d$ . A kernel  $K$  will be any measurable function associated with a sequence  $(b_n)_{n \geq 1}$  of bandwidths.

$$\hat{C}_n(t) = \sum_{\xi \in \mathbf{X} \cap \Lambda_n} \frac{f(\xi, t - \xi, \mathbf{X} \setminus \{\xi\})}{|\Lambda_n \cap (\Lambda_n + (t - 2\xi))|} \quad (3.6)$$

meanwhile being an estimator of  $\beta^* \bar{C}(o, t)$ . The function  $f$  is a nonnegative measurable function defined for all  $w, w' \in \mathbb{R}^d$ ,  $\mathbf{x} \in N_{lf}$ , by

$$f(w, w', \mathbf{x}) = \mathbb{1}(\mathbf{x} \cap B(w, R) = \emptyset, \mathbf{x} \cap B(w', R) = \emptyset)$$

and we also introduce the following function

$$\bar{C}(o, t) = \mathbb{E}[f(o, t, \mathbf{X})] = \mathbb{P}(\mathbf{X} \cap B(o, R) = \emptyset, \mathbf{X} \cap B(t, R) = \emptyset). \quad (3.7)$$

## 4 Uniform strong consistency

The proof of uniform strong consistency of the kernel-type estimator  $\hat{J}_n(t)$ , requires more delicate conditions. We have to impose certain restriction on the kernel function  $K$  and the sequence of bandwidths  $(b_n)_{n \geq 1}$ .

**Condition  $K(d, s)$  :**

The kernel function  $K : \mathbb{R}^d \rightarrow [0, \infty)$  is bounded with bounded support, and satisfies:

$$\int_{\mathbb{R}^d} K(\xi) d\xi = 1,$$

and for  $s > 1$

$$\int_{\mathbb{R}^d} \xi_1^{a_1} \dots \xi_d^{a_d} K(\xi) d\xi = 0$$

with  $a_1, \dots, a_d \in \mathbb{N}^*$  such that  $0 < \sum_{i=1}^d a_i < s$ .

**Condition  $K(L)$  :**

The kernel function  $K$  satisfies a Lipschitz condition, i.e. there exists a constant  $L > 0$  such that

$$|K(u) - K(v)| \leq L \|u - v\| \quad \text{for any } \xi, \eta \in \mathbb{R}^d.$$

**Condition  $C(b_n)$  :**

The sequence of bandwidths  $(b_n)_{n \geq 1}$  is a decreasing sequence of positive real numbers satisfying

$$\lim_{n \rightarrow \infty} b_n^d = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n^d n^d = \infty.$$

These assumptions are more than enough to guarantee rates of convergence of mean and a uniform strong consistency of the kernel-type estimator  $\widehat{J}_n(t)$ .

Now, we establish thereafter interesting results for establishing the uniform strong consistency of the estimator  $\widehat{V}_n(t)$  whose proofs are given in Section 5.

**Theorem 1.** *Let  $\mathbf{X}$  be a stationary pairwise interaction point process in  $\mathbb{R}^d$  with Papangelou conditional intensity (3.1) satisfying condition (3.2). Let the kernel function  $K$  satisfy the Conditions  $K(d, s)$  and  $K(L)$ . Let the bandwidths  $(b_n)_{n \geq 1}$  satisfy the Condition  $C(b_n)$ . Further, assume that  $V(t)\bar{C}(o, t)$  has bounded and continuous partial derivatives of order  $s$  on for any fixed compact set  $K_0 \subset \mathbb{R}^d$ . Then,*

$$\sup_{t \in K_0} |\widehat{V}_n(t) - \beta^* V(t)| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

The proof of uniform strong consistency of the nonparametric estimator  $\widehat{V}_n(t)$  is based on some asymptotic properties of the estimators  $\widehat{C}_n(t)$  and  $\widehat{J}_n(t)$ . For this last, we exploring the spatial dependence of point processes. We first study uniform strong consistency (including a convergence rate) of the estimator  $\widehat{J}_n(t)$  by considering  $\phi$ -mixing random fields. Secondly We focus more precisely on estimation issues in the case of a strong mixing ( $\alpha$ -mixing) random fields, we study uniform strong consistency of the kernel-type estimator  $\widehat{J}_n(t)$  and uniform rate of convergence are also given. It will be revealed that the class of Gibbs point processes under Dobrushin uniqueness conditions (Dobrushin [9]; Georgii [11]; Guyon [12]), satisfies these mixing conditions.

