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# FLATNESS FOR A STRONGLY DEGENERATE 1-D PARABOLIC EQUATION

IVÁN MOYANO

ABSTRACT. We consider the degenerate equation

$$\partial_t f(t, x) - \partial_x (x^\alpha \partial_x f)(t, x) = 0,$$

on the unit interval  $x \in (0, 1)$ , in the strongly degenerate case  $\alpha \in [1, 2)$  with adapted boundary conditions at  $x = 0$  and boundary control at  $x = 1$ . We use the flatness approach to construct explicit controls in some Gevrey classes steering the solution from any initial datum  $f_0 \in L^2(0, 1)$  to zero in any time  $T > 0$ .

**Keywords**— partial differential equations; degenerate parabolic equation; boundary control; null-controllability; motion planning; flatness.

## 1. INTRODUCTION

We consider the following control system

$$(1.1) \quad \begin{cases} \partial_t f(t, x) - \partial_x (x^\alpha \partial_x f)(t, x) = 0, & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x f)(t, x)|_{x=0} = 0, & t \in (0, T), \\ f(t, 1) = u(t), & t \in (0, T), \\ f(0, x) = f_0(x), & x \in (0, 1), \end{cases}$$

where the state is the solution  $f(t, x)$  and the control is the function  $u(t)$ . The parameter  $\alpha \in [1, 2)$  is fixed through the whole article.

The aim of this work is to construct explicit controls  $u$  for the null-controllability of system (1.1) in finite time  $T > 0$ , using the flatness method.

**1.1. Main result.** We will make use of the Gevrey class of functions.

**DEFINITION 1.1.** Let  $s \in \mathbb{R}^+$  and  $t_1, t_2 \in \mathbb{R}$  with  $t_1 < t_2$ . A function  $h \in \mathcal{C}^\infty([t_1, t_2])$  is said to be Gevrey of order  $s$  if

$$\exists M, R > 0 \text{ such that } \sup_{t_1 \leq r \leq t_2} |h^{(n)}(r)| \leq \frac{M(n!)^s}{R^n}.$$

We then write  $h \in \mathcal{G}^s([t_1, t_2])$ .

Before stating the main result, we have to recall the notion of weak solutions of the inhomogeneous system (1.1).

**DEFINITION 1.2** (Weak solutions). Let  $f_0 \in L^2(0, 1)$ ,  $T > 0$  and  $u \in H^1(0, T)$ . A weak solution of system (1.1) is a function  $f \in \mathcal{C}^0([0, T]; L^2(0, 1))$

such that for every  $t' \in [0, T]$  and for every

$$(1.2) \quad \psi \in \mathcal{C}^1([0, t']; L^2(0, 1)) \cap \mathcal{C}^0([0, t']; H^2(0, 1))$$

such that

$$(1.3) \quad (x^\alpha \partial_x) \psi(t, x)|_{x=0} = \psi(t, 1) = 0, \quad \forall t \in [0, t'],$$

one has

$$\begin{aligned} & \int_0^{t'} \int_0^1 f(t, x) (\partial_t \psi + \partial_x (x^\alpha \partial_x \psi))(t, x) dt dx \\ &= \int_0^1 f(t', x) \psi(t', x) dx - \int_0^1 f_0(x) \psi(0, x) dx + \int_0^{t'} u(t) \partial_x \psi(t, 1) dt. \end{aligned}$$

As we show in Section 2 (see Corollary 2.2), system (1.1) has a unique weak solution under suitable assumptions. Our main result is the following.

**THEOREM 1.3.** *Let  $f_0 \in L^2(0, 1)$ ,  $T > 0$ ,  $\tau \in (0, T)$  and  $s \in (1, 2)$ . Then, there exists a flat output  $y \in \mathcal{G}^s([\tau, T])$  such that the control*

$$(1.4) \quad u(t) = \begin{cases} 0, & \text{if } t \in [0, \tau], \\ \sum_{k=0}^{\infty} \frac{y^{(k)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k (j + \frac{\alpha-1}{2-\alpha})}, & \text{if } t \in (\tau, T], \end{cases}$$

steers to zero at time  $T$  the weak solution of system (1.1). Furthermore, the control  $u$  belongs to  $\mathcal{G}^s([0, T])$ .

## 1.2. Previous work.

1.2.1. *Null-controllability.* The null-controllability of system

$$\begin{cases} \partial_t f(t, x) - \partial_x (x^\alpha \partial_x) f(t, x) = 1_\omega(x) v(t, x), & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x) f(t, x)|_{x=0} = 0, & t \in (0, T), \\ f(t, 1) = 0, & t \in (0, T), \\ f(0, x) = f_0(x), & x \in (0, 1), \end{cases}$$

where  $\omega \subset (0, 1)$ , has been studied by P. Cannarsa, P. Martinez and J. Van-costenoble in [8]. Their strategy relies on appropriate Carleman estimates. To deal with the degeneracy at  $\{x = 0\}$ , they use an adequate functional framework that we recall in Section 2, and Hardy-type inequalities.

The null-controllability of system (1.1) is a consequence of the internal null-controllability and the extension principle, since the control is located on  $\{x = 1\}$ , away from the degeneracy. The interest of the present article is to provide explicit controls.

In the case of a control located on  $\{x = 0\}$ , an approximate controllability result for  $\alpha \in [0, 1)$  has been proven by P. Cannarsa, J. Tort and M. Yamamoto in [10] using Carleman estimates. The exact controllability was later proven by M. Gueye in [13] again in the weakly degenerate case  $\alpha \in [0, 1)$  by using the transmutation method.

Other related one-dimensional problems have been treated: see [6, 7, 2], see [5] for a non-divergence setting, see [20] for a system with a singular potential. A multi-dimensional case has been studied in [9].

1.2.2. *The flatness method.* The main interest of the flatness method is to provide explicit controls. It has been developed for finite-dimensional systems (see [12]) and then generalised to some infinite-dimensional systems; see [17] for the heat equation on a cylindrical domain with boundary control, [18] for one-dimensional parabolic equations with varying coefficients and [19] for the one-dimensional Schrödinger equation. However, the strongly degenerate case  $\alpha \in [1, 2)$  considered in Theorem 1.3 does not belong to the class concerned in [18]. Our goal is to adapt the flatness method to this case.

1.3. **Open questions and perspectives.** The flatness method may also be successful on similar equations, for instance in non-divergence form as in [5]. For the time being, this is an open problem.

1.4. **Structure of the article.** In Section 2 we recall a well-posedness result and the functional framework. In Section 3 we derive, thanks to an heuristic method, an explicit solution of system (1.1) consisting on a formal series development. We prove its convergence, provided that the corresponding flat output is in a Gevrey class. In Section 4 we discuss the spectral analysis of the associated stationary problem. In Section 5 we study the regularising effect of system (1.1) when  $u = 0$ . In Section 6 we construct an appropriate flat output steering the solution of (1.1) to zero, which concludes the proof of Theorem 1.3. Finally, we give in Appendices A and B a brief account of some results concerning the Gamma and Bessel functions needed in the proofs.

