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# RIGID AFFINE SURFACES WITH ISOMORPHIC $\mathbb{A}^2$ -CYLINDERS

ADRIEN DUBOULOZ

ABSTRACT. We construct families of smooth affine surfaces with pairwise non isomorphic  $\mathbb{A}^1$ -cylinders but whose  $\mathbb{A}^2$ -cylinders are all isomorphic. These arise as complements of cuspidal hyperplane sections of smooth projective cubic surfaces.

## INTRODUCTION

The Zariski Cancellation Problem, which asks whether two, say smooth affine, algebraic varieties  $X$  and  $Y$  with isomorphic cylinders  $X \times \mathbb{A}^n$  and  $Y \times \mathbb{A}^n$  for some  $n \geq 1$  are isomorphic themselves, has been studied very actively during the past decades culminating recently with a negative solution in dimension 3 and positive characteristic for the case  $X = \mathbb{A}^3$  [7]. The situation in the complex case, and more generally over any algebraically closed field of characteristic zero, is more contrasted: cancellation is known to hold for curves [1] and for  $\mathbb{A}^2$  [6], but many counter-examples in every dimension higher or equal to 2 have been discovered (see [14] for a survey), inspired by the two pioneering constructions of Hochster [8] and Danielewski [2].

Essentially all known families are counter-examples to the cancellation of 1-dimensional cylinders which arise from the existence of nontrivial decompositions of certain locally trivial  $\mathbb{A}^2$ -bundles over a base scheme  $Z$ . Namely, Hochster type constructions rely on the existence of non free, 1-stably free, projective modules which in geometric term correspond to non trivial decompositions of the trivial bundle  $Z \times \mathbb{A}^{r+1} \rightarrow Z$  into a trivial  $\mathbb{A}^1$ -bundle over a nontrivial vector bundle  $E \rightarrow Z$  of rank  $r \geq 1$ . For every such bundle, the varieties  $X = E$  and  $Y = Z \times \mathbb{A}^r$  have isomorphic cylinders  $X \times \mathbb{A}^1$  and  $Y \times \mathbb{A}^1$ , and one then gets a counter-example to the cancellation problem whenever  $X$  and  $Y$ , which by definition are non isomorphic as schemes over  $Z$ , are actually non isomorphic as abstract algebraic varieties [10]. In contrast, Danielewski type constructions usually involve non trivial decompositions of a principal homogeneous  $\mathbb{G}_a^2$ -bundle  $W \rightarrow Z$  into pairs  $W \rightarrow X \rightarrow Z$  and  $W \rightarrow Y \rightarrow Z$  consisting of trivial  $\mathbb{G}_a$ -bundles over nontrivial  $\mathbb{G}_a$ -bundles  $X \rightarrow Z$  and  $Y \rightarrow Z$  with affine total spaces, with the property that  $W$  is isomorphic to the fiber product  $W = X \times_Z Y$ . The isomorphism  $X \times \mathbb{A}^1 \simeq W \simeq Y \times \mathbb{A}^1$  is granted by definition, and similarly as in the previous type of construction, one obtains counter-examples to the cancellation problem as soon as  $X$  and  $Y$  are not isomorphic as abstract varieties.

Non-cancellation phenomena with respect to higher dimensional cylinders are more mysterious. In fact, it seems for instance that not even a single explicit example of a pair of non-isomorphic varieties  $X$  and  $Y$  which fail the  $\mathbb{A}^2$ -cancellation property in a minimal way, in the sense that  $X \times \mathbb{A}^2$  and  $Y \times \mathbb{A}^2$  are isomorphic while  $X \times \mathbb{A}^1$  and  $Y \times \mathbb{A}^1$  are still non isomorphic, is known so far. In this article, we fill this gap by constructing a positive dimensional moduli of smooth affine surfaces which fail the  $\mathbb{A}^2$ -cancellation property minimally. That such varieties exist was certainly a natural expectation, and their existence is therefore neither really surprising, nor probably exciting in itself due to the abundance of simpler counter-examples to the cancellation problem. Their interest lies rather in the fact that they provide additional insight on the algebro-geometric properties that a variety should satisfy in order to fail cancellation.

Indeed, it follows from Iitaka-Fujita strong Cancellation Theorem [9] that a smooth affine variety  $X$  which fails cancellation has negative logarithmic Kodaira dimension, a property conjecturally equivalent in dimension higher or equal to 3 to the fact that  $X$  is covered by images of the affine line and equivalent for surfaces to the existence of an  $\mathbb{A}^1$ -fibration  $\pi : X \rightarrow C$  over a smooth curve [12], i.e. a flat fibration with general fibers isomorphic to the affine line. In the particular case of the cancellation problem for 1-dimensional cylinders, a further striking discovery of Makar-Limanov is that the existence of nontrivial actions of the additive group  $\mathbb{G}_a$  on  $X$  is a necessary condition for non-cancellation. Namely, Makar-Limanov semi-rigidity theorem [11] (see also [5, Proposition 9.23]) asserts that if  $X$  is rigid, i.e. does not admit any nontrivial  $\mathbb{G}_a$ -action, then the projection  $\text{pr}_X : X \times \mathbb{A}^1 \rightarrow X$  is invariant under all  $\mathbb{G}_a$ -actions on  $X$ . As a consequence, if either  $X$  or  $Y$  is rigid then every isomorphism between  $X \times \mathbb{A}^1$  and  $Y \times \mathbb{A}^1$  descends to an isomorphism between  $X$  and  $Y$ . Combined with Fieseler's topological