#### 4.1 Rates of uniform strong consistency for $\widehat{J}_n(t)$ under uniformly strong mixing random fields

To obtain consistent estimators it is necessary to make assumptions on the asymptotic independence process observed. The mixing hypotheses are the most used. In this section we discuss sufficient conditions which ensure uniform strong consistency for the estimator  $\widehat{J}_n(t)$  under uniformly strong mixing random fields. We use Kahane-Khintchine inequalities in Orlicz spaces induced by exponential Young functions for stationary  $\phi$ -mixing real random fields which are bounded or satisfy some finite exponential moment condition. Given two sub- $\sigma$ -algebras  $\mathcal{U}$  and  $\mathcal{V}$  of  $\mathcal{F}$ . The  $\phi$ -mixing coefficients had been introduced by Ibragimov [14] and can be defined by:

$$\phi(\mathcal{U}, \mathcal{V}) = \sup\{\|\mathbb{P}(V|\mathcal{U}) - \mathbb{P}(V)\|_\infty, V \in \mathcal{V}\}.$$

Denote by  $|\Gamma|$  the cardinality of any subset  $\Gamma$  of  $\mathbb{Z}^d$ . In the sequel, we consider the uniform strong mixing coefficients  $\phi_{1,\infty}(n)$  defined for each positive integer  $n$  by

$$\phi_{1,\infty}(n) = \sup \{ \phi((\sigma(X_k), \mathcal{F}_\Gamma), k \in \mathbb{Z}^d, \Gamma \subset \mathbb{Z}^d, \rho(\Gamma, \{k\}) \geq n) \},$$

where  $\mathcal{F}_\Gamma = \sigma(X_i, i \in \Gamma)$ . The distance  $\rho$  is defined by  $\rho = \min\{|i - j| : i \in \Gamma_1, j \in \Gamma_2\}$ . We say that the random field  $(X_i)_{i \in \mathbb{Z}^d}$  is  $\phi$ -mixing if  $\lim_{n \rightarrow \infty} \phi_{1,\infty}(n) = 0$ .

Now, we assume that the domain  $\Lambda_n$  is divided into a fixed number of subdomains as follows

$$\Lambda_n = \cup_{i \in \Gamma_n} \Lambda_i,$$

all the  $\Lambda_i$  are disjoint with the same volume  $|\Lambda_i|$  increasing up to  $+\infty$  and assume that  $\Gamma_n$  increases towards  $\mathbb{Z}^d$ . Setting  $X_i = X \cap \Lambda_i$ ,  $i \in \Gamma_n$  be a real random field. We will consider estimation of  $\beta^{*2}V(t)\bar{C}(o, t)$  from  $\hat{J}_n(t)$ , where the process is observed in  $\cup_{i \in \Gamma_n} \Lambda_i$  as

$$\hat{J}_n(t) = \sum_{i \in \Gamma_n} \sum_{\xi, \eta \in \mathbf{X} \cap \Lambda_i}^{\neq} \frac{\mathbb{1}(\eta - \xi \in B(o, R)) f(\xi, \eta, \mathbf{X} \setminus \{\xi, \eta\}) K\left(\frac{\eta - \xi - t}{b_n}\right)}{b_n^d |\Lambda_n \cap (\Lambda_n + (\eta - \xi))|}.$$

Put

$$J_{n,i} = \sum_{\xi, \eta \in \mathbf{X} \cap \Lambda_i}^{\neq} \mathbb{1}(\eta - \xi \in B(o, R)) f(\xi, \eta, \mathbf{X} \setminus \{\xi, \eta\}) K\left(\frac{\eta - \xi - t}{b_n}\right)$$

we note that:  $\bar{J}_{n,i} = J_{n,i} - \mathbb{E} J_{n,i}$  and  $S_n = \sum_{i \in \Gamma_n} \bar{J}_{n,i}$ .

**Theorem 2.** *Let  $\mathbf{X}$  be a stationary pairwise interaction point process in  $\mathbb{R}^d$  with Papangelou conditional intensity (3.1). Let the kernel function  $K$  satisfy Condition  $K(L)$ . Then, for any fixed compact set  $K_0 \subset \mathbb{R}^d$ , we have:*

1) *If there exists  $0 < q < 2$  such that  $\bar{J}_{n,0} \in \mathbb{L}_{\psi_{\theta(q)}}$  and under condition*

$$\sum_{k \in \mathbb{Z}^d} \sqrt{\phi_{1,\infty}(|k|)} < \infty. \quad (4.8)$$

*Then*

$$\sup_{t \in K_0} |\hat{J}_n(t) - \mathbb{E}(\hat{J}_n(t))| = \mathcal{O}_{a.s.} \left( \frac{(\log n)^{1/q}}{(b_n \sqrt{n})^d} \right) \quad \text{as } n \rightarrow \infty.$$