1.5. **Notation.** Since all the functions appearing in the article are real-valued, we omit any explicit mention by writing, for instance,  $L^2(0, 1)$  instead of  $L^2((0, 1); \mathbb{R})$ . If  $h \in \mathcal{C}^k([t_1, t_2])$ , for some  $t_1, t_2 \in \mathbb{R}$  with  $t_1 < t_2$  and  $k \in \mathbb{N}^*$ , we will denote by  $h'(t)$  and  $h''(t)$  its first and second derivatives and by  $h^{(n)}(t)$ , for every  $n \in \mathbb{N}$ ,  $2 < n \leq k$ , the  $n$ -th derivative.

If  $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$  are two real-valued functions and  $\mu \in \overline{\mathbb{R}}$ , we will write  $h_1 \sim h_2$  as  $x \rightarrow \mu$  to denote that  $\lim_{t \rightarrow \mu} \frac{h_1(t)}{h_2(t)} = 1$ .

We will denote by  $\langle \cdot, \cdot \rangle$  the inner product in  $L^2(0, 1)$ .

## 2. WELL-POSEDNESS

We consider, for  $T > 0$  and  $f_0 \in L^2(0, 1)$ , the following system

$$(2.5) \quad \begin{cases} \partial_t f(t, x) - \partial_x (x^\alpha \partial_x) f(t, x) = h(t, x), & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x) f(t, x)|_{x=0} = 0, & t \in (0, T), \\ f(t, 1) = 0, & t \in (0, T), \\ f(0, x) = f_0(x), & x \in (0, 1). \end{cases}$$

We recall below a well-posedness result for system (2.5) proven originally in [7]. The strategy of the proof consists in a semigroup approach and the introduction of adequate weighted Sobolev spaces, that we recall below. We refer to [7, 4] for further details.

We introduce the weighted Sobolev space

$$H_\alpha^1(0, 1) := \left\{ f \in L^2(0, 1); f \text{ is loc. absolutely continuous on } (0, 1], \right. \\ \left. x^{\frac{\alpha}{2}} f' \in L^2(0, 1) \text{ and } f(1) = 0 \right\},$$

endowed with the norm

$$\|f\|_{H_\alpha^1(0,1)}^2 := \|f\|_{L^2(0,1)}^2 + \|x^{\frac{\alpha}{2}} f'\|_{L^2(0,1)}^2, \quad \forall f \in H_\alpha^1(0, 1).$$

We remark that  $H_\alpha^1(0, 1)$  is a Hilbert space with the scalar product

$$(2.6) \quad \langle f, g \rangle_{H_\alpha^1} := \int_0^1 f(x)g(x) dx + \int_0^1 x^\alpha f'(x)g'(x) dx, \quad \forall f, g \in H_\alpha^1(0, 1).$$

**PROPOSITION 2.1** ([7], Proposition 3.2 and Theorem 3.1). *Let*

$$(2.7) \quad \begin{cases} D(A) := \{f \in H_\alpha^1(0, 1); x^\alpha f' \in H^1(0, 1)\}, \\ Af := -(x^\alpha f')'. \end{cases}$$

*Then,  $A : D(A) \rightarrow L^2(0, 1)$  is a closed self-adjoint positive operator with dense domain. As a consequence,  $A$  is the infinitesimal generator of a strongly continuous semigroup, and for any  $f_0 \in L^2(0, 1)$ , and  $h \in L^2((0, T) \times (0, 1))$  there exists a unique weak solution of system (2.5), i.e., a function  $f \in \mathcal{C}^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H_\alpha^1(0, 1))$  such that*

$$f(t) = S(t)f_0 + \int_0^t S(t-s)h(s) ds, \quad \text{in } L^2(0, 1), \quad \forall t \in [0, T].$$

As a consequence, using classical arguments (see for instance [11, Section 2.5.3]), we deduce the following result.

**COROLLARY 2.2.** *Let  $T > 0$ ,  $f_0 \in L^2(0, 1)$  and  $u \in H^1(0, T)$ . Then, system (1.1) has a unique weak solution (see Definition 1.2).*

*Proof.* Let  $f_0 \in L^2(0, 1)$ ,  $u \in H^1(0, T)$  and

$$\theta(x) := x^2, \quad x \in [0, 1].$$

We consider the system

$$\begin{cases} (\partial_t - \partial_x(x^\alpha \partial_x))g(t, x) = H(t, x), & (t, x) \in (0, T) \times (0, 1), \\ (x^\alpha \partial_x)g(t, x)|_{x=0} = 0, & t \in (0, T), \\ g(t, 1) = 0, & t \in (0, T), \\ g(0, x) = f_0(x) - u(0)\theta(x), & x \in (0, 1), \end{cases}$$

with

$$H(t, x) := -u'(t)\theta(x) - u(t)A\theta(x), \quad \forall (t, x) \in (0, T) \times (0, 1).$$

Since  $H \in L^2((0, T) \times (0, 1))$ , by Proposition 2.1 there exists a unique weak solution  $g \in \mathcal{C}^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H_\alpha^1(0, 1))$  of this problem. We set

$$f(t, x) := g(t, x) + u(t)\theta(x).$$

Then, using the integral formulation associated to  $g$ , one shows that  $f$  is a weak solution of system (1.1) in the sense of Definition 1.2.

The uniqueness follows since, if  $f_1$  and  $f_2$  are weak solutions of (1.1), then  $f_1 - f_2$  is the unique weak solution of system (2.5) with  $h \equiv 0$ , and then by Proposition 2.1,  $f_1 - f_2 = 0$ .  $\square$

### 3. EXPLICIT SOLUTION

3.1. **Heuristics.** We consider the following formal expansion

$$f(t, x) = \sum_{k=0}^{\infty} c_{2k}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}, \quad \forall (t, x) \in (0, T) \times (0, 1).$$

where  $(c_{2k}(t))_{k \in \mathbb{N}}$  is a sequence of real numbers. We formally have

$$\begin{aligned} \partial_x (x^\alpha \partial_x f)(t, x) &= \sum_{k=0}^{\infty} c_{2(k+1)}(t) (2-\alpha)^2 (k+1) \left[ k+1 + \frac{\alpha-1}{2-\alpha} \right] \left(x^{1-\frac{\alpha}{2}}\right)^{2k}, \\ \partial_t f(t, x) &= \sum_{k=0}^{\infty} c'_{2k}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}. \end{aligned}$$

If  $f$  solves (1.1), then the following recurrence relation holds

$$c_{2(k+1)}(t) = \frac{c'_{2k}(t)}{(2-\alpha)^2 (k+1) \left(k+1 + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall k \in \mathbb{N}.$$

Choosing a flat output  $c_0(t) := y(t)$  and iterating, we readily have

$$c_{2k}(t) = \frac{y^{(k)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall t \in (0, T), \forall k \in \mathbb{N}.$$

This gives a formal solution of (1.1),

$$(3.8) \quad f(t, x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)},$$

and a control given by  $u(t) = f(t, 1)$ , which is

$$(3.9) \quad u(t) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)}.$$

**3.2. Pointwise solutions.** The goal of this section is to introduce a notion of pointwise solution of system (1.1) to give a sense to the heuristics made in the previous section.

