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ON A VARIETY RELATED TO THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA.

JEAN-YVES CHARBONNEL

ABSTRACT. For a reductive Lie algebra over an algebraically closed field of characteristic zero, we consider a Borel subgroup B of its adjoint group, a Cartan subalgebra contained in the Lie algebra of B and the closure X of its orbit under B in the Grassmannian. The variety X plays an important role in the study of the commuting variety. In this note, we prove that X is Gorenstein with rational singularities.

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1. INTRODUCTION

In this note, the base field \mathbb{k} is algebraically closed of characteristic 0, \mathfrak{g} is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. As usual, \mathfrak{b} denotes a Borel subalgebra of \mathfrak{g} , \mathfrak{h} a Cartan subalgebra of \mathfrak{g} , contained in \mathfrak{b} , and B the normalizer of \mathfrak{b} in G .

1.1. **Main results.** Let X be the closure in $\text{Gr}_\ell(\mathfrak{g})$ of the orbit of \mathfrak{h} under the action of B . By a well known result, $G.X$ is the closure in $\text{Gr}_\ell(\mathfrak{g})$ of the orbit of \mathfrak{h} under the action of G . By [Ri79], the commuting variety of \mathfrak{g} is the image by the canonical projection of the restriction to $G.X$ of the canonical vector bundle of rank 2ℓ over $\text{Gr}_\ell(\mathfrak{g})$. So X and $G.X$ play an important role in the study of the commuting variety. As it is explained in [CZ16], X and $G.X$ play the same role for the so called generalized commuting varieties and the so called generalized isospectral commuting varieties. The main result of this note is the following theorem:

Theorem 1.1. *The variety X is Gorenstein with rational singularities.*

An induction is used to prove this theorem. So we introduce the categories \mathcal{C}'_t and \mathcal{C}_t with \mathfrak{t} a commutative Lie algebra of finite dimension. Their objects are nilpotent Lie algebras of finite dimension, normalized by \mathfrak{t} with additional conditions analogous to those of the action of \mathfrak{h} in \mathfrak{u} . In particular the minimal dimension

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of the objects in \mathcal{C}_t is the dimension of t and an object of dimension $\dim t$ is a commutative algebra. The category \mathcal{C}_t is a full subcategory of \mathcal{C}'_t . For α in \mathcal{C}'_t , we consider the solvable Lie algebra $\mathfrak{r} := t + \alpha$ and R the adjoint group of \mathfrak{r} . Denoting by X_R the closure in $\text{Gr}_{\dim t}(\mathfrak{r})$ of the orbit of t under R , we prove by induction on $\dim \alpha$ the following theorem:

Theorem 1.2. *The variety X_R is normal and Cohen-Macaulay.*

The result for the category \mathcal{C}'_t is easily deduced from the result for the category \mathcal{C}_t by Corollary 2.2. One of the key argument in the proof is the consideration of the fixed points under the action of R in X_R . As a matter of fact, since the closure of all orbit under R in X_R contains a fixed point, X_R is Cohen-Macaulay if so are the fixed points by openness of the set of Cohen-Macaulay points. Then, by Serre's normality criterion, it suffices to prove that X_R is smooth in codimension 1. For that purpose the consideration of the restriction to X_R of the tautological vector bundle of rank $\dim t$ over $\text{Gr}_{\dim t}(\mathfrak{r})$ is very useful.

For the study of the fixed points, we introduce Property **(P)** and Property **(P₁)** for the objects of \mathcal{C}'_t :

- Property **(P)** for α in \mathcal{C}'_t says that for V in X_R , contained in the centralizer \mathfrak{r}^s of an element s of t , V is in the closure of the orbit of t under the centralizer R^s of s in R ,
- Property **(P₁)** for α in \mathcal{C}'_t says that for V in X_R normalized by t and such that $V \cap t$ is the center of \mathfrak{r} , then the non zero weights of t in V are linearly independent.

Property **(P₁)** for α results from Property **(P)** for α and Property **(P)** for α results from Property **(P₁)** for α and Property **(P)** for the objects of \mathcal{C}'_t of dimension smaller than $\dim \alpha$. So, the main result for the objects of \mathcal{C}'_t is the following proposition:

Proposition 1.3. *The objects of \mathcal{C}'_t have Property **(P)**.*

From this proposition, we deduce some structure property for the points of X_R .

The second part of Theorem 1.1, that is Gorensteinness property and Rational singularities, is obtained by considering a subcategory $\mathcal{C}_{t,*}$ of \mathcal{C}_t . This category is defined by an additional condition on the objects. The main point for α in $\mathcal{C}_{t,*}$ is the following result:

Proposition 1.4. *Let $k \geq 2$ be an integer. Denote by $\mathcal{E}^{(k)}$ the R -equivariant vector subbundle of $X_R \times \mathfrak{r}^k$ whose fiber at t is t^k . Then there exists on the smooth locus of $\mathcal{E}^{(k)}$ a regular differential form of top degree without zero.*

From Proposition 1.4 and Theorem 1.2, we deduce that $\mathcal{E}^{(k)}$ and X_R are Gorenstein with rational singularities.

This note is organized as follows. In Section 2, categories \mathcal{C}'_t and \mathcal{C}_t are introduced for some space t . In particular, u is an object of \mathcal{C}_t . In Subsection 2.3, we define Property **(P)** for the objects of \mathcal{C}'_t and we deduce some result on the structure of points of X_R . In Subsection 2.4, we define Property **(P₁)** for the objects of \mathcal{C}'_t and we prove that Property **(P₁)** is a consequence of Property **(P)**. In Subsection 2.5, we give some geometric constructions to prove Property **(P)** by induction on the dimension of α . At last, in Subsection 2.6, we prove Proposition 1.3. In particular, the proof of [CZ16, Lemma 4.4,(i)] is completed. In Section 3, we are interested in the singular locus of X_R . In Subsection 3.3, regularity in codimension 1 is proved with some additional properties analogous to those of [CZ16, Section 3]. Moreover, the constructions of Subsection 2.5 are used to prove the results by induction on the dimension of α . In Section 4, Cohen-Macaulayness property is proved by induction. In Section 5, the category $\mathcal{C}_{t,*}$ is introduced and Proposition 1.4 is proved. Then with some results given in the appendix, we finish the proof of Theorem 1.1.

1.2. **Notations.** • An algebraic variety is a reduced scheme over \mathbb{k} of finite type. For X an algebraic variety, its smooth locus is denoted by X_{sm} .

• Set $\mathbb{k}^* := \mathbb{k} \setminus \{0\}$. For V a vector space, its dual is denoted by V^* .

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X , denote by \overline{Y} the closure of Y in X . For Y an open subset of the algebraic variety X , Y is called a *big open subset* if the codimension of $X \setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X , its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, \mathcal{O}_X is its structural sheaf, $\mathbb{k}[X]$ is the algebra of regular functions on X , $\mathbb{k}(X)$ is the field of rational functions on X when X is irreducible and Ω_X is the sheaf of regular differential forms of top degree on X when X is smooth and irreducible.

• If E is a subset of a vector space V , denote by $\text{span}(E)$ the vector subspace of V generated by E . The grassmannian of all d -dimensional subspaces of V is denoted by $\text{Gr}_d(V)$.

• For \mathfrak{a} a Lie algebra, V a subspace of \mathfrak{a} and x in \mathfrak{a} , V^x denotes the centralizer of x in V . For A a subgroup of the group of automorphisms of \mathfrak{a} , A^x denotes the centralizer of x in A . An element x of \mathfrak{g} is regular if \mathfrak{g}^x has dimension ℓ and the set of regular elements of \mathfrak{g} is denoted by $\mathfrak{g}_{\text{reg}}$.

• The Lie algebra of an algebraic torus is also called a torus. In this note, a torus denoted by a gothic letter means the Lie algebra of an algebraic torus.

• For \mathfrak{a} a Lie algebra, the Lie algebra of derivations of \mathfrak{a} is denoted by $\text{Der}(\mathfrak{a})$. By definition $\text{Der}(\mathfrak{a})$ is the Lie algebra of the group $\text{Aut}(\mathfrak{a})$ of the automorphisms of \mathfrak{a} .

• Let \mathfrak{b} be a Borel subalgebra of \mathfrak{g} , \mathfrak{h} a Cartan subalgebra of \mathfrak{g} contained in \mathfrak{b} and \mathfrak{u} the nilpotent radical of \mathfrak{b} .

2. ON SOLVABLE ALGEBRAS

Let \mathfrak{t} be a vector space of positive dimension d . Denote by $\tilde{\mathcal{C}}_{\mathfrak{t}}$ the subcategory of the category of finite dimensional Lie algebras whose objects are finite dimensional nilpotent Lie algebras \mathfrak{a} such that there exists a morphism

$$\mathfrak{t} \xrightarrow{\varphi_{\mathfrak{a}}} \text{Der}(\mathfrak{a})$$

whose image is the Lie algebra of a subtorus of $\text{Aut}(\mathfrak{a})$. For \mathfrak{a} and \mathfrak{a}' in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, a morphism ψ from \mathfrak{a} to \mathfrak{a}' is a morphism of Lie algebras such that $\psi \circ \varphi_{\mathfrak{a}}(t) = \varphi_{\mathfrak{a}'}(t) \circ \psi$ for all t in \mathfrak{t} . For x in \mathfrak{t} , x is a semisimple derivation of \mathfrak{a} . Denote by $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$ the set of weights of \mathfrak{t} in \mathfrak{a} . Let $\mathcal{C}'_{\mathfrak{t}}$ be the full subcategory of objects \mathfrak{a} of $\tilde{\mathcal{C}}_{\mathfrak{t}}$ verifying the following conditions:

- (1) 0 is not in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$,
- (2) for α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, the weight space of weight α has dimension 1,
- (3) for α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, $\mathbb{k}\alpha \cap (\mathcal{R}_{\mathfrak{t},\mathfrak{a}} \setminus \{\alpha\})$ is empty.

For \mathfrak{a} in $\mathcal{C}'_{\mathfrak{t}}$ and \mathfrak{a}' a subalgebra of \mathfrak{a} , invariant under the adjoint action of \mathfrak{t} , \mathfrak{a}' is in $\mathcal{C}'_{\mathfrak{t}}$. Denote by $\mathcal{C}_{\mathfrak{t}}$ the full subcategory of objects \mathfrak{a} of $\mathcal{C}'_{\mathfrak{t}}$ such that $\varphi_{\mathfrak{a}}$ is an embedding. For example \mathfrak{u} is in $\mathcal{C}_{\mathfrak{t}}$.

For \mathfrak{a} in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, denote by $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ the solvable algebra $\mathfrak{t} + \mathfrak{a}$, $\pi_{\mathfrak{t},\mathfrak{a}}$ the quotient morphism from $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ to \mathfrak{t} , $R_{\mathfrak{t},\mathfrak{a}}$ the adjoint group of $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$, $A_{\mathfrak{t},\mathfrak{a}}$ the connected closed subgroup of $R_{\mathfrak{t},\mathfrak{a}}$ whose Lie algebra is $\text{ad } \mathfrak{a}$, $X_{R_{\mathfrak{t},\mathfrak{a}}}$ the closure in $\text{Gr}_d(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$ of the orbit of \mathfrak{t} under $R_{\mathfrak{t},\mathfrak{a}}$ and $\mathcal{E}_{\mathfrak{t},\mathfrak{a}}$ the restriction to $X_{R_{\mathfrak{t},\mathfrak{a}}}$ of the tautological vector bundle over $\text{Gr}_d(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$. The variety $X_{R_{\mathfrak{t},\mathfrak{a}}}$ is called *the main variety related to $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$* . For α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, let \mathfrak{a}^{α} be the weight space of weight α under the action of \mathfrak{t} in \mathfrak{a} .

In the following subsections, a vector space \mathfrak{t} of positive dimension d and an object \mathfrak{a} of \mathcal{C}'_t are fixed. We set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \quad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}}, \quad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \quad R := R_{\mathfrak{t},\mathfrak{a}}, \quad A := A_{\mathfrak{t},\mathfrak{a}}, \quad n := \dim \mathfrak{a}.$$

Let \mathfrak{z} be the orthogonal complement of \mathcal{R} in \mathfrak{t} and $d^\#$ its codimension in \mathfrak{t} . Then $n \geq d^\#$.

2.1. General remarks on \mathcal{C}'_t . For x in \mathfrak{r} , we say that x is semisimple if so is $\text{ad } x$ and x is nilpotent if so is $\text{ad } x$. For \mathfrak{s} a commutative subalgebra of \mathfrak{r} , we say that \mathfrak{s} is a torus if $\text{ad } \mathfrak{s}$ is the Lie algebra of a subtorus of $\text{GL}(\mathfrak{r})$.

Lemma 2.1. *Let x be in \mathfrak{r} and \mathfrak{s} a commutative subalgebra of \mathfrak{r} .*

- (i) *The center of \mathfrak{r} is equal to \mathfrak{z} .*
- (ii) *The element x is semisimple if and only if $R.x \cap \mathfrak{t}$ is not empty.*
- (iii) *The element x is nilpotent if and only if x is in $\mathfrak{z} + \mathfrak{a}$.*
- (iv) *The algebra \mathfrak{a} is in \mathcal{C}'_t if and only if $\mathfrak{z} \cap \mathfrak{a} = \{0\}$. In this case, x has a unique decomposition $x = x_s + x_n$ with $[x_s, x_n] = 0$, x_s semisimple and x_n nilpotent.*
- (v) *The algebra \mathfrak{s} is a torus if and only if $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and $\pi(\mathfrak{s})$ is a subtorus of \mathfrak{t} . In this case, \mathfrak{s} and $\pi(\mathfrak{s})$ are conjugate under R .*

Proof. By definition $\text{ad } \mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ is an algebraic solvable subalgebra of $\text{gl}(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$ and $\text{ad } \mathfrak{t}$ is a maximal subtorus of $\text{ad } \mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$.

- (i) Let \mathfrak{z}' be the center of \mathfrak{r} . As $[\mathfrak{t}, \mathfrak{z}'] = \{0\}$,

$$\mathfrak{z}' = \mathfrak{z} \cap \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{z}' \cap \mathfrak{a}^\alpha.$$

So, by Condition (1), \mathfrak{z}' is contained in \mathfrak{t} . For t in \mathfrak{t} , t is in \mathfrak{z}' if and only if $\alpha(t) = 0$ for all α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, whence $\mathfrak{z}' = \mathfrak{z}$.

(ii) As the elements of \mathfrak{t} are semisimple by definition, the condition is sufficient since the set of semisimple elements of \mathfrak{r} is invariant under the adjoint action of R . Suppose that x is semisimple. By [Hu95, Ch. VII], for some g in R , $\text{Ad } g(x)$ is in $\text{ad } \mathfrak{t}$, whence $g(x)$ is in \mathfrak{t} by (i).

(iii) As $\text{ad } \mathfrak{a}$ is the set of nilpotent elements of $\text{ad } \mathfrak{r}$, x is in $\mathfrak{z} + \mathfrak{a}$ if and only if it is nilpotent by (i).

(iv) By definition, \mathfrak{z} is the kernel of $\varphi_{\mathfrak{a}}$. Hence $\mathfrak{z} = \{0\}$ if and only if \mathfrak{a} is in \mathcal{C}'_t . As $\text{ad } \mathfrak{r}$ is an algebraic subalgebra of $\text{gl}(\mathfrak{r})$, it contains the components of the Jordan decomposition of $\text{ad } x$. As a result, when \mathfrak{a} is in \mathcal{C}'_t , x has a unique decomposition $x = x_s + x_n$ with $[x_s, x_n] = 0$, x_s semisimple and x_n nilpotent.

(v) Suppose that \mathfrak{s} is a torus. By (i), $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and by [Hu95, Ch. VII], for some g in R , $\text{ad } g(\mathfrak{s})$ is contained in $\text{ad } \mathfrak{t}$ since $\text{ad } \mathfrak{t}$ is a maximal torus of $\text{ad } \mathfrak{r}$. Then, by (i), $g(\mathfrak{s})$ is a subtorus of \mathfrak{t} . Moreover, $g(\mathfrak{s}) = \pi(\mathfrak{s})$ since $g(y) - y$ is in \mathfrak{a} for all y in \mathfrak{r} . Conversely, if $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and $\pi(\mathfrak{s})$ is a subtorus of \mathfrak{t} , $\text{ad } \mathfrak{s}$ is conjugate to the subtorus $\text{ad } \pi(\mathfrak{s})$ of $\text{ad } \mathfrak{t}$ by [Hu95, Ch. VII] so that \mathfrak{s} and $\pi(\mathfrak{s})$ are conjugate under R . \square

Denoting by $\mathfrak{t}^\#$ a complement to \mathfrak{z} in \mathfrak{t} , \mathfrak{a} is an object of $\mathcal{C}'_{\mathfrak{t}^\#}$ since $\varphi_{\mathfrak{a}}(\mathfrak{t}) = \varphi_{\mathfrak{a}}(\mathfrak{t}^\#)$ and the restriction of $\varphi_{\mathfrak{a}}$ to $\mathfrak{t}^\#$ is injective. Set $\mathfrak{r}^\# := \mathfrak{t}^\# + \mathfrak{a}$ and denote by $R^\#$ the adjoint group of $\mathfrak{r}^\#$. Let $X_{R^\#}$ be the closure in $\text{Gr}_{d^\#}(\mathfrak{r}^\#)$ of the orbit of $\mathfrak{t}^\#$ under $R^\#$.

Corollary 2.2. *All element of X_R is a commutative algebra containing \mathfrak{z} . Moreover, the map*

$$X_{R^\#} \longrightarrow X_R, \quad V \longmapsto V \oplus \mathfrak{z}$$

is an isomorphism.

Proof. As the set of commutative subalgebras of dimension d of \mathfrak{r} is a closed subset of $\text{Gr}_d(\mathfrak{r})$ containing \mathfrak{t} and invariant under R , all element of X_R is a commutative algebra. According to Lemma 2.1(i), all element of $R \cdot \mathfrak{t}$ contains \mathfrak{z} and so does all element of X_R . For g in R , denote by \bar{g} the image of g in $R^\#$ by the restriction morphism. Then

$$g(\mathfrak{t}) = \bar{g}(\mathfrak{t}^\#) + \mathfrak{z} \quad \text{and} \quad \bar{g}(\mathfrak{t}^\#) = g(\mathfrak{t}) \cap \mathfrak{r}^\#.$$

Hence the map

$$X_{R^\#} \longrightarrow X_R, \quad V \longmapsto V \oplus \mathfrak{z}$$

is an isomorphism whose inverse is the map $V \mapsto V \cap \mathfrak{r}^\#$. \square

For \mathfrak{a} of dimension $d^\#$, $\mathcal{R} := \{\beta_1, \dots, \beta_{d^\#}\}$, and for I subset of $\{1, \dots, d^\#\}$, denote $X_{R,I}$ the image of \mathbb{k}^I by the map

$$\mathbb{k}^I \longrightarrow X_R, \quad (z_i, i \in I) \longmapsto \mathfrak{z} \oplus \text{span}(\{t_i + z_i x_i, i \in I\}) \oplus \bigoplus_{i \notin I} \alpha^{\beta_i}$$

with x_i in α^{β_i} for $i = 1, \dots, d^\#$ and $t_1, \dots, t_{d^\#}$ in \mathfrak{t} such that $\beta_i(t_j) = \delta_{i,j}$ for $1 \leq i, j \leq d^\#$, with $\delta_{i,j}$ the Kronecker symbol.

Lemma 2.3. *Suppose that \mathfrak{a} has dimension $d^\#$. Denote by $\beta_1, \dots, \beta_{d^\#}$ the elements of \mathcal{R} .*

- (i) *The algebra \mathfrak{a} is commutative.*
- (ii) *The set X_R is the union of $X_{R,I}$, $I \subset \{1, \dots, d^\#\}$.*

Proof. (i) As \mathfrak{z} has codimension $d^\#$ in \mathfrak{t} , $\beta_1, \dots, \beta_{d^\#}$ are linearly independent. Hence for $i \neq j$, $\beta_i + \beta_j$ is not in \mathcal{R} . As a result, \mathfrak{a} is commutative.

(ii) According to Corollary 2.2, we can suppose $d = d^\#$ so that t_1, \dots, t_d is the dual basis of β_1, \dots, β_d . For I subset of $\{1, \dots, d\}$, denote by I' the complement to I in $\{1, \dots, d\}$ and $\mathfrak{z}_{I'}$ the orthogonal complement to β_i , $i \in I'$ in \mathfrak{t} and set:

$$V_I := \mathfrak{z}_{I'} \oplus \bigoplus_{i \in I'} \alpha^{\beta_i}.$$

By (i), for i in I ,

$$\exp(z_1 \text{ad } x_1 + \dots + z_d \text{ad } x_d)(t_i) = t_i - z_i x_i.$$

Hence $X_{R,I}$ is the orbit of V_I under A and its closure in X_R is the union of $X_{R,J}$, $J \subset I$. As a result, X_R is the union of $X_{R,I}$, $I \subset \{1, \dots, d\}$ since $X_{R,\{1, \dots, d\}}$ is the orbit of \mathfrak{t} under A . \square

2.2. On some subsets of \mathcal{R} . For α in \mathcal{R} , let x_α be in $\alpha^\alpha \setminus \{0\}$. For Λ subset of \mathcal{R} , denote by \mathfrak{t}_Λ the intersection of the kernels of its elements and set:

$$\mathfrak{a}_\Lambda := \bigoplus_{\alpha \in \Lambda} \alpha^\alpha \quad \text{and} \quad \mathfrak{r}_\Lambda := \mathfrak{t} \oplus \mathfrak{a}_\Lambda.$$

When Λ has only one element α , set $\mathfrak{t}_\alpha := \mathfrak{t}_\Lambda$.

Definition 2.4. Let Λ be a subset of \mathcal{R} . We say that Λ is a complete subset of \mathcal{R} if it contains all element of \mathcal{R} whose kernel contains \mathfrak{t}_Λ

For Λ complete subset of \mathcal{R} , \mathfrak{a}_Λ is a subalgebra of \mathfrak{a} and \mathfrak{r}_Λ is a subalgebra of \mathfrak{r} . In particular, \mathfrak{a}_Λ is in \mathcal{C}'_1 . In this case, denote by R_Λ the connected closed subgroup of R whose Lie algebra is $\text{ad } \mathfrak{r}_\Lambda$.

Lemma 2.5. *Let Λ be a complete subset of \mathcal{R} , strictly contained in \mathcal{R} . Then \mathfrak{a}_Λ is contained in an ideal \mathfrak{a}' of \mathfrak{r} of dimension $\dim \mathfrak{a} - 1$ and contained in \mathfrak{a} .*

Proof. As Λ is complete and strictly contained in \mathcal{R} , \mathfrak{a}_Λ is a subalgebra of \mathfrak{r} , strictly contained in \mathfrak{a} . Then, by Lie's Theorem, there is a sequence

$$\mathfrak{a}_\Lambda = \mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_m = \mathfrak{a}$$

of subalgebras of \mathfrak{r} such that \mathfrak{a}_i is an ideal of codimension 1 of \mathfrak{a}_{i+1} for $i = 0, \dots, m-1$, whence the lemma. \square

For s in \mathfrak{t} , denote by Λ_s the subset of elements of \mathcal{R} whose kernel contains s .

Lemma 2.6. *Let s be in \mathfrak{t} .*

- (i) *The centralizer \mathfrak{r}^s of s in \mathfrak{r} is the direct sum of \mathfrak{t} and \mathfrak{a}_{Λ_s} .*
- (ii) *The center of \mathfrak{r}^s is equal to \mathfrak{t}_{Λ_s} .*

Proof. By definition, Λ_s is a complete subset of \mathcal{R} . Let x be in \mathfrak{r} . Then x has a unique decomposition

$$x = x_0 + \sum_{\alpha \in \mathcal{R}} c_\alpha x_\alpha$$

with x_0 in \mathfrak{t} and $c_\alpha, \alpha \in \mathcal{R}$ in \mathbb{k} .

(i) Since s is in \mathfrak{t} , x is in \mathfrak{r}^s if and only if $c_\alpha = 0$ for $\alpha \in \mathcal{R} \setminus \Lambda_s$, whence the assertion.

(ii) The algebra \mathfrak{a}_{Λ_s} is in \mathcal{C}'_t and \mathfrak{t}_{Λ_s} is the orthogonal complement to Λ_s in \mathfrak{t} . So, by (i) and Lemma 2.1(i), \mathfrak{t}_{Λ_s} is the center of \mathfrak{r}^s . \square

2.3. Property (P) for objects of \mathcal{C}_t . Let \mathbf{T} be the connected closed subgroup of R whose Lie algebra is $\text{ad } \mathfrak{t}$. For s in \mathfrak{t} , denote by X_R^s the subset of elements of X_R contained in \mathfrak{r}^s and $\overline{R^s \cdot \mathfrak{t}}$ the closure in $\text{Gr}_d(\mathfrak{r})$ of the orbit of \mathfrak{t} under R^s . Then $\overline{R^s \cdot \mathfrak{t}}$ is contained in X_R^s .

