



**HAL**  
open science

# Automorphism groups of Koras–Russell threefolds of the second kind

Charlie Petitjean

► **To cite this version:**

Charlie Petitjean. Automorphism groups of Koras–Russell threefolds of the second kind. *Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry*, 2016, 57 (3), pp.599-605. 10.1007/s13366-016-0282-x . hal-01179229

**HAL Id: hal-01179229**

**<https://hal.science/hal-01179229v1>**

Submitted on 22 Jul 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# AUTOMORPHISM GROUPS OF KORAS-RUSSELL THREEFOLDS OF THE SECOND KIND

CHARLIE PETITJEAN

ABSTRACT. We determine the automorphism groups of Koras-Russell threefolds of the second kind. In particular we show that these groups are semi-direct products of two subgroups, one given by the multiplicative group and the other isomorphic to a polynomial ring in two variables with the addition law. We also show that these groups are generated by algebraic subgroups isomorphic to  $\mathbb{G}_m$  and  $\mathbb{G}_a$ .

## 1. INTRODUCTION

In this article, we study automorphism groups of a family of affine threefolds, called *Koras-Russell threefolds of the second kind*. These threefolds are smooth, contractible affine varieties which first appeared in the work of Koras and Russell on the linearization problem for action of the multiplicative group  $\mathbb{G}_m$  on  $\mathbb{A}^3$  ( see [Ka-K-ML-R, Ka-ML, K-R, ML1]). They can be roughly classified into three types according to the richness of the additive group action  $\mathbb{G}_a$  on them.

Koras-Russell threefolds of the first kind are defined by equations of the form:  $\{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\}$  in  $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ , where  $2 \leq d$ ,  $2 \leq \alpha_2 \leq \alpha_3$  with  $\gcd(\alpha_2, \alpha_3) = 1$ . These varieties admit several  $\mathbb{G}_a$ -actions.

Koras-Russell threefolds of the second kind are defined by equations of the form:  $\{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\}$  in  $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ , where  $2 \leq d$ ,  $1 \leq l$ ,  $2 \leq \alpha_2 \leq \alpha_3$  with  $\gcd(\alpha_2, d) = \gcd(\alpha_2, \alpha_3) = 1$ . These varieties admit essentially one  $\mathbb{G}_a$ -action ( see the second section). Such varieties are called *semi-rigid* ( see [A, 4.1.2]).

The other Koras-Russell threefolds which will be called of third kind do not admit non trivial actions of the additive group; such varieties are called rigid.

Although initially used only as part of the proof of linearization of  $\mathbb{G}_m$ -actions on  $\mathbb{A}^3$ , these varieties are now of interest in their own right as exotic hypersurfaces and also as counter-example of the cancellation property for example ( see [D, D-MJ-P, MJ, D-MJ-P1]).

By design all Koras-Russell threefolds admit hyperbolic  $\mathbb{G}_m$ -actions with a unique fixed point. The questions considered here are: what is the structure of their automorphism groups, what are the actions of the additive and multiplicative groups and more generally what are the algebraic subgroups.

The problem of describing the group of polynomial automorphisms of affine spaces is a classical subject in algebraic geometry. In dimension two, the theorem of Jung and Van der Kulk [J, VK] shows that every automorphism of the

---

2000 *Mathematics Subject Classification.* 14R10; 14R20.

*Key words and phrases.* Automorphism groups of affine varieties; Koras-Russell threefolds.

polynomial ring in two variables can be decomposed into the product of affine automorphisms and de Jonquieres automorphisms and in addition we have a structure of amalgamated product of these two groups. However, there is no similar theorem to describe the automorphism groups of the polynomial ring in  $n$  variables for  $n \geq 3$ . It is therefore interesting to have a description of the automorphism groups of three dimensional algebraic varieties which resemble  $\mathbb{A}^3$ , in the sense that they are smooth rational and contractible.

We recall that the determination of the automorphism groups of an affine variety  $X$  is equivalent to that of its ring of regular functions  $\mathbb{C}[X]$ . We denote for simplicity by  $Aut(X)$  the group  $Aut_{\mathbb{C}}(\mathbb{C}[X])$  of  $\mathbb{C}$ -automorphisms of  $\mathbb{C}[X]$ .

