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# Some Notes on Flat Polynomials

e. H. el Abdalaoui and M. G. Nadkarni

ABSTRACT. Connection of flat polynomials with some spectral questions in ergodic theory is discussed. A necessary condition for a sequence of polynomials of the type  $\frac{1}{\sqrt{N}}(1 + \sum_{j=1}^{N-1} z^{n_j})$  to be flat in almost everywhere sense is given, which contrasts with a similar necessary condition for a sequence of polynomials to be ultraflat. This connects the problem of existence of flat sequence (in a.e. sense) of such polynomials with some problems in combinatorial number theory. This is a revised version of the earlier paper with the same title.

## 1. Introduction

A sequence  $P_j, j = 1, 2, \dots$  of trigonometric polynomials of  $L^2$  norm one is said to be flat if the sequence  $|P_j|, j = 1, 2, \dots$  of their absolute values converges to the constant function 1 in some sense. The sense of convergence varies according to the situation. Littlewood problem requires that the convergence be in the sup norm and the individual polynomials in the sequence have coefficients of same absolute value [20],[18], [26], [4],[24], [6],[7], [8],[3]. When the convergence required is uniform, the sequence of polynomial is often called ultraflat. In problems connected with Barker sequences, the  $L^4$  norm of the polynomials is required to be close to 1 [12]. Our interest in flat sequence of polynomials comes from spectral questions about rank one transformations in ergodic theory where the polynomials are required to have nonnegative coefficients and their  $L^1$  norms close to one or they converge in absolute value to 1 almost everywhere. It is an open question if such a flat sequence of polynomials exists in a non-trivial sense [5],[15], [1], [17], [23]. In these notes we give, among other results, a necessary condition for a sequence of absolute values

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of such polynomials to converge almost everywhere to 1. This in turn connects this problem with some question in combinatorial number theory (see Section 5).

## 2. Ultraflat Sequence of Polynomials

DEFINITION 2.1. Let  $S^1$  denote the circle group and let  $dz$  denote the normalized Lebesgue measure on it. A sequence  $P_n, n = 1, 2, \dots$  of analytic trigonometric polynomials with  $L^2(S^1, dz)$  norm 1 and their constant terms positive, is said to be ultraflat if  $|P_j(z)| \rightarrow 1$  uniformly as  $j \rightarrow \infty$ . It is said to be flat a.e. ( $dz$ ) if  $|P_n(z)|$  converges to 1 a.e. ( $dz$ ).

The sequence  $P_j(z) = 1, j = 1, 2, \dots$  is obviously ultraflat. More generally let

$$P_j(z) = 1 + Z_j(z), j = 1, 2, \dots$$

where each  $Z_j$  is an analytic trigonometric polynomial with zero constant term and such that  $\|Z_j\|_\infty \rightarrow 0$  as  $j \rightarrow \infty$ . Then  $|\frac{P_j(z)}{\|P_j\|_2}, j = 1, 2, \dots$  is a sequence of ultraflat polynomials which we call a perturbation of the sequence of constant ultraflat polynomials  $P_j, j = 1, 2, \dots$ .

Let  $P(z)$  be a polynomial and let  $E$  denote its set of its zeros strictly inside the unit disk,  $F$  the set of zeros of  $P$  on or outside unit circle. Let

$$B(z) = \gamma \prod_{\alpha \in E} \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad Q(z) = \bar{\gamma} \prod_{\alpha \in E} (1 - \bar{\alpha}z) \prod_{\alpha \in F} (z - \alpha),$$

where  $\gamma$  is a constant of absolute value 1 such that constant term of  $Q(z)$  is positive. The function  $B(z)$  which is of absolute value 1 on  $S^1$  is called the inner part of  $P(z)$  and  $Q(z)$  the outer part of  $P(z)$ . We note that  $P = BQ$ . This factoring of  $P$  is in fact Beurling's factoring of an  $H^2$  function applied to the polynomial  $P$ . A function of the form  $B$  is called finite Blaschke product.

PROPOSITION 2.2. Given any sequence  $P_j, j = 1, 2, \dots$  of ultraflat polynomials, their outer parts  $Q_j, j = 1, 2, \dots$  form a sequence of ultraflat polynomials which is a perturbation of the constant ultraflat sequence. Moreover for all  $j, |P_j(z)| = |Q_j(z)|$  on  $S^1$ .

PROOF. That  $|P_j(z)| = |Q_j(z)|, j = 1, 2, \dots$  follows from the construction of inner and outer factors of  $P_j$ . Since  $|P_j|, j = 1, 2, \dots$  converges to 1 uniformly, we may assume without loss of generality that  $P_j$ 's, hence  $Q_j$ 's, do not vanish on  $S^1$ . Also, being outer,  $Q_j$ 's have no zeros inside the the unit disk. Therefore, for each  $j, \log |Q_j|$  is the real part of the holomorphic function  $\log Q_j$  on an open set containing the closed unit disk. By the mean value property of harmonic function

we see that

$$\log |Q_j(0)| = \int_{S^1} \log |Q_j(z)| dz \rightarrow 0 \text{ as } j \rightarrow \infty,$$

since  $|P_j(z)| \rightarrow 1$  uniformly as  $j \rightarrow \infty$ . Also, by construction  $Q_j(0)$  is positive, we see that  $Q_j(0) = |Q_j(0)| \rightarrow 1$  as  $j \rightarrow \infty$ . Clearly then  $Q_j, j = 1, 2, \dots$  is a perturbation of the sequence of constant ultraflat polynomials.  $\square$

Despite rather trivial nature of a sequence of ultraflat polynomials when divided by their inner factors, their importance stems from the following questions raised by J. E. Littlewood [20].

- (1) Does there exist a sequence of ultraflat polynomials  $P_j, j = 1, 2, \dots$  such that for each  $j$ , the coefficients of  $P_j$  are all equal in absolute value?
- (2) Can these coefficients in addition be real ?

J-P. Kahane [18] has answered the first question in the affirmative. The second question remains open. Also flat sequence of polynomials, in particular ultraflat sequence of polynomials, appear naturally in discussion of some spectral questions in ergodic theory. The papers of Bourgain [5] and M. Guenais [15] are the two early papers connecting  $L^1(S^1, dz)$  flatness with spectral questions. Solution of some of these problems depends of the existence of certain kind of flat sequence of polynomials [1] (see Remarks ).

Kahane's solution can be viewed in the following way: there is an ultraflat sequence of outer polynomials  $Q_j, j = 1, 2, \dots$  which when multiplied by appropriate inner functions yields an ultraflat sequence of polynomials  $P_j, j = 1, 2, \dots$  such that for each  $j$ , the coefficients of  $P_j$  are equal in absolute value.

It may seem natural to conjecture, in the light of the proposition above, that if  $P_j, j = 1, 2, \dots$  is a flat sequence a.e. ( $dz$ ) then the constant terms of their outer parts converge to 1. This however is false since for any given  $\lambda, -\infty < \lambda \leq 0$ , it is possible to give a sequence  $P_j, j = 1, 2, \dots$  of polynomials which is flat a.e. ( $dz$ ) and such that  $\int_{S^1} \log |P_j(z)| dz \rightarrow \lambda$  as  $j \rightarrow \infty$ ; the sequence  $Q_j, j = 1, 2, \dots$  of their outer parts will be flat a.e. ( $dz$ ) with the same property.

We will derive here a necessary condition for a sequence of polynomials to be ultraflat.