Definition 2.7. Say that \mathfrak{a} has Property (P) if X_R^s is equal to $\overline{R^s \cdot \mathfrak{t}}$ for all s in \mathfrak{t} .

By Corollary 2.2, \mathfrak{a} has Property (P) if and only if the object \mathfrak{a} of $\mathcal{C}_{\mathfrak{t}^\#}$ has Property (P).

Lemma 2.8. *If \mathfrak{a} has dimension $d^\#$, then \mathfrak{a} has Property (P).*

Proof. According to Corollary 2.2, we can suppose $d = d^\#$. Denote by β_1, \dots, β_d the elements of \mathcal{R} . Then β_1, \dots, β_d is a basis of \mathfrak{t}^* . Let t_1, \dots, t_d be the dual basis, s in \mathfrak{t} and V in X_R^s . By Lemma 2.3(ii), for some subset I of $\{1, \dots, d\}$, V is in $X_{R,I}$. Then for some $(z_i, i \in I)$,

$$V = \text{span}(\{t_i + z_i x_i \mid i \in I\}) \oplus \bigoplus_{i \in I'} \alpha^{\beta_i}$$

with I' the complement to I in $\{1, \dots, d\}$ and x_i in α^{β_i} for $i = 1, \dots, d$. Setting

$$I'' := I' \cup \{i \in I \mid z_i \neq 0\},$$

for i in $\{1, \dots, d\}$, i is in I'' if and only if $\beta_i(s) = 0$. So, by Lemma 2.5(i),

$$\mathfrak{r}^s = \mathfrak{t} \oplus \bigoplus_{i \in I''} \alpha^{\beta_i}.$$

Then by Lemma 2.3(ii), V is in $\overline{R^s \cdot \mathfrak{t}}$. \square

By definition, an *algebraic subalgebra* \mathfrak{k} of \mathfrak{r} is the semi-direct product of a torus \mathfrak{s} contained in \mathfrak{k} and $\mathfrak{k} \cap \mathfrak{a}$.

Lemma 2.9. *Suppose that \mathfrak{a} has Property (P). Let V be in X_R , x in V and y in \mathfrak{r} such that $\text{ad } y$ is the semisimple component of $\text{ad } x$. Then the center of \mathfrak{r}^y is contained in V .*

Proof. By Corollary 2.2, we can suppose a in \mathcal{C}_t so that y is the semisimple component of x by Lemma 2.1(iv). By Lemma 2.1(ii), for some g in R , $g(y)$ is in \mathfrak{t} . Denote by $\mathfrak{z}_{g(y)}$ the center of $\mathfrak{r}^{g(y)}$. By Lemma 2.6(ii), $\mathfrak{z}_{g(y)}$ is contained in \mathfrak{t} . As V is a commutative algebra, $g(V)$ is in $X_R^{g(y)}$. So, by Property (P), $\mathfrak{z}_{g(y)}$ is contained in $g(V)$ since $\mathfrak{z}_{g(y)}$ is in $k(\mathfrak{t})$ for all k in $R^{g(y)}$, whence the lemma. \square

Corollary 2.10. *Suppose that \mathfrak{a} has Property (P). Let V be in X_R . Then V is a commutative algebraic subalgebra of \mathfrak{r} and for some subset Λ of \mathcal{R} , the biggest torus contained in V is conjugate to \mathfrak{t}_Λ under R .*

Proof. According to Corollary 2.2, V is a commutative subalgebra of \mathfrak{r} and we can suppose $d = d^\#$. Let \mathfrak{s} be the set of semisimple elements of V . Then \mathfrak{s} is a subspace of V . By Lemma 2.9, V contains the semisimple components of its elements so that V is the direct sum of \mathfrak{s} and $V \cap \mathfrak{a}$. Let s be in \mathfrak{s} such that the center of \mathfrak{r}^s has maximal dimension. After conjugation by an element of R , we can suppose that s is in \mathfrak{t} . By Lemma 2.6(ii), \mathfrak{t}_{Λ_s} is the center of \mathfrak{r}^s . Hence, by Lemma 2.9, \mathfrak{t}_{Λ_s} is contained in \mathfrak{s} . Suppose that the inclusion is strict. A contradiction is expected. Let s' be in $\mathfrak{s} \setminus \mathfrak{t}_{\Lambda_s}$. Since V is contained in \mathfrak{r}^s , for some g in R^s , $g(s')$ is in \mathfrak{t} . Moreover, $g(\mathfrak{s})$ is the set of semisimple elements of $g(V)$ and \mathfrak{t}_{Λ_s} is contained in $g(\mathfrak{s})$. Denoting by Λ' the set of elements of Λ_s whose kernel contains $g(s')$, for some z in \mathbb{k}^* , Λ' is the set of elements of \mathcal{R} such that $\alpha(s + zg(s')) = 0$. By Lemma 2.9, $\mathfrak{t}_{\Lambda'}$ is contained in $g(V)$. So, by minimality of $|\Lambda_s|$, $\Lambda' = \Lambda_s$ and $g(s')$ is in \mathfrak{t}_{Λ_s} , whence the contradiction since $g(s')$ is in $g(\mathfrak{s}) \setminus \mathfrak{t}_{\Lambda_s}$. As a result, $\mathfrak{t}_{\Lambda_s} = \mathfrak{s}$ and $V = \mathfrak{t}_{\Lambda_s} + V \cap \mathfrak{a}$, whence the corollary. \square

2.4. Fixed points in X_R under \mathbf{T} and R . For V subspace of dimension d of \mathfrak{r} , denote by \mathcal{R}_V the set of elements β of \mathcal{R} such that α^β is contained in V , r_V the rank of \mathcal{R}_V and \mathfrak{z}_V its orthogonal complement in \mathfrak{t} so that $\dim \mathfrak{z}_V = d - r_V$. As $\text{Gr}_d(\mathfrak{r})$ and X_R are projective varieties, the actions of \mathbf{T} and R in these varieties have fixed points since \mathbf{T} and R are connected and solvable.

Definition 2.11. We say that \mathfrak{a} has Property (P₁) if for V fixed point under \mathbf{T} in X_R such that $V \cap \mathfrak{t} = \mathfrak{z}$, $r_V = |\mathcal{R}_V|$.

Lemma 2.12. *Suppose that \mathfrak{a} has Property (P). Let V be in $\text{Gr}_d(\mathfrak{r})$.*

(i) *The element V is a fixed point under \mathbf{T} in X_R if and only if V is a commutative subalgebra of \mathfrak{r} and*

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta.$$

In this case, $r_V = |\mathcal{R}_V|$.

(ii) *The element V is a fixed point under R in X_R if and only if V is a commutative ideal of \mathfrak{r} and \mathfrak{z} is the orthogonal complement of \mathcal{R}_V in \mathfrak{t} . In this case, $r_V = |\mathcal{R}_V| = d^\#$.*

Proof. If V is a fixed point under \mathbf{T} ,

$$V = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta.$$

(i) Suppose that V is a fixed point under \mathbf{T} in $X_R \setminus \{\mathfrak{t}\}$. Then \mathcal{R}_V is not empty. Let s be an element of \mathfrak{z}_V such that $\beta(s) \neq 0$ if β is not a linear combination of elements of \mathcal{R}_V . Then V is contained in \mathfrak{r}^s . So, by Property (P), V is in $\overline{R^s \cdot \mathfrak{t}}$. By Lemma 2.6(i), \mathfrak{z}_V is the center of \mathfrak{r}^s . Hence \mathfrak{z}_V is contained in V and $\mathfrak{z}_V = V \cap \mathfrak{t}$ since $V \cap \mathfrak{t}$ is contained in \mathfrak{z}_V . As a result, \mathfrak{z}_V has dimension $d - |\mathcal{R}_V|$ and $r_V = |\mathcal{R}_V|$.

Conversely, suppose that V is a commutative algebra and

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta.$$

Set:

$$\mathfrak{a}_V := \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta, \quad \mathfrak{r}_V := \mathfrak{t} \oplus \mathfrak{a}_V.$$

Then \mathfrak{a}_V is a commutative Lie algebra and \mathfrak{a}_V is in \mathcal{C}'_t . Moreover, \mathfrak{z}_V is the center of \mathfrak{r}_V by Lemma 2.1(i). By Lemma 2.3(ii), V is in the closure of the orbit of \mathfrak{t} under the action of the adjoint group of \mathfrak{r}_V in $\text{Gr}_d(\mathfrak{r}_V)$, whence the assertion.

(ii) The element V of $\text{Gr}_d(\mathfrak{r})$ is a fixed point under R if and only if V is an ideal of \mathfrak{r} . So, by (i), the condition is sufficient. Suppose that V is a fixed point under the action of R in X_R . By (i),

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta.$$

As V is an ideal of \mathfrak{r} , \mathfrak{z}_V is contained in the kernel of all elements of \mathcal{R} so that $\mathfrak{z}_V = \mathfrak{z}$. In particular, $|\mathcal{R}_V| = d^\#$ and the elements of \mathcal{R}_V are linearly independent. \square

2.5. On some varieties related to X_R . Let α' be an ideal of \mathfrak{r} of dimension $\dim \alpha - 1$ and contained in α . As a subalgebra of α normalized by \mathfrak{t} , α' is in \mathcal{C}'_t . Denote by \mathfrak{r}' the subalgebra $\mathfrak{t} + \alpha'$ of \mathfrak{r} , A' and R' the connected closed subgroups of R whose Lie algebras are $\text{ad } \alpha'$ and $\text{ad } \mathfrak{r}'$ respectively. Let $X_{R'}$ be the closure in $\text{Gr}_d(\mathfrak{r})$ of the orbit of \mathfrak{t} under R' and α the element of \mathcal{R} such that

$$\alpha = \alpha' \oplus \alpha^\alpha.$$

For δ in \mathcal{R} denote again by δ the character of \mathbf{T} whose differential at the identity is $\text{ad } x \mapsto \delta(x)$.

Setting:

$$\mathfrak{G}_{d-1,d,d,d+1} := \text{Gr}_{d-1}(\mathfrak{r}) \times \text{Gr}_d(\mathfrak{r}) \times \text{Gr}_d(\mathfrak{r}) \times \text{Gr}_{d+1}(\mathfrak{r}) \quad \text{and} \quad \mathfrak{G}_{d-1,d,d+1} := \text{Gr}_{d-1}(\mathfrak{r}) \times \text{Gr}_d(\mathfrak{r}) \times \text{Gr}_{d+1}(\mathfrak{r}),$$

denote by θ_α and θ'_α the maps

$$\mathbb{k} \times A' \xrightarrow{\theta_\alpha} \mathfrak{G}_{d-1,d,d,d+1}, \quad (z, g) \mapsto (g \cdot \mathfrak{t}_\alpha, g \cdot \mathfrak{t}, g \exp(\text{zad } x_\alpha) \cdot \mathfrak{t}, g \cdot (\mathfrak{t} + \alpha^\alpha)),$$

$$A' \xrightarrow{\theta'_\alpha} \mathfrak{G}_{d-1,d,d+1}, \quad g \mapsto (g \cdot \mathfrak{t}_\alpha, g \cdot \mathfrak{t}, g \cdot (\mathfrak{t} + \alpha^\alpha)).$$

Let I_α and S_α be the closures in $\text{Gr}_{d-1}(\mathfrak{r})$ and $\text{Gr}_{d+1}(\mathfrak{r})$ of the orbits of \mathfrak{t}_α and $\mathfrak{t} + \alpha^\alpha$ under A' respectively.

Lemma 2.13. *Let Γ and Γ' be the closures in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$ of the images of θ_α and θ'_α .*

(i) *The varieties Γ and Γ' have dimension n and $n - 1$ respectively. Moreover, they are invariant under the diagonal actions of R' in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$.*

(ii) *The image of Γ by the first, second, third and fourth projections are equal to I_α , $X_{R'}$, X_R , S_α respectively.*

(iii) *The set Γ' is the image of Γ by the projection*

$$\mathfrak{G}_{d-1,d,d,d+1} \xrightarrow{\varpi} \mathfrak{G}_{d-1,d,d+1}, \quad (V_1, V', V, W) \mapsto (V_1, V', W).$$

(iv) *For all (V_1, V', V, W) in Γ , V_1 is contained in $V' \cap V$ and $V' + V$ is contained in W .*

(v) *Let (V_1, V', V, W) be in Γ such that V' is contained in $\mathfrak{t}_\alpha + \alpha'$. Then W is contained in $\mathfrak{t}_\alpha + \alpha$.*

(vi) *Let (V_1, V', V, W) be in Γ such that V' is not contained in $\mathfrak{t}_\alpha + \alpha$. Then W is not commutative.*

Proof. (i) The maps θ_α and θ'_α are injective since \mathfrak{t} is the normalizer of \mathfrak{t} in \mathfrak{r} by Condition (1), whence $\dim \Gamma = n$ and $\dim \Gamma' = n - 1$. For (z, g, k) in $\mathbb{k} \times A' \times A'$, $\theta_\alpha(z, kg) = k \cdot \theta_\alpha(z, g)$ and $\theta'_\alpha(kg) = k \cdot \theta'_\alpha(g)$. Hence

Γ and Γ' are invariant under the diagonal action of A' in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$. Let k be in \mathbf{T} . For all (z, g) in $\mathbb{k} \times A'$,

$$\begin{aligned} kg.t_\alpha &= kgk^{-1}(t_\alpha), & kg.t &= kgk^{-1}(t), \\ kg.(t + \alpha^\alpha) &= kgk^{-1}.(t + \alpha^\alpha), & kg \exp(\text{zad } x_\alpha).t &= kgk^{-1} \exp(\text{zad } x_\alpha).t \end{aligned}$$

so that the images of θ_α and θ'_α are invariant under \mathbf{T} , whence the assertion.

(ii) Since $\text{Gr}_d(r)$, $\text{Gr}_{d-1}(r)$, $\text{Gr}_{d+1}(r)$ are projective varieties, the images of Γ by the first, second, third and fourth projections are closed subsets of their target varieties. Since the image of θ_α is contained in the closed subset $I_\alpha \times X_{R'} \times X_R \times S_\alpha$ of $\mathfrak{G}_{d-1,d,d,d+1}$, they are contained in I_α , $X_{R'}$, X_R and S_α respectively. By definition, $R'.t_\alpha$, $R'.t$ and $R'.(t + \alpha^\alpha)$ are contained in the images of Γ by the first, second and fourth projections and $R.t$ is contained in the image of Γ by the third projection since A is the image of $\mathbb{k} \times A'$ by the map $(z, g) \mapsto g \exp(\text{zad } x_\alpha)$, whence the assertion.

(iii) As $\text{Gr}_d(r)$ is a projective variety, $\varpi(\Gamma)$ is a closed subset of $\mathfrak{G}_{d-1,d,d+1}$ containing the image of θ'_α since $\varpi \circ \theta_\alpha(z, g) = \theta'_\alpha(g)$ for all (z, g) in $\mathbb{k} \times A'$. Moreover, Γ is contained in $\varpi^{-1}(\Gamma')$, whence $\Gamma' = \varpi(\Gamma)$.

(iv) The subset $\widetilde{\Gamma}$ of elements (V_1, V', V, W) of $\mathfrak{G}_{d-1,d,d,d+1}$ such that V_1 is contained in V' and V and such that V' and V are contained in W , is closed. For all (z, g) in $\mathbb{k} \times A'$,

$$g \exp(\text{zad } x_\alpha).(t + \alpha^\alpha) = g.(t + \alpha^\alpha).$$

Hence the image of θ_α and Γ are contained in $\widetilde{\Gamma}$ so that V_1 and $V + V'$ are contained in $V' \cap V$ and W respectively for all (V_1, V', V, W) in Γ .

(v) Denote by Γ_* the closure in $\text{Gr}_d(r) \times \text{Gr}_{d+1}(r)$ of the image of the map

$$A' \xrightarrow{\theta_{\alpha,*}} \text{Gr}_d(r) \times \text{Gr}_{d+1}(r), \quad g \mapsto (g(t), g(t + \alpha^\alpha)).$$

For all (T_1, T', T, T_2) in the image of θ_α , (T', T_2) is in the image of $\theta_{\alpha,*}$. Then Γ_* is the image of Γ by the projection

$$\mathfrak{G}_{d-1,d,d,d+1} \longrightarrow \text{Gr}_d(r) \times \text{Gr}_{d+1}(r), \quad (T_1, T', T, T_2) \mapsto (T', T_2).$$

Denote by τ the quotient morphism

$$r \xrightarrow{\tau} r/\alpha' = t + \alpha^\alpha.$$

For g in A' and x in r , $\tau \circ g(x) = \tau(x)$. Set:

$$X := \{(g, t, z, z', v, w) \in A' \times t_\alpha \times \mathbb{k}^2 \times r' \times r; | v = g(zs + t), w = g(zs + t + z'x_\alpha)\}$$

and denote by Y the closure in $r' \times r$ of the image of X by the canonical projection

$$A' \times t_\alpha \times \mathbb{k}^2 \times r' \times r \longrightarrow r' \times r.$$

As for all (g, t, z, z', v, w) in X ,

$$\tau(v) = zs + t \quad \text{and} \quad \tau(w) = zs + t + z'x_\alpha,$$

$$\alpha \circ \pi \circ \tau(v) = \alpha \circ \pi \circ \tau(w)$$

for all (v, w) in Y . By definition, for all (T, T') in Γ_* , $T \times T'$ is contained in Y . By hypothesis, V' is contained in the kernel of $\alpha \circ \pi$ and (V', W) is in Γ_* . Hence W is contained in the kernel of $\alpha \circ \pi$.

(vi) Denote by Γ'_* the closure in $\mathfrak{G}_{d-1,d,d,d+1} \times \text{Gr}_1(r)$ of the image of the map

$$\mathbb{k} \times A' \xrightarrow{\theta'_{\alpha,*}} \mathfrak{G}_{d-1,d,d,d+1} \times \text{Gr}_1(r), \quad (z, g) \mapsto (\theta_\alpha(z, g), g(\alpha^\alpha))$$

and Γ'_{**} the closure in $\text{Gr}_d(\mathfrak{r}') \times \text{Gr}_1(\mathfrak{r})$ of the image of the map

$$A' \longrightarrow \text{Gr}_d(\mathfrak{r}') \times \text{Gr}_1(\mathfrak{r}), \quad g \mapsto (g(\mathfrak{t}), g(\mathfrak{a}^\alpha)).$$

For all (T_1, T', T, T_2, T'_2) in the image of $\theta'_{\alpha, *}$, $T' + T'_2$ is contained in T_2 . Then so is it for all (T_1, T', T, T_2, T'_2) in Γ'_* . As $\mathfrak{G}_{d-1, d, d, d+1}$ and $\text{Gr}_1(\mathfrak{r})$ are projective varieties, Γ and Γ'_{**} are the images of Γ'_* by the projections

$$\mathfrak{G}_{d-1, d, d, d+1} \times \text{Gr}_1(\mathfrak{r}) \longrightarrow \mathfrak{G}_{d-1, d, d, d+1}, \quad (T_1, T', T, T_2, T'_2) \mapsto (T_1, T', T, T_2),$$

$$\mathfrak{G}_{d-1, d, d, d+1} \times \text{Gr}_1(\mathfrak{r}) \longrightarrow \text{Gr}_d(\mathfrak{r}') \times \text{Gr}_1(\mathfrak{r}), \quad (T_1, T', T, T_2, T'_2) \mapsto (T', T'_2)$$

respectively.

Set:

$$X' := \{(g, t, z, v, w) \in A' \times \mathfrak{t} \times \mathbb{k} \times \mathfrak{r}' \times \mathfrak{r} \mid v = g(t), w = g(zx_\alpha)\}$$

and denote by Y' the closure in $\mathfrak{r}' \times \mathfrak{r}$ of the image of X' by the canonical projection

$$A' \times \mathfrak{t} \times \mathbb{k} \times \mathfrak{r}' \times \mathfrak{r} \longrightarrow \mathfrak{r}' \times \mathfrak{r}.$$

As for all (g, t, z, v, w) in X' ,

$$[v, w] = g([t, zx_\alpha]) = \alpha(t)g(zx_\alpha) = \alpha \circ \pi(v)w,$$

$[v, w] = \alpha \circ \pi(v)w$ for all (v, w) in Y' . By definition, for all (T, T') in Γ'_{**} , $T \times T'$ is contained in Y' . For some W' in $\text{Gr}_1(\mathfrak{r})$, (V_1, V', V, W, W') is in Γ'_* . By hypothesis, V' is not contained in the kernel of $\alpha \circ \pi$. Hence, for some v in V' and w in $W \setminus \{0\}$, $\alpha \circ \pi(v) \neq 0$ and $[v, w] = \alpha \circ \pi(v)w$. \square

Corollary 2.14. *Suppose that \mathfrak{a}' has Property **(P)**. Let s be in \mathfrak{t} such that \mathfrak{r}^s is contained in \mathfrak{a}' and (V_1, V', V, W) be in Γ such that V is contained in \mathfrak{r}^s and $[s, V']$ is contained in V' .*

(i) *If W is not commutative then $V' = V$ and V is in $\overline{R^s \cdot \mathfrak{t}}$.*

(ii) *Suppose that for some v in \mathfrak{a} , $s + v$ is in V . Then $V' = V$ and V is in $\overline{R^s \cdot \mathfrak{t}}$.*

Proof. By Lemma 2.13(ii), V and V' are in X_R and $X_{R'}$ respectively.

(i) If $V' = V$, V is in $\overline{R^s \cdot \mathfrak{t}}$ by Property **(P)** for \mathfrak{a}' . Suppose $V' \neq V$. A contradiction is expected. Then, by Lemma 2.13(iv), for some x and y in W ,

$$V = V_1 \oplus \mathbb{k}x, \quad V' = V_1 \oplus \mathbb{k}y, \quad W = V_1 \oplus \mathbb{k}x \oplus \mathbb{k}y.$$

Moreover, as V is contained in \mathfrak{r}^s and $[s, V'] \subset V'$, W is contained in \mathfrak{r}' and we can choose y so that $[s, y] \in \mathbb{k}y$. Since V and V' are commutative subalgebras of \mathfrak{r} , $[x, y] \neq 0$. We have two cases to consider:

(a,1) V' is contained in \mathfrak{r}^s ,

(a,2) V' is not contained in \mathfrak{r}^s .

(a,1) By Property **(P)** for \mathfrak{a}' , s is in V' . Hence $s = ty + v$ for some in (t, v) in $\mathbb{k} \times V_1$. As V is a commutative subalgebra of \mathfrak{r}^s , containing V_1 and x ,

$$0 = [x, s] = t[x, y].$$

Then $s = v$ is in V_1 , whence a contradiction since $\alpha(s) \neq 0$ and V_1 is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ by Lemma 2.13(ii).

(a,2) For some a in \mathbb{k}^* , $[s, y] = ay$. Then y is in \mathfrak{a}' and V' is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ since so is V_1 . As a result, by Lemma 2.13(v), V and W are contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ since V is contained in \mathfrak{a}' . As $[s, [x, y]] = a[x, y]$, $[x, y] = by$ for some b in \mathbb{k}^* since the eigenspace of eigenvalue a of the restriction of $\text{ad } s$ to V' is generated by y . Then $\text{ad } x$ is not nilpotent. Let x_s be in \mathfrak{r}' such that $\text{ad } x_s$ is the semisimple component of $\text{ad } x$. Then x_s is in $\mathfrak{t}_\alpha + \mathfrak{a}'$, $[s, x_s] = 0$ and $[x_s, V_1] = \{0\}$ since $[s, x] = 0$ and $[x, V_1] = \{0\}$. Moreover, $[ax_s - bs, y] = 0$. Then $ax_s - bs$ is a semisimple element of \mathfrak{r}' such that $[ax_s - bs, V'] = \{0\}$. As it is conjugate under R' to an

element of \mathfrak{t} by Lemma 2.1(ii), by Property **(P)** for α' , $ax_s - bs$ is in V' , whence a contradiction since V' is contained in $\mathfrak{t}_\alpha + \alpha'$ and $ax_s - bs$ is not in $\mathfrak{t}_\alpha + \alpha'$.