We define

$$\mathcal{C}_\alpha^2(0, 1) := \{f \in \mathcal{C}^0([0, 1]) \cap \mathcal{C}^2((0, 1)) \text{ such that } x^\alpha f'(x) \in \mathcal{C}^0([0, 1])\}.$$

**DEFINITION 3.1** (Pointwise solution). *Let  $t_1, t_2 \in \mathbb{R}$  with  $t_1 < t_2$ . Let  $f_{t_1} \in \mathcal{C}^0(0, 1)$  and  $u \in \mathcal{C}^0([t_1, t_2])$ . A pointwise solution of system*

$$(3.10) \quad \begin{cases} \partial_t f(t, x) - \partial_x(x^\alpha \partial_x f)(t, x) = 0, & (t, x) \in (t_1, t_2) \times (0, 1), \\ x^\alpha \partial_x f(t, x)|_{x=0} = 0, & t \in (t_1, t_2), \\ f(t, 1) = u(t), & t \in (t_1, t_2), \\ f(t_1, x) = f_{t_1}(x), & x \in (0, 1), \end{cases}$$

is a function  $f \in \mathcal{C}^0([t_1, t_2] \times [0, 1]) \cap \mathcal{C}^1((t_1, t_2) \times (0, 1))$  such that

- (1)  $f(t, \cdot) \in \mathcal{C}_\alpha^2(0, 1)$ ,  $\forall t \in (t_1, t_2)$ ,
- (2)  $\partial_t f - \partial_x(x^\alpha \partial_x f) = 0$  pointwisely in  $(t_1, t_2) \times (0, 1)$ ,
- (3)  $\lim_{x \rightarrow 0^+} x^\alpha \partial_x f(t, x) = 0$ ,  $\forall t \in (t_1, t_2)$ ,
- (4)  $f(t, 1) = u(t)$ ,  $\forall t \in (t_1, t_2)$ ,
- (5)  $f(t_1, x) = f_{t_1}(x)$ ,  $\forall x \in (0, 1)$ .

**REMARK 3.2.** *The usual energy argument proves that, given  $u \in \mathcal{C}^0([t_1, t_2])$ , the pointwise solution of system (3.10) is unique. We observe that, changing parameters adequately in Definition 1.2 a pointwise solution of (3.10) is also a weak solution.*

**3.3. Convergence.** The goal of this section is the proof of the following result.

**PROPOSITION 3.3.** *Let  $t_1, t_2 \in \mathbb{R}$ , with  $t_1 < t_2$ . If  $y \in \mathcal{G}^s([t_1, t_2])$  for some  $s \in (0, 2)$ , then*

- (1) *the control  $u$  given by (3.9) is well defined and belongs to  $\mathcal{G}^s([t_1, t_2])$ ,*
- (2) *the function given by (3.8) is a pointwise solution (see Definition 3.1) of system (3.10) in  $(t_1, t_2) \times (0, 1)$  with  $u$  given by (3.9) and initial datum*

$$f_{t_1}(x) := \sum_{k=1}^{\infty} \frac{y^{(k)}(t_1) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall x \in [0, 1].$$

*Proof.* Let  $M, R > 0$  be such that  $|y^{(n)}(t)| \leq \frac{Mn!^s}{R^n}$ , for any  $n \in \mathbb{N}$ ,  $t \in [t_1, t_2]$ .

**Step 1:** We prove that  $u$  is well defined and belongs to  $\mathcal{C}^\infty([t_1, t_2])$ .

For any  $t \in [t_1, t_2]$ ,  $k \in \mathbb{N}^*$ , we have, as  $\frac{\alpha-1}{2-\alpha} \geq 0$ ,

$$\frac{|y^{(k)}(t)|}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)} \leq \frac{Mk!^s}{R^k (2-\alpha)^{2k} k!^2} = \frac{M}{R^k (2-\alpha)^{2k} k!^{2-s}}.$$

Hence, the series in (3.9) converges uniformly w.r.t.  $t \in [t_1, t_2]$  and  $u \in \mathcal{C}^0([t_1, t_2])$ . Furthermore, for any  $n \in \mathbb{N}^*$ , the function  $\xi_{n,k}(t) := \frac{y^{(k+n)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^k (j + \frac{\alpha-1}{2-\alpha})}$  satisfies

$$|\xi_{n,k}(t)| \leq \frac{M(k+n)!^s}{R^{n+k}(2-\alpha)^{2k} k!^2}, \quad \forall t \in [t_1, t_2], k, n \in \mathbb{N}.$$

Thus,  $\sum_k \xi_{n,k}(t)$  converges uniformly w.r.t  $t \in [t_1, t_2]$ . Whence,  $u \in \mathcal{C}^\infty([t_1, t_2])$  and for every  $n \in \mathbb{N}$ ,  $t \in [t_1, t_2]$ ,  $u^{(n)}(t) = \sum_{k=0}^\infty \xi_{n,k}(t)$ .

**Step 2:** We prove that  $u$  is Gevrey of order  $s$ .

Let  $n \in \mathbb{N}$ . We deduce from last inequality that

$$\begin{aligned} |u^{(n)}(t)| &\leq \sum_{k=0}^\infty \frac{M(k+n)!^s}{R^{n+k}(2-\alpha)^{2k} k!^2} \\ (3.11) \quad &\leq M \left[ \sum_{k=0}^\infty \frac{1}{(k!)^{2-s}} \left( \frac{2^s}{R(2-\alpha)^2} \right)^k \right] \left( \frac{2^s}{R} \right)^n n!^s, \end{aligned}$$

where we have used (A.41). The D'Alembert criterium for entire series shows that, whenever  $s \in (0, 2)$ , the series above converges, which shows that  $u \in \mathcal{G}^s([t_1, t_2])$ .

**Step 3:** We show that the function  $f$  given by (3.8) is well defined and  $f \in \mathcal{C}^0([t_1, t_2] \times [0, 1]) \cap \mathcal{C}^1((t_1, t_2) \times (0, 1))$ .

Let, for every  $k \in \mathbb{N}$ ,

$$f_k(t, x) := \frac{y^{(k)}(t) \left( x^{1-\frac{\alpha}{2}} \right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left( j + \frac{\alpha-1}{2-\alpha} \right)}, \quad \forall (t, x) \in [t_1, t_2] \times [0, 1].$$

Then,

$$|f_k(t, x)| \leq \frac{M}{k!^{2-s}} \left( \frac{1}{R(2-\alpha)} \right)^k, \quad \forall (t, x) \in [t_1, t_2] \times [0, 1].$$

This proves that  $\sum_k f_k$  converges uniformly w.r.t.  $(t, x) \in [t_1, t_2] \times [0, 1]$ . Thus,  $f \in \mathcal{C}^0([t_1, t_2] \times [0, 1])$ .