Consider an analytic trigonometric polynomial  $P(z)$ , with  $n$  terms, of  $L^2(S^1, dz)$  norm 1,  $P(0) > 0$ . Then for all  $z \in S^1$ ,  $1 - \epsilon \leq |P(z)|^2 \leq 1 + \epsilon$  where  $\epsilon = \sup_{z \in S^1} ||P(z)|^2 - 1|$ . For any continuous  $f$  on  $S^1$ ,

$$(1 - \epsilon) \int_{S^1} |f|^2 dz \leq \int_{S^1} |f(z)|^2 |P(z)|^2 dz \leq (1 + \epsilon) \int_{S^1} |f(z)|^2 dz;$$

in particular, if  $f(z) = \sum_{j=1}^k z^{m_j}$ , a sum of characters of  $S^1$ , then

$$(1 - \epsilon)k \leq \sum_{i=1}^k \sum_{j=1}^k \int_{S^1} z^{m_i - m_j} |P(z)|^2 dz \leq (1 + \epsilon)k, \quad (1)$$

Now

$$1 - \epsilon \leq |P(z)|^2 = 1 + \sum_{\substack{j=-N \\ j \neq 0}}^N b_j z^{n_j} \leq 1 + \epsilon,$$

for some suitable non-zero  $b_j = \overline{b_{-j}}$ , and integers  $n_j = -n_{-j}$ ,  $-N \leq j \leq N, j \neq 0$ . Note that  $N \leq n(n-1)$ . Putting  $z = 1$  we get

$$\begin{aligned} -\epsilon &\leq \sum_{\substack{j=-N \\ j \neq 0}}^N b_j \leq \epsilon \\ \left( \sum_{\substack{j=-N \\ j \neq 0}}^N b_j \right)^2 &\leq \epsilon^2 < \epsilon, \end{aligned} \quad (2)$$

Consider now the functions  $z^{n_j} - \overline{b_j}$ ,  $-N \leq j \leq N, j \neq 0$ . The gram matrix of these vectors in  $L^2(S^1, |P(z)|^2 dz)$  has entries

$$\int_{S^1} z^{n_i - n_j} |P(z)|^2 dz - \overline{b_i} b_j.$$

Sum of these entries, denoted by  $r = r(P)$ , can be seen to satisfy (by equations (1) and (2) above):

$$2N(1 - \epsilon) - \epsilon \leq r \leq 2N(1 + \epsilon) + \epsilon$$

Thus the sum of the entries of the gram matrix in question is of order  $N$  and diagonal entry of the  $i$ th row is  $1 - |b_i|^2$ .

We record this calculation as:

**PROPOSITION 2.3.** Let  $P(z)$  be an analytic polynomial of  $L^2(S^1, dz)$  norm 1 and let  $|P(z)|^2 = 1 + \sum_{j=-N, j \neq 0}^N b_j z^{n_j}, \forall j, b_j \neq 0$ . Let  $r(P)$  denote the sum of the entries of the gram matrix of the random variable  $z^{n_j} - \overline{b_j}, -N \leq j \leq N, j \neq 0$ . Then

$$2N(1 - \epsilon) - \epsilon \leq r \leq 2N(1 + \epsilon) + \epsilon$$

where  $\epsilon = \sup_{z \in S^1} ||P(z)|^2 - 1|$ .

**COROLLARY 2.4.** Let  $P_n, n = 1, 2, \dots$  be a sequence of polynomials of  $L^2(S^1, dz)$  norm 1. Let

$$|P_n(z)|^2 = 1 + \sum_{j=-N_n, j \neq 0}^{N_n} b_{j,n} z^{k_{j,n}}, \forall j, b_{j,n} \neq 0.$$

Then

(a) if  $P_n, n = 1, 2, \dots$  are uniformly bounded then so are the ratios  $\frac{r(P_n)}{N_n}, n = 1, 2, \dots$ ,

(b) if the sequence  $P_n, n = 1, 2, \dots$  is ultraflat then  $\frac{r(P_n)}{2N_n} \rightarrow 1$  as  $n \rightarrow \infty$ . In particular this holds for any ultraflat sequence of Kahane polynomials.

The Gauss-Fresnel polynomials and Hardy-Littlewood polynomials are defined respectively as follows

$$G_N(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} g(k)z^k, \text{ where } g(k) = \exp\left(\frac{\pi i k^2}{N}\right),$$

$$H_N(z) = \frac{1}{\sqrt{N}} \left(1 + \sum_{k=1}^{N-1} v(k)z^k\right), \text{ where } v(k) = \exp\left(2\pi i (ck \ln(k))\right).$$

Our terminology is due to the fact that the first polynomials are connected to the Gaussian sums and the Fresnel integral and the second are studied by Hardy-Littlewood in [16].

Furthermore, it is well known that the Gauss-Fresnel polynomials and Hardy-littlewood polynomials verify

$$\left|G_N(e^{2\pi i \theta})\right| \leq 3C\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right), \forall \theta \in [0, 1), \quad (3)$$

and

$$\left|H_N(e^{2\pi i \theta})\right| \leq K, \forall \theta \in [0, 1) \quad (4).$$

where  $C$  and  $K$  are constants independent of  $N$ .

These inequalities follow as an application of the van der Corput method. A nice account on this method can be found in [28, p.61-67], [27, p.31-37], [14], [22, p.15-18]. We present the proof of inequalities (3) and (4). The principal ingredient in the proof is the following lemma due to van der Corput [29, p.199], [22, p.15-18].

LEMMA 2.5. *Suppose that  $f$  is a real valued function with two continuous derivatives on  $[a, b]$ . Suppose also that there is some  $\rho > 0$  such that*

$$|f''(u)| \geq \rho, \forall u \in [a, b].$$

Then,

$$\left|\sum_{a \leq n \leq b} \exp(2\pi i f(n))\right| \leq \left(|f'(b) - f'(a)| + 2\right) \left(\frac{4}{\sqrt{\rho}} + 3\right).$$

It suffice now to take in the first case  $f(u) = u\theta + \frac{u^2}{2N}$  with  $a = 0$ ,  $b = N - 1$ , and in the second case  $f(u) = cu \ln(u) + u\theta$  with  $[1, N] = \bigcup_{j=0}^{n-1} [2^j, 2^{j+1}] \cup [2^n, N]$ ,  $2^n \leq n < 2^{N+1}$ . We therefore have as a corollary of proposition 2.3

**THEOREM 2.6.** *The ratios  $\frac{r(G_N)}{N}$ ,  $\frac{r(H_N)}{N}$ ,  $N = 1, 2, \dots$  are bounded above.*

Newman in [24] established the  $L^1$ -flatness of the Gauss-Fresnel polynomials. Besides, Littlewood proved in [21] that the Gauss-Fresnel polynomials converge in measure to 1. But, since the  $P_N$ 's are bounded, Littlewood result implies convergence of  $\|P_N\|_1$  to 1 as  $N \rightarrow \infty$  hence Newman's result.

We further notice that Egorov theorem allows to see that for any sequence of polynomials  $(P_n(z))$  with  $L^2$  norm 1,  $|P_n(z)|$  converge almost everywhere to 1 if and only if  $\| |P_n(z)|^2 - 1 \|_1$  converge to 0.

**Remarks.** (1) Consider the linear fractional transformation  $f(z) = \frac{z-\alpha}{1-\alpha z}$  where  $\alpha$  is real positive and less than one. It maps  $S^1$  onto itself and has power series expansion  $-\alpha + \sum_{k=1}^{\infty} (1-\alpha)\alpha^k z^k$  whose sequence of partial sums  $S_n, n = 1, 2, \dots$  converges uniformly to  $f(z)$ . Thus the sequence of polynomials  $\frac{S_n}{\|S_n\|_2}, n = 1, 2, \dots$  is ultraflat.

(2) Definition: Call a sequence  $P_n(z) = a_{0,n} + a_{1,n}z + \dots + a_{k_n,n}z^{k_n}, n = 1, 2, \dots$  of analytic trigonometric polynomials, each  $P_n$  of  $L^2(S^1, dz)$  norm 1, trivial if  $\max\{|a_{j,n}| : 1 \leq j \leq k_n\} \rightarrow 1$  as  $n \rightarrow \infty$ .