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description of algebraic quotient morphisms of  $\mathbb{G}_a$ -actions on smooth complex affine surfaces [4], these results imply that a smooth affine surface which fails  $\mathbb{A}^1$ -cancellation must admit a nontrivial  $\mathbb{G}_a$ -action whose algebraic quotient morphism  $\pi : X \rightarrow X//\mathbb{G}_a = \text{Spec}(\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_a})$  is not a locally trivial  $\mathbb{A}^1$ -bundle. This holds of course for the two smooth surfaces  $xz - y(y+1) = 0$  and  $x^2z - y(y+1) = 0$  used by Danielewski in his celebrated counter-example, showing a posteriori that his construction was essentially optimal in this dimension. The rich families of existing counter-examples to  $\mathbb{A}^1$ -cancellation in dimension 2 lend support to the conjecture that every smooth affine surface which is neither rigid nor isomorphic to the total space of line bundle over an affine curve fails the  $\mathbb{A}^1$ -cancellation property.

In view of this conjecture, a smooth affine surface  $X$  which fails the  $\mathbb{A}^2$ -cancellation property in a minimal way must be simultaneously rigid and equipped with an  $\mathbb{A}^1$ -fibration  $\pi : X \rightarrow C$  over a smooth curve, and the well known fact that  $\mathbb{A}^1$ -fibrations over affine curves are algebraic quotient morphisms of nontrivial  $\mathbb{G}_a$ -actions implies further that  $C$  must be projective. This is precisely the case for the family of surfaces we construct in this article, a particular example being the smooth affine cubic surfaces

$$X = \{(-1 + \alpha \sqrt[3]{2}x_2 + \alpha \sqrt[3]{2}x_3)^3 + 8(x_1^3 + x_2^3 + x_3^3) = 0\} \text{ and } X' = \{(-1 + 2\alpha x_1 + \alpha \sqrt[3]{2}x_2 + \alpha \sqrt[3]{2}x_3)^3 + 8(x_1^3 + x_2^3 + x_3^3) = 0\}$$

in  $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x_1, x_2, x_3])$ , where  $\alpha = \exp(i\pi/3)$ , which are both rigid and equipped with an  $\mathbb{A}^1$ -fibration over  $\mathbb{P}^1$ . These arise as the complements in the Fermat cubic surface  $V = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\}$  in  $\mathbb{P}^3$  of the plane cuspidal cubics  $C = \{-(x_2 + x_3)^3 + 4(x_1^3 + x_2^3 + x_3^3) = 0\}$  and  $C' = \{-((\sqrt[3]{2})^2 x_1 + x_2 + x_3)^3 + 4(x_1^3 + x_2^3 + x_3^3) = 0\}$  obtained by intersecting  $V$  with its tangent hyperplane at the points  $p = [\alpha \sqrt[3]{2} : 0 : 1 : 1]$  and  $p' = [\alpha \sqrt[3]{2} : \sqrt[3]{2} : -1 : 1]$  respectively. The group of automorphisms of  $V$  being isomorphic to  $\mathbb{Z}_3 \times \mathfrak{S}_4$ , where  $\mathbb{Z}_3$  is the 3-torsion subgroup of  $\text{PGL}(4; \mathbb{C})$  and where  $\mathfrak{S}_4$  denotes the group of permutations of the variables, the fact that  $p$  and  $p'$  do not belong to a same  $\text{Aut}(V)$ -orbit implies that the pairs  $(V, C)$  and  $(V, C')$  are not isomorphic. Our main result just below then implies in turn that  $X$  and  $X'$  are non isomorphic, with isomorphic  $\mathbb{A}^2$ -cylinders  $X \times \mathbb{A}^2$  and  $X' \times \mathbb{A}^2$ .

**Theorem.** *Let  $(V_i, C_i)$ ,  $i = 1, 2$ , be non isomorphic pairs consisting of a smooth cubic surface  $V_i \subset \mathbb{P}^3$  and a cuspidal hyperplane section  $C_i = V_i \cap H_i$ . Then the affine surfaces  $X_i = V_i \setminus C_i$  are non isomorphic, with non isomorphic  $\mathbb{A}^1$ -cylinders  $X_i \times \mathbb{A}^1$  but with isomorphic  $\mathbb{A}^2$ -cylinders  $X_i \times \mathbb{A}^2$ ,  $i = 1, 2$ .*

As a consequence, all smooth affine surfaces arising as complements of cuspidal hyperplane sections of smooth projective cubic surfaces have isomorphic  $\mathbb{A}^2$ -cylinders. Noting that the projective closure in  $\mathbb{P}^3$  of the surface  $X_0 \subset \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z])$  with equation  $x^2y + y^2 + z^3 + 1 = 0$  is a smooth cubic surface intersecting the plane at infinity along the cuspidal cubic  $x^2y + z^3 = 0$ , we obtain the following:

