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FLATNESS FOR A STRONGLY DEGENERATE 1-D PARABOLIC EQUATION

IVÁN MOYANO

ABSTRACT. We consider the degenerate equation

$$\partial_t f(t, x) - \partial_x (x^\alpha \partial_x f)(t, x) = 0,$$

on the unit interval $x \in (0, 1)$, in the strongly degenerate case $\alpha \in [1, 2)$ with adapted boundary conditions at $x = 0$ and boundary control at $x = 1$. We use the flatness approach to construct explicit controls in some Gevrey classes steering the solution from any initial datum $f_0 \in L^2(0, 1)$ to zero in any time $T > 0$.

Keywords—partial differential equations; degenerate parabolic equation; boundary control; null-controllability; motion planning; flatness.

1. INTRODUCTION

We consider the following control system

$$(1.1) \quad \begin{cases} \partial_t f(t, x) - \partial_x (x^\alpha \partial_x f)(t, x) = 0, & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x f)(t, x)|_{x=0} = 0, & t \in (0, T), \\ f(t, 1) = u(t), & t \in (0, T), \\ f(0, x) = f_0(x), & x \in (0, 1), \end{cases}$$

where the state is the solution $f(t, x)$ and the control is the function $u(t)$. The parameter $\alpha \in [1, 2)$ is fixed through the whole article.

The aim of this work is to construct explicit controls u for the null-controllability of system (1.1) in finite time $T > 0$, using the flatness method.

1.1. Main result. We will make use of the Gevrey class of functions.

DEFINITION 1.1. Let $s \in \mathbb{R}^+$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. A function $h \in \mathcal{C}^\infty([t_1, t_2])$ is said to be Gevrey of order s if

$$\exists M, R > 0 \text{ such that } \sup_{t_1 \leq r \leq t_2} |h^{(n)}(r)| \leq \frac{M(n!)^s}{R^n}.$$

We then write $h \in \mathcal{G}^s([t_1, t_2])$.

Before stating the main result, we have to recall the notion of weak solutions of the inhomogeneous system (1.1).

DEFINITION 1.2 (Weak solutions). Let $f_0 \in L^2(0, 1)$, $T > 0$ and $u \in H^1(0, T)$. A weak solution of system (1.1) is a function $f \in \mathcal{C}^0([0, T]; L^2(0, 1))$

such that for every $t' \in [0, T]$ and for every

$$(1.2) \quad \psi \in \mathcal{C}^1([0, t']; L^2(0, 1)) \cap \mathcal{C}^0([0, t']; H^2(0, 1))$$

such that

$$(1.3) \quad (x^\alpha \partial_x) \psi(t, x)|_{x=0} = \psi(t, 1) = 0, \quad \forall t \in [0, t'],$$

one has

$$\begin{aligned} & \int_0^{t'} \int_0^1 f(t, x) (\partial_t \psi + \partial_x (x^\alpha \partial_x \psi)) (t, x) dt dx \\ &= \int_0^1 f(t', x) \psi(t', x) dx - \int_0^1 f_0(x) \psi(0, x) dx + \int_0^{t'} u(t) \partial_x \psi(t, 1) dt. \end{aligned}$$

As we show in Section 2 (see Corollary 2.2), system (1.1) has a unique weak solution under suitable assumptions. Our main result is the following.

THEOREM 1.3. *Let $f_0 \in L^2(0, 1)$, $T > 0$, $\tau \in (0, T)$ and $s \in (1, 2)$. Then, there exists a flat output $y \in \mathcal{G}^s([\tau, T])$ such that the control*

$$(1.4) \quad u(t) = \begin{cases} 0, & \text{if } t \in [0, \tau], \\ \sum_{k=0}^{\infty} \frac{y^{(k)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k (j + \frac{\alpha-1}{2-\alpha})}, & \text{if } t \in (\tau, T], \end{cases}$$

steers to zero at time T the weak solution of system (1.1). Furthermore, the control u belongs to $\mathcal{G}^s([0, T])$.

1.2. Previous work.

1.2.1. *Null-controllability.* The null-controllability of system

$$\begin{cases} \partial_t f(t, x) - \partial_x (x^\alpha \partial_x) f(t, x) = 1_\omega(x) v(t, x), & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x) f(t, x)|_{x=0} = 0, & t \in (0, T), \\ f(t, 1) = 0, & t \in (0, T), \\ f(0, x) = f_0(x), & x \in (0, 1), \end{cases}$$

where $\omega \subset (0, 1)$, has been studied by P. Cannarsa, P. Martinez and J. Van-costenoble in [8]. Their strategy relies on appropriate Carleman estimates. To deal with the degeneracy at $\{x = 0\}$, they use an adequate functional framework that we recall in Section 2, and Hardy-type inequalities.

The null-controllability of system (1.1) is a consequence of the internal null-controllability and the extension principle, since the control is located on $\{x = 1\}$, away from the degeneracy. The interest of the present article is to provide explicit controls.

In the case of a control located on $\{x = 0\}$, an approximate controllability result for $\alpha \in [0, 1)$ has been proven by P. Cannarsa, J. Tort and M. Yamamoto in [10] using Carleman estimates. The exact controllability was later proven by M. Gueye in [13] again in the weakly degenerate case $\alpha \in [0, 1)$ by using the transmutation method.

Other related one-dimensional problems have been treated: see [6, 7, 2], see [5] for a non-divergence setting, see [20] for a system with a singular potential. A multi-dimensional case has been studied in [9].

1.2.2. The flatness method. The main interest of the flatness method is to provide explicit controls. It has been developed for finite-dimensional systems (see [12]) and then generalised to some infinite-dimensional systems; see [17] for the heat equation on a cylindrical domain with boundary control, [18] for one-dimensional parabolic equations with varying coefficients and [19] for the one-dimensional Schrödinger equation. However, the strongly degenerate case $\alpha \in [1, 2)$ considered in Theorem 1.3 does not belong to the class concerned in [18]. Our goal is to adapt the flatness method to this case.

1.3. Open questions and perspectives. The flatness method may also be successful on similar equations, for instance in non-divergence form as in [5]. For the time being, this is an open problem.

1.4. Structure of the article. In Section 2 we recall a well-posedness result and the functional framework. In Section 3 we derive, thanks to an heuristic method, an explicit solution of system (1.1) consisting on a formal series development. We prove its convergence, provided that the corresponding flat output is in a Gevrey class. In Section 4 we discuss the spectral analysis of the associated stationary problem. In Section 5 we study the regularising effect of system (1.1) when $u = 0$. In Section 6 we construct an appropriate flat output steering the solution of (1.1) to zero, which concludes the proof of Theorem 1.3. Finally, we give in Appendices A and B a brief account of some results concerning the Gamma and Bessel functions needed in the proofs.