2) If  $\bar{J}_{n,0} \in \mathbb{L}^\infty$  and under condition

$$\sum_{k \in \mathbb{Z}^d} \phi_{1,\infty}(|k|) < \infty. \quad (4.9)$$

Then

$$\sup_{t \in K_0} |\hat{J}_n(t) - \mathbb{E}(\hat{J}_n(t))| = \mathcal{O}_{a.s.} \left( \frac{(\log n)^{1/2}}{(b_n \sqrt{n})^d} \right) \quad \text{as } n \rightarrow \infty.$$

**Corollary 1.** *If in addition the function  $V(t)\bar{C}(o,t)$  has bounded and continuous partial derivatives of order  $s$  on  $K_0 \subset \mathbb{R}^d$  and suppose that the Conditions  $K(d,s)$  and  $C(b_n)$  are satisfied. We have:*

1) *If there exists  $0 < q < 2$  such that  $\bar{J}_{n,0} \in \mathbb{L}_{\psi_{\theta(q)}}$  and under condition*

$$\sum_{k \in \mathbb{Z}^d} \sqrt{\phi_{1,\infty}(|k|)} < \infty.$$

Then

$$\sup_{t \in K_0} |\hat{J}_n(t) - \beta^{*2}V(t)\bar{C}(o,t)| = \mathcal{O}_{a.s.} \left( \frac{(\log n)^{1/q}}{(b_n \sqrt{n})^d} \right) + \mathcal{O}(b_n^s) \quad \text{as } n \rightarrow \infty.$$

2) If  $\bar{J}_{n,0} \in \mathbb{L}^\infty$  and under condition

$$\sum_{k \in \mathbb{Z}^d} \phi_{1,\infty}(|k|) < \infty.$$

Then

$$\sup_{t \in K_0} |\hat{J}_n(t) - \beta^{*2}V(t)\bar{C}(o,t)| = \mathcal{O}_{a.s.} \left( \frac{(\log n)^{1/2}}{(b_n \sqrt{n})^d} \right) + \mathcal{O}(b_n^s) \quad \text{as } n \rightarrow \infty.$$

## 4.2 Rates of uniform strong consistency for $\hat{J}_n(t)$ under strongly mixing random fields

In this section, we apply Theorem 2 to strongly mixing variables. For more details on strong mixing conditions, the reader should refer to the recent survey by Bradley [3]. The  $\alpha$ -mixing coefficient (or strong mixing coefficient) by Rosenblatt [30] between the sub- $\sigma$ -algebras  $\mathcal{U}$  and  $\mathcal{V}$  of  $\mathcal{F}$  is defined by

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(U \cap V) - \mathbb{P}(U)\mathbb{P}(V)|, U \in \mathcal{U}, V \in \mathcal{V}\}.$$

In the sequel, we consider the strong mixing coefficients  $\alpha_{1,\infty}(n)$  defined for each positive integer  $n$  by

$$\alpha_{1,\infty}(n) = \sup \{ \alpha(\sigma(X_k), \mathcal{F}_\Gamma), k \in \mathbb{Z}^d, \Gamma \subset \mathbb{Z}^d, \rho(\Gamma, \{k\}) \geq n \}.$$

where  $\mathcal{F}_\Gamma = \sigma(X_i, i \in \Gamma)$ . We say that the random field  $(X_i)_{i \in \mathbb{Z}^d}$  is strongly mixing if  $\lim_{n \rightarrow \infty} \alpha_{1,\infty}(n) = 0$ .

For this purpose a further condition of uniform strong consistency for kernel estimator is need.

**Theorem 3.** *Suppose that the Conditions  $K(d, s)$ ,  $K(L)$  and  $C(b_n)$  are satisfied. Further, assume that  $V(t)\bar{C}(o, t)$  has bounded and continuous partial derivatives of order  $s$  on  $K_0$ . If there exists  $p > 2$  such that  $\bar{J}_{n,0} \in \mathbb{L}^p$  and under condition*

$$\sum_{k \in \mathbb{Z}^d} \left( \int_0^{\alpha_{1,\infty}(|k|)} Q_{\bar{J}_0}^p(\zeta) d\zeta \right)^{2/p} < \infty \quad (4.10)$$

where for any  $\zeta \geq 0$ ,  $Q_{\bar{J}_0}(\zeta) = Q_{J_0 - \mathbb{E} J_0}(\zeta) = \inf \{ \delta > 0, |\mathbb{P}(|J_0 - \mathbb{E} J_0| > \delta) \leq \zeta \}$ .