If an ultraflat sequence  $P_n(z) = a_{0,n} + a_{1,n}z + \dots + a_{k_n,n}z^{k_n}, n = 1, 2, \dots$  has all its coefficients non-negative then it is necessarily trivial. This follows from the inequality (since  $a_{j,k}$ 's are non-negative):

$$1 = \sum_{j=0}^{k_n} a_{j,n}^2 \leq \sum_{j=0}^{k_n} a_{j,n} = P_n(1) \rightarrow 1$$

as  $n \rightarrow \infty$ , which is same as  $0 \leq \sum_{j=1}^{k_n} (a_{j,n}(1 - a_{j,n})) \rightarrow 0$  as  $n \rightarrow \infty$ .

Thus there are no nontrivial ultraflat sequences with all coefficients non-negative.

(3) Next we consider flat sequence in almost everywhere sense. If  $\phi$  is a singular inner function, and if  $S_n, n = 1, 2, \dots$  is the sequence of partial sums of its power series expansion, then  $\frac{|S_n(z)|}{\|S_n\|_2} \rightarrow 1$  a.e.  $(dz)$  as  $n \rightarrow \infty$ . The sequence  $\frac{S_n}{\|S_n\|_2}, n = 1, 2, \dots$  is therefore a flat sequence in a.e.  $(dz)$ . sense, but not an ultraflat sequence.

(4) Call a function  $\phi$  of absolute value 1 a.e.  $(dz)$  trivial if it is of the type  $cz^n$  for some integer  $n, c$  will then be necessarily of absolute value 1. We note that if  $\phi$

is a measurable function on  $S^1$  of absolute value one a.e.  $(dz)$  with all its Fourier coefficients non-negative, then  $\phi$  is necessarily trivial. This follows by comparing the Fourier coefficients of two sides of the identity  $1 = \phi\bar{\phi}$  a.e.. Indeed if  $\phi$  is non-trivial and has all its Fourier coefficients non-negative then  $\phi$  has at least two Fourier coefficients positive which in turn implies that  $\phi\bar{\phi}$  has at least two Fourier coefficients positive, which is a contradiction. It is therefore not possible to get a non-trivial flat sequence  $P_j, n = 1, 2, \dots$  in a.e. sense with coefficients of each  $P_j$  non-negative and such that that the sequence  $P_j, j = 1, 2, \dots$  itself converges to a function of absolute value a.e.  $(dz)$ .

(5) This brings us to one of the main open question about flat sequence of polynomials in a.e. sense, namely, whether there exists an non-trivial flat sequence of polynomials in almost everywhere sense with all coefficients non-negative? It is known that if there is such a sequence then there exists non-dissipative, ergodic non-singular transformations with simple Lebesgue spectrum for the associated unitary operator [1].

If  $P_n, n = 1, 2, \dots$  is a flat sequence of polynomials in a.e. sense then the constant term of  $|P_n|^2$  is one, but all other coefficients of  $|P_n|^2$  tend to zero uniformly as  $n \rightarrow \infty$ . Indeed if  $k \neq 0$  then

$$\begin{aligned} \left| \int_{S^1} z^k |P_n(z)|^2 dz \right| &= \left| \int_{S^1} z^k (|P_n(z)|^2 - 1) dz \right| \\ &\leq \int_{S^1} ||P_n(z)|^2 - 1| dz \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , since  $|P_n(z)|^2 \rightarrow 1$  as  $n \rightarrow \infty$ . This in turn implies that for a flat sequence  $P_n, n = 1, 2, \dots$  in a.e. sense with coefficients of all  $P_n$  non-negative, the second largest coefficient of  $P_n(z)$  tends to zero as  $n \rightarrow \infty$ , which means that all except possibly the largest coefficient of  $P_n$  tend to zero uniformly as  $n \rightarrow \infty$ . If the largest coefficient of  $P_n$  tends to 1 as  $n \rightarrow \infty$  then  $P_n, n = 1, 2, \dots$  is a trivial flat sequence. It is an open question whether there exists an a.e.  $(dz)$  flat sequence  $P_n, n = 1, 2, \dots$  with non-negative coefficients with largest of the coefficients of  $P_n, n = 1, 2, \dots$  uniformly bounded away from 1.

(6) Given a flat sequence of polynomials  $P_n, n = 1, 2, \dots$  in a.e. sense, we can form the sequence  $M_n, n = 1, 2, \dots$  of covariance matrices associated to it, and the sums  $r_n, n = 1, 2, \dots$  of entries of  $M_n, n = 1, 2, \dots$ . Since the sequence is flat in a.e. sense, we see, from the considerations above, that the diagonal terms of  $M_n, n = 1, 2, \dots$  tend to 1 while the off diagonal terms converge to 0 uniformly. Thus  $M_n, n = 1, 2, \dots$  converges to the infinite identity matrix entrywise, where, moreover, the off diagonal terms tend to zero uniformly. In what follows we will study the behavior of the sequence  $r_n, n = 1, 2, \dots$ , for the case when the sequence  $P_n, n = 1, 2, \dots$  is flat in a.e. sense. We will also bring into play a sequence  $C_n, n = 1, 2, \dots$  where, for each  $n$ ,  $C_n$  is the sum of the absolute values of entries



of  $M_n, n = 1, 2, \dots$

(7) Consider now the class B of polynomials of the type  $\frac{1}{\sqrt{m}} \left(1 + \sum_{j=1}^{m-1} z^{R_j}\right)$ . As Bourgain[5] has shown the spectrum of a measure preserving rank one transformation is given (up to possibly some discrete points) by a generalized Riesz product made out of such polynomials. It is not known if there is a measure preserving rank one transformation with simple Lebesgue spectrum. This is equivalent to the question if there exist a flat sequence of polynomials in a.e. sense from the class B[1]. We will give below a necessary condition for a sequence of polynomials from the class B to be flat a.e.  $(dz)$  which contrasts with the necessary condition for ultraflat sequences derived in section 1. Consider a sequence of distinct polynomials  $P_j, j = 1, 2, \dots$  of the type

$$P_j(z) = \frac{1}{\sqrt{m_j}} \left(1 + \sum_{k=1}^{m_j-1} z^{R_{k,j}}\right), \quad (3)$$

Such a sequence can not be ultraflat since  $P_j(1) = \sqrt{m_j} \rightarrow \infty$  as  $j \rightarrow \infty$ . As mentioned above it is not known if such a sequence can be flat a.e.  $(dz)$ . However, we will show in next section that if a sequence of polynomials  $P_j, j = 1, 2, \dots$  of this type converges to 1 a.e.  $(dz)$  then  $\frac{r_j}{N_j} \rightarrow \infty$  as  $j \rightarrow \infty$ . This will follow from a more general result we prove below (Theorem 4.3). We will need some ideas and results about generalized product [1] which we give the next section.

(8) An inequality due to D. J. Newman [24] is

$$\frac{\|f\|_2^2}{\|f\|_4^{\frac{4}{3}}} \leq \|f\|_1^{\frac{2}{3}},$$

which is obtained by applying Hölder's inequality as follows: Since

$$\int_{S^1} |f|^2 dz = \int_{S^1} |f|^{\frac{4}{3}} |f|^{\frac{2}{3}} dz,$$

we get the required inequality by applying Hölder's inequality with  $p = 3, q = \frac{3}{2}$ . This inequality immediately implies that if  $P_n, n = 1, 2, \dots$  is a sequence of polynomials of  $L^2(S^1, dz)$  norm 1, and if it is flat in  $L_4$ -norm, then it is flat in  $L_1$  sense, hence over a subsequence it is flat in a.e.  $(dz)$ -sense. However this sufficiency criterion for flat sequence of polynomials is not applicable to a sequence of polynomials from the class B since for any polynomial  $P$  from the class,  $\|P\|_4^4 \geq 2$  [10].

