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ON A VARIETY RELATED TO THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA.

JEAN-YVES CHARBONNEL

Abstract. For a reductive Lie algbera over an algbraically closed field of charasteristic zero, we consider a Borel subgroup B of its adjoint group, a Cartan subalgebra contained in the Lie algebra of B and the closure X of its orbit under B in the Grassmannian. The variety X plays an important role in the study of the commuting variety. In this note, we prove that X is Gorenstein with rational singularities.

Contents

1. Introduction 1
2. On solvable algebras 3
3. Solvable algebras and main varieties 17
4. Normality for solvable Lie algebras 22
5. Rational singularities for solvable Lie algebras 30
Appendix A. Rational Singularities 37
Appendix B. About singularities 39
References 39

1. Introduction

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and B the normalizer of $\mathfrak{b}$ in G.

1.1. **Main results.** Let X be the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of B. By a well known result, $G.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of G. By [Ri79], the commuting variety of $\mathfrak{g}$ is the image by the canonical projection of the restriction to $G.X$ of the canonical vector bundle of rank 2ℓ over $\mathrm{Gr}_\ell(\mathfrak{g})$. So X and $G.X$ play an important role in the study of the commuting variety. As it is explained in [CZ16], X and $G.X$ play the same role for the so called generalized commuting varieties and the so called generalized isospectral commuting varieties. The main result of this note is the following theorem:

Theorem 1.1. *The variety X is Gorenstein with rational singulatrities.*

An induction is used to prove this theorem. So we introduce the categories $\mathcal{C}'_{\mathfrak{t}}$ and $\mathcal{C}_{\mathfrak{t}}$ with $\mathfrak{t}$ a commutative Lie algebra of finite dimension. Their objects are nilpotent Lie algebras of finite dimension, normalized by $\mathfrak{t}$ with additional conditions analogous to those of the action of $\mathfrak{h}$ in $\mathfrak{u}$. In particular the minimal dimension

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of the objects in $\mathcal{C}_{\mathfrak{t}}$ is the dimension of $\mathfrak{t}$ and an object of dimension $\dim \mathfrak{t}$ is a commutative algebra. The category $\mathcal{C}_{\mathfrak{t}}$ is a full subcategory of $\mathcal{C}'_{\mathfrak{t}}$. For $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$, we consider the solvable Lie algebra $\mathfrak{r} := \mathfrak{t} + \mathfrak{a}$ and R the adjoint group of $\mathfrak{r}$. Denoting by X_R the closure in $\mathrm{Gr}_{\dim \mathfrak{t}}(\mathfrak{r})$ of the orbit of $\mathfrak{t}$ under R, we prove by induction on $\dim \mathfrak{a}$ the following theorem:

Theorem 1.2. *The variety X_R is normal and Cohen-Macaulay.*

The result for the category $\mathcal{C}'_{\mathfrak{t}}$ is easily deduced from the result for the category $\mathcal{C}_{\mathfrak{t}}$ by Corollary 2.2. One of the key argument in the proof is the consideration of the fixed points under the action of R in X_R. As a matter of fact, since the closure of all orbit under R in X_R contains a fixed point, X_R is Cohen-Macaulay if so are the fixed points by openness of the set of Cohen-Macaulay points. Then, by Serre's normality criterion, it suffices to prove that X_R is smooth in codimension 1. For that purpose the consideration of the restriction to X_R of the tautological vector bundle of rank $\dim \mathfrak{t}$ over $\mathrm{Gr}_{\dim \mathfrak{t}}(\mathfrak{r})$ is very useful.

For the study of the fixed points, we introduce Property (**P**) and Property ($\mathbf{P}_1$) for the objects of $\mathcal{C}'_{\mathfrak{t}}$:

- Property (**P**) for $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$ says that for V in X_R, contained in the centralizer $\mathfrak{r}^s$ of an element s of $\mathfrak{t}$, V is in the closure of the orbit of $\mathfrak{t}$ under the centralizer R^s of s in R,
- Property ($\mathbf{P}_1$) for $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$ says that for V in X_R normalized by $\mathfrak{t}$ and such that $V \cap \mathfrak{t}$ is the center of $\mathfrak{r}$, then the non zero weights of $\mathfrak{t}$ in V are linearly independent.

Property ($\mathbf{P}_1$) for $\mathfrak{a}$ results from Property (**P**) for $\mathfrak{a}$ and Property (**P**) for $\mathfrak{a}$ results from Property ($\mathbf{P}_1$) for $\mathfrak{a}$ and Property (**P**) for the objects of $\mathcal{C}'_{\mathfrak{t}}$ of dimension smaller than $\dim \mathfrak{a}$. So, the main result for the objects of $\mathcal{C}'_{\mathfrak{t}}$ is the following proposition:

Proposition 1.3. *The objects of $\mathcal{C}'_{\mathfrak{t}}$ have Property* (**P**).

From this proposition, we deduce some structure property for the points of X_R.

The second part of Theorem 1.1, that is Gorensteinness property and Rational singularities, is obtained by considering a subcategory $\mathcal{C}_{\mathfrak{t},*}$ of $\mathcal{C}_{\mathfrak{t}}$. This category is defined by an additional condition on the objects. The main point for $\mathfrak{a}$ in $\mathcal{C}_{\mathfrak{t},*}$ is the following result:

Proposition 1.4. *Let $k \geqslant 2$ be an integer. Denote by $\mathcal{E}^{(k)}$ the R-equivariant vector subbundle of $X_R \times \mathfrak{r}^k$ whose fiber at $\mathfrak{t}$ is $\mathfrak{t}^k$. Then there exists on the smooth locus of $\mathcal{E}^{(k)}$ a regular differential form of top degree without zero.*

From Proposition 1.4 and Theorem 1.2, we deduce that $\mathcal{E}^{(k)}$ and X_R are Gorenstein with rational singularities.

This note is organized as follows. In Section 2, categories $\mathcal{C}'_{\mathfrak{t}}$ and $\mathcal{C}_{\mathfrak{t}}$ are introduced for some space $\mathfrak{t}$. In particular, $\mathfrak{u}$ is an object of $\mathcal{C}_{\mathfrak{b}}$. In Subsection 2.3, we define Property (**P**) for the objects of $\mathcal{C}'_{\mathfrak{t}}$ and we deduce some result on the structure of points of X_R. In Subsection 2.4, we define Property ($\mathbf{P}_1$) for the objects of $\mathcal{C}'_{\mathfrak{t}}$ and we prove that Property $(\mathbf{P})_1$ is a consequence of Property (**P**). In Subsection 2.5, we give some geometric constructions to prove Property (**P**) by induction on the dimension of $\mathfrak{a}$. At last, in Subsection 2.6, we prove Proposition 1.3. In particular, the proof of [CZ16, Lemma 4.4,(i)] is completed. In Section 3, we are interested in the singular locus of X_R. In Subsection 3.3, regularity in codimension 1 is proved with some additional properties analogous to those of [CZ16, Section 3]. Moreover, the constructions of Subsection 2.5 are used to prove the results by induction on the dimension of $\mathfrak{a}$. In Section 4, Cohen-Macaulayness property is proved by induction. In Section 5, the category $\mathcal{C}_{\mathfrak{t},*}$ is introduced and Proposition 1.4 is proved. Then with some results given in the appendix, we finish the proof of Theorem 1.1.

1.2. **Notations.** • An algebraic variety is a reduced scheme over $\Bbbk$ of finite type. For X an algebraic variety, its smooth locus is denoted by X_{sm}.

• Set $\Bbbk^* := \Bbbk \setminus \{0\}$. For V a vector space, its dual is denoted by V^*.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X, its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\mathcal{O}_X$ is its structural sheaf, $\Bbbk[X]$ is the algebra of regular functions on X, $\Bbbk(X)$ is the field of rational functions on X when X is irreducible and Ω_X is the sheaf of regular differential forms of top degree on X when X is smooth and irreducible.

• If E is a subset of a vector space V, denote by $\mathrm{span}(E)$ the vector subspace of V generated by E. The grassmannian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$.

• For $\mathfrak{a}$ a Lie algebra,V a subspace of $\mathfrak{a}$ and x in $\mathfrak{a}$, V^x denotes the centralizer of x in V. For A a subgroup of the group of automorphisms of $\mathfrak{a}$, A^x denotes the centralizer of x in A. An element x of $\mathfrak{g}$ is regular if $\mathfrak{g}^x$ has dimension ℓ and the set of regular elements of $\mathfrak{g}$ is denoted by $\mathfrak{g}_{\mathrm{reg}}$.

• The Lie algebra of an algberaic torus is also called a torus. In this note, a torus denoted by a gothic letter means the Lie algebra of an algebraic torus.

• For $\mathfrak{a}$ a Lie algebra, the Lie algebra of derivations of $\mathfrak{a}$ is denoted by $\mathrm{Der}(\mathfrak{a})$. By definition $\mathrm{Der}(\mathfrak{a})$ is the Lie algebra of the group $\mathrm{Aut}(\mathfrak{a})$ of the automorphisms of $\mathfrak{a}$.

• Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$ and $\mathfrak{u}$ the nilpotent radical of $\mathfrak{b}$.

2. On solvable algebras

Let $\mathfrak{t}$ be a vector space of positive dimension d. Denote by $\tilde{\mathcal{C}}_{\mathfrak{t}}$ the subcategory of the category of finite dimensional Lie algebras whose objects are finite dimensional nilpotent Lie algebras $\mathfrak{a}$ such that there exists a morphism

$$\mathfrak{t} \xrightarrow{\varphi_{\mathfrak{a}}} \mathrm{Der}(\mathfrak{a})$$

whose image is the Lie algebra of a subtorus of $\mathrm{Aut}(\mathfrak{a})$. For $\mathfrak{a}$ and $\mathfrak{a}'$ in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, a morphism ψ from $\mathfrak{a}$ to $\mathfrak{a}'$ is a morphism of Lie algebras such that $\psi \circ \varphi_{\mathfrak{a}}(t) = \varphi_{\mathfrak{a}'}(t) \circ \psi$ for all t in $\mathfrak{t}$. For x in $\mathfrak{t}$, x is a semisimple derivation of $\mathfrak{a}$. Denote by $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$ the set of weights of $\mathfrak{t}$ in $\mathfrak{a}$. Let $\mathcal{C}'_{\mathfrak{t}}$ be the full subcategory of objects $\mathfrak{a}$ of $\tilde{\mathcal{C}}_{\mathfrak{t}}$ verifying the following conditions:

(1) 0 is not in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$,
(2) for α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, the weight space of weight α has dimension 1,
(3) for α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, $\Bbbk\alpha \cap (\mathcal{R}_{\mathfrak{t},\mathfrak{a}} \setminus \{\alpha\})$ is empty.

For $\mathfrak{a}$ in $\mathcal{C}'_{\mathfrak{t}}$ and $\mathfrak{a}'$ a subalgebra of $\mathfrak{a}$, invariant under the adjoint action of $\mathfrak{t}$, $\mathfrak{a}'$ is in $\mathcal{C}'_{\mathfrak{t}}$. Denote by $\mathcal{C}_{\mathfrak{t}}$ the full subcategory of objects $\mathfrak{a}$ of $\mathcal{C}'_{\mathfrak{t}}$ such that $\varphi_{\mathfrak{a}}$ is an embedding. For example $\mathfrak{u}$ is in $\mathcal{C}_{\mathfrak{h}}$.

For $\mathfrak{a}$ in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, denote by $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ the solvable algebra $\mathfrak{t} + \mathfrak{a}$, $\pi_{\mathfrak{t},\mathfrak{a}}$ the quotient morphism from $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ to $\mathfrak{t}$, $R_{\mathfrak{t},\mathfrak{a}}$ the adjoint group of $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$, $A_{\mathfrak{t},\mathfrak{a}}$ the connected closed subgroup of $R_{\mathfrak{t},\mathfrak{a}}$ whose Lie algebra is $\mathrm{ad}\,\mathfrak{a}$, $X_{R_{\mathfrak{t},\mathfrak{a}}}$ the closure in $\mathrm{Gr}_d(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$ of the orbit of $\mathfrak{t}$ under $R_{\mathfrak{t},\mathfrak{a}}$ and $\mathcal{E}_{\mathfrak{t},\mathfrak{a}}$ the restriction to $X_{R_{\mathfrak{t},\mathfrak{a}}}$ of the tautological vector bundle over $\mathrm{Gr}_d(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$. The variety $X_{R_{\mathfrak{t},\mathfrak{a}}}$ is called *the main variety related to* $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$. For α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, let $\mathfrak{a}^\alpha$ be the weight space of weight α under the action of $\mathfrak{t}$ in $\mathfrak{a}$.

In the following subsections, a vector space $\mathfrak{t}$ of positive dimension d and an object $\mathfrak{a}$ of $\mathcal{C}'_{\mathfrak{t}}$ are fixed. We set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

Let $\mathfrak{z}$ be the orthogonal complement of $\mathcal{R}$ in $\mathfrak{t}$ and $d^{\#}$ its codimension in $\mathfrak{t}$. Then $n \geqslant d^{\#}$.

2.1. **General remarks on $\mathcal{C}'_{\mathfrak{t}}$.** For x in $\mathfrak{r}$, we say that x is semisimple if so is $\operatorname{ad} x$ and x is nilpotent if so is $\operatorname{ad} x$. For $\mathfrak{s}$ a commutative subalgebra of $\mathfrak{r}$, we say that $\mathfrak{s}$ is a torus if $\operatorname{ad}\mathfrak{s}$ is the Lie algebra of a subtorus of $\mathrm{GL}(\mathfrak{r})$.

Lemma 2.1. *Let x be in $\mathfrak{r}$ and $\mathfrak{s}$ a commutative subalgebra of $\mathfrak{r}$.*

(i) *The center of $\mathfrak{r}$ is equal to $\mathfrak{z}$.*

(ii) *The element x is semisimple if and only if $R.x \cap \mathfrak{t}$ is not empty.*

(iii) *The element x is nilpotent if and only if x is in $\mathfrak{z} + \mathfrak{a}$.*

(iv) *The algebra $\mathfrak{a}$ is in $\mathcal{C}_{\mathfrak{t}}$ if and only if $\mathfrak{z} = \{0\}$. In this case, x has a unique decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$ with $[x_{\mathrm{s}}, x_{\mathrm{n}}] = 0$, x_{s} semisimple and x_{n} nilpotent.*

(v) *The algebra $\mathfrak{s}$ is a torus if and only if $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and $\pi(\mathfrak{s})$ is a subtorus of $\mathfrak{t}$. In this case, $\mathfrak{s}$ and $\pi(\mathfrak{s})$ are conjugate under R.*

Proof. By definition $\operatorname{ad}\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ is an algebraic solvable subalgebra of $\mathfrak{gl}(\mathfrak{r}_{\mathfrak{t},\mathfrak{a}})$ and $\operatorname{ad}\mathfrak{t}$ is a maximal subtorus of $\operatorname{ad}\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$.

(i) Let $\mathfrak{z}'$ be the center of $\mathfrak{r}$. As $[\mathfrak{t}, \mathfrak{z}'] = \{0\}$,

$$\mathfrak{z}' = \mathfrak{z}' \cap \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{z}' \cap \mathfrak{a}^{\alpha}.$$

So, by Condition (1), $\mathfrak{z}'$ is contained in $\mathfrak{t}$. For t in $\mathfrak{t}$, t is in $\mathfrak{z}'$ if and only if $\alpha(t) = 0$ for all α in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, whence $\mathfrak{z}' = \mathfrak{z}$.

(ii) As the elements of $\mathfrak{t}$ are semisimple by defintion, the condition is sufficient since the set of semisimple elements of $\mathfrak{r}$ is invariant under the adjoint action of R. Suppose that x is semisimple. By [Hu95, Ch. VII], for some g in R, $\operatorname{Ad} g(x)$ is in $\operatorname{ad}\mathfrak{t}$, whence $g(x)$ is in $\mathfrak{t}$ by (i).

(iii) As $\operatorname{ad}\mathfrak{a}$ is the set of nilpotent elements of $\operatorname{ad}\mathfrak{r}$, x is in $\mathfrak{z} + \mathfrak{a}$ if and only if it is nilpotent by (i).

(iv) By definition, $\mathfrak{z}$ is the kernel of $\varphi_{\mathfrak{a}}$. Hence $\mathfrak{z} = \{0\}$ if and only if $\mathfrak{a}$ is in $\mathcal{C}_{\mathfrak{t}}$. As $\operatorname{ad}\mathfrak{r}$ is an algebraic subalgebra of $\mathrm{gl}(\mathfrak{r})$, it contains the components of the Jordan decomposition of $\operatorname{ad} x$. As a result, when $\mathfrak{a}$ is in $\mathcal{C}_{\mathfrak{t}}$, x has a unique decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$ with $[x_{\mathrm{s}}, x_{\mathrm{n}}] = 0$, x_{s} semisimple and x_{n} nilpotent.

(v) Suppose that $\mathfrak{s}$ is a torus. By (i), $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and by [Hu95, Ch. VII], for some g in R, $\operatorname{ad} g(\mathfrak{s})$ is contained in $\operatorname{ad}\mathfrak{t}$ since $\operatorname{ad}\mathfrak{t}$ is a maximal torus of $\operatorname{ad}\mathfrak{r}$. Then, by (i), $g(\mathfrak{s})$ is a subtorus of $\mathfrak{t}$. Moreover, $g(\mathfrak{s}) = \pi(\mathfrak{s})$ since $g(y) - y$ is in $\mathfrak{a}$ for all y in $\mathfrak{r}$. Conversely, if $\mathfrak{s} \cap \mathfrak{a} = \{0\}$ and $\pi(\mathfrak{s})$ is a subtorus of $\mathfrak{t}$, $\operatorname{ad}\mathfrak{s}$ is conjugate to the subtorus $\operatorname{ad}\pi(\mathfrak{s})$ of $\operatorname{ad}\mathfrak{t}$ by [Hu95, Ch. VII] so that $\mathfrak{s}$ and $\pi(\mathfrak{s})$ are conjugate under R. □

Denoting by $\mathfrak{t}^{\#}$ a complement to $\mathfrak{z}$ in $\mathfrak{t}$, $\mathfrak{a}$ is an object of $\mathcal{C}_{\mathfrak{t}^{\#}}$ since $\varphi_{\mathfrak{a}}(\mathfrak{t}) = \varphi_{\mathfrak{a}}(\mathfrak{t}^{\#})$ and the restriction of $\varphi_{\mathfrak{a}}$ to $\mathfrak{t}^{\#}$ is injective. Set $\mathfrak{r}^{\#} := \mathfrak{t}^{\#} + \mathfrak{a}$ and denote by $R^{\#}$ the adjoint group of $\mathfrak{r}^{\#}$. Let $X_{R^{\#}}$ be the closure in $\mathrm{Gr}_{d^{\#}}(\mathfrak{r}^{\#})$ of the orbit of $\mathfrak{t}^{\#}$ under $R^{\#}$.

Corollary 2.2. *All element of X_R is a commutative algebra containing $\mathfrak{z}$. Moreover, the map*

$$X_{R^{\#}} \longrightarrow X_R\ , \qquad V \longmapsto V \oplus \mathfrak{z}$$

is an isomorphism.

Proof. As the set of commutative subalgebras of dimension d of $\mathfrak{r}$ is a closed subset of $\mathrm{Gr}_d(\mathfrak{r})$ containing $\mathfrak{t}$ and invariant under R, all element of X_R is a commutative algebra. According to Lemma 2.1(i), all element of $R.\mathfrak{t}$ contains $\mathfrak{z}$ and so does all element of X_R. For g in R, denote by $\overline{g}$ the image of g in $R^\#$ by the restriction morphism. Then

$$g(\mathfrak{t}) = \overline{g}(\mathfrak{t}^\#) + \mathfrak{z} \quad \text{and} \quad \overline{g}(\mathfrak{t}^\#) = g(\mathfrak{t}) \cap \mathfrak{r}^\#.$$

Hence the map

$$X_{R^\#} \longrightarrow X_R\ , \qquad V \longmapsto V \oplus \mathfrak{z}$$

is an isomorphism whose inverse is the map $V \mapsto V \cap \mathfrak{r}^\#$. □

For $\mathfrak{a}$ of dimension $d^\#$, $\mathcal{R} := \{\beta_1, \dots, \beta_{d^\#}\}$, and for I subset of $\{1, \dots, d^\#\}$, denote $X_{R,I}$ the image of $\Bbbk^I$ by the map

$$\Bbbk^I \longrightarrow X_R\ , \qquad (z_i,\ i \in I) \longmapsto \mathfrak{z} \oplus \mathrm{span}(\{t_i + z_i x_i,\ i \in I\}) \oplus \bigoplus_{i \notin I} \mathfrak{a}^{\beta_i}$$

with x_i in $\mathfrak{a}^{\beta_i}$ for $i = 1, \dots, d^\#$ and $t_1, \dots, t_{d^\#}$ in $\mathfrak{t}$ such that $\beta_i(t_j) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant d^\#$, with $\delta_{i,j}$ the Kronecker symbol.

Lemma 2.3. *Suppose that $\mathfrak{a}$ has dimension $d^\#$. Denote by $\beta_1, \dots, \beta_{d^\#}$ the elements of $\mathcal{R}$.*

(i) *The algebra $\mathfrak{a}$ is commutative.*

(ii) *The set X_R is the union of $X_{R,I},\ I \subset \{1, \dots, d^\#\}$.*

Proof. (i) As $\mathfrak{z}$ has codimension $d^\#$ in $\mathfrak{t}$, $\beta_1, \dots, \beta_{d^\#}$ are linearly independent. Hence for $i \neq j$, $\beta_i + \beta_j$ is not in $\mathcal{R}$. As a result, $\mathfrak{a}$ is commutative.

(ii) According to Corollary 2.2, we can suppose $d = d^\#$ so that $t_1, \dots, t_d$ is the dual basis of $\beta_1, \dots, \beta_d$. For I subset of $\{1, \dots, d\}$, denote by I' the complement to I in $\{1, \dots, d\}$ and $\mathfrak{z}_{I'}$ the orthogonal complement to $\beta_i,\ i \in I'$ in $\mathfrak{t}$ and set:

$$V_I := \mathfrak{z}_{I'} \oplus \bigoplus_{i \in I'} \mathfrak{a}^{\beta_i}.$$

By (i), for i in I,

$$\exp(z_1 \mathrm{ad}\, x_1 + \cdots + z_d \mathrm{ad}\, x_d)(t_i) = t_i - z_i x_i.$$

Hence $X_{R,I}$ is the orbit of V_I under A and its closure in X_R is the union of $X_{R,J},\ J \subset I$. As a result, X_R is the union of $X_{R,I},\ I \subset \{1, \dots, d\}$ since $X_{R,\{1,\dots,d\}}$ is the orbit of $\mathfrak{t}$ under A. □

2.2. **On some subsets of $\mathcal{R}$.** For α in $\mathcal{R}$, let x_α be in $\mathfrak{a}^\alpha \setminus \{0\}$. For Λ subset of $\mathcal{R}$, denote by $\mathfrak{t}_\Lambda$ the intersection of the kernels of its elements and set:

$$\mathfrak{a}_\Lambda := \bigoplus_{\alpha \in \Lambda} \mathfrak{a}^\alpha \quad \text{and} \quad \mathfrak{r}_\Lambda := \mathfrak{t} \oplus \mathfrak{a}_\Lambda.$$

When Λ has only one element α, set $\mathfrak{t}_\alpha := \mathfrak{t}_\Lambda$.

Definition 2.4. Let Λ be a subset of $\mathcal{R}$. We say that Λ is a complete subset of $\mathcal{R}$ if it contains all element of $\mathcal{R}$ whose kernel contains $\mathfrak{t}_\Lambda$

For Λ complete subset of $\mathcal{R}$, $\mathfrak{a}_\Lambda$ is a subalgebra of $\mathfrak{a}$ and $\mathfrak{r}_\Lambda$ is a subalgebra of $\mathfrak{r}$. In particular, $\mathfrak{a}_\Lambda$ is in $\mathcal{C}'_{\mathfrak{t}}$. In this case, denote by R_Λ the connected closed subgroup of R whose Lie algebra is $\mathrm{ad}\, \mathfrak{r}_\Lambda$.

Lemma 2.5. *Let Λ be a complete subset of $\mathcal{R}$, strictly contained in $\mathcal{R}$. Then $\mathfrak{a}_\Lambda$ is contained in an ideal $\mathfrak{a}'$ of $\mathfrak{r}$ of dimension* $\dim \mathfrak{a} - 1$ *and contained in $\mathfrak{a}$.*

Proof. As Λ is complete and strictly contained in $\mathcal{R}$, $\mathfrak{a}_\Lambda$ is a subalgebra of $\mathfrak{r}$, strictly contained in $\mathfrak{a}$. Then, by Lie's Theorem, there is a sequence

$$\mathfrak{a}_\Lambda = \mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_m = \mathfrak{a}$$

of subalgebras of $\mathfrak{r}$ such that $\mathfrak{a}_i$ is an ideal of codimension 1 of $\mathfrak{a}_{i+1}$ for $i = 0, \ldots, m-1$, whence the lemma. □

For s in $\mathfrak{t}$, denote by Λ_s the subset of elements of $\mathcal{R}$ whose kernel contains s.

Lemma 2.6. *Let s be in $\mathfrak{t}$.*

(i) *The centralizer $\mathfrak{r}^s$ of s in $\mathfrak{r}$ is the direct sum of $\mathfrak{t}$ and $\mathfrak{a}_{\Lambda_s}$.*

(ii) *The center of $\mathfrak{r}^s$ is equal to $\mathfrak{t}_{\Lambda_s}$.*

Proof. By definition, Λ_s is a complete subset of $\mathcal{R}$. Let x be in $\mathfrak{r}$. Then x has a unique decomposition

$$x = x_0 + \sum_{\alpha \in \mathcal{R}} c_\alpha x_\alpha$$

with x_0 in $\mathfrak{t}$ and $c_\alpha, \alpha \in \mathcal{R}$ in $\Bbbk$.

(i) Since s is in $\mathfrak{t}$, x is in $\mathfrak{r}^s$ if and only if $c_\alpha = 0$ for $\alpha \in \mathcal{R} \setminus \Lambda_s$, whence the assertion.