Assume that  $b_n = n^{-q_2}(\log n)^{q_1}$  for some  $q_1, q_2 > 0$ . Let  $a, b \geq 0$  be fixed and if  $a(p+d) - d^2/2 - q_2d > 1$  and  $b(p+d) + q_1d > 1$ . Then

$$\sup_{t \in K_0} |\hat{J}_n(t) - \beta^{*2}V(t)\bar{C}(o, t)| = \mathcal{O}_{a.s.} \left( \frac{n^a(\log n)^b}{(b_n\sqrt{n})^d} \right) + \mathcal{O}(b_n^s) \quad \text{as } n \rightarrow \infty.$$

## 5 Proofs

### 5.1 Proof of Theorem 2

*Proof.* Let  $(a_n)_{n \geq 1}$  be sequence of positive numbers going to zero. Following Carbon and al. [4], the compact set  $K_0$  can be covered by  $v_n$  cubes  $T_k$  having sides of length  $L_n = a_n b_n^{d+1}$  and center at  $c_k$ . Clearly there exists  $c > 0$ , such that  $v_n \leq c/L_n^d$ . We use the following classical decomposition

$$\begin{aligned} \sup_{t \in K_0} |\hat{J}_n(t) - \mathbb{E} \hat{J}_n(t)| &\leq \max_{1 \leq k \leq v_n} \sup_{t \in T_k} |\hat{J}_n(t) - \hat{J}_n(c_k)| \\ &\quad + \max_{1 \leq k \leq v_n} \sup_{t \in T_k} |\mathbb{E} \hat{J}_n(t) - \mathbb{E} \hat{J}_n(c_k)| \\ &\quad + \max_{1 \leq k \leq v_n} |\hat{J}_n(c_k) - \mathbb{E} \hat{J}_n(c_k)| \\ &= B_1 + B_2 + B_3. \end{aligned}$$

For any  $t \in T_k$ , by Lipschitz condition, we derive that there exists constant  $L > 0$  such that  $n$  sufficiently large

$$\left| \widehat{J}_n(t) - \widehat{J}_n(c_k) \right| \leq L a_n \frac{1}{c n^d} \sum_{\xi, \eta \in \mathbf{X} \cap \Lambda_n}^{\neq} \mathbb{1}(\eta - \xi \in B(o, R)) f(\xi, \eta, \mathbf{X} \setminus \{\xi, \eta\}).$$

Using the spatial ergodic theorem of Nguyen and Zessin [24] and by the iterated Georgii-Nguyen-Zessin formula (3.3) with

$$h(\xi, \eta, \mathbf{X}) = \mathbb{1}(\eta - \xi \in B(o, R)) f(\xi, \eta, \mathbf{X})$$

it easily follows

$$B_1 = \mathcal{O}_{a.s.}(a_n). \quad (5.11)$$

From  $L^1$ -version of ergodic theorem of Nguyen and Zessin [24], it follows

$$B_2 = \mathcal{O}_{a.s.}(a_n). \quad (5.12)$$

Now, we focus on the term  $B_3$ . For  $\varepsilon > 0$ , using Markov's inequality, we get

$$\begin{aligned} \mathbb{P} \left( |\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n \right) &= \mathbb{P} \left( |S_n| > \varepsilon a_n (n b_n)^d \right) \\ &\leq \exp \left[ - \left( \frac{\varepsilon a_n (n b_n)^d}{\|S_n\|_{\psi_{\theta(q)}}} + l_q \right)^q \right] \mathbb{E} \exp \left[ \left( \frac{|S_n|}{\|S_n\|_{\psi_{\theta(q)}}} + l_q \right)^q \right]. \end{aligned} \quad (5.13)$$

1) Therefore, we assume that there exists a real  $0 < q < 2$ , such that  $\bar{J}_{n,0} \in \mathbb{L}_{\psi_{\theta(q)}}$ . As a direct application of Kahane-Khintchine inequality of Corollary 2 (El Machkouri [18]) as cited in the appendix section, to the sequence  $\bar{J}_{n,i}$ ,  $i \in \Gamma_n$ , we obtain that:

$$\mathbb{P} \left( |S_n| > \varepsilon a_n (b_n n)^d \right) \leq (1 + e^{l_q^q}) \exp \left[ - \left( \frac{\varepsilon a_n (b_n n)^d}{M_1(q) \|\bar{J}_{n,0}\|_{\psi_{\theta(q)}}^2 (\sum_{i \in \Gamma_n} \tilde{b}_{i,q}(\bar{J}))^{1/2}} + l_q \right)^q \right]$$

where

$$\tilde{b}_{i,q}(\bar{J}) = 1 + C(q) \sum_{k \in V_0^1} \sqrt{\phi_{1,\infty}(|k|)},$$

$M_1(q)$  and  $C(q)$  are positive universal constants depending only on  $q$ .