### 3. Dissociated Polynomials and Generalized Riesz Products

Consider the following two products:

$$\begin{aligned} (1+z)(1+z) &= 1+z+z+z^2 = 1+2z+z^2, \\ (1+z)(1+z^2) &= 1+z+z^2+z^3. \end{aligned}$$

In the first case we group terms with the same power of  $z$ , while in the second case all the powers of  $z$  in the formal expansion are distinct. In the second case we say that the polynomials  $1+z$  and  $1+z^2$  are dissociated. More generally we say that a set of trigonometric polynomials is dissociated if in the formal expansion of product of any finitely many of them, the powers of  $z$  in the non-zero terms are all distinct [1].

If  $P(z) = \sum_{j=-m}^m a_j z^j, Q(z) = \sum_{j=-n}^n b_j z^j, m \leq n$ , are two trigonometric polynomials then for some  $N$ ,  $P(z)$  and  $Q(z^N)$  are dissociated. Indeed

$$P(z) \cdot Q(z^N) = \sum_{i=-m}^m \sum_{j=-n}^n a_i b_j z^{i+Nj}.$$

If we choose  $N > 2n$ , then we will have two exponents, say  $i+Nj$  and  $u+Nv$ , equal if and only if  $i-u = N(v-j)$  and since  $N$  is bigger than  $2n$ , this can happen if and only if  $i = u$  and  $j = v$ . More generally, given any sequence  $P_1, P_2, \dots$  of polynomials one can find integers  $1 = N_1 < N_2 < N_3 < \dots$ , such that  $P_1(z^{N_1}), P_2(z^{N_2}), P_3(z^{N_3}), \dots$  are dissociated. Note that since the map  $z \mapsto z^{N_i}$  is measure preserving, for any  $p > 0$  the  $L^p(S^1, dz)$  norm of  $P_i(z)$  and  $P_i(z^{N_i})$  remain the same, as also their logarithmic integrals, i.e.  $\int_{S^1} \log |P_i(z)| dz = \int_{S^1} \log |P_i(z^{N_i})| dz$ .

Now let  $P_1, P_2, \dots$  be a sequence of polynomials, each  $P_i$  being of  $L^2(S^1, dz)$  norm 1. Then the constant term of each  $|P_i(z)|^2$  is 1. If we choose  $1 = N_1 < N_2 < N_3 < \dots$  so that  $|P_1(z^{N_1})|^2, |P_2(z^{N_2})|^2, |P_3(z^{N_3})|^2, \dots$  are dissociated, then the constant term of each finite product

$$\prod_{j=1}^n |P_j(z^{N_j})|^2$$

is one so that each finite product integrates to 1 with respect to  $dz$ . Also, since  $|P_j(z^{N_j})|^2, j = 1, 2, \dots$  are dissociated, for any given  $k$ , the  $k$ -th Fourier coefficient of  $\prod_{j=1}^n |P_j(z^{N_j})|^2$  is either zero for all  $n$ , or, if it is non-zero for some  $n = n_0$  (say), then it remains the same for all  $n \geq n_0$ . Thus the measures  $(\prod_{j=1}^n |P_j(z^{N_j})|^2) dz, n = 1, 2, \dots$  admit a weak limit on  $S^1$ . It is called the generalized Riesz product of the polynomials  $|P_j(z^{N_j})|^2, j = 1, 2, \dots$ . Let  $\mu$  denote this measure. It is known [1] that  $\prod_{j=1}^k |P_j(z^{N_j})|, k = 1, 2, \dots$ , converge in  $L^1(S^1, dz)$  to  $\sqrt{\frac{d\mu}{dz}}$  as  $k \rightarrow \infty$ . It follows from this that if  $\prod_{j=1}^k |P_j(z^{N_j})|, k = 1, 2, \dots$  converge a.e. ( $dz$ ) to a finite positive value then  $\mu$  has a part which is equivalent to Lebesgue measure.

#### 4. Flat a.e.(dz) Sequence of Polynomials: Necessary Conditions

Consider a polynomial of norm 1 in  $L^2(S^1, dz)$ . Such a polynomial with  $m$  non-zero coefficients can be written as:

$$P(z) = \epsilon_0 \sqrt{p_0} + \epsilon_1 \sqrt{p_1} z^{R_1} + \cdots + \epsilon_{m-1} \sqrt{p_{m-1}} z^{R_{m-1}}, \quad (4)$$

where each  $p_i$  is positive and their sum is 1, and where  $\epsilon_i$ 's are complex numbers of absolute value 1. Such a  $P$  gives a probability measure  $|P(z)|^2 dz$  on the circle group which we denote by  $\nu$ . Now  $|P(z)|^2$  can be written as

$$|P(z)|^2 = 1 + \sum_{\substack{k=-N \\ k \neq 0}}^N a_k z^{n_k},$$

where each  $n_k$  is of the form  $R_i - R_j$ , and each  $a_k$  is a sum of terms of the type  $\epsilon_i \bar{\epsilon}_j \sqrt{p_i} \sqrt{p_j}$ ,  $i \neq j$ , with  $R_j - R_i = n_k$ ,  $a_k = \bar{a}_{-k}$ ,  $1 \leq k \leq N$ . We will write

$$L = \sum_{\substack{k=-N \\ k \neq 0}}^N a_k = |P(1)|^2 - 1.$$

Consider the special case when each  $\epsilon_i = 1$ . Then

$$L = \sum_{\substack{0 \leq i, j \leq m-1 \\ i \neq j}} \sqrt{p_i} \sqrt{p_j},$$

is a function of probability vectors  $(p_0, p_1, p_2, \dots, p_{m-1})$ , which attains its maximum value when each  $p_i = \frac{1}{m}$ , and the maximum value is  $\frac{m(m-1)}{m} = m-1$ .

We conclude therefore that  $|L| \leq m-1$ , irrespective of whether  $\epsilon_i$ 's all one or not. We also note that  $m-1 \leq N \leq m(m-1)$ . So, when  $p_i$ 's are all equal and  $= \frac{1}{m}$  we have

$$\frac{N}{L^2} \leq \frac{m}{m-1} \leq 2$$

for  $m \geq 2$ .

Note that, in general, if  $\frac{N}{L^2}$  is bounded then  $L$  can not be close to zero, which in turn implies that a sequence of such polynomials stays away from 1 in absolute value at  $z = 1$ , and so can not ultraflat.

For each  $k$ ,  $-N \leq k \leq N$ ,  $k \neq 0$ , let  $D_k$  denote the cardinality of the set of pairs  $(i, j)$ ,  $i \neq j$ ,  $-N \leq i, j \leq N$ ,  $i, j \neq 0$ , such that  $n_j - n_i = n_k$ . For each  $k$ ,  $D_k \leq 2N - 2|k| + 2 \leq m(m-1)$ , whence

$$\left| \sum_{\substack{k=-N \\ k \neq 0}}^N a_k D_k \right| \leq m(m-1) \sum_{\substack{k=-N \\ k \neq 0}}^N |a_k| < m^3$$

We write

$$A(P) = A = \sum_{\substack{k=-N \\ k \neq 0}}^N a_k D_k, \quad B(P) = B = \sum_{\substack{-N \leq i, j \leq N \\ 0 \neq i, j}} a_i a_j = (|P(1)|^2 - 1)^2.$$

Consider the random variables  $X(k) = z^{n_k} - \overline{a_k}$  with respect to the measure  $\nu$ . We write  $m(k, l) = \int_{S^1} X(k) \overline{X(l)} d\nu$ ,  $-N \leq k, l \leq N, k, l \neq 0$  and  $M$  for the correlation matrix with entries  $m(k, l)$ ,  $-N \leq k, l \leq N, k, l \neq 0$ . We call  $M$  the covariance matrix associated to  $|P(z)|^2$ . Since linear combination of  $X(k)$ ,  $-N \leq k \leq N, k \neq 0$ , can vanish at no more than a finite set in  $S^1$ , and,  $\nu$  is non discrete, the random variables  $X(k)$ ,  $-N \leq k \leq N, k \neq 0$  are linearly independent, whence the covariance matrix  $M$  is non-singular.