(ii) If $V = V'$, V is in $\overline{R^s \cdot \mathfrak{t}}$ by Property **(P)** for α' . Suppose $V \neq V'$. A contradiction is expected. As V is contained in r^s , $[s, v] = 0$. Let x and y be as in (i). As V_1 is contained in $\mathfrak{t}_\alpha + \alpha'$, $s + v$ is not in V_1 since $\alpha(s) \neq 0$. So we can choose $s + v = x$. By (i), W is commutative. Then $[s + v, y] = 0$ and $[\text{ad } s, \text{ad } y] = 0$ since $\text{ad } s$ is the semisimple component of $\text{ad}(s + v)$. Hence, by Lemma 2.1(i), $[s, y] = 0$ since $[s, y]$ is in \mathfrak{a} . As a result, V' is contained in r^s since so is V_1 . So, by Property **(P)** for α' , s is in V' and W is not commutative by Lemma 2.13(vi), whence a contradiction. \square

For (T_1, T', T_2) in Γ' , denote by Γ_{T_1, T', T_2} the subset of elements (T_1, T', T, T_2) of $\mathfrak{G}_{d-1, d, d, d+1}$ such that T is contained in T_2 and contains T_1 . Then Γ_{T_1, T', T_2} is a closed subvariety of $\mathfrak{G}_{d-1, d, d, d+1}$, isomorphic to $\mathbb{P}^1(\mathbb{k})$. Let (V_1, V', V, W) be a fixed point under \mathbf{T} of Γ .

Lemma 2.15. (i) *For some affine open neighborhood Ω of (V_1, V', W) in Γ' , Ω is invariant under \mathbf{T} .*

(ii) *For $i = 0, \dots, n - 2$, there exist Y_i and O_i such that*

- (a) *Y_i is an irreducible closed subset of dimension $n - 1 - i$ of Ω , containing (V_1, V', W) and invariant under \mathbf{T} ,*
- (b) *O_i is a locally closed subvariety of dimension $n - 1 - i$ of A' , invariant under \mathbf{T} by conjugation,*
- (c) *$\theta'_\alpha(O_i)$ is contained in Y_i and (V_1, V', V, W) is in the closure of $\theta_\alpha(\mathbb{k} \times O_i)$ in Γ .*

(iii) *There exist a smooth projective curve C , an action of \mathbf{T} on C , x_1, \dots, x_m in C and two morphisms*

$$C \setminus \{x_1, \dots, x_m\} \xrightarrow{\mu} A', \quad C \xrightarrow{\nu} \Gamma'$$

satisfying the following conditions:

- (a) *x_1, \dots, x_m are the fixed points under \mathbf{T} in C ,*
- (b) *for g in \mathbf{T} and x in $C \setminus \{x_1, \dots, x_m\}$, $\mu(g.x) = g\mu(x)g^{-1}$ and $\nu(g.x) = g.\nu(x)$,*
- (c) *$\nu(x_1) = (V_1, V', W)$,*
- (d) *(V_1, V', V, W) is in the closure of the image of $\mathbb{k} \times (C \setminus \{x_1, \dots, x_m\})$ by the map $(z, x) \mapsto \theta_\alpha(z, \mu(x))$.*

Proof. (i) As Γ' is a projective variety with a \mathbf{T} action and (V_1, V', W) is a fixed point under \mathbf{T} , there exists an affine open neighborhood Ω of (V_1, V', W) in Γ' , invariant under \mathbf{T} .

(ii) Prove the assertion by induction on i . For $i = 0$, $Y_i = \Omega$ and O_i is the inverse image of Ω by θ'_α . Suppose that Y_i and O_i are known. Let Y'_i be the closure in Γ of $\theta_\alpha(\mathbb{k} \times O_i)$. By Condition (c), Y'_i is invariant under \mathbf{T} and $\theta_\alpha(\mathbb{k} \times O_i)$ is a \mathbf{T} -invariant dense subset of Y'_i . So, it contains an \mathbf{T} -invariant dense open subset O'_i of Y'_i . As θ'_α is an orbital injective morphism, $\theta'_\alpha(O_i)$ is a dense open subset of Y_i . Set:

$$Z' := Y'_i \setminus O'_i, \quad Z := Y_i \setminus \theta_\alpha(O'_i), \quad Z_* := \Omega \cap (\varpi(Z) \cup Z')$$

Then Z_* is a \mathbf{T} -invariant closed subset of Y_i , containing (V_1, V', W) .

Denote by Z_{**} the union of irreducible components of dimension $\dim Y_i - 1$ of Z_* and I the union of the ideals of definition in $\mathbb{k}[Y_i]$ of the irreducible components of Z_{**} . Let p be in $\mathbb{k}[Y_i] \setminus I$, semiinvariant under \mathbf{T} and such that $p((V_1, V', W)) = 0$. Denote by Y'_{i+1} an irreducible component of the nullvariety of p in $Y'_i \cap \varpi^{-1}(\Omega)$, containing (V_1, V', V, W) and Y_{i+1} the closure in Ω of $\varpi(Y'_{i+1})$. Then Y_{i+1} has dimension $n - i - 1$ and its intersection with $\theta'_\alpha(O_i)$ is not empty so that $O_{i+1} := \theta'^{-1}_\alpha(O_i \cap \theta'_\alpha(O_i))$ is a nonempty locally closed subset of dimension $n - i - 1$ of A' . Moreover, Y_{i+1} and O_{i+1} are invariant under \mathbf{T} since p is semiinvariant under \mathbf{T} . As $\theta_\alpha(\mathbb{k} \times O_{i+1})$ is the intersection of Y'_{i+1} and $\theta_\alpha(\mathbb{k} \times O_i)$, it is dense in Y'_{i+1} so that (V_1, V', V, W) is in the closure of $\theta_\alpha(\mathbb{k} \times O_{i+1})$ and (V_1, V', W) is in Y_{i+1} .

(iii) Let $\overline{Y_{n-2}}$ be the closure of Y_{n-2} , C its normalization and ν the normalization morphism. Then C is a smooth projective curve. As Y_{n-2} is invariant under \mathbf{T} , so is $\overline{Y_{n-2}}$ and there is an action of \mathbf{T} on C such that ν is an equivariant morphism. As the restriction of θ'_α to O_{n-2} is an isomorphism onto a dense open subset of Y_{n-2} , the actions of \mathbf{T} on $\overline{Y_{n-2}}$ and C are not trivial since θ'_α is equivariant under the actions of \mathbf{T} . As a result, \mathbf{T} has an open orbit O_* in $\overline{Y_{n-2}}$ and $\overline{Y_{n-2}} \setminus O_*$ is the set of fixed points under \mathbf{T} of $\overline{Y_{n-2}}$ since $\overline{Y_{n-2}}$ has dimension 1. Hence the restriction of ν to $\nu^{-1}(O_*)$ is an isomorphism, $C \setminus \nu^{-1}(O_*)$ is finite, its elements are fixed under \mathbf{T} and there exists a \mathbf{T} -equivariant morphism μ from $\nu^{-1}(O_*)$ to A' such that $\theta'_\alpha \circ \mu = \nu$. As (V_1, V', W) is a fixed point under \mathbf{T} , for some x_1 in $C \setminus \nu^{-1}(O_*)$, $\nu(x_1) = (V_1, V', W)$ since (V_1, V', W) is in $\nu(C)$. Moreover, (V_1, V', V, W) is in the closure of the map

$$\mathbb{k} \times (C \setminus \nu^{-1}(O_*)) \longrightarrow \Gamma, \quad (z, x) \mapsto \theta_\alpha(z, \mu(x))$$

since it is in $\overline{\theta_\alpha(\mathbb{k} \times O_{n-2})}$. □

Denote by η the morphism

$$\mathbb{k} \times (C \setminus \{x_1, \dots, x_m\}) \xrightarrow{\eta} \Gamma, \quad (z, x) \mapsto \theta_\alpha(z, \mu(x))$$

and Δ the closure of the graph of η in $\mathbb{P}^1(\mathbb{k}) \times C \times \Gamma$. Let ν be the restriction to Δ of the canonical projection

$$\mathbb{P}^1(\mathbb{k}) \times C \times \Gamma \longrightarrow \mathbb{P}^1(\mathbb{k}) \times C$$

and for (z, x) in $\mathbb{P}^1(\mathbb{k}) \times C$, let $F_{z,x}$ be the subset of Γ such that $\{(z, x)\} \times F_{z,x}$ is the fiber of ν at (z, x) . We have an action of \mathbf{T} in $\mathbb{P}^1(\mathbb{k})$ given by

$$t.z := \begin{cases} \alpha(t)z & \text{if } z \in \mathbb{k}^* \\ z & \text{if } z \in \{0, \infty\} \end{cases}.$$

Lemma 2.16. *Let Δ_ν be the graph of ν .*

(i) *The set Δ_ν is the image of Δ by the map $(z, x, y) \mapsto (x, \varpi(y))$.*

(ii) *For t in \mathbf{T} and (z, x, y) in Δ , $t.(z, x, y) := (t.z, t.x, t.y)$ is in Δ .*

(iii) *For (z, x) in $\mathbb{P}^1(\mathbb{k}) \times C$, η is regular at (z, x) if and only if $F_{z,x}$ has dimension 0. In this case, $|F_{z,x}| = 1$.*

(iv) *For (z, x) in $\mathbb{P}^1(\mathbb{k}) \times C \setminus \{0, \infty\} \times \{x_1, \dots, x_m\}$, η is regular at (z, x) .*

(v) *For $i = 1, \dots, m$, there exists a regular map η_i from $\mathbb{P}^1(\mathbb{k})$ to Γ such that $\eta_i(z) = \eta(z, x_i)$ for all z in \mathbb{k}^* .*

Moreover, its image is contained in $\varpi^{-1}(\{\nu(x_i)\}) \cap \Gamma$.

Proof. (i) As $\mathbb{P}^1(\mathbb{k})$ and $\text{Gr}_d(r)$ are projective varieties, the image of Δ by the map $(z, x, y) \mapsto (x, \varpi(y))$ is a closed subset of $C \times \Gamma'$ contained in Δ_ν since $\varpi \circ \eta(z, x) = \nu(x)$ for all (z, x) in $\mathbb{k} \times (C \setminus \{x_1, \dots, x_m\})$, whence the assertion since the inverse image of Δ_ν by this map is a closed subset of $\mathbb{P}^1(\mathbb{k}) \times C \times \Gamma$ containing the graph of η .

(ii) From the equality

$$t \exp(\text{zad } x_\alpha) t^{-1} = \exp(\alpha(t) \text{zad } x_\alpha)$$

for all (t, z) in $\mathbf{T} \times \mathbb{k}$, we deduce the equality

$$t.\eta(z, x) = t.\theta_\alpha(z, \mu(x)) = \theta_\alpha(\alpha(t)z, \mu(t.x)) = \eta(t.z, t.x)$$

for all (t, z, x) in $\mathbf{T} \times \mathbb{k} \times (C \setminus \{x_1, \dots, x_m\})$ since θ_α and μ are \mathbf{T} -equivariant, whence the assertion.

(iii) As Γ is a projective variety, ν is a projective morphism. Moreover, it is birational since Δ is the closure of the graph of η . So, by Zariski's Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of ν are connected of dimension 0 or 1 since $\mathbb{P}^1(\mathbb{k}) \times C$ is normal of dimension 2. Let (z, x) be in $\mathbb{P}^1(\mathbb{k}) \times C$ such that $F_{z,x}$ dimension 0. There exists a neighborhood $O_{z,x}$ of (z, x) in $\mathbb{P}^1(\mathbb{k}) \times C$ such that F_y has dimension 0 for y

in $O_{z,x}$. In other words, the restriction of ν to $\nu^{-1}(O_{z,x})$ is a quasi finite morphism. Moreover, it is birational and surjective. So, again by Zariski's Main Theorem [Mu88, §9], it is an isomorphism. Hence η is regular at (z, x) . Conversely, if η is regular at (z, x) , $\eta(z, x)$ is an isolated point in $F_{z,x}$, whence $F_{z,x} = \{\eta(z, x)\}$ since $F_{z,x}$ is connected.

(iv) The variety $\mathbb{k} \times (C \setminus \{x_1, \dots, x_m\})$ is an open subset of the smooth variety $\mathbb{P}^1(\mathbb{k}) \times C$ and Γ is a projective variety. Hence η has a regular extension to a big open subset of $\mathbb{P}^1(\mathbb{k}) \times C$ by [Sh94, Ch. 6, Theorem 6.1]. By Condition (a) of Lemma 2.15(iii), $\{0, \infty\} \times \{x_1, \dots, x_m\}$ is the set of fixed points under \mathbf{T} in $\mathbb{P}^1(\mathbb{k}) \times C$ and by (ii), $t.\eta(z, x) = \eta(t.z, t.x)$ for all (t, z, x) in $\mathbf{T} \times \mathbb{P}^1(\mathbb{k}) \times (C \setminus \{x_1, \dots, x_m\})$. Hence η is regular on $\mathbb{P}^1(\mathbb{k}) \times C \setminus \{0, \infty\} \times \{x_1, \dots, x_m\}$.

(v) The restriction of η to $\mathbb{k}^* \times \{x_i\}$ is a regular map from a dense open subset of the smooth variety $\mathbb{P}^1(\mathbb{k}) \times \{x_i\}$ to the projective variety Γ . So, again by [Sh94, Ch. 6, Theorem 6.1], this map has regular extension to $\mathbb{P}^1(\mathbb{k}) \times \{x_i\}$, whence the assertion by (i). \square

Let I be the set of indices such that $\nu(x_i) = (V_1, V', W)$. Denote by S the image of Δ by the canonical projection $\mathbb{P}^1(\mathbb{k}) \times C \times \Gamma \longrightarrow \Gamma$, S_n its normalization, σ the normalization morphism, $S^{\mathbf{T}}$ and $S_n^{\mathbf{T}}$ the sets of fixed points under \mathbf{T} in S and S_n respectively. Set

$$C_* := \mathbb{P}^1(\mathbb{k}) \times C \setminus \{(0, \infty) \times \{x_1, \dots, x_m\}\}.$$

By Lemma 2.15(iv), η is a dominant morphism from C_* to S , whence a commutative diagram

$$\begin{array}{ccc} C_* & \xrightarrow{\eta_n} & S_n \\ & \searrow \eta & \downarrow \sigma \\ & & S \end{array}$$

since C_* is smooth. Let Δ_n be the closure in $\mathbb{P}^1(\mathbb{k}) \times C \times S_n$ of the graph of η_n and ν_2 the restriction to Δ_n of the canonical projection

$$\mathbb{P}^1(\mathbb{k}) \times C \times S_n \longrightarrow S_n .$$

Lemma 2.17. *Suppose $V' \neq V$ and V and V' contained in $\mathfrak{z} + \mathfrak{a}$.*

- (i) *The variety Δ is the image of Δ_n by the map $(z, x, y) \mapsto (z, x, \sigma(y))$.*
- (ii) *The morphism ν_2 is projective and birational.*
- (iii) *There exists a \mathbf{T} -equivariant morphism*

$$(S_n \setminus S_n^{\mathbf{T}}) \xrightarrow{\varphi} C_*$$

such that $\eta \circ \varphi$ is the restriction of σ to $S_n \setminus S_n^{\mathbf{T}}$.

- (iv) *For some i in I , $\eta_i(1)$ is not invariant under \mathbf{T} .*

Proof. (i) As S is a projective variety, so are S_n , $\mathbb{P}^1(\mathbb{k}) \times C \times S_n$, Δ_n and the image of Δ_n by the map $(z, x, y) \mapsto (z, x, \sigma(y))$, whence the assertion since the image of the graph of η_n by this map is the graph of η .

(ii) As Δ_n is projective so is ν_2 . Since θ_α is injective, so is the restriction of η to $\mathbb{k} \times (C \setminus \{x_1, \dots, x_m\})$. Hence ν_2 is birational.

(iii) By (ii) and Zariski's Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of ν_2 are connected. For y in $S_n \setminus S_n^{\mathbf{T}}$ and (z, x) in $\mathbb{P}^1(\mathbb{k}) \times C$ such that (z, x, y) is in Δ_n , $\varpi \circ \sigma(y) = \nu(x)$ by (i). If x is not in $\{x_1, \dots, x_m\}$, $\nu^{-1}(\varpi \circ \sigma(y)) = \{x\}$ by Condition (b) of Lemma 2.15(iii) and z is the element of \mathbb{k} such that $\theta_\alpha(z, \mu(x)) = \sigma(y)$. Suppose $x = x_i$ for some $i = 1, \dots, m$. Let z and z' be in \mathbb{k}^* such that (z, x_i, y) and (z', x_i, y) are in Δ_n . Then $(z, x_i, \sigma(y))$ and $(z', x_i, \sigma(y))$ are in Δ . By Lemma 2.16(iii) and (iv), $\sigma(y) = \eta(z, x_i) = \eta(z', x_i)$. For some t

in \mathbf{T} , $z' = t.z$ so that $t.\sigma(y) = \sigma(y)$. As y is not invariant under \mathbf{T} so is $\sigma(y)$ since the fibers of σ are finite. Hence the stabilizer of $\sigma(y)$ in \mathbf{T} is finite and so is the fiber of ν_2 at y . As a result, the restriction of ν_2 to $\Delta_n \setminus \mathbb{P}^1(\mathbb{k}) \times C \times S_n^{\mathbf{T}}$ is an injective morphism. So, again by Zariski's Main Theorem [Mu88, §9], this map is an isomorphism, whence a morphism

$$(S_n \setminus S_n^{\mathbf{T}}) \xrightarrow{\varphi} C_* .$$

Moreover, φ is \mathbf{T} -equivariant since so is ν_2 . For y in S_n such that $\sigma(y) = \eta(z, x)$ for some (z, x) in $\mathbb{k}^* \times (C \setminus \{x_1, \dots, x_m\})$, (z, x, y) is the unique element of Δ_n above y . Hence $\eta \circ \varphi = \sigma$.

(iv) Suppose that for all i in I , $\eta_i(1)$ is invariant under \mathbf{T} . A contradiction is expected. As $V \neq V'$, $V_1 = V \cap V'$ and $V + V' = W$ by Lemma 2.13(iv). Moreover, since V and V' are contained in $\mathfrak{z} + \mathfrak{a}$, for some β and γ in \mathcal{R} ,

$$V = V_1 + \mathfrak{a}^\beta \quad \text{and} \quad V' = V_1 + \mathfrak{a}^\gamma .$$

Then $\Gamma_{V_1, V', W}$ is invariant under \mathbf{T} . More precisely, $\Gamma_{V_1, V', W}$ is a union of one orbit of dimension 1 and the set $\{(V_1, V', V', W), (V_1, V', V, W)\}$ of fixed points. As a result, $\Gamma_{V_1, V', W} \cap S$ is equal to $\{(V_1, V', V', W), (V_1, V', V, W)\}$ or $\Gamma_{V_1, V', W}$ since S is invariant under \mathbf{T} . By Lemma 2.16,(ii) and (v), for i in I , $\eta_i(\mathbb{P}^1(\mathbb{k}))$ is equal to (V_1, V', V', W) or (V_1, V', V, W) since $\nu(x_i) = (V_1, V', W)$.

Suppose $\Gamma_{V_1, V', W} \cap S = \{(V_1, V', V', W), (V_1, V', V, W)\}$. By Lemma 2.16,(v) and (iii), for all i in I , η is regular at $(0, x_i)$ and (∞, x_i) since $\nu(x_i) = (V_1, V', W)$, whence

$$\lim_{z \rightarrow 0} \eta_i(0) = (V_1, V', V', W) \quad \text{and} \quad \lim_{z \rightarrow \infty} \eta_i(\infty) = (V_1, V', V, W),$$

whence a contradiction.

Suppose $\Gamma_{V_1, V', W} \cap S = \Gamma_{V_1, V', W}$. Let y be in S_n such that

$$\sigma(y) \in \Gamma_{V_1, V', W} \setminus \{(V_1, V', V', W), (V_1, V', V, W)\} .$$

By (iii), for some i in I and some z in \mathbb{k}^* , $\varphi(t.y) = (t.z, x_i)$ and $t.\sigma(y) = t.\eta(z, x_i) = t.\eta_i(z)$ for all t in \mathbf{T} , whence a contradiction since (V_1, V', V', W) and (V_1, V', V, W) are in $\overline{\mathbf{T}.\sigma(y)}$. \square

Corollary 2.18. *Let (V_1, V', V, W) be a fixed point under \mathbf{T} of Γ such that $V \cap \mathfrak{t} = V' \cap \mathfrak{t} = \mathfrak{z}$. Then $V' = V$.*

Proof. Suppose $V' \neq V$. A contradiction is expected. By Lemma 2.13(iv), $V_1 = V \cap V'$ and $W = V + V'$. As $V \cap \mathfrak{t} = V' \cap \mathfrak{t} = \mathfrak{z}$, V and V' are contained in $\mathfrak{z} + \mathfrak{a}$. So, for some β in \mathcal{R} and γ in $\mathcal{R} \setminus \{\alpha\}$,

$$V = V_1 \oplus \mathfrak{a}^\beta \quad \text{and} \quad V' = V_1 \oplus \mathfrak{a}^\gamma .$$

since (V_1, V', V, W) is invariant under \mathbf{T} . By Lemma 2.17(iv), for some i in I , $\eta_i(1)$ is not fixed under \mathbf{T} . Then, by Lemma 2.13(ii), $\eta_i(\mathbb{P}^1(\mathbb{k})) = \Gamma_{V_1, V', W}$. Denoting by $\eta_i(z)_3$ the third component of $\eta_i(z)$, for all z in $\mathbb{P}^1(\mathbb{k})$, V_1 is contained in $\eta_i(z)_3$ and $\eta_i(z)_3$ is contained in W . Hence for some a in \mathbb{k}^* ,

$$\eta_i(1)_3 = V_1 \oplus \mathbb{k}(x_\beta + ax_\gamma) \quad \text{and} \quad \eta_i(\alpha(t))_3 = V_1 \oplus \mathbb{k}(\beta(t)x_\beta + \gamma(t)ax_\gamma)$$

for all t in \mathbf{T} . For some t_1 and t_2 in \mathbf{T} , for all δ in \mathcal{R} , $\delta(t_1)$ and $\delta(t_2)$ are positive rational numbers and

$$\alpha(t_1) > 1, \quad \alpha(t_2) > 1, \quad \beta(t_1) < \gamma(t_1), \quad \beta(t_2) > \gamma(t_2).$$

Then

$$\lim_{k \rightarrow \infty} V_1 \oplus \mathbb{k}(\beta(t_1^k)x_\beta + \gamma(t_1^k)ax_\gamma) = V_1 \oplus \mathfrak{a}^\gamma, \quad \lim_{k \rightarrow \infty} V_1 \oplus \mathbb{k}(\beta(t_2^k)x_\beta + \gamma(t_2^k)ax_\gamma) = V_1 \oplus \mathfrak{a}^\beta,$$

$$\lim_{k \rightarrow \infty} \eta_i(\alpha(t_1^k)) = \lim_{k \rightarrow \infty} \eta_i(\alpha(t_2^k)) = \eta_i(\infty),$$

whence $V = V'$ and the contradiction. \square

2.6. Property (P) and Property (P₁). In this subsection we suppose that all objects of \mathcal{C}'_t of dimension smaller than n has Property (P). For V a fixed point of X_R under \mathbf{T} , denote by Λ_V the orthogonal complement to \mathfrak{z}_V in \mathcal{R} and set:

$$r_V := r_{\Lambda_V}, \quad R_V := R_{\Lambda_V}.$$

Lemma 2.19. *Let V be a fixed point under \mathbf{T} in X_R .*

(i) *The action of R_V in $\overline{R_V.V}$ has fixed points. For V_∞ such a point,*

$$V_\infty = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \alpha^\beta, \quad |\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|, \quad r_V \geq r_{V_\infty}.$$

(ii) *The set \mathcal{R}_V has rank at least $|\mathcal{R}_V| - 1$.*

(iii) *Suppose that α has Property (P₁). Then \mathcal{R}_V has rank $|\mathcal{R}_V|$.*

(iv) *If α has Property (P₁), for s in \mathfrak{t} such that V is contained in r^s , V is in $\overline{R^s.t}$.*

Proof. (i) As $\overline{R_V.V}$ is a projective variety and R_V is connected and solvable, R_V has fixed points in $\overline{R_V.V}$. Denote by V_∞ such a point. Since V is fixed under \mathbf{T} ,

$$V = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta.$$

Moreover, $V \cap \mathfrak{t}$ is contained in \mathfrak{z}_V since V is commutative. By Lemma 2.6(ii), \mathfrak{z}_V is the center of r_V . Hence $V \cap \mathfrak{t}$ is contained in all element of $R_V.V$. Moreover, all element of $R_V.V$ is contained in $V \cap \mathfrak{t} + \alpha_{\Lambda_V}$. Then

$$V_\infty = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \alpha^\beta,$$

whence $|\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|$. Since \mathcal{R}_{V_∞} is contained in Λ_V and $r_V = d - \dim \mathfrak{z}_V$, $r_V \geq r_{V_\infty}$.

