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# UNIFORM SEMIGROUP SPECTRAL ANALYSIS OF THE DISCRETE, FRACTIONAL & CLASSICAL FOKKER-PLANCK EQUATIONS

STÉPHANE MISCHLER AND ISABELLE TRISTANI

ABSTRACT. In this paper, we investigate the spectral analysis (from the point of view of semigroups) of discrete, fractional and classical Fokker-Planck equations. Discrete and fractional Fokker-Planck equations converge in some sense to the classical one. As a consequence, we first deal with discrete and classical Fokker-Planck equations in a same framework, proving uniform spectral estimates using a perturbation argument and an enlargement argument. Then, we do a similar analysis for fractional and classical Fokker-Planck equations using an argument of enlargement of the space in which the semigroup decays. We also handle another class of discrete Fokker-Planck equations which converge to the fractional Fokker-Planck one, we are also able to treat these equations in a same framework from the spectral analysis viewpoint, still with a semigroup approach and thanks to a perturbative argument combined with an enlargement one. Let us emphasize here that we improve the perturbative argument introduced in [7] and developed in [11], relaxing the hypothesis of the theorem, enlarging thus the class of operators which fulfills the assumptions required to apply it.

**Keywords:** Fokker-Planck equation; fractional Laplacian; spectral gap; exponential rate of convergence; long-time asymptotic; semigroup; dissipativity.

**AMS Subject Classification:** 47G20 Integro-differential operators; 35B40 Asymptotic behavior of solutions; 35Q84 Fokker-Planck equations.

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## 1. INTRODUCTION

**1.1. Model and main result.** In this paper, we investigate from a spectral analysis point of view some discrete and fractional Fokker-Planck equations. They are simple models for describing the time evolution of a density function  $f = f(t, x)$ ,  $t \geq 0$ ,  $x \in \mathbb{R}^d$ , of particles undergoing both diffusion and (harmonic) confinement mechanisms and write

$$(1.1) \quad \partial_t f = \Lambda_\varepsilon f = \mathcal{D}_\varepsilon f + \operatorname{div}(xf).$$

The diffusion term may be either a discrete diffusion (Section 2)

$$\mathcal{D}_\varepsilon(f) := \frac{1}{\varepsilon^2}(k_\varepsilon * f - f),$$

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for a convenient (centered, nonnegative, smooth and decaying fast enough) kernel  $k$ , with the usual notation  $k_\varepsilon(x) = k(x/\varepsilon)/\varepsilon^d$ ,  $\varepsilon > 0$ . It can also be a fractional diffusion (Section 3)

$$(1.2) \quad \mathcal{D}_\varepsilon(f)(x) := -(-\Delta)^{\frac{2-\varepsilon}{2}} f(x) = c_\varepsilon \int_{\mathbb{R}^d} \frac{f(y) - f(x) - \chi(x-y)(x-y) \cdot \nabla f(x)}{|x-y|^{d+2-\varepsilon}} dy,$$

with  $\varepsilon \in (0, 2)$ ,  $\chi$  centered in  $\mathcal{D}(\mathbb{R}^d)$  satisfying  $\mathbb{1}_{B(0,1)} \leq \chi \leq \mathbb{1}_{B(0,2)}$ , and a convenient renormalization constant  $c_\varepsilon > 0$ . Both families of equations are related to the classical Fokker-Planck equation, because in the limit  $\varepsilon \rightarrow 0$ , one may recover

$$\partial_t f = \Lambda_0 f = \Delta f + \operatorname{div}(xf).$$

In Section 4, the diffusion term is a fractional one, discrete for  $\varepsilon > 0$ :

$$\mathcal{D}_\varepsilon(f) := k_\varepsilon * f - \|k_\varepsilon\|_{L^1} f$$

where  $k_\varepsilon$  is another convenient kernel which converges towards the kernel of the fractional diffusion operator  $k_0 := c_\alpha |\cdot|^{-d-\alpha}$  for some fixed  $\alpha \in (0, 2)$ . In the limit  $\varepsilon \rightarrow 0$ , one may recover the fractional Fokker-Planck equation

$$\partial_t f = \Lambda_0 f = -(-\Delta)^{\alpha/2} f + \operatorname{div}(xf).$$

The main features of these equations are (expected to be) the same: they are mass preserving, positivity preserving, have a unique positive stationary state with unit mass and that stationary state is exponentially stable, in particular

$$(1.3) \quad f(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

for any solution associated to an initial datum  $f_0$  with vanishing mass. Such results can be obtained using different tools as the spectral analysis of self-adjoint operators, some (generalization of) Poincaré inequalities or logarithmic Sobolev inequalities as well as the Krein-Rutman theory for positive semigroup.

The aim of this paper is to initiate a kind of unified treatment of these equations and more importantly to establish that the convergence (1.3) is exponentially fast uniformly with respect to the diffusion term for a large class of initial data which are taken in a fixed (large) weighted Lebesgue or weighted Sobolev space  $X$ . Our approach is a semigroup approach in the spirit of the semigroup decomposition framework introduced by Mouhot in [10] and developed subsequently in [7, 4, 12, 6]. A typical result we are able to prove is the following.