Note that

$$m_{i,j} = \int_{S^1} z^{n_i - n_j} d\nu - \overline{a_i} a_j, \quad m_{i,i} = 1 - |a_i|^2$$

Let  $r(P) = r$  denote the sum of the entries of the matrix  $M$ . We have

$$\begin{aligned} r &= \sum_{\substack{k=-N \\ k \neq 0}}^N \sum_{\{i,j, n_i - n_j = n_k, i, j \neq 0\}} m_{i,j} + \sum_{\substack{k=-N \\ k \neq 0}}^N m_{k,k} \\ &= \sum_{\substack{k=-N \\ k \neq 0}}^N \sum_{\{i,j, n_i - n_j = n_k, i, j \neq 0\}} (a_k - \overline{a_i} a_j) + 2N - \sum_{\substack{i=-N \\ i \neq 0}}^N |a_i|^2 \\ &= \sum_{\substack{k=-N \\ k \neq 0}}^N a_k D_k + 2N - \sum_{\substack{-N \leq i, j \leq N \\ i, j \neq 0}} a_i \overline{a_j} \\ &= A + 2N + |L|^2 \end{aligned}$$

Since  $|A|$  is of order at most  $m^3$ ,  $N \leq \frac{1}{2}m(m-1)$ , and  $|L|^2$  are of order  $m^2$ , we see that  $r$  is of order at most  $m^3$ . We also note that the quantity  $C(P) = C = \sum_{\{(i,j), -N \leq i, j \leq N, i, j \neq 0\}} |m_{i,j}|$  is also of order at most  $m^3$ . Indeed

$$C \leq \sum_{\substack{k=-N \\ k \neq 0}}^N \left( D_k |a_k| + \sum_{\{(i,j): i-j=k, i, j \neq 0\}} |\overline{a_i} a_j| \right) + 2N,$$

which shows that  $C$  is of order at most  $m^3$ .

We will now consider a sequence  $P_j(z)$ ,  $j = 1, 2, \dots$  of polynomials, each  $P_n$  of  $L^2(S^1, dz)$  norm 1. The quantities  $A(P_j), C(P_j)$  etc will now written as  $A_j, C_j$

etc. It will follow from our considerations below that *if a sequence of polynomials  $P_j, j = 1, 2, \dots$  from the class  $B$  is flat then  $\frac{C(P_j)}{m_j^2} \rightarrow \infty$  as  $j \rightarrow \infty$*

We will first prove two singularity lemmas under similar looking condition using Peyrière's method, and then use them to derive similar looking necessary conditions for the some classes of polynomials to admit flat (a.e  $(dz)$ ) sequences. These classes include  $B$  but do not contain any ultraflat sequence of polynomials.

LEMMA 4.1. *If  $P_j(z), j = 1, 2, \dots$  is a sequence of polynomials of  $L^2(S^1, dz)$  norm 1 such that*

- (i) *the squares of their absolute values are dissociated*  
(ii)  *$\frac{N_j}{L_j^2}, j = 1, 2, \dots$  are bounded, (iii)  $\sum_{j=1}^{\infty} \min(1, \sqrt{\frac{N_j}{r_j}}) = \infty$  then the weak limit  $\mu$  of the measures  $\prod_{j=1}^n |P_j(z)|^2 dz, n = 1, 2, \dots$ , is singular to Lebesgue measure.*

PROOF. Write  $s_j = \min\left\{1, \sqrt{\frac{N_j}{r_j}}\right\}$ . Since  $\sum_{j=1}^{\infty} s_j = \infty$ , by Banach-Steinhaus theorem there is an  $l^2$  sequence  $\lambda_j, j = 1, 2, \dots$  of positive real numbers such that

$$\sum_{j=1}^{\infty} \lambda_j s_j = \infty. \quad (5)$$

Consequently, since  $\frac{N_j}{L_j^2}$ 's are assumed to be bounded,

$$\sum_{j=1}^{\infty} \frac{\lambda_j^2 s_j^2}{L_j^2} N_j < \infty \quad (6)$$

Let  $A_j = \{n_{k,j} : -N_j \leq k \leq N_j, k \neq 0\}$ . Let  $V_j$  be the  $1 \times A_j$  matrix with all entries equal to  $\frac{\lambda_j s_j}{L_j}$ . The squared Euclidean norm of this vector is  $\frac{\lambda_j^2 s_j^2}{L_j^2} \times 2N_j$  which when summed over  $j$  is convergent by equation (6) above. Let  $U_j$  be the  $1 \times A_j$  matrix with entries  $u(n_k, j) = a_{k,j}, -N_j \leq k \leq N_j, k \neq 0$ . Then the dot product  $U_j \cdot V_j = L_j \times \frac{\lambda_j s_j}{L_j}$  which diverges when summed over  $j$  by the choice of  $\lambda_j$ 's.

Let

$$\begin{aligned} f_n &= \sum_{j=1}^n \sum_{k \in A_j} \frac{\lambda_j s_j}{L_j} z^{n_k}, \\ g_n &= \sum_{j=1}^n \sum_{k \in A_j} \frac{\lambda_j s_j}{L_j} (X_j(k)). \\ &= \sum_{j=1}^n \sum_{k \in A_j} \frac{\lambda_k s_j}{L_j} (z^{n_k} - \overline{a_{k,j}}) \end{aligned}$$

Now, for  $m < n$ ,

$$\int_{S^1} |f_n - f_m|^2 dz = \sum_{j=m+1}^n \sum_{k,l \in A_j} \frac{\lambda_j^2 s_j^2}{L_j^2} \int_{S^1} z^{n_k - n_l} dz = \sum_{j=m+1}^n 2 \frac{\lambda_j^2 s_j^2}{L_j^2} N_j$$

$$\rightarrow 0 \text{ as } m, n \rightarrow \infty,$$

and under the assumption of dissociation of the polynomials  $|P_j|^2, j = 1, 2, \dots$ ,

$$\int_{S^1} |g_n - g_m|^2 d\mu = \sum_{j=m+1}^n \sum_{k,l \in A_j} \frac{\lambda_j^2 s_j^2}{L_j^2} m_j(k, l) \leq \sum_{j=m+1}^n \frac{\lambda_j^2 s_j^2}{L_j^2} r_j$$

(since  $s_j = \min \left\{ 1, \sqrt{\frac{N_j}{r_j}} \right\}$ , in case  $s_j = 1$ , we have  $N_j \geq r_j$ , otherwise  $s_j^2 = \frac{N_j}{r_j}$ .)

$$\leq \sum_{j=m+1}^n \frac{\lambda_j^2}{L_j^2} N_j \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

We conclude that  $f_n$ 's converge in  $L^2(S^1, dz)$  to a function whose norm is  $\sum_{j=1}^{\infty} 2 \frac{\lambda_j^2 s_j^2}{L_j^2} \times N_j$ , and  $g_n$ 's converge in  $L^2(S^1, d\mu)$  to a function whose norm is no more than  $\sum_{j=1}^{\infty} \frac{\lambda_j^2}{L_j^2} N_j$ . If  $\mu$  is not singular to  $dz$ , then there is a sequence of  $l_k, k = 1, 2, \dots$  of natural numbers and a  $z_0 \in S^1$  such that  $f_{l_k}(z_0), g_{l_k}(z_0), k = 1, 2, \dots$  converge to a finite limits, which in turn implies that

$$f_{l_k}(z_0) - g_{l_k}(z_0) = \sum_{j=1}^{l_k} \sum_{u \in A_j} \frac{\lambda_j s_j}{L_j} a_{u,j}$$

$$= \sum_{j=1}^{l_k} \lambda_j s_j$$

is convergent as  $k \rightarrow \infty$  contrary to equation (5). □

LEMMA 4.2. *If (i)  $L_j, j = 1, 2, \dots$  are uniformly bounded away from 0 (ii)  $\sum_{j=1}^{\infty} \frac{L_j^2}{C_j} = \infty$ , then  $\mu$  is singular to its translate  $\mu_u$  for every  $u \in S^1$  for which the sequence  $|P_j(u)| \rightarrow 1$ , as  $j \rightarrow \infty$ .*