We observe that  $\exists k_0 = k_0(\alpha) \in \mathbb{N}^*$  such that  $(2-\alpha)k_0 \geq 1$ . Then, for every  $k \geq k_0$ ,  $f_k(t, \cdot) \in \mathcal{C}^1([0, 1])$  and

$$\begin{aligned} |\partial_x f_k(t, x)| &= \left| \frac{y^{(k)}(t) 2k \left( 1 - \frac{\alpha}{2} \right) x^{-\frac{\alpha}{2}} \left( x^{1-\frac{\alpha}{2}} \right)^{2k-1}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left( j + \frac{\alpha-1}{2-\alpha} \right)} \right| \\ &\leq 2M \left( 1 - \frac{\alpha}{2} \right) \frac{k}{k!^{2-s}} \left( \frac{1}{R(2-\alpha)^2} \right)^k, \quad \forall x \in [0, 1], \end{aligned}$$

since  $(1 - \frac{\alpha}{2})(2k-1) - \frac{\alpha}{2} \geq 0$ . This proves that  $\sum_{k \geq k_0} \partial_x f_k$  converges uniformly w.r.t.  $(t, x) \in [t_1, t_2] \times [0, 1]$ . Thus,  $f(t, \cdot) \in$



$\mathcal{C}^1((0, 1])$  for every  $t \in [t_1, t_2]$ . Note that  $f$  may not be differentiable w.r.t.  $x$  at  $x = 0$  because of the finite number of terms  $\sum_{k=0}^{k_0} \partial_x f_k$ . Moreover,  $\partial_x f(t, x) = \sum_{k=0}^{\infty} \partial_x f_k(t, x)$  for every  $(t, x) \in (t_1, t_2) \times (0, 1)$ .

A similar argument shows that, for every  $x \in (0, 1)$ ,  $f(\cdot, x) \in \mathcal{C}^1(t_1, t_2)$  and

$$(3.12) \quad \partial_t f(t, x) = \sum_{k=0}^{\infty} \partial_t f_k(t, x), \quad \forall (t, x) \in (t_1, t_2) \times (0, 1).$$

Finally, since the partial derivatives of  $f$  exist and are continuous in  $(t_1, t_2) \times (0, 1)$ ,  $f \in \mathcal{C}^1((t_1, t_2) \times (0, 1))$ .

**Step 4:** We show that  $f(t, \cdot) \in \mathcal{C}_\alpha^2(0, 1)$ , for every  $t \in (t_1, t_2)$ .

Let  $k_1 = k_1(\alpha) \in \mathbb{N}^*$  such that  $k_1(2 - \alpha) \geq 2$ . Working as in Step 3, we see that  $\sum_{k \geq k_1} \partial_x^2 f_k$  converges uniformly w.r.t.  $(t, x) \in (t_1, t_2) \times (0, 1)$ . Thus,  $f(t, \cdot) \in \mathcal{C}^2(0, 1)$ ,  $\forall t \in (t_1, t_2)$ . Furthermore,

$$(3.13) \quad \partial_x (x^\alpha \partial_x f)(t, x) = \sum_{k=1}^{\infty} \frac{y^{(k)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2(k-1)}}{(2-\alpha)^{2(k-1)} (k-1)! \prod_{j=1}^{k-1} \left(j + \frac{\alpha-1}{2-\alpha}\right)}.$$

for every  $(t, x) \in (t_1, t_2) \times (0, 1)$ . From Step 3, we obtain

$$\begin{aligned} |x^\alpha \partial_x f(t, x)| &= \left| \sum_{k=1}^{\infty} \frac{y^{(k)}(t) 2k \left(1 - \frac{\alpha}{2}\right) x^{2k(1-\frac{\alpha}{2})+\alpha-1}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)} \right| \\ &\leq 2M \left(1 - \frac{\alpha}{2}\right) \sum_{k=1}^{\infty} \left[ \frac{k}{k!^{2-s}} \left(\frac{1}{R(2-\alpha)^2}\right)^k \right] x, \end{aligned}$$

for all  $(t, x) \in (t_1, t_2) \times (0, 1)$ , which implies, since  $\alpha \in [1, 2)$ , that

$$x^\alpha \partial_x f(t, x) \rightarrow 0, \quad \text{as } x \rightarrow 0^+.$$

Therefore,  $f(t, \cdot) \in \mathcal{C}_\alpha^2$ , for every  $t \in (t_1, t_2)$ .

**Step 5:** According to (3.12) and (3.13), a straightforward computation shows that the equation in (3.10) is satisfied.  $\square$

#### 4. SPECTRAL ANALYSIS

The goal of this section is to give the explicit expression of the eigenfunctions and eigenvalues of the spectral problem

$$(4.14) \quad \begin{cases} A\varphi(x) = \lambda\varphi(x), & x \in (0, 1), \\ (x^\alpha \varphi')|_{x=0} = \varphi(1) = 0, \end{cases}$$

where  $A$  is given by (2.7). We will make use of several results about Bessel functions recalled in Appendix B. From now on, we use the notation

$$(4.15) \quad \nu := \frac{\alpha - 1}{2 - \alpha}.$$

**PROPOSITION 4.1.** *Let*

$$(4.16) \quad \varphi_k(x) = \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} J_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}), \quad \forall x \in (0, 1), k \in \mathbb{N}^*.$$

*Then,*

- (1)  $\varphi_k \in D(A)$ ,  $\forall k \in \mathbb{N}^*$ ,
- (2)  $\varphi_k$  satisfies (4.14) with

$$(4.17) \quad \lambda_k := \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2, \quad \forall k \in \mathbb{N}^*,$$

- (3)  $(\varphi_k)_{k \in \mathbb{N}^*}$  is a Hilbert basis of  $L^2(0, 1)$ ,
- (4) for every  $f_0 \in L^2(0, 1)$  the solution of (2.5) with  $h = 0$  writes

$$(4.18) \quad f(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle f_0, \varphi_k \rangle \varphi_k \quad \text{in } L^2(0, 1), \forall t \in [0, T].$$

*Proof.* We will note for simplicity  $b_k := \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|}$  and  $\tilde{\varphi}_k := \frac{1}{b_k} \varphi_k$ , for every  $k \in \mathbb{N}^*$ .

**Step 1:** We prove that  $\varphi_k \in D(A)$ , for every  $k \in \mathbb{N}^*$  and that  $A\varphi_k - \lambda_k \varphi_k = 0$ .