(ii) By (i), we can suppose that V is invariant under R_V . By Lemma 2.5, α_{Λ_V} is contained in an ideal α' of r of dimension $\dim \alpha - 1$ and contained in α . We then use the notations of Lemma 2.13. Set $\Gamma_V := \varpi_3^{-1}(V)$. By Lemma 2.13(i), Γ_V is a projective variety invariant under R_V since so is V . Then R_V has a fixed point in Γ_V . Let (V_1, V', V, W) be such a point. As α' has Property (P), by Lemma 2.12(i),

$$V' = \mathfrak{z}_{V'} \oplus \bigoplus_{\beta \in \mathcal{R}_{V'}} \alpha^\beta.$$

and the elements of $\mathcal{R}_{V'}$ are linearly independent.

If $V' = V$ then $\mathcal{R}_{V'} = \mathcal{R}_V$ so that $r_V = r_{V'} = |\mathcal{R}_V|$. Suppose $V' \neq V$. Then, by Lemma 2.13(iv),

$$V_1 = \mathfrak{z}_{V'} \cap V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V \cap \mathcal{R}_{V'}} \alpha^\beta.$$

As V_1 has codimension 1 in V and V' , $\mathcal{R}_{V'} = \mathcal{R}_V$ or $\mathfrak{z}_{V'} = V \cap \mathfrak{t}$. In the first case, $r_V = |\mathcal{R}_V|$ and in the second case,

$$|\mathcal{R}_V \cap \mathcal{R}_{V'}| = |\mathcal{R}_V| - 1 = |\mathcal{R}_{V'}| - 1,$$

whence $r_V \geq |\mathcal{R}_V| - 1$ since the elements of $\mathcal{R}_{V'}$ are linearly independent.

(iii) Prove the assertion by induction on $\dim \mathfrak{z}_V$. If $\mathfrak{z}_V = \mathfrak{z}$, then $r_V = |\mathcal{R}_V|$ by Property (P₁). Suppose $\dim \mathfrak{z}_V = \dim \mathfrak{z} + 1$ and $V \cap \mathfrak{t} = \mathfrak{z}$. Then $|\mathcal{R}_V| = d^\#$ and $r_V = d^\# - 1$. By Property (P₁), it is impossible. Hence $V \cap \mathfrak{t} = \mathfrak{z}_V$ since $V \cap \mathfrak{t}$ is contained in \mathfrak{z}_V . As a result $r_V = |\mathcal{R}_V|$.

Suppose $\dim \mathfrak{z}_V \geq 2 + \dim \mathfrak{z}$, the assertion true for the integers smaller than $\dim \mathfrak{z}_V$ and $r_V < |\mathcal{R}_V|$. A contradiction is expected. By (ii), $V \cap \mathfrak{t}$ has dimension at least $\dim \mathfrak{z}_V - 1$. Then, for some α in \mathcal{R} , $V \cap \mathfrak{t}_\alpha$ is strictly contained in $V \cap \mathfrak{t}$. Let Λ be the orthogonal complement to $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ in \mathcal{R} . As $\overline{R_\Lambda.V}$ is a projective variety and R_Λ is connected, R_Λ has a fixed point in $\overline{R_\Lambda.V}$. Let V_∞ be such a point. By Lemma 2.6(ii),

$\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is the center of \mathfrak{r}_Λ . Hence $V \cap \mathfrak{t}_\alpha$ is contained in all element of $R_\Lambda.V$. Moreover, all element of $R_\Lambda.V$ is contained in $V \cap \mathfrak{t} + \mathfrak{a}_\Lambda$. As V_∞ is an ideal of \mathfrak{r}_Λ , $V \cap \mathfrak{t}$ is not contained in V_∞ since it is not contained in the kernel of α . Then

$$V_\infty = V \cap \mathfrak{t}_\alpha \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \alpha^\beta.$$

By (ii), $r_{V_\infty} \geq |\mathcal{R}_{V_\infty}| - 1$, whence

$$\dim \mathfrak{z}_{V_\infty} \leq \dim V \cap \mathfrak{t}_\alpha + 1 = \dim V \cap \mathfrak{t} < \dim \mathfrak{z}_V.$$

So, by induction hypothesis, $|\mathcal{R}_{V_\infty}| = r_{V_\infty}$ and $\mathfrak{z}_{V_\infty} = V \cap \mathfrak{t}_\alpha$. Since $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is the center of \mathfrak{r}_Λ , $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is contained in \mathfrak{z}_{V_∞} , whence

$$\dim \mathfrak{z}_V - 1 \leq \dim V \cap \mathfrak{t}_\alpha = \dim V \cap \mathfrak{t} - 1.$$

As a result, $\mathfrak{z}_V = V \cap \mathfrak{t}$ since $V \cap \mathfrak{t}$ is contained in \mathfrak{z}_V , whence a contradiction.

(iv) Suppose that \mathfrak{a} has Property (\mathbf{P}_1) . By (iii),

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta$$

and $r_V = |\mathcal{R}_V|$. As a result, the centralizer of V in \mathfrak{t} is equal to \mathfrak{z}_V . Set

$$\alpha'_V = \bigoplus_{\beta \in \mathcal{R}_V} \alpha^\beta, \quad \mathfrak{r}'_V := \mathfrak{t} + \alpha'_V.$$

Denote by R'_V the connected closed subgroup of R whose Lie algebra is $\text{ad} \mathfrak{r}'_V$. The algebra α'_V is in \mathcal{C}'_t and has dimension $d - \dim \mathfrak{z}_V$. Then, by Lemma 2.3(ii), V is in $\overline{R'_V.t}$, whence the assertion since \mathfrak{r}'_V is contained in \mathfrak{r}^s . \square

Corollary 2.20. *Suppose that \mathfrak{a} has Property (\mathbf{P}_1) . Then \mathfrak{a} has Property (\mathbf{P}) .*

Proof. Let V be in X_R and s in $\mathfrak{t} \setminus \mathfrak{z}$ such that V is contained in \mathfrak{r}^s . As $\overline{\mathbf{T}.V}$ is a projective variety and \mathbf{T} is a connected commutative group, \mathbf{T} has a fixed point in $\overline{\mathbf{T}.V}$. Let V_∞ be such a point. Since all element of $\mathbf{T}.V$ is contained in \mathfrak{r}^s , so is V_∞ . Then, by Lemma 2.19(iv), V_∞ is in $\overline{R^s.t}$. In particular, s is in V_∞ . Let E a complement to V_∞ in \mathfrak{r} , invariant under \mathbf{T} . The map

$$\text{Hom}_{\mathbb{K}}(V_\infty, E) \xrightarrow{\kappa} \text{Gr}_d(\mathfrak{r}), \quad \varphi \mapsto \kappa(\varphi) := \text{span}(\{v + \varphi(v) \mid v \in V_\infty\})$$

is an isomorphism onto an open neighborhood Ω_E of V_∞ in $\text{Gr}_d(\mathfrak{r})$. For φ in $\text{Hom}_{\mathbb{K}}(V_\infty, E)$ such that $\kappa(\varphi)$ is in $\mathbf{T}.V$, $\varphi(s)$ is in \mathfrak{a}^s . Then, for some g in \mathbf{T} and for some v in \mathfrak{a}^s , $s + v$ is in $g(V)$ and the semisimple component of $\text{ad}(s + v)$ is different from 0 since s is not in \mathfrak{z} . Let x be in \mathfrak{r}^s such that $\text{ad} x$ is the semisimple component of $\text{ad}(s + v)$. By Lemma 2.1(ii), for some k in R^s , $k(x)$ is in \mathfrak{t} . Then, by Corollary 2.14(ii), $kg(V)$ is in $\overline{R^{k(x)}.t}$. As $k(x)$ is not in \mathfrak{z} , $\alpha^{k(x)}$ is an object of \mathcal{C}'_t of dimension smaller than n . By hypothesis, $\alpha^{k(x)}$ has Property (\mathbf{P}) . Moreover, $kg(V)$ is contained in $\mathfrak{r}^s \cap \mathfrak{r}^{k(x)}$. Hence, by Property (\mathbf{P}) for $\alpha^{k(x)}$, $kg(V)$ is in $\overline{R^s.t}$, whence V is in $\overline{R^s.t}$ since kg is in R^s . \square

Proposition 2.21. *The objects of \mathcal{C}'_t have Property (\mathbf{P}) .*

Proof. Prove by induction on n that \mathfrak{a} has Property (\mathbf{P}) . By Lemma 2.8, it is true for $n = d^\#$. Suppose that it is true for the integers smaller than n . By Corollary 2.20, it remains to prove that \mathfrak{a} has Property (\mathbf{P}_1) .

Suppose that \mathfrak{a} has not Property (\mathbf{P}_1) . A contradiction is expected. For some fixed point V under \mathbf{T} in X_R such that $V \cap \mathfrak{t} = \mathfrak{z}$, $r_V \neq |\mathcal{R}_V|$. By Lemma 2.19(ii), $r_V = |\mathcal{R}_V| - 1$. Then the orthogonal complement of \mathcal{R}_V in \mathfrak{t} is generated by \mathfrak{z} and an element s in $\mathfrak{t} \setminus \mathfrak{z}$. In particular, V is contained in \mathfrak{r}^s . According to Lemma 2.5,

for some ideal α' of codimension 1 of \mathfrak{a} , normalized by \mathfrak{t} , α^s is contained in α' . Denote by α the element of \mathcal{R} such that

$$\mathfrak{a} = \alpha' \oplus \alpha^\alpha$$

and consider θ_α and Γ as in Subsection 2.5. Denote by Γ_V the set of elements of Γ whose image by the projection

$$\Gamma \longrightarrow \mathrm{Gr}_d(\mathfrak{r}), \quad (T_1, T', T, T_2) \longmapsto T$$

is equal to V . By Lemma 2.13(ii), Γ_V is not empty and it is invariant under \mathbf{T} by Lemma 2.13(i). As it is a projective variety, it has a fixed point under \mathbf{T} . Denote by (V_1, V', V, W) such a point. As α' has Property (\mathbf{P}) , it has Property (\mathbf{P}_1) by Lemma 2.12. Hence $r_{V'} = |\mathcal{R}_{V'}|$ and $V' \neq V$ since $r_V \neq \mathcal{R}_V$. Then, by Lemma 2.13(iv),

$$V_1 = V \cap V' \quad \text{and} \quad W = V' + V.$$

As a result, $V' \cap \mathfrak{t} = V \cap \mathfrak{t} = \mathfrak{z}$ since $\mathcal{R}_{V'} \neq \mathcal{R}_V$ and V_1 has codimension 1 in V and V' . Then $V' = V$ by Corollary 2.18, whence a contradiction. \square

The following corollary results from Proposition 2.21, Corollary 2.10 and Lemma 2.12.

Corollary 2.22. *Let V be in X_R .*

(i) *The space V is a commutative algebraic subalgebra of \mathfrak{r} and for some subset Λ of \mathcal{R} , the biggest torus contained in V is conjugate to \mathfrak{t}_Λ under R .*

(ii) *If V is a fixed point under R , then V is an ideal of \mathfrak{r} and the elements of \mathcal{R}_V are linearly independent.*

3. SOLVABLE ALGEBRAS AND MAIN VARIETIES

Let \mathfrak{t} be a vector space of positive dimension d and \mathfrak{a} in $\mathcal{C}_\mathfrak{t}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t}, \mathfrak{a}}, \quad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t}, \mathfrak{a}}, \quad \pi := \pi_{\mathfrak{t}, \mathfrak{a}}, \quad R := R_{\mathfrak{t}, \mathfrak{a}}, \quad A := A_{\mathfrak{t}, \mathfrak{a}}, \quad \mathcal{E} := \mathcal{E}_{\mathfrak{t}, \mathfrak{a}}, \quad n := \dim \mathfrak{a}.$$

In this section, we give some results on the singular locus of X_R . For α' in $\mathcal{C}_\mathfrak{t}$, denote by $X_{R_{\mathfrak{t}, \alpha'}, n}$ the subset of elements of $X_{R_{\mathfrak{t}, \alpha'}}$ contained in α' .

3.1. Subvarieties of X_R . Denote by $\mathcal{P}_c(\mathcal{R})$ the set of complete subsets of \mathcal{R} and for Λ in $\mathcal{P}_c(\mathcal{R})$ denote by X_{R_Λ} the closure in $\mathrm{Gr}_d(\mathfrak{r})$ of the orbit $R_\Lambda \mathfrak{t}$.

Proposition 3.1. *Let Z be an irreducible closed subset of X_R , invariant under R .*

(i) *For a well defined complete subset Λ of \mathcal{R} , all element of a dense open subset of Z is conjugate under R to the sum of \mathfrak{t}_Λ and a subspace of \mathfrak{a} .*

(ii) *All element of Z is contained in $\mathfrak{t}_\Lambda \oplus \mathfrak{a}$.*

(iii) *For some irreducible closed subset Z_Λ of X_{R_Λ} , invariant under R_Λ , $R \cdot Z_\Lambda$ is dense in Z .*

Proof. (i) For Λ in $\mathcal{P}_c(\mathcal{R})$, let Y_Λ be the subset of elements V of Z such that $\pi(V) = \mathfrak{t}_\Lambda$. Since Z is invariant under R , so is Y_Λ . Moreover, by Corollary 2.22(i),

$$\overline{Y_\Lambda} \subset Y_\Lambda \cup \bigcup_{\substack{\Lambda' \in \mathcal{P}_c(\mathcal{R}) \\ \Lambda' \supseteq \Lambda}} Y_{\Lambda'}.$$

According to Corollary 2.22(i), Z is the union of Y_Λ , $\Lambda \in \mathcal{P}_c(\mathcal{R})$. As a result, since \mathcal{R} is finite and Z is irreducible, for a well defined complete subset Λ of \mathcal{R} , Y_Λ contains a dense open subset of Z . By Lemma 2.1(v), all element of Y_Λ is conjugate under R to the sum of \mathfrak{t}_Λ and a subspace of \mathfrak{a} .

(ii) By (i), for all V in a dense subset of Z , V is contained in $\mathfrak{t}_\Lambda \oplus \mathfrak{a}$, whence the assertion.

(iii) Let Z_* be the subset of elements of Z , containing t_Λ . Denote by s an element of t_Λ such that $\alpha(s) \neq 0$ for all α in $\mathcal{R} \setminus \Lambda$. By Lemma 2.6(i),

$$r^s = t \oplus \alpha_\Lambda.$$

Hence Z_* is contained in X_{R_Λ} by Proposition 2.21. Moreover, Z_* is invariant under R_Λ since Z is invariant under R . By (i), $R.Z_*$ is dense in Z . So, for some irreducible component Z_Λ of Z_* , $R.Z_\Lambda$ is dense in Z . Moreover, Z_Λ is invariant under R_Λ since so is Z_* . \square

For Λ in $\mathcal{P}_c(\mathcal{R})$, denote by $t_\Lambda^\#$ a complement to t_Λ in t and set:

$$r_\Lambda^\# := t_\Lambda^\# + \alpha_\Lambda.$$

Let $R_\Lambda^\#$ be the adjoint group of $r_\Lambda^\#$ and $A_\Lambda^\#$ the connected closed subgroup of $R_\Lambda^\#$ whose Lie algebra is $\text{ad } \alpha_\Lambda$.

Lemma 3.2. *Let Λ be in $\mathcal{P}_c(\mathcal{R})$, nonempty and strictly contained in \mathcal{R} .*

(i) *The tori t_Λ and $t_\Lambda^\#$ have positive dimension and α_Λ is in $\mathcal{C}_{t_\Lambda^\#}$. Moreover,*

$$\dim \alpha_\Lambda - \dim t_\Lambda^\# \leq \dim \alpha - \dim t.$$

(ii) *The map $V \mapsto V \oplus t_\Lambda$ is an isomorphism from $X_{R_\Lambda^\#}$ onto X_{R_Λ} .*

Proof. Since Λ is a complete subset of \mathcal{R} strictly contained in \mathcal{R} , t_Λ has positive dimension and since Λ is not empty, t_Λ is strictly contained in t . By definition, Λ is the set of weights of t in α_Λ so that α_Λ is in \mathcal{C}'_t . Then α_Λ is in $\mathcal{C}_{t_\Lambda^\#}$ and Assertion (ii) results from Corollary 2.2.

By Lemma 2.1,(i) and (iv), \mathcal{R} generates t^* . Hence

$$|\Lambda| + \dim t_\Lambda \leq |\mathcal{R}|.$$

By Condition (2) of Section 2, α has dimension $|\mathcal{R}|$ and α_Λ has dimension $|\Lambda|$. As a result,

$$\dim \alpha - \dim t = |\mathcal{R}| - \dim t_\Lambda - \dim t_\Lambda^\# \geq \dim \alpha_\Lambda - \dim t_\Lambda^\#.$$

\square

3.2. Smooth points of X_R and commutators. Denote by t_{reg} the complement in t to the union of t_α , $\alpha \in \mathcal{R}$ and r_{reg} the set of elements x of r such that r^x has minimal dimension.

Lemma 3.3. (i) *The set t_{reg} is a dense open subset of t , contained in r_{reg} . Moreover, $R.t_{\text{reg}}$ is a dense open subset of r .*

(ii) *For all x in r_{reg} , r^x is in X_R .*

(iii) *The set r_{reg} is a big open subset of r .*

Proof. (i) By definition, t_{reg} is a dense open subset of t . According to Lemma 2.6(i), for x in t_{reg} , $r^x = t$. Then $R.x = A.x = x + \alpha$ since $A.x$ is a closed subset of $x + \alpha$ of dimension $\dim \alpha$. As a result, $R.t_{\text{reg}} = t_{\text{reg}} + \alpha$ is a dense open subset of r . Hence $R.t_{\text{reg}}$ is contained in r_{reg} since r^x is conjugate to t for all x in $R.t_{\text{reg}}$ and r_{reg} is a dense open subset of r .

(ii) By (i), for all x in r_{reg} , r^x has dimension d , whence a regular map

$$r_{\text{reg}} \xrightarrow{\theta} \text{Gr}_d(r), \quad x \mapsto r^x.$$

As a result, by (i), for all x in r_{reg} , r^x is in X_R .

(iii) Suppose that r_{reg} is not a big open subset of r . A contradiction is expected. Let Σ be an irreducible component of codimension 1 of $r \setminus r_{\text{reg}}$. Since $\Sigma \cap A.t_{\text{reg}}$ is empty, $\pi(\Sigma)$ is contained in t_α for some α in r . Then $\Sigma = t_\alpha + \alpha$ since Σ has codimension 1 in r . By Condition (3) of Section 2, for some s in t_α , $\gamma(s) \neq 0$ for

all γ in $\mathcal{R} \setminus \{\alpha\}$. Then $r^{s+x_\alpha} = t_\alpha + \alpha^\alpha$ so that $s + x_\alpha$ is in r_{reg} by (i) and Condition (2) of Section 2, whence the contradiction. \square

Denote by X'_R the image of θ .

Proposition 3.4. (i) *The complement to $R.t$ in X_R is equidimensional of dimension $\dim \alpha - 1$.*

(ii) *The set X'_R is a smooth open subset of X_R , containing $R.t$.*

Proof. (i) As R is solvable and $R.t$ is dense in X_R , $R.t$ is an affine open subset of X_R . So, by [EGAIV, Corollaire 21.12.7], $X_R \setminus R.t$ is equidimensional of dimension $\dim \alpha - 1$ since X_R has dimension $\dim \alpha$.

(ii) By definition, \mathcal{E} is the subvariety of elements (V, x) of $X_R \times r$ such that x is in V . Let Γ be the image of the graph of θ by the isomorphism

$$r \times \text{Gr}_d(r) \longrightarrow \text{Gr}_d(r) \times r, \quad (x, V) \longmapsto (V, x).$$

Then Γ is the intersection of \mathcal{E} and $X_R \times r_{\text{reg}}$. Since Γ is isomorphic to r_{reg} , Γ is a smooth open subset of \mathcal{E} whose image by the bundle projection is X'_R . As a result, X'_R is a smooth open subset of X_R by [MA86, Ch. 8, Theorem 23.7]. \square

For α in \mathcal{R} , set $V_\alpha := t_\alpha \oplus \alpha^\alpha$ and denote by θ_α the map

$$\mathbb{k} \xrightarrow{\theta_\alpha} \text{Gr}_d(r), \quad z \longmapsto \exp(z \text{ad } x_\alpha)(t),$$

By Condition (2) of Section 2, V_α has dimension d .

Lemma 3.5. *Let α be in \mathcal{R} . Set $X_{R,\alpha} := \overline{A.V_\alpha}$.*

(i) *The map θ_α has a regular extension to $\mathbb{P}^1(\mathbb{k})$ such that $\theta_\alpha(\infty) = V_\alpha$.*

(ii) *The variety $X_{R,\alpha}$ has dimension $\dim \alpha - 1$ and it is an irreducible component of $X_R \setminus R.t$.*

(iii) *The intersection $X_{R,\alpha} \cap X'_R$ is not empty.*

Proof. (i) Let h_α be in t such that $\alpha(h_\alpha) = 1$. Since X_R is a projective variety, the map θ_α has a regular extension to $\mathbb{P}^1(\mathbb{k})$ by [Sh94, Ch. 6, Theorem 6.1]. For z in \mathbb{k} ,

$$\theta_\alpha(z) = t_\alpha \oplus \mathbb{k}(h_\alpha - z x_\alpha).$$

Hence $\theta_\alpha(\infty) = V_\alpha$.

(ii) By (i), $X_{R,\alpha}$ is contained in X_R and its elements are contained in $t_\alpha \oplus \alpha$ so that $X_{R,\alpha}$ is contained in $X_R \setminus R.t$. By Condition (3) of Section 2, for γ in $\mathcal{R} \setminus \{\alpha\}$ and v in α^γ , $[t_\alpha, v] = \mathbb{k}v$ so that no element of α^γ normalizes V_α . As a result, the normalizer of V_α in r is equal to $t + \alpha^\alpha$ so that $X_{R,\alpha}$ has dimension $\dim \alpha - 1$. Hence $X_{R,\alpha}$ is an irreducible component $X_R \setminus R.t$.

(iii) According to Condition (3) of Section 2, for some s in t_α , $\gamma(s) \neq 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$. Then $V_\alpha = r^{s+x_\alpha}$ so that $s + x_\alpha$ is in r_{reg} , whence the assertion. \square

3.3. On the singular locus of X_R . In this subsection we suppose $\dim \alpha > d$ and we fix an ideal α' of codimension 1 in α , normalized by t and such that α' is in \mathcal{C}_t . For example, all ideal of r of dimension $\dim \alpha - 1$, contained in α and containing a fixed point under R in X_R is in \mathcal{C}_t by Corollary 2.22(ii). Set:

$$r' := r_{t,\alpha'} \quad \pi' := \pi_{t,\alpha'}, \quad R' := R_{t,\alpha'}, \quad A' := A_{t,\alpha'}, \quad \mathcal{R}' := \mathcal{R}_{t,\alpha'}.$$

Let α be in \mathcal{R} such that

$$\alpha = \alpha' \oplus \alpha^\alpha$$

and Γ as in Subsection 2.5. Denote by $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ the restrictions to Γ of the first, second, third, fourth projections. Let Z be an irreducible component of $X_{R,n}$. According to Lemma 2.13(ii), for some irreducible

component T of $\varpi_3^{-1}(Z)$, $\varpi_3(T) = Z$. Denote by Z' the image of T by ϖ_2 and by T_1 the image of T by the projection $\varpi_1 \times \varpi_4$. Then Z' and T_1 are irreducible closed subsets of $\text{Gr}_d(\mathfrak{r})$ and $\text{Gr}_{d-1}(\mathfrak{r}) \times \text{Gr}_{d+1}(\mathfrak{r})$ respectively. Let T_0 be the subset of elements (V_1, V', V, W) of T such that $V' = V$. Then T_0 is a closed subset of T . If $T_0 = T$, $Z' = Z$ and Z is contained in $X_{R',n}$. Otherwise, $O := T \setminus T_0$ is a dense open subset of T . According to Lemma 2.13(iv), for all (V_1, V', V, W) in O , $V_1 = V' \cap V$ and $V' + V = W$. Denote by O_1 an open subset of T_1 , contained and dense in $\varpi_1 \times \varpi_4(O)$.

