# RIGID AFFINE SURFACES WITH ISOMORPHIC $\mathbb{A}^{2}$-CYLINDERS 

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#### Abstract

We construct families of smooth affine surfaces with pairwise non isomorphic $\mathbb{A}^{1}$-cylinders but whose $\mathbb{A}^{2}$-cylinders are all isomorphic. These arise as complements of cuspidal hyperplane sections of smooth projective cubic surfaces.


## Introduction

The Zariski Cancellation Problem, which asks whether two, say smooth affine, algebraic varieties $X$ and $Y$ with isomorphic cylinders $X \times \mathbb{A}^{n}$ and $Y \times \mathbb{A}^{n}$ for some $n \geq 1$ are isomorphic themselves, has been studied very actively during the past decades culminating recently with a negative solution in dimension 3 and positive characteristic for the case $X=\mathbb{A}^{3}[7]$. The situation in the complex case, and more generally over any algebraically closed field of characteristic zero, is more contrasted: cancellation is known to hold for curves [1] and for $\mathbb{A}^{2}$ [6], but many counter-examples in every dimension higher or equal to 2 have been discovered (see [14] for a survey), inspired by the two pioneering constructions of Hochster [8] and Danielewski [2].

Essentially all known families are counter-examples to the cancellation of 1-dimensional cylinders which arise from the existence of nontrivial decompositions of certain locally trivial $\mathbb{A}^{2}$-bundles over a base scheme $Z$. Namely, Hochster type constructions rely on the existence of non free, 1-stably free, projective modules which in geometric term correspond to non trivial decompositions of the trivial bundle $Z \times \mathbb{A}^{r+1} \rightarrow Z$ into a trivial $\mathbb{A}^{1}$-bundle over a nontrivial vector bundle $E \rightarrow Z$ of rank $r \geq 1$. For every such bundle, the varieties $X=E$ and $Y=Z \times \mathbb{A}^{r}$ have isomorphic cylinders $X \times \mathbb{A}^{1}$ and $Y \times \mathbb{A}^{1}$, and one then gets a counter-example to the cancellation problem whenever $X$ and $Y$, which by definition are non isomorphic as schemes over $Z$, are actually non isomorphic as abstract algebraic varieties [10]. In contrast, Danielewski type constructions usually involve non trivial decompositions of a principal homogeneous $\mathbb{G}_{a}^{2}$-bundle $W \rightarrow Z$ into pairs $W \rightarrow X \rightarrow Z$ and $W \rightarrow Y \rightarrow Z$ consisting of trivial $\mathbb{G}_{a}$-bundles over nontrivial $\mathbb{G}_{a}$-bundles $X \rightarrow Z$ and $Y \rightarrow Z$ with affine total spaces, with the property that $W$ is isomorphic to the fiber product $W=X \times_{Z} Y$. The isomorphism $X \times \mathbb{A}^{1} \simeq W \simeq Y \times \mathbb{A}^{1}$ is granted by definition, and similarly as in the previous type of construction, one obtains counter-examples to the cancellation problem as soon as $X$ and $Y$ are not isomorphic as abstract varieties.

Non-cancellation phenomena with respect to higher dimensional cylinders are more mysterious. In fact, it seems for instance that not even a single explicit example of a pair of non-isomorphic varieties $X$ and $Y$ which fail the $\mathbb{A}^{2}$-cancellation property in a minimal way, in the sense that $X \times \mathbb{A}^{2}$ and $Y \times \mathbb{A}^{2}$ are isomorphic while $X \times \mathbb{A}^{1}$ and $Y \times \mathbb{A}^{1}$ are still non isomorphic, is known so far. In this article, we fill this gap by constructing a positive dimensional moduli of smooth affine surfaces which fail the $\mathbb{A}^{2}$-cancellation property minimally. That such varieties exist was certainly a natural expectation, and their existence is therefore neither really surprising, nor probably exciting in itself due to the abundance of simpler counter-examples to the cancellation problem. Their interest lies rather in the fact that they provide additional insight on the algebro-geometric properties that a variety should satisfy in order to fail cancellation.

Indeed, it follows from Iitaka-Fujita strong Cancellation Theorem [9] that a smooth affine variety $X$ which fails cancellation has negative logarithmic Kodaira dimension, a property conjecturally equivalent in dimension higher or equal to 3 to the fact that $X$ is covered by images of the affine line and equivalent for surfaces to the existence of an $\mathbb{A}^{1}$-fibration $\pi: X \rightarrow C$ over a smooth curve [12], i.e. a flat fibration with general fibers isomorphic to the affine line. In the particular case of the cancellation problem for 1-dimensional cylinders, a further striking discovery of Makar-Limanov is that the existence of nontrivial actions of the additive group $\mathbb{G}_{a}$ on $X$ is a necessary condition for non-cancellation. Namely, Makar-Limanov semi-rigidity theorem [11] (see also [5, Proposition 9.23]) asserts that if $X$ is rigid, i.e. does not admit any nontrivial $\mathbb{G}_{a}$-action, then the projection $\operatorname{pr}_{X}: X \times \mathbb{A}^{1} \rightarrow X$ is invariant under all $\mathbb{G}_{a}$-actions on $X$. As a consequence, if either $X$ or $Y$ is rigid then every isomorphism between $X \times \mathbb{A}^{1}$ and $Y \times \mathbb{A}^{1}$ descends to an isomorphism between $X$ and $Y$. Combined with Fieseler's topological