Let  $k \in \mathbb{N}^*$ . We observe that  $\varphi_k \in \mathcal{C}^\infty((0, 1]) \cap \mathcal{C}^0([0, 1])$ , for any  $k \in \mathbb{N}^*$  and  $x \in (0, 1)$ . We have

$$(4.19) \quad \tilde{\varphi}'_k(x) = \frac{1-\alpha}{2} x^{-\frac{1+\alpha}{2}} J_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}) + j_{\nu,k} \left(1 - \frac{\alpha}{2}\right) x^{\frac{1}{2}-\alpha} J'_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}).$$

Whence, using (B.48) and Lemma B.3, we deduce

$$x^{\frac{\alpha}{2}} \tilde{\varphi}'_k = (1-\alpha) O_{x \rightarrow 0^+} \left(x^{\frac{\alpha}{2}-1}\right) + O_{x \rightarrow 0^+} \left(x^{1-\frac{\alpha}{2}}\right).$$

It follows that  $x^{\frac{\alpha}{2}} \varphi'_k \in L^2(0, 1)$ . Thus  $\varphi_k \in H^1_{\alpha}(0, 1)$ . Moreover, from (4.19), a direct computation shows

$$(4.20) \quad \begin{aligned} (x^{\alpha} \tilde{\varphi}'_k)' &= - \left(\frac{1-\alpha}{2}\right)^2 x^{\frac{\alpha-3}{2}} J_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k} x^{-\frac{1}{2}} J'_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J''_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}). \end{aligned}$$

Then, evaluating equation (B.47) at  $z = j_{\nu,k}x^{1-\frac{\alpha}{2}}$  and multiplying by  $x^{\frac{\alpha-3}{2}}$ , it follows

$$\begin{aligned} &j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J''_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}) \\ &= -j_{\nu,k} x^{-\frac{1}{2}} J'_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}) - j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(\frac{\alpha-1}{2-\alpha}\right)^2 x^{\frac{\alpha-3}{2}} J_{\nu}(j_{\nu,k}x^{1-\frac{\alpha}{2}}). \end{aligned}$$

Substituting in (4.20), this gives

$$-(x^\alpha \tilde{\varphi}'_k)' = \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) = \lambda_k \tilde{\varphi}_k.$$

Then, we readily have  $(x^\alpha \tilde{\varphi}'_k)' \in H_\alpha^1(0,1) \subset L^2(0,1)$ . Thus,  $\varphi_k \in D(A)$ . Moreover,  $A\varphi_k = \lambda_k \varphi_k$ .

**Step 2:** We check the boundary condition of (4.14) at  $x = 0$ .

We observe first that the case  $\alpha = 1$  is straightforward. From (4.19), (B.48) and Lemma B.3, we have

$$|x^\alpha \tilde{\varphi}'_n(x)| = O(x^{\alpha-1}) \text{ as } x \rightarrow 0^+.$$

Then, it follows that  $\lim_{x \rightarrow 0^+} x^\alpha \tilde{\varphi}'_n(x) = 0$ . This shows, combined with Step 1, that  $\varphi_k$  satisfies (4.14).

**Step 3:** We prove that  $(\varphi_k)_{k \in \mathbb{N}^*}$  is an orthonormal family in  $L^2(0,1)$ .

Let  $n, m \in \mathbb{N}^*$ . Then, changing variables and using (B.46), we get

$$\begin{aligned} & \int_0^1 \varphi_n(x) \varphi_m(x) dx \\ &= (2-\alpha) \int_0^1 x^{1-\alpha} \frac{J_\nu(j_{\nu,n} x^{1-\frac{\alpha}{2}})}{|J_{\nu+1}(j_{\nu,n})|} \frac{J_\nu(j_{\nu,m} x^{1-\frac{\alpha}{2}})}{|J_{\nu+1}(j_{\nu,m})|} dx \\ &= \frac{2}{|J_{\nu+1}(j_{\nu,n})| |J_{\nu+1}(j_{\nu,m})|} \int_0^1 y J_\nu(j_{\nu,n} y) J_\nu(j_{\nu,m} y) dy = \delta_{n,m}, \end{aligned}$$

where  $\delta_{n,m}$  stands for the Kronecker delta.

**Step 4:** We prove that  $(\varphi_k)_{k \in \mathbb{N}^*}$  is a Hilbert basis of  $L^2(0,1)$  by checking the Bessel equality. Let  $f \in L^2(0,1)$  and let

$$(4.21) \quad a_k := \int_0^1 f(x) \varphi_k(x) dx, \quad \forall k \in \mathbb{N}^*.$$

Then, using Lemma B.1 and changing variables twice, we get

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k|^2 &= \sum_{k=1}^{\infty} \left| \int_0^1 f(x) \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) dx \right|^2 \\ &= \frac{2}{2-\alpha} \sum_{k=1}^{\infty} \left| \int_0^1 y^{\frac{\alpha-1}{2-\alpha} + \frac{1}{2}} f(y^{\frac{2}{2-\alpha}}) \frac{\sqrt{2y}}{|J_{\nu+1}(j_{\nu,k})|} J_\nu(j_{\nu,k} y) dy \right|^2 \\ &= \frac{2}{2-\alpha} \int_0^1 y^{\frac{2(\alpha-1)}{2-\alpha} + 1} |f(y^{\frac{2}{2-\alpha}})|^2 dy \\ &= \int_0^1 |f(z)|^2 dz = \|f\|_{L^2(0,1)}^2. \end{aligned}$$

**Step 5:** Finally, (4.18) is a consequence of [3, Theorem 8.2.3, pp.237–240].

□

## 5. REGULARISING EFFECT

We use the orthonormal basis obtained in Proposition 4.1 and some properties of Bessel functions to quantify the smoothing of the solution of system (1.1) when  $u \equiv 0$ .

**PROPOSITION 5.1.** *Let  $f_0 \in L^2(0, 1)$ ,  $T > 0$  and let  $f \in \mathcal{C}^0([0, T]; L^2(0, 1))$  be the unique weak solution of system (2.5) when  $h = 0$ , according to Proposition 2.1. Then, there exists  $Y \in \mathcal{C}^\infty((0, T])$  such that for every  $\sigma \in (0, T)$ ,*

$$Y \in \mathcal{G}^1([\sigma, T])$$

and

$$(5.22) \quad f(t, x) = \sum_{n=0}^{\infty} \frac{Y^{(n)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{(2-\alpha)^{2n} n! \prod_{j=1}^n \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall (t, x) \in [\sigma, T] \times [0, 1].$$

Moreover,  $f$  solves system (3.10) pointwisely (see Definition 3.1) in  $(\sigma, T) \times (0, 1)$  with  $u = 0$  and initial datum  $f_\sigma(x) = f(\sigma, x)$ .

*Proof.* Let  $\nu$  be given by (4.15) and  $a_k$  as in (4.21). Let  $\sigma \in (0, T)$  be fixed but arbitrary. Let  $t \in [\sigma, T]$  be fixed. By (4.18) and (B.43), we have, for a.e.  $x \in [0, 1]$ ,

$$(5.23) \quad \begin{aligned} f(t, x) &= \sum_{k=1}^{\infty} e^{-\lambda_k t} \frac{a_k \sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} J_\nu \left( j_{\nu,k} x^{1-\frac{\alpha}{2}} \right) \\ &= \sum_{k=1}^{\infty} e^{-\lambda_k t} \frac{a_k \sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+1+\nu)} \left( \frac{j_{\nu,k} x^{1-\frac{\alpha}{2}}}{2} \right)^{2n+\nu} \\ &= \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} B_{n,k}(t, x), \end{aligned}$$

where, for every  $(n, k) \in \mathbb{N} \times \mathbb{N}^*$ ,

$$B_{n,k}(t, x) := e^{-\lambda_k t} b_k \frac{(-1)^n j_{\nu,k}^{2n+\nu}}{n! \Gamma(n+1+\nu) 2^{2n+\nu}} \frac{\left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{|J_{\nu+1}(j_{\nu,k})|},$$

and  $b_k := a_k \sqrt{2-\alpha}$ ,  $\forall k \in \mathbb{N}^*$ .