**Corollary.** *Let  $X$  be a smooth affine surface isomorphic to the complement of a cuspidal hyperplane section of a smooth projective cubic surface. Then  $X \times \mathbb{A}^2$  is isomorphic to the affine cubic fourfold  $Z \subset \mathbb{A}^5 = \text{Spec}(\mathbb{C}[x, y, z][u, v])$  with equation  $x^2y + y^2 + z^3 + 1 = 0$ . Furthermore,  $X$  is isomorphic to the geometric quotient of a proper action of the group  $\mathbb{G}_a^2$  on  $Z$ .*

The scheme of the proof of the Theorem given in the next section is the following. The fact that the affine surfaces  $X_1$  and  $X_2$  are non-isomorphic follows from the non-isomorphy of the pairs  $(V_1, C_1)$  and  $(V_2, C_2)$  via an argument of classical birational geometry of projective cubic surfaces, which simultaneously renders the conclusion that  $X_1$  and  $X_2$  are rigid. The non isomorphy of the cylinders  $X_1 \times \mathbb{A}^1$  and  $X_2 \times \mathbb{A}^1$  is then a straightforward consequence Makar-Limanov's semi-rigidity Theorem.

The existence of an isomorphism between the  $\mathbb{A}^2$ -cylinders  $X_1 \times \mathbb{A}^2$  and  $X_2 \times \mathbb{A}^2$  is derived in two steps: the first one consists of another instance of a Danielewski fiber product trick argument, which provides a smooth affine threefold  $W$  equipped with simultaneous structures of line bundles  $\pi_1 : W \rightarrow X_1$  and  $\pi_2 : W \rightarrow X_2$  over  $X_1$  and  $X_2$ . But here, in contrast with the situation in Danielewski's counter-example, the fact that the  $\mathbb{A}^1$ -cylinders over  $X_1$  and  $X_2$  are not isomorphic implies that these two line bundles cannot be simultaneously trivial. Nevertheless, the crucial observation which enables a second step, reminiscent to Hochster construction, is that the pull-backs via the isomorphisms  $\pi_i^* : \text{Pic}(X_i) \rightarrow \text{Pic}(W)$  of the classes of these line bundles in the Picard groups of  $X_1$  and  $X_2$ , say  $L_1$  and  $L_2$ , coincide. Letting  $q : E \rightarrow W$  be a line bundle representing the common inverse in  $\text{Pic}(W)$  of  $\pi_1^* L_1 = \pi_2^* L_2$ , the composition  $\pi_i \circ q : E \rightarrow X_i$  is then a vector bundle of rank 2 isomorphic to the direct sum  $L_i \oplus L_i^\vee$ , where  $L_i^\vee$  denotes the dual of  $L_i$ , hence isomorphic to  $\det(E) \oplus \mathbb{A}_{X_i}^1 = (L_i \otimes L_i^\vee) \oplus \mathbb{A}_{X_i}^1 \simeq \mathbb{A}_{X_i}^1 \oplus \mathbb{A}_{X_i}^1$  by virtue of result of Pavaman Murthy [13] asserting that every vector bundle on a smooth affine surface birationally equivalent to a ruled surface is isomorphic to the direct sum of a trivial bundle with a line bundle.

$$\begin{array}{ccccc}
X_1 \times \mathbb{A}^2 & \xrightarrow{\sim} & E & \xleftarrow{\sim} & X_2 \times \mathbb{A}^2 \\
\downarrow \text{pr}_{S_1} & & \downarrow q & & \downarrow \text{pr}_{X_2} \\
& & W & & \\
& \swarrow \pi_1 & & \searrow \pi_2 & \\
X_1 & & & & X_2.
\end{array}$$

The construction of these isomorphisms suggests the following strengthening of the above conjecture characterizing smooth affine surfaces failing the  $\mathbb{A}^1$ -cancellation property, which would settle the question of the behavior of smooth affine surfaces under stabilization by affine spaces:

**Conjecture.** *A smooth affine surface  $X$  with negative logarithmic Kodaira dimension is either isomorphic to the total space of a line bundle over a curve, or it fails the  $\mathbb{A}^2$ -cancellation property. Furthermore, every non rigid  $X$  which fails the  $\mathbb{A}^2$ -cancellation property also fails the  $\mathbb{A}^1$ -cancellation property.*

## 1. PROOF OF THE THEOREM

**1.1. Rigid affine cubic surfaces.** Given a smooth cubic surface  $V \subset \mathbb{P}^3$  and a hyperplane section  $V \cap H$  consisting of an irreducible plane cuspidal cubic  $C$ , the restriction of the projection  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$  from the singular point  $p$  of  $C$  induces a rational map  $V \dashrightarrow \mathbb{P}^2$  of degree 2 with  $p$  as a unique proper base point. Its lift to the blow-up  $\alpha : Y \rightarrow V$  of  $V$  at  $p$  coincides with the morphism  $\theta : Y \rightarrow \mathbb{P}^2$  defined by the anti-canonical linear system  $|-K_Y|$  and it factors through a birational morphism  $Y \rightarrow Z$  to the anti-canonical model  $Z = \text{Proj}_{\mathbb{C}}(\bigoplus_{m \geq 0} H^0(Y, -mK_Y))$  of  $Y$ , followed by a Galois double cover  $Z \rightarrow \mathbb{P}^2$  ramified over a quartic curve. The nontrivial involution of the double cover  $Z \rightarrow \mathbb{P}^2$  lifts to a biregular involution of  $Y$  exchanging the proper transform of  $C$  and the exceptional divisor  $E$  of  $\alpha$ . This involution descends back to a birational map  $G_p : V \dashrightarrow V$ , called the *Geiser involution of  $V$  with center at  $p$* , which contracts  $C$  to  $p$  and restricts to a biregular involution  $j_p : X \rightarrow X$  of the affine complement  $X$  of  $C$  in  $V$ .