Let (V_1, W) be in O_1 . Denote by E a complement to V_1 in \mathfrak{r} and by E' a complement to W in \mathfrak{r} contained in E . Let κ be the map

$$\begin{aligned} & \text{Hom}_{\mathbb{k}}(V_1, W \cap E) \times \text{Hom}_{\mathbb{k}}(W, E') \xrightarrow{\kappa} \text{Gr}_{d-1}(\mathfrak{r}) \times \text{Gr}_{d+1}(\mathfrak{r}), \\ (\varphi, \psi) & \mapsto (\text{span}(\{v + \varphi(v) + \psi(v) + \psi \circ \varphi(v) \mid v \in V_1\}), \text{span}(\{v + \psi(v) \mid v \in W\})). \end{aligned}$$

Then κ is an isomorphism from its source to an open neighborhood of (V_1, W) in the subvariety of elements (W_1, W_2) of $\text{Gr}_{d-1}(\mathfrak{r}) \times \text{Gr}_{d+1}(\mathfrak{r})$ such that W_1 is contained in W_2 . Denote by Ω the inverse image by κ of the intersection of T_1 and the image of κ . Let (e_1, e_2) be a basis of $W \cap E$ and let κ_* be the map

$$\begin{aligned} & \Omega \times (\mathbb{k}^2 \setminus \{(0, 0)\}) \xrightarrow{\kappa_*} \text{Gr}_d(\mathfrak{r}), \\ (\varphi, \psi, x_1, x_2) & \mapsto \text{span}(\{v + \varphi(v) + \psi(v) + \psi \circ \varphi(v) \mid v \in V_1\} \cup \{x_1(e_1 + \psi(e_1)) + x_2(e_2 + \psi(e_2))\}). \end{aligned}$$

Lemma 3.6. *Suppose that O is not empty. Denote by $\tilde{\Omega}$ the image of κ_* and \tilde{Z} the closure of $\tilde{\Omega}$ in $\text{Gr}_d(\mathfrak{r})$.*

(i) *The intersections $\tilde{\Omega} \cap Z'$ and $\tilde{\Omega} \cap Z$ are dense in Z' and Z respectively. In particular Z' and Z are contained in \tilde{Z} .*

(ii) *For V in $\tilde{\Omega}$, there exists (V', V'') in $Z' \times Z$ such that*

$$V' \cap V'' \subset V, \quad V \subset V' + V'', \quad (V' \cap V'', V' + V'') \in \kappa(\Omega).$$

(iii) *Let F' be the fiber of κ_* at some element V of $\kappa_*(\Omega)$. Denote by F the subset of elements (φ, ψ) of Ω such that V contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Then $F' = F \times \mathbb{k}^*(x_1, x_2)$ for some (x_1, x_2) in $\mathbb{k}^2 \setminus \{(0, 0)\}$.*

(iv) *The varieties \tilde{Z} and Z have dimension at most $\dim Z' + 1$.*

Proof. (i) Since T is irreducible so are T_1 and Ω . Hence \tilde{Z} is irreducible. For some (V', V) in $Z' \times Z$, V_1 is contained in V' and V and V' and V are contained in W . Since $\kappa(\Omega)$ is an open neighbourhood of (V_1, W) in T_1 ,

$$\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T) \quad \text{and} \quad \varpi_3(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T)$$

are dense subsets of Z' and Z respectively. For all (φ, ψ) in Ω , all element of $\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\varphi, \psi)) \cap T)$ contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Hence all element of $\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T)$ is in the image of κ_* . As a result, $\tilde{\Omega} \cap Z'$ is dense in Z' and Z' is contained in \tilde{Z} . In the same way, $\tilde{\Omega} \cap Z$ is dense in Z and Z is contained in \tilde{Z} .

(ii) According to Lemma 2.13(iv), for all (V'_1, V', V, W') in O , $V'_1 = V' \cap V$ and $W' = V' + V$. By definition, $\kappa(\Omega)$ is contained in $\varpi_1 \times \varpi_4(O)$ and for V in $\tilde{\Omega}$, $V'_1 \subset V$ and $V \subset W'$ for some (V'_1, W') in $\kappa(\Omega)$, whence the assertion.

(iii) For (φ, ψ) in F and for (x_1, x_2) in $\mathbb{k}^2 \setminus \{(0, 0)\}$ such that

$$V = \kappa_*(\varphi, \psi, x_1, x_2),$$

the subset of elements (y_1, y_2) of \mathbb{k}^2 such that $(\varphi, \psi, y_1, y_2)$ is in F' is equal to $\mathbb{k}^*(x_1, x_2)$. Moreover, for all $(\varphi, \psi, y_1, y_2)$ in F' , (φ, ψ) is in F , whence the assertion.

(iv) In (iii), we can choose V such that F' has minimal dimension so that

$$\dim \tilde{Z} = \dim \Omega + 2 - (\dim F + 1) = \dim \Omega - \dim F + 1.$$

By (ii), for some V' in Z' , for all (φ, ψ) in F , V' contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. So, again by (iii) and (ii),

$$\dim Z' \geq \dim \Omega - \dim F,$$

whence $\dim \tilde{Z} \leq \dim Z' + 1$ and $\dim Z \leq \dim Z' + 1$ since Z is contained in \tilde{Z} by (i). \square

Proposition 3.7. *The variety $X_{R,n}$ has dimension at most $n - d$.*

Proof. Prove this by induction on n . According to Lemma 2.3(ii), it is true for $n - d = 0$. Suppose that $n - d$ is positive and that it is true for all integer smaller than $n - d$. In particular, $X_{R',n}$ has dimension at most $n - d - 1$. Let Z be an irreducible component of $X_{R,n}$. According to Lemma 2.13(ii), for some irreducible component T of $\varpi_3^{-1}(Z)$, $\varpi_3(T) = Z$. Denote by Z' the image of T by ϖ_2 . Let T_0 be the subset of elements (V_1, V', V, W) of T such that $V' = V$. Consider the following cases:

- (a) $T_0 = T$,
- (b) $T_0 \neq T$ and Z' is contained in $X_{R',n}$,
- (c) Z' is not contained in $X_{R',n}$.

(a) In this case, $Z' = Z$ and $\dim Z \leq n - d - 1$ by induction hypothesis.

(b) By induction hypothesis, $\dim Z' \leq n - d - 1$ and by Lemma 3.6(iv), $\dim Z \leq \dim Z' + 1$, whence $\dim Z \leq n - d$.

(c) In this case, $T_0 \neq T$, whence $\dim Z \leq \dim Z' + 1$ by Lemma 3.6(iv). Since Z is an irreducible component of $X_{R,n}$, Z is invariant under R . By Lemma 2.13(i), ϖ_2 and ϖ_3 are equivariant under the action of R' in Γ so that T and Z' are invariant under R' . For all (V_1, V', V, W) in $T \setminus T_0$, $V_1 = V' \cap V$. Hence all element of a dense open subset of Z' contains a subspace of dimension $d - 1$ of α' . Then, by Proposition 3.1, for some complete subset Λ of \mathcal{R}' such that t_Λ has dimension 1 and for some closed subset Z_Λ of $X_{R,\Lambda}$, $R'.Z_\Lambda$ is dense in Z' so that

$$\dim Z' \leq \dim Z_\Lambda + \dim \alpha' - \dim \alpha_\Lambda.$$

If $\dim \alpha_\Lambda - \dim t + 1 = n - d$, then $\Lambda = \mathcal{R}'$. In this case, since α' is in \mathcal{C}_t , Λ generates t^* . As t_Λ has dimension 1, it is impossible. As a result,

$$\dim Z_\Lambda \leq \dim \alpha_\Lambda - \dim t + 1 \quad \text{and} \quad \dim Z' \leq n - d$$

by Lemma 3.2 and induction hypothesis for α_Λ . Then $\dim Z \leq n - d + 1$. According to Lemma 3.6(i) and (iv), \tilde{Z} is an irreducible variety of dimension at most $\dim Z' + 1$, containing Z' and Z . If $\dim Z' = n - d$ and $\dim Z = n - d + 1$, then $Z = \tilde{Z}$. In particular, Z' is contained in Z . It is impossible since all element of Z is contained in α . As a result, $\dim Z \leq n - d$, whence the proposition. \square

Corollary 3.8. (i) *The irreducible components of $X_R \setminus R.t$ are the $X_{R,\alpha}$, $\alpha \in \mathcal{R}$.*

(ii) *The set X'_R is a smooth big open subset of X_R , containing $R.t$.*

Proof. According to Proposition 3.4(ii) and Lemma 3.5(iii), Assertion (ii) results from Assertion (i). Prove Assertion (i) by induction on $n = \dim \alpha$. For $n = 1$, $d = 1$ by Lemma 2.1(i) and (iv) so that X_R is the union of $R.t$ and α^α , whence Assertion (i) in this case. Suppose $n \geq 2$ and the assertion true for the integers smaller than n . By Lemma 2.1(i), Condition (2) and Condition (3) of Section 2, $d \geq 2$. According to Lemma 3.5(ii), for all α in \mathcal{R} , $X_{R,\alpha}$ is an irreducible component of $X_R \setminus R.t$. Let Z be an irreducible component of $X_R \setminus R.t$. By Proposition 3.4(i), Z has dimension $n - 1$. So, by Proposition 3.7, Z is not contained in $X_{R,n}$. Moreover,

Z is invariant under R . Then, by Proposition 3.1, for some complete subset Λ of \mathcal{R} , strictly contained in \mathcal{R} and for some irreducible closed subset Z_Λ of X_{R_Λ} , invariant under R_Λ , $R.Z_\Lambda$ is dense in Z . By Lemma 3.2, α_Λ is in $\mathcal{C}_{\mathfrak{t}_\Lambda^\#}$ and Z_Λ is the image of a closed subset Z'_Λ of $X_{R_\Lambda^\#}$, invariant by $R_\Lambda^\#$, by the map $V \mapsto V \oplus \mathfrak{t}_\Lambda$. Since Z_Λ is contained in Z , $Z'_\Lambda \cap R_\Lambda^\#.\mathfrak{t}_\Lambda^\#$ is empty. As Λ is strictly contained in \mathcal{R} , $\dim \alpha_\Lambda$ is smaller than n . So, by induction hypothesis, for some α in Λ , Z'_Λ is contained in $X_{R_\Lambda^\#, \alpha}$. As a result, Z_Λ and Z are contained in $X_{R, \alpha}$, whence $Z = X_{R, \alpha}$ since Z is an irreducible component of $X_R \setminus R.\mathfrak{t}$. \square

4. NORMALITY FOR SOLVABLE LIE ALGEBRAS

Let \mathfrak{t} be a vector space of positive dimension d and \mathfrak{a} in $\mathcal{C}_\mathfrak{t}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t}, \mathfrak{a}}, \quad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t}, \mathfrak{a}}, \quad \pi := \pi_{\mathfrak{t}, \mathfrak{a}}, \quad R := R_{\mathfrak{t}, \mathfrak{a}}, \quad A := A_{\mathfrak{t}, \mathfrak{a}}, \quad \mathcal{E} := \mathcal{E}_{\mathfrak{t}, \mathfrak{a}}, \quad n := \dim \mathfrak{a}.$$

The goal of the section is to prove that X_R is normal and Cohen-Macaulay.

4.1. The case $\dim \mathfrak{a} = \dim \mathfrak{t}$. By Condition (2) of Section 2, \mathcal{R} has d elements β_1, \dots, β_d linearly independent. Denote by t_1, \dots, t_d the dual basis in \mathfrak{t} . For $i = 1, \dots, d$, let v_i be a generator of \mathfrak{a}^{β_i} .

Lemma 4.1. *If $\dim \mathfrak{a} = \dim \mathfrak{t}$ then X_R is a smooth variety. Moreover, for all (z_1, \dots, z_d) in \mathbb{k}^d , the subspace generated by $v_1 + z_1 t_1, \dots, v_d + z_d t_d$ is in X_R .*

Proof. According to Lemma 2.3, \mathfrak{a} is in X_R and the map

$$\mathbb{k}^d \longrightarrow X_R, \quad (z_1, \dots, z_d) \longmapsto \text{span}(\{v_1 + z_1 t_1, \dots, v_d + z_d t_d\})$$

is an isomorphism onto an open neighborhood of \mathfrak{a} in X_R . Hence \mathfrak{a} is a smooth point of X_R . By Corollary 2.22, R has only one fixed point \mathfrak{a} in X_R . Since for all V in X_R , R has a fixed point in $\overline{R.V}$ and $X_{R_{\text{sm}}}$ is an open subset of X_R , invariant under R , $X_R = X_{R_{\text{sm}}}$. \square

4.2. Cohen-Macaulayness property for some algebras. Let A_* be an integral domain and a local commutative \mathbb{k} -algebra with maximal ideal \mathfrak{m} and u_1, \dots, u_s a regular sequence in A_* of elements of \mathfrak{m} . Let T_1, \dots, T_s be indeterminates. Set $B_s := A_*[T_1, \dots, T_s]$ and denote by P_s and P'_s the ideals of B_s generated by the sequences $u_j T_k - u_k T_j$, $1 \leq j, k \leq s$ and $u_j T_1 - u_1 T_j$, $1 \leq j \leq s$ respectively.

Lemma 4.2. *The ideal P_s is a prime ideal of B_s .*

Proof. For $s = 1$, $P_s = \{0\}$. Suppose $s \geq 2$. Let \tilde{P} be the ideal of $B_s[T_1^{-1}]$ generated by P_s . For $1 \leq j, k \leq s$,

$$T_1(u_j T_k - u_k T_j) = T_k(u_j T_1 - u_1 T_j) + T_j(u_1 T_k - u_k T_1).$$

Hence \tilde{P} is the ideal of $B_s[T_1^{-1}]$ generated by P'_s . Setting $S_j := T_j/T_1$ for $j = 2, \dots, s$, denote by C the polynomial algebra $A_*[S_2, \dots, S_s]$ over A_* so that $B_s[T_1^{-1}] = C[T_1, T_1^{-1}]$ and \tilde{P} is the ideal of $B_s[T_1^{-1}]$ generated by $u_j - u_1 S_j$, $j = 2, \dots, s$.

Claim 4.3. Let Q be the ideal of C generated by $u_j - u_1 S_j$, $j = 2, \dots, s$. Then Q is prime.

Proof. [Proof of Claim 4.3] Let \tilde{Q} be the ideal of $C[u_1^{-1}]$ generated by Q . Then \tilde{Q} is prime since it is generated by $u_j u_1^{-1} - S_j$, $j = 2, \dots, s$. As a result, for p and q in C such that pq is in Q , for some nonnegative integer m , $u_1^m p$ or $u_1^m q$ is in Q . So it remains to prove that for p in C , p is in Q if so is $u_1 p$.

Let p be in C such that $u_1 p$ is in Q . For some q_2, \dots, q_s in C ,

$$u_1 p = \sum_{j=2}^s q_j (u_j - u_1 S_j) \quad \text{whence} \quad \sum_{j=1}^s q_j u_j = 0 \quad \text{with} \quad q_1 := -(p + \sum_{j=2}^s q_j S_j).$$

By hypothesis, the sequence u_1, \dots, u_s is regular in C . So for some sequence $q_{j,k}$, $1 \leq j, k \leq s$ in C such that $q_{j,k} = -q_{k,j}$,

$$q_j = \sum_{k=1}^s q_{j,k} u_k$$

for $j = 1, \dots, s$. As a result,

$$\begin{aligned} u_1 p &= \sum_{j=2}^s \sum_{k=1}^s q_{j,k} u_k (u_j - u_1 S_j) \\ &= \sum_{j=2}^s q_{j,1} u_j u_1 - \sum_{j=2}^s \sum_{k=1}^s q_{j,k} u_k u_1 S_j \\ &= u_1 (\sum_{j=2}^s q_{j,1} (u_j - u_1 S_j) + \sum_{2 \leq j < k \leq s} q_{j,k} (u_j S_k - u_k S_j)). \end{aligned}$$

For $2 \leq j, k \leq s$,

$$u_j S_k - u_k S_j = (u_j - u_1 S_j) S_k - (u_k - u_1 S_k) S_j \in Q,$$

whence the claim. \square

By the claim, \tilde{P} is a prime ideal of $B_s[T_1^{-1}]$ since it is generated by Q . As a result for p and q in B_s such that pq is in P_s , for some nonnegative integer m , $T_1^m p$ or $T_1^m q$ is in P'_s since $T_1 P_s$ is contained in P'_s by the equality

$$T_1(u_j T_k - u_k T_j) = T_k(u_j T_1 - u_1 T_j) + T_j(u_1 T_k - u_k T_1)$$

for $1 \leq i, j \leq s$. So it remains to prove that for p in B_s , p is in P_s if $T_1 p$ is in P'_s .

Let p be in B_s such that $T_1 p$ is in P'_s . For some r_2, \dots, r_s in B_s ,

$$T_1 p = \sum_{j=2}^s r_j (u_j T_1 - u_1 T_j).$$

For $j = 2, \dots, s$, r_j has an expansion

$$r_j = r_{j,0} + T_1 r_{j,1}$$

with $r_{j,0}$ and $r_{j,1}$ in $B'_s := A_*[T_2, \dots, T_s]$ and B_s respectively. Set:

$$p' := p - \sum_{j=2}^s r_{j,1} (u_j T_1 - u_1 T_j).$$

Then

$$T_1 p' = \sum_{j=2}^s r_{j,0} (u_j T_1 - u_1 T_j)$$

so that the element

$$\sum_{j=2}^s r_{j,0} u_1 T_j \in B'_s$$

is divisible by T_1 in B_s , whence

$$\sum_{j=2}^s r_{j,0} T_j = 0.$$

As T_2, \dots, T_s is a regular sequence in B_s , for some sequence $r_{j,k,0}$, $2 \leq j, k \leq s$ in B_s such that $r_{j,k,0} = -r_{k,j,0}$ for all (j, k) ,

$$r_{j,0} = \sum_{k=2}^s r_{j,k,0} T_k$$

for $j = 2, \dots, s$. Then

$$T_1 p' = \sum_{2 \leq j, k \leq s} r_{j,k,0} T_k (u_j T_1 - u_1 T_j) = T_1 \sum_{2 \leq j < k \leq s} r_{j,k,0} (T_k u_j - T_j u_k).$$

As a result p' and p are in P_s , whence the lemma. \square

Denote by P'_s the ideal of B_s generated by P_{s-1} and $u_s T_1 - u_1 T_s$. Let \mathfrak{B}_s and \mathfrak{B}'_s be the quotients of B_s by P_s and P'_s respectively. The restrictions to A_* of the quotient morphisms $B_s \longrightarrow \mathfrak{B}'_s$ and $B_s \longrightarrow \mathfrak{B}_s$ are embeddings. For $j = 1, \dots, s$, denote again by T_j its images in \mathfrak{B}'_s and \mathfrak{B}_s by these morphisms.

Lemma 4.4. Denote by $\overline{P_s}$ the image in \mathfrak{B}'_s of P_s by the quotient morphism.

- (i) The intersection of $\overline{P_s}$ and $T_1 \mathfrak{B}'_s$ is equal to $\{0\}$.
- (ii) The \mathfrak{B}'_s -modules $T_1 \mathfrak{B}'_s$ and \mathfrak{B}_s are isomorphic.

Proof. Let a be in B_s such that $T_1 a$ is in P_s . According to Lemma 4.2, P_s is a prime ideal of B_s . Hence a is in P_s since T_1 is not in P_s . Moreover, for $j = 1, \dots, s$,

$$T_1(u_j T_s - u_s T_j) = T_s(u_j T_1 - u_1 T_j) + T_j(u_1 T_s - u_s T_1).$$

Hence $T_1 P_s$ is contained in P'_s . As a result, $\overline{P_s}$ is the kernel of the endomorphism $a \mapsto T_1 a$ of \mathfrak{B}'_s and the intersection of $\overline{P_s}$ and $T_1 \mathfrak{B}'_s$ is equal to $\{0\}$. As \mathfrak{B}_s is the quotient of \mathfrak{B}'_s by $\overline{P_s}$, the endomorphism $a \mapsto T_1 a$ defines through the quotient an isomorphism

$$\mathfrak{B}_s \longrightarrow T_1 \mathfrak{B}'_s$$

of \mathfrak{B}'_s -modules. \square

Let Q_s be the ideal of the polynomial algebra $A_*[T_2, \dots, T_s]$ generated by the sequence $u_i T_k - u_k T_i$, $2 \leq i, k \leq s$ and denote by $\mathfrak{B}_s^\#$ the quotient of $A_*[T_2, \dots, T_s]$ by Q_s .

Lemma 4.5. (i) The quotient of the algebra $\mathfrak{B}_s/T_1 \mathfrak{B}_s$ by the ideal generated by u_1 is equal to the quotient of $\mathfrak{B}_s^\#$ by the ideal generated by u_1 .

- (ii) The canonical map $A_* \longrightarrow \mathfrak{B}_s/T_1 \mathfrak{B}_s$ is an embedding.
- (iii) The ideal of $\mathfrak{B}_s/T_1 \mathfrak{B}_s$ generated by u_1 is isomorphic to A_*

Proof. Denote by Q'_s the ideal of B_s generated by P_s and T_1 .

(i) As the ideal of B_s generated by Q'_s and u_1 is equal to the ideal generated by u_1 , T_1 and Q_s , $\mathfrak{B}_s^\#/u_1 \mathfrak{B}_s^\#$ is equal to the quotient of $\mathfrak{B}_s/T_1 \mathfrak{B}_s$ by the ideal generated by u_1 .

(ii) Since the intersection of A_* and Q'_s is equal to $\{0\}$, the canonical map $A_* \longrightarrow \mathfrak{B}_s/T_1 \mathfrak{B}_s$ is an embedding.

(iii) For $k = 2, \dots, s$, $u_1 T_k$ is in Q'_s . Hence $u_1 B_s$ is contained in the sum of $u_1 A_*$ and Q'_s . As a result, $u_1 A_*$ is equal to $u_1 \mathfrak{B}_s/T_1 \mathfrak{B}_s$ by (ii), whence the assertion since A_* is an integral domain. \square

Proposition 4.6. Suppose that A_* is Cohen-Macaulay.

- (i) The algebra \mathfrak{B}_s is an integral domain and a Cohen-Macaulay algebra of dimension $\dim A_* + 1$.
- (ii) For a_1, \dots, a_m regular sequence in A_* of elements of \mathfrak{m} an for \mathfrak{p} prime ideal of \mathfrak{B}_s containing it, a_1, \dots, a_m is a regular sequence in the localization of \mathfrak{B}_s at \mathfrak{p} .

Proof. (i) Prove the assertion by induction on s . As \mathfrak{B}_1 is the polynomial algebra $A_*[T_1]$, the assertion is true for $s = 1$ since A_* an integral domain and a Cohen-Macaulay algebra. Suppose the assertion true for $s - 1$. By induction hypothesis, $\mathfrak{B}_{s-1}[T_s]$ is an integral domain and a Cohen-Macaulay algebra as a polynomial algebra over \mathfrak{B}_{s-1} and its dimension is equal to $\dim A_* + 2$. As a result, \mathfrak{B}'_s is Cohen-Macaulay

of dimension $\dim A_* + 1$ as the quotient of the integral domain and a Cohen-Macaulay algebra $\mathfrak{B}_{s-1}[T_s]$ by the ideal generated by $T_s u_1 - T_1 u_s$. As \mathfrak{B}_s is the quotient of \mathfrak{B}'_s by \overline{P}_s , \mathfrak{B}_s has dimension at most $\dim A_* + 1$. By Lemma 4.2, \mathfrak{B}_s is an integral domain so that $\mathfrak{B}_s/T_1 \mathfrak{B}_s$ has dimension at most $\dim A_*$.