The natural correspondence between an affine variety and its ring of regular functions gives a particular interpretation for  $\mathbb{G}_a$ -actions. Indeed, the set of all  $\mathbb{G}_a$ -actions on  $X$  is in one-to-one correspondence with the set of all *locally nilpotent derivations* on  $\mathbb{C}[X]$  (see [F, ML2]). We recall that a derivation  $\partial$  on  $\mathbb{C}[X]$  is called a locally nilpotent derivation if for any  $f \in \mathbb{C}[X]$ , there exists  $n \in \mathbb{Z}_{\geq 0}$  such that  $\partial^n(f) = 0$ . The set of all locally nilpotent derivations on  $\mathbb{C}[X]$  is denoted by  $LND(\mathbb{C}[X])$ . The intersection of the kernels of all locally nilpotent derivation on  $\mathbb{C}[X]$ ,  $ML(X) := \bigcap_{\partial \in LND(\mathbb{C}[X])} ker(\partial)$ , is called the *Makar-Limanov invariant* of  $X$ . It is a subring of the ring of regular functions  $\mathbb{C}[X]$  which is invariant by all automorphisms. When this subring is non trivial, that is, not equal to  $\mathbb{C}$  or  $\mathbb{C}[X]$ , this fact can be used to study the automorphism groups.

In the case of the affine space it is trivial, that is, it is equal to  $\mathbb{C}$ , thus the Makar-Limanov invariant does not provide any information on the automorphism group of  $\mathbb{A}^n$ . The Koras-Russell threefolds of the third kind admit a Makar-Limanov invariant given by their ring of regular functions  $\mathbb{C}[X]$  [Ka-ML, 8.3]. Once again the approach of using the Makar-Limanov does not give any information on the automorphism groups.

The study of the automorphism groups of varieties of the first kind has been described by Moser-Jauslin in [MJ]. In this case, the Makar-Limanov invariant is given by  $\mathbb{C}[x]$  [Ka-ML, 8.3], thus these varieties are not isomorphic to  $\mathbb{A}^3$  but admit several  $\mathbb{G}_a$ -actions. We have the following sequence of inclusion of rings:

$$\mathbb{C}[x] \subset \mathbb{C}[x, z, t] \subset \mathbb{C}[X] \subset \mathbb{C}[x^{\pm 1}, z, t] \subset \mathbb{C}(x, z, t).$$

Moreover, the polynomial defining the hypersurface is homogeneous for the following linear  $\mathbb{G}_m$ -action on  $\mathbb{A}^4$ :

$$\lambda \cdot (x, y, z, t) \rightarrow (\lambda^{\alpha_2 \alpha_3} x, \lambda^{-(d-1)\alpha_2 \alpha_3} y, \lambda^{\alpha_3} z, \lambda^{\alpha_2} t),$$

and so  $\mathbb{G}_m$  is one subgroup of the automorphism groups of these varieties. One of the results in [MJ] is the following :

**Theorem (MJ).** *Let  $X$  be a Koras-Russell threefold of the first kind. The automorphism group is isomorphic to  $Aut(X) \simeq \mathcal{A}_1 \rtimes \mathbb{G}_m$ , where  $\mathcal{A}_1$  is the subgroup of  $\mathbb{C}[x]$ -automorphisms of  $\mathbb{C}[x][z, t]$  which are congruent to the identity modulo  $(x)$ , and which stabilize the ideal  $I = (x^d, z^{\alpha_3} + t^{\alpha_2} + x)$ .*

In particular, the subalgebras  $\mathbb{C}[x, z, t]$  and  $\mathbb{C}[x]$  are stable by any automorphism and any element of  $\mathcal{A}_1$  can be lifted to element of  $Aut(X)$ .