Thus, using the Dobrushin's uniqueness condition (see Dobrushin [9]; Georgii [11]; Guyon [12]), the condition (2) is satisfied and from the stationary of  $\mathbf{X}$  and by definition of the norm  $\|\cdot\|_{\psi_{\theta(q)}}$ , we infer that there exists a constant  $C > 0$  such that

$$\mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) \leq (1 + e^{l_q^q}) \exp \left[ - \frac{\varepsilon^q a_n^q (b_n \sqrt{n})^{dq}}{C^q} \right].$$

Inserting  $a_n = (\log n)^{1/q}/(b_n\sqrt{n})^d$ , we see after a short calculation that

$$\mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) \leq (1 + e^{l_q^q}) \exp \left[ - \frac{\varepsilon^q \log n}{C^q} \right].$$

From last equality we find that

$$\sup_{t \in K_0} \mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) \leq (1 + e^{l_q^q}) \exp \left[ - \frac{\varepsilon^q \log n}{C^q} \right]. \quad (5.14)$$

Since

$$\mathbb{P}(|B_3| > \varepsilon a_n) \leq v_n \sup_{t \in K_0} \mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n),$$

using (5.14), it follows with Borel-Cantelli's lemma

$$\mathbb{P}(\limsup_{n \rightarrow \infty} |B_3| > \varepsilon a_n) = 0$$

which completes the first assertion of Theorem 2.

2) By reformulating Markov's inequality (5.13) and by of Corollary 2 (El Machkouri [18]) as cited in the appendix section in the case  $\bar{J}_{n,0} \in \mathbb{L}^\infty$  and for  $\varepsilon$  a real positive, we obtain

$$\begin{aligned} \mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) &= \mathbb{P}(|S_n| > \varepsilon a_n (b_n n)^d) \\ &\leq (1 + e^{l_q^q}) \exp \left[ - \left( \frac{\varepsilon a_n (b_n n)^d}{M_2(q) \|\bar{J}_{n,0}\|_\infty (\sum_{i \in \Gamma_n} \tilde{b}_{i,\infty}(\bar{J}))^{1/2}} + l_q \right)^q \right] \end{aligned}$$

where

$$\tilde{b}_{i,\infty}(\bar{J}) = 1 + 2 \sum_{k \in V_0^1} \phi_{1,\infty}(|k|).$$

In analogy to the above proof, we derive under condition (4.9) and if  $a_n = (\log n)^{1/2}/(b_n\sqrt{n})^d$ , that there exists a constant  $C > 0$  such that

$$\mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) \leq 2 \exp \left[ - \frac{\varepsilon^2 \log n}{C^2} \right].$$

We may write that,

$$\sup_{t \in K_0} \mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) \leq 2 \exp \left[ - \frac{\varepsilon^2 \log n}{C^2} \right]. \quad (5.15)$$

Since

$$\mathbb{P}(|B_3| > \varepsilon a_n) \leq v_n \sup_{t \in K_0} \mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n),$$

using (5.15), it follows with Borel-Cantelli's lemma

$$\mathbb{P}(\limsup_{n \rightarrow \infty} |B_3| > \varepsilon a_n) = 0.$$

We conclude the proof of Theorem 2. ■

## 5.2 Proof of Corollary 1

*Proof.* Using the iterated Georgii-Nguyen-Zessin formula (3.3) with

$$h(\xi, \eta, \mathbf{X}) = \frac{\mathbb{1}(\eta - \xi \in B(o, R))}{b_n^d |\Lambda_n \cap (\Lambda_n + (\eta - \xi))|} f(\xi, \eta, \mathbf{X}) K\left(\frac{\eta - \xi - t}{b_n}\right),$$

we have

$$\mathbb{E} \widehat{J}_n(t) = \mathbb{E} \int_{\Lambda_n^2} \frac{\mathbb{1}(\eta - \xi \in B(o, R))}{b_n^d |\Lambda_n \cap (\Lambda_n + (\eta - \xi))|} f(\xi, \eta, \mathbf{X}) K\left(\frac{\eta - \xi - t}{b_n}\right) \lambda(\xi, \eta, \mathbf{X}) d\xi d\eta.$$

We remember the second order Papangelou conditional intensity by:

$$\lambda(\xi, \eta, \mathbf{X}) = \lambda(\xi, \mathbf{X}) \lambda(\eta, \mathbf{X} \cup \{\xi\}) \quad \text{for any } \xi, \eta \in \mathbb{R}^d.$$