By induction hypothesis again, $\mathfrak{B}_s^\#$ is an integral domain and a Cohen-Macaulay algebra of dimension $\dim A_* + 1$. Hence $\mathfrak{B}_s^\#/u_1 \mathfrak{B}_s^\#$ is Cohen-Macaulay of dimension $\dim A_*$. According to Lemma 4.5, we have a short exact sequence

$$0 \longrightarrow A_* \longrightarrow \mathfrak{B}_s/T_1 \mathfrak{B}_s \longrightarrow \mathfrak{B}_s^\#/u_1 \mathfrak{B}_s^\# \longrightarrow 0 .$$

Hence the algebra $\mathfrak{B}_s/T_1 \mathfrak{B}_s$ is Cohen-Macaulay of dimension $\dim A_*$ since A_* and $\mathfrak{B}_s^\#/u_1 \mathfrak{B}_s^\#$ are Cohen-Macaulay of dimension $\dim A_*$ and $\mathfrak{B}_s/T_1 \mathfrak{B}_s$ has dimension at most $\dim A_*$. As a result, \mathfrak{B}_s has dimension $\dim A_* + 1$. As \mathfrak{B}_s is the quotient of \mathfrak{B}'_s by \overline{P}_s , we have a short exact sequence

$$0 \longrightarrow \overline{P}_s + T_1 \mathfrak{B}'_s \longrightarrow \mathfrak{B}'_s \longrightarrow \mathfrak{B}_s/T_1 \mathfrak{B}_s \longrightarrow 0 .$$

Then, setting $M := \overline{P}_s + T_1 \mathfrak{B}'_s$ and denoting by M_* the localization of M at a maximal ideal of \mathfrak{B}'_s , containing T_1 ,

$$\text{Ext}^j(\mathbb{k}, M_*) = 0$$

for $j \leq \dim A_*$ since \mathfrak{B}'_s and $\mathfrak{B}_s/T_1 \mathfrak{B}_s$ have dimension $\dim A_* + 1$ and $\dim A_*$. By Lemma 4.4(i), M is the direct sum \overline{P}_s and $T_1 \mathfrak{B}'_s$. So, denoting by $(T_1 \mathfrak{B}'_s)_*$ the localization of $T_1 \mathfrak{B}'_s$ at a maximal ideal of \mathfrak{B}'_s ,

$$\text{Ext}^j(\mathbb{k}, (T_1 \mathfrak{B}'_s)_*) = 0$$

for $j \leq \dim A_*$ since $(T_1 \mathfrak{B}'_s)_*$ is the localization of \mathfrak{B}'_s at this maximal ideal when it does not contain T_1 . As a result, by Lemma 4.4(ii), \mathfrak{B}_s is Cohen-Macaulay since it has dimension $\dim A_* + 1$.

(ii) Let \mathfrak{q} be a minimal prime ideal of \mathfrak{B}_s , containing a_1, \dots, a_m . Since A_* is embedded in \mathfrak{B}_s , $\mathfrak{q} \cap A_*$ is a prime ideal of A_* containing a_1, \dots, a_m . As A_* is Cohen-Macaulay and a_1, \dots, a_m is a regular sequence in A_* , $\mathfrak{q} \cap A_*$ has height at least m and $A_*/\mathfrak{q} \cap A_*$ has dimension at most $\dim A_* - m$ by [MA86, Ch. 6, Theorem 17.4]. Then $\mathfrak{B}_s/\mathfrak{q}$ has dimension at most $\dim A_* + 1 - m$ since the fraction field of $\mathfrak{B}_s/\mathfrak{q}$ is generated by the fraction field of $A_*/\mathfrak{q} \cap A_*$ and the image of T_1 by the quotient morphism $B_s \longrightarrow \mathfrak{B}_s/\mathfrak{q}$. As a result, by (i) and [MA86, Ch. 6, Theorem 17.4], \mathfrak{q} has height at least m . As a minimal prime ideal of \mathfrak{B}_s containing m elements, \mathfrak{q} has height at most m . Hence all minimal prime ideal of \mathfrak{B}_s , containing a_1, \dots, a_m , has height m . So, by (i) and [MA86, Ch. 6, Theorem 17.4], a_1, \dots, a_m is a regular sequence in the localization of \mathfrak{B}_s at \mathfrak{p} . \square

4.3. Normality and Cohen-Macaulayness property for X_R . Let V_0 be a fixed point under the action of R in X_R and β_1, \dots, β_d the elements of \mathcal{R}_{V_0} . By Corollary 2.22(ii), β_1, \dots, β_d is a basis of \mathfrak{t}^* . Let t_1, \dots, t_d be the dual basis. Denote by m the codimension of V_0 in \mathfrak{a} . According to Lie's Theorem, for $m > 0$, the elements $\gamma_1, \dots, \gamma_m$ of $\mathcal{R} \setminus \{\beta_1, \dots, \beta_d\}$ can be ordered so that

$$\mathfrak{a}_i := V_0 \oplus \mathfrak{a}^{\gamma_1} \oplus \dots \oplus \mathfrak{a}^{\gamma_i}$$

is an algebra of codimension $m - i$ of \mathfrak{a} for $i = 1, \dots, m$. Set:

$$\begin{aligned} \mathcal{R}' &:= \mathcal{R} \setminus \{\gamma_m\}, & \mathfrak{a}' &= \mathfrak{a}_{m-1}, & \mathfrak{r}' &:= \mathfrak{r}_{\mathfrak{t}, \mathfrak{a}'}, & \pi' &:= \pi_{\mathfrak{t}, \mathfrak{a}'}, & R' &:= R_{\mathfrak{t}, \mathfrak{a}'}, & A' &:= A_{\mathfrak{t}, \mathfrak{a}'}, \\ E &:= \bigoplus_{i=1}^m \mathfrak{a}^{\gamma_i}, & E' &:= E \cap \mathfrak{a}'. \end{aligned}$$

Denote by κ the map

$$\text{Hom}_{\mathbb{k}}(V_0, E \oplus \mathfrak{t}) \xrightarrow{\kappa} \text{Gr}_d(\mathfrak{r}), \quad \varphi \longmapsto \text{span}(\{v + \varphi(v) \mid v \in V_0\}).$$

Then κ is an isomorphism from $\text{Hom}_{\mathbb{k}}(V_0, E \oplus \mathfrak{t})$ onto an affine open neighbourhood of V_0 in $\text{Gr}_d(\mathfrak{r})$. Moreover, there is a short exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{k}}(V_0, \mathbb{k}x_{\gamma_m}) \longrightarrow \text{Hom}_{\mathbb{k}}(V_0, E \oplus \mathfrak{t}) \xrightarrow{p} \text{Hom}_{\mathbb{k}}(V_0, E' \oplus \mathfrak{t}) \longrightarrow 0.$$

Let Ω and Ω' be the inverse images by κ of the intersections of the image of κ with X_R and $X_{R'}$ respectively. For φ in Ω and $i = 1, \dots, d$

$$\varphi(v_i) = \sum_{j=1}^d z_{i,j}(\varphi)t_j + \sum_{j=1}^m a_{i,j}(\varphi)x_{\gamma_j}$$

so that the $z_{i,j}$'s, $1 \leq i, j \leq d$ and the $a_{i,j}$'s, $1 \leq i \leq d$ and $1 \leq j \leq m$ are regular functions on Ω .

Let ψ be the map

$$\mathbb{k} \times \Omega' \xrightarrow{\psi} X_R, \quad (s, \varphi) \mapsto \exp(\text{ad } x_{\gamma_m}) \cdot \kappa(\varphi).$$

Lemma 4.7. *Let O be the subset of elements (s, φ) of $\mathbb{k} \times \Omega'$ such that $\psi(s, \varphi)$ is in $\kappa(\Omega)$.*

(i) *The subset O of $\mathbb{k} \times \Omega'$ is open and contains $\{0\} \times \Omega'$.*

(ii) *The map*

$$O \xrightarrow{\bar{\psi}} \Omega, \quad (s, \varphi) \mapsto \kappa^{-1} \circ \psi(s, \varphi)$$

is a birational morphism from O to Ω . In particular, the function $(s, \varphi) \mapsto s$ is in $\mathbb{k}(\Omega)$.

Proof. (i) As $\kappa(\Omega)$ is an open neighborhood of V_0 in X_R , O is an open subset of $\mathbb{k} \times \Omega'$, containing $\{0\} \times \Omega'$ since ψ is a regular map such that $\psi(0, \varphi) = \kappa(\varphi)$ for all φ in Ω' .

(ii) Let Ω^c be the subset of elements φ of Ω such that $\kappa(\varphi)$ is in A.t. Then Ω^c is a dense open subset of Ω . Let O' be the inverse image of Ω^c by $\bar{\psi}$. Let (s, φ) and (s', φ') be in O' such that $\bar{\psi}(s, \varphi) = \bar{\psi}(s', \varphi')$, that is

$$\exp(\text{ad } x_{\gamma_m}) \cdot \kappa(\varphi) = \exp(\text{ad } x_{\gamma_m}) \cdot \kappa(\varphi') \quad \text{whence} \quad \exp((s - s') \text{ad } x_{\gamma_m}) \cdot \kappa(\varphi) = \kappa(\varphi').$$

According to the above notations, for $i = 1, \dots, d$,

$$\varphi(v_i) = \sum_{j=1}^d z_{i,j}(\varphi)t_j + \sum_{j=1}^{m-1} a_{i,j}(\varphi)x_{\gamma_j}.$$

Since $\kappa(\varphi)$ is in A.t.,

$$\det([z_{i,j}(\varphi), 1 \leq i, j \leq d]) \neq 0.$$

For $i = 1, \dots, d$,

$$\exp((s - s') \text{ad } x_{\gamma_m}) \left(\sum_{j=1}^d z_{i,j}(\varphi)t_j \right) = \sum_{j=1}^d z_{i,j}(\varphi)t_j - (s - s') \left(\sum_{j=1}^d z_{i,j}(\varphi)\gamma_m(t_j) \right) x_{\gamma_m}.$$

For some j , $\gamma_m(t_j) \neq 0$, whence $s = s'$ since $\kappa(\varphi')$ is contained in \mathfrak{r}' . As a result, the restriction of $\bar{\psi}$ to O' is injective, whence the assertion since $\bar{\psi}$ is a dominant morphism. \square

For $i = 1, \dots, d$ and γ in \mathfrak{t}^* , denote by $u_{i,\gamma}$ the function on Ω ,

$$u_{i,j} := z_{i,1}\gamma(t_1) + \dots + z_{i,d}\gamma(t_d).$$

Let \mathfrak{A} be the subalgebra of $\mathbb{k}[\Omega]$ generated by the functions $z_{i,j}$'s, $1 \leq i, j \leq d$ and $a_{i,j}$'s, $1 \leq i \leq d$ and $1 \leq j \leq m - 1$.

Lemma 4.8. *Let ι be the restriction morphism from Ω to Ω' .*

- (i) *The restriction of ι to \mathfrak{A} is an isomorphism onto $\mathbb{k}[\Omega']$.*
- (ii) *For $1 \leq i, j \leq d$, $u_{i,\gamma_m} a_{j,m} - u_{j,\gamma_m} a_{i,m}$ is equal to 0.*
- (iii) *For $i = 1, \dots, d$ and γ in \mathfrak{t}^* , if $\gamma(t_i) \neq 0$ then $u_{i,\gamma}$ is different from 0.*

Proof. (i) For $1 \leq i, j \leq d$, denote by $z'_{i,j}$ the restriction of $z_{i,j}$ to Ω' and for $1 \leq i \leq d$ and $1 \leq j \leq m-1$ denote by $a'_{i,j}$ the restriction of $a_{i,j}$ to Ω' . Since $\mathbb{k}[\Omega']$ is generated by the functions

$$z'_{i,j}, 1 \leq i, j \leq d \quad \text{and} \quad a'_{i,j}, 1 \leq i \leq d, 1 \leq j \leq m-1,$$

the restriction of ι to \mathfrak{A} is surjective. Let \mathfrak{p} be the kernel of the restriction of ι to \mathfrak{A} . It remains to prove $\mathfrak{p} = \{0\}$.

For $1 \leq i, j \leq d$ and $k = 1, \dots, m-1$, denote by $\bar{z}_{i,j}$ and $\bar{a}_{i,k}$ the functions on $\mathbb{k} \times \Omega'$ such that

$$\begin{aligned} & \exp(\text{sad } x_{\gamma_m})(v_i + \sum_{j=1}^d z'_{i,j}(\varphi)t_j + \sum_{k=1}^{m-1} a'_{i,k}(\varphi)x_{\gamma_k}) - \\ & (\sum_{j=1}^d \bar{z}_{i,j}(s, \varphi)t_j - \sum_{j=1}^d s z_{i,j}(\varphi)\gamma_m(t_j)x_{\gamma_m} + \sum_{k=1}^{m-1} \bar{a}_{i,k}(s, \varphi)x_{\gamma_k}) \in V_0. \end{aligned}$$

Then $\bar{z}_{i,j}$ and $\bar{a}_{i,k}$ are regular functions on $\mathbb{k} \times \Omega'$ as restrictions to $\mathbb{k} \times \Omega'$ of regular functions on $\mathbb{k} \times \text{Hom}(V_0, E' \oplus \mathfrak{t})$. Let $\bar{\mathfrak{A}}$ be the subalgebra of $\mathbb{k}[\Omega'][[s]]$ generated by the functions

$$\bar{z}_{i,j}, i, j = 1, \dots, d \quad \text{and} \quad \bar{a}_{i,k}, i = 1, \dots, d, k = 1, \dots, m-1.$$

Since $z'_{i,j}(\varphi) = \bar{z}_{i,j}(0, \varphi)$ and $a'_{i,k}(\varphi) = \bar{a}_{i,k}(0, \varphi)$ for all φ in Ω' , the restriction to $\bar{\mathfrak{A}}$ of the quotient morphism $\mathbb{k}[\Omega'][[s]] \longrightarrow \mathbb{k}[\Omega']$ is surjective. As a result, $\bar{\mathfrak{A}}$ has dimension n or $n-1$ since Ω' and $\mathbb{k}[\Omega'][[s]]$ have dimension $n-1$ and n respectively. As $\exp(\text{sad } x_{\gamma_m})(v_i)$ is not necessarily equal to v_i ,

$$p \circ \psi \neq (\bar{z}_{i,j}, \bar{a}_{i,j}, 1 \leq i \leq d, 1 \leq j \leq m-1).$$

Moreover, Ω' is contained in $p(\Omega)$ by Lemma 4.7(i) but the inclusion may be strict.

Claim 4.9. The algebra $\bar{\mathfrak{A}}$ has dimension $n-1$.

Proof. [Proof of Claim 4.9] There are two cases to consider:

- (1) for $i = 1, \dots, m-1$, $[\alpha^{\gamma_m}, \alpha^{\gamma_i}]$ is contained in V_0 ,
- (2) for some i in $\{1, \dots, m-1\}$, $[\alpha^{\gamma_m}, \alpha^{\gamma_i}]$ is not contained in V_0 .

In the first case, $\bar{\mathfrak{A}} = \mathbb{k}[\Omega']$. Otherwise, denote by j the biggest integer such that $[\alpha^{\gamma_m}, \alpha^{\gamma_j}]$ is not contained in V_0 and $a'_{i,j} \neq 0$ for some $i = 1, \dots, d$. Then, for some j' smaller than j , $\gamma_m + \gamma_j = \gamma_{j'}$. Furthermore, for $k < j$ such that $[\alpha^{\gamma_m}, \alpha^{\gamma_k}]$ is not contained in V_0 , $\gamma_m + \gamma_k$ is in $\mathcal{R} \setminus \{\gamma_{j'}, \dots, \gamma_m\}$. Then for $k \geq j'$ and $i = 1, \dots, d$, $a'_{i,k} = \bar{a}_{i,k}$ and for all (s, φ) in $\mathbb{k} \times \Omega'$,

$$\bar{a}_{i,j'}(s, \varphi) = a'_{i,j'}(\varphi) + s a'_{i,j}(\varphi).$$

As a result, by induction on $m-k$, for $i = 1, \dots, d$,

$$a'_{i,k} - \bar{a}_{i,k} \in s \bar{\mathfrak{A}}[s].$$

Hence $\mathbb{k}[\Omega'][[s]] = \bar{\mathfrak{A}}[s]$ and there exists a surjective morphism $\mathbb{k}[\Omega'] \longrightarrow \bar{\mathfrak{A}}$ so that $\bar{\mathfrak{A}}$ has dimension $n-1$. \square

According to Lemma 4.7(ii), the comorphism of $\overline{\psi}$ is an embedding of $\mathbb{k}[\Omega]$ into $\mathbb{k}[O]$ and from this embedding results an isomorphism from $\mathbb{k}(\Omega)$ onto $\mathbb{k}(\Omega')(s)$. Moreover, \mathfrak{A} is the image of \mathfrak{A} by this embedding so that \mathfrak{A} has dimension $n - 1$. As a result, $\mathfrak{p} = \{0\}$ since ι is surjective and Ω' has dimension $n - 1$.

(ii) Let φ be in Ω . Since $\kappa(\varphi)$ is a commutative algebra, for $1 \leq i, j \leq d$,

$$0 = [v_i + \varphi(v_i), v_j + \varphi(v_j)] = [v_i, \varphi(v_j)] + [\varphi(v_i), v_j] + [\varphi(v_i), \varphi(v_j)].$$

The component on x_{γ_m} of the right hand side is

$$\sum_{k=1}^d (z_{i,k} a_{j,m}(\varphi) - z_{j,k} a_{i,m}(\varphi)) [t_k, x_{\gamma_m}] = (u_{i,\gamma_m} a_{j,m} - u_{j,\gamma_m} a_{i,m})(\varphi) x_{\gamma_m},$$

whence the assertion.

(iii) Denote by R_0 the adjoint group of $r_0 := t + V_0$ and X_{R_0} the closure in $\text{Gr}_d(r_0)$ of $R_0.t$. Let Ω_0 be the inverse image of X_{R_0} by κ . According to Lemma 4.1, for $i, j = 1, \dots, d$, the restriction to Ω_0 of $z_{i,j}$ is equal to 0 if $j \neq i$, otherwise it is different from 0. As a result, for $i = 1, \dots, d$ and γ in t^* , the restriction of $u_{i,\gamma}$ to Ω_0 is equal to $\overline{z_{i,i}}\gamma(t_i)$ with $\overline{z_{i,i}}$ the restriction of $z_{i,i}$ to Ω_0 , whence the assertion. \square

For γ in t^* , set:

$$I_\gamma := \{j \in \{1, \dots, d\} \mid \gamma(t_j) \neq 0\}.$$

Proposition 4.10. *Denote by $\mathbb{k}[\Omega]_0$ the localization of $\mathbb{k}[\Omega]$ at 0.*

(i) *The local algebra $\mathbb{k}[\Omega]_0$ is Cohen-Macaulay.*

(ii) *For γ in t^* , $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in $\mathbb{k}[\Omega]_0$ of elements of its maximal ideal.*

Proof. Prove the proposition by induction on m . By Lemma 4.1, for $m = 0$, $\mathbb{k}[\Omega]$ is a polynomial algebra of dimension d , generated by $z_{1,1,0}, \dots, z_{d,d,0}$. Moreover, for $i = 1, \dots, d$ and γ in t^* , $u_{i,\gamma} = z_{i,i}\gamma(t_i)$, whence the proposition for $m = 0$. Suppose $m > 0$ and the proposition true for $m - 1$ and use the notations of Lemma 4.8.

According to Lemma 4.8(i) and the induction hypothesis, the localization \mathfrak{A}_* of \mathfrak{A} at 0 is Cohen-Macaulay and for γ in t^* , $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in \mathfrak{A}_* of elements of its maximal ideal. Denote by \mathfrak{B} the polynomial algebra $\mathfrak{A}_*[T_i, i \in I_\gamma]$ and by P the ideal of \mathfrak{B} generated by the sequence $u_{i,\gamma_m} T_j - u_{j,\gamma_m} T_i, (i, j) \in I_\gamma^2$. According to Condition (3) of Section 2, $s := |I_\gamma| \geq 2$. By Lemma 4.8(ii), $\mathbb{k}[\Omega]_0$ is a quotient of the localization at 0 of \mathfrak{B}/P and by Lemma 4.2, P is a prime ideal of \mathfrak{B} . By Proposition 4.6(i), \mathfrak{B}/P is an integral domain and a Cohen-Macaulay algebra of dimension n since $\mathbb{k}[\Omega']$ has dimension $n - 1$. Hence $\mathbb{k}[\Omega]_0$ is the localization of \mathfrak{B}/P at 0 since $\mathbb{k}[\Omega]_0$ is an integral domain of dimension n . As a result, $\mathbb{k}[\Omega]_0$ is Cohen-Macaulay and by Proposition 4.6(ii), for γ in t^* , the sequence $u_{i,\gamma}, i \in I_\gamma$ is regular in $\mathbb{k}[\Omega]_0$. \square

Theorem 4.11. *The variety X_R is normal and Cohen-Macaulay.*

Proof. By Corollary 3.8, X_R is smooth in codimension 1. So, by Serre's normality criterion [Bou98, §1, no 10, Théorème 4], it suffices to prove that X_R is Cohen-Macaulay. According to [MA86, Ch. 8, Theorem 24.5], the set of points x of X_R such that $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay, is open. For x in X_R , the closure in X_R of $R.x$ contains a fixed point. So it suffices to prove that for x a fixed point under the action of R in X_R , $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay. Let V_0 and Ω be as in Lemma 4.7. Then Ω is an affine open neighborhood of V_0 in X_R . By Proposition 4.10(i), $\mathcal{O}_{\Omega,0}$ is Cohen-Macaulay, whence the theorem since κ is an isomorphism from Ω onto an open neighborhood of V_0 in X_R and $\kappa(0) = V_0$. \square

4.4. **Nipotent cone and regular sequence in $\mathcal{O}_{\mathcal{E}}$.** Let β_1, \dots, β_d be a basis of \mathfrak{t}^* . For $i = 1, \dots, d$, denote again by β_i the element of \mathfrak{r}^* extending β_i and equal to 0 on \mathfrak{a} . For Λ a complete subset of \mathcal{R} , denote by $\mathfrak{t}_{\Lambda}^{\#}$ a complement to \mathfrak{t}_{Λ} in \mathfrak{t} and set

$$R'_{\Lambda} := R_{\mathfrak{t}_{\Lambda}^{\#}, \mathfrak{a}_{\Lambda}} \quad \text{and} \quad \mathcal{E}_{\Lambda} := \mathcal{E}_{\mathfrak{t}_{\Lambda}^{\#}, \mathfrak{a}_{\Lambda}}.$$

For Y closed subset of $X_{R'_{\Lambda}}$, denote by $\mathcal{E}_{\Lambda, Y}$ the restriction of \mathcal{E}_{Λ} to Y . Let \mathcal{N}'_{Λ} be the image of the map

$$\mathcal{E}_{\Lambda, X_{R'_{\Lambda}, n}} \longrightarrow \mathcal{E}, \quad (V, x) \longmapsto (V \oplus \mathfrak{t}_{\Lambda}, x)$$

and \mathcal{N}_{Λ} the closure in \mathcal{E} of $R \cdot \mathcal{N}'_{\Lambda}$.

Lemma 4.12. *For $i = 1, \dots, d$, let $\tilde{\beta}_i$ be the function on \mathcal{E} defined by $\tilde{\beta}_i(V, x) = \beta_i(x)$. Denote by \mathcal{N} the nullvariety of $\tilde{\beta}_1, \dots, \tilde{\beta}_d$ in \mathcal{E} .*

- (i) *For all complete subset Λ of \mathcal{R} , \mathcal{N}_{Λ} is a subvariety of \mathcal{N} of dimension at most n .*
- (ii) *The variety \mathcal{N} is the union of \mathcal{N}_{Λ} , $\Lambda \in \mathcal{P}_c(\mathcal{R})$.*
- (iii) *The variety \mathcal{N} is equidimensional of dimension n .*

Proof. (i) Since \mathfrak{a} is the nullvariety of β_1, \dots, β_d in \mathfrak{r} , \mathcal{N} is the intersection of \mathcal{E} and $X_R \times \mathfrak{a}$. By definition \mathcal{N}'_{Λ} is contained in $X_R \times \mathfrak{a}$. Hence \mathcal{N}_{Λ} is contained in \mathcal{N} . By Proposition 3.7,

$$\dim \mathcal{N}'_{\Lambda} = \dim \mathfrak{t}_{\Lambda}^{\#} + \dim X_{R'_{\Lambda}, n} \leq \dim \mathfrak{a}_{\Lambda}.$$

Since the image of $X_{R'_{\Lambda}, n}$ by the map $V \mapsto V \oplus \mathfrak{t}_{\Lambda}$ is invariant by R_{Λ} ,

$$\dim \mathcal{N}_{\Lambda} \leq \dim \mathcal{N}'_{\Lambda} + \dim \mathfrak{a} - \dim \mathfrak{a}_{\Lambda} \leq \dim \mathfrak{a}.$$

(ii) Let ϖ_1 be the bundle projection of the vector bundle \mathcal{E} over X_R and τ_1 the restriction to \mathcal{E} of the projection $X_R \times \mathfrak{r} \longrightarrow \mathfrak{r}$. Let T be an irreducible component of \mathcal{N} . For all V in $\varpi_1(T)$, $\tau_1(\varpi_1^{-1}(V) \cap T)$ is a closed cone of \mathfrak{a} . Hence $\varpi_1(T) \times \{0\}$ is the intersection of T and $X_R \times \{0\}$ so that $\varpi_1(T)$ is a closed subset of X_R . Since \mathcal{N} is the intersection of \mathcal{E} and $X_R \times \mathfrak{a}$, \mathcal{N} and its irreducible components are invariant under R . As a result, $\varpi_1(T)$ is invariant under R and by Proposition 3.1, for some complete subset Λ of \mathcal{R} and for some closed subset of Z_{Λ} of $X_{R_{\Lambda}}$, $\varpi_1(T) = \overline{R \cdot Z_{\Lambda}}$. Moreover, by Lemma 3.2, for some closed subset Z'_{Λ} of $X_{R'_{\Lambda}, n}$, Z_{Λ} is the image of Z'_{Λ} by the map $V \mapsto V \oplus \mathfrak{t}_{\Lambda}$. As a result,

$$\mathcal{E}_{\Lambda, Z'_{\Lambda}} \subset \mathcal{E}_{\Lambda, X_{R'_{\Lambda}, n}} \quad \text{and} \quad \varpi_1^{-1}(Z_{\Lambda}) \cap X_R \times \mathfrak{a} \subset \mathcal{N}'_{\Lambda}.$$

Then T is contained in \mathcal{N}_{Λ} , whence the assertion by (i).