PROOF. By Banach-Steinhaus theorem there exist  $b_j, j = 1, 2, \dots$ , with their sum of absolute squares finite such that for each  $j, \frac{L_j}{C_j} b_j \geq 0$  and  $\sum_{j=1}^{\infty} \frac{L_j}{\sqrt{C_j}} b_j = \infty$ . Fix a  $v \in S^1$  such that  $|P_j(v)| \rightarrow 1$  as  $j \rightarrow \infty$ . Note that

$$\sum_{j=1}^{\infty} \left( \sum_{\substack{k=-N_j \\ k \neq 0}}^{N_j} a_j (1 - v^{n_{k,j}}) \right) = \sum_{j=1}^{\infty} \left( L_j - (|P_j(v)|^2 - 1) \right).$$

Since  $|P_j(v)|^2 \rightarrow 1$  as  $j \rightarrow \infty$ , the series  $\sum_{l=1}^{\infty} \left( \frac{L - (|P_j(v)|^2 - 1)}{\sqrt{C_j}} \right) b_j$  diverges. Let  $B_j$  be the  $1 \times 2N_j$  with all entries equal to  $\frac{b_j}{\sqrt{C_j}}, j = 1, 2, \dots$ . Then

$$(M_j B_j, B_j) = \frac{r_j |b_j|^2}{C_j} \leq |b_j|^2,$$

whence  $\sum_{j=1}^{\infty} (M_j B_j, B_j)$  is a finite sum, which in turn implies that the series in  $j$

$$\sum_{j=1}^{\infty} \sum_{\substack{k=-N_j \\ k \neq 0}}^{N_j} \frac{b_j}{\sqrt{C_j}} (z^{n_{k,j}} - a_{k,j})$$

converges a.e.  $(\mu)$  over a subsequence.

Consider now the translated measure  $\mu_v(\cdot) = \mu(v(\cdot))$ . We have

$$\int_{S^1} z^{n_{k,j}} d\mu_v = v^{-n_{k,j}} a_{k,j}.$$

The covariance matrix  $M_{v,j}$  of the random variables  $z^{n_{k,j}} - v^{-n_{k,j}} a_{k,j}, -N_j \leq k \leq N_j, k \neq 0$  with respect to the translated measure  $\mu_v$  has entries  $v^{-(n_{k,j} - n_{l,j})} M_{k,l}$ , which can be seen to be unitarily equivalent to  $M_j$ . Indeed,

$$M_{v,j} = U_j^{-1} M_j U_j,$$

where  $U_j$  is a  $2N_j \times 2N_j$  diagonal matrix with entries

$$v^{n_{-N_j,j}}, v^{n_{-N_j+1,j}}, \dots, v^{n_{-1,j}}, v^{n_{1,j}}, \dots, v^{n_{N_j-1,j}}, v^{n_{N_j,j}},$$

along the diagonal in that order.

We note that

$$\begin{aligned} & \sum_{j=1}^{\infty} (M_{v,j} B_j, B_j) \\ &= \sum_{j=1}^{\infty} \frac{r_{v,j}}{C_j} |b_j|^2 < \infty, \end{aligned}$$

where  $r_{v,j}$  is the sum of the entries of the of the matrix  $M_{v,j}, j = 1, 2, \dots$ . It is clear that for all  $j, |r_{v,j}| \leq C_j$ .

As before we conclude that the series

$$\sum_{j=1}^{\infty} \left( \sum_{\substack{k=-N_j \\ k \neq 0}}^{N_j} \frac{b_j}{\sqrt{C_j}} (z^{n_{k,j}} - v^{-n_{k,j}} a_{k,j}) \right)$$

converges a.e  $\mu_v$  over a subsequence subsequence.

If  $\mu$  and  $\mu_v$  are not mutually singular, then there exist an  $z_0 \in S^1$  and an increasing sequence of natural  $K_p, p = 1, 2, \dots$  of natural numbers such that the sequences (with  $p = 1, 2, \dots$ )

$$\sum_{j=1}^{K_p} \sum_{\substack{k=-N_j \\ k \neq 0}}^{N_j} \frac{b_j}{\sqrt{C_j}} (z_0^{n_{k,j}} - a_{k,j})$$

$$\sum_{j=1}^{K_p} \sum_{\substack{k=-N_j \\ k \neq 0}}^{N_j} \frac{b_j}{\sqrt{C_j}} (z_0^{n_{k,j}} - v^{-n_{k,j}} a_{k,j})$$

converge to a finite number as  $p \rightarrow \infty$ . The difference of the two partial sums is

$$\sum_{j=1}^{K_p} \sum_{\substack{k=-N_j \\ k \neq 0}}^{N_j} \frac{b_j}{\sqrt{C_j}} a_{k,j} (1 - v^{-n_{k,j}}),$$

which diverges as  $p \rightarrow \infty$ . The contradiction shows that  $\mu$  and  $\mu_v$  are singular.  $\square$

The following theorem is proved in [1].

**THEOREM 4.3.** *Let  $P_j, j = 1, 2, \dots$  be a sequence of non-constant polynomials of  $L^2(S^1, dz)$  norm 1 such that  $\lim_{j \rightarrow \infty} |P_j(z)| = 1$  a.e.  $(dz)$  then there exists a subsequence  $P_{j_k}, k = 1, 2, \dots$  and natural numbers  $l_1 < l_2 < \dots$  such that the polynomials  $P_{j_k}(z^{l_k}), k = 1, 2, \dots$  are dissociated and the infinite product  $\prod_{k=1}^{\infty} |P_{j_k}(z^{l_k})|^2$  has finite nonzero value a.e.  $(dz)$ .*

We are now in a position to give two similar looking but distinct necessary conditions for a sequence of polynomials in certain classes to be flat in a.e.  $(dz)$  sense. The class  $B$  is included in both these classes.

**THEOREM 4.4.** *(i) If  $\frac{N_j}{L_j^2}, j = 1, 2, \dots$  remain bounded and  $\lim_{j \rightarrow \infty} |P_j(z)| = 1$  a.e.  $(dz)$  then  $\frac{r_j}{N_j} \rightarrow \infty$  as  $j \rightarrow \infty$ . (ii) If  $L_j, j = 1, 2, \dots$  are uniformly bounded away from 0 and  $\lim_{j \rightarrow \infty} |P_j(z)| = 1$  a.e.  $(dz)$  then  $\frac{C_j}{m_j} \rightarrow \infty$  as  $j \rightarrow \infty$*

**PROOF.** Under the hypothesis of part (i) of the theorem, by theorem 4.3 we get a subsequence  $P_{j_k} = Q_k, k = 1, 2, \dots$  and natural numbers  $l_1 < l_2 < \dots$  such that the polynomials  $|Q_k(z^{l_k})|^2, k = 1, 2, \dots$  are dissociated and the infinite product  $\prod_{k=1}^{\infty} |Q_k(z^{l_k})|^2$  has finite non-zero limit a.e.  $(dz)$ . Also, since the absolute squared  $Q_k(z^{l_k})$ 's are dissociated, the measures  $\mu_n \stackrel{\text{def}}{=} \prod_{k=1}^n |Q_k(z^{l_k})|^2 dz$



converge weakly to a measure  $\mu$  on  $S^1$  for which  $\frac{d\mu}{dz} > 0$  a.e. ( $dz$ ), indeed

$$\frac{d\mu}{dz} = \prod_{k=1}^{\infty} |Q(z^{l_k})|^2 \text{ a.e.}(dz)$$

Since the map  $z \mapsto z^{l_k}$  preserves the Lebesgue measure on  $S^1$ , the  $m_{j_k}(u, v)$ 's for  $|P_{j_k}(z^{l_k})|^2 dz$  remains the same as for  $|P_{j_k}(z)|^2 dz$ . If  $\sum_{k=1}^{\infty} \sqrt{\frac{N_{j_k}}{r_{j_k}}} = \infty$ , then by Lemma 4.1  $\mu$  will be singular to ( $dz$ ) which is not true. So  $\sum_{k=1}^{\infty} \sqrt{\frac{N_{j_k}}{r_{j_k}}} < \infty$ . If  $\sqrt{\frac{N_{j_k}}{r_{j_k}}}, j = 1, 2, \dots$  does not tend to 0 as  $j \rightarrow \infty$  then over a subsequence these ratios remain bounded away from 0. But by the above considerations, over a further subsequence these ratios have a finite sum, which is a contradiction. So  $\frac{N_j}{r_j} \rightarrow 0$  as  $j \rightarrow \infty$