**Theorem 1.1** (rough version). *There exist  $\varepsilon_0 \in (0, 2)$ ,  $a < 0$  and  $C \geq 1$  such that:*

$$\|S_{\Lambda_\varepsilon}(t)f - \Pi_{\Lambda_\varepsilon,0} S_{\Lambda_\varepsilon}(t)f\|_X \leq C e^{at} \|f - \Pi_{\Lambda_\varepsilon,0} f\|_X \quad \forall t \geq 0, \forall \varepsilon \in [0, \varepsilon_0], \forall f \in X,$$

where  $X$  is (for instance) a  $L^1$  weighted space,  $S_{\Lambda_\varepsilon}(t) = e^{\Lambda_\varepsilon t}$  stands for the semigroup associated to the generator  $\Lambda_\varepsilon$  and  $\Pi_{\Lambda_\varepsilon,0}$  for the projector onto the null space of  $\Lambda_\varepsilon$ .

Theorem 1.1 generalizes to the discrete diffusion Fokker-Planck equation and to the discrete fractional Fokker-Planck equation similar results obtained for the classical Fokker-Planck equation in [4, 6] (Section 2) and for the fractional one in [12] (Section 4). It also makes uniform with respect to the fractional diffusion parameter the convergence results obtained for the fractional diffusion equation in [12] (Section 3). It is worth mentioning that there exists a huge literature on the long-time behaviour for the Fokker-Planck equation as well as (to a lesser extend) for the fractional Fokker-Planck equation. We refer to the references quoted in [4, 6, 12] for details. There also probably exist many papers on the discrete diffusion equation since it is strongly related to a standard random walk in  $\mathbb{R}^d$ , but we were not able to find any precise reference in this PDE context.

**1.2. Method of proof.** Let us explain our method. First, we may associate a semigroup  $S_{\Lambda_\varepsilon}$  to the evolution equation (1.1) in many Sobolev spaces, and that semigroup is mass preserving and strongly positive. In other words,  $S_{\Lambda_\varepsilon}$  is a Markov semigroup and it is then expected that there exists a unique positive and unit mass steady state  $G_\varepsilon$  to the equation (1.1). Next, we are able to establish that the semigroup splits as

$$(1.4) \quad S_{\Lambda_\varepsilon} = S_\varepsilon^1 + S_\varepsilon^2, \quad S_\varepsilon^1 \approx e^{tT_\varepsilon}, \quad T_\varepsilon \text{ finite dimensional}, \quad S_\varepsilon^2 = \mathcal{O}(e^{at}), \quad a < 0,$$

in these many weighted Sobolev spaces. The above decomposition of the semigroup is the main technical issue of the paper. It is obtained by introducing a convenient splitting

$$(1.5) \quad \Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$$

where  $\mathcal{B}_\varepsilon$  enjoys suitable dissipativity property and  $\mathcal{A}_\varepsilon$  enjoys some suitable  $\mathcal{B}_\varepsilon$ -power regularity (by analogy with the  $\mathcal{B}_\varepsilon$ -power compactness notion introduced by Voigt [13]). It is worth emphasizing that we are able to exhibit such a splitting with uniform (dissipativity, regularity) estimates with respect to the diffusion parameter  $\varepsilon \in [0, \varepsilon_0]$  in several weighted Sobolev spaces.

As a consequence of (1.4), we may indeed apply the Krein-Rutman theory developed in [9, 5] and exhibit such a unique positive and unit mass steady state  $G_\varepsilon$ . Of course for the classical and fractional Fokker-Planck equations the steady state is trivially given through an explicit formula (the Krein-Rutman theory is useless in that cases). A next direct consequence of the above spectral and semigroup decomposition (1.4) is that there is a spectral gap in the spectral set  $\Sigma(\Lambda_\varepsilon)$  of the generator  $\Lambda_\varepsilon$ , namely

$$(1.6) \quad \lambda_\varepsilon := \sup\{\Re \xi \in \Sigma(\Lambda_\varepsilon) \setminus \{0\}\} < 0,$$

and then that an exponential trend to equilibrium can be established, namely

$$(1.7) \quad \|S_{\Lambda_\varepsilon}(t)f_0 - G_\varepsilon\|_X \leq C_\varepsilon e^{at} \|f_0 - G_\varepsilon\|_X \quad \forall t \geq 0, \quad \forall \varepsilon \in [0, \varepsilon_0], \quad \forall a > \lambda_\varepsilon,$$

for any unit mass initial datum  $f_0 \in X$ .

Our next step consists in proving that the spectral gap (1.6) and the estimate (1.7) are uniform with respect to  $\varepsilon$ , more precisely, there exists  $\lambda^* < 0$  such that  $\lambda_\varepsilon \leq \lambda^*$  for any  $\varepsilon \in [0, \varepsilon_0]$  and  $C_\varepsilon$  can be chosen independent to  $\varepsilon \in [0, \varepsilon_0]$ .