**Lemma 1.** *Let  $X_i$  be the complements of cuspidal hyperplane sections  $C_i = V_i \cap H_i$  with respective singular points  $p_i$  of smooth cubic surfaces  $V_i \subset \mathbb{P}^3$ ,  $i = 1, 2$ . Then for every isomorphism  $\psi : X_1 \xrightarrow{\sim} X_2$ , the birational map  $\bar{\psi} : V_1 \dashrightarrow V_2$  extending  $\psi$  is either an isomorphism of pairs  $(V_1, C_1) \xrightarrow{\sim} (V_2, C_2)$  or it factors in a unique way as the composition of the Geiser involution  $G_{p_1} : V_1 \dashrightarrow V_1$  followed by an isomorphism of pairs  $(V_1, C_1) \xrightarrow{\sim} (V_2, C_2)$ . In particular,  $X_1$  and  $X_2$  are isomorphic if and only if so are the pairs  $(V_1, C_1)$  and  $(V_2, C_2)$ .*

*Proof.* Letting  $\alpha_i : Y_i \rightarrow V_i$  be the blow-up of  $V_i$  at  $p_i$ , with exceptional divisor  $E_i$ ,  $X_i$  is isomorphic to  $Y_i \setminus (C_i \cup E_i)$  where we identified  $C_i$  and its proper transform in  $Y_i$ . The birational map  $\bar{\psi} : V_1 \dashrightarrow V_2$  lifts to a birational  $\bar{\Psi} : Y_1 \dashrightarrow Y_2$  extending  $\psi$ , and the assertion is equivalent to the fact that  $\bar{\Psi}$  is an isomorphism of pairs  $(Y_1, C_1 \cup E_1) \xrightarrow{\sim} (Y_2, C_2 \cup E_2)$ . Since  $Y_1$  and  $Y_2$  are smooth with the same Picard rank  $\rho(Y_i) = 8$ , this holds provided that either  $\bar{\Psi}$  or  $\bar{\Psi}^{-1}$  is a morphism. So suppose for contradiction that  $\bar{\Psi}$  or  $\bar{\Psi}^{-1}$  are both strictly birational and let  $Y_1 \xrightarrow{\sigma_1} W \xrightarrow{\sigma_2} Y_2$  be the minimal resolution of  $\bar{\Psi}$ . Since  $Y_1$  and  $Y_2$  are smooth and  $\bar{\Psi}$  and  $\bar{\Psi}^{-1}$  are both strictly birational,  $\sigma_1$  consists of a non-empty sequence of blow-ups of smooth points whose centers lie over  $C_1 \cup E_1$ , while  $\sigma_2$  is a non-empty sequence of contractions of successive  $(-1)$ -curves on  $W$  supported on the total transform  $\sigma_1^{-1}(C_1 \cup E_1)$  of  $C_1 \cup E_1$ . Furthermore, the minimality assumption implies that the first curve contracted by  $\sigma_2$  is the proper transform in  $W$  of  $C_1$  or  $E_1$ . Since  $C_1$  and  $E_1$  are  $(-1)$ -curves in  $Y_1$ , the only possibility is thus that all successive centers of  $\sigma_1$  lie over  $E_1 \setminus C_1$  (resp.  $C_1 \setminus E_1$ ) and that the first curve contracted by  $\sigma_2$  is the proper transform of  $E_1$  (resp.  $C_1$ ). But since  $C_1$  and  $E_1$  are tangent in  $Y_1$ , so are their proper transforms in  $W$ , and then the image of  $C_1$  (resp.  $E_1$ ) by the contraction  $\tau : W \rightarrow W'$  of  $E_1$  (resp.  $C_1$ ) factoring  $\sigma_2 : W \rightarrow Y_2$  would be singular. Since it cannot be contracted at any further step, its image by  $\sigma_2$  would be a singular curve contained in  $Y_2 \setminus X_2 = C_2 \cup E_2$ , which is absurd.  $\square$

**Corollary 2.** *Let  $X$  be the complement of a cuspidal hyperplane section  $C$  of a smooth cubic surface  $V \subset \mathbb{P}^3$ . Then there exists a split exact sequence*

$$0 \rightarrow \text{Aut}(V, C) \rightarrow \text{Aut}(X) \rightarrow \{\text{id}_X, j_p\} \simeq \mathbb{Z}_2 \rightarrow 0,$$

where  $\text{Aut}(V, C)$  is the automorphism group of the pair  $(V, C)$  and  $j_p : X \xrightarrow{\sim} X$  is the biregular involution induced by the Geiser involution of  $V$  with center at the singular point  $p$  of  $C$ . In particular,  $\text{Aut}(X)$  is a finite group, isomorphic to  $\mathbb{Z}_2$  for a general smooth cubic surface  $V$ .