[^0]description of algebraic quotient morphisms of $\mathbb{G}_{a}$-actions on smooth complex affine surfaces [4], these results imply that a smooth affine surface which fails $\mathbb{A}^{1}$-cancellation must admit a nontrivial $\mathbb{G}_{a}$-action whose algebraic quotient morphism $\pi: X \rightarrow X / / \mathbb{G}_{a}=\operatorname{Spec}\left(\Gamma\left(X, \mathcal{O}_{X}\right)^{\mathbb{G}_{a}}\right)$ is not a locally trivial $\mathbb{A}^{1}$-bundle. This holds of course for the two smooth surfaces $x z-y(y+1)=0$ and $x^{2} z-y(y+1)=0$ used by Danielewski in his celebrated counter-example, showing a posteriori that his construction was essentially optimal in this dimension. The rich families of existing counter-examples to $\mathbb{A}^{1}$-cancellation in dimension 2 lend support to the conjecture that every smooth affine surface which is neither rigid nor isomorphic to the total space of line bundle over an affine curve fails the $\mathbb{A}^{1}$-cancellation property.

In view of this conjecture, a smooth affine surface $X$ which fails the $\mathbb{A}^{2}$-cancellation property in a minimal way must be simultaneously rigid and equipped with an $\mathbb{A}^{1}$-fibration $\pi: X \rightarrow C$ over a smooth curve, and the well known fact that $\mathbb{A}^{1}$-fibrations over affine curves are algebraic quotient morphisms of nontrivial $\mathbb{G}_{a}$-actions implies further that $C$ must be projective. This is precisely the case for the family of surfaces we construct in this article, a particular example being the smooth affine cubic surfaces
$X=\left\{\left(-1+\alpha \sqrt[3]{2} x_{2}+\alpha \sqrt[3]{2} x_{3}\right)^{3}+8\left(x_{1}^{3}+x_{2}^{3}+x_{3}^{3}\right)=0\right\}$ and $X^{\prime}=\left\{\left(-1+2 \alpha x_{1}+\alpha \sqrt[3]{2} x_{2}+\alpha \sqrt[3]{2} x_{3}\right)^{3}+8\left(x_{1}^{3}+x_{2}^{3}+x_{3}^{3}\right)=0\right\}$
in $\mathbb{A}^{3}=\operatorname{Spec}\left(\mathbb{C}\left[x_{1}, x_{2}, x_{3}\right]\right)$, where $\alpha=\exp (i \pi / 3)$, which are both rigid and equipped with an $\mathbb{A}^{1}$-fibration over $\mathbb{P}^{1}$. These arise as the complements in the Fermat cubic surface $V=\left\{x_{0}^{3}+x_{1}^{3}+x_{2}^{3}+x_{3}^{3}=0\right\}$ in $\mathbb{P}^{3}$ of the plane cuspidal cubics $C=\left\{-\left(x_{2}+x_{3}\right)^{3}+4\left(x_{1}^{3}+x_{2}^{3}+x_{3}^{3}\right)\right\}=0$ and $C^{\prime}=\left\{-\left((\sqrt[3]{2})^{2} x_{1}+x_{2}+x_{3}\right)^{3}+4\left(x_{1}^{3}+x_{2}^{3}+x_{3}^{3}\right)=0\right\}$ obtained by intersecting $V$ with its tangent hyperplane at the points $p=[\alpha \sqrt[3]{2}: 0: 1: 1]$ and $p^{\prime}=[\alpha \sqrt[3]{2}: \sqrt[3]{2}:-1: 1]$ respectively. The group of automorphisms of $V$ being isomorphic to $\mathbb{Z}_{3} \times \mathfrak{S}_{4}$, where $\mathbb{Z}_{3}$ is the 3-torsion subgroup of $\operatorname{PGL}(4 ; \mathbb{C})$ and where $\mathfrak{S}_{4}$ denotes the group of permutations of the variables, the fact that $p$ and $p^{\prime}$ do not belong to a same $\operatorname{Aut}(V)$-orbit implies that the pairs $(V, C)$ and $\left(V, C^{\prime}\right)$ are not isomorphic. Our main result just below then implies in turn that $X$ and $X^{\prime}$ are non isomorphic, with isomorphic $\mathbb{A}^{2}$-cylinders $X \times \mathbb{A}^{2}$ and $X^{\prime} \times \mathbb{A}^{2}$.

Theorem. Let $\left(V_{i}, C_{i}\right), i=1,2$, be non isomorphic pairs consisting of a smooth cubic surface $V_{i} \subset \mathbb{P}^{3}$ and a cuspidal hyperplane section $C_{i}=V_{i} \cap H_{i}$. Then the affine surfaces $X_{i}=V_{i} \backslash C_{i}$ are non isomorphic, with non isomorphic $\mathbb{A}^{1}$-cylinders $X_{i} \times \mathbb{A}^{1}$ but with isomorphic $\mathbb{A}^{2}$-cylinders $X_{i} \times \mathbb{A}^{2}, i=1,2$.