(ii) The algebra $\mathfrak{a}_{\Lambda_s}$ is in $\mathcal{C}'_{\mathfrak{t}}$ and $\mathfrak{t}_{\Lambda_s}$ is the orthogonal complement to Λ_s in $\mathfrak{t}$. So, by (i) and Lemma 2.1(i), $\mathfrak{t}_{\Lambda_s}$ is the center of $\mathfrak{r}^s$. □

2.3. **Property (P) for objects of $\mathcal{C}_{\mathfrak{t}}$.** Let $\mathbf{T}$ be the connected closed subgroup of R whose Lie algebra is $\mathrm{ad}\,\mathfrak{t}$. For s in $\mathfrak{t}$, denote by X_R^s the subset of elements of X_R contained in $\mathfrak{r}^s$ and $\overline{R^s.\mathfrak{t}}$ the closure in $\mathrm{Gr}_d(\mathfrak{r})$ of the orbit of $\mathfrak{t}$ under R^s. Then $\overline{R^s.\mathfrak{t}}$ is contained in X_R^s.

Definition 2.7. Say that $\mathfrak{a}$ has Property (**P**) if X_R^s is equal to $\overline{R^s.\mathfrak{t}}$ for all s in $\mathfrak{t}$.

By Corollary 2.2, $\mathfrak{a}$ has Property (**P**) if and only if the object $\mathfrak{a}$ of $\mathcal{C}_{\mathfrak{t}^\#}$ has Property (**P**).

Lemma 2.8. *If $\mathfrak{a}$ has dimension $d^\#$, then $\mathfrak{a}$ has Property* (**P**).

Proof. According to Corollary 2.2, we can suppose $d = d^\#$. Denote by $\beta_1, \ldots, \beta_d$ the elements of $\mathcal{R}$. Then $\beta_1, \ldots, \beta_d$ is a basis of $\mathfrak{t}^*$. Let $t_1, \ldots, t_d$ be the dual basis, s in $\mathfrak{t}$ and V in X_R^s. By Lemma 2.3(ii), for some subset I of $\{1, \ldots, d\}$, V is in $X_{R,I}$. Then for some $(z_i,\ i \in I)$,

$$V = \mathrm{span}(\{t_i + z_i x_i\ i \in I\}) \oplus \bigoplus_{i \in I'} \mathfrak{a}^{\beta_i}$$

with I' the complement to I in $\{1, \ldots, d\}$ and x_i in $\mathfrak{a}^{\beta_i}$ for $i = 1, \ldots, d$. Setting

$$I'' := I' \cup \{i \in I \mid z_i \neq 0\},$$

for i in $\{1, \ldots, d\}$, i is in I'' if and only if $\beta_i(s) = 0$. So, by Lemma 2.5(i),

$$\mathfrak{r}^s = \mathfrak{t} \oplus \bigoplus_{i \in I''} \mathfrak{a}^{\beta_i}.$$

Then by Lemma 2.3(ii), V is in $\overline{R^s.\mathfrak{t}}$. □

By definition, an *algebraic subalgebra* $\mathfrak{k}$ of $\mathfrak{r}$ is the semi-direct product of a torus $\mathfrak{s}$ contained in $\mathfrak{k}$ and $\mathfrak{k} \cap \mathfrak{a}$.

Lemma 2.9. *Suppose that $\mathfrak{a}$ has Property* (**P**). *Let V be in X_R, x in V and y in $\mathfrak{r}$ such that $\mathrm{ad}\,y$ is the semisimple component of $\mathrm{ad}\,x$. Then the center of $\mathfrak{r}^y$ is contained in V.*

Proof. By Corollary 2.2, we can suppose $\mathfrak{a}$ in $\mathcal{C}_{\mathfrak{t}}$ so that y is the semisimple component of x by Lemma 2.1(iv). By Lemma 2.1(ii), for some g in R, $g(y)$ is in $\mathfrak{t}$. Denote by $\mathfrak{z}_{g(y)}$ the center of $\mathfrak{r}^{g(y)}$. By Lemma 2.6(ii), $\mathfrak{z}_{g(y)}$ is contained in $\mathfrak{t}$. As V is a commutative algebra, $g(V)$ is in $X_R^{g(y)}$. So, by Property (**P**), $\mathfrak{z}_{g(y)}$ is contained in $g(V)$ since $\mathfrak{z}_{g(y)}$ is in $k(\mathfrak{t})$ for all k in $R^{g(y)}$, whence the lemma. □

Corollary 2.10. *Suppose that $\mathfrak{a}$ has Property* (**P**). *Let V be in X_R. Then V is a commutative algebraic subalgebra of $\mathfrak{r}$ and for some subset Λ of $\mathcal{R}$, the biggest torus contained in V is conjugate to $\mathfrak{t}_\Lambda$ under R.*

Proof. According to Corollary 2.2, V is a commutative subalgebra of $\mathfrak{r}$ and we can suppose $d = d^{\#}$. Let $\mathfrak{s}$ be the set of semisimple elements of V. Then $\mathfrak{s}$ is a subspace of V. By Lemma 2.9, V contains the semisimple components of its elements so that V is the direct sum of $\mathfrak{s}$ and $V \cap \mathfrak{a}$. Let s be in $\mathfrak{s}$ such that the center of $\mathfrak{r}^s$ has maximal dimension. After conjugation by an element of R, we can suppose that s is in $\mathfrak{t}$. By Lemma 2.6(ii), $\mathfrak{t}_{\Lambda_s}$ is the center of $\mathfrak{r}^s$. Hence, by Lemma 2.9, $\mathfrak{t}_{\Lambda_s}$ is contained in $\mathfrak{s}$. Suppose that the inclusion is strict. A contradiction is expected. Let s' be in $\mathfrak{s} \setminus \mathfrak{t}_{\Lambda_s}$. Since V is contained in $\mathfrak{r}^s$, for some g in R^s, $g(s')$ is in $\mathfrak{t}$. Moreover, $g(\mathfrak{s})$ is the set of semisimple elements of $g(V)$ and $\mathfrak{t}_{\Lambda_s}$ is contained in $g(\mathfrak{s})$. Denoting by Λ' the set of elements of Λ_s whose kernel contains $g(s')$, for some z in $\Bbbk^*$, Λ' is the set of elements of $\mathcal{R}$ such that $\alpha(s + zg(s')) = 0$. By Lemma 2.9, $\mathfrak{t}_{\Lambda'}$ is contained in $g(V)$. So, by minimality of $|\Lambda_s|$, $\Lambda' = \Lambda_s$ and $g(s')$ is in $\mathfrak{t}_{\Lambda_s}$, whence the contradiction since $g(s')$ is in $g(\mathfrak{s}) \setminus \mathfrak{t}_{\Lambda_s}$. As a result, $\mathfrak{t}_{\Lambda_s} = \mathfrak{s}$ and $V = \mathfrak{t}_{\Lambda_s} + V \cap \mathfrak{a}$, whence the corollary. □

2.4. **Fixed points in X_R under $\mathbf{T}$ and R.** For V subspace of dimension d of $\mathfrak{r}$, denote by $\mathcal{R}_V$ the set of elements β of $\mathcal{R}$ such that $\mathfrak{a}^\beta$ is contained in V, r_V the rank of $\mathcal{R}_V$ and $\mathfrak{z}_V$ its orthogonal complement in $\mathfrak{t}$ so that $\dim \mathfrak{z}_V = d - r_V$. As $\mathrm{Gr}_d(\mathfrak{r})$ and X_R are projective varieties, the actions of $\mathbf{T}$ and R in these varieties have fixed points since $\mathbf{T}$ and R are connected and solvable.

Definition 2.11. We say that $\mathfrak{a}$ has Property ($\mathbf{P}_1$) if for V fixed point under $\mathbf{T}$ in X_R such that $V \cap \mathfrak{t} = \mathfrak{z}$, $r_V = |\mathcal{R}_V|$.

Lemma 2.12. *Suppose that $\mathfrak{a}$ has Property* (**P**). *Let V be in* $\mathrm{Gr}_d(\mathfrak{r})$.

(i) *The element V is a fixed point under $\mathbf{T}$ in X_R if and only if V is a commutative subalgebra of $\mathfrak{r}$ and*

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta.$$

In this case, $r_V = |\mathcal{R}_V|$.

(ii) *The element V is a fixed point under R in X_R if and only if V is a commutative ideal of $\mathfrak{r}$ and $\mathfrak{z}$ is the orthogonal complement of $\mathcal{R}_V$ in $\mathfrak{t}$. In this case, $r_V = |\mathcal{R}_V| = d^{\#}$.*

Proof. If V is a fixed point under $\mathbf{T}$,

$$V = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta.$$

(i) Suppose that V is a fixed point under $\mathbf{T}$ in $X_R \setminus \{\mathfrak{t}\}$. Then $\mathcal{R}_V$ is not empty. Let s be an element of $\mathfrak{z}_V$ such that $\beta(s) \neq 0$ if β is not a linear combination of elements of $\mathcal{R}_V$. Then V is contained in $\mathfrak{r}^s$. So, by Property (**P**), V is in $\overline{R^s.\mathfrak{t}}$. By Lemmma 2.6(i), $\mathfrak{z}_V$ is the center of $\mathfrak{r}^s$. Hence $\mathfrak{z}_V$ is contained in V and $\mathfrak{z}_V = V \cap \mathfrak{t}$ since $V \cap \mathfrak{t}$ is contained in $\mathfrak{z}_V$. As a result, $\mathfrak{z}_V$ has dimension $d - |\mathcal{R}_V|$ and $r_V = |\mathcal{R}_V|$.

Conversely, suppose that V is a commutative algebra and

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta.$$

Set:

$$\mathfrak{a}_V := \bigoplus_{\beta\in\mathcal{R}_V} \mathfrak{a}^\beta, \qquad \mathfrak{r}_V := \mathfrak{t} \oplus \mathfrak{a}_V.$$

Then $\mathfrak{a}_V$ is a commutative Lie algebra and $\mathfrak{a}_V$ is in $\mathcal{C}'_{\mathfrak{t}}$. Moreover, $\mathfrak{z}_V$ is the center of $\mathfrak{r}_V$ by Lemma 2.1(i). By Lemma 2.3(ii), V is in the closure of the orbit of $\mathfrak{t}$ under the action of the adjoint group of $\mathfrak{r}_V$ in $\mathrm{Gr}_d(\mathfrak{r}_V)$, whence the assertion.

(ii) The element V of $\mathrm{Gr}_d(\mathfrak{r})$ is a fixed point under R if and only if V is an ideal of $\mathfrak{r}$. So, by (i), the condition is sufficient. Suppose that V is a fixed point under the action of R in X_R. By (i),

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta\in\mathcal{R}_V} \mathfrak{a}^\beta.$$

As V is an ideal of $\mathfrak{r}$, $\mathfrak{z}_V$ is contained in the kernel of all elements of $\mathcal{R}$ so that $\mathfrak{z}_V = \mathfrak{z}$. In particular, $|\mathcal{R}_V| = d^\#$ and the elements of $\mathcal{R}_V$ are linearly independent. □

2.5. **On some varieties related to X_R.** Let $\mathfrak{a}'$ be an ideal of $\mathfrak{r}$ of dimension $\dim\mathfrak{a} - 1$ and contained in $\mathfrak{a}$. As a subalgebra of $\mathfrak{a}$ normalized by $\mathfrak{t}$, $\mathfrak{a}'$ is in $\mathcal{C}'_{\mathfrak{t}}$. Denote by $\mathfrak{r}'$ the subalgebra $\mathfrak{t}+\mathfrak{a}'$ of $\mathfrak{r}$, A' and R' the connected closed subgroups of R whose Lie algebras are $\mathrm{ad}\,\mathfrak{a}'$ and $\mathrm{ad}\,\mathfrak{r}'$ respectively. Let $X_{R'}$ be the closure in $\mathrm{Gr}_d(\mathfrak{r})$ of the orbit of $\mathfrak{t}$ under R' and α the element of $\mathcal{R}$ such that

$$\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}^\alpha.$$

For δ in $\mathcal{R}$ denote again by δ the character of $\mathbf{T}$ whose differential at the identity is $\mathrm{ad}\,x \longmapsto \delta(x)$.

Setting:

$$\mathfrak{G}_{d-1,d,d,d+1} := \mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}) \quad \text{and} \quad \mathfrak{G}_{d-1,d,d+1} := \mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}),$$

denote by θ_α and θ'_α the maps

$$\Bbbk \times A' \xrightarrow{\theta_\alpha} \mathfrak{G}_{d-1,d,d,d+1}\ , \quad (z,g) \longmapsto (g.\mathfrak{t}_\alpha, g.\mathfrak{t}, g\exp(z\mathrm{ad}\,x_\alpha).\mathfrak{t}, g.(\mathfrak{t}+\mathfrak{a}^\alpha)),$$

$$A' \xrightarrow{\theta'_\alpha} \mathfrak{G}_{d-1,d,d+1}\ , \quad g \longmapsto (g.\mathfrak{t}_\alpha, g.\mathfrak{t}, g.(\mathfrak{t}+\mathfrak{a}^\alpha)).$$

Let I_α and S_α be the closures in $\mathrm{Gr}_{d-1}(\mathfrak{r})$ and $\mathrm{Gr}_{d+1}(\mathfrak{r})$ of the orbits of $\mathfrak{t}_\alpha$ and $\mathfrak{t}+\mathfrak{a}^\alpha$ under A' respectively.

Lemma 2.13. *Let Γ and Γ' be the closures in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$ of the images of θ_α and θ'_α.*

(i) *The varieties Γ and Γ' have dimension n and $n-1$ respectively. Moreover, they are invariant under the diagonal actions of R' in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$.*

(ii) *The image of Γ by the first, second, third and fourth projections are equal to I_α, $X_{R'}$, X_R, S_α respectively.*

(iii) *The set Γ' is the image of Γ by the projection*

$$\mathfrak{G}_{d-1,d,d,d+1} \xrightarrow{\varpi} \mathfrak{G}_{d-1,d,d+1}\ , \qquad (V_1,V',V,W) \longmapsto (V_1,V',W).$$

(iv) *For all (V_1,V',V,W) in Γ, V_1 is contained in $V'\cap V$ and $V'+V$ is contained in W.*

(v) *Let (V_1,V',V,W) be in Γ such that V' is contained in $\mathfrak{t}_\alpha+\mathfrak{a}'$. Then W is contained in $\mathfrak{t}_\alpha+\mathfrak{a}$.*

(vi) *Let (V_1,V',V,W) be in Γ such that V' is not contained in $\mathfrak{t}_\alpha+\mathfrak{a}$. Then W is not commutative.*

Proof. (i) The maps θ_α and θ'_α are injective since $\mathfrak{t}$ is the normalizer of $\mathfrak{t}$ in $\mathfrak{r}$ by Condition (1), whence $\dim\Gamma = n$ and $\dim\Gamma' = n-1$. For (z,g,k) in $\Bbbk\times A'\times A'$, $\theta_\alpha(z,kg) = k.\theta_\alpha(z,g)$ and $\theta'_\alpha(kg) = k.\theta'_\alpha(g)$. Hence

Γ and Γ' are invariant under the diagonal action of A' in $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathfrak{G}_{d-1,d,d+1}$. Let k be in $\mathbf{T}$. For all (z,g) in $\Bbbk \times A'$,

$$kg.\mathfrak{t}_\alpha = kgk^{-1}(\mathfrak{t}_\alpha), \qquad kg.\mathfrak{t} = kgk^{-1}(\mathfrak{t}),$$

$$kg.(\mathfrak{t}+\mathfrak{a}^\alpha) = kgk^{-1}.(\mathfrak{t}+\mathfrak{a}^\alpha), \qquad kg\exp(z\mathrm{ad}\,x_\alpha).\mathfrak{t} = kgk^{-1}\exp(z\alpha(k)\mathrm{ad}\,x_\alpha).\mathfrak{t}$$

so that the images of θ_α and θ'_α are invariant under $\mathbf{T}$, whence the assertion.

(ii) Since $\mathrm{Gr}_d(\mathfrak{r})$, $\mathrm{Gr}_{d-1}(\mathfrak{r})$, $\mathrm{Gr}_{d+1}(\mathfrak{r})$ are projective varieties, the images of Γ by the first, second, third and fourth projections are closed subsets of their target varieties. Since the image of θ_α is contained in the closed subset $I_\alpha \times X_{R'} \times X_R \times S_\alpha$ of $\mathfrak{G}_{d-1,d,d,d+1}$, they are contained in I_α, $X_{R'}$, X_R and S_α respectively. By definition, $R'.\mathfrak{t}_\alpha$, $R'.\mathfrak{t}$ and $R'.(\mathfrak{t}+\mathfrak{a}^\alpha)$ are contained in the images of Γ by the first, second and fourth projections and $R.\mathfrak{t}$ is contained in the image of Γ by the third projection since A is the image of $\Bbbk \times A'$ by the map $(z,g) \mapsto g\exp(z\mathrm{ad}\,x_\alpha)$, whence the assertion.

(iii) As $\mathrm{Gr}_d(\mathfrak{r})$ is a projective variety, $\varpi(\Gamma)$ is a closed subset of $\mathfrak{G}_{d-1,d,d+1}$ containing the image of θ'_α since $\varpi\circ\theta_\alpha(z,g) = \theta'_\alpha(g)$ for all (z,g) in $\Bbbk \times A'$. Moreover, Γ is contained in $\varpi^{-1}(\Gamma')$, whence $\Gamma' = \varpi(\Gamma)$.

(iv) The subset $\widetilde{\Gamma}$ of elements (V_1, V', V, W) of $\mathfrak{G}_{d-1,d,d,d+1}$ such that V_1 is contained in V' and V and such that V' and V are contained in W, is closed. For all (z,g) in $\Bbbk \times A'$,

$$g\exp(z\mathrm{ad}\,x_\alpha).(\mathfrak{t}+\mathfrak{a}^\alpha) = g.(\mathfrak{t}+\mathfrak{a}^\alpha).$$

Hence the image of θ_α and Γ are contained in $\widetilde{\Gamma}$ so that V_1 and $V+V'$ are contained in $V' \cap V$ and W respectively for all (V_1, V', V, W) in Γ.

(v) Denote by Γ_* the closure in $\mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r})$ of the image of the map

$$A' \xrightarrow{\theta_{\alpha,*}} \mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}) \ , \qquad g \longmapsto (g(\mathfrak{t}), g(\mathfrak{t}+\mathfrak{a}^\alpha)).$$

For all (T_1, T', T, T_2) in the image of θ_α, (T', T_2) is in the image of $\theta_{\alpha,*}$. Then Γ_* is the image of Γ by the projection

$$\mathfrak{G}_{d-1,d,d,d+1} \longrightarrow \mathrm{Gr}_d(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}) \ , \qquad (T_1, T', T, T_2) \longmapsto (T', T_2).$$

Denote by τ the quotient morphism

$$\mathfrak{r} \xrightarrow{\tau} \mathfrak{r}/\mathfrak{a}' = \mathfrak{t}+\mathfrak{a}^\alpha \ .$$

For g in A' and x in $\mathfrak{r}$, $\tau\circ g(x) = \tau(x)$. Set:

$$X := \{(g,t,z,z',v,w) \in A' \times \mathfrak{t}_\alpha \times \Bbbk^2 \times \mathfrak{r}' \times \mathfrak{r};\mid v = g(zs+t),\ w = g(zs+t+z'x_\alpha)\}$$

and denote by Y the closure in $\mathfrak{r}' \times \mathfrak{r}$ of the image of X by the canonical projection

$$A' \times \mathfrak{t}_\alpha \times \Bbbk^2 \times \mathfrak{r}' \times \mathfrak{r} \longrightarrow \mathfrak{r}' \times \mathfrak{r} \ .$$

As for all (g,t,z,z',v,w) in X,

$$\tau(v) = zs+t \quad \text{and} \quad \tau(w) = zs+t+z'x_\alpha,$$

$$\alpha\circ\pi\circ\tau(v) = \alpha\circ\pi\circ\tau(w)$$

for all (v,w) in Y. By definition, for all (T,T') in Γ_*, $T \times T'$ is contained in Y. By hypothesis, V' is contained in the kernel of $\alpha\circ\pi$ and (V', W) is in Γ_*. Hence W is contained in the kernel of $\alpha\circ\pi$.

(vi) Denote by Γ'_* the closure in $\mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r})$ of the image of the map

$$\Bbbk \times A' \xrightarrow{\theta'_{\alpha,*}} \mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r}) \ , \qquad (z,g) \longmapsto (\theta_\alpha(z,g), g(\mathfrak{a}^\alpha))$$

and Γ'_{**} the closure in $\mathrm{Gr}_d(\mathfrak{r}') \times \mathrm{Gr}_1(\mathfrak{r})$ of the image of the map

$$A' \longrightarrow \mathrm{Gr}_d(\mathfrak{r}') \times \mathrm{Gr}_1(\mathfrak{r}) \ , \qquad g \longmapsto (g(\mathfrak{t}), g(\mathfrak{a}^\alpha).$$

For all (T_1, T', T, T_2, T'_2) in the image of $\theta'_{\alpha,*}$, $T' + T'_2$ is contained in T_2. Then so is it for all (T_1, T', T, T_2, T'_2) in Γ'_*. As $\mathfrak{G}_{d-1,d,d,d+1}$ and $\mathrm{Gr}_1(\mathfrak{r})$ are projective varieties, Γ and Γ'_{**} are the images of Γ'_* by the projections

$$\mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r}) \longrightarrow \mathfrak{G}_{d-1,d,d,d+1} \ , \qquad (T_1, T', T, T_2, T'_2) \longmapsto (T_1, T', T, T_2),$$

$$\mathfrak{G}_{d-1,d,d,d+1} \times \mathrm{Gr}_1(\mathfrak{r}) \longrightarrow \mathrm{Gr}_d(\mathfrak{r}') \times \mathrm{Gr}_1(\mathfrak{r}) \ , \qquad (T_1, T', T, T_2, T'_2) \longmapsto (T', T'_2)$$

respectively.

Set:

$$X' := \{(g, t, z, v, w) \in A' \times \mathfrak{t} \times \Bbbk \times \mathfrak{r}' \times \mathfrak{r} \mid v = g(t),\ w = g(zx_\alpha)\}$$

and denote by Y' the closure in $\mathfrak{r}' \times \mathfrak{r}$ of the image of X' by the canonical projection

$$A' \times \mathfrak{t} \times \Bbbk \times \mathfrak{r}' \times \mathfrak{r} \longrightarrow \mathfrak{r}' \times \mathfrak{r} \ .$$

As for all (g, t, z, v, w) in X',

$$[v, w] = g([t, zx_\alpha]) = \alpha(t)g(zx_\alpha) = \alpha{\circ}\pi(v)w,$$

$[v, w] = \alpha{\circ}\pi(v)w$ for all (v, w) in Y'. By definition, for all (T, T') in Γ'_{**}, $T \times T'$ is contained in Y'. For some W' in $\mathrm{Gr}_1(\mathfrak{r})$, (V_1, V', V, W, W') is in Γ'_*. By hypothesis, V' is not contained in the kernel of $\alpha{\circ}\pi$. Hence, for some v in V' and w in $W \setminus \{0\}$, $\alpha{\circ}\pi(v) \neq 0$ and $[v, w] = \alpha{\circ}\pi(v)w$. □

Corollary 2.14. *Suppose that $\mathfrak{a}'$ has Property* (**P**). *Let s be in $\mathfrak{t}$ such that $\mathfrak{r}^s$ is contained in $\mathfrak{a}'$ and (V_1, V', V, W) be in Γ such that V is contained in $\mathfrak{r}^s$ and $[s, V']$ is contained in V'.*

(i) *If W is not commutative then $V' = V$ and V is in $\overline{R^s.\mathfrak{t}}$.*

(ii) *Suppose that for some v in $\mathfrak{a}$, $s + v$ is in V. Then $V' = V$ and V is in $\overline{R^s.\mathfrak{t}}$.*

Proof. By Lemma 2.13(ii), V and V' are in X_R and $X_{R'}$ respectively.

(i) If $V' = V$, V is in $\overline{R^s.\mathfrak{t}}$ by Property (**P**) for $\mathfrak{a}'$. Suppose $V' \neq V$. A contradiction is expected. Then, by Lemma 2.13(iv), for some x and y in W,

$$V = V_1 \oplus \Bbbk x, \quad V' = V_1 \oplus \Bbbk y, \quad W = V_1 \oplus \Bbbk x \oplus \Bbbk y.$$

Moreover, as V is contained in $\mathfrak{r}^s$ and $[s, V'] \subset V'$, W is contained in $\mathfrak{r}'$ and we can choose y so that $[s, y] \in \Bbbk y$. Since V and V' are commutative subalgebras of $\mathfrak{r}$, $[x, y] \neq 0$. We have two cases to consider:

(a,1) V' is contained in $\mathfrak{r}^s$,

(a,2) V' is not contained in $\mathfrak{r}^s$.

(a,1) By Property (**P**) for $\mathfrak{a}'$, s is in V'. Hence $s = ty + v$ for some in (t, v) in $\Bbbk \times V_1$. As V is a commutative subalgebra of $\mathfrak{r}^s$, containing V_1 and x,

$$0 = [x, s] = t[x, y].$$

Then $s = v$ is in V_1, whence a contradiction since $\alpha(s) \neq 0$ and V_1 is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ by Lemma 2.13(ii).

(a,2) For some a in $\Bbbk^*$, $[s, y] = ay$. Then y is in $\mathfrak{a}'$ and V' is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ since so is V_1. As a result, by Lemma 2.13(v), V and W are contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ since V is contained in $\mathfrak{a}'$. As $[s, [x, y]] = a[x, y]$, $[x, y] = by$ for some b in $\Bbbk^*$ since the eigenspace of eigenvalue a of the restriction of $\mathrm{ad}\, s$ to V' is generated by y. Then $\mathrm{ad}\, x$ is not nilpotent. Let x_{s} be in $\mathfrak{r}'$ such that $\mathrm{ad}\, x_{\mathrm{s}}$ is the semisimple component of $\mathrm{ad}\, x$. Then x_{s} is in $\mathfrak{t}_\alpha + \mathfrak{a}'$, $[s, x_s] = 0$ and $[x_{\mathrm{s}}, V_1] = \{0\}$ since $[s, x] = 0$ and $[x, V_1] = \{0\}$. Moreover, $[ax_{\mathrm{s}} - bs, y] = 0$. Then $ax_{\mathrm{s}} - bs$ is a semisimple element of $\mathfrak{r}'$ such that $[ax_{\mathrm{s}} - bs, V'] = \{0\}$. As it is conjugate under R' to an

element of $\mathfrak{t}$ by Lemma 2.1(ii), by Property (**P**) for $\mathfrak{a}'$, $ax_{\mathrm{s}} - bs$ is in V', whence a contradiction since V' is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$ and $ax_{\mathrm{s}} - bs$ is not in $\mathfrak{t}_\alpha + \mathfrak{a}'$.