Part (ii) of the theorem is proved similarly, applying Lemma 4.2 this time.  $\square$

**Remarks.** Over a subsequence the Gauss-Fresnel polynomials  $G_N, N = 1, 2, \dots$  are uniformly bounded away from 1 in absolute value at  $z = 1$ . Over a further subsequence they converge to 1 in absolute value a.e. ( $dz$ ) since they are  $L^1(S^1, dz)$  flat. By Theorem 4.4 (ii) we see that the ratios  $\frac{C(G_N)}{N^2}, N = 1, 2, \dots$  are unbounded (see Theorem 2.6).

## 5. Connection with combinatorial number theory

In this section we discuss the ratios  $\frac{C}{m^2}$  for the class  $B$ . In particular we give a sequence  $P_j, j = 1, 2, \dots$  from this class for which  $\frac{C(P_j)}{m_j^2}, j = 1, 2, \dots$  diverges but  $P_j, j = 1, 2, \dots$  is not flat in a.e. ( $dz$ ) sense.

For a given polynomial  $P(z) = \frac{1}{\sqrt{m}}(1 + z^{R_1} + z^{R_2} + \dots + z^{R_{m-1}})$  of class  $B$ , with

$$|P(z)|^2 = 1 + \sum_{j=1}^N a_j(z^{n_j} + z^{-n_j}),$$

we know that  $\frac{C(P)}{m^2}$  has the same order as  $\frac{2 \sum_{j=1}^N a_j D_j}{m^2}$ . However just ensuring that each  $D_j$  receives maximum possible value, namely  $N - j$ , is not enough to ensure that  $2 \sum_{j=1}^N a_j D_j$  is large in comparison with  $m^2$ . For consider the case when for each  $j, R_j = j$ , so that

$$P(z) = \frac{1}{\sqrt{m}}(1 + z + z^2 + \dots + z^{m-1})$$

$$|P(z)|^2 = 1 + \frac{1}{m} \sum_{j=1}^{m-1} (m-j)(z^j + z^{-j})$$

Now each  $D_j = m - j$ , so that

$$2 \sum_{j=1}^{m-1} a_j D_j = \frac{1}{m} \sum_{j=1}^{m-1} (m-j)^2 = \frac{1}{m} \sum_{j=1}^{m-1} j^2 = \frac{(m+1)(2m+1)}{6}$$

which is of order  $m^2$ .

One can ensure  $C$  large in comparison with  $m^2$  if each  $D_j$  has its maximum possible value, namely,  $N - j$ , and  $N$  is of higher order than  $m$ . Using some combinatorial number theory one can arrange this.

Let  $R$  be a natural number  $> 2$  and let  $m \geq 2$  be a natural number  $\leq R$ . Write  $R_0 = 0$ . Let  $R_0 < R_1 < R_2 < \dots < R_{m-1} = R$  be a set of  $m$  integers between 0 and  $R$ . Denote it by  $S$ . Note that 0 and  $R$  are in  $S$ . Let

$$T = \{(R_i, R_j) : i < j, 0 \leq i, j \leq m-1\}.$$

Let  $[0, R]$  be the set of integers from 0 to  $R$ . Define  $F : T \rightarrow [0, R]$  as follows:

$$F(R_i, R_j) = R_j - R_i, (R_i, R_j) \in T.$$

Since  $F$  depends on  $S$ , we will write  $F_S$  for  $F$  when needed.

For any  $r \in [0, R]$ , write  $d(r) = |F^{-1}(r)| =$  the cardinality of  $F^{-1}(r)$ . Note that  $F$  is one-one if and only of  $d(r) \leq 1$  for all  $r$ . If  $F$  is one-one, then  $S$  is called Sidon subset of  $[0, R]$ . As pointed out to us by R. Balasubramanian, if  $S$  is a Sidon set, then  $F$  can not be onto (unless  $R \leq 6$ ). This is a consequence of a well known result of Erdős and Turan [13] which says, that if  $S \subset [1, R]$  is a Sidon set then  $|S|$  is at most  $R^{\frac{1}{2}} + R^{\frac{1}{4}} + 1$ , see [19]. So, if  $S$  is a Sidon set then the cardinality  $|T|$  of  $T$  is  $\frac{1}{2} |S| (|S| - 1) < R$ .

Let  $M(S) = \max_{i \in [0, R]} d(i)$ . The quantities  $M(S)$  and  $|F_S(T)|$  are in some sense 'balanced' in that if one is large the other is small, and  $M(S) |F_S(T)|$  seems to be of order  $|S|^2$ . This is clearly true when  $S$  is a Sidon set and the other extreme case when  $S = [0, m-1]$

Let

$$\lambda(R) = \min \left\{ |S| : S \subset [0, R], F_S \text{ is onto} \right\}.$$

It can be shown that  $\lambda(R) \leq 2[R^{\frac{1}{2}}] + 2$ , where  $[R^{\frac{1}{2}}]$  denotes the integral part of  $R^{\frac{1}{2}}$ . We show that  $\frac{C}{m^2}$  is not bounded over the class  $B$ . For a given positive integer  $R > 2$  choose  $S \subset [0, R]$  of cardinality  $\lambda(R)$  and such that  $F_S(T) = [0, R]$ . Let  $m$  denote  $\lambda(R)$ , let  $R_0 < R_1 < \dots < R_{m-1} = R$  be the set  $S$ . Let

$$P(z) = \frac{1}{\sqrt{m}} (1 + z^{R_1} + z^{R_2} + \dots + z^{R_{m-1}})$$

$$|P(z)|^2 = 1 + \frac{1}{m} \sum_{j=1}^R d(j) (z^j + z^{-j})$$

Now

$$\begin{aligned} C(P) \geq A(P) &= 2 \sum_{j=1}^R \frac{1}{m} d(j) D_j > 2 \sum_{j=1}^R \frac{1}{m} D_j \\ &= \sum_{j=1}^R \frac{1}{m} (R-j) = \frac{1}{m} \frac{1}{2} (R-1)R \\ &\geq \frac{1}{16} (R-1)R^{\frac{1}{2}}, \end{aligned}$$

since  $m = \lambda(R) \leq 2\sqrt{R} + 2$

Clearly, therefore,  $\frac{C}{m^2}$  is unbounded over the class  $B$ .

We do not know if one can choose, for each  $R$ , a suitable Sidon set in  $S_R \subset [0, R]$ , with  $0, R \in S_R$ , such that ratios  $\frac{C(P_{R, S_R})}{|S_R|^2}$ ,  $R = 1, 2, \dots$  are unbounded, where  $P_{R, S_R}$  is the polynomial in class  $B$  with frequencies in  $S_R$ , and if such a sequence of polynomials can in addition be flat in a.e.  $(dz)$  sense.

We now give an example of a sequence  $P_j, j = 1, 2, \dots$  from the class  $B$  for which  $\frac{C(P_j)}{m_j^2} \rightarrow \infty$  but the sequence  $P_j, j = 1, 2, \dots$  is not flat in a.e  $(dz)$  sense.