As a consequence, all smooth affine surfaces arising as complements of cuspidal hyperplane sections of smooth projective cubic surfaces have isomorphic $\mathbb{A}^{2}$-cylinders. Noting that the projective closure in $\mathbb{P}^{3}$ of the surface $X_{0} \subset \mathbb{A}^{3}=\operatorname{Spec}(\mathbb{C}[x, y, z])$ with equation $x^{2} y+y^{2}+z^{3}+1=0$ is a smooth cubic surface intersecting the plane at infinity along the cuspidal cubic $x^{2} y+z^{3}=0$, we obtain the following:

Corollary. Let $X$ be a smooth affine surface isomorphic to the complement of a cuspidal hyperplane section of a smooth projective cubic surface. Then $X \times \mathbb{A}^{2}$ is isomorphic to the affine cubic fourfold $Z \subset \mathbb{A}^{5}=\operatorname{Spec}(\mathbb{C}[x, y, z][u, v])$ with equation $x^{2} y+y^{2}+z^{3}+1=0$. Furthermore, $X$ is isomorphic to the geometric quotient of a proper action of the group $\mathbb{G}_{a}^{2}$ on $Z$.

The scheme of the proof of the Theorem given in the next section is the following. The fact that the affine surfaces $X_{1}$ and $X_{2}$ are non-isomorphic follows from the non-isomorphy of the pairs $\left(V_{1}, C_{1}\right)$ and ( $V_{2}, C_{2}$ ) via an argument of classical birational geometry of projective cubic surfaces, which simultaneously renders the conclusion that $X_{1}$ and $X_{2}$ are rigid. The non isomorphy of the cylinders $X_{1} \times \mathbb{A}^{1}$ and $X_{2} \times \mathbb{A}^{1}$ is then a straightforward consequence Makar-Limanov's semi-rigidity Theorem.

The existence of an isomorphism between the $\mathbb{A}^{2}$-cylinders $X_{1} \times \mathbb{A}^{2}$ and $X_{2} \times \mathbb{A}^{2}$ is derived in two steps: the first one consists of another instance of a Danielewski fiber product trick argument, which provides a smooth affine threefold $W$ equipped with simultaneous structures of line bundles $\pi_{1}: W \rightarrow X_{1}$ and $\pi_{2}: W \rightarrow X_{2}$ over $X_{1}$ and $X_{2}$. But here, in contrast with the situation in Danielewski's counter-example, the fact that the $\mathbb{A}^{1}$-cylinders over $X_{1}$ and $X_{2}$ are not isomorphic implies that these two line bundles cannot be simultaneously trivial. Nevertheless, the crucial observation which enables a second step, reminiscent to Hochster construction, is that the pull-backs via the isomorphisms $\pi_{i}^{*}: \operatorname{Pic}\left(X_{i}\right) \rightarrow \operatorname{Pic}(W)$ of the classes of these lines bundles in the Picard groups of $X_{1}$ and $X_{2}$, say $L_{1}$ and $L_{2}$, coincide. Letting $q: E \rightarrow W$ be a line bundle representing the common inverse in $\operatorname{Pic}(W)$ of $\pi_{1}^{*} L_{1}=\pi_{2}^{*} L_{2}$, the composition $\pi_{i} \circ q: E \rightarrow X_{i}$ is then a vector bundle of rank 2 isomorphic to the direct sum $L_{i} \oplus L_{i}^{\vee}$, where $L_{i}^{\vee}$ denotes the dual of $L_{i}$, hence isomorphic to $\operatorname{det}(E) \oplus \mathbb{A}_{X_{i}}^{1}=\left(L_{i} \otimes L_{i}^{\vee}\right) \oplus \mathbb{A}_{X_{i}}^{1} \simeq \mathbb{A}_{X_{i}}^{1} \oplus \mathbb{A}_{X_{i}}^{1}$ by virtue of result of Pavaman Murthy [13] asserting that every vector bundle on a smooth affine surface birationaly equivalent to a ruled surface is isomorphic to the direct sum of a trivial bundle with a line bundle.


The construction of these isomorphisms suggests the following strengthening of the above conjecture characterizing smooth affine surfaces failing the $\mathbb{A}^{1}$-cancellation property, which would settle the question of the behavior of smooth affine surfaces under stabilization by affine spaces:

Conjecture. A smooth affine surface $X$ with negative logarithmic Kodaira dimension is either isomorphic to the total space of a line bundle over a curve, or it fails the $\mathbb{A}^{2}$-cancellation property. Furthermore, every non rigid $X$ which fails the $\mathbb{A}^{2}$-cancellation property also fails the $\mathbb{A}^{1}$-cancellation property.

## 1. Proof of the theorem

1.1. Rigid affine cubic surfaces. Given a smooth cubic surface $V \subset \mathbb{P}^{3}$ and a hyperplane section $V \cap H$ consisting of an irreducible plane cuspidal cubic $C$, the restriction of the projection $\mathbb{P}^{3} \rightarrow \mathbb{P}^{2}$ from the singular point $p$ of $C$ induces a rational map $V \rightarrow \mathbb{P}^{2}$ of degree 2 with $p$ as a unique proper base point. Its lift to the blow-up $\alpha: Y \rightarrow V$ of $V$ at $p$ coincides with the morphism $\theta: Y \rightarrow \mathbb{P}^{2}$ defined by the anti-canonical linear system $\left|-K_{Y}\right|$ and it factors through a birational morphism $Y \rightarrow Z$ to the anti-canonical model $Z=\operatorname{Proj}_{\mathbb{C}}\left(\bigoplus_{m \geq 0} H^{0}\left(Y,-m K_{Y}\right)\right)$ of $Y$, followed by a Galois double cover $Z \rightarrow \mathbb{P}^{2}$ ramified over a quartic curve. The nontrivial involution of the double cover $Z \rightarrow \mathbb{P}^{2}$ lifts to a biregular involution of $Y$ exchanging the proper transform of $C$ and the exceptional divisor $E$ of $\alpha$. This involution descends back to a birational map $G_{p}: V \rightarrow V$, called the Geiser involution of $V$ with center at $p$, which contracts $C$ to $p$ and restricts to a biregular involution $j_{p}: X \rightarrow X$ of the affine complement $X$ of $C$ in $V$.