(ii) If $V = V'$, V is in $\overline{R^s.\mathfrak{t}}$ by Property (**P**) for $\mathfrak{a}'$. Suppose $V \neq V'$. A contradiction is expected. As V is contained in $\mathfrak{r}^s$, $[s, v] = 0$. Let x and y be as in (i). As V_1 is contained in $\mathfrak{t}_\alpha + \mathfrak{a}'$, $s + v$ is not in V_1 since $\alpha(s) \neq 0$. So we can choose $s+v = x$. By (i), W is commutative. Then $[s+v, y] = 0$ and $[\mathrm{ad}\, s, \mathrm{ad}\, y] = 0$ since $\mathrm{ad}\, s$ is the semisimple component of $\mathrm{ad}(s + v)$. Hence, by Lemma 2.1(i), $[s, y] = 0$ since $[s, y]$ is in $\mathfrak{a}$. As a result, V' is contained in $\mathfrak{r}^s$ since so is V_1. So, by Property (**P**) for $\mathfrak{a}'$, s is in V' and W is not commutative by Lemma 2.13(vi), whence a contradiction. □

For (T_1, T', T_2) in Γ', denote by Γ_{T_1,T',T_2} the subset of elements (T_1, T', T, T_2) of $\mathfrak{G}_{d-1,d,d,d+1}$ such that T is contained in T_2 and contains T_1. Then Γ_{T_1,T',T_2} is a closed subvariety of $\mathfrak{G}_{d-1,d,d,d+1}$, isomorphic to $\mathbb{P}^1(\Bbbk)$. Let (V_1, V', V, W) be a fixed point under $\mathbf{T}$ of Γ.

Lemma 2.15. (i) *For some affine open neighborhood Ω of (V_1, V', W) in Γ', Ω is invariant under $\mathbf{T}$.*

(ii) *For $i = 0, \dots, n-2$, there exist Y_i and O_i such that*

(a) *Y_i is an irreducible closed subset of dimension $n - 1 - i$ of Ω, containing (V_1, V', W) and invariant under $\mathbf{T}$,*

(b) *O_i is a locally closed subvariety of dimension $n - 1 - i$ of A', invariant under $\mathbf{T}$ by conjugation,*

(c) *$\theta'_\alpha(O_i)$ is contained in Y_i and (V_1, V', V, W) is in the closure of $\theta_\alpha(\Bbbk \times O_i)$ in Γ.*

(iii) *There exist a smooth projective curve C, an action of $\mathbf{T}$ on C, $x_1, \dots, x_m$ in C and two morphisms*

$$C \setminus \{x_1, \dots, x_m\} \xrightarrow{\ \mu\ } A' , \qquad C \xrightarrow{\ \nu\ } \Gamma'$$

satisfying the following conditions:

(a) *$x_1, \dots, x_m$ are the fixed points under $\mathbf{T}$ in C,*

(b) *for g in $\mathbf{T}$ and x in $C \setminus \{x_1, \dots, x_m\}$, $\mu(g.x) = g\mu(x)g^{-1}$ and $\nu(g.x) = g.\nu(x)$,*

(c) *$\nu(x_1) = (V_1, V', W)$,*

(d) *(V_1, V', V, W) is in the closure of the image of $\Bbbk \times (C \setminus \{x_1, \dots, x_m\})$ by the map $(z, x) \longmapsto \theta_\alpha(z, \mu(x))$.*

Proof. (i) As Γ' is a projective variety with a $\mathbf{T}$ action and (V_1, V', W) is a fixed point under $\mathbf{T}$, there exists an affine open neighborhood Ω of (V_1, V', W) in Γ', invariant under $\mathbf{T}$.

(ii) Prove the assertion by induction on i. For $i = 0$, $Y_i = \Omega$ and O_i is the inverse image of Ω by θ'_α. Suppose that Y_i and O_i are known. Let Y'_i be the closure in Γ of $\theta_\alpha(\Bbbk \times O_i)$. By Condition (c), Y'_i is invariant under $\mathbf{T}$ and $\theta_\alpha(\Bbbk \times O_i)$ is a $\mathbf{T}$-invariant dense subset of Y'_i. So, it contains an $\mathbf{T}$-invariant dense open subset O'_i of Y'_i. As θ'_α is an orbital injective morphism, $\theta'_\alpha(O_i)$ is a dense open subset of Y_i. Set:

$$Z' := Y'_i \setminus O'_i, \quad Z := Y_i \setminus \theta_\alpha(O'_i), \quad Z_* := \Omega \cap (\varpi(Z) \cup Z').$$

Then Z_* is a $\mathbf{T}$-invariant closed subset of Y_i, containing (V_1, V', W).

Denote by Z_{**} the union of irreducible components of dimension $\dim Y_i - 1$ of Z_* and I the union of the ideals of definition in $\Bbbk[Y_i]$ of the irreducible components of Z_{**}. Let p be in $\Bbbk[Y_i] \setminus I$, semiinvariant under $\mathbf{T}$ and such that $p((V_1, V', W)) = 0$. Denote by Y'_{i+1} an irreducible component of the nullvariety of p in $Y'_i \cap \varpi^{-1}(\Omega)$, containing (V_1, V', V, W) and Y_{i+1} the closure in Ω of $\varpi(Y'_{i+1})$. Then Y_{i+1} has dimension $n - i - 1$ and its intersection with $\theta'_\alpha(O_i)$ is not empty so that $O_{i+1} := {\theta'_\alpha}^{-1}(O_i \cap \theta'_\alpha(O_i))$ is a nonempty locally closed subset of dimension $n - i - 1$ of A'. Moreover, Y_{i+1} and O_{i+1} are invariant under $\mathbf{T}$ since p is semiinvariant under $\mathbf{T}$. As $\theta_\alpha(\Bbbk \times O_{i+1})$ is the intersection of Y'_{i+1} and $\theta_\alpha(\Bbbk \times O_i)$, it is dense in Y'_{i+1} so that (V_1, V', V, W) is in the closure of $\theta_\alpha(\Bbbk \times O_{i+1})$ and (V_1, V', W) is in Y_{i+1}.

(iii) Let $\overline{Y_{n-2}}$ be the closure of Y_{n-2}, C its normalization and ν the normalization morphism. Then C is a smooth projective curve. As Y_{n-2} is invariant under $\mathbf{T}$, so is $\overline{Y_{n-2}}$ and there is an action of $\mathbf{T}$ on C such that ν is an equivariant morphism. As the restriction of θ'_α to O_{n-2} is an isomorphism onto a dense open subset of Y_{n-2}, the actions of $\mathbf{T}$ on $\overline{Y_{n-2}}$ and C are not trivial since θ'_α is equivariant under the actions of $\mathbf{T}$. As a result, $\mathbf{T}$ has an open orbit O_* in $\overline{Y_{n-2}}$ and $\overline{Y_{n-2}} \setminus O_*$ is the set of fixed points under $\mathbf{T}$ of $\overline{Y_{n-2}}$ since $\overline{Y_{n-2}}$ has dimension 1. Hence the restriction of ν to $\nu^{-1}(O_*)$ is an isomorphism, $C \setminus \nu^{-1}(O_*)$ is finite, its elements are fixed under $\mathbf{T}$ and there exists a $\mathbf{T}$-equivariant morphism μ from $\nu^{-1}(O_*)$ to A' such that $\theta'_\alpha \circ \mu = \nu$. As (V_1, V', W) is a fixed point under $\mathbf{T}$, for some x_1 in $C \setminus \nu^{-1}(O_*)$, $\nu(x_1) = (V_1, V', W)$ since (V_1, V', W) is in $\nu(C)$. Moreover, (V_1, V', V, W) is in the closure of the map

$$\Bbbk \times (C \setminus \nu^{-1}(O_*)) \longrightarrow \Gamma\ , \qquad (z, x) \longmapsto \theta_\alpha(z, \mu(x))$$

since it is in $\overline{\theta_\alpha(\Bbbk \times O_{n-2})}$. □

Denote by η the morphism

$$\Bbbk \times (C \setminus \{x_1, \dots, x_m\}) \xrightarrow{\ \eta\ } \Gamma\ , \qquad (z, x) \longmapsto \theta_\alpha(z, \mu(x))$$

and Δ the closure of the graph of η in $\mathbb{P}^1(\Bbbk) \times C \times \Gamma$. Let υ be the restriction to Δ of the canonical projection

$$\mathbb{P}^1(\Bbbk) \times C \times \Gamma \longrightarrow \mathbb{P}^1(\Bbbk) \times C$$

and for (z, x) in $\mathbb{P}^1(\Bbbk) \times C$, let $F_{z,x}$ be the subset of Γ such that $\{(z, x)\} \times F_{z,x}$ is the fiber of υ at (z, x). We have an action of $\mathbf{T}$ in $\mathbb{P}^1(\Bbbk)$ given by

$$t.z := \begin{cases} \alpha(t)z & \text{if} \quad z \in \Bbbk^* \\ z & \text{if} \quad z \in \{0, \infty\} \end{cases} .$$

Lemma 2.16. *Let Δ_ν be the graph of ν.*

(i) *The set Δ_ν is the image of Δ by the map $(z, x, y) \mapsto (x, \varpi(y))$.*

(ii) *For t in $\mathbf{T}$ and (z, x, y) in Δ, $t.(z, x, y) := (t.z, t.x, t.y)$ is in Δ.*

(iii) *For (z, x) in $\mathbb{P}^1(\Bbbk) \times C$, η is regular at (z, x) if and only if $F_{z,x}$ has dimension 0. In this case, $|F_{z,x}| = 1$.*

(iv) *For (z, x) in $\mathbb{P}^1(\Bbbk) \times C \setminus \{0, \infty\} \times \{x_1, \dots, x_m\}$, η is regular at (z, x).*

(v) *For $i = 1, \dots, m$, there exists a regular map η_i from $\mathbb{P}^1(\Bbbk)$ to Γ such that $\eta_i(z) = \eta(z, x_i)$ for all z in $\Bbbk^*$. Moreover, its image is contained in $\varpi^{-1}(\{\nu(x_i)\}) \cap \Gamma$.*

Proof. (i) As $\mathbb{P}^1(\Bbbk)$ and $\mathrm{Gr}_d(\mathfrak{r})$ are projective varieties, the image of Δ by the map $(z, x, y) \mapsto (x, \varpi(y))$ is a closed subset of $C \times \Gamma'$ contained in Δ_ν since $\varpi \circ \eta(z, x) = \nu(x)$ for all (z, x) in $\Bbbk \times (C \setminus \{x_1, \dots, x_m\})$, whence the assertion since the inverse image of Δ_ν by this map is a closed subset of $\mathbb{P}^1(\Bbbk) \times C \times \Gamma$ containing the graph of η.

(ii) From the equality

$$t \exp(z \operatorname{ad} x_\alpha) t^{-1} = \exp(\alpha(t) z \operatorname{ad} x_\alpha)$$

for all (t, z) in $\mathbf{T} \times \Bbbk$, we deduce the equality

$$t.\eta(z, x) = t.\theta_\alpha(z, \mu(x)) = \theta_\alpha(\alpha(t)z, \mu(t.x) = \eta(t.z, t.x)$$

for all (t, z, x) in $\mathbf{T} \times \Bbbk \times (C \setminus \{x_1, \dots, x_m\})$ since θ_α and μ are $\mathbf{T}$-equivariant, whence the assertion.

(iii) As Γ is a projective variety, υ is a projective morphism. Moreover, it is birational since Δ is the closure of the graph of η. So, by Zariski's Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of υ are connected of dimension 0 or 1 since $\mathbb{P}^1(\Bbbk) \times C$ is normal of dimension 2. Let (z, x) be in $\mathbb{P}^1(\Bbbk) \times C$ such that $F_{z,x}$ dimension 0. There exists a neighborhood $O_{z,x}$ of (z, x) in $\mathbb{P}^1(\Bbbk) \times C$ such that F_y has dimension 0 for y

in $O_{z,x}$. In other words, the restriction of υ to $\upsilon^{-1}(O_{z,x})$ is a quasi finite morphism. Moreover, it is birational and surjective. So, again by Zariski's Main Theorem [Mu88, §9], it is an isomorphism. Hence η is regular at (z, x). Conversely, if η is regular at (z, x), $\eta(z, x)$ is an isolated point in $F_{z,x}$, whence $F_{z,x} = \{\eta(z, x)\}$ since $F_{z,x}$ is connected.

(iv) The variety $\Bbbk\times(C\setminus\{x_1, \dots, x_m\})$ is an open subset of the smooth variety $\mathbb{P}^1(\Bbbk)\times C$ and Γ is a projective variety. Hence η has a regular extension to a big open subset of $\mathbb{P}^1(\Bbbk) \times C$ by [Sh94, Ch. 6, Theorem 6.1]. By Condition (a) of Lemma 2.15(iii), $\{0, \infty\} \times \{x_1, \dots, x_m\}$ is the set of fixed points under $\mathbf{T}$ in $\mathbb{P}^1(\Bbbk) \times C$ and by (ii), $t.\eta(z, x) = \eta(t.z, t.x)$ for all (t, z, x) in $\mathbf{T} \times \mathbb{P}^1(\Bbbk) \times (C \setminus \{x_1, \dots, x_m\})$. Hence η is regular on $P^1(\Bbbk) \times C \setminus \{0, \infty\} \times \{x_1, \dots, x_m\}$.

(v) The restriction of η to $\Bbbk^* \times \{x_i\}$ is a regular map from a dense open subset of the smooth variety $\mathbb{P}^1(\Bbbk) \times \{x_i\}$ to the projective variety Γ. So, again by [Sh94, Ch. 6, Theorem 6.1], this map has regular extension to $\mathbb{P}^1(\Bbbk) \times \{x_i\}$, whence the assertion by (i). □

Let I be the set of indices such that $\nu(x_i) = (V_1, V', W)$. Denote by S the image of Δ by the canonical projection $\mathbb{P}^1(\Bbbk) \times C \times \Gamma \longrightarrow \Gamma$, S_{n} its normalization, σ the normalization morphism, $S^{\mathbf{T}}$ and $S_{\mathrm{n}}^{\mathbf{T}}$ the sets of fixed points under $\mathbf{T}$ in S and S_{n} respectively. Set

$$C_* := \mathbb{P}^1(\Bbbk) \times C \setminus \{(0, \infty\} \times \{x_1, \dots, x_m\}.$$

By Lemma 2.15(iv), η is a dominant morphism from C_* to S, whence a commutative diagram

$$\begin{tikzcd} C_* \arrow[r, "\eta_{\mathrm{n}}"] \arrow[dr, "\eta"'] & S_{\mathrm{n}} \arrow[d, "\sigma"] \\ & S \end{tikzcd}$$

since C_* is smooth. Let Δ_{n} be the closure in $\mathbb{P}^1(\Bbbk) \times C \times S_{\mathrm{n}}$ of the graph of η_{n} and υ_2 the restriction to Δ_{n} of the canonical projection

$$\mathbb{P}^1(\Bbbk) \times C \times S_{\mathrm{n}} \longrightarrow S_{\mathrm{n}} .$$

Lemma 2.17. *Suppose $V' \neq V$ and V and V' contained in $\mathfrak{z} + \mathfrak{a}$.*

(i) *The variety Δ is the image of Δ_{n} by the map $(z, x, y) \mapsto (z, x, \sigma(y))$.*

(ii) *The morphism υ_2 is projective and birational.*

(iii) *There exists a $\mathbf{T}$-equivariant morphism*

$$(S_{\mathrm{n}} \setminus S_{\mathrm{n}}^{\mathbf{T}}) \xrightarrow{\varphi} C_*$$

such that $\eta\circ\varphi$ is the restriction of σ to $S_{\mathrm{n}} \setminus S_{\mathrm{n}}^{\mathbf{T}}$.

(iv) *For some i in I, $\eta_i(1)$ is not invariant under $\mathbf{T}$.*

Proof. (i) As S is a projective variety, so are S_{n}, $\mathbb{P}^1(\Bbbk) \times C \times S_{\mathrm{n}}$, Δ_{n} and the image of Δ_{n} by the map $(z, x, y) \mapsto (z, x, \sigma(y))$, whence the assertion since the image of the graph of η_{n} by this map is the graph of η.

(ii) As Δ_{n} is projective so is υ_2. Since θ_α is injective, so is the restriction of η to $\Bbbk \times (C \setminus \{x_1, \dots, x_m\})$. Hence υ_2 is birational.

(iii) By (ii) and Zariski's Main Theorem [H77, Ch. III, Corollary 11.4], the fibers of υ_2 are connected. For y in $S_{\mathrm{n}} \setminus S_{\mathrm{n}}^{\mathbf{T}}$ and (z, x) in $\mathbb{P}^1(\Bbbk)\times C$ such that (z, x, y) is in Δ_{n}, $\varpi\circ\sigma(y) = \nu(x)$ by (i). If x is not in $\{x_1, \dots, x_m\}$, $\nu^{-1}(\varpi\circ\sigma(y)) = \{x\}$ by Condition (b) of Lemma 2.15(iii) and z is the element of $\Bbbk$ such that $\theta_\alpha(z, \mu(x)) = \sigma(y)$. Suppose $x = x_i$ for some $i = 1, \dots, m$. Let z and z' be in $\Bbbk^*$ such that (z, x_i, y) and (z', x_i, y) are in Δ_{n}. Then $(z, x_i, \sigma(y))$ and $(z', x_i, \sigma(y))$ are in Δ. By Lemma 2.16,(iii) and (iv), $\sigma(y) = \eta(z, x_i) = \eta(z', x_i)$. For some t

in $\mathbf{T}$, $z' = t.z$ so that $t.\sigma(y) = \sigma(y)$. As y is not invariant under $\mathbf{T}$ so is $\sigma(y)$ since the fibers of σ are finite. Hence the stabilizer of $\sigma(y)$ in $\mathbf{T}$ is finite and so is the fiber of υ_2 at y. As a result, the restriction of υ_2 to $\Delta_{\mathrm{n}} \setminus \mathbb{P}^1(\Bbbk) \times C \times S_{\mathrm{n}}^{\mathbf{T}}$ is an injective morphism. So, again by Zariski's Main Theorem [Mu88, §9], this map is an isomorphism, whence a morphism

$$(S_{\mathrm{n}} \setminus S_{\mathrm{n}}^{\mathbf{T}}) \xrightarrow{\quad\varphi\quad} C_* \ .$$

Moreover, φ is $\mathbf{T}$-equivariant since so is υ_2. For y in S_{n} such that $\sigma(y) = \eta(z, x)$ for some (z, x) in $\Bbbk^* \times (C \setminus \{x_1, \ldots, x_m\})$, (z, x, y) is the unique element of Δ_{n} above y. Hence $\eta \circ \varphi = \sigma$.

(iv) Suppose that for all i in I, $\eta_i(1)$ is invariant under $\mathbf{T}$. A contradiction is expected. As $V \neq V'$, $V_1 = V \cap V'$ and $V + V' = W$ by Lemma 2.13(iv). Moreover, since V and V' are contained in $\mathfrak{z} + \mathfrak{a}$, for some β and γ in $\mathcal{R}$,

$$V = V_1 + \mathfrak{a}^{\beta} \quad \text{and} \quad V' = V_1 + \mathfrak{a}^{\gamma}.$$

Then $\Gamma_{V_1,V',W}$ is invariant under $\mathbf{T}$. More precisely, $\Gamma_{V_1,V',W}$ is a union of one orbit of dimension 1 and the set $\{(V_1, V', V', W), (V_1, V', V, W)\}$ of fixed points. As a result, $\Gamma_{V_1,V',W} \cap S$ is equal to $\{(V_1, V', V', W), (V_1, V', V, W)\}$ or $\Gamma_{V_1,V',W}$ since S is invariant under $\mathbf{T}$. By Lemma 2.16,(ii) and (v), for i in I, $\eta_i(\mathbb{P}^1(\Bbbk))$ is equal to (V_1, V', V', W) or (V_1, V', V, W) since $\nu(x_i) = (V_1, V', W)$.

Suppose $\Gamma_{V_1,V',W} \cap S = \{(V_1, V', V', W), (V_1, V', V, W)\}$. By Lemma 2.16,(v) and (iii), for all i in I, η is regular at $(0, x_i)$ and (∞, x_i) since $\nu(x_i) = (V_1, V', W)$, whence

$$\lim_{z \to 0} \eta_i(0) = (V_1, V', V', W) \quad \text{and} \quad \lim_{z \to \infty} \eta_i(\infty) = (V_1, V', V, W),$$

whence a contradiction.

Suppose $\Gamma_{V_1,V',W} \cap S = \Gamma_{V_1,V',W}$. Let y be in S_{n} such that

$$\sigma(y) \in \Gamma_{V_1,V',W} \setminus \{(V_1, V', V', W), (V_1, V', V, W)\}.$$

By (iii), for some i in I and some z in $\Bbbk^*$, $\varphi(t.y) = (t.z, x_i)$ and $t.\sigma(y) = t.\eta(z, x_i) = t.\eta_i(z)$ for all t in $\mathbf{T}$, whence a contradiction since (V_1, V', V', W) and (V_1, V', V, W) are in $\overline{\mathbf{T}.\sigma(y)}$. □

Corollary 2.18. *Let (V_1, V', V, W) be a fixed point under $\mathbf{T}$ of Γ such that $V \cap \mathfrak{t} = V' \cap \mathfrak{t} = \mathfrak{z}$. Then $V' = V$.*

Proof. Suppose $V' \neq V$. A contradiction is expected. By Lemma 2.13(iv), $V_1 = V \cap V'$ and $W = V + V'$. As $V \cap \mathfrak{t} = V' \cap \mathfrak{t} = \mathfrak{z}$, V and V' are contained in $\mathfrak{z} + \mathfrak{a}$. So, for some β in $\mathcal{R}$ and γ in $\mathcal{R} \setminus \{\alpha\}$,

$$V = V_1 \oplus \mathfrak{a}^{\beta} \quad \text{and} \quad V' = V_1 \oplus \mathfrak{a}^{\gamma}.$$

since (V_1, V', V, W) is invariant under $\mathbf{T}$. By Lemma 2.17(iv), for some i in I, $\eta_i(1)$ is not fixed under $\mathbf{T}$. Then, by Lemma 2.13(ii), $\eta_i(\mathbb{P}^1(\Bbbk)) = \Gamma_{V_1,V',W}$. Denoting by $\eta_i(z)_3$ the third component of $\eta_i(z)$, for all z in $\mathbb{P}^1(\Bbbk)$, V_1 is contained in $\eta_i(z)_3$ and $\eta_i(z)_3$ is contained in W. Hence for some a in $\Bbbk^*$,

$$\eta_i(1)_3 = V_1 \oplus \Bbbk(x_\beta + a x_\gamma) \quad \text{and} \quad \eta_i(\alpha(t))_3 = V_1 \oplus \Bbbk(\beta(t) x_\beta + \gamma(t) a x_\gamma)$$

for all t in $\mathbf{T}$. For some t_1 and t_2 in $\mathbf{T}$, for all δ in $\mathcal{R}$, $\delta(t_1)$ and $\delta(t_2)$ are positive rational numbers and

$$\alpha(t_1) > 1, \qquad \alpha(t_2) > 1, \qquad \beta(t_1) < \gamma(t_1), \qquad \beta(t_2) > \gamma(t_2).$$

Then

$$\lim_{k \to \infty} V_1 \oplus \Bbbk(\beta(t_1^k) x_\beta + \gamma(t_1^k) a x_\gamma) = V_1 \oplus \mathfrak{a}^{\gamma}, \quad \lim_{k \to \infty} V_1 \oplus \Bbbk(\beta(t_2^k) x_\beta + \gamma(t_2^k) a x_\gamma) = V_1 \oplus \mathfrak{a}^{\beta},$$

$$\lim_{k \to \infty} \eta_i(\alpha(t_1^k) = \lim_{k \to \infty} \eta_i(\alpha(t_2^k) = \eta_i(\infty),$$

whence $V = V'$ and the contradiction. □

2.6. **Property (P) and Property ($\mathbf{P}_1$).** In this subsection we suppose that all objects of $\mathcal{C}'_{\mathfrak{t}}$ of dimension smaller than n has Property (**P**). For V a fixed point of X_R under $\mathbf{T}$, denote by Λ_V the orthogonal complement to $\mathfrak{z}_V$ in $\mathcal{R}$ and set:

$$\mathfrak{r}_V := \mathfrak{r}_{\Lambda_V}, \qquad R_V := R_{\Lambda_V}.$$

Lemma 2.19. *Let V be a fixed point under $\mathbf{T}$ in X_R.*

(i) *The action of R_V in $\overline{R_V.V}$ has fixed points. For V_∞ such a point,*

$$V_\infty = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \mathfrak{a}^\beta, \quad |\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|, \quad \mathfrak{r}_V \geqslant \mathfrak{r}_{V_\infty}.$$

(ii) *The set $\mathcal{R}_V$ has rank at least $|\mathcal{R}_V| - 1$.*

(iii) *Suppose that $\mathfrak{a}$ has Property ($\mathbf{P}_1$). Then $\mathcal{R}_V$ has rank $|\mathcal{R}_V|$.*

(iv) *If $\mathfrak{a}$ has Property ($\mathbf{P}_1$), for s in $\mathfrak{t}$ such that V is contained in $\mathfrak{r}^s$, V is in $\overline{R^s.\mathfrak{t}}$.*

Proof. (i) As $\overline{R_V.V}$ is a projective variety and R_V is connected and solvable, R_V has fixed points in $\overline{R_V.V}$. Denote by V_∞ such a point. Since V is fixed under $\mathbf{T}$,

$$V = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta.$$

Moreover, $V \cap \mathfrak{t}$ is contained in $\mathfrak{z}_V$ since V is commutative. By Lemma 2.6(ii), $\mathfrak{z}_V$ is the center of $\mathfrak{r}_V$. Hence $V \cap \mathfrak{t}$ is contained in all element of $R_V.V$. Moreover, all element of $R_V.V$ is contained in $V \cap \mathfrak{t} + \mathfrak{a}_{\Lambda_V}$. Then

$$V_\infty = V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \mathfrak{a}^\beta,$$

whence $|\mathcal{R}_V| = |\mathcal{R}_{V_\infty}|$. Since $\mathcal{R}_{V_\infty}$ is contained in Λ_V and $\mathfrak{r}_V = d - \dim \mathfrak{z}_V$, $\mathfrak{r}_V \geqslant \mathfrak{r}_{V_\infty}$.