(iii) By (i) and (ii), since \mathcal{R} is finite, the irreducible components of \mathcal{N} have dimension at most n . As the nullvariety of d functions on the irreducible variety \mathcal{E}_{X_R} , the irreducible components of \mathcal{N} have dimension at least n , whence the assertion. \square

For x in \mathcal{E} , denote by I_x the subset of elements i of $\{1, \dots, d\}$ such that $\tilde{\beta}_i(x) = 0$.

Corollary 4.13. *For all x in \mathcal{E} , the sequence $\tilde{\beta}_i, i \in I_x$ is regular in $\mathcal{O}_{\mathcal{E}, x}$.*

Proof. According to Lemma 4.12, for all subset I of $\{1, \dots, d\}$, the nullvariety of $\tilde{\beta}_i, i \in I$ in \mathcal{E} is equidimensional of dimension $n + d - |I|$. By Theorem 4.11 and Lemma B.1(iii), \mathcal{E} is Cohen-Macaulay as a vector bundle over a Cohen-Macaulay variety, whence the corollary by [MA86, Ch. 6, Theorem 17.4]. \square

5. RATIONAL SINGULARITIES FOR SOLVABLE LIE ALGEBRAS

Let \mathfrak{t} be a vector space of positive dimension d . Denote by $\mathcal{C}_{\mathfrak{t},*}$ the full subcategory of $\mathcal{C}_{\mathfrak{t}}$ whose objects \mathfrak{a} satisfy the following condition:

- (4) there exist regular maps $\varepsilon_1, \dots, \varepsilon_d$ from $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ to $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ such that $\varepsilon_1(x), \dots, \varepsilon_d(x)$ is a basis of $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}^x$ for all x in $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}\text{reg}}$.

According to [Ko63, Theorem 9], \mathfrak{u} is in $\mathcal{C}_{\mathfrak{b},*}$.

Lemma 5.1. *Let \mathfrak{a} be in $\mathcal{C}_{\mathfrak{t},*}$ and \mathfrak{a}' an ideal of $\mathfrak{t} + \mathfrak{a}$, contained in \mathfrak{a} and containing a fixed point under the action of $R_{\mathfrak{t},\mathfrak{a}}$ in $X_{R_{\mathfrak{t},\mathfrak{a}}}$. Then \mathfrak{a}' is in $\mathcal{C}_{\mathfrak{t},*}$.*

Proof. Set $\mathfrak{r} := \mathfrak{t} + \mathfrak{a}$ and $\mathfrak{r}' := \mathfrak{t} + \mathfrak{a}'$. According to Corollary 2.22(ii), \mathfrak{a}' is in $\mathcal{C}_{\mathfrak{t}}$ since it is in $\mathcal{C}'_{\mathfrak{t}}$. Set $\mathfrak{t}_{\text{reg}} := \mathfrak{r}_{\text{reg}} \cap \mathfrak{t}$. As $\mathcal{R}_{\mathfrak{t},\mathfrak{a}'}$ is contained in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, $\mathfrak{t}_{\text{reg}}$ is contained in $\mathfrak{r}'_{\text{reg}}$ by Lemma 3.3(i). Then $\mathfrak{r}'_{\text{reg}}$ is contained in $\mathfrak{r}_{\text{reg}}$ and for all x in $A_{\mathfrak{t},\mathfrak{a}'\text{reg}}$, $\mathfrak{r}^x = \mathfrak{r}'^x$ since $A_{\mathfrak{t},\mathfrak{a}'\text{reg}}$ is a dense open subset of \mathfrak{r}' by Lemma 3.3(i). So, for all regular map ε from \mathfrak{r} to \mathfrak{r} such that $[x, \varepsilon(x)] = 0$ for all x in \mathfrak{r} , $\varepsilon(x)$ is in \mathfrak{r}' for all x in \mathfrak{r}' , whence the lemma. \square

Let \mathfrak{a} be in $\mathcal{C}_{\mathfrak{t},*}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \quad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \quad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \quad R := R_{\mathfrak{t},\mathfrak{a}}, \quad A := A_{\mathfrak{t},\mathfrak{a}}, \quad \mathcal{E} := \mathcal{E}_{\mathfrak{t},\mathfrak{a}}, \quad n := \dim \mathfrak{a}.$$

The goal of the section is to prove that X_R is Gorenstein with rational singularities.

For k positive integer, set:

$$\mathcal{E}^{(k)} := \{(u, x_1, \dots, x_k) \in X_R \times \mathfrak{r}^k \mid u \ni x_1, \dots, u \ni x_k\}$$

and denote by $\mathfrak{X}_{R,k}$ the image of $\mathcal{E}^{(k)}$ by the projection

$$(u, x_1, \dots, x_k) \mapsto (x_1, \dots, x_k).$$

Since X_R is a projective variety, $\mathfrak{X}_{R,k}$ is a closed subset of \mathfrak{r}^k , invariant under the diagonal action of R in \mathfrak{r}^k .

5.1. Differential forms on some smooth open subsets of $\mathfrak{X}_{R,k}$. For $j = 1, \dots, k$, let $V_j^{(k)}$ be the subset of elements of $\mathfrak{X}_{R,k}$ whose j -th component is in $\mathfrak{r}_{\text{reg}}$.

Lemma 5.2. *For $j = 1, \dots, k$, $V_j^{(k)}$ is a smooth open subset of $\mathfrak{X}_{R,k}$. Moreover, $\Omega_{V_j^{(k)}}$ has a global section without zero.*

Proof. Denoting by σ_j the automorphism of \mathfrak{r}_k which permutes the first and the j -th component, $\mathfrak{X}_{R,k}$ is invariant under σ_j and $\sigma_j(V_1^{(k)}) = V_j^{(k)}$ so that we can suppose $j = 1$. Moreover, for $k = 1$, $\mathfrak{X}_{R,k} = \mathfrak{r}$ so that we can suppose $k \geq 2$. By definition, $V_1^{(k)}$ is the intersection of $\mathfrak{r}_{\text{reg}} \times \mathfrak{r}^{k-1}$ and $\mathfrak{X}_{R,k}$. Hence $V_1^{(k)}$ is an open subset of $\mathfrak{X}_{R,k}$ since $\mathfrak{r}_{\text{reg}}$ is an open subset of \mathfrak{r} .

Let $\varepsilon_1, \dots, \varepsilon_d$ satisfying Condition (4) with respect to \mathfrak{r} . Let θ be the map

$$\mathfrak{r}_{\text{reg}} \times M_{k-1,d}(\mathbb{k}) \xrightarrow{\theta} \mathfrak{r}^k, \quad (x, a_{i,j}, 2 \leq i \leq k, 1 \leq j \leq d) \mapsto (x, \sum_{j=1}^d a_{i,j} \varepsilon_j(x)).$$

Since for all (x, x_2, \dots, x_k) in $V_1^{(k)}$, x_2, \dots, x_k are in \mathfrak{r}^x , θ is a bijective map onto $V_1^{(k)}$. The open subset $\mathfrak{r}_{\text{reg}}$ has a cover by open subsets V such that for some e_1, \dots, e_n in \mathfrak{r} , $\varepsilon_1(x), \dots, \varepsilon_d(x), e_1, \dots, e_n$ is a basis of \mathfrak{r} for all x in V . Then there exist regular functions $\varphi_1, \dots, \varphi_d$ on $V \times \mathfrak{r}$ such that

$$v - \sum_{j=1}^d \varphi_j(x, v) \varepsilon_j(x) \in \text{span}(\{e_1, \dots, e_n\})$$

for all (x, v) in $V \times \mathfrak{r}$, so that the restriction of θ to $V \times M_{k-1,d}(\mathbb{k})$ is an isomorphism onto $\mathfrak{X}_{R,k} \cap V \times \mathfrak{r}^{k-1}$ whose inverse is

$$(x_1, \dots, x_k) \mapsto (x_1, ((\varphi_1(x_1, x_i), \dots, \varphi_d(x_1, x_i)), i = 2, \dots, k))$$

As a result, θ is an isomorphism and $V_1^{(k)}$ is a smooth variety. Since $\mathfrak{r}_{\text{reg}}$ is a smooth open subset of the vector space \mathfrak{r} , there exists a regular differential form ω of top degree on $\mathfrak{r}_{\text{reg}} \times M_{k-1,\ell}(\mathbb{k})$, without zero. Then $\theta_*(\omega)$ is a regular differential form of top degree on $V_1^{(k)}$, without zero. \square

For $k \geq 2$ set:

$$V^{(k)} := V_1^{(k)} \cup V_2^{(k)} \quad \text{and} \quad V_{1,2}^{(k)} := V_1^{(k)} \cap V_2^{(k)}.$$

For $2 \leq k' \leq k$, the projection

$$\mathfrak{r}^k \longrightarrow \mathfrak{r}^{k'}, \quad (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{k'})$$

induces the projection

$$\mathfrak{X}_{R,k} \longrightarrow \mathfrak{X}_{R,k'}, \quad V_j^{(k)} \longrightarrow V_j^{(k')}$$

for $j = 1, \dots, k'$.

Lemma 5.3. *Suppose $k \geq 2$. Let ω be a regular differential form of top degree on $V_1^{(k)}$, without zero. Denote by ω' its restriction to $V_{1,2}^{(k)}$.*

- (i) *For φ in $\mathbb{k}[V_1^{(k)}]$, if φ has no zero then φ is in \mathbb{k}^* .*
- (ii) *For some invertible element ψ of $\mathbb{k}[V_{1,2}^{(2)}]$, $\omega' = \psi\sigma_{2*}(\omega')$.*
- (iii) *The function $\psi(\psi\sigma_2)$ on $V_{1,2}^{(k)}$ is equal to 1.*

Proof. The existence of ω results from Lemma 5.2.

(i) According to Lemma 5.2, there is an isomorphism θ from $\mathfrak{r}_{\text{reg}} \times M_{k-1,d}(\mathbb{k})$ onto $V_1^{(k)}$. Since φ is invertible, $\varphi \circ \theta$ is an invertible element of $\mathbb{k}[\mathfrak{r}_{\text{reg}}]$. According to Lemma 3.3(iii), $\mathbb{k}[\mathfrak{r}_{\text{reg}}] = \mathbb{k}[\mathfrak{r}]$. Hence φ is in \mathbb{k}^* .

(ii) The open subset $V_{1,2}^{(k)}$ is invariant under σ_2 so that ω' and $\sigma_{2*}(\omega')$ are regular differential forms of top degree on $V_{1,2}^{(k)}$, without zero. Then for some invertible element ψ of $\mathbb{k}[V_{1,2}^{(k)}]$, $\omega' = \psi\sigma_{2*}(\omega')$. Let O_2 be the set of elements $(x, a_{i,j}, 1 \leq i \leq k-1, 1 \leq j \leq d)$ of $\mathfrak{r}_{\text{reg}} \times M_{k-1,d}(\mathbb{k})$ such that

$$a_{1,1}\varepsilon_1(x) + \dots + a_{1,\ell}\varepsilon_\ell(x) \in \mathfrak{r}_{\text{reg}}.$$

Then O_2 is the inverse image of $V_{1,2}^{(k)}$ by θ . As a result, $\mathbb{k}[V_{1,2}^{(k)}]$ is a polynomial algebra over $\mathbb{k}[V_{1,2}^{(2)}]$ since for $k = 2$, O_2 is the inverse image by θ of $V_{1,2}^{(2)}$. Hence ψ is in $\mathbb{k}[V_{1,2}^{(2)}]$ since ψ is invertible.

(iii) Since the restriction of σ_2 to $V_{1,2}^{(k)}$ is an involution,

$$\sigma_{2*}(\omega') = (\psi \circ \sigma_2)\omega' = (\psi \circ \sigma_2)\psi\sigma_{2*}(\omega'),$$

whence $(\psi \circ \sigma_2)\psi = 1$. \square

Corollary 5.4. *The function ψ is invariant under the action of R in $V_{1,2}^{(k)}$ and for some sequence $m_\alpha, \alpha \in \mathcal{R}$ in \mathbb{Z} ,*

$$\psi(x_1, \dots, x_k) = \pm \prod_{\alpha \in \mathcal{R}} (\alpha(x_1)\alpha(x_2)^{-1})^{m_\alpha},$$

for all (x_1, \dots, x_k) in $\mathfrak{t}_{\text{reg}}^2 \times \mathfrak{t}^{k-2}$.

Proof. First of all, since $V_1^{(k)}$ and $V_2^{(k)}$ are invariant under the action of R in $\mathfrak{X}_{R,k}$, so is $V_{1,2}^{(k)}$. Let g be in R . As ω has no zero, $g.\omega = p_g\omega$ for some invertible element p_g of $\mathbb{k}[V_1^{(k)}]$. By Lemma 5.3(i), p_g is in \mathbb{k}^* . Since σ_2 is a R -equivariant isomorphism from $V_1^{(k)}$ onto $V_2^{(k)}$,

$$g.\sigma_{2*}(\omega) = p_g\sigma_{2*}(\omega) \quad \text{and} \quad p_g\omega' = g.\omega' = (g.\psi)g.\sigma_{2*}(\omega') = p_g(g.\psi)\sigma_{2*}(\omega'),$$

whence $g.\psi = \psi$.

The open subset t_{reg}^2 of t^2 is the complement to the nullvariety of the function

$$(x, y) \longmapsto \prod_{\alpha \in \mathcal{R}} \alpha(x)\alpha(y).$$

Then, by Lemma 5.3(ii), for some a in \mathbb{k}^* and for some sequences $m_\alpha, n_\alpha, \alpha \in \mathcal{R}$ and $n_\alpha, \alpha \in \mathcal{R}$ in \mathbb{Z} ,

$$\psi(x_1, \dots, x_k) = a \prod_{\alpha \in \mathcal{R}} \alpha(x_1)^{m_\alpha} \alpha(x_2)^{n_\alpha},$$

for all (x_1, \dots, x_k) in $t_{\text{reg}}^2 \times t^{k-2}$. By Lemma 5.3(iii),

$$a^2 \prod_{\alpha \in \mathcal{R}} \alpha(x)^{m_\alpha + n_\alpha} \alpha(y)^{m_\alpha + n_\alpha} = 1,$$

for all (x, y) in t_{reg}^2 . Hence $a^2 = 1$ and $m_\alpha + n_\alpha = 0$ for all α in \mathcal{R} . \square

According to Lemma 3.5(i), for α in \mathcal{R} , θ_α is a bijective regular map from $\mathbb{P}^1(\mathbb{k})$ onto the closed subset Z_α of X_R such that $\theta_\alpha(\infty) = V_\alpha$. Recall that x_α is a generator of α^α and h_α is an element of t such that $\alpha(h_\alpha) = 1$. Denote by t'_α the subset of elements x of t_α such that $\gamma(x) \neq 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$. According to Condition (3) of Section 2, t'_α is a dense open subset of t_α . Let $x_{-\alpha}$ be in r^* orthogonal to $t + \alpha^\gamma$ for all γ in $\mathcal{R} \setminus \{\alpha\}$ and such that $x_{-\alpha}(x_\alpha) = 1$.

Lemma 5.5. *Suppose $k \geq 2$. Let α be in \mathcal{R} , x_0 and y_0 in t'_α . Set:*

$$E := \mathbb{k}x_0 \oplus \mathbb{k}h_\alpha \oplus \alpha^\alpha, \quad E_* := x_0 \oplus \mathbb{k}h_\alpha \oplus \alpha^\alpha, \quad E_{*,1} := x_0 \oplus \mathbb{k}h_\alpha \oplus (\alpha^\alpha \setminus \{0\}), \quad E_{*,2} = y_0 \oplus \mathbb{k}h_\alpha \oplus (\alpha^\alpha \setminus \{0\}).$$

- (i) For x in E_* , r^x is contained in $t_\alpha + E$.
- (ii) For V subspace of dimension d of $t_\alpha + E$, V is in X_R if and only if it is in Z_α .
- (iii) The intersection of $E_{*,1} \times E_{*,2}$ and $\mathfrak{X}_{R,2}$ is the nullvariety of the function

$$(x, y) \longmapsto x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y)$$

on $E_{*,1} \times E_{*,2}$.

Proof. (i) If x is regular semisimple, its component on h_α is different from 0 so that $r^x = \theta_\alpha(z)$ for some z in \mathbb{k} . Suppose that x is not regular semisimple. Then x is in $x_0 + \alpha^\alpha$. Hence r^x is contained in $t_\alpha + E$ since so is r^{x_0} .

(ii) All element of Z_α is contained in $t_\alpha + E$. Let V be an element of X_R , contained in $t_\alpha + E$. According to Corollary 2.22(i), V is an algebraic commutative subalgebra of dimension d of r . By (i), $V = \theta_\alpha(z)$ for some z in \mathbb{k} if V is in $A.t$. Otherwise, x_α is in V . Then $V = \theta_\alpha(\infty)$ since $\theta_\alpha(\infty)$ is the centralizer of x_α in $t_\alpha + E$.

(iii) Let (x, y) be in $E_{*,1} \times E_{*,2} \cap \mathfrak{X}_{R,2}$. By definition, for some V in X_R , x and y are in V . By (i) and (ii), $V = \theta_\alpha(z)$ for some z in $\mathbb{P}^1(\mathbb{k})$. For z in \mathbb{k} ,

$$x = x_0 + s(h_\alpha - zx_\alpha) \quad \text{and} \quad y = y_0 + s'(h_\alpha - zx_\alpha)$$

for some s, s' in \mathbb{k} , whence the equality of the assertion. For $z = \infty$,

$$x = x_0 + sx_\alpha \quad \text{and} \quad y = y_0 + s'x_\alpha$$

for some s, s' in \mathbb{k} so that $\alpha(x) = \alpha(y) = 0$. Conversely, let (x, y) be in $E_{*,1} \times E_{*,2}$ such that

$$x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y) = 0.$$

If $\alpha(x) = 0$ then $\alpha(y) = 0$ and x and y are in $V_\alpha = \theta_\alpha(\infty)$. If $\alpha(x) \neq 0$, then $\alpha(y) \neq 0$ and

$$x \in \theta_\alpha\left(-\frac{x_{-\alpha}(x)}{\alpha(x)}\right) \quad \text{and} \quad y \in \theta_\alpha\left(-\frac{x_{-\alpha}(x)}{\alpha(x)}\right),$$

whence the assertion. □

Set $V^{(1)} := r_{\text{reg}}$.

Proposition 5.6. *For k positive integer, there exists on $V^{(k)}$ a regular differential form of top degree without zero.*

Proof. For $k = 1$, it is true since r_{reg} is an open subset of the vector sapce r . So we can suppose $k \geq 2$. According to Corollary 5.4, it suffices to prove $m_\alpha = 0$ for all α in \mathcal{R} . Indeed, if so, by Corollary 5.4, $\psi = \pm 1$ on the open subset $R.(t_{\text{reg}}^2 \times t^{k-2})$ of $V^{(k)}$ so that $\psi = \pm 1$ on $V_{1,2}^{(k)}$. Then, by Lemma 5.3(ii), ω and $\pm\sigma_{2*}(\omega)$ have the same restriction to $V_{1,2}^{(k)}$ so that there exists a regular differential form of top degree $\tilde{\omega}$ on $V^{(k)}$ whose restrictions to $V_1^{(k)}$ and $V_2^{(k)}$ are ω and $\pm\sigma_{2*}(\omega)$ respectively. Moreover, $\tilde{\omega}$ has no zero since so has ω .

Since ψ is in $\mathbb{k}[V_{1,2}^{(2)}]$ by Lemma 5.3(ii), we can suppose $k = 2$. Let α be in \mathcal{R} , $E, E_*, E_{*,1}, E_{*,2}$ as in Lemma 5.3. Suppose $m_\alpha \neq 0$. A contradiction is expected. The restriction of ψ to $E_{*,1} \times E_{*,2} \cap V_{1,2}^{(2)}$ is given by

$$\psi(x, y) = ax_{-\alpha}(x)^m x_{-\alpha}(y)^n,$$

with a in \mathbb{k}^* and (m, n) in \mathbb{Z}^2 since ψ is an invertible element of $\mathbb{k}[V_{1,2}^{(2)}]$. According to Lemma 5.3(iii), $n = -m$ and $a = \pm 1$. Interchanging the role of x and y , we can suppose m in \mathbb{N} . For (x, y) in $E_{*,1} \times E_{*,2} \cap V_{1,2}^{(2)}$ such that $\alpha(x) \neq 0, \alpha(y) \neq 0$ and

$$\psi(x, y) = \pm x_{-\alpha}(x)^m \left(\frac{x_{-\alpha}(x)\alpha(y)}{\alpha(x)}\right)^{-m} = \pm \alpha(x)^m \alpha(y)^{-m}.$$

As a result, by Corollary 5.4, for x in $x_0 + \mathbb{k}^*h_\alpha$ and y in $y_0 + \mathbb{k}^*h_\alpha$,

$$(1) \quad \pm \alpha(x)^m \alpha(y)^{-m} = \pm \prod_{\gamma \in \mathcal{R}} \gamma(x)^{m_\gamma} \gamma(y)^{-m_\gamma}.$$

For γ in \mathcal{R} ,

$$\gamma(x) = \gamma(x_0) + \alpha(x)\gamma(h_\alpha) \quad \text{and} \quad \gamma(y) = \gamma(y_0) + \alpha(y)\gamma(h_\alpha).$$

Since m is in \mathbb{N} ,

$$(2) \quad \begin{aligned} & \pm \alpha(x)^m \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma > 0}} (\gamma(y_0) + \alpha(y)\gamma(h_\alpha))^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma < 0}} (\gamma(x_0) + \alpha(x)\gamma(h_\alpha))^{-m_\gamma} = \\ & \pm \alpha(y)^m \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma > 0}} (\gamma(x_0) + \alpha(x)\gamma(h_\alpha))^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma < 0}} (\gamma(y_0) + \alpha(y)\gamma(h_\alpha))^{-m_\gamma}. \end{aligned}$$

For m_α positive, the terms of lowest degree in $(\alpha(x), \alpha(y))$ of left and right sides are

$$\pm \alpha(x)^m \alpha(y)^{m_\alpha} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma > 0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma < 0}} \gamma(x_0)^{-m_\gamma} \quad \text{and} \quad \pm \alpha(y)^m \alpha(x)^{m_\alpha} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma > 0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma < 0}} \gamma(y_0)^{-m_\gamma}$$

respectively and for m_α negative, the terms of lowest degree in $(\alpha(x), \alpha(y))$ of left and right sides are

$$\pm \alpha(x)^{m+m_\alpha} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma > 0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma < 0}} \gamma(x_0)^{-m_\gamma} \quad \text{and} \quad \pm \alpha(y)^{m+m_\alpha} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma > 0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma < 0}} \gamma(y_0)^{-m_\gamma}$$

respectively. From the equality of these terms, we deduce $m = \pm m_\alpha$ and

$$\prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma > 0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma < 0}} \gamma(x_0)^{-m_\gamma} = \pm \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma > 0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \setminus \{\alpha\} \\ m_\gamma < 0}} \gamma(y_0)^{-m_\gamma}.$$

Since the last equality does not depend on the choice of x_0 and y_0 in t'_α , this equality remains true for all (x_0, y_0) in $t_\alpha \times t_\alpha$. As a result, as the degrees in $\alpha(x)$ of the left and right sides of Equality (2) are the same,

$$(3) \quad m - \sum_{\substack{\gamma \in \mathcal{R} \\ m_\gamma < 0 \text{ and } \gamma(h_\alpha) \neq 0}} m_\gamma = \sum_{\substack{\gamma \in \mathcal{R} \\ m_\gamma > 0 \text{ and } \gamma(h_\alpha) \neq 0}} m_\gamma.$$

Suppose $m = m_\alpha$. By Equality (1),

$$\prod_{\gamma \in \mathcal{R} \setminus \{\alpha\}} \gamma(x)^{m_\gamma} \gamma(y)^{-m_\gamma} = \pm 1.$$