A first way to get such uniform bounds is just to have in at least one Hilbert space  $E_\varepsilon \subset L^1(\mathbb{R}^d)$  the estimate

$$\forall f \in \mathcal{D}(\mathbb{R}^d), \quad \langle f \rangle = \int_{\mathbb{R}^d} f \, dx = 0, \quad (\Lambda_\varepsilon f, f)_{E_\varepsilon} \leq \lambda^* \|f\|_{E_\varepsilon}^2,$$

and then (1.7) essentially follows from the fact that the splitting (1.5) is true with operators which are uniformly bounded with respect to  $\varepsilon \in [0, \varepsilon_0]$ . It is the strategy we use in the case of the fractional diffusion (Section 3) and the work has already been made in [12] except for the simple but fundamental observation that the fractional diffusion operator is uniformly bounded (and converges to the classical diffusion operator) when it is suitable (re)scaled.

A second way to get the desired uniform estimate is to use a perturbation argument. Observing that, in the discrete cases (Sections 2 and 4),

$$\forall \varepsilon \in [0, \varepsilon_0], \quad \Lambda_\varepsilon - \Lambda_0 = \mathcal{O}(\varepsilon),$$

for a suitable operator norm, we are able to deduce that  $\varepsilon \mapsto \lambda_\varepsilon$  is a continuous function at 0, from which we readily conclude. We use here again that the considered models converge to the classical or the fractional Fokker-Planck equation. In other words, the discrete models can be seen as (singular) perturbations to the limit equations and our analyze takes advantage of such a property in order to capture the asymptotic behaviour of the related spectral objects (spectrum, spectral projector, ...) in order to get the above uniform spectral decomposition. This kind of perturbative method has been introduced in [7], improved in [11]. In Section 4, we again improve it in the sense that we are able to relax the assumptions needed to use such an argument, some of the assumptions are only required to be satisfied on the limit operator ( $\varepsilon = 0$ ).

### 1.3. Comments and possible extensions.

*Motivations.* The main motivation of the present work is rather theoretical and methodological. Spectral gap and semigroup estimates in large Lebesgue spaces have been established both for Boltzmann like equations and Fokker-Planck like equations in a series of recent papers [10, 7, 4, 9, 2, 1, 12, 6, 8]. The proofs are based on a splitting of the generator method as here and previously explained, but the appropriate splitting are rather different for the two kinds of models. The operator  $\mathcal{A}_\varepsilon$  is a multiplication (0-order) operator for a Fokker-Planck equation while it is an integral (-1-order) operator for a Boltzmann equation. More importantly, the fundamental and necessary regularizing effect is given by the action of the semigroup  $S_{\mathcal{B}_\varepsilon}$  for the Fokker-Planck equation while it is given by the action of the operator  $\mathcal{A}_\varepsilon$  for the Boltzmann equation. Let us underline here that in Section 4, we exhibit a new splitting for fractional Fokker-Planck like operators (different from the one introduced in [12]) in the spirit of Boltzmann like operators (the operator  $\mathcal{A}_\varepsilon$  is an integral operator whereas it was a multiplication operator in [12] and in Section 3). Our purpose is precisely to show that all these equations can be handled in the same framework, by exhibiting a suitable and compatible splitting (1.5) which does not blow up and such that the time indexed family of operators  $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$  (or some iterated convolution products of that one) have a good regularizing property which is uniform in the singular limit  $\varepsilon \rightarrow 0$ .

*Probability interpretation.* The discrete and fractional Fokker-Planck equations are the evolution equations satisfied by the law of the stochastic process which is solution to the SDE

$$dX_t = -X_t dt - d\mathcal{L}_t^\varepsilon,$$

where  $\mathcal{L}_t^\varepsilon$  is the Levy (jump) process associated to  $k_\varepsilon/\varepsilon^2$  or  $c_\varepsilon/|z|^{d+2-\varepsilon}$ . For two trajectories  $X_t$  and  $Y_t$  to the above SDE associated to some initial datum  $X_0$  and  $Y_0$ , and  $p \in [1, 2)$ , we have

$$d|X_t - Y_t|^p = -p|X_t - Y_t|^{p-1} dt,$$

from which we deduce

$$\mathbf{E}(|X_t - Y_t|^p) \leq e^{-pt} \mathbf{E}(|X_0 - Y_0|^p), \quad \forall t \geq 0.$$

Denoting by  $f_\varepsilon(t)$  the law of  $X_t$  and  $G_\varepsilon$  the law of the stable process  $Y_t$ , we classically deduce the Wasserstein distance estimate

$$(1.8) \quad W_p(f_\varepsilon(t), G_\varepsilon) \leq e^{-t} W_p(f_0, G_\varepsilon), \quad \forall t \geq 0.$$