Here we apply a similar approach for automorphism groups of Koras-Russell threefolds of the second kind. In this case, the Makar-Limanov invariant is equal

to  $\mathbb{C}[x, z]$  [Ka-ML, 8.3]. In particular these varieties are *semi-rigid* (see [A, 4.1.2]), that is,  $\text{LND}(X) = \ker(\partial) \cdot \partial$  for some non trivial locally nilpotent derivation  $\partial \in \text{LND}(X)$ . In order to study  $\text{Aut}(X)$ , we first note that  $X$  is rational, and furthermore that  $\mathbb{C}[X]$  is generated by four elements  $x, y, z, t$  which satisfy the relation  $x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0$ . Let  $f$  be the polynomial  $x^d + z^{\alpha_2}$ . The fraction field of  $\mathbb{C}[X]$  is generated by  $\mathbb{C}(x, z, t)$  where  $y = -\frac{x+t^{\alpha_3}}{(f)^l}$ . In other words any element of  $\text{Aut}(X)$  is determined by the image of  $x, z$  and  $t$ . We have the following sequence of inclusion :

$$\mathbb{C}[x, z] \subset \mathbb{C}[x, z, t] \subset \mathbb{C}[X] \subset \mathbb{C}[x, z, t, f^{-1}] \subset \mathbb{C}(x, z, t).$$

Moreover, the polynomial defining the hypersurface is homogeneous for the following linear  $\mathbb{G}_m$ -action on  $\mathbb{A}^4$ :

$$\lambda \cdot (x, y, z, t) \rightarrow (\lambda^{\alpha_2 \alpha_3} x, \lambda^{-(dl-1)\alpha_2 \alpha_3} y, \lambda^{d\alpha_3} z, \lambda^{\alpha_2} t),$$

and so  $\mathbb{G}_m$  is isomorphic to a subgroup of the automorphism groups of these varieties. The goal of this article is to determine completely the automorphism groups.

**Theorem.** *Let  $X$  be a Koras-Russell threefold of the second kind. The automorphism group is isomorphic to  $\text{Aut}(X) \simeq \mathcal{A} \rtimes \mathbb{G}_m$ , where  $\mathcal{A}$  is the subgroup of  $\text{Aut}(\mathbb{C}[x, z, t])$  whose elements fix  $\mathbb{C}[x, z]$  and send  $t$  to  $t + (x^d + z^{\alpha_2})^l p(x, z)$  for some polynomial  $p(x, z) \in \mathbb{C}[x, z]$ . The group  $\mathcal{A}$  is isomorphic to  $(\mathbb{C}[x, z], +)$ .*

In particular the subalgebras  $\mathbb{C}[x, z, t]$ ,  $\mathbb{C}[x]$  (that is  $\text{ML}(X)$ ) and  $\mathbb{C}[z]$  are stable by every automorphism and every element of  $\mathcal{A}$  can be lifted to a unique element of  $\text{Aut}(X)$ .

## 2. PROOF OF THE THEOREM

Let  $X$  be a Koras-Russell threefolds of the second kind given by:

$$\{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\} \subset \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]).$$

The Makar-Limanov invariant of  $X$  is equal to  $\mathbb{C}[x, z]$ . We will use this fact to completely determine the actions of the additive group  $\mathbb{G}_a$  on  $X$ . Let  $\partial \in \text{LND}(\mathbb{C}[X])$  be the irreducible derivation given by:

$$\partial := \alpha_3 t^{\alpha_3-1} \frac{\partial}{\partial y} - (x^d + z^{\alpha_2})^l \frac{\partial}{\partial t}.$$

This derivation is obtained by considering the Jacobian determinant:

$$\text{Jac}_{\mathbb{C}[x, z]}(P, \cdot) = \begin{vmatrix} \frac{\partial P}{\partial y} & \frac{\partial \cdot}{\partial y} \\ \frac{\partial P}{\partial t} & \frac{\partial \cdot}{\partial t} \end{vmatrix} \text{ with } P(x, y, z, t) = x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3}.$$

Thus  $\partial(x) = 0$ ,  $\partial(z) = 0$ ,  $\partial(t) = -(x^d + z^{\alpha_2})^l$  and  $\partial(y) = \alpha_3 t^{\alpha_3-1}$  and the kernel of  $\partial$  is  $\mathbb{C}[x, z] = \text{ML}(X)$ , preserved by the automorphisms of  $\mathbb{C}[X]$ . As these hypersurfaces are semi-rigid (see [A, 4.1.2]) any other locally nilpotent derivation can be written in the following form:  $\partial_q = q(x, z)\partial$  with  $q(x, z) \in \text{ML}(X) = \mathbb{C}[x, z]$ . In this case  $\partial$  is the unique irreducible element of  $\text{LND}(X)$  up to multiplication by a constant. For any  $\varphi \in \text{Aut}(X)$ ,  $\partial_\varphi := \varphi^{-1} \circ \partial \circ \varphi$  is also an irreducible element of  $\text{LND}(\mathbb{C}[X])$  [F, corollary 2.3]. Therefore, there exists  $a \in \mathbb{C}^*$  such that  $\partial_\varphi = a\partial$ . Moreover as the kernel of  $\partial$  is equal to  $\mathbb{C}[x, z]$  we obtain that  $\varphi(\mathbb{C}[x, z]) = \mathbb{C}[x, z]$ .