(ii) By (i), we can suppose that V is invariant under R_V. By Lemma 2.5, $\mathfrak{a}_{\Lambda_V}$ is contained in an ideal $\mathfrak{a}'$ of $\mathfrak{r}$ of dimension $\dim \mathfrak{a} - 1$ and contained in $\mathfrak{a}$. We then use the notations of Lemma 2.13. Set $\Gamma_V := \varpi_3^{-1}(V)$. By Lemma 2.13(i), Γ_V is a projective variety invariant under R_V since so is V. Then R_V has a fixed point in Γ_V. Let (V_1, V', V, W) be such a point. As $\mathfrak{a}'$ has Property (**P**), by Lemma 2.12(i),

$$V' = \mathfrak{z}_{V'} \oplus \bigoplus_{\beta \in \mathcal{R}_{V'}} \mathfrak{a}^\beta.$$

and the elements of $\mathcal{R}_{V'}$ are linearly independent.

If $V' = V$ then $\mathcal{R}_{V'} = \mathcal{R}_V$ so that $\mathfrak{r}_V = \mathfrak{r}_{V'} = |\mathcal{R}_V|$. Suppose $V' \neq V$. Then, by Lemma 2.13(iv),

$$V_1 = \mathfrak{z}_{V'} \cap V \cap \mathfrak{t} \oplus \bigoplus_{\beta \in \mathcal{R}_V \cap \mathcal{R}_{V'}} \mathfrak{a}^\beta.$$

As V_1 has codimension 1 in V and V', $\mathcal{R}_{V'} = \mathcal{R}_V$ or $\mathfrak{z}_{V'} = V \cap \mathfrak{t}$. In the first case, $\mathfrak{r}_V = |\mathcal{R}_V|$ and in the second case,

$$|\mathcal{R}_V \cap \mathcal{R}_{V'}| = |\mathcal{R}_V| - 1 = |\mathcal{R}_{V'}| - 1,$$

whence $\mathfrak{r}_V \geqslant |\mathcal{R}_V| - 1$ since the elements of $\mathcal{R}_{V'}$ are linearly independent.

(iii) Prove the assertion by induction on $\dim \mathfrak{z}_V$. If $\mathfrak{z}_V = \mathfrak{z}$, then $r_V = |\mathcal{R}_V|$ by Property ($\mathbf{P}_1$). Suppose $\dim \mathfrak{z}_V = \dim \mathfrak{z} + 1$ and $V \cap \mathfrak{t} = \mathfrak{z}$. Then $|\mathcal{R}_V| = d^\#$ and $r_V = d^\# - 1$. By Property ($\mathbf{P}_1$), it is impossible. Hence $V \cap \mathfrak{t} = \mathfrak{z}_V$ since $V \cap \mathfrak{t}$ is contained in $\mathfrak{z}_V$. As a result $r_V = |\mathcal{R}_V|$.

Suppose $\dim \mathfrak{z}_V \geqslant 2 + \dim \mathfrak{z}$, the assertion true for the integers smaller than $\dim \mathfrak{z}_V$ and $r_V < |\mathcal{R}_V|$. A contradiction is expected. By (ii), $V \cap \mathfrak{t}$ has dimension at least $\dim \mathfrak{z}_V - 1$. Then, for some α in $\mathcal{R}$, $V \cap \mathfrak{t}_\alpha$ is strictly contained in $V \cap \mathfrak{t}$. Let Λ be the orthogonal complement to $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ in $\mathcal{R}$. As $\overline{R_\Lambda.V}$ is a projective variety and R_Λ is connected, R_Λ has a fixed point in $\overline{R_\Lambda.V}$. Let V_∞ be such a point. By Lemma 2.6(ii),

$\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is the center of $\mathfrak{r}_\Lambda$. Hence $V \cap \mathfrak{t}_\alpha$ is contained in all element of $R_\Lambda.V$. Moreover, all element of $R_\Lambda.V$ is contained in $V \cap \mathfrak{t} + \mathfrak{a}_\Lambda$. As V_∞ is an ideal of $\mathfrak{r}_\Lambda$, $V \cap \mathfrak{t}$ is not contained in V_∞ since it is not contained in the kernel of α. Then

$$V_\infty = V \cap \mathfrak{t}_\alpha \oplus \bigoplus_{\beta \in \mathcal{R}_{V_\infty}} \mathfrak{a}^\beta.$$

By (ii), $r_{V_\infty} \geqslant |\mathcal{R}_{V_\infty}| - 1$, whence

$$\dim \mathfrak{z}_{V_\infty} \leqslant \dim V \cap \mathfrak{t}_\alpha + 1 = \dim V \cap \mathfrak{t} < \dim \mathfrak{z}_V.$$

So, by induction hypothesis, $|\mathcal{R}_{V_\infty}| = \mathrm{r}_{V_\infty}$ and $\mathfrak{z}_{V_\infty} = V \cap \mathfrak{t}_\alpha$. Since $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is the center of $\mathfrak{r}_\Lambda$, $\mathfrak{z}_V \cap \mathfrak{t}_\alpha$ is contained in $\mathfrak{z}_{V_\infty}$, whence

$$\dim \mathfrak{z}_V - 1 \leqslant \dim V \cap \mathfrak{t}_\alpha = \dim V \cap \mathfrak{t} - 1.$$

As a result, $\mathfrak{z}_V = V \cap \mathfrak{t}$ since $V \cap \mathfrak{t}$ is contained in $\mathfrak{z}_V$, whence a contradiction.

(iv) Suppose that $\mathfrak{a}$ has Property $(\mathbf{P}_1)$. By (iii),

$$V = \mathfrak{z}_V \oplus \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta$$

and $\mathrm{r}_V = |\mathcal{R}_V|$. As a result, the centralizer of V in $\mathfrak{t}$ is equal to $\mathfrak{z}_V$. Set

$$\mathfrak{a}'_V = \bigoplus_{\beta \in \mathcal{R}_V} \mathfrak{a}^\beta, \qquad \mathfrak{r}'_V := \mathfrak{t} + \mathfrak{a}'_V.$$

Denote by R'_V the connected closed subgroup of R whose Lie algebra is $\mathrm{ad}\,\mathfrak{r}'_V$. The algebra $\mathfrak{a}'_V$ is in $\mathcal{C}'_\mathfrak{t}$ and has dimension $d - \dim \mathfrak{z}_V$. Then, by Lemma 2.3(ii), V is in $\overline{R'_V.\mathfrak{t}}$, whence the assertion since $\mathfrak{r}'_V$ is contained in $\mathfrak{r}^s$. □

Corollary 2.20. *Suppose that* $\mathfrak{a}$ *has Property* $(\mathbf{P}_1)$. *Then* $\mathfrak{a}$ *has Property* $(\mathbf{P})$.

Proof. Let V be in X_R and s in $\mathfrak{t} \setminus \mathfrak{z}$ such that V is contained in $\mathfrak{r}^s$. As $\overline{\mathbf{T}.V}$ is a projective variety and $\mathbf{T}$ is a connected commutative group, $\mathbf{T}$ has a fixed point in $\overline{\mathbf{T}.V}$. Let V_∞ be such a point. Since all element of $\mathbf{T}.V$ is contained in $\mathfrak{r}^s$, so is V_∞. Then, by Lemma 2.19(iv), V_∞ is in $\overline{R^s.\mathfrak{t}}$. In particular, s is in V_∞. Let E a complement to V_∞ in $\mathfrak{r}$, invariant under $\mathbf{T}$. The map

$$\mathrm{Hom}_\Bbbk(V_\infty, E) \xrightarrow{\kappa} \mathrm{Gr}_d(\mathfrak{r}), \qquad \varphi \longmapsto \kappa(\varphi) := \mathrm{span}(\{v + \varphi(v) \mid v \in V_\infty\})$$

is an isomorphism onto an open neighborhood Ω_E of V_∞ in $\mathrm{Gr}_d(\mathfrak{r})$. For φ in $\mathrm{Hom}_\Bbbk(V_\infty, E)$ such that $\kappa(\varphi)$ is in $\mathbf{T}.V$, $\varphi(s)$ is in $\mathfrak{a}^s$. Then, for some g in $\mathbf{T}$ and for some v in $\mathfrak{a}^s$, $s + v$ is in $g(V)$ and the semisimple component of $\mathrm{ad}(s + v)$ is different from 0 since s is not in $\mathfrak{z}$. Let x be in $\mathfrak{r}^s$ such that $\mathrm{ad}\,x$ is the semisimple component of $\mathrm{ad}(s + v)$. By Lemma 2.1(ii), for some k in R^s, $k(x)$ is in $\mathfrak{t}$. Then, by Corollary 2.14(ii), $kg(V)$ is in $\overline{R^{k(x)}.\mathfrak{t}}$. As $k(x)$ is not in $\mathfrak{z}$, $\mathfrak{a}^{k(x)}$ is an object of $\mathcal{C}'_\mathfrak{t}$ of dimension smaller than n. By hypothesis, $\mathfrak{a}^{k(x)}$ has Property $(\mathbf{P})$. Moreover, $kg(V)$ is contained in $\mathfrak{r}^s \cap \mathfrak{r}^{k(x)}$. Hence, by Property $(\mathbf{P})$ for $\mathfrak{a}^{k(x)}$, $kg(V)$ is in $\overline{R^s.\mathfrak{t}}$, whence V is in $\overline{R^s.\mathfrak{t}}$ since kg is in R^s. □

Proposition 2.21. *The objects of* $\mathcal{C}'_\mathfrak{t}$ *have Property* $(\mathbf{P})$.

Proof. Prove by induction on n that $\mathfrak{a}$ has Property $(\mathbf{P})$. By Lemma 2.8, it is true for $n = d^\#$. Suppose that it is true for the integers smaller than n. By Corollary 2.20, it remains to prove that $\mathfrak{a}$ has Property $(\mathbf{P}_1)$.

Suppose that $\mathfrak{a}$ has not Property $(\mathbf{P}_1)$. A contradiction is expected. For some fixed point V under $\mathbf{T}$ in X_R such that $V \cap \mathfrak{t} = \mathfrak{z}$, $r_V \neq |\mathcal{R}_V|$. By Lemma 2.19(ii), $r_V = |\mathcal{R}_V| - 1$. Then the orthogonal complement of $\mathcal{R}_V$ in $\mathfrak{t}$ is generated by $\mathfrak{z}$ and an element s in $\mathfrak{t} \setminus \mathfrak{z}$. In particular, V is contained in $\mathfrak{r}^s$. According to Lemma 2.5,

for some ideal $\mathfrak{a}'$ of codimension 1 of $\mathfrak{a}$, normalized by $\mathfrak{t}$, $\mathfrak{a}^s$ is contained in $\mathfrak{a}'$. Denote by α the element of $\mathcal{R}$ such that

$$\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}^\alpha$$

and consider θ_α and Γ as in Subsection 2.5. Denote by Γ_V the set of elements of Γ whose image by the projection

$$\Gamma \longrightarrow \mathrm{Gr}_d(\mathfrak{r}) \ , \qquad (T_1, T', T, T_2) \longmapsto T$$

is equal to V. By Lemma 2.13(ii), Γ_V is not empty and it is invariant under $\mathbf{T}$ by Lemma 2.13(i). As it is a projective variety, it has a fixed point under $\mathbf{T}$. Denote by (V_1, V', V, W) such a point. As $\mathfrak{a}'$ has Property (**P**), it has Property ($\mathbf{P}_1$) by Lemma 2.12. Hence $r_{V'} = |\mathcal{R}_{V'}|$ and $V' \neq V$ since $r_V \neq \mathcal{R}_V$. Then, by Lemma 2.13(iv),

$$V_1 = V \cap V' \quad \text{and} \quad W = V' + V.$$

As a result, $V' \cap \mathfrak{t} = V \cap \mathfrak{t} = \mathfrak{z}$ since $\mathcal{R}_{V'} \neq \mathcal{R}_V$ and V_1 has codimension 1 in V and V'. Then $V' = V$ by Corollary 2.18, whence a contradiction. □

The following corollary results from Proposition 2.21, Corollary 2.10 and Lemma 2.12.

Corollary 2.22. *Let V be in X_R.*

(i) *The space V is a commutative algebraic subalgebra of $\mathfrak{r}$ and for some subset Λ of $\mathcal{R}$, the biggest torus contained in V is conjugate to $\mathfrak{t}_\Lambda$ under R.*

(ii) *If V is a fixed point under R, then V is an ideal of $\mathfrak{r}$ and the elements of $\mathcal{R}_V$ are linearly independent.*

3. Solvable algebras and main varieties

Let $\mathfrak{t}$ be a vector space of positive dimension d and $\mathfrak{a}$ in $\mathcal{C}_\mathfrak{t}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad \mathcal{E} := \mathcal{E}_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

In this section, we give some results on the singular locus of X_R. For $\mathfrak{a}'$ in $\mathcal{C}_\mathfrak{t}$, denote by $X_{R_{\mathfrak{t},\mathfrak{a}'},\mathfrak{n}}$ the subset of elements of $X_{R_{\mathfrak{t},\mathfrak{a}'}}$ contained in $\mathfrak{a}'$.

3.1. **Subvarieties of X_R.** Denote by $\mathcal{P}_c(\mathcal{R})$ the set of complete subsets of $\mathcal{R}$ and for Λ in $\mathcal{P}_c(\mathcal{R})$ denote by X_{R_Λ} the closure in $\mathrm{Gr}_d(\mathfrak{r})$ of the orbit $R_\Lambda.\mathfrak{t}$.

Proposition 3.1. *Let Z be an irreducible closed subset of X_R, invariant under R.*

(i) *For a well defined complete subset Λ of $\mathcal{R}$, all element of a dense open subset of Z is conjugate under R to the sum of $\mathfrak{t}_\Lambda$ and a subspace of $\mathfrak{a}$.*

(ii) *All element of Z is contained in $\mathfrak{t}_\Lambda \oplus \mathfrak{a}$.*

(iii) *For some irreducible closed subset Z_Λ of X_{R_Λ}, invariant under R_Λ, $R.Z_\Lambda$ is dense in Z.*

Proof. (i) For Λ in $\mathcal{P}_c(\mathcal{R})$, let Y_Λ be the subset of elements V of Z such that $\pi(V) = \mathfrak{t}_\Lambda$. Since Z is invariant under R, so is Y_Λ. Moreover, by Corollary 2.22(i),

$$\overline{Y_\Lambda} \subset Y_\Lambda \cup \bigcup_{\substack{\Lambda' \in \mathcal{P}_c(\mathcal{R}) \\ \Lambda' \supsetneq \Lambda}} Y_{\Lambda'}.$$

According to Corollary 2.22(i), Z is the union of $Y_\Lambda, \Lambda \in \mathcal{P}_c(\mathcal{R})$. As a result, since $\mathcal{R}$ is finite and Z is irreducible, for a well defined complete subset Λ of $\mathcal{R}$, Y_Λ contains a dense open subset of Z. By Lemma 2.1(v), all element of Y_Λ is conjugate under R to the sum of $\mathfrak{t}_\Lambda$ and a subspace of $\mathfrak{a}$.

(ii) By (i), for all V in a dense subset of Z, V is contained in $\mathfrak{t}_\Lambda \oplus \mathfrak{a}$, whence the assertion.

(iii) Let Z_* be the subset of elements of Z, containing $\mathfrak{t}_\Lambda$. Denote by s an element of $\mathfrak{t}_\Lambda$ such that $\alpha(s) \neq 0$ for all α in $\mathcal{R} \setminus \Lambda$. By Lemma 2.6(i),

$$\mathfrak{r}^s = \mathfrak{t} \oplus \mathfrak{a}_\Lambda.$$

Hence Z_* is contained in X_{R_Λ} by Proposition 2.21. Moreover, Z_* is invariant under R_Λ since Z is invariant under R. By (i), $R.Z_*$ is dense in Z. So, for some irreducible component Z_Λ of Z_*, $R.Z_\Lambda$ is dense in Z. Moreover, Z_Λ is invariant under R_Λ since so is Z_*. □

For Λ in $\mathcal{P}_c(\mathcal{R})$, denote by $\mathfrak{t}^\#_\Lambda$ a complement to $\mathfrak{t}_\Lambda$ in $\mathfrak{t}$ and set:

$$\mathfrak{r}^\#_\Lambda := \mathfrak{t}^\#_\Lambda + \mathfrak{a}_\Lambda.$$

Let $R^\#_\Lambda$ be the adjoint group of $\mathfrak{r}^\#_\Lambda$ and $A^\#_\Lambda$ the connected closed subgroup of $R^\#_\Lambda$ whose Lie algebra is $\operatorname{ad}\mathfrak{a}_\Lambda$.

Lemma 3.2. *Let Λ be in $\mathcal{P}_c(\mathcal{R})$, nonempty and strictly contained in $\mathcal{R}$.*

(i) *The tori $\mathfrak{t}_\Lambda$ and $\mathfrak{t}^\#_\Lambda$ have positive dimension and $\mathfrak{a}_\Lambda$ is in $\mathcal{C}_{\mathfrak{t}^\#_\Lambda}$. Moreover,*

$$\dim \mathfrak{a}_\Lambda - \dim \mathfrak{t}^\#_\Lambda \leqslant \dim \mathfrak{a} - \dim \mathfrak{t}.$$

(ii) *The map $V \mapsto V \oplus \mathfrak{t}_\Lambda$ is an isomorphism from $X_{R^\#_\Lambda}$ onto X_{R_Λ}.*

Proof. Since Λ is a complete subset of $\mathcal{R}$ strictly contained in $\mathcal{R}$, $\mathfrak{t}_\Lambda$ has positive dimension and since Λ is not empty, $\mathfrak{t}_\Lambda$ is strictly contained in $\mathfrak{t}$. By definition, Λ is the set of weights of $\mathfrak{t}$ in $\mathfrak{a}_\Lambda$ so that $\mathfrak{a}_\Lambda$ is in $\mathcal{C}'_{\mathfrak{t}}$. Then $\mathfrak{a}_\Lambda$ is in $\mathcal{C}_{\mathfrak{t}^\#_\Lambda}$ and Assertion (ii) results from Corollary 2.2.

By Lemma 2.1,(i) and (iv), $\mathcal{R}$ generates $\mathfrak{t}^*$. Hence

$$|\Lambda| + \dim \mathfrak{t}_\Lambda \leqslant |\mathcal{R}|.$$

By Condition (2) of Section 2, $\mathfrak{a}$ has dimension $|\mathcal{R}|$ and $\mathfrak{a}_\Lambda$ has dimension $|\Lambda|$. As a result,

$$\dim \mathfrak{a} - \dim \mathfrak{t} = |\mathcal{R}| - \dim \mathfrak{t}_\Lambda - \dim \mathfrak{t}^\#_\Lambda \geqslant \dim \mathfrak{a}_\Lambda - \dim \mathfrak{t}^\#_\Lambda.$$

□

3.2. **Smooth points of X_R and commutators.** Denote by $\mathfrak{t}_{\mathrm{reg}}$ the complement in $\mathfrak{t}$ to the union of $\mathfrak{t}_\alpha, \alpha \in \mathcal{R}$ and $\mathfrak{r}_{\mathrm{reg}}$ the set of elements x of $\mathfrak{r}$ such that $\mathfrak{r}^x$ has minimal dimension.

Lemma 3.3. (i) *The set $\mathfrak{t}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{t}$, contained in $\mathfrak{r}_{\mathrm{reg}}$. Moreover, $R.\mathfrak{t}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{r}$.*

(ii) *For all x in $\mathfrak{r}_{\mathrm{reg}}$, $\mathfrak{r}^x$ is in X_R.*

(iii) *The set $\mathfrak{r}_{\mathrm{reg}}$ is a big open subset of $\mathfrak{r}$.*

Proof. (i) By definition, $\mathfrak{t}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{t}$. According to Lemma 2.6(i), for x in $\mathfrak{t}_{\mathrm{reg}}$, $\mathfrak{r}^x = \mathfrak{t}$. Then $R.x = A.x = x + \mathfrak{a}$ since $A.x$ is a closed subset of $x + \mathfrak{a}$ of dimension $\dim \mathfrak{a}$. As a result, $R.\mathfrak{t}_{\mathrm{reg}} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{a}$ is a dense open subset of $\mathfrak{r}$. Hence $R.\mathfrak{t}_{\mathrm{reg}}$ is contained in $\mathfrak{r}_{\mathrm{reg}}$ since $\mathfrak{r}^x$ is conjugate to $\mathfrak{t}$ for all x in $R.\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{r}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{r}$.

(ii) By (i), for all x in $\mathfrak{r}_{\mathrm{reg}}$, $\mathfrak{r}^x$ has dimension d, whence a regular map

$$\mathfrak{r}_{\mathrm{reg}} \xrightarrow{\theta} \mathrm{Gr}_d(\mathfrak{r}) \ , \qquad x \longmapsto \mathfrak{r}^x.$$

As a result, by (i), for all x in $\mathfrak{r}_{\mathrm{reg}}$, $\mathfrak{r}^x$ is in X_R.

(iii) Suppose that $\mathfrak{r}_{\mathrm{reg}}$ is not a big open subset of $\mathfrak{r}$. A contradiction is expected. Let Σ be an irreducible component of codimension 1 of $\mathfrak{r} \setminus \mathfrak{r}_{\mathrm{reg}}$. Since $\Sigma \cap A.\mathfrak{t}_{\mathrm{reg}}$ is empty, $\pi(\Sigma)$ is contained in $\mathfrak{t}_\alpha$ for some α in $\mathfrak{r}$. Then $\Sigma = \mathfrak{t}_\alpha + \mathfrak{a}$ since Σ has codimension 1 in $\mathfrak{r}$. By Condition (3) of Section 2, for some s in $\mathfrak{t}_\alpha$, $\gamma(s) \neq 0$ for

all γ in $\mathcal{R} \setminus \{\alpha\}$. Then $\mathfrak{r}^{s+x_\alpha} = \mathfrak{t}_\alpha + \mathfrak{a}^\alpha$ so that $s + x_\alpha$ is in $\mathfrak{r}_{\mathrm{reg}}$ by (i) and Condition (2) of Section 2, whence the contradiction. □

Denote by X'_R the image of θ.

Proposition 3.4. (i) *The complement to $R.\mathfrak{t}$ in X_R is equidimensional of dimension* $\dim \mathfrak{a} - 1$.

(ii) *The set X'_R is a smooth open subset of X_R, containing $R.\mathfrak{t}$.*

Proof. (i) As R is solvable and $R.\mathfrak{t}$ is dense in X_R, $R.\mathfrak{t}$ is an affine open subset of X_R. So, by [EGAIV, Corollaire 21.12.7], $X_R \setminus R.\mathfrak{t}$ is equidimensional of dimension $\dim \mathfrak{a} - 1$ since X_R has dimension $\dim \mathfrak{a}$.

(ii) By definition, $\mathcal{E}$ is the subvariety of elements (V, x) of $X_R \times \mathfrak{r}$ such that x is in V. Let Γ be the image of the graph of θ by the isomorphism

$$\mathfrak{r} \times \mathrm{Gr}_d(\mathfrak{r}) \longrightarrow \mathrm{Gr}_d(\mathfrak{r}) \times \mathfrak{r}\ , \qquad (x, V) \longmapsto (V, x).$$

Then Γ is the intersection of $\mathcal{E}$ and $X_R \times \mathfrak{r}_{\mathrm{reg}}$. Since Γ is isomorphic to $\mathfrak{r}_{\mathrm{reg}}$, Γ is a smooth open subset of $\mathcal{E}$ whose image by the bundle projection is X'_R. As a result, X'_R is a smooth open subset of X_R by [MA86, Ch. 8, Theorem 23.7]. □

For α in $\mathcal{R}$, set $V_\alpha := \mathfrak{t}_\alpha \oplus \mathfrak{a}^\alpha$ and denote by θ_α the map

$$\Bbbk \xrightarrow{\theta_\alpha} \mathrm{Gr}_d(\mathfrak{r})\ , \qquad z \longmapsto \exp(z \mathrm{ad}\, x_\alpha)(\mathfrak{t}),$$

By Condition (2) of Section 2, V_α has dimension d.

Lemma 3.5. *Let α be in $\mathcal{R}$. Set $X_{R,\alpha} := \overline{A.V_\alpha}$.*

(i) *The map θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$ such that $\theta_\alpha(\infty) = V_\alpha$.*

(ii) *The variety $X_{R,\alpha}$ has dimension* $\dim \mathfrak{a} - 1$ *and it is an irreducible component of $X_R \setminus R.\mathfrak{t}$.*

(iii) *The intersection $X_{R,\alpha} \cap X'_R$ is not empty.*

Proof. (i) Let h_α be in $\mathfrak{t}$ such that $\alpha(h_\alpha) = 1$. Since X_R is a projective variety, the map θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$ by [Sh94, Ch. 6, Theorem 6.1]. For z in $\Bbbk$,

$$\theta_\alpha(z) = \mathfrak{t}_\alpha \oplus \Bbbk(h_\alpha - z x_\alpha).$$

Hence $\theta_\alpha(\infty) = V_\alpha$.

(ii) By (i), $X_{R,\alpha}$ is contained in X_R and its elements are contained in $\mathfrak{t}_\alpha \oplus \mathfrak{a}$ so that $X_{R,\alpha}$ is contained in $X_R \setminus R.\mathfrak{t}$. By Condition (3) of Section 2, for γ in $\mathcal{R} \setminus \{\alpha\}$ and v in $\mathfrak{a}^\gamma$, $[\mathfrak{t}_\alpha, v] = \Bbbk v$ so that no element of $\mathfrak{a}^\gamma$ normalizes V_α. As a result, the normalizer of V_α in $\mathfrak{r}$ is equal to $\mathfrak{t} + \mathfrak{a}^\alpha$ so that $X_{R,\alpha}$ has dimension $\dim \mathfrak{a} - 1$. Hence $X_{R,\alpha}$ is an irreducible component $X_R \setminus R.\mathfrak{t}$.