Estimate (1.8) has to be compared with (1.7). While the proof of (1.8) is just straightforward, the proof of (1.7) is not. In particular, for  $p = 1$ , the Kantorovich-Rubinstein Theorem says that (1.8) is equivalent to the estimate

$$(1.9) \quad \|f_\varepsilon(t) - G_\varepsilon\|_{(W^{1,\infty}(\mathbb{R}^d))'} \leq e^{-t} \|f_0 - G_\varepsilon\|_{(W^{1,\infty}(\mathbb{R}^d))'}, \quad \forall t \geq 0.$$

Estimates (1.8) and (1.9) have to be compared with (1.7). Proceeding in a similar way as in [9, 6] it is likely that the spectral gap estimate (1.9) can be extended (by “shrinkage of the space”) to a weighted Lebesgue space framework and then to get the estimate in Theorem 1.1 for any  $a \in (-1, 0)$ .

*Trotter-Kato.* From the Trotter-Kato formula

$$S_{\Lambda_\varepsilon} - S_{\Lambda_0} = S_{\Lambda_\varepsilon} * (\Lambda_\varepsilon - \Lambda_0) S_{\Lambda_0}$$

and the two observations

$$D(\Lambda_0^{1/4}) \subset D(\Lambda_\varepsilon) \subset D(\Lambda_0), \quad \|\Lambda_\varepsilon - \Lambda_0\|_{D(\Lambda_0^3) \rightarrow X} = \mathcal{O}(\varepsilon),$$

we should deduce

$$\|S_{\Lambda_\varepsilon} - S_{\Lambda_0}\|_{D(\Lambda_0^2) \rightarrow X} = \mathcal{O}(\varepsilon).$$

We believe that these arguments can be made rigorous and then that the same analysis we have performed here should make possible to improve the above estimate into

$$\sup_{t \geq 0} \|S_{\Lambda_\varepsilon}(t) - S_{\Lambda_0}(t)\|_{\mathcal{B}(X)} e^{-at} = \mathcal{O}(\varepsilon).$$

*Singular kernel and other confinement term.* We also believe that a similar analysis can be handle with more singular kernels than the ones considered here, the typical example should be  $k(z) = (\delta_{-1} + \delta_1)/2$  in dimension  $d = 1$ , and for confinement term different from the harmonic confinement considered here, including other forces or discrete confinement term. In order to perform such an analysis one could use some trick developed in [9] in order to handle the equal mitosis (which uses one more iteration of the convolution product of the time indexed family of operators  $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ ).

*Linearized and nonlinear equations.* We also believe that a similar analysis can be adapted to nonlinear equations. The typical example we have in mind is the Landau grazing collision limit of the Boltzmann equation. One can then expect to get an exponential trend of solutions to its associated Maxwellian equilibrium which is uniform with respect to the considered model (Boltzmann equation with and without Grad's cutoff and Landau equation).

*Kinetic like models.* A more challenging issue would be to extend the uniform asymptotic analysis to the Langevin SDE or the kinetic Fokker-Planck equation by using some idea developed in [1] which make possible to connect (from a spectral analysis point of view) the parabolic-parabolic Keller-Segel equation to the parabolic-elliptic Keller-Segel equation. The next step should be to apply the theory to the Navier-Stokes diffusion limit of the (in)elastic Boltzmann equation. These more technical problems will be investigated in next works.

**1.4. Outline of the paper.** Let us describe the plan of the paper. In each section, we treat a family of equations in a uniform framework, from a spectral analysis viewpoint with a semigroup approach. In Section 2, we deal with the discrete and classical Fokker-Planck equations. Section 3 is dedicated to the analysis of the fractional and classical Fokker-Planck equations. Finally, Section 4 is devoted to the study of the discrete and fractional Fokker-Planck equations.

**1.5. Notations.** For a (measurable) moment function  $m : \mathbb{R}^d \rightarrow \mathbb{R}_+$ , we define the norms

$$\|f\|_{L^p(m)} := \|f m\|_{L^p(\mathbb{R}^d)}, \quad \|f\|_{W^{k,p}(m)}^p := \sum_{i=0}^k \|\partial^i f\|_{L^p(m)}^p, \quad k \geq 1,$$

and the associated weighted Lebesgue and Sobolev spaces  $L^p(m)$  and  $W^{k,p}(m)$ , we denote  $H^k(m) = W^{k,2}(m)$  for  $k \geq 1$ . We also use the shorthand  $L_r^p$  and  $W_r^{1,p}$  for the Lebesgue and Sobolev spaces  $L^p(m)$  and  $W^{1,p}(m)$  when  $m(x) = \langle x \rangle^r$ ,  $\langle x \rangle := (1 + |x|^2)^{1/2}$ .

From now on, we fix a polynomial weight  $m(x) := \langle x \rangle^q$  with  $q > 0$ , the range of admissible  $q$  will be specified in each section.

Throughout this paper, we will use the same notation  $C$  for positive constants that may change from line to line. Moreover, the notation  $A \approx B$  shall mean that there exist two positive constants  $C_1, C_2$  such that  $C_1 A \leq B \leq C_2 A$ .