- ❶ Any automorphism  $\varphi$  of  $X$  induces an automorphism  $\hat{\varphi}$  of  $\mathbb{C}[x, z]$  satisfying:
- i) the ideal  $(f)$  generated by  $f$  is preserved
  - ii) there is  $\mu_{\hat{\varphi}} \in \mathbb{C}^*$  such that  $\hat{\varphi}(x) = (\mu_{\hat{\varphi}})^{\alpha_2}x$  and  $\hat{\varphi}(z) = (\mu_{\hat{\varphi}})^d z$ .

The part i) comes from the fact that  $\varphi$  must preserve the Makar-Limanov invariant  $ML(X) = \mathbb{C}[x, z]$ , which is also the kernel of  $\partial$ . Moreover, let  $\pi$  be the projection on the coordinates  $(x, z)$ :

$$(2.1) \quad \begin{array}{ccc} \pi : & X & \rightarrow \mathbb{A}^2 = \text{Spec}(\mathbb{C}[x_0, z_0]) \\ & (x, y, z, t) & \rightarrow (x, z), \end{array}$$

and let  $f_0 := (x_0^d + z_0^{\alpha_2})$  for any point  $(x_0, z_0) \in \mathbb{A}^2$ . Then  $\pi^{-1}(x_0, z_0)$  is isomorphic to  $\mathbb{A}^1$  if  $f_0 \neq 0$  or  $f_0 = 0$  and  $x = 0$  but it is isomorphic to  $\alpha_3$  copies of  $\mathbb{A}^1$  otherwise. Thus the cuspidal curve must be preserved by  $\hat{\varphi}$ , that is, there exists  $\lambda_{\hat{\varphi}} \in \mathbb{C}^*$  such that  $\hat{\varphi}(f) = \lambda_{\hat{\varphi}} f$ .

The only automorphism  $\psi$  of  $\mathbb{C}[x, z]$  which is congruent to the identity modulo  $f$  is indeed the identity. To see this, note that if such an automorphism exists, it would stabilise all plane curves defined by the level sets of  $f$ . Suppose that  $C$  is the zero set of the polynomial  $f - c$ , where  $c$  is a non-zero constant. Then  $C$  is the open set of a smooth compact Riemann surface of genus  $\geq 1$ . By Hurwitz's theorem (see [H]), the automorphism group of  $C$  is finite, and in fact the automorphism  $\psi$  must be of finite order. This implies in particular that  $\psi$  is conjugate to a linear action by [J, VK]. By considering the linear part, since  $\psi$  is congruent to the identity modulo  $f$ , we have that  $\psi$  is conjugate to the identity, and thus is in fact the identity.

As  $\hat{\varphi} \in \text{Aut}(\mathbb{C}[x, z])$  satisfies  $\hat{\varphi}(f) = \lambda_{\hat{\varphi}} f$  there is  $\mu_{\hat{\varphi}} \in \mathbb{C}^*$  such that  $\hat{\varphi}(x) = (\mu_{\hat{\varphi}})^{\alpha_2}x$  modulo  $f$  and  $\hat{\varphi}(z) = (\mu_{\hat{\varphi}})^d z$  modulo  $f$ . By composition with the linear automorphism  $\varphi_0: \varphi_0(x) = (\mu_{\hat{\varphi}})^{-\alpha_2}x$  and  $\varphi_0(z) = (\mu_{\hat{\varphi}})^{-d}z$ , we obtain  $\varphi_0 \circ \hat{\varphi} = \psi \equiv \text{id}$ . This proves part ii).