(iii) According to Condition (3) of Section 2, for some s in $\mathfrak{t}_\alpha$, $\gamma(s) \neq 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$. Then $V_\alpha = \mathfrak{r}^{s+x_\alpha}$ so that $s + x_\alpha$ is in $\mathfrak{r}_{\mathrm{reg}}$, whence the assertion. □

3.3. **On the singular locus of X_R.** In this subsection we suppose $\dim \mathfrak{a} > d$ and we fix an ideal $\mathfrak{a}'$ of codimension 1 in $\mathfrak{a}$, normalized by $\mathfrak{t}$ and such that $\mathfrak{a}'$ is in $\mathcal{C}_\mathfrak{t}$. For example, all ideal of $\mathfrak{r}$ of dimension $\dim \mathfrak{a} - 1$, contained in $\mathfrak{a}$ and containing a fixed point under R in X_R is in $\mathcal{C}_\mathfrak{t}$ by Corollary 2.22(ii). Set:

$$\mathfrak{r}' := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}'} \qquad \pi' := \pi_{\mathfrak{t},\mathfrak{a}'}, \qquad R' := R_{\mathfrak{t},\mathfrak{a}'}, \qquad A' := A_{\mathfrak{t},\mathfrak{a}'}, \qquad \mathcal{R}' := \mathcal{R}_{\mathfrak{t},\mathfrak{a}'}.$$

Let α be in $\mathcal{R}$ such that

$$\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}^\alpha$$

and Γ as in Subsection 2.5. Denote by ϖ_1, ϖ_2, ϖ_3, ϖ_4 the restrictions to Γ of the first, second, third, fourth projections. Let Z be an irreducible component of $X_{R,\mathfrak{n}}$. According to Lemma 2.13(ii), for some irreducible

component T of $\varpi_3^{-1}(Z)$, $\varpi_3(T) = Z$. Denote by Z' the image of T by ϖ_2 and by T_1 the image of T by the projection $\varpi_1 \times \varpi_4$. Then Z' and T_1 are irreducible closed subsets of $\mathrm{Gr}_d(\mathfrak{r})$ and $\mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r})$ respectively. Let T_0 be the subset of elements (V_1, V', V, W) of T such that $V' = V$. Then T_0 is a closed subset of T. If $T_0 = T$, $Z' = Z$ and Z is contained in $X_{R',\mathrm{n}}$. Otherwise, $O := T \setminus T_0$ is a dense open subset of T. According to Lemma 2.13(iv), for all (V_1, V', V, W) in O, $V_1 = V' \cap V$ and $V' + V = W$. Denote by O_1 an open subset of T_1, contained and dense in $\varpi_1 \times \varpi_4(O)$.

Let (V_1, W) be in O_1. Denote by E a complement to V_1 in $\mathfrak{r}$ and by E' a complement to W in $\mathfrak{r}$ contained in E. Let κ be the map

$$\begin{aligned} \mathrm{Hom}_{\Bbbk}(V_1, W \cap E) \times \mathrm{Hom}_{\Bbbk}(W, E') &\xrightarrow{\kappa} \mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r}) \ , \\ (\varphi, \psi) &\longmapsto (\mathrm{span}(\{v + \varphi(v) + \psi(v) + \psi\circ\varphi(v) \mid v \in V_1\}), \mathrm{span}(\{v + \psi(v) \mid v \in W\})). \end{aligned}$$

Then κ is an isomorphism from its source to an open neighborhood of (V_1, W) in the subvariety of elements (W_1, W_2) of $\mathrm{Gr}_{d-1}(\mathfrak{r}) \times \mathrm{Gr}_{d+1}(\mathfrak{r})$ such that W_1 is contained in W_2. Denote by Ω the inverse image by κ of the intersection of T_1 and the image of κ. Let (e_1, e_2) be a basis of $W \cap E$ and let κ_* be the map

$$\begin{aligned} \Omega \times (\Bbbk^2 \setminus \{(0,0)\}) &\xrightarrow{\kappa_*} \mathrm{Gr}_d(\mathfrak{r}) \ , \\ (\varphi, \psi, x_1, x_2) &\longmapsto \mathrm{span}(\{v + \varphi(v) + \psi(v) + \psi\circ\varphi(v) \mid v \in V_1\} \cup \{x_1(e_1 + \psi(e_1)) + x_2(e_2 + \psi(e_2))\}). \end{aligned}$$

Lemma 3.6. *Suppose that O is not empty. Denote by $\widetilde{\Omega}$ the image of κ_* and $\tilde{Z}$ the closure of $\widetilde{\Omega}$ in $\mathrm{Gr}_d(\mathfrak{r})$.*

(i) *The intersections $\widetilde{\Omega} \cap Z'$ and $\widetilde{\Omega} \cap Z$ are dense in Z' and Z respectively. In particular Z' and Z are contained in $\tilde{Z}$.*

(ii) *For V in $\widetilde{\Omega}$, there exists (V', V'') in $Z' \times Z$ such that*

$$V' \cap V'' \subset V, \qquad V \subset V' + V'', \qquad (V' \cap V'', V' + V'') \in \kappa(\Omega).$$

(iii) *Let F' be the fiber of κ_* at some element V of $\kappa_*(\Omega)$. Denote by F the subset of elements (φ, ψ) of Ω such that V contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Then $F' = F \times \Bbbk^*(x_1, x_2)$ for some (x_1, x_2) in $\Bbbk^2 \setminus \{(0,0)\}$.*

(iv) *The varieties $\tilde{Z}$ and Z have dimension at most* $\dim Z' + 1$.

Proof. (i) Since T is irreducible so are T_1 and Ω. Hence $\tilde{Z}$ is irreducible. For some (V', V) in $Z' \times Z$, V_1 is contained in V' and V and V' and V are contained in W. Since $\kappa(\Omega)$ is an open neighbourhood of (V_1, W) in T_1,

$$\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T) \quad \text{and} \quad \varpi_3(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T)$$

are dense subsets of Z' and Z respectively. For all (φ, ψ) in Ω, all element of $\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\varphi, \psi)) \cap T)$ contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. Hence all element of $\varpi_2(\varpi_1 \times \varpi_4^{-1}(\kappa(\Omega)) \cap T)$ is in the image of κ_*. As a result, $\widetilde{\Omega} \cap Z'$ is dense in Z' and Z' is contained in $\tilde{Z}$. In the same way, $\widetilde{\Omega} \cap Z$ is dense in Z and Z is contained in $\tilde{Z}$.

(ii) According to Lemma 2.13(iv), for all (V_1', V', V, W') in O, $V_1' = V' \cap V$ and $W' = V' + V$. By definition, $\kappa(\Omega)$ is contained in $\varpi_1 \times \varpi_4(O)$ and for V in $\widetilde{\Omega}$, $V_1' \subset V$ and $V \subset W'$ for some (V_1', W') in $\kappa(\Omega)$, whence the assertion.

(iii) For (φ, ψ) in F and for (x_1, x_2) in $\Bbbk^2 \setminus \{(0,0)\}$ such that

$$V = \kappa_*(\varphi, \psi, x_1, x_2),$$

the subset of elements (y_1, y_2) of $\Bbbk^2$ such that $(\varphi, \psi, y_1, y_2)$ is in F' is equal to $\Bbbk^*.(x_1, x_2)$. Moreover, for all $(\varphi, \psi, y_1, y_2)$ in F', (φ, ψ) is in F, whence the assertion.

(iv) In (iii), we can choose V such that F' has minimal dimension so that

$$\dim \tilde{Z} = \dim \Omega + 2 - (\dim F + 1) = \dim \Omega - \dim F + 1.$$

By (ii), for some V' in Z', for all (φ, ψ) in F, V' contains the first component of $\kappa(\varphi, \psi)$ and is contained in the second component of $\kappa(\varphi, \psi)$. So, again by (iii) and (ii),

$$\dim Z' \geqslant \dim \Omega - \dim F,$$

whence $\dim \tilde{Z} \leqslant \dim Z' + 1$ and $\dim Z \leqslant \dim Z' + 1$ since Z is contained in $\tilde{Z}$ by (i). □

Proposition 3.7. *The variety $X_{R,\mathfrak{n}}$ has dimension at most $n - d$.*

Proof. Prove this by induction on n. According to Lemma 2.3(ii), it is true for $n - d = 0$. Suppose that $n - d$ is positive and that it is true for all integer smaller than $n - d$. In particular, $X_{R',\mathfrak{n}}$ has dimension at most $n - d - 1$. Let Z be an irreducible component of $X_{R,\mathfrak{n}}$. According to Lemma 2.13(ii), for some irreducible component T of $\varpi_3^{-1}(Z)$, $\varpi_3(T) = Z$. Denote by Z' the image of T by ϖ_2. Let T_0 be the subset of elements (V_1, V', V, W) of T such that $V' = V$. Consider the following cases:

(a) $T_0 = T$,
(b) $T_0 \neq T$ and Z' is contained in $X_{R',\mathfrak{n}}$,
(c) Z' is not contained in $X_{R',\mathfrak{n}}$.

(a) In this case, $Z' = Z$ and $\dim Z \leqslant n - d - 1$ by induction hypothesis.

(b) By induction hypothesis, $\dim Z' \leqslant n - d - 1$ and by Lemma 3.6(iv), $\dim Z \leqslant \dim Z' + 1$, whence $\dim Z \leqslant n - d$.

(c) In this case, $T_0 \neq T$, whence $\dim Z \leqslant \dim Z' + 1$ by Lemma 3.6(iv). Since Z is an irreducible component of $X_{R,\mathfrak{n}}$, Z is invariant under R. By Lemma 2.13(i), ϖ_2 and ϖ_3 are equivariant under the action of R' in Γ so that T and Z' are invariant under R'. For all (V_1, V', V, W) in $T \setminus T_0$, $V_1 = V' \cap V$. Hence all element of a dense open subset of Z' contains a subspace of dimension $d - 1$ of $\mathfrak{a}'$. Then, by Proposition 3.1, for some complete subset Λ of $\mathcal{R}'$ such that $\mathfrak{t}_\Lambda$ has dimension 1 and for some closed subset Z_Λ of X_{R_Λ}, $R'.Z_\Lambda$ is dense in Z' so that

$$\dim Z' \leqslant \dim Z_\Lambda + \dim \mathfrak{a}' - \dim \mathfrak{a}_\Lambda.$$

If $\dim \mathfrak{a}_\Lambda - \dim \mathfrak{t} + 1 = n - d$, then $\Lambda = \mathcal{R}'$. In this case, since $\mathfrak{a}'$ is in $\mathcal{C}_\mathfrak{t}$, Λ generates $\mathfrak{t}^*$. As $\mathfrak{t}_\Lambda$ has dimension 1, it is impossible. As a result,

$$\dim Z_\Lambda \leqslant \dim \mathfrak{a}_\Lambda - \dim \mathfrak{t} + 1 \quad \text{and} \quad \dim Z' \leqslant n - d$$

by Lemma 3.2 and induction hypothesis for $\mathfrak{a}_\Lambda$. Then $\dim Z \leqslant n - d + 1$. According to Lemma 3.6,(i) and (iv), $\tilde{Z}$ is an irreducible variety of dimension at most $\dim Z' + 1$, containing Z' and Z. If $\dim Z' = n - d$ and $\dim Z = n - d + 1$, then $Z = \tilde{Z}$. In particular, Z' is contained in Z. It is impossible since all element of Z is contained in $\mathfrak{a}$. As a result, $\dim Z \leqslant n - d$, whence the proposition. □

Corollary 3.8. (i) *The irreducible components of $X_R \setminus R.\mathfrak{t}$ are the $X_{R,\alpha}, \alpha \in \mathcal{R}$.*

(ii) *The set X'_R is a smooth big open subset of X_R, containing $R.\mathfrak{t}$.*

Proof. According to Proposition 3.4(ii) and Lemma 3.5(iii), Assertion (ii) results from Assertion (i). Prove Assertion (i) by induction on $n = \dim \mathfrak{a}$. For $n = 1$, $d = 1$ by Lemma 2.1,(i) and (iv) so that X_R is the union of $R.\mathfrak{t}$ and $\mathfrak{a}^\alpha$, whence Assertion (i) in this case. Suppose $n \geqslant 2$ and the assertion true for the integers smaller than n. By Lemma 2.1(i), Condition (2) and Condition (3) of Section 2, $d \geqslant 2$. According to Lemma 3.5(ii), for all α in $\mathcal{R}$, $X_{R,\alpha}$ is an irreducible component of $X_R \setminus R.\mathfrak{t}$. Let Z be an irreducible component of $X_R \setminus R.\mathfrak{t}$. By Proposition 3.4(i), Z has dimension $n - 1$. So, by Proposition 3.7, Z is not contained in $X_{R,\mathfrak{n}}$. Moreover,

Z is invariant under R. Then, by Proposition 3.1, for some complete subset Λ of $\mathcal{R}$, strictly contained in $\mathcal{R}$ and for some irreducible closed subset Z_Λ of X_{R_Λ}, invariant under R_Λ, $R.Z_\Lambda$ is dense in Z. By Lemma 3.2, $\mathfrak{a}_\Lambda$ is in $\mathcal{C}_{\mathfrak{t}^\#_\Lambda}$ and Z_Λ is the image of a closed subset Z'_Λ of $X_{R^\#_\Lambda}$, invariant by $R^\#_\Lambda$, by the map $V \mapsto V \oplus \mathfrak{t}_\Lambda$. Since Z_Λ is contained in Z, $Z'_\Lambda \cap R^\#_\Lambda.\mathfrak{t}^\#_\Lambda$ is empty. As Λ is strictly contained in $\mathcal{R}$, $\dim \mathfrak{a}_\Lambda$ is smaller than n. So, by induction hypothesis, for some α in Λ, Z'_Λ is contained in $X_{R^\#_\Lambda,\alpha}$. As a result, Z_Λ and Z are contained in $X_{R,\alpha}$, whence $Z = X_{R,\alpha}$ since Z is an irreducible component of $X_R \setminus R.\mathfrak{t}$. □

4. Normality for solvable Lie algebras

Let $\mathfrak{t}$ be a vector space of positive dimension d and $\mathfrak{a}$ in $\mathcal{C}_\mathfrak{t}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad \mathcal{E} := \mathcal{E}_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

The goal of the section is to prove that X_R is normal and Cohen-Macaulay.

4.1. **The case** $\dim \mathfrak{a} = \dim \mathfrak{t}$**.** By Condition (2) of Section 2, $\mathcal{R}$ has d elements $\beta_1, \dots, \beta_d$ linearly independent. Denote by $t_1, \dots, t_d$ the dual basis in $\mathfrak{t}$. For $i = 1, \dots, d$, let v_i be a generator of $\mathfrak{a}^{\beta_i}$.

Lemma 4.1. *If* $\dim \mathfrak{a} = \dim \mathfrak{t}$ *then* X_R *is a smooth variety. Moreover, for all* $(z_1, \dots, z_d)$ *in* $\Bbbk^d$*, the subspace generated by* $v_1 + z_1 t_1, \dots, v_d + z_d t_d$ *is in* X_R*.*

Proof. According to Lemma 2.3, $\mathfrak{a}$ is in in X_R and the map

$$\Bbbk^d \longrightarrow X_R\ , \qquad (z_1, \dots, z_d) \longmapsto \operatorname{span}(\{v_1 + z_1 t_1, \dots, v_d + z_d x_d\})$$

is an isomorphism onto an open neighborhood of $\mathfrak{a}$ in X_R. Hence $\mathfrak{a}$ is a smooth point of X_R. By Corollary 2.22, R has only one fixed point $\mathfrak{a}$ in X_R. Since for all V in X_R, R has a fixed point in $\overline{R.V}$ and $X_{R\mathrm{sm}}$ is an open subset of X_R, invariant under R, $X_R = X_{R\mathrm{sm}}$. □

4.2. **Cohen-Macaulayness property for some algebras.** Let A_* be an integral domain and a local commutative $\Bbbk$-algebra with maximal ideal $\mathfrak{m}$ and $u_1, \dots, u_s$ a regular sequence in A_* of elements of $\mathfrak{m}$. Let $T_1, \dots, T_s$ be indeterminates. Set $B_s := A_*[T_1, \dots, T_s]$ and denote by P_s and P'_s the ideals of B_s generated by the sequences $u_j T_k - u_k T_j$, $1 \leqslant j, k \leqslant s$ and $u_j T_1 - u_1 T_j$, $1 \leqslant j \leqslant s$ respectively.

Lemma 4.2. *The ideal* P_s *is a prime ideal of* B_s*.*

Proof. For $s = 1$, $P_s = \{0\}$. Suppose $s \geqslant 2$. Let $\tilde{P}$ be the ideal of $B_s[T_1^{-1}]$ generated by P_s. For $1 \leqslant j, k \leqslant s$,

$$T_1(u_j T_k - u_k T_j) = T_k(u_j T_1 - u_1 T_j) + T_j(u_1 T_k - u_k T_1).$$

Hence $\tilde{P}$ is the ideal of $B_s[T_1^{-1}]$ generated by P'_s. Setting $S_j := T_j/T_1$ for $j = 2, \dots, s$, denote by C the polynomial algebra $A_*[S_2, \dots, S_s]$ over A_* so that $B_s[T_1^{-1}] = C[T_1, T_1^{-1}]$ and $\tilde{P}$ is the ideal of $B_s[T_1^{-1}]$ generated by $u_j - u_1 S_j$, $j = 2, \dots, s$.

Claim 4.3. Let Q be the ideal of C generated by $u_j - u_1 S_j$, $j = 2, \dots, s$. Then Q is prime.

Proof. [Proof of Claim 4.3] Let $\tilde{Q}$ be the ideal of $C[u_1^{-1}]$ generated by Q. Then $\tilde{Q}$ is prime since it is generated by $u_j u_1^{-1} - S_j$, $j = 2, \dots, s$. As a result, for p and q in C such that pq is in Q, for some nonnegative integer m, $u_1^m p$ or $u_1^m q$ is in Q. So it remains to prove that for p in C, p is in Q if so is $u_1 p$.

Let p be in C such that $u_1 p$ is in Q. For some $q_2, \dots, q_s$ in C,

$$u_1 p = \sum_{j=2}^{s} q_j(u_j - u_1 S_j) \quad \text{whence} \quad \sum_{j=1}^{s} q_j u_j = 0 \quad \text{with} \quad q_1 := -(p + \sum_{j=2}^{s} q_j S_j).$$

By hypothesis, the sequence $u_1, \ldots, u_s$ is regular in C. So for some sequence $q_{j,k}, 1 \leqslant j, k \leqslant s$ in C such that $q_{j,k} = -q_{k,j}$,

$$q_j = \sum_{k=1}^{s} q_{j,k} u_k$$

for $j = 1, \ldots, s$. As a result,

$$\begin{aligned} u_1 p = & \quad \textstyle\sum_{j=2}^{s} \sum_{k=1}^{s} q_{j,k} u_k (u_j - u_1 S_j) \\ = & \quad \textstyle\sum_{j=2}^{s} q_{j,1} u_j u_1 - \sum_{j=2}^{s} \sum_{k=1}^{s} q_{j,k} u_k u_1 S_j \\ = & \quad u_1 (\textstyle\sum_{j=2}^{s} q_{j,1} (u_j - u_1 S_j) + \sum_{2 \leqslant j < k \leqslant s} q_{j,k} (u_j S_k - u_k S_j)). \end{aligned}$$

For $2 \leqslant j, k \leqslant s$,

$$u_j S_k - u_k S_j = (u_j - u_1 S_j) S_k - (u_k - u_1 S_k) S_j \in Q,$$

whence the claim. □

By the claim, $\tilde{P}$ is a prime ideal of $B_s[T_1^{-1}]$ since it is generated by Q. As a result for p and q in B_s such that pq is in P_s, for some nonnegative integer m, $T_1^m p$ or $T_1^m q$ is in P'_s since $T_1 P_s$ is contained in P'_s by the equality

$$T_1(u_j T_k - u_k T_j) = T_k(u_j T_1 - u_1 T_j) + T_j(u_1 T_k - u_k T_1)$$

for $1 \leqslant i, j \leqslant s$. So it remains to prove that for p in B_s, p is in P_s if $T_1 p$ is in P'_s.

Let p be in B_s such that $T_1 p$ is in P'_s. For some $r_2, \ldots, r_s$ in B_s,

$$T_1 p = \sum_{j=2}^{s} r_j (u_j T_1 - u_1 T_j).$$

For $j = 2, \ldots, s$, r_j has an expansion

$$r_j = r_{j,0} + T_1 r_{j,1}$$

with $r_{j,0}$ and $r_{j,1}$ in $B'_s := A_*[T_2, \ldots, T_s]$ and B_s respectively. Set:

$$p' := p - \sum_{j=2}^{s} r_{j,1} (u_j T_1 - u_1 T_j).$$

Then

$$T_1 p' = \sum_{j=2}^{s} r_{j,0} (u_j T_1 - u_1 T_j)$$

so that the element

$$\sum_{j=2}^{s} r_{j,0} u_1 T_j \in B'_s$$

is divisible by T_1 in B_s, whence

$$\sum_{j=2}^{s} r_{j,0} T_j = 0.$$

As $T_2, \ldots, T_s$ is a regular sequence in B_s, for some sequence $r_{j,k,0}, 2 \leqslant j, k \leqslant s$ in B_s such that $r_{j,k,0} = -r_{k,j,0}$ for all (j, k),

$$r_{j,0} = \sum_{k=2}^{s} r_{j,k,0} T_k$$

for $j = 2, \dots, s$. Then

$$T_1 p' = \sum_{2 \leqslant j,k \leqslant s} r_{j,k,0} T_k (u_j T_1 - u_1 T_j) = T_1 \sum_{2 \leqslant j < k \leqslant s} r_{j,k,0} (T_k u_j - T_j u_k).$$

As a result p' and p are in P_s, whence the lemma. □

Denote by P''_s the ideal of B_s generated by P_{s-1} and $u_s T_1 - u_1 T_s$. Let $\mathfrak{B}_s$ and $\mathfrak{B}'_s$ be the quotients of B_s by P_s and P''_s respectively. The restrictions to A_* of the quotient morphisms $B_s \longrightarrow \mathfrak{B}'_s$ and $B_s \longrightarrow \mathfrak{B}_s$ are embeddings. For $j = 1, \dots, s$, denote again by T_j its images in $\mathfrak{B}'_s$ and $\mathfrak{B}_s$ by these morphisms.

Lemma 4.4. *Denote by $\overline{P_s}$ the image in $\mathfrak{B}'_s$ of P_s by the quotient morphism.*

(i) *The intersection of $\overline{P_s}$ and $T_1 \mathfrak{B}'_s$ is equal to* $\{0\}$.

(ii) *The $\mathfrak{B}'_s$-modules $T_1 \mathfrak{B}'_s$ and $\mathfrak{B}_s$ are isomorphic.*

Proof. Let a be in B_s such that $T_1 a$ is in P_s. According to Lemma 4.2, P_s is a prime ideal of B_s. Hence a is in P_s since T_1 is not in P_s. Moreover, for $j = 1, \dots, s$,

$$T_1 (u_j T_s - u_s T_j) = T_s (u_j T_1 - u_1 T_j) + T_j (u_1 T_s - u_s T_1).$$

Hence $T_1 P_s$ is contained in P''_s. As a result, $\overline{P_s}$ is the kernel of the endomorphism $a \mapsto T_1 a$ of $\mathfrak{B}'_s$ and the intersection of $\overline{P_s}$ and $T_1 \mathfrak{B}'_s$ is equal to $\{0\}$. As $\mathfrak{B}_s$ is the quotient of $\mathfrak{B}'_s$ by $\overline{P_s}$, the endomorphism $a \mapsto T_1 a$ defines through the quotient an isomorphism

$$\mathfrak{B}_s \longrightarrow T_1 \mathfrak{B}'_s$$

of $\mathfrak{B}'_s$-modules. □

Let Q_s be the ideal of the polynomial algebra $A_*[T_2, \dots, T_s]$ generated by the sequence $u_i T_k - u_k T_i$, $2 \leqslant i, k \leqslant s$ and denote by $\mathfrak{B}^{\#}_s$ the quotient of $A_*[T_2, \dots, T_s]$ by Q_s.

Lemma 4.5. (i) *The quotient of the algebra $\mathfrak{B}_s / T_1 \mathfrak{B}_s$ by the ideal generated by u_1 is equal to the quotient of $\mathfrak{B}^{\#}_s$ by the ideal generated by u_1.*

(ii) *The canonical map $A_* \longrightarrow \mathfrak{B}_s / T_1 \mathfrak{B}_s$ is an embedding.*

(iii) *The ideal of $\mathfrak{B}_s / T_1 \mathfrak{B}_s$ generated by u_1 is isomorphic to A_**

Proof. Denote by Q'_s the ideal of B_s generated by P_s and T_1.

(i) As the ideal of B_s generated by Q'_s and u_1 is equal to the ideal generated by u_1, T_1 and Q_s, $\mathfrak{B}^{\#}_s / u_1 \mathfrak{B}^{\#}_s$ is equal to the quotient of $\mathfrak{B}_s / T_1 \mathfrak{B}_s$ by the ideal generated by u_1.

(ii) Since the intersection of A_* and Q'_s is equal to $\{0\}$, the canonical map $A_* \longrightarrow \mathfrak{B}_s / T_1 \mathfrak{B}_s$ is an embedding.

(iii) For $k = 2, \dots, s$, $u_1 T_k$ is in Q'_s. Hence $u_1 B_s$ is contained in the sum of $u_1 A_*$ and Q'_s. As a result, $u_1 A_*$ is equal to $u_1 \mathfrak{B}_s / T_1 \mathfrak{B}_s$ by (ii), whence the assertion since A_* is an integral domain. □

Proposition 4.6. *Suppose that A_* is Cohen-Macaulay.*

(i) *The algebra $\mathfrak{B}_s$ is an integral domain and a Cohen-Macaulay algebra of dimension* $\dim A_* + 1$.

(ii) *For $a_1, \dots, a_m$ regular sequence in A_* of elements of $\mathfrak{m}$ an for $\mathfrak{p}$ prime ideal of $\mathfrak{B}_s$ containing it, $a_1, \dots, a_m$ is a regular sequence in the localization of $\mathfrak{B}_s$ at $\mathfrak{p}$.*

Proof. (i) Prove the assertion by induction on s. As $\mathfrak{B}_1$ is the polynomial algebra $A_*[T_1]$, the assertion is true for $s = 1$ since A_* an integral domain and a Cohen-Macaulay algebra. Suppose the assertion true for $s - 1$. By induction hypothesis, $\mathfrak{B}_{s-1}[T_s]$ is an integral domain and a Cohen-Macaulay algebra as a polynomial algebra over $\mathfrak{B}_{s-1}$ and its dimension is equal to $\dim A_* + 2$. As a result, $\mathfrak{B}'_s$ is Cohen-Macaulay

of dimension $\dim A_* + 1$ as the quotient of the integral domain and a Cohen-Macaulay algebra $\mathfrak{B}_{s-1}[T_s]$ by the ideal generated by $T_s u_1 - T_1 u_s$. As $\mathfrak{B}_s$ is the quotient of $\mathfrak{B}'_s$ by $\overline{P_s}$, $\mathfrak{B}_s$ has dimension at most $\dim A_* + 1$. By Lemma 4.2, $\mathfrak{B}_s$ is an integral domain so that $\mathfrak{B}_s/T_1\mathfrak{B}_s$ has dimension at most $\dim A_*$.