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## 2. FROM DISCRETE TO CLASSICAL FOKKER-PLANCK EQUATION

In the sequel, we consider a kernel  $k \in W^{2,1}(\mathbb{R}^d) \cap L_{2q+3}^1$  satisfying the centered condition

$$(2.1) \quad \int_{\mathbb{R}^d} k(x) \begin{pmatrix} 1 \\ x \\ x \otimes x \end{pmatrix} dx = \begin{pmatrix} 1 \\ 0 \\ 2I_d \end{pmatrix},$$

as well as the positivity condition: there exist  $\kappa, r > 0$  such that

$$(2.2) \quad k \geq \kappa \mathbf{1}_{B(0,r)}.$$

Let us notice that assumptions made on  $k$  imply

$$(2.3) \quad \widehat{k^2}(\xi) \leq C \frac{1 - \widehat{k}(\xi)}{|\xi|^2}, \quad \forall \xi \in \mathbb{R}^d$$

for some constant  $C > 0$ .

We define  $k_\varepsilon(x) := 1/\varepsilon^d k(x/\varepsilon)$ ,  $x \in \mathbb{R}^d$  for  $\varepsilon > 0$ , and we consider the discrete and classical Fokker-Planck equations

$$(2.4) \quad \begin{cases} \partial_t f = \frac{1}{\varepsilon^2} (k_\varepsilon * f - f) + \operatorname{div}_x(xf) =: \Lambda_\varepsilon f, & \varepsilon > 0 \\ \partial_t f = \Delta_x f + \operatorname{div}_x(xf) =: \Lambda_0 f. \end{cases}$$

The main result of the section reads as follows.

**Theorem 2.1.** *Assume  $q > d/2 + 5$  and consider a kernel  $k \in W^{2,1}(\mathbb{R}^d) \cap L^1_{2q+3}$  which satisfies (2.1) and (2.2).*

(1) *For any  $\varepsilon > 0$ , there exists a positive and unit mass normalized steady state  $G_\varepsilon \in L^1_q(\mathbb{R}^d)$  to the discrete Fokker-Planck equation (2.4).*

(2) *There exist an explicit constant  $a_0 < 0$  and a constant  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ , the semigroup  $S_{\Lambda_\varepsilon}(t)$  associated to the discrete Fokker-Planck equation (2.4) satisfies: for any  $f \in L^1_q$  and any  $a > a_0$ ,*

$$\|S_{\Lambda_\varepsilon}(t)f - G_\varepsilon\langle f \rangle\|_{L^1_q} \leq C_a e^{at} \|f - G_\varepsilon\langle f \rangle\|_{L^1_q}, \quad \forall t \geq 0$$

for some explicit constant  $C_a \geq 1$ . In particular, the spectrum  $\Sigma(\Lambda_\varepsilon)$  of  $\Lambda_\varepsilon$  satisfies the separation property  $\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_0} = \{0\}$  in  $L^1_q$ .

The method of the proof consists in introducing a suitable splitting  $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ , in establishing some dissipativity and regularity properties on  $\mathcal{B}_\varepsilon$  and  $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$  and finally to apply the Krein-Rutman theory revisited in [9, 5] and the perturbation theory developed in [7, 11, 5].

**2.1. Splitting of  $\Lambda_\varepsilon$ .** We recall that  $\chi \in \mathcal{D}(\mathbb{R}^d)$  is centered and satisfies  $\mathbb{1}_{B(0,1)} \leq \chi \leq \mathbb{1}_{B(0,2)}$ , we define  $\chi_R$  by  $\chi_R(x) := \chi(x/R)$  for  $R > 0$  and we denote  $\chi_R^c := 1 - \chi_R$ .

We define the splitting of  $\Lambda_\varepsilon$  for  $\varepsilon \geq 0$  as follows.

*Splitting of  $\Lambda_\varepsilon$  for  $\varepsilon > 0$ .* We define

$$\mathcal{A}_\varepsilon f := M \chi_R (k_\varepsilon * f)$$

and

$$\mathcal{B}_\varepsilon f := \left( \frac{1}{\varepsilon^2} - M \right) (k_\varepsilon * f - f) + M \chi_R^c (k_\varepsilon * f - f) + \operatorname{div}(xf) - M \chi_R f,$$

for some constants  $M, R$  to be chosen later. One can notice that  $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ .

*Splitting of  $\Lambda_0$ .* We define  $\mathcal{A}_0 f := M \chi_R f$  and  $\mathcal{B}_0 f := \Lambda_0 f - M \chi_R f$  so that  $\Lambda_0 = \mathcal{A}_0 + \mathcal{B}_0$ .