Lemma 1. Let $X_{i}$ be the complements of cuspidal hyperplanes sections $C_{i}=V_{i} \cap H_{i}$ with respective singular points $p_{i}$ of smooth cubic surfaces $V_{i} \subset \mathbb{P}^{3}, i=1,2$. Then for every isomorphism $\psi: X_{1} \xrightarrow{\sim} X_{2}$, the birational map $\bar{\psi}: V_{1} \rightarrow V_{2}$ extending $\psi$ is either an isomorphism of pairs $\left(V_{1}, C_{1}\right) \xrightarrow{\sim}\left(V_{2}, C_{2}\right)$ or it factors in a unique way as the composition of the Geiser involution $G_{p_{1}}: V_{1} \rightarrow V_{1}$ followed by an isomorphism of pairs $\left(V_{1}, C_{1}\right) \xrightarrow{\sim}\left(V_{2}, C_{2}\right)$. In particular, $X_{1}$ and $X_{2}$ are isomorphic if and only if so are the pairs $\left(V_{1}, C_{1}\right)$ and $\left(V_{2}, C_{2}\right)$.

Proof. Letting $\alpha_{i}: Y_{i} \rightarrow V_{i}$ be the blow-up of $V_{i}$ at $p_{i}$, with exceptional divisor $E_{i}, X_{i}$ is isomorphic to $Y_{i} \backslash\left(C_{i} \cup E_{i}\right)$ where we identified $C_{i}$ and its proper transform in $Y_{i}$. The birational map $\bar{\psi}: V_{1} \rightarrow V_{2}$ lifts to a birational $\bar{\Psi}=\alpha_{2}^{-1} \circ \bar{\psi} \circ \alpha_{1}: Y_{1} \rightarrow Y_{2}$ extending $\psi$, and the assertion is equivalent to the fact that $\bar{\Psi}$ is an isomorphism of pairs $\left(Y_{1}, C_{1} \cup E_{1}\right) \xrightarrow{\sim}\left(Y_{2}, C_{2} \cup E_{2}\right)$. Since $Y_{1}$ and $Y_{2}$ are smooth with the same Picard rank $\rho\left(Y_{i}\right)=8$, this holds provided that either $\bar{\Psi}$ or $\bar{\Psi}^{-1}$ is a morphism. So suppose for contradiction that $\bar{\Psi}$ or $\bar{\Psi}^{-1}$ are both strictly birational and let $Y_{1} \stackrel{\sigma_{1}}{\leftarrow} W \xrightarrow{\sigma_{2}} Y_{2}$ be the minimal resolution of $\bar{\Psi}$. Since $Y_{1}$ and $Y_{2}$ are smooth and $\bar{\Psi}$ and $\bar{\Psi}^{-1}$ are both strictly birational, $\sigma_{1}$ consists of a non-empty sequence of blow-ups of smooth points whose centers lie over $C_{1} \cup E_{1}$, while $\sigma_{2}$ is a non-empty sequence of contractions of successives ( -1 )-curves on $W$ supported on the total transform $\sigma_{1}^{-1}\left(C_{1} \cup E_{1}\right)$ of $C_{1} \cup E_{1}$. Furthermore, the minimality assumption implies that the first curve contracted by $\sigma_{2}$ is the proper transform in $W$ of $C_{1}$ or $E_{1}$. Since $C_{1}$ and $E_{1}$ are ( -1 )-curves in $Y_{1}$, the only possibility is thus that all successive centers of $\sigma_{1}$ lie over $E_{1} \backslash C_{1}$ (resp. $C_{1} \backslash E_{1}$ ) and that the first curve contracted by $\sigma_{2}$ is the proper transform of $E_{1}$ (resp. $C_{1}$ ). But since $C_{1}$ and $E_{1}$ are tangent in $Y_{1}$, so are their proper transforms in $W$, and then the image of $C_{1}$ (resp. $E_{1}$ ) by the contraction $\tau: W \rightarrow W^{\prime}$ of $E_{1}$ (resp. $C_{1}$ ) factoring $\sigma_{2}: W \rightarrow Y_{2}$ would be singular. Since it cannot be contracted at any further step, its image by $\sigma_{2}$ would be a singular curve contained in $Y_{2} \backslash X_{2}=C_{2} \cup E_{2}$, which is absurd.

Corollary 2. Let $X$ be the complement of a cuspidal hyperplane section $C$ of a smooth cubic surface $V \subset \mathbb{P}^{3}$. Then there exists a split exact sequence

$$
0 \rightarrow \operatorname{Aut}(V, C) \rightarrow \operatorname{Aut}(X) \rightarrow\left\{\operatorname{id}_{X}, j_{p}\right\} \simeq \mathbb{Z}_{2} \rightarrow 0
$$

where $\operatorname{Aut}(V, C)$ is the automorphism group of the pair $(V, C)$ and $j_{p}: X \xrightarrow{\sim} X$ is the biregular involution induced by the Geiser involution of $V$ with center at the singular point $p$ of $C$. In particular, Aut $(X)$ is a finite group, isomorphic to $\mathbb{Z}_{2}$ for a general smooth cubic surface $V$.