By induction hypothesis again, $\mathfrak{B}^\#_s$ is an integral domain and a Cohen- Macaulay algebra of dimension $\dim A_* + 1$. Hence $\mathfrak{B}^\#_s/u_1\mathfrak{B}^\#_s$ is Cohen-Macaulay of dimension $\dim A_*$. According to Lemma 4.5, we have a short exact sequence

$$0 \longrightarrow A_* \longrightarrow \mathfrak{B}_s/T_1\mathfrak{B}_s \longrightarrow \mathfrak{B}^\#_s/u_1\mathfrak{B}^\#_s \longrightarrow 0\ .$$

Hence the algebra $\mathfrak{B}_s/T_1\mathfrak{B}_s$ is Cohen-Macaulay of dimension $\dim A_*$ since A_* and $\mathfrak{B}^\#_s/u_1\mathfrak{B}^\#_s$ are Cohen-Macaulay of dimension $\dim A_*$ and $\mathfrak{B}_s/T_1\mathfrak{B}_s$ has dimension at most $\dim A_*$. As a result, $\mathfrak{B}_s$ has dimension $\dim A_* + 1$. As $\mathfrak{B}_s$ is the quotient of $\mathfrak{B}'_s$ by $\overline{P_s}$, we have a short exact sequence

$$0 \longrightarrow \overline{P_s} + T_1\mathfrak{B}'_s \longrightarrow \mathfrak{B}'_s \longrightarrow \mathfrak{B}_s/T_1\mathfrak{B}_s \longrightarrow 0\ .$$

Then, setting $M := \overline{P_s} + T_1\mathfrak{B}'_s$ and denoting by M_* the localization of M at a maximal ideal of $\mathfrak{B}'_s$, containing T_1,

$$\mathrm{Ext}^j(\Bbbk, M_*) = 0$$

for $j \leqslant \dim A_*$ since $\mathfrak{B}'_s$ and $\mathfrak{B}_s/T_1\mathfrak{B}_s$ have dimension $\dim A_* + 1$ and $\dim A_*$. By Lemma 4.4(i), M is the direct sum $\overline{P_s}$ and $T_1\mathfrak{B}'_s$. So, denoting by $(T_1\mathfrak{B}'_s)_*$ the localization of $T_1\mathfrak{B}'_s$ at a maximal ideal of $\mathfrak{B}'_s$,

$$\mathrm{Ext}^j(\Bbbk, (T_1\mathfrak{B}'_s)_*) = 0$$

for $j \leqslant \dim A_*$ since $(T_1\mathfrak{B}'_s)_*$ is the localization of $\mathfrak{B}'_s$ at this maximal ideal when it does not contain T_1. As a result, by Lemma 4.4(ii), $\mathfrak{B}_s$ is Cohen-Macaulay since it has dimension $\dim A_* + 1$.

(ii) Let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{B}_s$, containing $a_1, \dots, a_m$. Since A_* is embedded in $\mathfrak{B}_s$, $\mathfrak{q} \cap A_*$ is a prime ideal of A_* containing $a_1, \dots, a_m$. As A_* is Cohen-Macaulay and $a_1, \dots, a_m$ is a regular sequence in A_*, $\mathfrak{q} \cap A_*$ has height at least m and $A_*/\mathfrak{q} \cap A_*$ has dimension at most $\dim A_* - m$ by [MA86, Ch. 6, Theorem 17.4]. Then $\mathfrak{B}_s/\mathfrak{q}$ has dimension at most $\dim A_* + 1 - m$ since the fraction field of $\mathfrak{B}_s/\mathfrak{q}$ is generated by the fraction field of $A_*/\mathfrak{q} \cap A_*$ and the image of T_1 by the quotient morphism $B_s \longrightarrow \mathfrak{B}_s/\mathfrak{q}$. As a result, by (i) and [MA86, Ch. 6, Theorem 17.4], $\mathfrak{q}$ has height at least m. As a minimal prime ideal of $\mathfrak{B}_s$ containing m elements, $\mathfrak{q}$ has height at most m. Hence all minimal prime ideal of $\mathfrak{B}_s$, containing $a_1, \dots, a_m$, has height m. So, by (i) and [MA86, Ch. 6, Theorem 17.4], $a_1, \dots, a_m$ is a regular sequence in the localization of $\mathfrak{B}_s$ at $\mathfrak{p}$. □

4.3. **Normality and Cohen-Macaulayness property for X_R.** Let V_0 be a fixed point under the action of R in X_R and $\beta_1, \dots, \beta_d$ the elements of $\mathcal{R}_{V_0}$. By Corollary 2.22(ii), $\beta_1, \dots, \beta_d$ is a basis of $\mathfrak{t}^*$. Let $t_1, \dots, t_d$ be the dual basis. Denote by m the codimension of V_0 in $\mathfrak{a}$. According to Lie's Theorem, for $m > 0$, the elements $\gamma_1, \dots, \gamma_m$ of $\mathcal{R} \setminus \{\beta_1, \dots, \beta_d\}$ can be ordered so that

$$\mathfrak{a}_i := V_0 \oplus \mathfrak{a}^{\gamma_1} \oplus \cdots \oplus \mathfrak{a}^{\gamma_i}$$

is an algebra of codimension $m - i$ of $\mathfrak{a}$ for $i = 1, \dots, m$. Set:

$$\mathcal{R}' := \mathcal{R} \setminus \{\gamma_m\}, \qquad \mathfrak{a}' = \mathfrak{a}_{m-1}, \qquad \mathfrak{r}' := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}'}, \qquad \pi' := \pi_{\mathfrak{t},\mathfrak{a}'}, \qquad R' := R_{\mathfrak{t},\mathfrak{a}'}, \qquad A' := A_{\mathfrak{t},\mathfrak{a}'},$$

$$E := \bigoplus_{i=1}^{m} \mathfrak{a}^{\gamma_i}, \qquad E' := E \cap \mathfrak{a}'.$$

Denote by κ the map

$$\mathrm{Hom}_{\Bbbk}(V_0, E \oplus \mathfrak{t}) \xrightarrow{\ \kappa\ } \mathrm{Gr}_d(\mathfrak{r})\ , \qquad \varphi \longmapsto \mathrm{span}(\{v + \varphi(v) \mid v \in V_0\}).$$

Then κ is an isomorphism from $\mathrm{Hom}_{\Bbbk}(V_0, E \oplus \mathfrak{t})$ onto an affine open neighbourhood of V_0 in $\mathrm{Gr}_d(\mathfrak{r})$. Moreover, there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\Bbbk}(V_0, \Bbbk x_{\gamma_m}) \longrightarrow \mathrm{Hom}_{\Bbbk}(V_0, E \oplus \mathfrak{t}) \xrightarrow{p} \mathrm{Hom}_{\Bbbk}(V_0, E' \oplus \mathfrak{t}) \longrightarrow 0 \ .$$

Let Ω and Ω' be the inverse images by κ of the intersections of the image of κ with X_R and $X_{R'}$ respectively. For φ in Ω and $i = 1, \dots, d$

$$\varphi(v_i) = \sum_{i=1}^{d} z_{i,j}(\varphi) t_j + \sum_{j=1}^{m} a_{i,j}(\varphi) x_{\gamma_j}$$

so that the $z_{i,j}$'s, $1 \leqslant i, j \leqslant d$ and the $a_{i,j}$'s, $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$ are regular functions on Ω.

Let ψ be the map

$$\Bbbk \times \Omega' \xrightarrow{\ \psi\ } X_R \ , \qquad (s, \varphi) \longmapsto \exp(s \mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi).$$

Lemma 4.7. *Let O be the subset of elements (s, φ) of $\Bbbk \times \Omega'$ such that $\psi(s, \varphi)$ is in $\kappa(\Omega)$.*

(i) *The subset O of $\Bbbk \times \Omega'$ is open and contains $\{0\} \times \Omega'$.*

(ii) *The map*

$$O \xrightarrow{\ \overline{\psi}\ } \Omega \ , \qquad (s, \varphi) \longmapsto \kappa^{-1} \circ \psi(s, \varphi)$$

is a birational morphism from O to Ω. In particular, the function $(s, \varphi) \mapsto s$ is in $\Bbbk(\Omega)$.

Proof. (i) As $\kappa(\Omega)$ is an open neighborhood of V_0 in X_R, O is an open subset of $\Bbbk \times \Omega'$, containing $\{0\} \times \Omega'$ since ψ is a regular map such that $\psi(0, \varphi) = \kappa(\varphi)$ for all φ in Ω'.

(ii) Let Ω^c be the subset of elements φ of Ω such that $\kappa(\varphi)$ is in $A.\mathfrak{t}$. Then Ω^c is a dense open subset of Ω. Let O' be the inverse image of Ω^c by $\overline{\psi}$. Let (s, φ) and (s', φ') be in O' such that $\overline{\psi}(s, \varphi) = \overline{\psi}(s', \varphi')$, that is

$$\exp(s \mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi) = \exp(s' \mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi') \quad \text{whence} \quad \exp((s - s') \mathrm{ad}\, x_{\gamma_m}).\kappa(\varphi) = \kappa(\varphi').$$

According to the above notations, for $i = 1, \dots, d$,

$$\varphi(v_i) = \sum_{j=1}^{d} z_{i,j}(\varphi) t_j + \sum_{j=1}^{m-1} a_{i,j}(\varphi) x_{\gamma_j}.$$

Since $\kappa(\varphi)$ is in $A.\mathfrak{t}$,

$$\det([z_{i,j}(\varphi), 1 \leqslant i, j \leqslant d]) \neq 0.$$

For $i = 1, \dots, d$,

$$\exp((s - s') \mathrm{ad}\, x_{\gamma_m})(\sum_{j=1}^{d} z_{i,j}(\varphi) t_j) = \sum_{j=1}^{d} z_{i,j}(\varphi) t_j - (s - s')(\sum_{j=1}^{d} z_{i,j}(\varphi) \gamma_m(t_j)) x_{\gamma_m}.$$

For some j, $\gamma_m(t_j) \neq 0$, whence $s = s'$ since $\kappa(\varphi')$ is contained in $\mathfrak{r}'$. As a result, the restriction of $\overline{\psi}$ to O' is injective, whence the assertion since $\overline{\psi}$ is a dominant morphism. □

For $i = 1, \dots, d$ and γ in $\mathfrak{t}^*$, denote by $u_{i,\gamma}$ the function on Ω,

$$u_{i,j} := z_{i,1} \gamma(t_1) + \cdots + z_{i,d} \gamma(t_d).$$

Let $\mathfrak{A}$ be the subalgebra of $\Bbbk[\Omega]$ generated by the functions $z_{i,j}$'s, $1 \leqslant i, j \leqslant d$ and $a_{i,j}$'s, $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m - 1$.

Lemma 4.8. *Let ι be the restriction morphism from Ω to Ω'.*

(i) *The restriction of ι to $\mathfrak{A}$ is an isomorphism onto $\Bbbk[\Omega']$.*

(ii) *For $1 \leqslant i, j \leqslant d$, $u_{i,\gamma_m} a_{j,m} - u_{j,\gamma_m} a_{i,m}$ is equal to 0.*

(iii) *For $i = 1, \dots, d$ and γ in $\mathfrak{t}^*$, if $\gamma(t_i) \neq 0$ then $u_{i,\gamma}$ is different from 0.*

Proof. (i) For $1 \leqslant i, j \leqslant d$, denote by $z'_{i,j}$ the restriction of $z_{i,j}$ to Ω' and for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m-1$ denote by $a'_{i,j}$ the restriction of $a_{i,j}$ to Ω'. Since $\Bbbk[\Omega']$ is generated by the functions

$$z'_{i,j},\ 1 \leqslant i, j \leqslant d \quad \text{and} \quad a'_{i,j},\ 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant m-1,$$

the restriction of ι to $\mathfrak{A}$ is surjective. Let $\mathfrak{p}$ be the kernel of the restriction of ι to $\mathfrak{A}$. It remains to prove $\mathfrak{p} = \{0\}$.

For $1 \leqslant i, j \leqslant d$ and $k = 1, \dots, m-1$, denote by $\overline{z}_{i,j}$ and $\overline{a}_{i,k}$ the functions on $\Bbbk \times \Omega'$ such that

$$\exp(s \operatorname{ad} x_{\gamma_m})(v_i + \sum_{j=1}^{d} z'_{i,j}(\varphi) t_j + \sum_{k=1}^{m-1} a'_{i,k}(\varphi) x_{\gamma_k}) -$$
$$(\sum_{j=1}^{d} \overline{z}_{i,j}(s,\varphi) t_j - \sum_{j=1}^{d} s z_{i,j}(\varphi) \gamma_m(t_j) x_{\gamma_m} + \sum_{k=1}^{m-1} \overline{a}_{i,k}(s,\varphi) x_{\gamma_k}) \in V_0.$$

Then $\overline{z}_{i,j}$ and $\overline{a}_{i,k}$ are regular functions on $\Bbbk \times \Omega'$ as restrictions to $\Bbbk \times \Omega'$ of regular functions on $\Bbbk \times \operatorname{Hom}(V_0, E' \oplus \mathfrak{t})$. Let $\overline{\mathfrak{A}}$ be the subalgebra of $\Bbbk[\Omega'][s]$ generated by the functions

$$\overline{z}_{i,j}, i, j = 1, \dots, d \quad \text{and} \quad \overline{a}_{i,k}, i = 1, \dots, d, k = 1, \dots, m-1.$$

Since $z'_{i,j}(\varphi) = \overline{z}_{i,j}(0,\varphi)$ and $a'_{i,k}(\varphi) = \overline{a}_{i,k}(0,\varphi)$ for all φ in Ω', the restriction to $\overline{\mathfrak{A}}$ of the quotient morphism $\Bbbk[\Omega'][s] \longrightarrow \Bbbk[\Omega']$ is surjective. As a result, $\overline{\mathfrak{A}}$ has dimension n or $n-1$ since Ω' and $\Bbbk[\Omega'][s]$ have dimension $n-1$ and n respectively. As $\exp(s \operatorname{ad} x_{\gamma_m})(v_i)$ is not necessarily equal to v_i,

$$p \circ \psi \neq (\overline{z}_{i,j}, \overline{a}_{i,j},\ 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant m-1).$$

Moreover, Ω' is contained in $p(\Omega)$ by Lemma 4.7(i) but the inclusion may be strict.

Claim 4.9. The algebra $\overline{\mathfrak{A}}$ has dimension $n-1$.

Proof. [Proof of Claim 4.9] There are two cases to consider:

(1) for $i = 1, \dots, m-1$, $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_i}]$ is contained in V_0,

(2) for some i in $\{1, \dots, m-1\}$, $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_i}]$ is not contained in V_0.

In the first case, $\overline{\mathfrak{A}} = \Bbbk[\Omega']$. Otherwise, denote by j the biggest integer such that $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_j}]$ is not contained in V_0 and $a'_{i,j} \neq 0$ for some $i = 1, \dots, d$. Then, for some j' smaller than j, $\gamma_m + \gamma_j = \gamma_{j'}$. Furthermore, for $k < j$ such that $[\mathfrak{a}^{\gamma_m}, \mathfrak{a}^{\gamma_k}]$ is not contained in V_0, $\gamma_m + \gamma_k$ is in $\mathcal{R} \setminus \{\gamma_{j'}, \dots, \gamma_m\}$. Then for $k \geqslant j'$ and $i = 1, \dots, d$, $a'_{i,k} = \overline{a}_{i,k}$ and for all (s,φ) in $\Bbbk \times \Omega'$,

$$\overline{a}_{i,j'}(s,\varphi) = a'_{i,j'}(\varphi) + s a'_{i,j}(\varphi).$$

As a result, by induction on $m-k$, for $i = 1, \dots, d$,

$$a'_{i,k} - \overline{a}_{i,k} \in s\overline{\mathfrak{A}}[s].$$

Hence $\Bbbk[\Omega'][s] = \overline{\mathfrak{A}}[s]$ and there exists a surjective morphism $\Bbbk[\Omega'] \longrightarrow \overline{\mathfrak{A}}$ so that $\overline{\mathfrak{A}}$ has dimension $n-1$. □

According to Lemma 4.7(ii), the comorphism of $\overline{\psi}$ is an embedding of $\Bbbk[\Omega]$ into $\Bbbk[O]$ and from this embedding results an isomorphism from $\Bbbk(\Omega)$ onto $\Bbbk(\Omega')(s)$. Moreover, $\overline{\mathfrak{A}}$ is the image of $\mathfrak{A}$ by this embedding so that $\mathfrak{A}$ has dimension $n-1$. As a result, $\mathfrak{p} = \{0\}$ since ι is surjective and Ω' has dimension $n-1$.

(ii) Let φ be in Ω. Since $\kappa(\varphi)$ is a commutative algebra, for $1 \leqslant i, j \leqslant d$,

$$0 = [v_i + \varphi(v_i), v_j + \varphi(v_j)] = [v_i, \varphi(v_j)] + [\varphi(v_i), v_j] + [\varphi(v_i), \varphi(v_j)].$$

The component on x_{γ_m} of the right hand side is

$$\sum_{k=1}^{d} (z_{i,k} a_{j,m}(\varphi) - z_{j,k} a_{i,m}(\varphi))[t_k, x_{\gamma_m}] = (u_{i,\gamma_m} a_{j,m} - u_{j,\gamma_m} a_{i,m})(\varphi) x_{\gamma_m},$$

whence the assertion.

(iii) Denote by R_0 the adjoint group of $\mathfrak{r}_0 := \mathfrak{t} + V_0$ and X_{R_0} the closure in $\mathrm{Gr}_d(\mathfrak{r}_0)$ of $R_0.\mathfrak{t}$. Let Ω_0 be the inverse image of X_{R_0} by κ. According to Lemma 4.1, for $i, j = 1, \ldots, d$, the restriction to Ω_0 of $z_{i,j}$ is equal to 0 if $j \neq i$, otherwise it is different from 0. As a result, for $i = 1, \ldots, d$ and γ in $\mathfrak{t}^*$, the restriction of $u_{i,\gamma}$ to Ω_0 is equal to $\overline{z_{i,i}}\gamma(t_i)$ with $\overline{z_{i,i}}$ the restriction of $z_{i,i}$ to Ω_0, whence the assertion. □

For γ in $\mathfrak{t}^*$, set:

$$I_\gamma := \{j \in \{1, \ldots, d\} \mid \gamma(t_j) \neq 0\}.$$

Proposition 4.10. *Denote by $\Bbbk[\Omega]_0$ the localization of $\Bbbk[\Omega]$ at 0.*

(i) *The local algebra $\Bbbk[\Omega]_0$ is Cohen-Macaulay.*

(ii) *For γ in $\mathfrak{t}^*$, $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in $\Bbbk[\Omega]_0$ of elements of its maximal ideal.*

Proof. Prove the proposition by induction on m. By Lemma 4.1, for $m = 0$, $\Bbbk[\Omega]$ is a polynomial algebra of dimension d, generated by $z_{1,1,0}, \ldots, z_{d,d,0}$. Moreover, for $i = 1, \ldots, d$ and γ in $\mathfrak{t}^*$, $u_{i,\gamma} = z_{i,i}\gamma(t_i)$, whence the proposition for $m = 0$. Suppose $m > 0$ and the proposition true for $m-1$ and use the notations of Lemma 4.8.

According to Lemma 4.8(i) and the induction hypothesis, the localization $\mathfrak{A}_*$ of $\mathfrak{A}$ at 0 is Cohen-Macaulay and for γ in $\mathfrak{t}^*$, $u_{i,\gamma}, i \in I_\gamma$ is a regular sequence in $\mathfrak{A}_*$ of elements of its maximal ideal. Denote by $\mathfrak{B}$ the polynomial algebra $\mathfrak{A}_*[T_i, i \in I_{\gamma_m}]$ and by P the ideal of $\mathfrak{B}$ generated by the sequence $u_{i,\gamma_m} T_j - u_{j,\gamma_m} T_i, (i, j) \in I_{\gamma_m}^2$. According to Condition (3) of Section 2, $s := |I_{\gamma_m}| \geqslant 2$. By Lemma 4.8(ii), $\Bbbk[\Omega]_0$ is a quotient of the localization at 0 of $\mathfrak{B}/P$ and by Lemma 4.2, P is a prime ideal of $\mathfrak{B}$. By Proposition 4.6(i), $\mathfrak{B}/P$ is an integral domain and a Cohen-Macaulay algebra of dimension n since $\Bbbk[\Omega']$ has dimension $n-1$. Hence $\Bbbk[\Omega]_0$ is the localization of $\mathfrak{B}/P$ at 0 since $\Bbbk[\Omega]_0$ is an integral domain of dimension n. As a result, $\Bbbk[\Omega]_0$ is Cohen-Macaulay and by Proposition 4.6(ii), for γ in $\mathfrak{t}^*$, the sequence $u_{i,\gamma}, i \in I_\gamma$ is regular in $\Bbbk[\Omega]_0$. □

Theorem 4.11. *The variety X_R is normal and Cohen-Macaulay.*

Proof. By Corollary 3.8, X_R is smooth in codimension 1. So, by Serre's normality criterion [Bou98, §1,no 10, Théorème 4], it suffices to prove that X_R is Cohen-Macaulay. According to [MA86, Ch. 8, Theorem 24.5], the set of points x of X_R such that $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay, is open. For x in X_R, the closure in X_R of $R.x$ contains a fixed point. So it suffices to prove that for x a fixed point under the action of R in X_R, $\mathcal{O}_{X_R,x}$ is Cohen-Macaulay. Let V_0 and Ω be as in Lemma 4.7. Then Ω is an affine open neighborhood of V_0 in X_R. By Proposition 4.10(i), $\mathcal{O}_{\Omega,0}$ is Cohen-Macaulay, whence the theorem since κ is an isomorphism from Ω onto an open neighborhood of V_0 in X_R and $\kappa(0) = V_0$. □

4.4. **Nipotent cone and regular sequence in $\mathcal{O}_{\mathcal{E}}$.** Let $\beta_1, \dots, \beta_d$ be a basis of $\mathfrak{t}^*$. For $i = 1, \dots, d$, denote again by β_i the element of $\mathfrak{r}^*$ extending β_i and equal to 0 on $\mathfrak{a}$. For Λ a complete subset of $\mathcal{R}$, denote by $\mathfrak{t}^{\#}_{\Lambda}$ a complement to $\mathfrak{t}_{\Lambda}$ in $\mathfrak{t}$ and set

$$R'_{\Lambda} := R_{\mathfrak{t}^{\#}_{\Lambda}, \mathfrak{a}_{\Lambda}} \quad \text{and} \quad \mathcal{E}_{\Lambda} := \mathcal{E}_{\mathfrak{t}^{\#}_{\Lambda}, \mathfrak{a}_{\Lambda}}.$$

For Y closed subset of $X_{R'_{\Lambda}}$, denote by $\mathcal{E}_{\Lambda,Y}$ the restriction of $\mathcal{E}_{\Lambda}$ to Y. Let $\mathcal{N}'_{\Lambda}$ be the image of the map

$$\mathcal{E}_{\Lambda, X_{R'_{\Lambda},\mathfrak{n}}} \longrightarrow \mathcal{E}\ , \qquad (V, x) \longmapsto (V \oplus \mathfrak{t}_{\Lambda}, x)$$

and $\mathcal{N}_{\Lambda}$ the closure in $\mathcal{E}$ of $R.\mathcal{N}'_{\Lambda}$.

Lemma 4.12. *For $i = 1, \dots, d$, let $\tilde{\beta}_i$ be the function on $\mathcal{E}$ defined by $\tilde{\beta}_i(V, x) = \beta_i(x)$. Denote by $\mathcal{N}$ the nullvariety of $\tilde{\beta}_1, \dots, \tilde{\beta}_d$ in $\mathcal{E}$.*

(i) *For all complete subset Λ of $\mathcal{R}$, $\mathcal{N}_{\Lambda}$ is a subvariety of $\mathcal{N}$ of dimension at most n.*

(ii) *The varieyt $\mathcal{N}$ is the union of $\mathcal{N}_{\Lambda}, \Lambda \in \mathcal{P}_c(\mathcal{R})$.*

(iii) *The variety $\mathcal{N}$ is equidimensional of dimension n.*

Proof. (i) Since $\mathfrak{a}$ is the nullvariety of $\beta_1, \dots, \beta_d$ in $\mathfrak{r}$, $\mathcal{N}$ is the intersection of $\mathcal{E}$ and $X_R \times \mathfrak{a}$. By definition $\mathcal{N}'_{\Lambda}$ is contained in $X_R \times \mathfrak{a}$. Hence $\mathcal{N}_{\Lambda}$ is contained in $\mathcal{N}$. By Proposition 3.7,

$$\dim \mathcal{N}'_{\Lambda} = \dim \mathfrak{t}^{\#}_{\Lambda} + \dim X_{R'_{\Lambda},\mathfrak{n}} \leqslant \dim \mathfrak{a}_{\Lambda}.$$

Since the image of $X_{R'_{\Lambda},\mathfrak{n}}$ by the map $V \mapsto V \oplus \mathfrak{t}_{\Lambda}$ is invariant by R_{Λ},

$$\dim \mathcal{N}_{\Lambda} \leqslant \dim \mathcal{N}'_{\Lambda} + \dim \mathfrak{a} - \dim \mathfrak{a}_{\Lambda} \leqslant \dim \mathfrak{a}.$$

(ii) Let ϖ_1 be the bundle projection of the vector bundle $\mathcal{E}$ over X_R and τ_1 the restriction to $\mathcal{E}$ of the projection $X_R \times \mathfrak{r} \longrightarrow \mathfrak{r}$. Let T be an irreducible component of $\mathcal{N}$. For all V in $\varpi_1(T)$, $\tau_1(\varpi_1^{-1}(V) \cap T)$ is a closed cone of $\mathfrak{a}$. Hence $\varpi_1(T) \times \{0\}$ is the intersection of T and $X_R \times \{0\}$ so that $\varpi_1(T)$ is a closed subset of X_R. Since $\mathcal{N}$ is the intersection of $\mathcal{E}$ and $X_R \times \mathfrak{a}$, $\mathcal{N}$ and its irreducible components are invariant under R. As a result, $\varpi_1(T)$ is invariant under R and by Proposition 3.1, for some complete subset Λ of $\mathcal{R}$ and for some closed subset of Z_{Λ} of $X_{R_{\Lambda}}$, $\varpi_1(T) = \overline{R.Z_{\Lambda}}$. Moreover, by Lemma 3.2, for some closed subset Z'_{Λ} of $X_{R'_{\Lambda},\mathfrak{n}}$, Z_{Λ} is the image of Z'_{Λ} by the map $V \mapsto V \oplus \mathfrak{t}_{\Lambda}$. As a result,

$$\mathcal{E}_{\Lambda, Z'_{\Lambda}} \subset \mathcal{E}_{\Lambda, X_{R'_{\Lambda},\mathfrak{n}}} \quad \text{and} \quad \varpi_1^{-1}(Z_{\Lambda}) \cap X_R \times \mathfrak{a} \subset \mathcal{N}'_{\Lambda}.$$

Then T is contained in $\mathcal{N}_{\Lambda}$, whence the assertion by (i).