**2.2. Convergences  $\mathcal{A}_\varepsilon \rightarrow \mathcal{A}_0$  and  $\mathcal{B}_\varepsilon \rightarrow \mathcal{B}_0$ .**

**Lemma 2.2.** *Consider  $s \in \mathbb{N}$ . The following convergences hold:*

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathcal{B}(H^{s+1}(m), H^s(m))} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{and} \quad \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathcal{B}(H^{s+3}(m), H^s(m))} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

*Proof. Step 1.* We first deal with  $\mathcal{A}_\varepsilon$  in the case  $s = 0$ :

$$\|\mathcal{A}_\varepsilon f - \mathcal{A}_0 f\|_{L^2(m)} = \|M \chi_R (k_\varepsilon * f - f)\|_{L^2} \leq C \|k_\varepsilon * f - f\|_{L^2} \leq \eta(\varepsilon) \|f\|_{H^1}, \quad \eta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Concerning the first derivative, writing that

$$\partial_x(\mathcal{A}_\varepsilon f - \mathcal{A}_0 f) = M (\partial_x \chi_R) (k_\varepsilon * f - f) + M \chi_R (k_\varepsilon * \partial_x f - \partial_x f)$$

and using that  $\partial_x \chi_R$  is uniformly bounded as well as  $\chi_R$ , we obtain the result. We omit the details of the proof for higher order derivatives.

*Step 2.* In order to prove the second part of the result, we are going to prove that

$$\|\Lambda_\varepsilon - \Lambda_0\|_{\mathcal{B}(H^{s+3}(m), H^s(m))} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

First, let us remark that  $(\Lambda_\varepsilon - \Lambda_0)f = 1/\varepsilon^2(k_\varepsilon * f - f) - \Delta f$ . Using (2.1), we have

$$\Lambda_\varepsilon f(x) - \Lambda_0 f(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} k_\varepsilon(x-y)(f(y) - f(x)) dy - \Delta f(x).$$

We now write a Taylor development of  $f$  between  $x$  and  $y$ :

$$\begin{aligned} f(y) - f(x) &= (y-x) \cdot \nabla f(x) + \frac{1}{2} D^2 f(x)(y-x, y-x) \\ &\quad + \frac{1}{2} \int_0^1 (1-s)^2 D^3 f(x+s(y-x))(y-x, y-x, y-x) ds, \end{aligned}$$

the first involving the gradient of  $f$  will give no contribution using (2.1). Performing a change of variables, we obtain:

$$\begin{aligned} &\Lambda_\varepsilon f(x) - \Lambda_0 f(x) \\ &= \int_{\mathbb{R}^d} k(z) \left( \frac{1}{2} D^2 f(x)(z, z) + \frac{\varepsilon}{2} \int_0^1 (1-s)^2 D^3 f(x+s\varepsilon z)(z, z, z) ds \right) dz - \Delta f(x). \end{aligned}$$

Using that

$$(2.5) \quad D^2 f(x)(z, z) = \sum_{i=1}^n z_i^2 \frac{\partial^2 f(x)}{\partial x_i^2} + \sum_{i \neq j} z_i z_j \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

and (2.1), we deduce that the first term on the right-hand side of (2.5) will be canceled by  $\Delta f(x)$  and that the second one vanishes. It thus implies

$$\Lambda_\varepsilon f(x) - \Lambda_0 f(x) = \frac{\varepsilon}{2} \int_{\mathbb{R}^d} k(z) \int_0^1 (1-s)^2 D^3 f(x+s\varepsilon z)(z, z, z) ds dz.$$

Consequently, using (2.1), Jensen inequality with the probability measure  $\mathbf{1}_{[0,1]}(s) ds$  and performing a change of variable, we get

$$\begin{aligned} &\|\Lambda_\varepsilon - \Lambda_0\|_{L^2(m)} \\ &\leq C \varepsilon \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k(z) \int_0^1 (1-s)^2 D^3 f(x+s\varepsilon z)(z, z, z) ds dz \right)^2 m^2(x) dx \right)^{1/2} \\ &\leq C \varepsilon \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(z) |z|^3 \int_0^1 |D^3 f(x+s\varepsilon z)|^2 m^2(x+s\varepsilon z) m^2(s\varepsilon z) ds dz dx \right)^{1/2} \\ &\leq C \varepsilon \left( \int_{\mathbb{R}^d} |D^3 f(x)|^2 m^2(x) dx \right)^{1/2} \left( \int_{\mathbb{R}^d} k(z) |z|^3 m^2(z) dz \right)^{1/2} \\ &\leq C \varepsilon \|f\|_{H^3(m)} \xrightarrow{\varepsilon \rightarrow 0} 0, \end{aligned}$$

where we have used  $k \in L^1_{2q+3}$  and this concludes the proof of the second part in the case  $s = 0$ . Since the operator  $\partial_x$  commutes with  $\Lambda_\varepsilon - \Lambda_0$ , there is no need here to write the proof for  $s > 0$ .  $\square$

### 2.3. Uniform boundedness of $\mathcal{A}_\varepsilon$ .

**Lemma 2.3.** *For any  $p \in [1, \infty]$ ,  $s \geq 0$  and any weight function  $\nu \geq 1$ , the operator  $\mathcal{A}_\varepsilon$  is bounded from  $W^{s,p}$  into  $W^{s,p}(\nu)$  with a norm which does not depend on  $\varepsilon$ .*