*Proof.* We view  $\text{Aut}(V, C)$  as a subgroup of  $\text{Aut}(X)$  via the homomorphism which associates to every automorphism of  $V$  preserving  $C$ , hence  $X$ , its restriction to  $X$ . Since by virtue of the previous lemma, the extension of every automorphism  $\varphi$  of  $X$  to a birational self-map  $\bar{\varphi} : V \dashrightarrow V$  is either an automorphism of the pair  $(V, C)$  or the composition of the Geiser involution  $G_p : V \dashrightarrow V$  with an automorphism of this pair, the first assertion follows. The second assertion is a consequence of the fact that the automorphism group  $\text{Aut}(V)$  of a smooth cubic surface  $V$  is always finite, actually trivial for a general such surface.  $\square$

The following proposition provides the first part of the proof of the theorem:

**Proposition 3.** *Let  $X_i$  be the complements of cuspidal hyperplanes sections  $C_i = V_i \cap H_i$  of smooth cubic surfaces  $V_i \subset \mathbb{P}^3$ ,  $i = 1, 2$ . If the pairs  $(V_1, C_1)$  and  $(V_2, C_2)$  are not isomorphic then the  $\mathbb{A}^1$ -cylinders  $X_1 \times \mathbb{A}^1$  and  $X_2 \times \mathbb{A}^1$  are not isomorphic.*

*Proof.* The rigidity of  $X_i$  asserted by Corollary 2 implies by virtue of [5, Proposition 9.23] that the Makar-Limanov invariant  $\text{ML}(X_i \times \mathbb{A}^1)$  of  $X_i \times \mathbb{A}^1$  is equal to the sub-algebra  $\Gamma(X_i, \mathcal{O}_{X_i})$  of  $\Gamma(X_i, \mathcal{O}_{X_i})[t] = \Gamma(X_i \times \mathbb{A}^1, \mathcal{O}_{X_i \times \mathbb{A}^1})$ . Since every isomorphism between two algebras induces an isomorphism between their Makar-Limanov invariants, it follows that every isomorphism  $X_1 \times \mathbb{A}^1 \xrightarrow{\sim} X_2 \times \mathbb{A}^1$  descends to a unique isomorphism  $\psi : X_1 \xrightarrow{\sim} X_2$  making the following diagram commutative

$$\begin{array}{ccc} X_1 \times \mathbb{A}^1 & \xrightarrow{\sim} & X_2 \times \mathbb{A}^1 \\ \text{pr}_{X_1} \downarrow & & \downarrow \text{pr}_{X_2} \\ X_1 & \xrightarrow{\psi} & X_2. \end{array}$$

On the other hand, the hypothesis that the pairs  $(V_1, C_1)$  and  $(V_2, C_2)$  are not isomorphic combined with Lemma 1, implies that  $X_1$  is not isomorphic to  $X_2$  and so,  $X_1 \times \mathbb{A}^1$  is not isomorphic to  $X_2 \times \mathbb{A}^1$ .  $\square$

**1.2. Isomorphisms between  $\mathbb{A}^2$ -cylinders.** As explained above, the first step of the construction is a Danielewski fiber product trick creating a smooth affine threefold  $W$  which is simultaneously the total space of a line bundle over  $X_1$  and  $X_2$ . To setup such a fiber product argument, we first construct a certain smooth algebraic space  $\delta : B \rightarrow \mathbb{P}^1$  with the property that every complement  $X$  of an irreducible cuspidal hyperplane section  $C$  of a smooth cubic surface  $V \subset \mathbb{P}^3$  admits the structure of an étale locally trivial  $\mathbb{A}^1$ -bundle  $\rho : X \rightarrow B$ .

1.2.1. Letting  $\mathbb{P}^1 = \text{Proj}(\mathbb{C}[z_0, z_1])$ , the algebraic space  $\delta : B \rightarrow \mathbb{P}^1$  is obtained by the following gluing procedure:

1) We let  $U_\infty = \mathbb{P}^1 \setminus \{0\} \simeq \text{Spec}(\mathbb{C}[w_\infty])$ , where  $w_\infty = z_1/z_0$ , and we let  $\delta_\infty : B_\infty \rightarrow U_\infty$  be the scheme isomorphic to affine line with a 6-fold origin obtained by gluing six copies  $\delta_{\infty, i} : U_{\infty, i} \xrightarrow{\sim} U_\infty$ ,  $i = 1, \dots, 6$  of  $U_\infty$ , by the identity outside the points  $\infty_i = \delta_{\infty, i}^{-1}(\infty)$ .