1.5. Notation. Since all the functions appearing in the article are real-valued, we omit any explicit mention by writing, for instance, $L^2(0, 1)$ instead of $L^2((0, 1); \mathbb{R})$. If $h \in \mathcal{C}^k([t_1, t_2])$, for some $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $k \in \mathbb{N}^*$, we will denote by $h'(t)$ and $h''(t)$ its first and second derivatives and by $h^{(n)}(t)$, for every $n \in \mathbb{N}$, $2 < n \leq k$, the n -th derivative.

If $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$ are two real-valued functions and $\mu \in \overline{\mathbb{R}}$, we will write $h_1 \sim h_2$ as $x \rightarrow \mu$ to denote that $\lim_{t \rightarrow \mu} \frac{h_1(t)}{h_2(t)} = 1$.

We will denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(0, 1)$.

2. WELL-POSEDNESS

We consider, for $T > 0$ and $f_0 \in L^2(0, 1)$, the following system

$$(2.5) \quad \begin{cases} \partial_t f(t, x) - \partial_x (x^\alpha \partial_x) f(t, x) = h(t, x), & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x) f(t, x)|_{x=0} = 0, & t \in (0, T), \\ f(t, 1) = 0, & t \in (0, T), \\ f(0, x) = f_0(x), & x \in (0, 1). \end{cases}$$

We recall below a well-posedness result for system (2.5) proven originally in [7]. The strategy of the proof consists in a semigroup approach and the introduction of adequate weighted Sobolev spaces, that we recall below. We refer to [7, 4] for further details.

We introduce the weighted Sobolev space

$$H_\alpha^1(0, 1) := \left\{ f \in L^2(0, 1); f \text{ is loc. absolutely continuous on } (0, 1], \right. \\ \left. x^{\frac{\alpha}{2}} f' \in L^2(0, 1) \text{ and } f(1) = 0 \right\},$$

endowed with the norm

$$\|f\|_{H_\alpha^1(0,1)}^2 := \|f\|_{L^2(0,1)}^2 + \|x^{\frac{\alpha}{2}} f'\|_{L^2(0,1)}^2, \quad \forall f \in H_\alpha^1(0, 1).$$

We remark that $H_\alpha^1(0, 1)$ is a Hilbert space with the scalar product

$$(2.6) \quad \langle f, g \rangle_{H_\alpha^1} := \int_0^1 f(x)g(x) dx + \int_0^1 x^\alpha f'(x)g'(x) dx, \quad \forall f, g \in H_\alpha^1(0, 1).$$

PROPOSITION 2.1 ([7], Proposition 3.2 and Theorem 3.1). *Let*

$$(2.7) \quad \begin{cases} D(A) := \{ f \in H_\alpha^1(0, 1); x^\alpha f' \in H^1(0, 1) \}, \\ Af := -(x^\alpha f')'. \end{cases}$$

Then, $A : D(A) \rightarrow L^2(0, 1)$ is a closed self-adjoint positive operator with dense domain. As a consequence, A is the infinitesimal generator of a strongly continuous semigroup, and for any $f_0 \in L^2(0, 1)$, and $h \in L^2((0, T) \times (0, 1))$ there exists a unique weak solution of system (2.5), i.e., a function $f \in \mathcal{C}^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H_\alpha^1(0, 1))$ such that

$$f(t) = S(t)f_0 + \int_0^t S(t-s)h(s) ds, \quad \text{in } L^2(0, 1), \quad \forall t \in [0, T].$$

As a consequence, using classical arguments (see for instance [11, Section 2.5.3]), we deduce the following result.

COROLLARY 2.2. *Let $T > 0$, $f_0 \in L^2(0, 1)$ and $u \in H^1(0, T)$. Then, system (1.1) has a unique weak solution (see Definition 1.2).*

Proof. Let $f_0 \in L^2(0, 1)$, $u \in H^1(0, T)$ and

$$\theta(x) := x^2, \quad x \in [0, 1].$$

We consider the system

$$\begin{cases} (\partial_t - \partial_x(x^\alpha \partial_x))g(t, x) = H(t, x), & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x)g(t, x)|_{x=0} = 0, & t \in (0, T), \\ g(t, 1) = 0, & t \in (0, T), \\ g(0, x) = f_0(x) - u(0)\theta(x), & x \in (0, 1), \end{cases}$$

with

$$H(t, x) := -u'(t)\theta(x) - u(t)A\theta(x), \quad \forall (t, x) \in (0, T) \times (0, 1).$$

Since $H \in L^2((0, T) \times (0, 1))$, by Proposition 2.1 there exists a unique weak solution $g \in \mathcal{C}^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H_\alpha^1(0, 1))$ of this problem. We set

$$f(t, x) := g(t, x) + u(t)\theta(x).$$

Then, using the integral formulation associated to g , one shows that f is a weak solution of system (1.1) in the sense of Definition 1.2.

The uniqueness follows since, if f_1 and f_2 are weak solutions of (1.1), then $f_1 - f_2$ is the unique weak solution of system (2.5) with $h \equiv 0$, and then by Proposition 2.1, $f_1 - f_2 = 0$. \square

3. EXPLICIT SOLUTION

3.1. Heuristics. We consider the following formal expansion

$$f(t, x) = \sum_{k=0}^{\infty} c_{2k}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}, \quad \forall (t, x) \in (0, T) \times (0, 1).$$

where $(c_{2k}(t))_{k \in \mathbb{N}}$ is a sequence of real numbers. We formally have

$$\begin{aligned} \partial_x (x^\alpha \partial_x f)(t, x) &= \sum_{k=0}^{\infty} c_{2(k+1)}(t) (2-\alpha)^2 (k+1) \left[k+1 + \frac{\alpha-1}{2-\alpha}\right] \left(x^{1-\frac{\alpha}{2}}\right)^{2k}, \\ \partial_t f(t, x) &= \sum_{k=0}^{\infty} c'_{2k}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}. \end{aligned}$$

If f solves (1.1), then the following recurrence relation holds

$$c_{2(k+1)}(t) = \frac{c'_{2k}(t)}{(2-\alpha)^2 (k+1) \left(k+1 + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall k \in \mathbb{N}.$$

Choosing a flat output $c_0(t) := y(t)$ and iterating, we readily have

$$c_{2k}(t) = \frac{y^{(k)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall t \in (0, T), \forall k \in \mathbb{N}.$$

This gives a formal solution of (1.1),

$$(3.8) \quad f(t, x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)},$$

and a control given by $u(t) = f(t, 1)$, which is

$$(3.9) \quad u(t) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)}.$$

3.2. Pointwise solutions. The goal of this section is to introduce a notion of pointwise solution of system (1.1) to give a sense to the heuristics made in the previous section.