*Proof.* For any  $f \in L^p(\nu)$ , we have

$$\|\mathcal{A}_\varepsilon f\|_{L^p(\nu)} \leq C \|k_\varepsilon * f\|_{L^p} \leq C \|k_\varepsilon\|_{L^1} \|f\|_{L^p}.$$

thanks to the Young inequality. We conclude that  $\mathcal{A}_\varepsilon$  is bounded from  $L^p$  into  $L^p(\nu)$  by observing that  $\|k_\varepsilon\|_{L^1} = \|k\|_{L^1} = 1$ . The proof for the case  $s > 0$  is similar and it is thus skipped.  $\square$



#### 2.4. Uniform dissipativity properties of $\mathcal{B}_\varepsilon$ .

**Lemma 2.4.** *We suppose that  $q > d/2$ . For any  $a > d/2 - q$ , there exist  $\varepsilon_0 > 0$ ,  $M \geq 0$  and  $R \geq 0$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ ,  $\mathcal{B}_\varepsilon - a$  is dissipative in  $L^2(m)$ .*

*Proof.* We consider  $a > d/2 - q$ . We are going to estimate the integral  $\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f) f m^2$  for  $\varepsilon > 0$  which can be split into several pieces:

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f) f m^2 &= \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d} (k_\varepsilon * f - f) f m^2 + \int_{\mathbb{R}^d} M \chi_R^c (k_\varepsilon * f - f) f m^2 \\ &\quad + \int_{\mathbb{R}^d} \operatorname{div}(xf) f m^2 - \int_{\mathbb{R}^d} M \chi_R f^2 m^2 \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

We fix  $\varepsilon_1 > 0$  such that  $M \leq 1/(2\varepsilon_1^2)$  and consider  $\varepsilon \in (0, \varepsilon_1]$ .

We first deal with  $T_1$  performing a classical computation and using that  $\int_{\mathbb{R}^d} k_\varepsilon = 1$ :

$$\begin{aligned} T_1 &= \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) (f(y) - f(x)) f(x) m^2(x) dy dx \\ &= -\frac{1}{2} \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))^2 k_\varepsilon(x-y) m^2(x) dy dx \\ &\quad + \frac{1}{2} \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} (f^2(y) - f^2(x)) k_\varepsilon(x-y) m^2(x) dy dx \\ &\leq \frac{1}{2} \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} (m^2(y) - m^2(x)) k_\varepsilon(x-y) f^2(x) dy dx \end{aligned}$$

where we have performed a change of variables to get the last inequality. We then write a Taylor development of  $m^2$  between  $x$  and  $y$ :

$$m^2(y) - m^2(x) = (y-x) \cdot \nabla m^2(x) + \frac{1}{2} D^2 m^2(x + \theta(y-x))(y-x, y-x)$$

for some  $\theta \in (0, 1)$ . The term involving the gradient of  $m^2$  will give no contribution because of (2.1) and using that

$$|D^2 m^2(x + \theta(y-x))(y-x, y-x)| \leq C |x-y|^2 \langle x \rangle^{2q-2} \langle x-y \rangle^{2q-2},$$

and that  $k \in L_{2q}^1$ , we obtain

$$\begin{aligned} (2.6) \quad T_1 &\leq C \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) |x-y|^2 \langle x-y \rangle^{2q-2} dy f^2(x) \langle x \rangle^{2q-2} dx \\ &\leq C \left( \frac{1}{\varepsilon^2} - M \right) \varepsilon^2 \int_{\mathbb{R}^d} k(z) |z|^2 \langle z \rangle^{2q-2} dz \int_{\mathbb{R}^d} f^2(x) \langle x \rangle^{2q-2} dx \\ &\leq C \int_{\mathbb{R}^d} f^2(x) \langle x \rangle^{2q-2} dx. \end{aligned}$$

We now treat the second term  $T_2$ :

$$\begin{aligned} T_2 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} M \chi_R^c(x) k_\varepsilon(x-y) f(x) f(y) m^2(x) dy dx - \int_{\mathbb{R}^d} M \chi_R^c(x) f^2(x) m^2(x) dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} M \chi_R^c(x) k_\varepsilon(x-y) f^2(x) m^2(x) dy dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} M \chi_R^c(x) k_\varepsilon(x-y) f^2(y) m^2(x) dy dx - \int_{\mathbb{R}^d} M \chi_R^c(x) f^2(x) m^2(x) dx \\ &=: T_{21} + T_{22} + T_{23}. \end{aligned}$$

To estimate  $T_{21}$ , we use again the fact that  $\int_{\mathbb{R}^d} k_\varepsilon = 1$  to get

$$(2.7) \quad T_{21} \leq \frac{M}{2} \int_{\mathbb{R}^d} \chi_R^c f^2 m^2.$$

Then, to estimate  $T_{22}$ , we first perform a change of variable:

$$T_{22} = \frac{M}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k(z) \chi_R^c(y + \varepsilon z) m^2(y + \varepsilon z) dz f^2(y) dy.$$