2) We  $U_0 = \mathbb{P}^1 \setminus \{\infty\} \simeq \text{Spec}(\mathbb{C}[w_0])$ , where  $w_0 = z_0/z_1$ , we let  $\xi : \tilde{U}_0 \simeq \mathbb{A}^1 = \text{Spec}(\mathbb{C}[\tilde{w}_0]) \rightarrow U_0 \simeq \text{Spec}(\mathbb{C}[w_0])$ ,  $\tilde{w}_0 \mapsto w_0 = \tilde{w}_0^3$  be the triple Galois cover totally ramified over 0 and étale elsewhere, and we let  $\tilde{\delta}_0 : \tilde{B}_0 \rightarrow \tilde{U}_0$  be the scheme isomorphic to the affine line with 3-fold origin obtained by gluing three copies  $\tilde{U}_{0,1}$ ,  $\tilde{U}_{0,\omega}$  and  $\tilde{U}_{0,\omega^2}$  of  $\tilde{U}_0$  by the identity outside their respective origins  $\tilde{0}_{0,1}$ ,  $\tilde{0}_{0,\omega}$  and  $\tilde{0}_{0,\omega^2}$ . The action of the Galois group  $\mu_3 = \{1, \omega, \omega^2\}$  of complex third roots of unity of the covering  $\xi$  lifts to fixed point free action on  $\tilde{B}_0$  given locally by  $\tilde{U}_{0,\eta} \ni \tilde{z}_0 \mapsto \omega \tilde{z}_0 \in \tilde{U}_{0,\omega\eta}$ . Since the latter has trivial isotropies, a geometric quotient exists in the category of algebraic spaces in the form an étale locally trivial  $\mu_3$ -bundle  $\tilde{B}_0 \rightarrow \tilde{B}_0/\mu_3 = B_0$  over a certain algebraic space  $B_0$ . Furthermore, the  $\mu_3$ -equivariant morphism  $\tilde{\delta}_0 : \tilde{B}_0 \rightarrow \tilde{U}_0$  descends to a morphism  $\delta_0 : B_0 \rightarrow U_0//\mu_3 \simeq U_0$  restricting to an isomorphism over  $U_0 \setminus \{0\}$  and totally ramified over  $\{0\}$ , with ramification index 3.

3) Finally,  $\delta : B \rightarrow \mathbb{P}^1$  is obtained by gluing  $\delta_\infty : B_\infty \rightarrow U_\infty$  and  $\delta_0 : B_0 \rightarrow U_0$  along the open sub-schemes  $\delta_\infty^{-1}(U_0 \cap U_\infty) \simeq \text{Spec}(\mathbb{C}[w_\infty^{\pm 1}])$  and  $\delta_0^{-1}(U_0 \cap U_\infty) \simeq \text{Spec}(\mathbb{C}[w_\infty^{\pm 1}])$  by the isomorphism  $w_\infty \mapsto w_0 = w_\infty^{-1}$ .

*Remark 4.* Letting  $p_0$  be the unique closed point of  $B$  over  $0 \in U_0 \subset \mathbb{P}^1$ , we have  $\delta^{-1}(0) = 3p_0$  while the restriction of  $\delta$  over  $U_0 \setminus \{0\}$  is an isomorphism. This implies that  $B$  is not a scheme, for otherwise the restriction of  $\delta$  over  $U_0$  would be an isomorphism by virtue of Zariski Main Theorem. In fact,  $p_0$  is a point which does not admit any affine open neighborhood  $V$ : otherwise the inverse image of  $V$  by the finite morphism  $\tilde{B}_0 \rightarrow \tilde{B}_0/\mu_3 = B_0$  would be an affine open sub-scheme of  $\tilde{B}_0$  containing the three points  $\tilde{0}_{0,1}$ ,  $\tilde{0}_{0,\omega}$  and  $\tilde{0}_{0,\omega^2}$  which is impossible.

1.2.2. Since the automorphism group of  $\mathbb{A}^1$  is the affine group  $\text{Aff}_1 = \mathbb{G}_m \times \mathbb{G}_a$ , every étale locally trivial  $\mathbb{A}^1$ -bundle  $\rho : S \rightarrow B$  is an affine-linear bundle whose isomorphism class is determined by an element in the non-abelian cohomology group  $H_{\text{ét}}^1(B, \text{Aff}_1)$ . Equivalently  $\rho : S \rightarrow B$  is a principal homogeneous bundle under the action of a line bundle  $L \rightarrow B$ , considered as a locally constant group scheme over  $B$  for the group law induced by the addition of germs of sections, whose class in  $\text{Pic}(B) \simeq H_{\text{ét}}^1(B, \mathbb{G}_m)$  coincides with the image of the isomorphism class of  $\rho : S \rightarrow B$  in  $H_{\text{ét}}^1(B, \text{Aff}_1)$  by the map  $H_{\text{ét}}^1(B, \text{Aff}_1) \rightarrow H_{\text{ét}}^1(B, \mathbb{G}_m)$  in the long exact sequence of non-abelian cohomology

$$0 \rightarrow H^0(B, \mathbb{G}_a) \rightarrow H^0(B, \text{Aff}_1) \rightarrow H^0(B, \text{Aff}_1) \rightarrow H_{\text{ét}}^1(B, \mathbb{G}_a) \rightarrow H_{\text{ét}}^1(B, \text{Aff}_1) \rightarrow H_{\text{ét}}^1(B, \mathbb{G}_m)$$

associated to the short exact sequence  $0 \rightarrow \mathbb{G}_a \rightarrow \text{Aff}_1 \rightarrow \mathbb{G}_m \rightarrow 0$ . Isomorphism classes of principal homogeneous under a given line bundle  $L \rightarrow B$  are in turn classified by the cohomology group  $H_{\text{ét}}^1(B, L)$ .