We define

$$\mathcal{C}_\alpha^2(0, 1) := \{f \in \mathcal{C}^0([0, 1]) \cap \mathcal{C}^2((0, 1)) \text{ such that } x^\alpha f'(x) \in \mathcal{C}^0([0, 1])\}.$$

DEFINITION 3.1 (Pointwise solution). *Let $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Let $f_{t_1} \in \mathcal{C}^0(0, 1)$ and $u \in \mathcal{C}^0([t_1, t_2])$. A pointwise solution of system*

$$(3.10) \quad \begin{cases} \partial_t f(t, x) - \partial_x(x^\alpha \partial_x f)(t, x) = 0, & (t, x) \in (t_1, t_2) \times (0, 1), \\ x^\alpha \partial_x f(t, x)|_{x=0} = 0, & t \in (t_1, t_2), \\ f(t, 1) = u(t), & t \in (t_1, t_2), \\ f(t_1, x) = f_{t_1}(x), & x \in (0, 1), \end{cases}$$

is a function $f \in \mathcal{C}^0([t_1, t_2] \times [0, 1]) \cap \mathcal{C}^1((t_1, t_2) \times (0, 1))$ such that

- (1) $f(t, \cdot) \in \mathcal{C}_\alpha^2(0, 1)$, $\forall t \in (t_1, t_2)$,
- (2) $\partial_t f - \partial_x(x^\alpha \partial_x f) = 0$ pointwisely in $(t_1, t_2) \times (0, 1)$,
- (3) $\lim_{x \rightarrow 0^+} x^\alpha \partial_x f(t, x) = 0$, $\forall t \in (t_1, t_2)$,
- (4) $f(t, 1) = u(t)$, $\forall t \in (t_1, t_2)$,
- (5) $f(t_1, x) = f_{t_1}(x)$, $\forall x \in (0, 1)$.

REMARK 3.2. *The usual energy argument proves that, given $u \in \mathcal{C}^0([t_1, t_2])$, the pointwise solution of system (3.10) is unique. We observe that, changing parameters adequately in Definition 1.2 a pointwise solution of (3.10) is also a weak solution.*

3.3. Convergence. The goal of this section is the proof of the following result.

PROPOSITION 3.3. *Let $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$. If $y \in \mathcal{G}^s([t_1, t_2])$ for some $s \in (0, 2)$, then*

- (1) *the control u given by (3.9) is well defined and belongs to $\mathcal{G}^s([t_1, t_2])$,*
- (2) *the function given by (3.8) is a pointwise solution (see Definition 3.1) of system (3.10) in $(t_1, t_2) \times (0, 1)$ with u given by (3.9) and initial datum*

$$f_{t_1}(x) := \sum_{k=1}^{\infty} \frac{y^{(k)}(t_1) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall x \in [0, 1].$$

Proof. Let $M, R > 0$ be such that $|y^{(n)}(t)| \leq \frac{Mn!^s}{R^n}$, for any $n \in \mathbb{N}$, $t \in [t_1, t_2]$.

Step 1: We prove that u is well defined and belongs to $\mathcal{C}^\infty([t_1, t_2])$.

For any $t \in [t_1, t_2]$, $k \in \mathbb{N}^*$, we have, as $\frac{\alpha-1}{2-\alpha} \geq 0$,

$$\frac{|y^{(k)}(t)|}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)} \leq \frac{Mk!^s}{R^k (2-\alpha)^{2k} k!^2} = \frac{M}{R^k (2-\alpha)^{2k} k!^{2-s}}.$$

Hence, the series in (3.9) converges uniformly w.r.t. $t \in [t_1, t_2]$ and $u \in \mathcal{C}^0([t_1, t_2])$. Furthermore, for any $n \in \mathbb{N}^*$, the function $\xi_{n,k}(t) := \frac{y^{(k+n)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k (j + \frac{\alpha-1}{2-\alpha})}$ satisfies

$$|\xi_{n,k}(t)| \leq \frac{M(k+n)!^s}{R^{n+k}(2-\alpha)^{2k} k!^2}, \quad \forall t \in [t_1, t_2], k, n \in \mathbb{N}.$$

Thus, $\sum_k \xi_{n,k}(t)$ converges uniformly w.r.t. $t \in [t_1, t_2]$. Whence, $u \in \mathcal{C}^\infty([t_1, t_2])$ and for every $n \in \mathbb{N}$, $t \in [t_1, t_2]$, $u^{(n)}(t) = \sum_{k=0}^\infty \xi_{n,k}(t)$.

Step 2: We prove that u is Gevrey of order s .

Let $n \in \mathbb{N}$. We deduce from last inequality that

$$\begin{aligned} |u^{(n)}(t)| &\leq \sum_{k=0}^\infty \frac{M(k+n)!^s}{R^{n+k}(2-\alpha)^{2k} k!^2} \\ (3.11) \quad &\leq M \left[\sum_{k=0}^\infty \frac{1}{(k!)^{2-s}} \left(\frac{2^s}{R(2-\alpha)^2} \right)^k \right] \left(\frac{2^s}{R} \right)^n n!^s, \end{aligned}$$

where we have used (A.41). The D'Alembert criterium for entire series shows that, whenever $s \in (0, 2)$, the series above converges, which shows that $u \in \mathcal{G}^s([t_1, t_2])$.

Step 3: We show that the function f given by (3.8) is well defined and $f \in \mathcal{C}^0([t_1, t_2] \times [0, 1]) \cap \mathcal{C}^1((t_1, t_2) \times (0, 1))$.

Let, for every $k \in \mathbb{N}$,

$$f_k(t, x) := \frac{y^{(k)}(t) \left(x^{1-\frac{\alpha}{2}} \right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha} \right)}, \quad \forall (t, x) \in [t_1, t_2] \times [0, 1].$$

Then,

$$|f_k(t, x)| \leq \frac{M}{k!^{2-s}} \left(\frac{1}{R(2-\alpha)} \right)^k, \quad \forall (t, x) \in [t_1, t_2] \times [0, 1].$$

This proves that $\sum_k f_k$ converges uniformly w.r.t. $(t, x) \in [t_1, t_2] \times [0, 1]$. Thus, $f \in \mathcal{C}^0([t_1, t_2] \times [0, 1])$.