Proof. We view $\operatorname{Aut}(V, C)$ as a subgroup of $\operatorname{Aut}(X)$ via the homomorphism which associates to every automorphism of $V$ preserving $C$, hence $X$, its restriction to $X$. Since by virtue of the previous lemma, the extension of every automorphism $\varphi$ of $X$ to a birational self-map $\bar{\varphi}: V \rightarrow V$ is either an automorphism of the pair ( $V, C$ ) or the composition of the Geiser involution $G_{p}: V \rightarrow V$ with an automorphism of this pair, the first assertion follows. The second assertion is a consequence of the fact that the automorphism group $\operatorname{Aut}(V)$ of a smooth cubic surface $V$ is always finite, actually trivial for a general such surface.

The following proposition provides the first part of the proof of the theorem:
Proposition 3. Let $X_{i}$ be the complements of cuspidal hyperplanes sections $C_{i}=V_{i} \cap H_{i}$ of smooth cubic surfaces $V_{i} \subset \mathbb{P}^{3}, i=1,2$. If the pairs $\left(V_{1}, C_{1}\right)$ and $\left(V_{2}, C_{2}\right)$ are not isomorphic then the $\mathbb{A}^{1}$-cylinders $X_{1} \times \mathbb{A}^{1}$ and $X_{2} \times \mathbb{A}^{1}$ are not isomorphic.

Proof. The rigidity of $X_{i}$ asserted by Corollary 2 implies by virtue of [5, Proposition 9.23] that the Makar-Limanov invariant $\operatorname{ML}\left(X_{i} \times \mathbb{A}^{1}\right)$ of $X_{i} \times \mathbb{A}^{1}$ is equal to the sub-algebra $\Gamma\left(X_{i}, \mathcal{O}_{X_{i}}\right)$ of $\Gamma\left(X_{i}, \mathcal{O}_{X_{i}}\right)[t]=\Gamma\left(X_{i} \times \mathbb{A}^{1}, \mathcal{O}_{X_{i} \times \mathbb{A}^{1}}\right)$. Since every isomorphism between two algebras induces an isomorphism between their Makar-Limanov invariants, it follows that every isomorphism $X_{1} \times \mathbb{A}^{1} \xrightarrow{\sim} X_{2} \times \mathbb{A}^{1}$ descends to a unique isomorphism $\psi: X_{1} \xrightarrow{\sim} X_{2}$ making the following diagram commutative


On the other hand, the hypothesis that the pairs $\left(V_{1}, C_{1}\right)$ and $\left(V_{2}, C_{2}\right)$ are not isomorphic combined with Lemma 1, implies that $X_{1}$ is not isomorphic to $X_{2}$ and so, $X_{1} \times \mathbb{A}^{1}$ is not isomorphic to $X_{2} \times \mathbb{A}^{1}$.
1.2. Isomorphisms between $\mathbb{A}^{2}$-cylinders. As explained above, the first step of the construction is a Danielewski fiber product trick creating a smooth affine threefold $W$ which is simultaneously the total space of a line bundle over $X_{1}$ and $X_{2}$. To setup such a fiber product argument, we first construct a certain smooth algebraic space $\delta: B \rightarrow \mathbb{P}^{1}$ with the property that every complement $X$ of an irreducible cuspidal hyperplane section $C$ of a smooth cubic surface $V \subset \mathbb{P}^{3}$ admits the structure of an étale locally trivial $\mathbb{A}^{1}$-bundle $\rho: X \rightarrow B$.
1.2.1. Letting $\mathbb{P}^{1}=\operatorname{Proj}\left(\mathbb{C}\left[z_{0}, z_{1}\right]\right)$, the algebraic space $\delta: B \rightarrow \mathbb{P}^{1}$ is obtained by the following gluing procedure:

1) We let $U_{\infty}=\mathbb{P}^{1} \backslash\{0\} \simeq \operatorname{Spec}\left(\mathbb{C}\left[w_{\infty}\right]\right)$, where $w_{\infty}=z_{1} / z_{0}$, and we let $\delta_{\infty}: B_{\infty} \rightarrow U_{\infty}$ be the scheme isomorphic to affine line with a 6 -fold origin obtained by gluing six copies $\delta_{\infty, i}: U_{\infty, i} \xrightarrow{\sim} U_{\infty}, i=1, \ldots, 6$ of $U_{\infty}$, by the identity outside the points $\infty_{i}=\delta_{\infty, i}^{-1}(\infty)$.
2) We $U_{0}=\mathbb{P}^{1} \backslash\{\infty\} \simeq \operatorname{Spec}\left(\mathbb{C}\left[w_{0}\right]\right)$, where $w_{0}=z_{0} / z_{1}$, we let $\xi: \tilde{U}_{0} \simeq \mathbb{A}^{1}=\operatorname{Spec}\left(\mathbb{C}\left[\tilde{w}_{0}\right]\right) \rightarrow U_{0} \simeq \operatorname{Spec}\left(\mathbb{C}\left[w_{0}\right]\right)$, $\tilde{w}_{0} \mapsto w_{0}=\tilde{w}_{0}^{3}$ be the triple Galois cover totally ramified over 0 and étale elsewhere, and we let $\tilde{\delta}_{0}: \tilde{B}_{0} \rightarrow \tilde{U}_{0}$ be the scheme isomorphic to the affine line with 3 -fold origin obtained by gluing three copies $\tilde{U}_{0,1}, \tilde{U}_{0, \omega}$ and $\tilde{U}_{0, \omega^{2}}$ of $\tilde{U}_{0}$ by the identity outside their respective origins $\tilde{0}_{0,1}, \tilde{0}_{0, \omega}$ and $\tilde{0}_{0, \omega^{2}}$. The action of the Galois group $\mu_{3}=\left\{1, \omega, \omega^{2}\right\}$ of complex third roots of unity of the covering $\xi$ lifts to fixed point free action on $\tilde{B}_{0}$ given locally by $\tilde{U}_{0, \eta} \ni \tilde{z}_{0} \mapsto \omega \tilde{z}_{0} \in \tilde{U}_{0, \omega \eta}$. Since the latter has trivial isotropies, a geometric quotient exists in the category of algebraic spaces in the form an étale locally trivial $\mu_{\tilde{B}_{3}}$-bundle $\tilde{U}_{0} \rightarrow \tilde{B}_{0} / \mu_{3}=B_{0}$ over a certain algebraic space $B_{0}$. Furthermore, the $\mu_{3}$-equivariant morphism $\tilde{\delta}_{0}: \tilde{B}_{0} \rightarrow \tilde{U}_{0}$ descends to a morphism $\delta_{0}: B_{0} \rightarrow \tilde{U}_{0} / / \mu_{3} \simeq U_{0}$ restricting to an isomorphism over $U_{0} \backslash\{0\}$ and totally ramified over $\{0\}$, with ramification index 3 .
3) Finally, $\delta: B \rightarrow \mathbb{P}^{1}$ is obtained by gluing $\delta_{\infty}: B_{\infty} \rightarrow U_{\infty}$ and $\delta_{0}: B_{0} \rightarrow U_{0}$ along the open sub-schemes $\delta_{\infty}^{-1}\left(U_{0} \cap U_{\infty}\right) \simeq \operatorname{Spec}\left(\mathbb{C}\left[w_{\infty}^{ \pm 1}\right]\right)$ and $\delta_{0}^{-1}\left(U_{0} \cap U_{\infty}\right) \simeq \operatorname{Spec}\left(\mathbb{C}\left[w_{\infty}^{ \pm 1}\right]\right)$ by the isomorphism $w_{\infty} \mapsto w_{0}=w_{\infty}^{-1}$.
Remark 4. Letting $p_{0}$ be the unique closed point of $B$ over $0 \in U_{0} \subset \mathbb{P}^{1}$, we have $\delta^{-1}(0)=3 p_{0}$ while the restriction of $\delta$ over $U_{0} \backslash\{0\}$ is an isomorphism. This implies that $B$ is not a scheme, for otherwise the restriction of $\delta$ over $U_{0}$ would be an isomorphism by virtue of Zariski Main Theorem. In fact, $p_{0}$ is a point which does not admit any affine open neighborhood $V$ : otherwise the inverse image of $V$ by the finite morphism $\tilde{B}_{0} \rightarrow \tilde{B}_{0} / \mu_{3}=B_{0}$ would be an affine open sub-scheme of $\tilde{B}_{0}$ containing the three points $\tilde{0}_{0,1}, \tilde{0}_{0, \omega}$ and $\tilde{0}_{0, \omega^{2}}$ which is impossible.
1.2.2. Since the automorphism group of $\mathbb{A}^{1}$ is the affine group $A f f=\mathbb{G}_{m} \ltimes \mathbb{G}_{a}$, every étale locally trivial $\mathbb{A}^{1}$ bundle $\rho: S \rightarrow B$ is an affine-linear bundle whose isomorphy class is determined by an element in the non-abelian cohomology group $H_{\text {êt }}^{1}\left(B, \mathrm{Aff}_{1}\right)$. Equivalently $\rho: S \rightarrow B$ is a principal homogeneous bundle under the action of a line bundle $L \rightarrow B$, considered as a locally constant group scheme over $B$ for the group law induced by the addition of germs of sections, whose class in $\operatorname{Pic}(B) \simeq H_{\text {ét }}^{1}\left(B, \mathbb{G}_{m}\right)$ coincides with the image of the isomorphy class of $\rho: S \rightarrow B$ in $H_{\text {et }}^{1}\left(B, \operatorname{Aff}_{1}\right)$ by the map $H_{\text {et }}^{1}\left(B, \mathrm{Aff}_{1}\right) \rightarrow H_{\text {ett }}^{1}\left(B, \mathbb{G}_{m}\right)$ in the long exact sequence of non-abelian cohomology

$$
0 \rightarrow H^{0}\left(B, \mathbb{G}_{a}\right) \rightarrow H^{0}\left(B, \operatorname{Aff}_{1}\right) \rightarrow H^{0}\left(B, \operatorname{Aff}_{1}\right) \rightarrow H_{\text {êt }}^{1}\left(B, \mathbb{G}_{a}\right) \rightarrow H_{\text {ét }}^{1}\left(B, \operatorname{Aff}_{1}\right) \rightarrow H_{\text {ét }}^{1}\left(B, \mathbb{G}_{m}\right)
$$

associated to the short exact sequence $0 \rightarrow \mathbb{G}_{a} \rightarrow \mathrm{Aff}_{1} \rightarrow \mathbb{G}_{m} \rightarrow 0$. Isomorphy classes of principal homogeneous under a given line bundle $L \rightarrow B$ are are in turn classified by the cohomology group $H_{\text {ett }}^{1}(B, L)$.