**Proposition 5.** *The complement  $X$  of a cuspidal hyperplane section  $C$  of a smooth cubic surface  $V \subset \mathbb{P}^3$  admits an  $\mathbb{A}^1$ -fibration  $f : X \rightarrow \mathbb{P}^1$  which factors through a principal homogeneous bundle  $\rho : X \rightarrow B$  under the action of the cotangent line bundle  $\gamma : \Omega_B^1 \rightarrow B$  of  $B$ .*

*Proof.* Since  $C$  is an anti-canonical divisor on  $V$ , it follows from adjunction formula that every line on  $V$  intersects  $C$  transversally in a unique point. The image of  $C$  by the contraction  $\tau : V \rightarrow \mathbb{P}^2$  of any 6-tuple of disjoint lines,  $L_1, \dots, L_6$ , on  $V$  is therefore a rational cuspidal cubic containing the images  $q_i = \tau(L_i)$ ,  $i = 1, \dots, 6$ , in its regular locus. The rational pencil  $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  generated by  $\tau(C)$  and three times its tangent line  $T$  at its singular point lifts to a rational pencil  $\bar{f} : V \dashrightarrow \mathbb{P}^1$  whose restriction to  $X$  is an  $\mathbb{A}^1$ -fibration  $f : X \rightarrow \mathbb{P}^1$  with two degenerate fibers: one irreducible of multiplicity three consisting of the intersection of the proper transform of  $T$  with  $X$ , and a reduced one consisting of the disjoint union of the curves  $L_i \cap X \simeq \mathbb{A}^1$ ,  $i = 1, \dots, 6$ . Choosing homogeneous coordinates  $[z_0 : z_1]$  on  $\mathbb{P}^1$  in such a way that 0 and  $\infty$  are the respective images of  $T$  and  $C$  by  $\bar{f}$ , the same argument as in [3, §4] implies that  $f : X \rightarrow \mathbb{P}^1$  factors through an étale locally trivial  $\mathbb{A}^1$ -bundle  $\rho : X \rightarrow B$ . Letting  $\gamma : L \rightarrow B$  be the line bundle under which  $\rho : X \rightarrow B$  becomes a principal homogeneous bundle, it follows from the relative cotangent exact sequence

$$0 \rightarrow \rho^* \Omega_B^1 \rightarrow \Omega_X^1 \rightarrow \Omega_{X/B}^1 \simeq \rho^* L^\vee \rightarrow 0$$

of  $\rho$  that  $\det \Omega_X^1 \simeq \rho^*(\Omega_B^1 \otimes L^\vee)$ . Since  $\det \Omega_X^1$  is trivial as  $C$  is an anti-canonical divisor on  $V$  and since  $\rho^* : \text{Pic}(B) \rightarrow \text{Pic}(X)$  is an isomorphism because  $\rho : X \rightarrow B$  is a locally trivial  $\mathbb{A}^1$ -bundle, we conclude that  $L \simeq \Omega_B^1$ .  $\square$

*Remark 6.* By construction of  $\delta : B \rightarrow \mathbb{P}^1$ , we have  $\delta^{-1}(0) = 3p_0$  and  $\delta^{-1}(\infty) = \sum_{i=1}^6 \infty_i$ . The Picard group  $\text{Pic}(B)$  of  $B$  is thus isomorphic to  $\mathbb{Z}^6$  generated by the classes of the lines bundle  $\mathcal{O}_B(p_0)$ ,  $\mathcal{O}_B(\infty_i)$ ,  $i = 1, \dots, 6$ , with the unique relation  $\mathcal{O}_B(3p_0) = \mathcal{O}_B(\sum_{i=1}^6 \infty_i)$ . Furthermore, since  $\delta$  is étale except at  $p_0$  where it has ramification index 3, we deduce from the ramification formula for the morphism  $\delta : B \rightarrow \mathbb{P}^1$  that the cotangent bundle  $\gamma : \Omega_B^1 \rightarrow B$  of  $B$  is isomorphic to

$$\delta^* \Omega_{\mathbb{P}^1} \otimes_{\mathcal{O}_B} \mathcal{O}_B(2p_0) \simeq \delta^*(\mathcal{O}_{\mathbb{P}^1}(-\{0\} - \{\infty\})) \otimes_{\mathcal{O}_B} \mathcal{O}_B(2p_0) \simeq \mathcal{O}_B(-p_0 - \sum_{i=1}^6 \infty_i).$$

1.2.3. Now we are ready for the second step of the construction, which completes the proof of the theorem. Letting  $X_i$ ,  $i = 1, 2$ , be the complements of irreducible plane cuspidal hyperplanes sections  $C_i = V_i \cap H_i$  of smooth cubic surfaces  $V_i \subset \mathbb{P}^3$ , Proposition 5 asserts the existence of principal homogeneous bundles  $\rho_i : X_i \rightarrow B$  under the action of the cotangent line bundle  $\gamma : \Omega_B^1 \rightarrow B$  of  $B$ . The fiber product  $W = X_1 \times_B X_2$  inherits via the first and second projections respectively the structure of a principal homogeneous bundle  $\pi_i : W \rightarrow X_i$  under  $\rho_i^* \Omega_B^1$ ,  $i = 1, 2$ . Since  $X_i$  is affine, the vanishing of  $H_{\text{ét}}^1(X_i, \rho_i^* \Omega_B^1)$  implies that these bundles are both trivial, yielding isomorphisms  $\rho_1^* \Omega_B^1 \simeq W \simeq \rho_2^* \Omega_B^1$ . Letting  $q : E \rightarrow W$  be the pull-back of the dual  $(\Omega_B^1)^\vee$  of  $\Omega_B^1$  by the morphism  $\rho_1 \circ \pi_1 = \rho_2 \circ \pi_2 : W \rightarrow B$ ,  $\pi_i \circ q : E \rightarrow X_i$  is a vector bundle over  $X_i$  isomorphic to the direct sum of  $\rho_i^* \Omega_B^1$  and