We observe that $\exists k_0 = k_0(\alpha) \in \mathbb{N}^*$ such that $(2-\alpha)k_0 \geq 1$. Then, for every $k \geq k_0$, $f_k(t, \cdot) \in \mathcal{C}^1([0, 1])$ and

$$\begin{aligned} |\partial_x f_k(t, x)| &= \left| \frac{y^{(k)}(t) 2k \left(1 - \frac{\alpha}{2} \right) x^{-\frac{\alpha}{2}} \left(x^{1-\frac{\alpha}{2}} \right)^{2k-1}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha} \right)} \right| \\ &\leq 2M \left(1 - \frac{\alpha}{2} \right) \frac{k}{k!^{2-s}} \left(\frac{1}{R(2-\alpha)^2} \right)^k, \quad \forall x \in [0, 1], \end{aligned}$$

since $(1 - \frac{\alpha}{2})(2k-1) - \frac{\alpha}{2} \geq 0$. This proves that $\sum_{k \geq k_0} \partial_x f_k$ converges uniformly w.r.t. $(t, x) \in [t_1, t_2] \times [0, 1]$. Thus, $f(t, \cdot) \in$

$\mathcal{C}^1((0, 1])$ for every $t \in [t_1, t_2]$. Note that f may not be differentiable w.r.t. x at $x = 0$ because of the finite number of terms $\sum_{k=0}^{k_0} \partial_x f_k$. Moreover, $\partial_x f(t, x) = \sum_{k=0}^{\infty} \partial_x f_k(t, x)$ for every $(t, x) \in (t_1, t_2) \times (0, 1)$.

A similar argument shows that, for every $x \in (0, 1)$, $f(\cdot, x) \in \mathcal{C}^1(t_1, t_2)$ and

$$(3.12) \quad \partial_t f(t, x) = \sum_{k=0}^{\infty} \partial_t f_k(t, x), \quad \forall (t, x) \in (t_1, t_2) \times (0, 1).$$

Finally, since the partial derivatives of f exist and are continuous in $(t_1, t_2) \times (0, 1)$, $f \in \mathcal{C}^1((t_1, t_2) \times (0, 1))$.

Step 4: We show that $f(t, \cdot) \in \mathcal{C}_{\alpha}^2(0, 1)$, for every $t \in (t_1, t_2)$.

Let $k_1 = k_1(\alpha) \in \mathbb{N}^*$ such that $k_1(2 - \alpha) \geq 2$. Working as in Step 3, we see that $\sum_{k \geq k_1} \partial_x^2 f_k$ converges uniformly w.r.t. $(t, x) \in (t_1, t_2) \times (0, 1)$. Thus, $f(t, \cdot) \in \mathcal{C}^2(0, 1)$, $\forall t \in (t_1, t_2)$. Furthermore,

$$(3.13) \quad \partial_x (x^{\alpha} \partial_x f)(t, x) = \sum_{k=1}^{\infty} \frac{y^{(k)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2(k-1)}}{(2-\alpha)^{2(k-1)}(k-1)! \prod_{j=1}^{k-1} \left(j + \frac{\alpha-1}{2-\alpha}\right)}.$$

for every $(t, x) \in (t_1, t_2) \times (0, 1)$. From Step 3, we obtain

$$\begin{aligned} |x^{\alpha} \partial_x f(t, x)| &= \left| \sum_{k=1}^{\infty} \frac{y^{(k)}(t) 2k \left(1 - \frac{\alpha}{2}\right) x^{2k(1-\frac{\alpha}{2})+\alpha-1}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)} \right| \\ &\leq 2M \left(1 - \frac{\alpha}{2}\right) \sum_{k=1}^{\infty} \left[\frac{k}{k!^{2-s}} \left(\frac{1}{R(2-\alpha)^2}\right)^k \right] x, \end{aligned}$$

for all $(t, x) \in (t_1, t_2) \times (0, 1)$, which implies, since $\alpha \in [1, 2)$, that

$$x^{\alpha} \partial_x f(t, x) \rightarrow 0, \quad \text{as } x \rightarrow 0^+.$$

Therefore, $f(t, \cdot) \in \mathcal{C}_{\alpha}^2$, for every $t \in (t_1, t_2)$.

Step 5: According to (3.12) and (3.13), a straightforward computation shows that the equation in (3.10) is satisfied. \square

4. SPECTRAL ANALYSIS

The goal of this section is to give the explicit expression of the eigenfunctions and eigenvalues of the spectral problem

$$(4.14) \quad \begin{cases} A\varphi(x) = \lambda\varphi(x), & x \in (0, 1), \\ (x^{\alpha}\varphi')|_{x=0} = \varphi(1) = 0, \end{cases}$$

where A is given by (2.7). We will make use of several results about Bessel functions recalled in Appendix B. From now on, we use the notation

$$(4.15) \quad \nu := \frac{\alpha - 1}{2 - \alpha}.$$

PROPOSITION 4.1. *Let*

$$(4.16) \quad \varphi_k(x) = \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}), \quad \forall x \in (0, 1), k \in \mathbb{N}^*.$$

Then,

- (1) $\varphi_k \in D(A)$, $\forall k \in \mathbb{N}^*$,
- (2) φ_k satisfies (4.14) with

$$(4.17) \quad \lambda_k := \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2, \quad \forall k \in \mathbb{N}^*,$$

- (3) $(\varphi_k)_{k \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(0, 1)$,
- (4) for every $f_0 \in L^2(0, 1)$ the solution of (2.5) with $h = 0$ writes

$$(4.18) \quad f(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle f_0, \varphi_k \rangle \varphi_k \quad \text{in } L^2(0, 1), \forall t \in [0, T].$$

Proof. We will note for simplicity $b_k := \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|}$ and $\tilde{\varphi}_k := \frac{1}{b_k} \varphi_k$, for every $k \in \mathbb{N}^*$.

Step 1: We prove that $\varphi_k \in D(A)$, for every $k \in \mathbb{N}^*$ and that $A\varphi_k - \lambda_k \varphi_k = 0$.