❷ Now consider automorphisms of  $\mathbb{C}[X]$  which fixed  $\mathbb{C}[x, z]$  and we focus on the image of  $t$  by any element  $\varphi \in \text{Aut}(X)$ . We prove first that  $\varphi(t)$  is of the form  $at + h(x, z)$  with  $a \in \mathbb{C}^*$ .

As  $\varphi(f) = f$  where  $f = (x^d + z^{\alpha_2})$  we apply  $\partial_{\varphi}$  to the variable  $t$ :  $\partial_{\varphi}(t) = \varphi^{-1} \circ \partial \circ \varphi(t)$  thus:

$$\begin{aligned} \varphi \circ (a\partial(t)) &= \partial \circ \varphi(t) \\ \varphi(-af^l) &= \partial \circ \varphi(t) \\ -af^l &= \partial \circ \varphi(t). \end{aligned}$$

This means that  $\partial \circ \varphi(t) = \partial(at)$  and thus  $\varphi(t) - at \in \ker(\partial) = \mathbb{C}[x, z]$ .

Secondly we prove that  $a$  is a  $\alpha_3$ -th root of the unity and  $h(x, z) \in (f^l)$ . Let  $J \subset \mathbb{C}[X]$  be the ideal generated by  $(f^l)$ , and let  $I = J \cap \mathbb{C}[x, z, t]$ . Then  $I = (f^l, x + t^{\alpha_3}) \subset \mathbb{C}[x, z, t]$ . Indeed,  $f^l$  and  $x + t^{\alpha_3}$  are in  $I$  thus  $(f^l, x + t^{\alpha_3}) \subset I$ . Now suppose  $Q \in I$ . Since  $Q \in J$ , there exists  $P \in \mathbb{C}[X]$  such that  $Q = f^l P$ . Any  $P \in \mathbb{C}[X]$  can be decomposed in a unique way as follows  $P = \sum_{i=0}^n p_i(x, z, t)y^i$  such that  $f^l$  does not divide  $p_i$  if  $i \geq 1$  since  $y = -(x + t^{\alpha_3})f^{-l}$ . This gives  $Q = f^l p_0 + \sum_{i=1}^n p_i(x, z, t)(x + t^{\alpha_3})y^{i-1}$ . Now since  $Q \in \mathbb{C}[x, z, t]$ , the polynomial  $p_i(x, z, t) = 0$  for  $i \geq 2$  and then  $Q = f^l p_0 + p_1(x, z, t)(x + t^{\alpha_3})$ .

Consider the ideal  $I = (f^l, x + t^{\alpha_3}) \subset \mathbb{C}[x, z, t]$  which is preserved by  $\varphi$ , thus  $\varphi(x + t^{\alpha_3}) \in I$ . There exists  $b(x, z, t) = \sum_{i=0}^n b_i(x, z)t^i$  and  $c(x, z, t) = \sum_{i=0}^m c_i(x, z)t^i$  such that:

$$\varphi(x + t^{\alpha_3}) = x + (at + h(x, z))^{\alpha_3} = \left( \sum_{i=0}^n c_i(x, z)t^i \right) (x + t^{\alpha_3}) + \left( \sum_{i=0}^m b_i(x, z)t^i \right) f^l.$$

In addition, one can assume that for all  $i$ ,  $f^l$  does not divide  $c_i(x, z)$ . Considering the highest degree in the variable  $t$  on both side, that is the coefficient of  $t^{\alpha_3}$ . If  $n \geq 1$ , then  $f^l$  divides  $c_n(x, z)$ , which contradicts the assumption. Thus  $n = 0$ , and  $c_0(x, z)$  is congruent to  $a^{\alpha_3}$  modulo  $f^l$ . We can therefore suppose  $c_0(x, z)$  equals  $a^{\alpha_3}$ , by adding the appropriate term to  $b(x, z, t)$ . Moreover, the equality above implies that  $h(x, z, t)$  is in the ideal  $(x, z, t)$ . Now considering the coefficient of  $x$  on both sides, we see that  $c_0(x, z) = 1$ . This implies:

$$\varphi(x + t^{\alpha_3}) - (x + t^{\alpha_3}) = h(x, z) \left[ \sum_{k=1}^{\alpha_3-1} \binom{\alpha_3}{k} (at)^k h(x, z)^{\alpha_3-1-k} \right] = \sum_{j=1}^m b_j(x, z) t^j f^l,$$

and considering the coefficient of  $t$  on both sides, we see that  $f^l$  divides  $h(x, z)$ . So  $h(x, z)$  is in the ideal generated by  $f^l$  in  $\mathbb{C}[x, z]$ , that is  $h(x, z) = (x^d + z^{\alpha_2})^l p(x, z)$ .