Now, using the finite range property (3.2) for each function  $\lambda(\xi, \mathbf{X})$  and  $\lambda(\eta, \mathbf{X} \cup \{\xi\})$ . This implies that

$$\begin{aligned} \lambda(\xi, \mathbf{X}) &= \lambda(\xi, \mathbf{X} \cap B(\xi, R)) \\ &= \beta^* \quad \text{when } \mathbf{X} \cap B(\xi, R) = \emptyset \end{aligned}$$

and

$$\begin{aligned} \lambda(\eta, \mathbf{X} \cup \{\xi\}) &= \lambda(\eta, (\mathbf{X} \cap B(\eta, R)) \cup \{\xi\}) \\ &= \beta^* V(\eta - \xi) \quad \text{when } \mathbf{X} \cap B(\eta, R) = \emptyset. \end{aligned}$$

In this way we obtain by stationarity of  $\mathbf{X}$

$$\mathbb{E} \widehat{J}_n(t) - \beta^{*2} V(t) \bar{C}(o, t) = \beta^{*2} \int K(z) (\bar{C}(o, b_n z + t) V(b_n z + t) - V(t) \bar{C}(o, t)) dz$$

where  $\bar{C}(o, t)$  defined throug (3.7).

By Taylor expansion of the integrand in neighborhood of  $t$  and making use of Conditions  $K(d, s)$ ,  $C(b_n)$  and the the function  $G(t)\bar{F}(o, t)$  has bounded and continuous partial derivatives of order  $s$  of in  $B^o(t, \delta)$  (for some  $\delta > 0$ ), we get the following rate of convergence

$$\mathbb{E} \widehat{J}_n(t) - \beta^{*2} V(t) \bar{C}(o, t) = \mathcal{O}(b_n^s) \quad \text{as } n \rightarrow \infty. \quad (5.16)$$

By the Theorem 2, we complete the proof.  $\blacksquare$

### 5.3 Proof of Theorem 3

*Proof.* Let  $p > 2$  be fixed, such that  $\bar{J}_{n,0} \in \mathbb{L}^p$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} \mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) &= \mathbb{P}(|S_n| > \varepsilon a_n (b_n n)^d) \\ &\leq \frac{\varepsilon^{-p} \mathbb{E} |S_n|^p}{a_n^p (b_n n)^{pd}} \\ &\leq \frac{\varepsilon^{-p}}{a_n^p (b_n n)^{pd}} \left( 2p \sum_{i \in \Gamma_n} c_i(\bar{J}) \right)^{p/2}. \end{aligned}$$

The last inequality follows from a Marcinkiewicz-Zygmund type inequality by Dedecker [6] (see Proposition 1 as cited in the appendix section), where

$$c_i(\bar{J}) = \|\bar{J}_i\|_p^2 + \sum_{k \in V_i^1} \|\bar{J}_k E_{|k-i|}(\bar{J}_i)\|_{\frac{p}{2}}.$$

For  $\alpha$ -mixing random fields, we control  $\sum_{k \in V_i^1} \|\bar{J}_k E_{|k-i|}(\bar{J}_i)\|_{\frac{p}{2}}$  by combining Rio's inequality [28], we obtain

$$\|\bar{J}_k E_{|k|}(\bar{J}_0)\|_{\frac{p}{2}} \leq 4 \left( \int_0^{\alpha_{1,\infty}(|k|)} Q_{\bar{J}_0}^p(\zeta) d\zeta \right)^{2/p}.$$

Therefore we deduce that under condition (4.10), there is a constant  $\kappa > 0$  and from the stationary of  $\mathbf{X}$  such that

$$\mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) \leq \frac{\kappa \varepsilon^{-p}}{a_n^p b_n^p (\sqrt{n})^{pd}}.$$

We conclude

$$\begin{aligned} \mathbb{P}(|B_3| > \varepsilon a_n) &\leq v_n \sup_{t \in K_0} \mathbb{P}(|\widehat{J}_n(t) - \mathbb{E} \widehat{J}_n(t)| > \varepsilon a_n) \\ &\leq v_n \frac{\kappa \varepsilon^{-p}}{a_n^p (b_n \sqrt{n})^{pd}}. \end{aligned}$$

Since  $v_n \leq c/L_n^d$  and  $L_n = a_n b_n^{1+d}$ , then for  $a_n = n^a (\log n)^b / (b_n \sqrt{n})^d$ , it follows

$$\begin{aligned} \mathbb{P}(|B_3| > \varepsilon a_n) &\leq \frac{\kappa \varepsilon^{-p}}{n^{a(p+d)-d^2/2} (\log n)^{b(p+d)} b_n^d} \\ &\leq \frac{\kappa \varepsilon^{-p}}{n^{a(p+d)-d^2/2-q_2 d} (\log n)^{b(p+d)+q_1 d}}. \end{aligned}$$