Let $k \in \mathbb{N}^*$. We observe that $\varphi_k \in \mathcal{C}^\infty((0, 1]) \cap \mathcal{C}^0([0, 1])$, for any $k \in \mathbb{N}^*$ and $x \in (0, 1)$. We have

$$(4.19) \quad \tilde{\varphi}'_k(x) = \frac{1-\alpha}{2} x^{-\frac{1+\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) + j_{\nu,k} \left(1 - \frac{\alpha}{2}\right) x^{\frac{1}{2}-\alpha} J'_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}).$$

Whence, using (B.48) and Lemma B.3, we deduce

$$x^{\frac{\alpha}{2}} \tilde{\varphi}'_k = (1-\alpha) O_{x \rightarrow 0^+} \left(x^{\frac{\alpha}{2}-1}\right) + O_{x \rightarrow 0^+} \left(x^{1-\frac{\alpha}{2}}\right).$$

It follows that $x^{\frac{\alpha}{2}} \varphi'_n \in L^2(0, 1)$. Thus $\varphi_k \in H_\alpha^1(0, 1)$. Moreover, from (4.19), a direct computation shows

$$(4.20) \quad \begin{aligned} (x^\alpha \tilde{\varphi}'_k)' &= - \left(\frac{1-\alpha}{2}\right)^2 x^{\frac{\alpha-3}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k} x^{-\frac{1}{2}} J'_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J''_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}). \end{aligned}$$

Then, evaluating equation (B.47) at $z = j_{\nu,k} x^{1-\frac{\alpha}{2}}$ and multiplying by $x^{\frac{\alpha-3}{2}}$, it follows

$$\begin{aligned} &j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J''_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &= -j_{\nu,k} x^{-\frac{1}{2}} J'_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) - j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(\frac{\alpha-1}{2-\alpha}\right)^2 x^{\frac{\alpha-3}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}). \end{aligned}$$

Substituting in (4.20), this gives

$$-(x^\alpha \tilde{\varphi}'_k)' = \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) = \lambda_k \tilde{\varphi}_k.$$

Then, we readily have $(x^\alpha \tilde{\varphi}'_k)' \in H_\alpha^1(0,1) \subset L^2(0,1)$. Thus, $\varphi_k \in D(A)$. Moreover, $A\varphi_k = \lambda_k \varphi_k$.

Step 2: We check the boundary condition of (4.14) at $x = 0$.

We observe first that the case $\alpha = 1$ is straightforward. From (4.19), (B.48) and Lemma B.3, we have

$$|x^\alpha \tilde{\varphi}'_n(x)| = O_{x \rightarrow 0^+}(x^{\alpha-1}).$$

Then, it follows that $\lim_{x \rightarrow 0^+} x^\alpha \tilde{\varphi}'_n(x) = 0$. This shows, combined with Step 1, that φ_k satisfies (4.14).

Step 3: We prove that $(\varphi_k)_{k \in \mathbb{N}^*}$ is an orthonormal family in $L^2(0,1)$.

Let $n, m \in \mathbb{N}^*$. Then, changing variables and using (B.46), we get

$$\begin{aligned} & \int_0^1 \varphi_n(x) \varphi_m(x) dx \\ &= (2-\alpha) \int_0^1 x^{1-\alpha} \frac{J_\nu(j_{\nu,n} x^{1-\frac{\alpha}{2}})}{|J_{\nu+1}(j_{\nu,n})|} \frac{J_\nu(j_{\nu,m} x^{1-\frac{\alpha}{2}})}{|J_{\nu+1}(j_{\nu,m})|} dx \\ &= \frac{2}{|J_{\nu+1}(j_{\nu,n})| |J_{\nu+1}(j_{\nu,m})|} \int_0^1 y J_\nu(j_{\nu,n} y) J_\nu(j_{\nu,m} y) dy = \delta_{n,m}, \end{aligned}$$

where $\delta_{n,m}$ stands for the Kronecker delta.

Step 4: We prove that $(\varphi_k)_{k \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(0,1)$ by checking the Bessel equality. Let $f \in L^2(0,1)$ and let

$$(4.21) \quad a_k := \int_0^1 f(x) \varphi_k(x) dx, \quad \forall k \in \mathbb{N}^*.$$

Then, using Lemma B.1 and changing variables twice, we get

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k|^2 &= \sum_{k=1}^{\infty} \left| \int_0^1 f(x) \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) dx \right|^2 \\ &= \frac{2}{2-\alpha} \sum_{k=1}^{\infty} \left| \int_0^1 y^{\frac{\alpha-1}{2-\alpha} + \frac{1}{2}} f(y^{\frac{2}{2-\alpha}}) \frac{\sqrt{2y}}{|J_{\nu+1}(j_{\nu,k})|} J_\nu(j_{\nu,k} y) dy \right|^2 \\ &= \frac{2}{2-\alpha} \int_0^1 y^{\frac{2(\alpha-1)}{2-\alpha} + 1} \left| f(y^{\frac{2}{2-\alpha}}) \right|^2 dy \\ &= \int_0^1 |f(z)|^2 dz = \|f\|_{L^2(0,1)}^2. \end{aligned}$$

Step 5: Finally, (4.18) is a consequence of [3, Theorem 8.2.3, pp.237–240].

□

5. REGULARISING EFFECT

We use the orthonormal basis obtained in Proposition 4.1 and some properties of Bessel functions to quantify the smoothing of the solution of system (1.1) when $u \equiv 0$.

PROPOSITION 5.1. *Let $f_0 \in L^2(0, 1)$, $T > 0$ and let $f \in \mathcal{C}^0([0, T]; L^2(0, 1))$ be the unique weak solution of system (2.5) when $h = 0$, according to Proposition 2.1. Then, there exists $Y \in \mathcal{C}^\infty((0, T])$ such that for every $\sigma \in (0, T)$,*

$$Y \in \mathcal{G}^1([\sigma, T])$$

and

$$(5.22) \quad f(t, x) = \sum_{n=0}^{\infty} \frac{Y^{(n)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{(2-\alpha)^{2n} n! \prod_{j=1}^n \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall (t, x) \in [\sigma, T] \times [0, 1].$$

Moreover, f solves system (3.10) pointwisely (see Definition 3.1) in $(\sigma, T) \times (0, 1)$ with $u = 0$ and initial datum $f_\sigma(x) = f(\sigma, x)$.

Proof. Let ν be given by (4.15) and a_k as in (4.21). Let $\sigma \in (0, T)$ be fixed but arbitrary. Let $t \in [\sigma, T]$ be fixed. By (4.18) and (B.43), we have, for a.e. $x \in [0, 1]$,

$$\begin{aligned} f(t, x) &= \sum_{k=1}^{\infty} e^{-\lambda_k t} \frac{a_k \sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} J_\nu \left(j_{\nu,k} x^{1-\frac{\alpha}{2}}\right) \\ &= \sum_{k=1}^{\infty} e^{-\lambda_k t} \frac{a_k \sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+1+\nu)} \left(\frac{j_{\nu,k} x^{1-\frac{\alpha}{2}}}{2}\right)^{2n+\nu} \\ (5.23) \quad &= \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} B_{n,k}(t, x), \end{aligned}$$

where, for every $(n, k) \in \mathbb{N} \times \mathbb{N}^*$,

$$B_{n,k}(t, x) := e^{-\lambda_k t} b_k \frac{(-1)^n j_{\nu,k}^{2n+\nu}}{n! \Gamma(n+1+\nu) 2^{2n+\nu}} \frac{\left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{|J_{\nu+1}(j_{\nu,k})|},$$

and $b_k := a_k \sqrt{2-\alpha}$, $\forall k \in \mathbb{N}^*$.