$\rho_i^*(\Omega_B^1)^\vee$ :

$$\begin{array}{ccccc}
X_1 \times \mathbb{A}^2 \simeq_{X_1} \rho_1^* \Omega_B^1 \oplus \rho_1^*(\Omega_B^1)^\vee & \xrightarrow{\sim} & E & \xleftarrow{\sim} & \rho_2^* \Omega_B^1 \oplus \rho_2^*(\Omega_B^1)^\vee \simeq_{X_2} X_1 \times \mathbb{A}^2 \\
\downarrow & & \downarrow q & & \downarrow \\
\rho_1^* \Omega_B^1 & \xrightarrow{\sim} & X_1 \times_B X_2 & \xleftarrow{\sim} & \rho_2^* \Omega_B^1 \\
\downarrow & \nearrow \pi_1 & & \searrow \pi_2 & \downarrow \\
X_1 & & & & X_2 \\
& \searrow \rho_1 & & \nearrow \rho_2 & \\
& & B & & 
\end{array}$$

So by virtue of [13, Theorem 3.1],  $E$  is isomorphic as a vector bundle over  $X_i$  to  $\det(\rho_i^* \Omega_B^1 \oplus \rho_i^*(\Omega_B^1)^\vee) \oplus \mathbb{A}_{X_i}^1 \simeq \mathbb{A}_{X_i}^1 \oplus \mathbb{A}_{X_i}^1$  providing the desired isomorphisms  $X_1 \times \mathbb{A}^2 \simeq E \simeq X_2 \times \mathbb{A}^2$ .

**Example 7.** Let  $V \subset \mathbb{P}^3$  be a general smooth cubic surface and let  $\Delta \subset V$  be the curve consisting of points  $p$  of  $V$  at which the projective tangent hyperplane  $T_p V \subset \mathbb{P}^3$  of  $V$  at  $p$  intersects  $V$  along a cuspidal cubic. Let  $\mathcal{V} = \Delta \times V$  and let  $\mathcal{C} \subset \mathcal{V}$  be relatively ample Cartier divisor with respect to  $\text{pr}_\Delta : \mathcal{V} \rightarrow \Delta$  whose fiber  $\mathcal{C}_p$  over every point  $p \in \Delta$  is equal to the intersection  $C_p = V \cap T_p V$ . Since  $\text{Aut}(V)$  is trivial, the pairs  $(V, C_p)$ ,  $p \in \Delta$ , are pairwise non isomorphic, and so  $\Theta = \text{pr}_\Delta|_{\mathcal{X}} : \mathcal{X} = \mathcal{V} \setminus \mathcal{C} \rightarrow \Delta$  is a family of pairwise non isomorphic rigid smooth affine surfaces whose  $\mathbb{A}^2$ -cylinders are all isomorphic.

*Remark 8.* The  $\mathbb{A}^2$ -cylinder  $X \times \mathbb{A}^2$  over the complement  $X$  of a cuspidal hyperplane section  $C$  of a smooth cubic surface  $V$  is *flexible in codimension 1*, that is, for every closed point  $p$  outside a possible empty closed subset  $Z \subset X \times \mathbb{A}^2$  of codimension at least two, the tangent space  $T_{X \times \mathbb{A}^2, p}$  of  $X \times \mathbb{A}^2$  at  $p$  is spanned by tangent vectors to orbits of algebraic  $\mathbb{G}_a$ -actions on  $X \times \mathbb{A}^2$ . This can be seen as follows: one first constructs by a similar procedure as in [3, §3.2] a flexible mate  $S$  for  $X$ , in the form of smooth affine surface flexible in codimension 1 admitting an  $\mathbb{A}^1$ -fibration  $\pi : S \rightarrow \mathbb{P}^1$  which factors through a principal homogeneous bundle  $\tilde{\pi} : S \rightarrow B$  under the action of a certain line bundle  $\gamma' : L \rightarrow B$ . The fiber product  $S \times_B X$  is then a smooth affine threefold which is simultaneously isomorphic to the total spaces of the line bundles  $\tilde{\pi}^* \Omega_B^1$  and  $\rho^* L$  over  $S$  and  $X$  via the first and second projection respectively. Since  $S$  is flexible in codimension 1, it follows from [3, Lemma 2.3] that  $S \times_B X$  and the total space  $F \rightarrow S \times_B X$  of the pull-back of  $L^\vee$  by the morphism  $\tilde{\pi} \circ \text{pr}_S = \rho \circ \text{pr}_X$  are both flexible in codimension 1. By construction,  $F$  is a vector bundle of rank 2 over  $X$ , isomorphic to  $\rho^*(L \oplus L^\vee)$  hence to the trivial vector bundle  $X \times \mathbb{A}^2$  by virtue of [13, Theorem 3.1].

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