(iii) By (i) and (ii), since $\mathcal{R}$ is finite, the irreducible components of $\mathcal{N}$ have dimension at most n. As the nullvariety of d functions on the irreducible variety $\mathcal{E}_{X_R}$, the irreducible components of $\mathcal{N}$ have dimension at least n, whence the assertion. □

For x in $\mathcal{E}$, denote by I_x the subset of elements i of $\{1, \dots, d\}$ such that $\tilde{\beta}_i(x) = 0$.

Corollary 4.13. *For all x in $\mathcal{E}$, the sequence $\tilde{\beta}_i, i \in I_x$ is regular in $\mathcal{O}_{\mathcal{E},x}$.*

Proof. According to Lemma 4.12, for all subset I of $\{1, \dots, d\}$, the nullvariety of $\tilde{\beta}_i, i \in I$ in $\mathcal{E}$ is equidimensional of dimension $n + d - |I|$. By Theorem 4.11 and Lemma B.1(iii), $\mathcal{E}$ is Cohen-Macaulay as a vector bundle over a Cohen-Macaulay variety, whence the corollary by [MA86, Ch. 6, Theorem 17.4]. □

5. Rational singularities for solvable Lie algebras

Let $\mathfrak{t}$ be a vector space of positive dimension d. Denote by $\mathcal{C}_{\mathfrak{t},*}$ the full subcategory of $\mathcal{C}_{\mathfrak{t}}$ whose objects $\mathfrak{a}$ satisfy the following condition:

(4) there exist regular maps $\varepsilon_1, \dots, \varepsilon_d$ from $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ to $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}$ such that $\varepsilon_1(x), \dots, \varepsilon_d(x)$ is a basis of $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}}^x$ for all x in $\mathfrak{r}_{\mathfrak{t},\mathfrak{a}\mathrm{reg}}$.

According to [Ko63, Theorem 9], $\mathfrak{u}$ is in $\mathcal{C}_{\mathfrak{h},*}$.

Lemma 5.1. *Let $\mathfrak{a}$ be in $\mathcal{C}_{\mathfrak{t},*}$ and $\mathfrak{a}'$ an ideal of $\mathfrak{t}+\mathfrak{a}$, contained in $\mathfrak{a}$ and containing a fixed point under the action of $R_{\mathfrak{t},\mathfrak{a}}$ in $X_{R_{\mathfrak{t},\mathfrak{a}}}$. Then $\mathfrak{a}'$ is in $\mathcal{C}_{\mathfrak{t},*}$.*

Proof. Set $\mathfrak{r} := \mathfrak{t}+\mathfrak{a}$ and $\mathfrak{r}' := \mathfrak{t}+\mathfrak{a}'$. According to Corollary 2.22(ii), $\mathfrak{a}'$ is in $\mathcal{C}_{\mathfrak{t}}$ since it is in $\mathcal{C}'_{\mathfrak{t}}$. Set $\mathfrak{t}_{\mathrm{reg}} := \mathfrak{r}_{\mathrm{reg}} \cap \mathfrak{t}$. As $\mathcal{R}_{\mathfrak{t},\mathfrak{a}'}$ is contained in $\mathcal{R}_{\mathfrak{t},\mathfrak{a}}$, $\mathfrak{t}_{\mathrm{reg}}$ is contained in $\mathfrak{r}'_{\mathrm{reg}}$ by Lemma 3.3(i). Then $\mathfrak{r}'_{\mathrm{reg}}$ is contained in $\mathfrak{r}_{\mathrm{reg}}$ and for all x in $A_{\mathfrak{t},\mathfrak{a}'}.\mathfrak{t}_{\mathrm{reg}}$, $\mathfrak{r}^x = \mathfrak{r}'^x$ since $A_{\mathfrak{t},\mathfrak{a}'}.\mathfrak{t}_{\mathrm{reg}}$ is a dense open subset of $\mathfrak{r}'$ by Lemma 3.3(i). So, for all regular map ε from $\mathfrak{r}$ to $\mathfrak{r}$ such that $[x, \varepsilon(x)] = 0$ for all x in $\mathfrak{r}$, $\varepsilon(x)$ is in $\mathfrak{r}'$ for all x in $\mathfrak{r}'$, whence the lemma. □

Let $\mathfrak{a}$ be in $\mathcal{C}_{\mathfrak{t},*}$. Set:

$$\mathcal{R} := \mathcal{R}_{\mathfrak{t},\mathfrak{a}}, \qquad \mathfrak{r} := \mathfrak{r}_{\mathfrak{t},\mathfrak{a}} \qquad \pi := \pi_{\mathfrak{t},\mathfrak{a}}, \qquad R := R_{\mathfrak{t},\mathfrak{a}}, \qquad A := A_{\mathfrak{t},\mathfrak{a}}, \qquad \mathcal{E} := \mathcal{E}_{\mathfrak{t},\mathfrak{a}}, \qquad n := \dim \mathfrak{a}.$$

The goal of the section is to prove that X_R is Gorenstein with rational singularities.

For k positive integer, set:

$$\mathcal{E}^{(k)} := \{(u, x_1, \dots, x_k) \in X_R \times \mathfrak{r}^k \mid u \ni x_1, \dots, u \ni x_k\}$$

and denote by $\mathfrak{X}_{R,k}$ the image of $\mathcal{E}^{(k)}$ by the projection

$$(u, x_1, \dots, x_k) \longmapsto (x_1, \dots, x_k).$$

Since X_R is a projective variety, $\mathfrak{X}_{R,k}$ is a closed subset of $\mathfrak{r}^k$, invariant under the diagonal action of R in $\mathfrak{r}^k$.

5.1. **Differential forms on some smooth open subsets of $\mathfrak{X}_{R,k}$.** For $j = 1, \dots, k$, let $V_j^{(k)}$ be the subset of elements of $\mathfrak{X}_{R,k}$ whose j-th component is in $\mathfrak{r}_{\mathrm{reg}}$.

Lemma 5.2. *For $j = 1, \dots, k$, $V_j^{(k)}$ is a smooth open subset of $\mathfrak{X}_{R,k}$. Moreover, $\Omega_{V_j^{(k)}}$ has a global section without zero.*

Proof. Denoting by σ_j the automorphism of $\mathfrak{r}_k$ which permutes the first and the j-th component, $\mathfrak{X}_{R,k}$ is invariant under σ_j and $\sigma_j(V_1^{(k)}) = V_j^{(k)}$ so that we can suppose $j = 1$. Moreover, for $k = 1$, $\mathfrak{X}_{R,k} = \mathfrak{r}$ so that we can suppose $k \geqslant 2$. By definition, $V_1^{(k)}$ is the intersection of $\mathfrak{r}_{\mathrm{reg}} \times \mathfrak{r}^{k-1}$ and $\mathfrak{X}_{R,k}$. Hence $V_1^{(k)}$ is an open susbet of $\mathfrak{X}_{R,k}$ since $\mathfrak{r}_{\mathrm{reg}}$ is an open subset of $\mathfrak{r}$.

Let $\varepsilon_1, \dots, \varepsilon_d$ satisfying Condition (4) with respect to $\mathfrak{r}$. Let θ be the map

$$\mathfrak{r}_{\mathrm{reg}} \times \mathrm{M}_{k-1,d}(\Bbbk) \xrightarrow{\quad\theta\quad} \mathfrak{r}^k, \qquad (x, a_{i,j}, 2 \leqslant i \leqslant k, 1 \leqslant j \leqslant d) \longmapsto (x, \sum_{j=1}^{d} a_{i,j}\varepsilon_j(x)).$$

Since for all $(x, x_2, \dots, x_k)$ in $V_1^{(k)}$, $x_2, \dots, x_k$ are in $\mathfrak{r}^x$, θ is a bijective map onto $V_1^{(k)}$. The open subset $\mathfrak{r}_{\mathrm{reg}}$ has a cover by open subsets V such that for some $e_1, \dots, e_n$ in $\mathfrak{r}$, $\varepsilon_1(x), \dots, \varepsilon_d(x), e_1, \dots, e_n$ is a basis of $\mathfrak{r}$ for all x in V. Then there exist regular functions $\varphi_1, \dots, \varphi_d$ on $V \times \mathfrak{r}$ such that

$$v - \sum_{j=1}^{\ell} \varphi_j(x, v)\varepsilon_j(x) \in \mathrm{span}(\{e_1, \dots, e_n\})$$

for all (x, v) in $V \times \mathfrak{r}$, so that the restriction of θ to $V \times \mathrm{M}_{k-1,d}(\Bbbk)$ is an isomorphism onto $\mathfrak{X}_{R,k} \cap V \times \mathfrak{r}^{k-1}$ whose inverse is

$$(x_1, \dots, x_k) \longmapsto (x_1, ((\varphi_1(x_1, x_i), \dots, \varphi_d(x_1, x_i)), i = 2, \dots, k))$$

As a result, θ is an isomorphism and $V_1^{(k)}$ is a smooth variety. Since $\mathfrak{r}_{\mathrm{reg}}$ is a smooth open subset of the vector space $\mathfrak{r}$, there exists a regular differential form ω of top degree on $\mathfrak{r}_{\mathrm{reg}} \times \mathrm{M}_{k-1,\ell}(\Bbbk)$, without zero. Then $\theta_*(\omega)$ is a regular differential form of top degree on $V_1^{(k)}$, without zero. □

For $k \geqslant 2$ set:

$$V^{(k)} := V_1^{(k)} \cup V_2^{(k)} \quad \text{and} \quad V_{1,2}^{(k)} := V_1^{(k)} \cap V_2^{(k)}.$$

For $2 \leqslant k' \leqslant k$, the projection

$$\mathfrak{r}^k \longrightarrow \mathfrak{r}^{k'}, \qquad (x_1, \dots, x_k) \longmapsto (x_1, \dots, x_{k'})$$

induces the projection

$$\mathfrak{X}_{R,k} \longrightarrow \mathfrak{X}_{R,k'}, \qquad V_j^{(k)} \longrightarrow V_j^{(k')}$$

for $j = 1, \dots, k'$.

Lemma 5.3. *Suppose $k \geqslant 2$. Let ω be a regular differential form of top degree on $V_1^{(k)}$, without zero. Denote by ω' its restriction to $V_{1,2}^{(k)}$.*

(i) *For φ in $\Bbbk[V_1^{(k)}]$, if φ has no zero then φ is in $\Bbbk^*$.*

(ii) *For some invertible element ψ of $\Bbbk[V_{1,2}^{(2)}]$, $\omega' = \psi\sigma_{2*}(\omega')$.*

(iii) *The function $\psi(\psi\circ\sigma_2)$ on $V_{1,2}^{(k)}$ is equal to 1.*

Proof. The existence of ω results from Lemma 5.2.

(i) According to Lemma 5.2, there is an isomorphism θ from $\mathfrak{r}_{\mathrm{reg}} \times \mathrm{M}_{k-1,d}(\Bbbk)$ onto $V_1^{(k)}$. Since φ is invertible, $\varphi\circ\theta$ is an invertible element of $\Bbbk[\mathfrak{r}_{\mathrm{reg}}]$. According to Lemma 3.3(iii), $\Bbbk[\mathfrak{r}_{\mathrm{reg}}] = \Bbbk[\mathfrak{r}]$. Hence φ is in $\Bbbk^*$.

(ii) The open subset $V_{1,2}^{(k)}$ is invariant under σ_2 so that ω' and $\sigma_{2*}(\omega')$ are regular differential forms of top degree on $V_{1,2}^{(k)}$, without zero. Then for some invertible element ψ of $\Bbbk[V_{1,2}^{(k)}]$, $\omega' = \psi\sigma_{2*}(\omega')$. Let O_2 be the set of elements $(x, a_{i,j}, 1 \leqslant i \leqslant k-1, 1 \leqslant j \leqslant d)$ of $\mathfrak{r}_{\mathrm{reg}} \times \mathrm{M}_{k-1,d}(\Bbbk)$ such that

$$a_{1,1}\varepsilon_1(x) + \cdots + a_{1,\ell}\varepsilon_\ell(x) \in \mathfrak{r}_{\mathrm{reg}}.$$

Then O_2 is the inverse image of $V_{1,2}^{(k)}$ by θ. As a result, $\Bbbk[V_{1,2}^{(k)}]$ is a polynomial algebra over $\Bbbk[V_{1,2}^{(2)}]$ since for $k = 2$, O_2 is the inverse image by θ of $V_{1,2}^{(2)}$. Hence ψ is in $\Bbbk[V_{1,2}^{(2)}]$ since ψ is invertible.

(iii) Since the restriction of σ_2 to $V_{1,2}^{(k)}$ is an involution,

$$\sigma_{2*}(\omega') = (\psi\circ\sigma_2)\omega' = (\psi\circ\sigma_2)\psi\sigma_{2*}(\omega'),$$

whence $(\psi\circ\sigma_2)\psi = 1$. □

Corollary 5.4. *The function ψ is invariant under the action of R in $V_{1,2}^{(k)}$ and for some sequence $m_\alpha, \alpha \in \mathcal{R}$ in $\mathbb{Z}$,*

$$\psi(x_1, \dots, x_k) = \pm \prod_{\alpha\in\mathcal{R}} (\alpha(x_1)\alpha(x_2)^{-1})^{m_\alpha},$$

for all $(x_1, \dots, x_k)$ in $\mathfrak{t}_{\mathrm{reg}}^2 \times \mathfrak{t}^{k-2}$.

Proof. First of all, since $V_1^{(k)}$ and $V_2^{(k)}$ are invariant under the action of R in $\mathfrak{X}_{R,k}$, so is $V_{1,2}^{(k)}$. Let g be in R. As ω has no zero, $g.\omega = p_g\omega$ for some invertible element p_g of $\Bbbk[V_1^{(k)}]$. By Lemma 5.3(i), p_g is in $\Bbbk^*$. Since σ_2 is a R-equivariant isomorphism from $V_1^{(k)}$ onto $V_2^{(k)}$,

$$g.\sigma_{2*}(\omega) = p_g\sigma_{2*}(\omega) \quad \text{and} \quad p_g\omega' = g.\omega' = (g.\psi)g.\sigma_{2*}(\omega') = p_g(g.\psi)\sigma_{2*}(\omega'),$$

whence $g.\psi = \psi$.

The open subset $\mathfrak{t}^2_{\mathrm{reg}}$ of $\mathfrak{t}^2$ is the complement to the nullvariety of the function

$$(x, y) \longmapsto \prod_{\alpha\in\mathcal{R}} \alpha(x)\alpha(y).$$

Then, by Lemma 5.3(ii), for some a in $\Bbbk^*$ and for some sequences $m_\alpha, \alpha \in \mathcal{R}$ and $n_\alpha, \alpha \in \mathcal{R}$ in $\mathbb{Z}$,

$$\psi(x_1, \dots, x_k) = a \prod_{\alpha\in\mathcal{R}} \alpha(x_1)^{m_\alpha}\alpha(x_2)^{n_\alpha},$$

for all $(x_1, \dots, x_k)$ in $\mathfrak{t}^2_{\mathrm{reg}} \times \mathfrak{t}^{k-2}$. By Lemma 5.3(iii),

$$a^2 \prod_{\alpha\in\mathcal{R}} \alpha(x)^{m_\alpha+n_\alpha}\alpha(y)^{m_\alpha+n_\alpha} = 1,$$

for all (x, y) in $\mathfrak{t}^2_{\mathrm{reg}}$. Hence $a^2 = 1$ and $m_\alpha + n_\alpha = 0$ for all α in $\mathcal{R}$. □

According to Lemma 3.5(i), for α in $\mathcal{R}$, θ_α is a bijective regular map from $\mathbb{P}^1(\Bbbk)$ onto the closed subset Z_α of X_R such that $\theta_\alpha(\infty) = V_\alpha$. Recall that x_α is a generator of $\mathfrak{a}^\alpha$ and h_α is an element of $\mathfrak{t}$ such that $\alpha(h_\alpha) = 1$. Denote by $\mathfrak{t}'_\alpha$ the subset of elements x of $\mathfrak{t}_\alpha$ such that $\gamma(x) \neq 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$. According to Condition (3) of Section 2, $\mathfrak{t}'_\alpha$ is a dense open subset of $\mathfrak{t}_\alpha$. Let $x_{-\alpha}$ be in $\mathfrak{r}^*$ orthogonal to $\mathfrak{t} + \mathfrak{a}^\gamma$ for all γ in $\mathcal{R} \setminus \{\alpha\}$ and such that $x_{-\alpha}(x_\alpha) = 1$.

Lemma 5.5. *Suppose $k \geqslant 2$. Let α be in $\mathcal{R}$, x_0 and y_0 in $\mathfrak{t}'_\alpha$. Set:*

$$E := \Bbbk x_0 \oplus \Bbbk h_\alpha \oplus \mathfrak{a}^\alpha, \quad E_* := x_0 \oplus \Bbbk h_\alpha \oplus \mathfrak{a}^\alpha, \quad E_{*,1} := x_0 \oplus \Bbbk h_\alpha \oplus (\mathfrak{a}^\alpha \setminus \{0\}), \quad E_{*,2} = y_0 \oplus \Bbbk h_\alpha \oplus (\mathfrak{a}^\alpha \setminus \{0\}).$$

(i) *For x in E_*, $\mathfrak{r}^x$ is contained in $\mathfrak{t}_\alpha + E$.*

(ii) *For V subspace of dimension d of $\mathfrak{t}_\alpha + E$, V is in X_R if and only if it is in Z_α.*

(iii) *The intersection of $E_{*,1} \times E_{*,2}$ and $\mathfrak{X}_{R,2}$ is the nullvariety of the function*

$$(x, y) \longmapsto x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y)$$

on $E_{*,1} \times E_{*,2}$.

Proof. (i) If x is regular semisimple, its component on h_α is different from 0 so that $\mathfrak{r}^x = \theta_\alpha(z)$ for some z in $\Bbbk$. Suppose that x is not regular semisimple. Then x is in $x_0 + \mathfrak{a}^\alpha$. Hence $\mathfrak{r}^x$ is contained in $\mathfrak{t}_\alpha + E$ since so is $\mathfrak{r}^{x_0}$.

(ii) All element of Z_α is contained in $\mathfrak{t}_\alpha + E$. Let V be an element of X_R, contained in $\mathfrak{t}_\alpha + E$. According to Corollary 2.22(i), V is an algebraic commutative subalgebra of dimension d of $\mathfrak{r}$. By (i), $V = \theta_\alpha(z)$ for some z in $\Bbbk$ if V is in $A.\mathfrak{t}$. Otherwise, x_α is in V. Then $V = \theta_\alpha(\infty)$ since $\theta_\alpha(\infty)$ is the centralizer of x_α in $\mathfrak{t}_\alpha + E$.

(iii) Let (x, y) be in $E_{*,1} \times E_{*,2} \cap \mathfrak{X}_{R,2}$. By definition, for some V in X_R, x and y are in V. By (i) and (ii), $V = \theta_\alpha(z)$ for some z in $\mathbb{P}^1(\Bbbk)$. For z in $\Bbbk$,

$$x = x_0 + s(h_\alpha - zx_\alpha) \quad \text{and} \quad y = y_0 + s'(h_\alpha - zx_\alpha)$$

for some s, s' in $\Bbbk$, whence the equality of the assertion. For $z = \infty$,

$$x = x_0 + sx_\alpha \quad \text{and} \quad y = y_0 + s'x_\alpha$$

for some s, s' in $\Bbbk$ so that $\alpha(x) = \alpha(y) = 0$. Conversely, let (x, y) be in $E_{*,1} \times E_{*,2}$ such that

$$x_{-\alpha}(y)\alpha(x) - x_{-\alpha}(x)\alpha(y) = 0.$$

If $\alpha(x) = 0$ then $\alpha(y) = 0$ and x and y are in $V_\alpha = \theta_\alpha(\infty)$. If $\alpha(x) \neq 0$, then $\alpha(y) \neq 0$ and

$$x \in \theta_\alpha(-\frac{x_{-\alpha}(x)}{\alpha(x)}) \quad \text{and} \quad y \in \theta_\alpha(-\frac{x_{-\alpha}(x)}{\alpha(x)}),$$

whence the assertion. □

Set $V^{(1)} := \mathfrak{r}_{\mathrm{reg}}$.

Proposition 5.6. *For k positive integer, there exists on $V^{(k)}$ a regular differential form of top degree without zero.*

Proof. For $k = 1$, it is true since $\mathfrak{r}_{\mathrm{reg}}$ is an open subset of the vector sapce $\mathfrak{r}$. So we can suppose $k \geqslant 2$. According to Corollary 5.4, it suffices to prove $m_\alpha = 0$ for all α in $\mathcal{R}$. Indeed, if so, by Corollary 5.4, $\psi = \pm 1$ on the open subset $R.(\mathfrak{t}^2_{\mathrm{reg}} \times \mathfrak{t}^{k-2})$ of $V^{(k)}$ so that $\psi = \pm 1$ on $V^{(k)}_{1,2}$. Then, by Lemma 5.3(ii), ω and $\pm\sigma_{2*}(\omega)$ have the same restriction to $V^{(k)}_{1,2}$ so that there exists a regular differential form of top degree $\tilde{\omega}$ on $V^{(k)}$ whose restrictions to $V^{(k)}_1$ and $V^{(k)}_2$ are ω and $\pm\sigma_{2*}(\omega)$ respectively. Moreover, $\tilde{\omega}$ has no zero since so has ω.

Since ψ is in $\Bbbk[V^{(2)}_{1,2}]$ by Lemma 5.3(ii), we can suppose $k = 2$. Let α be in $\mathcal{R}$, E, E_*,$E_{*,1}$, $E_{*,2}$ as in Lemma 5.3. Suppose $m_\alpha \neq 0$. A contradiction is expected. The restriction of ψ to $E_{*,1} \times E_{*,2} \cap V^{(2)}_{1,2}$ is given by

$$\psi(x, y) = a x_{-\alpha}(x)^m x_{-\alpha}(y)^n,$$

with a in $\Bbbk^*$ and (m, n) in $\mathbb{Z}^2$ since ψ is an invertible element of $\Bbbk[V^{(2)}_{1,2}]$. According to Lemma 5.3(iii), $n = -m$ and $a = \pm 1$. Interchanging the role of x and y, we can suppose m in $\mathbb{N}$. For (x, y) in $E_{*,1} \times E_{*,2} \cap V^{(2)}_{1,2}$ such that $\alpha(x) \neq 0$, $\alpha(y) \neq 0$ and

$$\psi(x, y) = \pm x_{-\alpha}(x)^m (\frac{x_{-\alpha}(x)\alpha(y)}{\alpha(x)})^{-m} = \pm\alpha(x)^m \alpha(y)^{-m}.$$

As a result, by Corollary 5.4, for x in $x_0 + \Bbbk^* h_\alpha$ and y in $y_0 + \Bbbk^* h_\alpha$,

$$\pm\alpha(x)^m \alpha(y)^{-m} = \pm \prod_{\gamma \in \mathcal{R}} \gamma(x)^{m_\gamma} \gamma(y)^{-m_\gamma}. \tag{1}$$

For γ in $\mathcal{R}$,

$$\gamma(x) = \gamma(x_0) + \alpha(x)\gamma(h_\alpha) \quad \text{and} \quad \gamma(y) = \gamma(y_0) + \alpha(y)\gamma(h_\alpha).$$

Since m is in $\mathbb{N}$,

$$\begin{aligned} \pm\alpha(x)^m \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma > 0}} (\gamma(y_0) + \alpha(y)\gamma(h_\alpha))^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma < 0}} (\gamma(x_0) + \alpha(x)\gamma(h_\alpha))^{-m_\gamma} = \\ \pm\alpha(y)^m \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma > 0}} (\gamma(x_0) + \alpha(x)\gamma(h_\alpha))^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R} \\ m_\gamma < 0}} (\gamma(y_0) + \alpha(y)\gamma(h_\alpha))^{-m_\gamma}. \end{aligned} \tag{2}$$

For m_α positive, the terms of lowest degree in $(\alpha(x), \alpha(y))$ of left and right sides are

$$\pm\alpha(x)^m \alpha(y)^{m_\alpha} \prod_{\substack{\gamma \in \mathcal{R}\setminus\{\alpha\} \\ m_\gamma > 0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R}\setminus\{\alpha\} \\ m_\gamma < 0}} \gamma(x_0)^{-m_\gamma} \quad \text{and} \quad \pm\,\alpha(y)^m \alpha(x)^{m_\alpha} \prod_{\substack{\gamma \in \mathcal{R}\setminus\{\alpha\} \\ m_\gamma > 0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma \in \mathcal{R}\setminus\{\alpha\} \\ m_\gamma < 0}} \gamma(y_0)^{-m_\gamma}$$

respectively and for m_α negative, the terms of lowest degree in $(\alpha(x), \alpha(y))$ of left and right sides are

$$\pm\alpha(x)^{m+m_\alpha} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(x_0)^{-m_\gamma} \quad \text{and} \quad \pm\,\alpha(y)^{m+m_\alpha} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(y_0)^{-m_\gamma}$$

respectively. From the equality of these terms, we deduce $m = \pm m_\alpha$ and

$$\prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(y_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(x_0)^{-m_\gamma} = \pm \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma>0}} \gamma(x_0)^{m_\gamma} \prod_{\substack{\gamma\in\mathcal{R}\setminus\{\alpha\}\\ m_\gamma<0}} \gamma(y_0)^{-m_\gamma}.$$