Step 1: We show that

$$(5.24) \quad \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} |B_{n,k}(t, x)| \right) < \infty, \quad \forall x \in [0, 1].$$

Indeed, since $\lambda_k > 0$, we have for every $(n, k) \in \mathbb{N} \times \mathbb{N}^*$ and $x \in [0, 1]$,

$$|B_{n,k}(t, x)| \leq \frac{|b_k| j_{\nu,k}^{2n+\nu} e^{-\lambda_k \sigma}}{2^{2n+\nu} n! \Gamma(n+1+\nu) |J_{\nu+1}(j_{\nu,k})|}$$

$$(5.25) \quad \leq \frac{C_1 |b_k| e^{-\lambda_k \sigma} j_{\nu,k}^{2n+\nu+\frac{1}{2}}}{2^{2n} n! \Gamma(n+1+\nu)},$$

for a constant $C_1 > 0$, using Lemma B.4.

We fix $n \in \mathbb{N}$ and we define the function $h_n^\alpha \in \mathcal{C}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ by

$$h_n^\alpha(x) := e^{-(1-\frac{\alpha}{2})^2 x^2 \sigma} x^{2n+\nu+\frac{1}{2}}, \quad \forall x \in [0, +\infty),$$

which satisfies that

$$(5.26) \quad \frac{d}{dx} h_n^\alpha(x) > 0, \quad \forall x \in (0, N_n^\alpha) \quad \text{and} \quad \frac{d}{dx} h_n^\alpha(x) < 0, \quad \forall x \in (N_n^\alpha, \infty),$$

where $N_n^\alpha := \frac{2}{2-\alpha} \sqrt{\frac{1}{\sigma} \left(n + \frac{\alpha}{4(2-\alpha)} \right)}$. Hence, from (5.25) and (4.17),

$$(5.27) \quad \sum_{k=1}^{\infty} |B_{n,k}(t, x)| \leq \frac{C_1 \sup_k |b_k|}{2^{2n} n! \Gamma(n+1+\nu)} \sum_{k=1}^{\infty} h_n^\alpha(j_{\nu,k})$$

Introducing $K_n^\alpha := \sup \{k \in \mathbb{N}^*; j_{\nu,k} \leq N_n^\alpha\}$, we write

$$(5.28) \quad \sum_{k=1}^{\infty} h_n^\alpha(j_{\nu,k}) = h_n^\alpha(j_{\nu, K_n^\alpha}) + h_n^\alpha(j_{\nu, K_n^\alpha+1}) + \sum_{k \in \mathbb{N}^* - \{K_n^\alpha, K_n^\alpha+1\}} h_n^\alpha(j_{\nu,k})$$

On one hand, we have

$$(5.29) \quad \begin{aligned} h_n^\alpha(j_{\nu, K_n^\alpha}) + h_n^\alpha(j_{\nu, K_n^\alpha+1}) &\leq 2h_n^\alpha(N_n^\alpha) \\ &\leq 2e^{-(n+\frac{\alpha}{4(2-\alpha)})} \left(n + \frac{\alpha}{4(2-\alpha)} \right)^{n+\frac{\alpha}{4(2-\alpha)}} \left[\frac{1}{\sigma} \left(\frac{2}{2-\alpha} \right)^2 \right]^{n+\frac{\alpha}{4(2-\alpha)}} \\ &\leq C_2 \Gamma \left(n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2} \right) \left[\frac{1}{\sigma} \left(\frac{2}{2-\alpha} \right)^2 \right]^{n+\frac{\alpha}{4(2-\alpha)}}, \end{aligned}$$

for a constant $C_2 > 0$, using Lemma A.1 with $a = 1, b = \frac{1}{2}$. On the other hand, using (5.26), we write

$$\begin{aligned} &\sum_{k \in \mathbb{N}^* - \{K_n^\alpha, K_n^\alpha+1\}} h_n^\alpha(j_{\nu,k}) \leq \\ &\leq \sum_{k=1}^{K_n^\alpha-1} \frac{1}{j_{\nu,k+1} - j_{\nu,k}} \int_{j_{\nu,k}}^{j_{\nu,k+1}} h_n^\alpha(x) dx + \sum_{k=K_n^\alpha+1}^{\infty} \frac{1}{j_{\nu,k} - j_{\nu,k-1}} \int_{j_{\nu,k-1}}^{j_{\nu,k}} h_n^\alpha(x) dx \\ &\leq \sup_{k \in \mathbb{N}^*} \left\{ \frac{1}{j_{\nu,k+1} - j_{\nu,k}} \right\} \left(\int_{j_{\nu,1}}^{j_{\nu, K_n^\alpha}} h_n^\alpha(x) dx + \int_{j_{\nu, K_n^\alpha+1}}^{\infty} h_n^\alpha(x) dx \right) \\ &\leq C_3 \int_0^{\infty} h_n^\alpha(x) dx, \end{aligned}$$

for a constant $C_3 > 0$, using (B.45). Moreover, we have

$$\begin{aligned}
 \int_0^\infty h_n^\alpha(x) dx &= \int_0^\infty e^{-(1-\frac{\alpha}{2})^2 x^2 \sigma} x^{2n+\frac{\alpha}{2(2-\alpha)}} dx \\
 &= \int_0^\infty e^{-t} \left(\frac{2}{2-\alpha} \sqrt{\frac{t}{\sigma}} \right)^{2n+\frac{\alpha}{2(2-\alpha)}} \frac{1}{2\sqrt{\sigma t}} \left(\frac{2}{2-\alpha} \right) dt \\
 &= \frac{1}{2} \left[\frac{1}{\sqrt{\sigma}} \left(\frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}+1} \int_0^\infty e^{-t} t^{n+\frac{\alpha}{4(2-\alpha)}-\frac{1}{2}} dt \\
 &= \frac{1}{2} \left[\frac{1}{\sqrt{\sigma}} \left(\frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}+1} \Gamma \left(n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2} \right),
 \end{aligned}$$

where we have used (A.38) with $p = n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2}$. Hence, combining this with (5.28) and (5.29), we get

$$\sum_{k=1}^\infty h_n^\alpha(j_{\nu,k}) \leq \left(C_2 + \frac{C_3}{\sqrt{\sigma}(2-\alpha)} \right) \left[\frac{1}{\sqrt{\sigma}} \left(\frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}} \Gamma \left(n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2} \right),$$

which, according to (5.27), implies

$$\sum_{k=1}^\infty |B_{n,k}(t, x)| \leq C_4 \left[\frac{1}{\sqrt{\sigma}} \left(\frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}} \frac{\Gamma \left(n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2} \right)}{2^{2n} n! \Gamma(n + \nu + 1)}.$$

Henceforth, the D'Alembert criterium for entire series gives (5.24).