Proposition 5. The complement $X$ of a cuspidal hyperplane section $C$ of a smooth cubic surface $V \subset \mathbb{P}^{3}$ admits an $\mathbb{A}^{1}$-fibration $f: X \rightarrow \mathbb{P}^{1}$ which factors through a principal homogeneous bundle $\rho: X \rightarrow B$ under the action of the cotangent line bundle $\gamma: \Omega_{B}^{1} \rightarrow B$ of $B$.

Proof. Since $C$ is an anti-canonical divisor on $V$, it follows from adjunction formula that every line on $V$ intersects $C$ transversally in a unique point. The image of $C$ by the contraction $\tau: V \rightarrow \mathbb{P}^{2}$ of any 6 -tuple of disjoint lines, $L_{1}, \ldots, L_{6}$, on $V$ is therefore a rational cuspidal cubic containing the images $q_{i}=\tau\left(L_{i}\right), i=1, \ldots, 6$, in its regular locus. The rational pencil $\mathbb{P}^{2} \rightarrow \mathbb{P}^{1}$ generated by $\tau(C)$ and three times its tangent line $T$ at its singular point lifts to a rational pencil $\bar{f}: V \rightarrow \mathbb{P}^{1}$ whose restriction to $X$ is an $\mathbb{A}^{1}$-fibration $f: X \rightarrow \mathbb{P}^{1}$ with two degenerate fibers: one irreducible of multiplicity three consisting of the intersection of the proper transform of $T$ with $X$, and a reduced one consisting of the disjoint union of the curves $L_{i} \cap X \simeq \mathbb{A}^{1}, i=1, \ldots, 6$. Choosing homogeneous coordinates $\left[z_{0}: z_{1}\right]$ on $\mathbb{P}^{1}$ in such a way that 0 and $\infty$ are the respective images of $T$ and $C$ by $\bar{f}$, the same argument as in $[3, \S 4]$ implies that $f: X \rightarrow \mathbb{P}^{1}$ factors through an étale locally trivial $\mathbb{A}^{1}$-bundle $\rho: X \rightarrow B$. Letting $\gamma: L \rightarrow B$ be the line bundle under which $\rho: X \rightarrow B$ becomes a principal homogeneous bundle, it follows from the relative cotangent exact sequence

$$
0 \rightarrow \rho^{*} \Omega_{B}^{1} \rightarrow \Omega_{X}^{1} \rightarrow \Omega_{X / B}^{1} \simeq \rho^{*} L^{\vee} \rightarrow 0
$$

of $\rho$ that $\operatorname{det} \Omega_{X}^{1} \simeq \rho^{*}\left(\Omega_{B}^{1} \otimes L^{\vee}\right)$. Since $\operatorname{det} \Omega_{X}^{1}$ is trivial as $C$ is an anti-canonical divisor on $V$ and since $\rho^{*}: \operatorname{Pic}(B) \rightarrow$ $\operatorname{Pic}(X)$ is an isomorphism because $\rho: X \rightarrow B$ is a locally trivial $\mathbb{A}^{1}$-bundle, we conclude that $L \simeq \Omega_{B}^{1}$.

Remark 6. By construction of $\delta: B \rightarrow \mathbb{P}^{1}$, we have $\delta^{-1}(0)=3 p_{0}$ and $\delta^{-1}(\infty)=\sum_{i=1}^{6} \infty_{i}$. The Picard group Pic $(B)$ of $B$ is thus isomorphic to $\mathbb{Z}^{6}$ generated by the classes of the lines bundle $\mathcal{O}_{B}\left(p_{0}\right), \mathcal{O}_{B}\left(\infty_{i}\right), i=1, \ldots, 6$, with the unique relation $\mathcal{O}_{B}\left(3 p_{0}\right)=\mathcal{O}_{B}\left(\sum_{i=1}^{6} \infty_{i}\right)$. Furthermore, since $\delta$ is étale except at $p_{0}$ where it has ramification index 3 , we deduce from the ramification formula for the morphism $\delta: B \rightarrow \mathbb{P}^{1}$ that the cotangent bundle $\gamma: \Omega_{B}^{1} \rightarrow B$ of $B$ is isomorphic to

$$
\delta^{*} \Omega_{\mathbb{P}^{1}} \otimes \mathcal{O}_{B} \mathcal{O}_{B}\left(2 p_{0}\right) \simeq \delta^{*}\left(\mathcal{O}_{\mathbb{P}^{1}}(-\{0\}-\{\infty\})\right) \otimes \mathcal{O}_{B} \mathcal{O}_{B}\left(2 p_{0}\right) \simeq \mathcal{O}_{B}\left(-p_{0}-\sum_{i=1}^{6} \infty_{i}\right)
$$