Since this equality does not depend on the choice of x_0 and y_0 in t'_α , it holds for all (x, y) in $t_{\text{reg}} \times t_{\text{reg}}$. Hence $m_\gamma = 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$ and $m = 0$ by Equality (3). It is impossible since $m_\alpha \neq 0$. Hence $m = -m_\alpha$. Then, by Equality (1)

$$\prod_{\gamma \in \mathcal{R} \setminus \{\alpha\}} \gamma(x)^{m_\gamma} \gamma(y)^{-m_\gamma} = \pm \alpha(x)^{2m} \alpha(y)^{-2m}.$$

Since this equality does not depend on the choice of x_0 and y_0 in t'_α , it holds for all (x, y) in $t_{\text{reg}} \times t_{\text{reg}}$. Then $m = 0$, whence the contradiction. \square

5.2. Rational singularities and Gorensteinness of X_R . For Y subvariety of $\text{Gr}_d(r)$, denote by \mathcal{E}_Y the restriction to Y of the tautological vector bundle of rank d over $\text{Gr}_d(r)$. In particular, for Y contained in X_R , \mathcal{E}_Y is a subvariety of \mathcal{E} . For k positive integer, denote by τ_k and ϖ_k the restrictions to $\mathcal{E}^{(k)}$ of the canonical projections

$$X_R \times r^k \xrightarrow{\tau_k} r^k \quad \text{and} \quad X_R \times r^k \xrightarrow{\varpi_k} X_R.$$

Lemma 5.7. (i) *The morphism τ_k is a projective and birational morphism onto $\mathfrak{X}_{R,k}$.*

(ii) *The sets $V^{(k)}$ and $\tau_k^{-1}(V^{(k)})$ are smooth open subsets of $\mathfrak{X}_{R,k}$ and $\mathcal{E}^{(k)}$. Moreover, for $k \geq 2$, they are big open subsets of $\mathfrak{X}_{R,k}$ and $\mathcal{E}^{(k)}$.*

(iii) *The restriction of τ_k to $\tau_k^{-1}(V^{(k)})$ is an isomorphism onto $V^{(k)}$.*

Proof. Since X_R is a projective variety, τ_k is projective and its image is $\mathfrak{X}_{R,k}$ by definition. For (x_1, \dots, x_k) in $V^{(k)}$ and (u, x_1, \dots, x_k) in $\tau_k^{-1}((x_1, \dots, x_k))$, $u = r^{x_1}$ if x_1 is in r_{reg} and $u = r^{x_2}$ if x_2 is in r_{reg} . As a result, the restriction of τ_k to $\tau_k^{-1}(V^{(k)})$ is a bijective morphism onto $V^{(k)}$. Hence τ_k is a birational morphism and by Zariski's Main Theorem [Mu88, §9], this restriction is an isomorphism since $V^{(k)}$ is a smooth variety by Lemma 5.2. So it remains to prove that for $k \geq 2$, $\tau_k^{-1}(V^{(k)})$ is a big open subset of $\mathcal{E}^{(k)}$

Suppose that $\mathcal{E}^{(k)} \setminus \tau_k^{-1}(V^{(k)})$ has an irreducible component Σ of dimension $\dim \mathcal{E}^{(k)} - 1$. A contradiction is expected. Since $\mathcal{E}^{(k)}$ and $\tau_k^{-1}(V^{(k)})$ are invariant under the automorphisms of $X_R \times r^k$,

$$(u, x_1, \dots, x_k) \mapsto (u, tx_1, \dots, tx_k), \quad (t \in \mathbb{k}^*),$$

so is Σ . Then $\Sigma \cap X_R \times \{0\} = \varpi_k(\Sigma) \times \{0\}$ so that $\varpi_k(\Sigma)$ is a closed subset of X_R . Since $\dim \Sigma = \dim \mathcal{E}^{(k)} - 1$, $\dim \varpi_k(\Sigma) \geq \dim X_R - 1$. Suppose $\dim \Sigma = \dim X_R - 1$. Then for all u in $\varpi_k(\Sigma)$, $\{u\} \times u^k$ is in Σ . It is

impossible since for all u in a dense open subset of $\varpi_k(\Sigma)$, $u = r^x$ for some x in r_{reg} by Corollary 3.8. Hence $\varpi_k(\Sigma) = X_R$. Then for all u in a dense open subset of X'_R , $\{u\} \times u^k \cap \Sigma$ has codimension 1 in $\{u\} \times u^k$. Since the image of $\{u\} \times u^k \cap \Sigma$ by the projection

$$(u, x_1, \dots, x_k) \mapsto x_1$$

is not dense in u , for all x_1 in a dense open subset of its image, $\{u\} \times \{x_1\} \times u^{k-1}$ is contained in Σ , whence the contradiction since $u \cap r_{\text{reg}}$ is not empty. \square

By definition, $\mathcal{E}^{(k)}$ is the inverse image of X_R by the bundle projection of the vector bundle

$$\{u, x_1, \dots, x_k\} \in \text{Gr}_d(r) \times r^k \mid u \ni x_1, \dots, u \ni x_k$$

over $\text{Gr}_d(r)$ so that $\mathcal{E}^{(k)}$ is vector bundle of rank kd over X_R . In particular, $\mathcal{E}^{(1)} = \mathcal{E}$. According to [Hir64], there exists a desingularization Γ of X_R with morphism ρ such that the restriction of ρ to $\rho^{-1}(X_{R\text{sm}})$ is an isomorphism onto $X_{R\text{sm}}$. Let $\widetilde{\mathcal{E}}^{(1)}$ be the following fiber product

$$\begin{array}{ccc} \widetilde{\mathcal{E}}^{(1)} & \xrightarrow{\bar{\rho}} & \mathcal{E}^{(1)} \\ \downarrow & & \downarrow \varpi_1 \\ \Gamma & \xrightarrow{\rho} & X_R \end{array}$$

with $\bar{\rho}$ the restriction map. Then $\widetilde{\mathcal{E}}^{(1)}$ is a vector bundle of rank d over Γ . In particular, it is a smooth variety since Γ is smooth.

Let O be a trivialization open subset of the vector bundle $\mathcal{E}^{(1)}$ and let Φ_1 be a local trivialization over O of $\mathcal{E}^{(1)}$, whence the following commutative diagram

$$\begin{array}{ccc} \varpi_1^{-1}(O) & \xrightarrow{\Phi_1} & O \times \mathbb{k}^d \\ & \searrow \varpi_1 & \downarrow \text{pr}_1 \\ & & O \end{array}$$

Then O is a trivialization open subset of the vector bundle $\mathcal{E}^{(k)}$. The variety $\mathcal{E}^{(1)}$ is a closed subbundle of $\mathcal{E}^{(k)}$ over X_R and for some local trivialization Φ over O of $\mathcal{E}^{(k)}$, we have the following commutative diagram

$$\begin{array}{ccc} \varpi_k^{-1}(O) & \xrightarrow{\Phi} & O \times \mathbb{k}^{kd} \\ & \searrow \varpi_k & \downarrow \text{pr}_1 \\ & & O \end{array}$$

Φ_1 is the restriction of Φ to $\varpi_1^{-1}(O)$ and $\Phi(\varpi_1^{-1}(O)) = O \times \mathbb{k}^d \times \{0\}$.

Lemma 5.8. *Suppose $k \geq 2$. Denote by μ a generator of $\Omega_{\mathbb{k}^{kd}}$ and by $\tilde{\rho}$ the restriction of $\rho \times \text{id}_{\mathbb{k}^{kd}}$ to $\rho^{-1}(O) \times \mathbb{k}^{kd}$.*

- (i) *The sheaf $\Omega_{\mathcal{E}^{(k)}_{\text{sm}}}$ has a global section ω without zero.*
- (ii) *The sheaf $\Omega_{O_{\text{sm}}}$ has a global section ω_O without zero.*
- (iii) *For some p in $\mathbb{k}[O \times \mathbb{k}^{kd}] \setminus \{0\}$, $\tilde{\rho}^*(p(\omega_O \wedge \mu))$ has a regular extension to $\rho^{-1}(O) \times \mathbb{k}^{kd}$.*

Proof. (i) According to Proposition 5.6 and Lemma 5.7(iii), $\Omega_{\tau_k^{-1}(V^{(k)})}$ has a global section without zero. By Lemma 5.7(ii), $\tau_k^{-1}(V^{(k)})$ is a smooth big open subset of $\mathcal{E}^{(k)}$. So, by Lemma A.1, $\Omega_{\mathcal{E}^{(k)}_{\text{sm}}}$ has a global section without zero.

(ii) Since μ is a generator of $\Omega_{\mathbb{k}^{kd}}$, there exists a unique ν in $\mathbb{k}[\mathbb{k}^{kd}] \otimes_{\mathbb{k}} \Gamma(O_{\text{sm}}, \Omega_{O_{\text{sm}}})$ such that

$$\Phi_*(\omega|_{\varpi_k^{-1}(O_{\text{sm}})}) = \nu \wedge \mu.$$

Moreover, ν has no zero since so has ω . Let V be an affine open subset of O_{sm} such that the restriction of $\Omega_{O_{\text{sm}}}$ to V is locally free, generated by the local section ω_V . Then for some p_V in $\mathbb{k}[V \times \mathbb{k}^{kd}]$,

$$(4) \quad \Phi_*(\omega|_{\varpi_k^{-1}(V)}) = p_V \omega_V \wedge \mu.$$

Then p_V has no zero since so has $\nu \wedge \mu$. As a result, p_V is in $\mathbb{k}[V]$ and $p_V \omega_V$ is a local section of $\Omega_{O_{\text{sm}}}$ without zero. By the unicity of the decomposition (4), for two different affine open subsets V and V' as above, the differential forms $p_V \omega_V$ and $p_{V'} \omega_{V'}$ have the same restriction to $V \cap V'$. As a result, since such affine open subsets cover O_{sm} , for some global section ω_O of $\Omega_{O_{\text{sm}}}$,

$$\Phi_*(\omega|_{\varpi_k^{-1}(O_{\text{sm}})}) = \omega_O \wedge \mu.$$

Moreover, ω_O is unique and has no zero.

(iii) Let ω_1 be a generator of $\Omega_{\mathbb{k}^d}$ and let μ_1 be a generator of $\Omega_{\mathbb{k}^d}$. By (i), $\omega_O \wedge \mu_1$ is a global section of $\Omega_{O_{\text{sm}} \times \mathbb{k}^d}$, without zero. So for some regular function p on $O_{\text{sm}} \times \mathbb{k}^d$,

$$(5) \quad \Phi_{1*}((\tau_1)^*(\omega_1)|_{\varpi_1^{-1}(O_{\text{sm}})}) = p \omega_O \wedge \mu_1.$$

According to Theorem 4.11, X_R is normal. Then so is O and p has a regular extension to $O \times \mathbb{k}^d$. Denote again by p this extension. According to Equality (5), the differential form $\tilde{\rho}^*(p \omega_O \wedge \mu_1)$ on $\rho^{-1}(O_{\text{sm}}) \times \mathbb{k}^d$ has a regular extension to $\rho^{-1}(O) \times \mathbb{k}^d$. In fact, denoting by $\overline{\Phi}_1$ the local trivialization over $\rho^{-1}(O)$ of $\overline{\mathcal{E}}^{(1)}$ such that the following diagram

$$\begin{array}{ccc} (\varpi_1 \circ \overline{\rho}^{-1})(O) & \xrightarrow{\overline{\Phi}_1} & \rho^{-1}(O) \times \mathbb{k}^d \\ \overline{\rho} \downarrow & & \downarrow \tilde{\rho} \\ \varpi_1^{-1}(O) & \xrightarrow{\Phi_1} & O \times \mathbb{k}^d \end{array}$$

is commutative, it is the restriction to $\rho^{-1}(O_{\text{sm}}) \times \mathbb{k}^d$ of

$$\overline{\Phi}_{1*}((\tau_1 \circ \overline{\rho})^*(\omega_1)|_{(\varpi_1 \circ \overline{\rho}^{-1})^{-1}(O)}).$$

For some generator μ' of $\Omega_{\mathbb{k}^{(k-1)d}}$, $\mu = \mu_1 \wedge \mu'$ and $\mathbb{k}[O \times \mathbb{k}^d]$ is naturally embedded in $\mathbb{k}[O \times \mathbb{k}^{kd}]$. As a result, $\tilde{\rho}^*(p \omega_O \wedge \mu)$ has a regular extension to $\rho^{-1}(O) \times \mathbb{k}^{kd}$. \square

Proposition 5.9. *The variety X_R is Gorenstein with rational singularities.*

Proof. According to Theorem 4.11, X_R is normal and Cohen-Macaulay. Then by Lemma 5.8,(ii) and (iii) and Corollary A.5, $O \times \mathbb{k}^{kd}$ is Gorenstein with rational singularities. Then so is O by Lemma B.1,(i) and (ii). Since there is a cover of X_R by open subsets as O , X_R is Gorenstein with rational singularities. \square

As already mentioned, u is in $\mathcal{C}_{\mathfrak{b},*}$, whence Theorem 1.1 by Proposition 5.9.

APPENDIX A. RATIONAL SINGULARITIES

Let X be an affine irreducible normal variety.

Lemma A.1. *Let Y be a smooth big open subset of X .*

(i) *All regular differential form of top degree on Y has a unique regular extension to X_{sm} .*

(ii) *Suppose that ω is a regular differential form of top degree on Y , without zero. Then the regular extension of ω to X_{sm} has no zero.*

Proof. (i) Since $\Omega_{X_{\text{sm}}}$ is a locally free module of rank one, there is an affine open cover O_1, \dots, O_k of X_{sm} such that the restriction of $\Omega_{X_{\text{sm}}}$ to O_i is a free \mathcal{O}_{O_i} -module generated by some section ω_i . For $i = 1, \dots, k$, set $O'_i := O_i \cap Y$. Let ω be a regular differential form of top degree on Y . For $i = 1, \dots, k$, for some regular function a_i on O'_i , $a_i\omega_i$ is the restriction of ω to O'_i . As Y is a big open subset of X , O'_i is a big open subset of O_i . Hence a_i has a regular extension to O_i since O_i is normal. Denoting again by a_i this extension, for $1 \leq i, j \leq k$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O'_i \cap O'_j$ and $O_i \cap O_j$ since $\Omega_{X_{\text{sm}}}$ is torsion free as a locally free module. Let ω' be the global section of $\Omega_{X_{\text{sm}}}$ extending the $a_i\omega_i$'s. Then ω' is a regular extension of ω to X_{sm} and this extension is unique since Y is dense in X_{sm} and $\Omega_{X_{\text{sm}}}$ is torsion free.

(ii) Suppose that ω has no zero. Let Σ be the nullvariety of ω' in X_{sm} . If it is not empty, Σ has codimension 1 in X_{sm} . As Y is a big open subset of X , $\Sigma \cap X_{\text{sm}}$ is not empty if so is Σ . As a result, Σ is empty. \square

Denote by ι the canonical injection from X_{sm} into X .

Lemma A.2. *Suppose that $\Omega_{X_{\text{sm}}}$ has a global section ω without zero. Then the \mathcal{O}_X -module $\iota_*(\Omega_{X_{\text{sm}}})$ is free of rank 1. More precisely, the morphism θ :*

$$\mathcal{O}_X \xrightarrow{\theta} \iota_*(\Omega_{X_{\text{sm}}}), \quad \psi \mapsto \psi\omega$$

is an isomorphism.

Proof. For φ a local section of $\iota_*(\Omega_{X_{\text{sm}}})$ above the open subset U of X , for some regular function ψ on $U \cap X_{\text{sm}}$, $\psi\omega$ is the restriction of φ to $U \cap X_{\text{sm}}$. Since X is normal, so is U and $U \cap X_{\text{sm}}$ is a big open subset of U . Hence ψ has a regular extension to U . As a result, there exists a well defined morphism from $\iota_*(\Omega_{X_{\text{sm}}})$ to \mathcal{O}_X whose inverse is θ . \square

According to [Hir64], X has a desingularization Z with morphism τ such that the restriction of τ to $\tau^{-1}(X_{\text{sm}})$ is an isomorphism onto X_{sm} . For U open subset of X , denote by τ_U the restriction of τ to $\tau^{-1}(U)$.

Proposition A.3. *Suppose that X is Cohen-Macaulay and that there exists a morphism $\mu : \mathcal{O}_Z \longrightarrow \Omega_Z$ such that for some p in $\mathbb{k}[X]$, $\tau_*\mu$ is an isomorphism onto $p\tau_*(\Omega_Z)$. Then X has rational singularities.*

The following proof is the weak variation of the proof of [Hi91, Lemma 2.3].

Proof. Since Z and X are varieties over \mathbb{k} , we have the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\tau} & X \\ & \searrow p & \swarrow q \\ & \text{Spec}(\mathbb{k}) & \end{array}$$

According to [H66, V. §10.2], $p^1(\mathbb{k})$ and $q^1(\mathbb{k})$ are dualizing complexes over Z and X respectively. Furthermore, by [H66, VII, 3.4] or [Hi91, 4.3,(ii)], $p^1(\mathbb{k})[-\dim Z]$ equals Ω_Z . Set $\mathcal{D} := q^1(\mathbb{k})[-\dim Z]$ so that $\tau^1(\mathcal{D}) = \Omega_Z$ by [H66, VII, 3.4] or [Hi91, 4.3,(iv)]. In particular, \mathcal{D} is dualizing over X .

Since τ is a projective morphism, we have the isomorphism

$$(6) \quad R\tau_*(R\mathcal{H}om_Z(\Omega_Z, \Omega_Z)) \longrightarrow R\mathcal{H}om_X(R(\tau)_*(\Omega_Z), \mathcal{D})$$

by [H66, VII, 3.4] or [Hi91, 4.3,(iii)]. Since $H^i(R\mathcal{H}om_Z(\Omega_Z, \Omega_Z)) = \mathcal{O}_Z$ for $i = 0$ and 0 for $i > 0$, the left hand side of (6) can be identified with $R\tau_*(\mathcal{O}_Z)$.

According to Grauert-Riemenschneider Theorem [GR70], $R\tau_*(\Omega_Z)$ has only cohomology in degree 0. Since τ is projective and birational and Z is normal, $\tau_*(\mathcal{O}_Z) = \mathcal{O}_X$. So by assumption of the proposition,

$$R\tau_*(\Omega_Z) \approx \frac{1}{p}\mathcal{O}_X,$$

whence

$$R\mathcal{H}om_X(R(\tau)_*(\Omega_Z), \mathcal{D}) \approx p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}$$

and (6) can be rewritten as

$$(7) \quad R\tau_*(\mathcal{O}_Z) \approx p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}.$$

Since X is Cohen-Macaulay, \mathcal{D} has cohomology in only one degree. So, by flatness of the \mathcal{O}_X -module $p\mathcal{O}_X$, $p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}$ has cohomology in only one degree. As a result, by (7), $R^i\tau_*(\mathcal{O}_Z) = 0$ for $i > 0$, that is X has rational singularities. \square

Denote by \mathcal{M} the cohomology in degree 0 of \mathcal{D} .

Lemma A.4. *Suppose that X has rational singularities. Then the \mathcal{O}_X -modules $\tau_*(\Omega_Z)$ and \mathcal{M} are isomorphic. In particular, $\tau_*(\Omega_Z)$ has finite injective dimension.*

Proof. Since X has rational singularities, $R\tau_*(\mathcal{O}_Z) = \mathcal{O}_X$ and \mathcal{D} has only cohomology in degree 0. Moreover, by Grauert-Riemenschneider Theorem [GR70], $R\tau_*(\Omega_Z)$ has only cohomology in degree 0, whence $R\tau_*(\Omega_Z) = \tau_*(\Omega_Z)$. Then, by (6), we have the isomorphism

$$\mathcal{O}_X \longrightarrow \mathcal{H}om_X((\tau)_*(\Omega_Z), \mathcal{M}).$$

As \mathcal{D} is dualizing, we have the isomorphism

$$R\tau_*(\Omega_Z) \longrightarrow R\mathcal{H}om_X(R\mathcal{H}om_X(R\tau_*(\Omega_Z), \mathcal{D}), \mathcal{D})$$

whence the isomorphism $\tau_*(\Omega_Z) \longrightarrow \mathcal{M}$ by (6). As a result, $\tau_*(\Omega_Z)$ has finite injective dimension since so has \mathcal{M} . \square

Corollary A.5. *Let Y be a smooth big open subset of X . Suppose that the following conditions are verified:*

- (1) X is Cohen-Macaulay,
- (2) Ω_Y has a global section ω without zero,
- (3) for some global section ω_Z of Ω_Z and for some p in $\mathbb{k}[X] \setminus \{0\}$, the restriction of ω_Z to $\tau^{-1}(Y)$ is equal to $p\tau_Y^*(\omega)$.

Then X is Gorenstein with rational singularities. Moreover, its canonical module is free of rank 1.

Proof. According to Lemma A.1(ii), ω has a unique regular extension to X_{sm} and this extension has no zero. Denote again by ω this extension. Since Z is irreducible, $\tau^{-1}(Y)$ is dense in $\tau^{-1}(X_{\text{sm}})$ so that the restriction of ω_Z to $\tau^{-1}(X_{\text{sm}})$ is equal to $p\tau_{X_{\text{sm}}}^*(\omega)$ since Ω_Z has no torsion. Denote by μ the morphism

$$\mathcal{O}_Z \xrightarrow{\mu} \Omega_Z, \quad \varphi \longmapsto \varphi\omega_Z.$$

Let U be an open subset of X and ν a local section of $\tau_*(\Omega_Z)$ above U . Since ω has no zero and $\tau_{U_{\text{sm}}}$ is an isomorphism onto U_{sm} ,

$$\nu|_{\tau^{-1}(U_{\text{sm}})} = \tau_{U_{\text{sm}}}^*(\varphi\omega|_{U_{\text{sm}}})$$

for some φ in $\mathbb{k}[U_{\text{sm}}]$, whence

$$p\nu|_{\tau^{-1}(U_{\text{sm}})} = \varphi \circ \tau_{U_{\text{sm}}}(\omega_Z|_{\tau^{-1}(U_{\text{sm}})})$$

by Condition (3). Since X is normal, so is U and U_{sm} is a big open subset of U . Hence φ has a regular extension to U . Denoting again by φ this extension,

$$p\nu = \varphi \circ \tau_U(\omega_Z|_{\tau^{-1}(U)})$$

since Z is irreducible and Ω_Z has no torsion. As a result the morphism

$$\tau_*\mu : \tau_*(\mathcal{O}_Z) \longrightarrow p\tau_*(\Omega_Z)$$

is an isomorphism since it is obviously injective. So, by Proposition A.3, X has rational singularities. In particular, by [KK73, p.50], $\tau_*(\Omega_X) = \iota_*(\Omega_X)$. Then, by Lemma A.2, the canonical module of X is free of rank 1 and by Lemma A.4, X is Gorenstein. \square

APPENDIX B. ABOUT SINGULARITIES

In this section we recall a well known result. Let X be a variety and Y a vector bundle over X . Denote by τ the bundle projection.

Lemma B.1. (i) *If Y is Gorenstein, then X is Gorenstein.*

(ii) *The variety X has rational singularities if and only if so has Y .*

(iii) *If X is Cohen-Macaulay, then so is Y .*

Proof. Let y be in Y , $x := \tau(y)$. Denote by $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{Y,y}}$ the completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ respectively.

(i) Since Y is a vector bundle over X , $\widehat{\mathcal{O}_{Y,y}}$ is a ring of formal series over $\widehat{\mathcal{O}_{X,x}}$. By [Bru, Proposition 3.1.19,(c)], $\widehat{\mathcal{O}_{Y,y}}$ is Gorenstein. So, by [Bru, Proposition 3.1.19,(b)], $\widehat{\mathcal{O}_{X,x}}$ is Gorenstein. Then by [Bru, Proposition 3.1.19,(c)], $\mathcal{O}_{X,x}$ is Gorenstein, whence the assertion.

(ii) Since Y is a vector bundle over X , then there exists a cover of X by open subsets O , such that $\tau^{-1}(O)$ is isomorphic to $O \times \mathbb{k}^m$ with $m = \dim Y - \dim X$. According to [KK73, p.50], $O \times \mathbb{k}^m$ has rational singularities if and only if so has O , whence the assertion since a variety has rational singularities if and only if it has a cover by open subsets having rational singularities.

(iii) According to [MA86, Ch. 6, Theorem 17.7], a polynomial algebra over a Cohen-Macaulay algebra is Cohen-Macaulay. Hence for O open subset of X as in (ii), $\tau^{-1}(O)$ is Cohen-Macaulay, whence the assertion since there is a cover of Y by open subsets as $\tau^{-1}(O)$. \square

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