Since the last equality does not depend on the choice of x_0 and y_0 in $\mathfrak{t}'_\alpha$, this equality remains true for all (x_0, y_0) in $\mathfrak{t}_\alpha \times \mathfrak{t}_\alpha$. As a result, as the degrees in $\alpha(x)$ of the left and right sides of Equality (2) are the same,

$$m - \sum_{\substack{\gamma\in\mathcal{R}\\ m_\gamma<0 \text{ and } \gamma(h_\alpha)\neq 0}} m_\gamma = \sum_{\substack{\gamma\in\mathcal{R}\\ m_\gamma>0 \text{ and } \gamma(h_\alpha)\neq 0}} m_\gamma. \tag{3}$$

Suppose $m = m_\alpha$. By Equality (1),

$$\prod_{\gamma\in\mathcal{R}\setminus\{\alpha\}} \gamma(x)^{m_\gamma}\gamma(y)^{-m_\gamma} = \pm 1.$$

Since this equality does not depend on the choice of x_0 and y_0 in $\mathfrak{t}'_\alpha$, it holds for all (x, y) in $\mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}_{\mathrm{reg}}$. Hence $m_\gamma = 0$ for all γ in $\mathcal{R} \setminus \{\alpha\}$ and $m = 0$ by Equality (3). It is impossible since $m_\alpha \neq 0$. Hence $m = -m_\alpha$. Then, by Equality (1)

$$\prod_{\gamma\in\mathcal{R}\setminus\{\alpha\}} \gamma(x)^{m_\gamma}\gamma(y)^{-m_\gamma} = \pm\alpha(x)^{2m}\alpha(y)^{-2m}.$$

Since this equality does not depend on the choice of x_0 and y_0 in $\mathfrak{t}'_\alpha$, it holds for all (x, y) in $\mathfrak{t}_{\mathrm{reg}} \times \mathfrak{t}_{\mathrm{reg}}$. Then $m = 0$, whence the contradiction. □

5.2. **Rational singularities and Gorensteinness of X_R.** For Y subvariety of $\mathrm{Gr}_d(\mathfrak{r})$, denote by $\mathcal{E}_Y$ the restriction to Y of the tautological vector bundle of rank d over $\mathrm{Gr}_d(\mathfrak{r})$. In particular, for Y contained in X_R, $\mathcal{E}_Y$ is a subvariety of $\mathcal{E}$. For k positive integer, denote by τ_k and ϖ_k the restrictions to $\mathcal{E}^{(k)}$ of the canonical projections

$$X_R \times \mathfrak{r}^k \xrightarrow{\quad\tau_k\quad} \mathfrak{r}^k \quad \text{and} \quad X_R \times \mathfrak{r}^k \xrightarrow{\quad\varpi_k\quad} X_R\ .$$

Lemma 5.7. (i) *The morphism τ_k is a projective and birational morphism onto $\mathfrak{X}_{R,k}$.*

(ii) *The sets $V^{(k)}$ and $\tau_k^{-1}(V^{(k)})$ are smooth open subsets of $\mathfrak{X}_{R,k}$ and $\mathcal{E}^{(k)}$. Moreover, for $k \geqslant 2$, they are big open subsets of $\mathfrak{X}_{R,k}$ and $\mathcal{E}^{(k)}$.*

(iii) *The restriction of τ_k to $\tau_k^{-1}(V^{(k)})$ is an isomorphism onto $V^{(k)}$.*

Proof. Since X_R is a projective variety, τ_k is projective and its image is $\mathfrak{X}_{R,k}$ by definition. For $(x_1, \dots, x_k)$ in $V^{(k)}$ and $(u, x_1, \dots, x_k)$ in $\tau_k^{-1}((x_1, \dots, x_k))$, $u = \mathfrak{r}^{x_1}$ if x_1 is in $\mathfrak{r}_{\mathrm{reg}}$ and $u = \mathfrak{r}^{x_2}$ if x_2 is in $\mathfrak{r}_{\mathrm{reg}}$. As a result, the restriction of τ_k to $\tau_k^{-1}(V^{(k)})$ is a bijective morphism onto $V^{(k)}$. Hence τ_k is a birational morphism and by Zariski's Main Theorem [Mu88, §9], this restriction is an isomorphism since $V^{(k)}$ is a smooth variety by Lemma 5.2. So it remains to prove that for $k \geqslant 2$, $\tau_k^{-1}(V^{(k)})$ is a big open subset of $\mathcal{E}^{(k)}$

Suppose that $\mathcal{E}^{(k)} \setminus \tau_k^{-1}(V^{(k)})$ has an irreducible component Σ of dimension $\dim \mathcal{E}^{(k)} - 1$. A contradiction is expected. Since $\mathcal{E}^{(k)}$ and $\tau_k^{-1}(V_k)$ are invariant under the automorphisms of $X_R \times \mathfrak{r}^k$,

$$(u, x_1, \dots, x_k) \longmapsto (u, tx_1, \dots, tx_k), \qquad (t \in \Bbbk^*),$$

so is Σ. Then $\Sigma \cap X_R \times \{0\} = \varpi_k(\Sigma) \times \{0\}$ so that $\varpi_k(\Sigma)$ is a closed subset of X_R. Since $\dim \Sigma = \dim \mathcal{E}^{(k)} - 1$, $\dim \varpi_k(\Sigma) \geqslant \dim X_R - 1$. Suppose $\dim \Sigma = \dim X_R - 1$. Then for all u in $\varpi_k(\Sigma)$, $\{u\} \times u^k$ is in Σ. It is

impossible since for all u in a dense open subset of $\varpi_k(\Sigma)$, $u = \mathfrak{r}^x$ for some x in $\mathfrak{r}_{\mathrm{reg}}$ by Corollary 3.8. Hence $\varpi_k(\Sigma) = X_R$. Then for all u in a dense open subset of X'_R, $\{u\} \times u^k \cap \Sigma$ has codimension 1 in $\{u\} \times u^k$. Since the image of $\{u\} \times u^k \cap \Sigma$ by the projection

$$(u, x_1, \ldots, x_k) \longmapsto x_1$$

is not dense in u, for all x_1 in a dense open subset of its image, $\{u\} \times \{x_1\} \times u^{k-1}$ is contained in Σ, whence the contradiction since $u \cap \mathfrak{r}_{\mathrm{reg}}$ is not empty. □

By definition, $\mathcal{E}^{(k)}$ is the inverse image of X_R by the bundle projection of the vector bundle

$$\{u, x_1, \ldots, x_k) \in \mathrm{Gr}_d(\mathfrak{r}) \times \mathfrak{r}^k \mid u \ni x_1, \ldots, u \ni x_k\}$$

over $\mathrm{Gr}_d(\mathfrak{r})$ so that $\mathcal{E}^{(k)}$ is vector bundle of rank kd over X_R. In particular, $\mathcal{E}^{(1)} = \mathcal{E}$. According to [Hir64], there exists a desingulartization Γ of X_R with morphism ρ such that the restriction of ρ to $\rho^{-1}(X_{R\mathrm{sm}})$ is an isomorphism onto $X_{R\mathrm{sm}}$. Let $\widetilde{\mathcal{E}^{(1)}}$ be the following fiber product

$$\begin{tikzcd} \widetilde{\mathcal{E}^{(1)}} \arrow[r, "\overline{\rho}"] \arrow[d] & \mathcal{E}^{(1)} \arrow[d, "\varpi_1"] \\ \Gamma \arrow[r, "\rho"'] & X_R \end{tikzcd}$$

with $\overline{\rho}$ the restriction map. Then $\widetilde{\mathcal{E}^{(1)}}$ is a vector bundle of rank d over Γ. In particular, it is a smooth variety since Γ is smooth.

Let O be a trivialization open subset of the vector bundle $\mathcal{E}^{(1)}$ and let Φ_1 be a local trivialization over O of $\mathcal{E}^{(1)}$, whence the following commutative diagram

$$\begin{tikzcd} \varpi_1^{-1}(O) \arrow[r, "\Phi_1"] \arrow[dr, "\varpi_1"'] & O \times \Bbbk^d \arrow[d, "\mathrm{pr}_1"] \\ & O \end{tikzcd} .$$

Then O is a trivialization open subset of the vector bundle $\mathcal{E}^{(k)}$. The variety $\mathcal{E}^{(1)}$ is a closed subbundle of $\mathcal{E}^{(k)}$ over X_R and for some local trivialization Φ over O of $\mathcal{E}^{(k)}$, we have the following commutative diagram

$$\begin{tikzcd} \varpi_k^{-1}(O) \arrow[r, "\Phi"] \arrow[dr, "\varpi_k"'] & O \times \Bbbk^{kd} \arrow[d, "\mathrm{pr}_1"] \\ & O \end{tikzcd} ,$$

Φ_1 is the restriction of Φ to $\varpi_1^{-1}(O)$ and $\Phi(\varpi_1^{-1}(O)) = O \times \Bbbk^d \times \{0\}$.

Lemma 5.8. *Suppose $k \geqslant 2$. Denote by μ a generator of $\Omega_{\Bbbk^{kd}}$ and by $\tilde{\rho}$ the restriction of $\rho \times \mathrm{id}_{\Bbbk^{kd}}$ to $\rho^{-1}(O) \times \Bbbk^{kd}$.*

(i) *The sheaf $\Omega_{\mathcal{E}^{(k)}{}_{\mathrm{sm}}}$ has a global section ω without zero.*

(ii) *The sheaf $\Omega_{O_{\mathrm{sm}}}$ has a global section ω_O without zero.*

(iii) *For some p in $\Bbbk[O \times \Bbbk^{kd}] \setminus \{0\}$, $\tilde{\rho}^*(p(\omega_O \wedge \mu))$ has a regular extension to $\rho^{-1}(O) \times \Bbbk^{kd}$.*

Proof. (i) According to Proposition 5.6 and Lemma 5.7(iii), $\Omega_{\tau_k^{-1}(V^{(k)})}$ has a global section without zero. By Lemma 5.7(ii), $\tau_k^{-1}(V^{(k)})$ is a smooth big open subset of $\mathcal{E}^{(k)}$. So, by Lemma A.1, $\Omega_{\mathcal{E}^{(k)}{}_{\mathrm{sm}}}$ has a global section without zero.

(ii) Since μ is a generator of $\Omega_{\Bbbk^{kd}}$, there exists a unique ν in $\Bbbk[\Bbbk^{kd}] \otimes_{\Bbbk} \Gamma(O_{\mathrm{sm}}, \Omega_{O_{\mathrm{sm}}})$ such that

$$\Phi_*(\omega\,|_{\varpi_k^{-1}(O_{\mathrm{sm}})}) = \nu \wedge \mu.$$

Moreover, ν has no zero since so has ω. Let V be an affine open subset of O_{sm} such that the restriction of $\Omega_{O_{\mathrm{sm}}}$ to V is locally free, generated by the local section ω_V. Then for some p_V in $\Bbbk[V \times \Bbbk^{kd}]$,

$$\Phi_*(\omega\,|_{\varpi_k^{-1}(V)}) = p_V \omega_V \wedge \mu. \tag{4}$$

Then p_V has no zero since so has $\nu \wedge \mu$. As a result, p_V is in $\Bbbk[V]$ and $p_V\omega_V$ is a local section of $\Omega_{O_{\mathrm{sm}}}$ without zero. By the unicity of the decomposition (4), for two different affine open subsets V and V' as above, the differential forms $p_V\omega_V$ and $p_{V'}\omega_{V'}$ have the same restriction to $V \cap V'$. As a result, since such affine open subsets cover O_{sm}, for some global section ω_O of $\Omega_{O_{\mathrm{sm}}}$,

$$\Phi_*(\omega\,|_{\varpi_k^{-1}(O_{\mathrm{sm}})}) = \omega_O \wedge \mu.$$

Moreover, ω_O is unique and has no zero.

(iii) Let ω_1 be a generator of Ω_{r} and let μ_1 be a generator of $\Omega_{\Bbbk^d}$. By (i), $\omega_O \wedge \mu_1$ is a global section of $\Omega_{O_{\mathrm{sm}} \times \Bbbk^d}$, without zero. So for some regular function p on $O_{\mathrm{sm}} \times \Bbbk^d$,

$$\Phi_{1*}((\tau_1)^*(\omega_1)\,|_{\varpi_1^{-1}(O_{\mathrm{sm}})}) = p\omega_O \wedge \mu_1. \tag{5}$$

According to Theorem 4.11, X_R is normal. Then so is O and p has a regular extension to $O \times \Bbbk^d$. Denote again by p this extension. According to Equality (5), the differential form $\tilde{\rho}^*(p\omega_O \wedge \mu_1)$ on $\rho^{-1}(O_{\mathrm{sm}}) \times \Bbbk^d$ has a regular extension to $\rho^{-1}(O) \times \Bbbk^d$. In fact, denoting by $\overline{\Phi_1}$ the local trivialization over $\rho^{-1}(O)$ of $\widetilde{\mathcal{E}^{(1)}}$ such that the following diagram

$$\begin{tikzcd}
(\varpi_1 \circ \overline{\rho}^{-1})(O) \arrow[r, "\overline{\Phi_1}"] \arrow[d, "\overline{\rho}"'] & \rho^{-1}(O) \times \Bbbk^d \arrow[d, "\tilde{\rho}"] \\
\varpi_1^{-1}(O) \arrow[r, "\Phi_1"'] & O \times \Bbbk^d
\end{tikzcd}$$

is commutative, it is the restriction to $\rho^{-1}(O_{\mathrm{sm}}) \times \Bbbk^d$ of

$$\overline{\Phi_1}_*((\tau_1 \circ \overline{\rho})^*(\omega_1)\,|_{(\varpi_1 \circ \overline{\rho}^{-1})^{-1}(O)}).$$

For some generator μ' of $\Omega_{\Bbbk^{(k-1)d}}$, $\mu = \mu_1 \wedge \mu'$ and $\Bbbk[O \times \Bbbk^d]$ is naturally embedded in $\Bbbk[O \times \Bbbk^{kd}]$. As a result, $\tilde{\rho}^*(p\omega_O \wedge \mu)$ has a regular extension to $\rho^{-1}(O) \times \Bbbk^{kd}$. □

Proposition 5.9. *The variety X_R is Gorenstein with rational singularities.*

Proof. According to Theorem 4.11, X_R is normal and Cohen-Macaulay. Then by Lemma 5.8,(ii) and (iii) and Corollary A.5, $O \times \Bbbk^{kd}$ is Gorenstein with rational singularities. Then so is O by Lemma B.1,(i) and (ii). Since there is a cover of X_R by open subsets as O, X_R is Gorenstein with rational singularities. □

As already mentioned, $\mathfrak{u}$ is in $\mathcal{C}_{\mathfrak{h},*}$, whence Theorem 1.1 by Proposition 5.9.

Appendix A. Rational Singularities

Let X be an affine irreducible normal variety.

Lemma A.1. *Let Y be a smooth big open subset of X.*

(i) *All regular differential form of top degree on Y has a unique regular extension to X_{sm}.*

(ii) *Suppose that ω is a regular differential form of top degree on Y, without zero. Then the regular extension of ω to X_{sm} has no zero.*

Proof. (i) Since $\Omega_{X_{\mathrm{sm}}}$ is a locally free module of rank one, there is an affine open cover $O_1, \dots, O_k$ of X_{sm} such that the restriction of $\Omega_{X_{\mathrm{sm}}}$ to O_i is a free $\mathcal{O}_{O_i}$-module generated by some section ω_i. For $i = 1, \dots, k$, set $O'_i := O_i \cap Y$. Let ω be a regular differential form of top degree on Y. For $i = 1, \dots, k$, for some regular function a_i on O'_i, $a_i\omega_i$ is the restriction of ω to O'_i. As Y is a big open subset of X, O'_i is a big open subset of O_i. Hence a_i has a regular extension to O_i since O_i is normal. Denoting again by a_i this extension, for $1 \leqslant i, j \leqslant k$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O'_i \cap O'_j$ and $O_i \cap O_j$ since $\Omega_{X_{\mathrm{sm}}}$ is torsion free as a locally free module. Let ω' be the global section of $\Omega_{X_{\mathrm{sm}}}$ extending the $a_i\omega_i$'s. Then ω' is a regular extension of ω to X_{sm} and this extension is unique since Y is dense in X_{sm} and $\Omega_{X_{\mathrm{sm}}}$ is torsion free.

(ii) Suppose that ω has no zero. Let Σ be the nullvariety of ω' in X_{sm}. If it is not empty, Σ has codimension 1 in X_{sm}. As Y is a big open subset of X, $\Sigma \cap X_{\mathrm{sm}}$ is not empty if so is Σ. As a result, Σ is empty. □

Denote by ι the canonical injection from X_{sm} into X.

Lemma A.2. *Suppose that $\Omega_{X_{\mathrm{sm}}}$ has a global section ω without zero. Then the $\mathcal{O}_X$-module $\iota_*(\Omega_{X_{\mathrm{sm}}})$ is free of rank* 1. *More precisely, the morphism θ:*

$$\mathcal{O}_X \xrightarrow{\theta} \iota_*(\Omega_{X_{\mathrm{sm}}}) \ , \qquad \psi \longmapsto \psi\omega$$

is an isomorphism.

Proof. For φ a local section of $\iota_*(\Omega_{X_{\mathrm{sm}}})$ above the open subset U of X, for some regular function ψ on $U \cap X_{\mathrm{sm}}$, $\psi\omega$ is the restriction of φ to $U \cap X_{\mathrm{sm}}$. Since X is normal, so is U and $U \cap X_{\mathrm{sm}}$ is a big open subset of U. Hence ψ has a regular extension to U. As a result, there exists a well defined morphism from $\iota_*(\Omega_{X_{\mathrm{sm}}})$ to $\mathcal{O}_X$ whose inverse is θ. □

According to [Hir64], X has a desingularization Z with morphism τ such that the restriction of τ to $\tau^{-1}(X_{\mathrm{sm}})$ is an isomorphism onto X_{sm}. For U open subset of X, denote by τ_U the restriction of τ to $\tau^{-1}(U)$.

Proposition A.3. *Suppose that X is Cohen-Macaulay and that there exists a morphism $\mu : \mathcal{O}_Z \longrightarrow \Omega_Z$ such that for some p in $\Bbbk[X]$, $\tau_*\mu$ is an isomorphism onto $p\tau_*(\Omega_Z)$. Then X has rational singularities.*

The following proof is the weak variation of the proof of [Hi91, Lemma 2.3].

Proof. Since Z and X are varieties over $\Bbbk$, we have the commutative diagram

$$\begin{tikzcd} Z \arrow[rr, "\tau"] \arrow[dr, "p"'] & & X \arrow[dl, "q"] \\ & \mathrm{Spec}(\Bbbk) & \end{tikzcd} .$$

According to [H66, V. §10.2], $p^!(\Bbbk)$ and $q^!(\Bbbk)$ are dualizing complexes over Z and X respectively. Furthermore, by [H66, VII, 3.4] or [Hi91, 4.3,(ii)], $p^!(\Bbbk)[-\dim Z]$ equals Ω_Z. Set $\mathcal{D} := q^!(\Bbbk)[-\dim Z]$ so that $\tau^!(\mathcal{D}) = \Omega_Z$ by [H66, VII, 3.4] or [Hi91, 4.3,(iv)]. In particular, $\mathcal{D}$ is dualizing over X.

Since τ is a projective morphism, we have the isomorphism

$$\mathrm{R}\tau_*(\mathrm{R}\mathscr{H}om_Z(\Omega_Z,\Omega_Z)) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}(\tau)_*(\Omega_Z),\mathcal{D}) \tag{6}$$

by [H66, VII, 3.4] or [Hi91, 4.3,(iii)]. Since $\mathrm{H}^i(\mathrm{R}\mathscr{H}om_Z(\Omega_Z,\Omega_Z)) = \mathcal{O}_Z$ for $i = 0$ and 0 for $i > 0$, the left hand side of (6) can be identified with $\mathrm{R}\tau_*(\mathcal{O}_Z)$.

According to Grauert-Riemenschneider Theorem [GR70], $\mathrm{R}\tau_*(\Omega_Z)$ has only cohomology in degree 0. Since τ is projective and birational and Z is normal, $\tau_*(\mathcal{O}_Z) = \mathcal{O}_X$. So by assumption of the proposition,

$$\mathrm{R}\tau_*(\Omega_Z) \approx \frac{1}{p}\mathcal{O}_X,$$

whence

$$\mathrm{R}\mathscr{H}om_X(\mathrm{R}(\tau)_*(\Omega_Z),\mathcal{D}) \approx p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}$$

and (6) can be rewritten as

$$\mathrm{R}\tau_*(\mathcal{O}_Z) \approx p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}. \tag{7}$$

Since X is Cohen-Macaulay, $\mathcal{D}$ has cohomology in only one degree. So, by flatness of the $\mathcal{O}_X$-module $p\mathcal{O}_X$, $p\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}$ has cohomology in only one degree. As a result, by (7), $\mathrm{R}^i\tau_*(\mathcal{O}_Z) = 0$ for $i > 0$, that is X has rational singularities. □

Denote by $\mathcal{M}$ the cohomology in degree 0 of $\mathcal{D}$.

Lemma A.4. *Suppose that X has rational singularities. Then the $\mathcal{O}_X$-modules $\tau_*(\Omega_Z)$ and $\mathcal{M}$ are isomorphic. In particular, $\tau_*(\Omega_Z)$ has finite injective dimension.*

Proof. Since X has rational singularities, $\mathrm{R}\tau_*(\mathcal{O}_Z) = \mathcal{O}_X$ and $\mathcal{D}$ has only cohomology in degree 0. Moreover, by Grauert-Riemenschneider Theorem [GR70], $\mathrm{R}\tau_*(\Omega_Z)$ has only cohomology in degree 0, whence $R\tau_*(\Omega_Z) = \tau_*(\Omega_Z)$. Then, by (6), we have the isomorphism

$$\mathcal{O}_X \longrightarrow \mathscr{H}om_X((\tau)_*(\Omega_Z),\mathcal{M}) \ .$$

As $\mathcal{D}$ is dualizing, we have the isomorphism

$$R\tau_*(\Omega_Z) \longrightarrow \mathrm{R}\mathscr{H}om_X(\mathrm{R}\mathscr{H}om_X(R\tau_*(\Omega_Z),\mathcal{D}),\mathcal{D})$$

whence the isomorphism $\tau_*(\Omega_Z) \longrightarrow \mathcal{M}$ by (6). As a result, $\tau_*(\Omega_Z)$ has finite injective dimension since so has $\mathcal{M}$. □

Corollary A.5. *Let Y be a smooth big open subset of X. Suppose that the following conditions are verified:*

1. *X is Cohen-Macaulay,*
2. *Ω_Y has a global section ω without zero,*
3. *for some global section ω_Z of Ω_Z and for some p in $\Bbbk[X] \setminus \{0\}$, the restriction of ω_Z to $\tau^{-1}(Y)$ is equal to $p\tau_Y^*(\omega)$.*

Then X is Gorenstein with rational singularities. Moreover, its canonical module is free of rank 1.

Proof. According to Lemma A.1(ii), ω has a unique regular extension to X_{sm} and this extension has no zero. Denote again by ω this extension. Since Z is irreducible, $\tau^{-1}(Y)$ is dense in $\tau^{-1}(X_{\mathrm{sm}})$ so that the restriction of ω_Z to $\tau^{-1}(X_{\mathrm{sm}})$ is equal to $p\tau_{X_{\mathrm{sm}}}^*(\omega)$ since Ω_Z has no torsion. Denote by μ the morphism

$$\mathcal{O}_Z \xrightarrow{\ \mu\ } \Omega_Z \ , \qquad \varphi \longmapsto \varphi\omega_Z.$$

Let U be an open subset of X and ν a local section of $\tau_*(\Omega_Z)$ above U. Since ω has no zero and $\tau_{U_{\rm sm}}$ is an isomorphism onto $U_{\rm sm}$,

$$\nu\,|_{\tau^{-1}(U_{\rm sm})} = \tau^*_{U_{\rm sm}}(\varphi\omega\,|_{U_{\rm sm}})$$

for some φ in $\Bbbk[U_{\rm sm}]$, whence

$$p\nu\,|_{\tau^{-1}(U_{\rm sm})} = \varphi\circ\tau_{U_{\rm sm}}(\omega_Z\,|_{\tau^{-1}(U_{\rm sm})})$$

by Condition (3). Since X is normal, so is U and $U_{\rm sm}$ is a big open subset of U. Hence φ has a regular extension to U. Denoting again by φ this extension,

$$p\nu = \varphi\circ\tau_U(\omega_Z\,|_{\tau^{-1}(U)})$$

since Z is irreducible and Ω_Z has no torsion. As a result the morphism

$$\tau_*\mu:\ \tau_*(\mathcal{O}_Z) \longrightarrow p\tau_*(\Omega_Z)$$

is an isomorphism since it is obviously injective. So, by Proposition A.3, X has rational singularities. In particular, by [KK73, p.50], $\tau_*(\Omega_X) = \iota_*(\Omega_X)$. Then, by Lemma A.2, the canonical module of X is free of rank 1 and by Lemma A.4, X is Gorenstein. □

Appendix B. About singularities

In this section we recall a well known result. Let X be a variety and Y a vector bundle over X. Denote by τ the bundle projection.

Lemma B.1. (i) *If Y is Gorenstein, then X is Gorenstein.*

(ii) *The variety X has rational singularities if and only if so has Y.*

(iii) *If X is Cohen-Macaualay, then so is Y.*

Proof. Let y be in Y, $x := \tau(y)$. Denote by $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{Y,y}}$ the completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ respectively.

(i) Since Y is a vector bundle over X, $\widehat{\mathcal{O}_{Y,y}}$ is a ring of formal series over $\widehat{\mathcal{O}_{X,x}}$. By [Bru, Proposition 3.1.19,(c)], $\widehat{\mathcal{O}_{Y,y}}$ is Gorenstein. So, by [Bru, Proposition 3.1.19,(b)], $\widehat{\mathcal{O}_{X,x}}$ is Gorenstein. Then by [Bru, Proposition 3.1.19,(c)], $\mathcal{O}_{X,x}$ is Gorenstein, whence the assertion.

(ii) Since Y is a vector bundle over X, then there exists a cover of X by open subsets O, such that $\tau^{-1}(O)$ is isomorphic to $O\times\Bbbk^m$ with $m = \dim Y - \dim X$. According to [KK73, p.50], $O\times\Bbbk^m$ has rational singularities if and only if so has O, whence the assertion since a variety has rational singularities if and only it has a cover by open subsets having rational singularities.

(iii) According to [MA86, Ch. 6, Theorem 17.7], a polynomial algebra over a Cohen-Macaulay algebra is Cohen-Macaulay. Hence for O open subset of X as in (ii), $\tau^{-1}(O)$ is Cohen-Macaulay, whence the assertion since there is a cover of Y by open subsets as $\tau^{-1}(O)$. □

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