⊛ If  $\psi \in \text{Aut}(\mathbb{C}[x, z, t])$  is in  $\mathcal{A}$  then  $\psi$  extends in a unique way to an automorphism in  $\text{Aut}(X)$ , via the computation:

$$(2.2) \quad \psi(y) = \psi \left( \frac{-(x + t^{\alpha_3})}{f^l} \right) = \frac{-(x + (t + f^l p(x, z))^{\alpha_3})}{f^l} = y + H(x, z, t).$$

In particular  $\mathcal{A}$  can be identified with a normal subgroup of  $\text{Aut}(X)$ . By part 2 and the  $\mathbb{G}_m$ -action given in the introduction every automorphism  $\varphi$  can be expressed in a unique way as a composition of an element of  $\mathcal{A}$  and an element of the  $\mathbb{G}_m$ -action. In conclusion  $\text{Aut}(X) = \mathcal{A} \rtimes \mathbb{G}_m$ .

We recall several results of [D-MJ-P1, proposition 3.1] in order to compare with that obtained as corollaries of the theorem of this article. In the case of Koras-Russell threefolds of the first kind the following results hold: every automorphism of  $X$  extends to an automorphism of  $\mathbb{A}^4$ , the group  $\text{Aut}(X)$  admits 4 orbits with in particular the origin  $(0, 0, 0, 0)$  as fixed point and the subgroup generated by all  $\mathbb{G}_a$  and  $\mathbb{G}_m$  actions is strictly smaller than  $\text{Aut}(X)$ .

In the case of Koras-Russell threefolds of the second kind, let  $\pi$  be the projection defined in 2.1 and consider a partition of  $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[x_0, z_0])$  by the origin and the  $\mathbb{G}_m$ -stable curves  $C_{\alpha, \beta} = \{\alpha x_0^d + \beta z_0^{\alpha_2} = 0\} \setminus (0, 0)$ , with  $[\alpha : \beta] \in \mathbb{P}^1$ . Then  $\pi^{-1}(0, 0)$  admits two orbits given by  $(0, 0, 0, 0)$  and the line  $\{x = z = t = 0\}$  minus the point  $(0, 0, 0, 0)$ . For every curve  $C_{\alpha, \beta}$  if  $[\alpha : \beta] \neq [1, 1]$  then  $\pi^{-1}(C_{\alpha, \beta}) \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$  considering  $y = -\frac{x+t^{\alpha_3}}{(f)^l}$  and if  $[\alpha : \beta] = [1, 1]$  then  $x = t^{\alpha_3}$  thus any choice of  $t$  determines  $x$  and gives  $\{t^{d\alpha_3} + z^{\alpha_2} = 0\} \setminus (0, 0) \simeq \mathbb{A}_*^1$  and  $\mathcal{A}$  acts trivially on  $\pi^{-1}(C_{1,1}) \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$ .

**Corollary 1.** *The automorphism group acts on  $X$  with an infinite number of orbits:*

- a) *One fixed point  $(0, 0, 0, 0)$ .*
- b) *The line  $\{x = z = t = 0\}$  minus the point  $(0, 0, 0, 0)$  isomorphic to  $\mathbb{A}_*^1$ .*
- c) *An infinite number of orbits  $\pi^{-1}(C_{\alpha, \beta}) \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$  for  $[\alpha : \beta] \neq [1, 1]$ .*

d) An infinite number of orbits isomorphic to  $\mathbb{A}_*^1$ , whose union is  $\pi^{-1}(C_{1,1})$ .