Let

$$P_j(z) = \frac{1}{\sqrt{2j}} \left( \sum_{i=0}^{j-1} z^i + \sum_{i=1}^j z^{ij} \right) = \frac{1}{\sqrt{2j}} \frac{1-z^j}{1-z} + \frac{1}{\sqrt{2j}} \frac{1-z^{j^2}}{1-z^j},$$

then clearly, for a given  $z \neq 1$ ,  $P_j(z) \rightarrow 0$  over every subsequence  $j_n, n = 1, 2, \dots$  over which  $z^{j_n}, n = 1, 2, \dots$  stays uniformly away from 1, whence  $P_j(z), j = 1, 2, \dots$  is not a flat sequence in a.e.  $(dz)$  sense.

Note that the set  $S_j$  of indices in  $P_j$  is the interval of integers  $[0, j-1]$  together with the integers  $j, 2j, \dots, (j-1)j, j^2$ . Also  $S_j$  has  $2j$  elements, and,  $F_{S_j}(T_j) = [1, j^2]$ . So, for each  $j$ ,  $|P_j(z)|^2$  admits all the frequencies from 1 to  $j^2$ , whence, as seen above,  $\frac{C(P_j)}{j^2} \rightarrow \infty$  as  $j \rightarrow \infty$ .

The equality  $F_{S_j}(T_j) = [0, j^2]$  in fact shows that  $\lambda(j^2) \leq 2j$  from which it is easy to deduce that for any  $j$ ,  $\lambda(j) \leq 2\sqrt{j} + 2$ . We note that

$$j^2 < \frac{1}{2} ((\sqrt{2})j + 1)(\sqrt{2})j,$$

however we do not know if  $\lambda(j^2) \leq [(\sqrt{2})j] + k$  for some positive integer  $k$ , independent of  $j$ . Also not known is how small the the largest  $d(i)$  can be when  $S$  is of cardinality  $\lambda(j^2)$ , and  $F_S(T) = [1, j^2]$  ?

We give below some probabilistic considerations which need further investigation. Let  $R > 2$  be an integer, and let  $S \subset [0, R^2]$  of cardinality  $2R$ , with  $0, R \in S$ . Let  $\Omega_R$  denote the the collection of all such subsets  $S$  in  $[0, R^2]$ . Cardinality of  $\Omega_R$

is  $\binom{R^2-1}{2R-2}$ . Equip  $\Omega$  with uniform distribution, denoted by  $\mathbb{P}_R$ . Let  $P(R, S)$  denote the polynomial of class  $B$  with frequencies in  $S$ . For a fixed  $\epsilon > 0$ , one can consider  $E(\epsilon, R) = \mathbb{P}_R(\{S : \| |P(R, S)|^2 - 1 \|_1 > \epsilon\})$ . If for every  $\epsilon > 0$ ,  $E(\epsilon, R) \rightarrow 0$  as  $R \rightarrow \infty$ , we will have a probabilistic proof of the existence of a sequence flat polynomials (in a.e. ( $dz$ ) sense) in the class  $B$ .

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## References

- [1] E. H. el Abdalaoui and M. Nadkarni, *Calculus of Generalized Riesz Products*, Contemporary Mathematics(AMS) 631 (2014), pp. 145-180.
- [2] E. H. el Abdalaoui and M. Nadkarni, *A non-singular transformation whose Spectrum has Lebesgue component of multiplicity one*, Ergodic Theory and Dynamical Systems, available on CJO2014. doi:10.1017/etds.2014.85.
- [3] C. Aistleitner, *On a problem of Bourgain concerning the  $L^1$ -norm of exponential sums*, Math. Z., 275 (2013), no. 3-4, 681-688.
- [4] E. Bombieri & J. Bourgain, *On Kahane's ultraflat polynomials*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 3, 627-703.
- [5] J. Bourgain, *On the spectral type of Ornstein class one transformations*, Isr. J. Math. ,**84** (1993), 53-63.
- [6] P. Borwein, M. J. Mossinghoff, *Barker sequences and flat polynomials*, Number Theory and Polynomials, 71-88, Lond. Math. Soc. Lecture Notes Series, 352, Cambridge Univ Press, Cambridge, 2008.
- [7] P. Borwein, M. J. Mossinghoff, *Wiefrich pairs and Barker sequences II*, LMS J. Comput. Math., 17 (1) (2014), 24-32.
- [8] P. Borwein, T. Erdélyi and F. Littmann, *Polynomials with coefficients from a finite set*, Trans. Amer. Math. Soc. 360 (2008), 5145-5154.
- [9] B. Brown, *Singular infinitely divisible distributions whose characteristic functions vanish at infinity*, Math. Proc. Cambridge Philos. Soc. 82 (1977), no. 2, 277-287.
- [10] J. Chidamberswamy, *On the Mean Modulus of Trigonometric Polynomials and a Conjecture of S. Chowla*, Proc. of Amer. Math. Soc. **36** Number I, November 1972.
- [11] J. R. Choksi and M. G. Nadkarni, *The maximal spectral type of rank one transformation*, Can. Math. Bull., **37** (1) (1994), 29-36.
- [12] T. Downarowich, Y. Lacroix, *Merit Factors and Morse Sequences* Theoretical Computer Science, (209) (1998), 377-387.
- [13] P. Erdős and P. Turan, *On a Problem of Sidon in Additive Number Theory and and some related problems* J. London. Math. Soc. **16**(1941) 212-215.
- [14] S. W. Graham, S. W & G. Kolesnik, *G. van der Corput's method of exponential sums*, London Mathematical Society Lecture Note Series, 126. Cambridge University Press, Cambridge, 1991.
- [15] M. Guenais, *Morse cocycles and simple Lebesgue spectrum*, Ergodic Theory Dynam. Systems, **19** (1999), no. 2, 437-446.
- [16] G.H. Hardy and J. E. Littlewood, *Some Problems of Diophantine Approximation, A Remarkable Trigonometrical Series*, Proc. of the National Acad., of Sciences, (2) 1916, 583-586.
- [17] B. Host, J.-F. Méla, F. Parreau, *Non-singular transformations and spectral analysis of measures*, Bull. Soc. math. France **119** (1991), 33-90.
- [18] J. -P. Kahane, *Sur les polynômes à coefficients unimodulaires*, Bull. London Math. Soc. 12, 1980, 321-342.

- [19] B. Lindström, *An Inequality for  $B_2$  Sequences*, J. Comb. Th., **6** 211-212 (1969).
- [20] Littlewood, J. E. (1966). *On polynomials  $\sum^n z^m, \sum_{m=0}^n e^{\alpha_i} z^m, z = e^{\theta i}$* , J. London Math. Soc. 41, 1966, 367-376.
- [21] J. E. Littlewood, *On the mean values of certain trigonometric polynomials. II*. Illinois J. Math., (6) 1962, 1-39.
- [22] L. Kuipers & H. Niederreiter, *Uniform Distribution of Sequences*, Dover Publishing, 2006.
- [23] M. G. Nadkarni, *Spectral theory of dynamical systems*, Birkhäuser, Cambridge, MA, 1998.
- [24] D. J. Newman, *An  $L^1$  extremal problem for polynomials*, Proc. Amer. Math. Soc. 16 1965 1287-1290.
- [25] J. Peyrière, *Étude de quelques propriétés des produits de Riesz*, Ann. Inst. Fourier, Grenoble **25**, 2 (1975), 127-169.
- [26] H. Queffelec and B. Saffari, *On Bernstein's Inequality and Kahane's Ultraflat Polynomials*, J. of Fourier Ana. and App. Volume 2, Number 6, 1996, 519-582.
- [27] R. Salem, *Essais sur les séries trigonométriques*, Hermann & Cie, éditeurs, 1940.
- [28] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.
- [29] A. Zygmund, *Trigonometric series* vol. II, second ed., Cambridge Univ. Press, Cambridge, 1959.

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