For  $a(p+d) - d^2/2 - q_2 d > 1$  and  $b(p+d) + q_1 d > 1$ , we have

$$\sum_{n \geq 1} \mathbb{P}(|B_3| > \varepsilon a_n) < \infty.$$

We conclude the proof of Theorem 3 by combining (5.11), (5.12) and (5.16). ■

## 5.4 Proof of Theorem 1

*Proof.* For  $t \in K_0$ , we assume that

$$\exists \alpha_1 > 0, \quad \beta^* \bar{C}(o, t) \geq \alpha_1 \quad \text{and} \quad \exists \alpha_2 > 0, \quad \beta^{*2} V(t) \bar{C}(o, t) \leq \alpha_2,$$

then,

$$|\hat{V}_n(t) - \beta^* V(t)| \leq \frac{|\hat{J}_n(t) - \beta^{*2} V(t) \bar{C}(o, t)|}{\alpha_1 - |\hat{C}_n(t) - \beta^* \bar{C}(o, t)|} + \alpha_2 \alpha_1^{-1} |\hat{C}_n(t) - \beta^* \bar{C}(o, t)|. \quad (5.17)$$

Additionally assume that  $\mathbb{P}$  is ergodic, the estimator (3.6) turn out to be unbiased estimator of  $\beta^* \bar{C}(o, t)$  and strongly consistent as  $n$  tends infinity, since a classical ergodic theorem for spatial point processes obtained in [24]. This implies the following: as  $n \rightarrow \infty$ ,

$$|\hat{C}_n(t) - \beta^* \bar{C}(o, t)| \xrightarrow{a.s.} 0. \quad (5.18)$$

Using the monotony of functions  $\tilde{C}$  and  $\hat{C}_n$ , we can approach the functions  $\tilde{C}$  and  $\hat{C}_n$  by their values in a finite number of points. Bringing this remark and the result (5.18), we have as  $n \rightarrow \infty$ ,

$$\sup_{t \in K_0} |\hat{C}_n(t) - \beta^* \bar{C}(o, t)| \longrightarrow 0 \quad (5.19)$$

almost surely.

By Corollary 1 or Theorem 3, the expression(5.19) and the expression (5.17), we complete the proof. ■

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## 6 Appendix

we collect some preliminary corollary and proposition, which are used in the proofs of our main results. Consider the following assumption :

$$\exists q \in ]0, 2[ \quad \exists c > 0 \quad \mathbb{E}[\exp(c|X_0|^{\theta(q)})] < +\infty \quad (6.20)$$

where  $\theta(q) = 2q/(2 - q)$  for any  $0 < q < 2$ .

**Corollary 2.** *(El Machkouri [18]) Let  $(X_i)_{i \in \mathbb{Z}^d}$  be a zero mean stationary real random field which satisfies the assumption (6.20) for some  $0 < q < 2$ . For any family  $(a_i)_{i \in \mathbb{Z}^d}$  of real numbers and any finite subset  $\Gamma$  de  $\mathbb{Z}^d$ ,*

$$\left\| \sum_{i \in \Gamma} a_i X_i \right\|_{\psi_q} \leq M_1(q) \|X_0\|_{\psi_{\theta(q)}} \left( \sum_{i \in \Gamma} |a_i| \tilde{b}_{i,q}(X) \right)^{1/2}$$

where

$$\tilde{b}_{i,q}(X) = |a_i| + C(q) \sum_{k \in V_0^1} |a_{k+i}| \sqrt{\phi_{1,\infty}(|k|)}.$$

If  $(X_i)_{i \in \mathbb{Z}^d}$  is bounded then for any  $0 < q \leq 2$ , there exists a universal positive constant  $M_2(q)$  depending only on  $q$  such that any family  $(a_i)_{i \in \mathbb{Z}^d}$  of real numbers and any finite subset  $\Gamma$  de  $\mathbb{Z}^d$ ,

$$\left\| \sum_{i \in \Gamma} a_i X_i \right\|_{\psi_q} \leq M_2(q) \|X_0\|_{\infty} \left( \sum_{i \in \Gamma} |a_i| \tilde{b}_{i,\infty}(X) \right)^{1/2}$$

where

$$\tilde{b}_{i,\infty}(X) = |a_i| + 2 \sum_{k \in V_0^1} |a_{k+i}| \phi_{1,\infty}(|k|).$$

**Proposition 1.** *(Dedecker, 2001) Let  $(X_i)_{i \in \mathbb{Z}^d}$  be a zero mean real random field and  $\Gamma$  be a finite subset of  $\mathbb{Z}^d$ . For any  $p > 2$ ,*

$$\left\| \sum_{i \in \Gamma} X_i \right\|_p \leq \left( 2p \sum_{i \in \Gamma} c_i(X) \right)^{1/2}$$

where

$$c_i(X) = \|X_i\|_p^2 + \sum_{k \in V_i^1} \|X_k E_{|k-i|}(X_i)\|_{\frac{p}{2}}.$$