1.2.3. Now we are ready for the second step of the construction, which completes the proof of the theorem. Letting $X_{i}, i=1,2$, be the complements of irreducible plane cuspidal hyperplanes sections $C_{i}=V_{i} \cap H_{i}$ of smooth cubic surfaces $V_{i} \subset \mathbb{P}^{3}$, Proposition 5 asserts the existence of principal homogeneous bundles $\rho_{i}: X_{i} \rightarrow B$ under the action of the cotangent line bundle $\gamma: \Omega_{B}^{1} \rightarrow B$ of $B$. The fiber product $W=X_{1} \times{ }_{B} X_{2}$ inherits via the first and second projections respectively the structure of a principal homogeneous bundle $\pi_{i}: W \rightarrow X_{i}$ under $\rho_{i}^{*} \Omega_{B}^{1}$, $i=1,2$. Since $X_{i}$ is affine, the vanishing of $H_{e \text { et }}^{1}\left(X_{i}, \rho_{i}^{*} \Omega_{B}^{1}\right)$ implies that these bundles are both trivial, yielding isomorphisms $\rho_{1}^{*} \Omega_{B}^{1} \simeq W \simeq \rho_{2}^{*} \Omega_{B}^{1}$. Letting $q: E \rightarrow W$ be the pull-back of the dual $\left(\Omega_{B}^{1}\right)^{\vee}$ of $\Omega_{B}^{1}$ by the morphism $\rho_{1} \circ \pi_{1}=\rho_{2} \circ \pi_{2}: W \rightarrow B, \pi_{i} \circ q: E \rightarrow S_{i}$ is a vector bundle over $X_{i}$ isomorphic to the direct sum of $\rho_{i}^{*} \Omega_{B}^{1}$ and
$\rho_{i}^{*}\left(\Omega_{B}^{1}\right)^{\vee}:$


So by virtue of [13, Theorem 3.1], $E$ is isomorphic as a vector bundle over $X_{i}$ to $\operatorname{det}\left(\rho_{i}^{*} \Omega_{B}^{1} \oplus \rho_{i}^{*}\left(\Omega_{B}^{1}\right)^{\vee}\right) \oplus \mathbb{A}_{X_{i}}^{1} \simeq$ $\mathbb{A}_{X_{i}}^{1} \oplus \mathbb{A}_{X_{i}}^{1}$ providing the desired isomorphisms $X_{1} \times \mathbb{A}^{2} \simeq E \simeq X_{2} \times \mathbb{A}^{2}$.
Example 7. Let $V \subset \mathbb{P}^{3}$ be a general smooth cubic surface and let $\Delta \subset V$ be the curve consisting of points $p$ of $V$ at which the projective tangent hyperplane $T_{p} V \subset \mathbb{P}^{3}$ of $V$ at $p$ intersects $V$ along a cuspidal cubic. Let $\mathcal{V}=\Delta \times V$ and let $\mathcal{C} \subset \mathcal{V}$ be relatively ample Cartier divisor with respect to $\mathrm{pr}_{\Delta}: \mathcal{V} \rightarrow \Delta$ whose fiber $\mathcal{C}_{p}$ over every point $p \in \Delta$ is equal to the intersection $C_{p}=V \cap T_{p} V$. Since $\operatorname{Aut}(V)$ is trivial, the pairs $\left(V, C_{p}\right), p \in \Delta$, are pairwise non isomorphic, and so $\Theta=\operatorname{pr}_{\Delta} \mid \mathcal{X}: \mathcal{X}=\mathcal{V} \backslash \mathcal{C} \rightarrow \Delta$ is a family of pairwise non isomorphic rigid smooth affine surfaces whose $\mathbb{A}^{2}$-cylinders are all isomorphic.

Remark 8. The $\mathbb{A}^{2}$-cylinder $X \times \mathbb{A}^{2}$ over the complement $X$ of a cuspidal hyperplane section $C$ of a smooth cubic surface $V$ is flexible in codimension 1, that is, for every closed point $p$ outside a possible empty closed subset $Z \subset X \times \mathbb{A}^{2}$ of codimension at least two, the tangent space $T_{X \times \mathbb{A}^{2}, p}$ of $X \times \mathbb{A}^{2}$ at $p$ is spanned by tangent vectors to orbits of algebraic $\mathbb{G}_{a}$-actions on $X \times \mathbb{A}^{2}$. This can be seen as follows: one first constructs by a similar procedure as in $[3, \S 3.2]$ a flexible mate $S$ for $X$, in the form of smooth affine surface flexible in codimension 1 admitting an $\mathbb{A}^{1}$-fibration $\pi: S \rightarrow \mathbb{P}^{1}$ which factors through a principal homogeneous bundle $\tilde{\pi}: S \rightarrow B$ under the action of a certain line bundle $\gamma^{\prime}: L \rightarrow B$. The fiber product $S \times_{B} X$ is then a smooth affine threefold which is simultaneously isomorphic to the total spaces of the line bundles $\tilde{\pi}^{*} \Omega_{B}^{1}$ and $\rho^{*} L$ over $S$ and $X$ via the first and second projection respectively. Since $S$ is flexible in codimension 1, it follows from [3, Lemma 2.3] that $S \times_{B} X$ and the total space $F \rightarrow S \times_{B} X$ of the pull-back of $L^{\vee}$ by the morphism $\tilde{\pi} \circ \operatorname{pr}_{S}=\rho \circ \operatorname{pr}_{X}$ are both flexible in codimension 1. By construction, $F$ is a vector bundle of rank 2 over $X$, isomorphic to $\rho^{*}\left(L \oplus L^{\vee}\right)$ hence to the trivial vector bundle $X \times \mathbb{A}^{2}$ by virtue of [13, Theorem 3.1].

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[^0]:    2010 Mathematics Subject Classification. 14R05, 14R25, 14E07.
    Key words and phrases. Rigid surfaces, Cancellation Problem.
    This research was partialy supported by ANR Grant "BirPol" ANR-11-JS01-004-01.