**Step 1:** We show that

$$(5.24) \quad \sum_{n=0}^{\infty} \left( \sum_{k=1}^{\infty} |B_{n,k}(t, x)| \right) < \infty, \quad \forall x \in [0, 1].$$

Indeed, since  $\lambda_k > 0$ , we have for every  $(n, k) \in \mathbb{N} \times \mathbb{N}^*$  and  $x \in [0, 1]$ ,

$$|B_{n,k}(t, x)| \leq \frac{|b_k| j_{\nu,k}^{2n+\nu} e^{-\lambda_k \sigma}}{2^{2n+\nu} n! \Gamma(n+1+\nu) |J_{\nu+1}(j_{\nu,k})|}$$

$$(5.25) \quad \leq \frac{C_1 |b_k| e^{-\lambda_k \sigma} j_{\nu,k}^{2n+\nu+\frac{1}{2}}}{2^{2n} n! \Gamma(n+1+\nu)},$$

for a constant  $C_1 > 0$ , using Lemma [B.4](#).

We fix  $n \in \mathbb{N}$  and we define the function  $h_n^\alpha \in \mathcal{C}^\infty(\mathbb{R}^+; \mathbb{R}^+)$  by

$$h_n^\alpha(x) := e^{-(1-\frac{\alpha}{2})^2 x^2 \sigma} x^{2n+\nu+\frac{1}{2}}, \quad \forall x \in [0, +\infty),$$

which satisfies that

$$(5.26) \quad \frac{d}{dx} h_n^\alpha(x) > 0, \quad \forall x \in (0, N_n^\alpha) \quad \text{and} \quad \frac{d}{dx} h_n^\alpha(x) < 0, \quad \forall x \in (N_n^\alpha, \infty),$$

where  $N_n^\alpha := \frac{2}{2-\alpha} \sqrt{\frac{1}{\sigma} \left( n + \frac{\alpha}{4(2-\alpha)} \right)}$ . Hence, from [\(5.25\)](#) and [\(4.17\)](#),

$$(5.27) \quad \sum_{k=1}^{\infty} |B_{n,k}(t, x)| \leq \frac{C_1 \sup_k |b_k|}{2^{2n} n! \Gamma(n+1+\nu)} \sum_{k=1}^{\infty} h_n^\alpha(j_{\nu,k})$$

Introducing  $K_n^\alpha := \sup \{k \in \mathbb{N}^*; j_{\nu,k} \leq N_n^\alpha\}$ , we write

$$(5.28) \quad \sum_{k=1}^{\infty} h_n^\alpha(j_{\nu,k}) = h_n^\alpha(j_{\nu, K_n^\alpha}) + h_n^\alpha(j_{\nu, K_n^\alpha+1}) + \sum_{k \in \mathbb{N}^* - \{K_n^\alpha, K_n^\alpha+1\}} h_n^\alpha(j_{\nu,k})$$

On one hand, we have

$$(5.29) \quad \begin{aligned} h_n^\alpha(j_{\nu, K_n^\alpha}) + h_n^\alpha(j_{\nu, K_n^\alpha+1}) &\leq 2h_n^\alpha(N_n^\alpha) \\ &\leq 2e^{-\left(n+\frac{\alpha}{4(2-\alpha)}\right)} \left(n + \frac{\alpha}{4(2-\alpha)}\right)^{n+\frac{\alpha}{4(2-\alpha)}} \left[\frac{1}{\sigma} \left(\frac{2}{2-\alpha}\right)^2\right]^{n+\frac{\alpha}{4(2-\alpha)}} \\ &\leq C_2 \Gamma\left(n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2}\right) \left[\frac{1}{\sigma} \left(\frac{2}{2-\alpha}\right)^2\right]^{n+\frac{\alpha}{4(2-\alpha)}}, \end{aligned}$$

for a constant  $C_2 > 0$ , using Lemma [A.1](#) with  $a = 1, b = \frac{1}{2}$ . On the other hand, using [\(5.26\)](#), we write

$$\begin{aligned} &\sum_{k \in \mathbb{N}^* - \{K_n^\alpha, K_n^\alpha+1\}} h_n^\alpha(j_{\nu,k}) \leq \\ &\leq \sum_{k=1}^{K_n^\alpha-1} \frac{1}{j_{\nu,k+1} - j_{\nu,k}} \int_{j_{\nu,k}}^{j_{\nu,k+1}} h_n^\alpha(x) dx + \sum_{k=K_n^\alpha+1}^{\infty} \frac{1}{j_{\nu,k} - j_{\nu,k-1}} \int_{j_{\nu,k-1}}^{j_{\nu,k}} h_n^\alpha(x) dx \\ &\leq \sup_{k \in \mathbb{N}^*} \left\{ \frac{1}{j_{\nu,k+1} - j_{\nu,k}} \right\} \left( \int_{j_{\nu,1}}^{j_{\nu, K_n^\alpha}} h_n^\alpha(x) dx + \int_{j_{\nu, K_n^\alpha+1}}^{\infty} h_n^\alpha(x) dx \right) \\ &\leq C_3 \int_0^{\infty} h_n^\alpha(x) dx, \end{aligned}$$

for a constant  $C_3 > 0$ , using (B.45). Moreover, we have

$$\begin{aligned}
 \int_0^\infty h_n^\alpha(x) dx &= \int_0^\infty e^{-(1-\frac{\alpha}{2})^2 x^2 \sigma} x^{2n+\frac{\alpha}{2(2-\alpha)}} dx \\
 &= \int_0^\infty e^{-t} \left( \frac{2}{2-\alpha} \sqrt{\frac{t}{\sigma}} \right)^{2n+\frac{\alpha}{2(2-\alpha)}} \frac{1}{2\sqrt{\sigma t}} \left( \frac{2}{2-\alpha} \right) dt \\
 &= \frac{1}{2} \left[ \frac{1}{\sqrt{\sigma}} \left( \frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}+1} \int_0^\infty e^{-t} t^{n+\frac{\alpha}{4(2-\alpha)}-\frac{1}{2}} dt \\
 &= \frac{1}{2} \left[ \frac{1}{\sqrt{\sigma}} \left( \frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}+1} \Gamma \left( n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2} \right),
 \end{aligned}$$

where we have used (A.38) with  $p = n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2}$ . Hence, combining this with (5.28) and (5.29), we get

$$\sum_{k=1}^\infty h_n^\alpha(j_{\nu,k}) \leq \left( C_2 + \frac{C_3}{\sqrt{\sigma(2-\alpha)}} \right) \left[ \frac{1}{\sqrt{\sigma}} \left( \frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}} \Gamma \left( n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2} \right),$$

which, according to (5.27), implies

$$\sum_{k=1}^\infty |B_{n,k}(t, x)| \leq C_4 \left[ \frac{1}{\sqrt{\sigma}} \left( \frac{2}{2-\alpha} \right) \right]^{2n+\frac{\alpha}{2(2-\alpha)}} \frac{\Gamma \left( n + \frac{\alpha}{4(2-\alpha)} + \frac{1}{2} \right)}{2^{2n} n! \Gamma(n + \nu + 1)}.$$

Henceforth, the D'Alembert criterium for entire series gives (5.24).