## References

- [1] A. Baddeley, R. Turner, J. Møller, and M. Hazelton. Residual analysis for spatial point processes. *Journal of the Royal Statistical Society, Series B*, 67:1–35, 2005.
- [2] D. Bosq. La statistique non paramétrique des processus. *Rend. Sem. Mat. Univers. Politecn. Torino*, 45(2):1–24, 1987.
- [3] R. Bradley. Introduction to Strong Mixing Conditions. *Kendrick Press*, 3:1–3, 2007.
- [4] M. Carbon, M. Hallin, and B. Wu. Kernel density estimation for random fields. *Nonparametric Statist*, 6:157–170, 1996.
- [5] D. Daley and D. Vere-Jones. *An introduction to the Theory of Point Processes*. Springer Verlag, New York, 1988.
- [6] J. Dedecker. Exponential inequalities and functional central limit theorems for random fields. *ESAIM Probability and Statistics*, 5:77–104, 2001.
- [7] M. Delecroix. *Thèse de Doctorat d’Etat, Université de Lille 1*, 1987.
- [8] P.J. Diggle. *Statistical analysis of spatial point patterns*. Academic Press, London, 2003.
- [9] R.L. Dobrushin. The description of a random field by means of conditional probabilities and condition of its regularity. *Theory of Probability and its Applications*, 13:197–224, 1968.
- [10] P. Doukhan and S. Louhichi. A new weak dependence condition and application to moment inequalities. *Stochastic Processes and their Applications*, 84:313–342, 1999.
- [11] H.O. Georgii. *Gibbs measures and Phase Transitions*. Walter de Gruyter, Berlin, 1988.
- [12] X. Guyon. *Random Fields on a Network: Modeling, Statistics and Applications*. Springer, New York, 1995.
- [13] L. Heinrich and E. Liescher. Strong convergence of kernel estimators for product densities of absolutely regular point processes. *S. J. Nonparametric Statistics*, 1:65–96, 1997.

- [14] I.A.Ibragimov. Some limit theorems for stationary processes. *Theory of Probability and Applications*, 7(4):349–382, 1962.
- [15] I.A. Ibragimov and Yu.V. Linnik. Independent and Stationary Sequences of Random Variables. *Wolters-Noordhoff, Groningen*, 1971.
- [16] M. A. Krasnosel'skii and Y. B. Rutickii. *Convex Functions and Orlicz Spaces*, P. Noordhoff LTD-Groningen-The Netherlands, 1961.
- [17] A.B. Lawson. *Statistical Methods in Spatial Epidemiology*. Wiley, New York, 2001.
- [18] M. El Machkouri. Kahane-khintchine inequalities and functional central limit theorem for stationary real random fields. *Stochastic Processes and Their Applications*, 120:285–299, 2002.
- [19] B. Matérn. *Spatial Variation: Stochastic Models and their Applications to Some Problems in Forest Surveys and Other Sampling Investigations*. 2nd. Springer, Heidelberg, 1986.
- [20] D.L. McLeish. A maximal inequality and dependent strong laws. *Annals of Probability*, 3(5):829–839, 1975.
- [21] J. Møller and R. Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes*. Chapman and Hall/CRC, Boca Raton, 2004.
- [22] J. Møller and R.P. Waagepetersen. Modern statistics for spatial point processes. *Scandinavian Journal of Statistics*, 34(4):643–684, 2007.
- [23] J. Neyman and E.L. Scott. Statistical Approach to Problems of Cosmology. *Journal of the Royal Statistical Society, series B*, 20:1–43, 1958.
- [24] X.X. Nguyen and H. Zessin. Ergodic theorems for Spatial Process. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 48:133–158, 1979.
- [25] F. Papangelou. The conditional intensity of general point processes and application to line processes . *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 28:207–227, 1974.
- [26] F. Papangelou. The Armenian Connection: Reminiscences from a time of interactions . *Contemporary mathematical analysis*, 44(1):14–19, 2009.
- [27] M. Pelidrad. Properties of uniform consistency of the kernel estimators of density and of regression functions under dependence assumptions. *Reidel Publishing Company*, 33-43, 1987.

- [28] E. Rio. Covariance for strongly mixing processes. *Annales l'IHP*, 29(4):587–597, 1993.
- [29] B. Ripley and F. Kelly. Markov point processes. *Journal of the London Mathematical Society*, 15:188–192, 1977.
- [30] M. Rosenblatt. A central limit theorem and a strong mixing condition. *Nat. Acad. Sci. USA*, 42(1):43–47, 1956.
- [31] R.J. Serfling. Contributions to central limit theory for dependent variables. *Annals of Statistics Mathematics*, 39(4):1158–1175, 1968.