Step 2: We find $Y \in \mathcal{G}^1([\sigma, T])$ such that (5.22) holds.

Thanks to Fubini's theorem, (5.23) and (A.39), we may write

$$f(t, x) = \sum_{n=0}^\infty \frac{y_n(t) \left(x^{1-\frac{\alpha}{2}} \right)^{2n}}{(2-\alpha)^{2n} n! \prod_{j=1}^n (j + \nu)},$$

where, for every $n \in \mathbb{N}$,

$$y_n(t) := \frac{(-1)^n \sqrt{2-\alpha} \left(1 - \frac{\alpha}{2} \right)^{2n}}{2^\nu \Gamma \left(\frac{1}{2-\alpha} \right)} \sum_{k=1}^\infty a_k e^{-\lambda_k t} \frac{j_{\nu,k}^{2n+\nu}}{|J_{\nu+1}(j_{\nu,k})|}, \quad \forall t \in [\sigma, T],$$

and ν is given by (4.15). Putting

$$(5.30) \quad Y(t) := \frac{\sqrt{2-\alpha}}{2^\nu \Gamma \left(\frac{1}{2-\alpha} \right)} \sum_{k=1}^\infty \frac{a_k j_{\nu,k}^\nu}{|J_{\nu+1}(j_{\nu,k})|} e^{-(1-\frac{\alpha}{2})^2 j_{\nu,k}^2 t}, \quad t \in [\sigma, T],$$

we have that, since $\sigma > 0$, Y is analytic in $[\sigma, T]$. Moreover,

$$Y^{(n)}(t) = y_n(t), \quad \forall t \in [\sigma, T], \forall n \in \mathbb{N}.$$

Hence, we obtain (5.22) with this choice. Since $\sigma \in (0, T)$ is arbitrary, we have in addition that $Y \in \mathcal{C}^\infty((0, T])$.

Furthermore, applying Proposition 3.3 to (5.22) with $t_1 = \sigma$ and $t_2 = T$, we deduce that f solves (1.1) pointwisely in $(\sigma, T) \times (0, 1)$ with $u = 0$ and $f_\sigma(x) = f(\sigma, x)$.

□

6. CONSTRUCTION OF THE CONTROL

Let $s \in \mathbb{R}$ with $s > 1$. The function (see [17, Section 2] and [21, Theorem 11.2, p.48])

$$(6.31) \quad \phi_s(t) := \begin{cases} 1, & \text{if } t \leq 0, \\ \frac{e^{-(1-t)^{-\frac{1}{s-1}}}}{e^{-(1-t)^{-\frac{1}{s-1}}} + e^{-t^{-\frac{1}{s-1}}}}, & \text{if } 0 < t < 1, \\ 0, & \text{if } t \geq 1, \end{cases}$$

belongs to $\mathcal{G}^s([0, 1])$ and satisfies

$$(6.32) \quad \phi_s(0) = 1, \phi_s(1) = 0, \quad \phi_s^{(i)}(0) = \phi_s^{(i)}(1) = 0, \forall i \in \mathbb{N}^*.$$

Proof of Theorem 1.3. Let $f_0 \in L^2(0, 1)$, $T > 0$. Let f and Y be given by Proposition 5.1.

We pick $\tau \in (0, T)$, $s \in (1, 2)$ and we set the flat output

$$y(t) := \phi_s\left(\frac{t - \tau}{T - \tau}\right) Y(t), \quad \forall t \in (0, T],$$

which belongs to $\mathcal{C}^\infty(0, T)$. Moreover, for every $\sigma \in (0, T)$, $y \in \mathcal{G}^s([\sigma, T])$, as it is a product of two functions in $\mathcal{G}^s([\sigma, T])$. We define accordingly the function

$$\tilde{f}(t, x) := \sum_{k=1}^{\infty} \frac{y^{(n)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{(2-\alpha)^{2n} n! \prod_{j=1}^n \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall (t, x) \in (0, T] \times [0, 1],$$

and the control

$$(6.33) \quad u(t) = \begin{cases} 0, & t \in [0, \tau], \\ \tilde{f}(t, 1), & t \in (\tau, T]. \end{cases}$$

Since $y \in \mathcal{G}^s([\sigma, T])$ for some $s \in (1, 2)$, Proposition 3.3 shows that

$$(6.34) \quad \forall \sigma \in (0, T), \tilde{f} \text{ is the pointwise solution of (3.10) with } t_1 = \sigma, t_2 = T, f_{t_1} = f(\sigma, \cdot) \text{ and (6.33).}$$

As a consequence of (6.32), we have

$$(6.35) \quad \begin{aligned} y(t) &= Y(t), \quad \forall t \in (0, \tau], \\ y(T) &= 0. \end{aligned}$$

Whence, $\tilde{f}(t, x) = f(t, x)$, for every $(t, x) \in (0, \tau) \times (0, 1)$. Thus, as $f \in \mathcal{C}^0([0, T]; L^2(0, 1))$, we deduce

$$(6.36) \quad \tilde{f} \in \mathcal{C}^0([0, T]; L^2(0, 1)),$$

$$(6.37) \quad \tilde{f}(0) = f_0 \quad \text{in } L^2(0, 1).$$

We have to check that \tilde{f} is the weak solution of system (1.1) on $(0, T)$. To do so, and according to Definition 1.2, let $t' \in (0, T)$ and let ψ satisfying (1.2) and (1.3). Then, by (6.34) and since a pointwise solution is a weak solution (see Remark 3.2), we have, for every $\sigma > 0$,

$$\begin{aligned} & \int_{\sigma}^{t'} \int_0^1 \tilde{f}(t, x) (\partial_t \psi + \partial_x (x^\alpha \partial_x \psi)) (t, x) dt dx \\ &= \int_0^1 \tilde{f}(t', x) \psi(t', x) dx - \int_0^1 \tilde{f}(\sigma, x) \psi(\sigma, x) dx + \int_{\sigma}^{t'} u(t) (x^\alpha \partial_x \psi) (t, 1) dt. \end{aligned}$$

Then, from (6.33), (6.36), (6.37) and (1.2), taking $\sigma \rightarrow 0^+$, we get the conclusion.

Finally, by construction (6.35) implies that $\tilde{f}(T, x) = 0$, for every $x \in (0, 1)$. \square

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APPENDIX A. SOME PROPERTIES OF THE GAMMA FUNCTION

For any $p \in \mathbb{R}^+$, the Gamma function is defined (see [1, 6.1.1, p.254]) by

$$(A.38) \quad \Gamma(p) := \int_0^\infty e^{-t} t^{p-1} dt,$$

which is a monotone increasing function on $(0, \infty)$. Furthermore, (see [1, 6.1.15, p.256])

$$(A.39) \quad \Gamma(x+1) = x\Gamma(x), \quad \forall x \in (0, \infty).$$

We have the following asymptotics of the Gamma function.