**Step 2:** We find  $Y \in \mathcal{G}^1([\sigma, T])$  such that (5.22) holds.

Thanks to Fubini's theorem, (5.23) and (A.39), we may write

$$f(t, x) = \sum_{n=0}^\infty \frac{y_n(t) \left( x^{1-\frac{\alpha}{2}} \right)^{2n}}{(2-\alpha)^{2n} n! \prod_{j=1}^n (j + \nu)},$$

where, for every  $n \in \mathbb{N}$ ,

$$y_n(t) := \frac{(-1)^n \sqrt{2-\alpha} \left( 1 - \frac{\alpha}{2} \right)^{2n}}{2^\nu \Gamma \left( \frac{1}{2-\alpha} \right)} \sum_{k=1}^\infty a_k e^{-\lambda_k t} \frac{j_{\nu,k}^{2n+\nu}}{|J_{\nu+1}(j_{\nu,k})|}, \quad \forall t \in [\sigma, T],$$

and  $\nu$  is given by (4.15). Putting

$$(5.30) \quad Y(t) := \frac{\sqrt{2-\alpha}}{2^\nu \Gamma \left( \frac{1}{2-\alpha} \right)} \sum_{k=1}^\infty \frac{a_k j_{\nu,k}^\nu}{|J_{\nu+1}(j_{\nu,k})|} e^{-(1-\frac{\alpha}{2})^2 j_{\nu,k}^2 t}, \quad t \in [\sigma, T],$$

we have that, since  $\sigma > 0$ ,  $Y$  is analytic in  $[\sigma, T]$ . Moreover,

$$Y^{(n)}(t) = y_n(t), \quad \forall t \in [\sigma, T], \forall n \in \mathbb{N}.$$

Hence, we obtain (5.22) with this choice. Since  $\sigma \in (0, T)$  is arbitrary, we have in addition that  $Y \in \mathcal{C}^\infty((0, T])$ .

Furthermore, applying Proposition 3.3 to (5.22) with  $t_1 = \sigma$  and  $t_2 = T$ , we deduce that  $f$  solves (1.1) pointwisely in  $(\sigma, T) \times (0, 1)$  with  $u = 0$  and  $f_\sigma(x) = f(\sigma, x)$ .

□

## 6. CONSTRUCTION OF THE CONTROL

Let  $s \in \mathbb{R}$  with  $s > 1$ . The function (see [17, Section 2] and [21, Theorem 11.2, p.48])

$$(6.31) \quad \phi_s(t) := \begin{cases} 1, & \text{if } t \leq 0, \\ \frac{e^{-(1-t)^{-\frac{1}{s-1}}}}{e^{-(1-t)^{-\frac{1}{s-1}}} + e^{-t^{-\frac{1}{s-1}}}}, & \text{if } 0 < t < 1, \\ 0, & \text{if } t \geq 1, \end{cases}$$

belongs to  $\mathcal{G}^s([0, 1])$  and satisfies

$$(6.32) \quad \phi_s(0) = 1, \phi_s(1) = 0, \quad \phi_s^{(i)}(0) = \phi_s^{(i)}(1) = 0, \forall i \in \mathbb{N}^*.$$

*Proof of Theorem 1.3.* Let  $f_0 \in L^2(0, 1)$ ,  $T > 0$ . Let  $f$  and  $Y$  be given by Proposition 5.1.

We pick  $\tau \in (0, T)$ ,  $s \in (1, 2)$  and we set the flat output

$$y(t) := \phi_s\left(\frac{t - \tau}{T - \tau}\right) Y(t), \quad \forall t \in (0, T],$$

which belongs to  $\mathcal{C}^\infty(0, T)$ . Moreover, for every  $\sigma \in (0, T)$ ,  $y \in \mathcal{G}^s([\sigma, T])$ , as it is a product of two functions in  $\mathcal{G}^s([\sigma, T])$ . We define accordingly the function

$$\tilde{f}(t, x) := \sum_{k=1}^{\infty} \frac{y^{(k)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^k \left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall (t, x) \in (0, T] \times [0, 1],$$

and the control

$$(6.33) \quad u(t) = \begin{cases} 0, & t \in [0, \tau], \\ \tilde{f}(t, 1), & t \in (\tau, T]. \end{cases}$$

Since  $y \in \mathcal{G}^s([\sigma, T])$  for some  $s \in (1, 2)$ , Proposition 3.3 shows that

$$(6.34) \quad \forall \sigma \in (0, T), \tilde{f} \text{ is the pointwise solution of (3.10) with } t_1 = \sigma, t_2 = T, f_{t_1} = f(\sigma, \cdot) \text{ and (6.33)}.$$

As a consequence of (6.32), we have

$$(6.35) \quad \begin{aligned} y(t) &= Y(t), \quad \forall t \in (0, \tau], \\ y(T) &= 0. \end{aligned}$$

Whence,  $\tilde{f}(t, x) = f(t, x)$ , for every  $(t, x) \in (0, \tau) \times (0, 1)$ . Thus, as  $f \in \mathcal{C}^0([0, T]; L^2(0, 1))$ , we deduce

$$(6.36) \quad \tilde{f} \in \mathcal{C}^0([0, T]; L^2(0, 1)),$$

$$(6.37) \quad \tilde{f}(0) = f_0 \quad \text{in } L^2(0, 1).$$

We have to check that  $\tilde{f}$  is the weak solution of system (1.1) on  $(0, T)$ . To do so, and according to Definition 1.2, let  $t' \in (0, T)$  and let  $\psi$  satisfying (1.2) and (1.3). Then, by (6.34) and since a pointwise solution is a weak solution (see Remark 3.2), we have, for every  $\sigma > 0$ ,

$$\begin{aligned} & \int_{\sigma}^{t'} \int_0^1 \tilde{f}(t, x) (\partial_t \psi + \partial_x (x^\alpha \partial_x \psi))(t, x) dt dx \\ &= \int_0^1 \tilde{f}(t', x) \psi(t', x) dx - \int_0^1 \tilde{f}(\sigma, x) \psi(\sigma, x) dx + \int_{\sigma}^{t'} u(t) (x^\alpha \partial_x \psi)(t, 1) dt. \end{aligned}$$

Then, from (6.33), (6.36), (6.37) and (1.2), taking  $\sigma \rightarrow 0^+$ , we get the conclusion.