We have proved that there was not only the Makar-Limanov invariant  $\mathbb{C}[x, z]$  which was preserved by any automorphism of  $\mathbb{C}[X]$  but also  $\mathbb{C}[x, z, t]$  since for any automorphism  $\varphi$  of  $\mathbb{C}[X]$ ,  $\varphi(t) = at + (x^d + z^{\alpha_2})^l p(x, z)$  and as the image of  $y$  is determined by that of the other variable. Using the same argument as in the computation of the equation 2.2, we see immediately that every automorphism of  $X$  comes from the restriction of an automorphism of  $\text{Aut}(\mathbb{A}^4)$ .

For every  $\varphi \in \mathcal{A}$  we have:

$$\varphi(\cdot) = \exp(p\partial) = \sum_{k=0}^{\infty} \frac{p(x, z)^k \partial^k(\cdot)}{k!}.$$

In particular  $\varphi$  belongs to an algebraic subgroup of  $\text{Aut}(X)$  isomorphic to  $\mathbb{G}_a$ . Thus the following holds.

**Corollary 2.** *The automorphism group of  $X$  is generated by  $\mathbb{G}_a$ -actions and  $\mathbb{G}_m$ -actions.*

#### REFERENCES

- [A] B. Alhajjar. The Filtration Induced by a Locally Nilpotent Derivation. (arXiv:1310.3784)
- [D] A. Dubouloz. The cylinder over the Koras-Russell cubic threefold has a trivial Makar-Limanov invariant. *Transform. Groups* 14 (2009), no. 3, 531–539.
- [D-MJ-P] A. Dubouloz, L. Moser-Jauslin, P.-M. Poloni . Inequivalent embeddings of the Koras-Russell cubic 3-fold. *Michigan Math. J.* 59 (2010), no. 3, 679–694.
- [D-MJ-P1] A. Dubouloz, L. Moser-Jauslin, P.-M. Poloni . Automorphisms in Birational and Affine Geometry. Proceedings of conference in Levico Terme, Italy, October, 2012, Springer Proceedings in Mathematics and Statistics, (2014).
- [F] G. Freudenburg. Algebraic theory of locally nilpotent derivations. *Encyclopaedia of Mathematical Sciences*, 136. Invariant Theory and Algebraic Transformation Groups, VII. Springer-Verlag, Berlin, 2006.
- [H] A. Hurwitz, Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.* 41 (1892), no. 3, 403–442.
- [J] H. Jung, Uber ganze birationale Transformationen der Ebene. *J. Reine Angew. Math.* 184, (1942). 161–174.
- [K-R] M. Koras, P. Russell, Contractible threefolds and  $\mathbb{C}^*$ -actions on  $\mathbb{C}^3$ , *J. Algeb. Geom.* 6 (1997) 671–695.
- [Ka-ML] S. Kaliman, L. Makar-Limanov, On the Russell–Koras contractible threefolds, *J. Algeb. Geom.* 6(2) (1997) 247–268.
- [Ka-K-ML-R] S. Kaliman, M. Koras, L. Makar-Limanov, P. Russell,  $\mathbb{C}^*$ -actions on  $\mathbb{C}^3$  are linearizable, *Electron. Res. Announc. Am. Math. Soc.* 3 (1997) 63–71.
- [MJ] L. Moser-Jauslin. Automorphism groups of Koras-Russell threefolds of the first kind. *Affine algebraic geometry*, 261–270, CRM Proc. Lecture Notes, 54, Amer. Math. Soc., Providence, RI, 2011.
- [ML1] L. Makar-Limanov. On the hypersurface  $x + x^2y + z^2 + t^3 = 0$  in  $\mathbb{C}^4$  or a  $\mathbb{C}^3$ -like threefold which is not  $\mathbb{C}^3$ . *Israel J. Math.* 96 (1996), part B, 419–429.
- [ML2] L. Makar-Limanov, Locally nilpotent derivations, a new ring invariant and applications , available at [www.math.wayne.edu/~lml/lmlnotes.pdf](http://www.math.wayne.edu/~lml/lmlnotes.pdf)
- [VK] W. Van der Kulk. On polynomial rings in two variables. *Nieuw Arch. Wiskunde* (3) 1, (1953). 33–41.

CHARLIE PETITJEAN, INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UNIVERSITÉ DE BOURGOGNE FRANCHE-COMTÉ, 9 AVENUE ALAIN SAVARY, BP 47870, 21078 DIJON CEDEX, FRANCE  
*E-mail address:* charlie.petitjean@u-bourgogne.fr