LEMMA A.1 ([1, 6.1.39]). *Let $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Then,*

$$(A.40) \quad \Gamma(ax+b) \underset{x \rightarrow \infty}{\sim} \sqrt{2\pi} e^{-ax} (ax)^{ax+b-\frac{1}{2}}.$$

We show an inequality used in Proposition 3.3.

LEMMA A.2.

$$(A.41) \quad (n+k)! \leq 2^{k+n} n! k!, \quad \forall n, k \in \mathbb{N}.$$

Proof. Let us observe first that

$$(A.42) \quad (2n)! \leq 2^{2n} n!^2, \quad \forall n \in \mathbb{N}.$$

This inequality follows by induction, since, for every $n \in \mathbb{N}$,

$$\begin{aligned} (2(n+1))! &= (2n)!(2n+1)(2n+2) \\ &\leq (2n)! 2^2 (n+1)^2 \leq 2^{2(n+1)} (n+1)!. \end{aligned}$$

To show (A.41), we assume, w.l.o.g., that $n < k$. Then, using (A.42),

$$\begin{aligned} (n+k)! &= (2n)! \prod_{j=1}^{k-n} (2n+j) \\ &\leq (2n)! 2^{k-n} \prod_{j=1}^{k-n} (n+j) \leq 2^{n+k} n! k!. \end{aligned}$$

□

APPENDIX B. SOME PROPERTIES OF BESSEL FUNCTIONS

Let $\nu \in \mathbb{R}$. The Bessel function of order ν of the first kind is ([1, 9.1.10, p.360])

$$(B.43) \quad J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad \forall z \in [0, \infty).$$

We denote by $\{j_{\nu,n}\}_{n \in \mathbb{N}^*}$ the increasing sequence of zeros of J_ν , which are real for any $\nu \geq 0$ and enjoy the following properties (see [1, 9.5.2, p.370] and [15, Proposition 7.8, p.135]).

$$(B.44) \quad \nu < j_{\nu,n} < j_{\nu,n+1}, \quad \forall n \in \mathbb{N}^*,$$

$$(B.45) \quad j_{\nu,n+1} - j_{\nu,n} \rightarrow \pi, \quad \text{as } n \rightarrow \infty.$$

We also have the integral formula ([1, 11.4.5, p.485])

$$(B.46) \quad \int_0^1 y J_\nu(j_{\nu,n}y) J_\nu(j_{\nu,m}y) dy = \frac{1}{2} |J_{\nu+1}(j_{\nu,n})|^2 \delta_{n,m}, \quad \forall n, m \in \mathbb{N}^*.$$

This allows to show the following.

LEMMA B.1. [14, p.40] *Let $\nu \geq 0$. The family $\{w_n\}_{n \in \mathbb{N}^*}$ defined by*

$$w_n(z) := \frac{\sqrt{2z}}{|J_{\nu+1}(j_{\nu,n})|} J_\nu(j_{\nu,n}z), \quad \forall z \in (0, 1),$$

is an orthonormal basis of $L^2(0, 1)$. In particular, if $f \in L^2(0, 1)$ and $d_n := \int_0^1 f(z) w_n(z) dz$, $\forall n \in \mathbb{N}^$, then $\|f\|_{L^2(0,1)}^2 = \sum_{n=1}^{\infty} |d_n|^2$.*

We recall that $\forall \nu \in \mathbb{R}$, the Bessel function J_ν satisfies the following differential equation (see [1, 9.1.1, p.358])

$$(B.47) \quad z^2 J_\nu''(z) + z J_\nu'(z) + (z^2 - \nu^2) J_\nu(z) = 0, \quad \forall z \in (0, +\infty),$$

and the recurrence relation (see [1, 9.1.27, p.361]),

$$(B.48) \quad 2J_\nu'(z) = J_{\nu-1}(z) + J_{\nu+1}(z), \quad \forall z \in (0, +\infty).$$

Asymptotic behaviour. We recall the asymptotic behaviour of J_ν for large arguments and near zero.

LEMMA B.2. [15, Lemma 7.2, p.129] *For any $\nu \in \mathbb{R}$,*

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O_{z \rightarrow \infty}\left(\frac{1}{z\sqrt{z}}\right).$$

LEMMA B.3. [1, 9.1.7, p.360] *For any $\nu \in \mathbb{R} \setminus \{-\mathbb{N}^*\}$,*

$$J_\nu(z) \underset{z \rightarrow 0}{\sim} \frac{z^\nu}{2^\nu \Gamma(\nu + 1)}.$$

The following asymptotic result is important in the proof of Proposition 5.1. We give the proof for the sake of completeness.

LEMMA B.4. *Let $\nu \in \mathbb{R}^+$. Then,*

$$(B.49) \quad \sqrt{j_{\nu,k}} |J_{\nu+1}(j_{\nu,k})| = \sqrt{\frac{2}{\pi}} + O_{k \rightarrow \infty}\left(\frac{1}{j_{\nu,k}}\right).$$

In particular, there exists a constant $C_1 > 0$ such that for all $k \in \mathbb{N}^$,*

$$\frac{1}{|J_{\nu+1}(j_{\nu,k})|} \leq C_1 \sqrt{j_{\nu,k}}.$$

Proof. Using Lemma B.2, for $\nu + 1$ and $x = j_{\nu,k}$,

$$\begin{aligned} \sqrt{j_{\nu,k}} |J_{\nu+1}(j_{\nu,k})| &= \sqrt{\frac{2}{\pi}} \left| \cos\left(j_{\nu,k} - \frac{\pi(\nu+1)}{2} - \frac{\pi}{4}\right) \right| + O_{k \rightarrow \infty}\left(\frac{1}{j_{\nu,k}}\right) \\ &= \sqrt{\frac{2}{\pi}} \left| \sin\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \right| + O_{k \rightarrow \infty}\left(\frac{1}{j_{\nu,k}}\right). \end{aligned}$$

Using again Lemma B.2 with ν and $x = j_{\nu,k}$, we have that

$$\cos\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) = O_{k \rightarrow \infty}\left(\frac{1}{j_{\nu,k}}\right),$$

which gives

$$\left| \sin\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \right| = \sqrt{1 + O_{k \rightarrow \infty}\left(\frac{1}{j_{\nu,k}^2}\right)} = 1 + O_{k \rightarrow \infty}\left(\frac{1}{j_{\nu,k}}\right)$$

and then (B.49). □

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