Finally, by construction (6.35) implies that  $\tilde{f}(T, x) = 0$ , for every  $x \in (0, 1)$ . □

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#### APPENDIX A. SOME PROPERTIES OF THE GAMMA FUNCTION

For any  $p \in \mathbb{R}^+$ , the Gamma function is defined (see [1, 6.1.1, p.254]) by

$$(A.38) \quad \Gamma(p) := \int_0^{\infty} e^{-t} t^{p-1} dt,$$

which is a monotone increasing function on  $(0, \infty)$ . Furthermore, (see [1, 6.1.15, p.256])

$$(A.39) \quad \Gamma(x+1) = x\Gamma(x), \quad \forall x \in (0, \infty).$$

We have the following asymptotics of the Gamma function.

**LEMMA A.1** ([1, 6.1.39]). *Let  $a \in \mathbb{R}^+$  and  $b \in \mathbb{R}$ . Then,*

$$(A.40) \quad \Gamma(ax+b) \underset{x \rightarrow \infty}{\sim} \sqrt{2\pi} e^{-ax} (ax)^{ax+b-\frac{1}{2}}.$$

We show an inequality used in Proposition 3.3.

**LEMMA A.2.**

$$(A.41) \quad (n+k)! \leq 2^{k+n} n! k!, \quad \forall n, k \in \mathbb{N}.$$

*Proof.* Let us observe first that

$$(A.42) \quad (2n)! \leq 2^{2n} n!^2, \quad \forall n \in \mathbb{N}.$$

This inequality follows by induction, since, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} (2(n+1))! &= (2n)!(2n+1)(2n+2) \\ &\leq (2n)! 2^2 (n+1)^2 \leq 2^{2(n+1)} (n+1)!. \end{aligned}$$



To show (A.41), we assume, w.l.o.g., that  $n < k$ . Then, using (A.42),

$$\begin{aligned} (n+k)! &= (2n)! \prod_{j=1}^{k-n} (2n+j) \\ &\leq (2n)! 2^{k-n} \prod_{j=1}^{k-n} (n+j) \leq 2^{n+k} n! k!. \end{aligned}$$

□

## APPENDIX B. SOME PROPERTIES OF BESSEL FUNCTIONS

Let  $\nu \in \mathbb{R}$ . The Bessel function of order  $\nu$  of the first kind is ([1, 9.1.10, p.360])

$$(B.43) \quad J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad \forall z \in [0, \infty).$$

We denote by  $\{j_{\nu,n}\}_{n \in \mathbb{N}^*}$  the increasing sequence of zeros of  $J_\nu$ , which are real for any  $\nu \geq 0$  and enjoy the following properties (see [1, 9.5.2, p.370] and [15, Proposition 7.8, p.135]).

$$(B.44) \quad \nu < j_{\nu,n} < j_{\nu,n+1}, \quad \forall n \in \mathbb{N}^*,$$

$$(B.45) \quad j_{\nu,n+1} - j_{\nu,n} \rightarrow \pi, \quad \text{as } n \rightarrow \infty.$$

We also have the integral formula ([1, 11.4.5, p.485])

$$(B.46) \quad \int_0^1 y J_\nu(j_{\nu,n}y) J_\nu(j_{\nu,m}y) dy = \frac{1}{2} |J_{\nu+1}(j_{\nu,n})|^2 \delta_{n,m}, \quad \forall n, m \in \mathbb{N}^*.$$

This allows to show the following.

**LEMMA B.1.** [14, p.40] *Let  $\nu \geq 0$ . The family  $\{w_n\}_{n \in \mathbb{N}^*}$  defined by*

$$w_n(z) := \frac{\sqrt{2z}}{|J_{\nu+1}(j_{\nu,n})|} J_\nu(j_{\nu,n}z), \quad \forall z \in (0, 1),$$

*is an orthonormal basis of  $L^2(0, 1)$ . In particular, if  $f \in L^2(0, 1)$  and  $d_n := \int_0^1 f(z) w_n(z) dz$ ,  $\forall n \in \mathbb{N}^*$ , then  $\|f\|_{L^2(0,1)}^2 = \sum_{n=1}^{\infty} |d_n|^2$ .*

We recall that  $\forall \nu \in \mathbb{R}$ , the Bessel function  $J_\nu$  satisfies the following differential equation (see [1, 9.1.1, p.358])

$$(B.47) \quad z^2 J_\nu''(z) + z J_\nu'(z) + (z^2 - \nu^2) J_\nu(z) = 0, \quad \forall z \in (0, +\infty),$$

and the recurrence relation (see [1, 9.1.27, p.361]),

$$(B.48) \quad 2J_\nu'(z) = J_{\nu-1}(z) + J_{\nu+1}(z), \quad \forall z \in (0, +\infty).$$

**Asymptotic behaviour.** We recall the asymptotic behaviour of  $J_\nu$  for large arguments and near zero.

**LEMMA B.2.** [15, Lemma 7.2, p.129] *For any  $\nu \in \mathbb{R}$ ,*

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \underset{z \rightarrow \infty}{O}\left(\frac{1}{z\sqrt{z}}\right).$$

**LEMMA B.3.** [1, 9.1.7, p.360] *For any  $\nu \in \mathbb{R} \setminus \{-\mathbb{N}^*\}$ ,*

$$J_\nu(z) \underset{z \rightarrow 0}{\sim} \frac{z^\nu}{2^\nu \Gamma(\nu + 1)}.$$

The following asymptotic result is important in the proof of Proposition 5.1. We give the proof for the sake of completeness.

**LEMMA B.4.** *Let  $\nu \in \mathbb{R}^+$ . Then,*

$$(B.49) \quad \sqrt{j_{\nu,k}} |J_{\nu+1}(j_{\nu,k})| = \sqrt{\frac{2}{\pi}} + \underset{k \rightarrow \infty}{O}\left(\frac{1}{j_{\nu,k}}\right).$$

*In particular, there exists a constant  $C_1 > 0$  such that for all  $k \in \mathbb{N}^*$ ,*

$$\frac{1}{|J_{\nu+1}(j_{\nu,k})|} \leq C_1 \sqrt{j_{\nu,k}}.$$

*Proof.* Using Lemma B.2, for  $\nu + 1$  and  $x = j_{\nu,k}$ ,

$$\begin{aligned} \sqrt{j_{\nu,k}} |J_{\nu+1}(j_{\nu,k})| &= \sqrt{\frac{2}{\pi}} \left| \cos\left(j_{\nu,k} - \frac{\pi(\nu+1)}{2} - \frac{\pi}{4}\right) \right| + \underset{k \rightarrow \infty}{O}\left(\frac{1}{j_{\nu,k}}\right) \\ &= \sqrt{\frac{2}{\pi}} \left| \sin\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \right| + \underset{k \rightarrow \infty}{O}\left(\frac{1}{j_{\nu,k}}\right). \end{aligned}$$

Using again Lemma B.2 with  $\nu$  and  $x = j_{\nu,k}$ , we have that

$$\cos\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) = \underset{k \rightarrow \infty}{O}\left(\frac{1}{j_{\nu,k}}\right),$$

which gives

$$\left| \sin\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \right| = \sqrt{1 + \underset{k \rightarrow \infty}{O}\left(\frac{1}{j_{\nu,k}^2}\right)} = 1 + \underset{k \rightarrow \infty}{O}\left(\frac{1}{j_{\nu,k}}\right)$$

and then (B.49). □

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