Using the mean value theorem, we deduce that there exists  $\theta, \theta' \in (0, 1)$  such that

$$\chi_R^c(y + \varepsilon z) = \chi_R^c(y) + \varepsilon z \cdot \nabla \chi_R^c(y + \theta \varepsilon z), \quad m^2(y + \varepsilon z) = m^2(y) + \varepsilon z \cdot \nabla m^2(y + \theta' \varepsilon z).$$

We then use the fact that  $|\nabla \chi_R^c| \leq C_R$  in  $\mathbb{R}^d$ . It implies that

$$T_{22} \leq \frac{M}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k(z) (\chi_R^c(y) + \varepsilon |z| C_R) (m^2(y) + \varepsilon |z| \cdot |\nabla m^2(y + \theta' \varepsilon z)|) dz f^2(y) dy.$$

Then, because of (2.1) and the fact that  $|\nabla m^2(y + \theta' \varepsilon z)| \leq C \langle y \rangle^{2q-1} \langle z \rangle^{2q-1}$ , since  $k \in L_{2q+3}^1$ , we conclude that

$$(2.8) \quad T_{22} \leq M C_R \kappa_\varepsilon \int_{\mathbb{R}^d} f^2 m^2 + \frac{M}{2} \int_{\mathbb{R}^d} \chi_R^c f^2 m^2, \quad \kappa_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Putting together (2.7), (2.8) and the contribution of the term  $T_{23}$ , it yields

$$(2.9) \quad T_2 \leq M C_R \kappa_\varepsilon \int_{\mathbb{R}^d} f^2 m^2, \quad \kappa_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0.$$

As far as  $T_3$  is concerned, we just perform an integration by parts:

$$(2.10) \quad \begin{aligned} T_3 &= \int_{\mathbb{R}^d} \operatorname{div}(x f) f m^2 \\ &= d \int_{\mathbb{R}^d} f^2 m^2 + \int_{\mathbb{R}^d} x \cdot \nabla f f m^2 \\ &= d \int_{\mathbb{R}^d} f^2 m^2 - \frac{1}{2} \int_{\mathbb{R}^d} f^2 \operatorname{div}(x m^2) \\ &= \int_{\mathbb{R}^d} f^2(x) m^2(x) \left( \frac{d}{2} - \frac{q|x|^2}{\langle x \rangle^2} \right) dx. \end{aligned}$$

The estimates (2.6), (2.9) and (2.10) together give

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{B}_\varepsilon f f m^2 &\leq \int_{\mathbb{R}^d} f^2 m^2 \left( C \langle x \rangle^{-2} + \frac{d}{2} - \frac{q|x|^2}{\langle x \rangle^2} + M C_R \kappa_\varepsilon - M \chi_R \right) \\ &= \int_{\mathbb{R}^d} f^2 m^2 (\psi_R^\varepsilon - M \chi_R), \end{aligned}$$

where we have denoted

$$(2.11) \quad \psi_R^\varepsilon(x) := C \langle x \rangle^{-2} + \frac{d}{2} - \frac{q|x|^2}{\langle x \rangle^2} + M C_R \kappa_\varepsilon \xrightarrow{|x| \rightarrow \infty, \varepsilon \rightarrow 0} d/2 - q.$$

We can thus choose  $M \geq 0$ ,  $R \geq 0$  and  $\varepsilon_0 \leq \varepsilon_1$  such that

$$\forall x \in \mathbb{R}^d, \quad \psi_R^\varepsilon(x) \leq a.$$

As a conclusion, for such a choice of constants, we obtain that for any  $\varepsilon \in (0, \varepsilon_0]$ ,

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon - a) f f m^2 \leq 0$$

and we refer to [4, 6] for the proof in the case  $\varepsilon = 0$ .  $\square$

**Lemma 2.5.** *For any  $a > -q$ , there exist  $\varepsilon_0 > 0$ ,  $M \geq 0$  and  $R \geq 0$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ ,  $\mathcal{B}_\varepsilon - a$  is dissipative in  $L^1(m)$ .*

*Proof.* We estimate the integral

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{B}_\varepsilon f (\text{sign} f) m &= \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d} (k_\varepsilon * f - f) (\text{sign} f) m + \int_{\mathbb{R}^d} M \chi_R^c (k_\varepsilon * f - f) (\text{sign} f) m \\ &\quad + \int_{\mathbb{R}^d} \text{div}(xf) (\text{sign} f) m - \int_{\mathbb{R}^d} M \chi_R |f| m \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

We omit the details of the proof which is very similar to the one of Lemma 2.4. We have

$$T_1 \leq C \int_{\mathbb{R}^d} |f|(x) \langle x \rangle^{q-2} dx, \quad T_2 \leq M C_R \kappa_\varepsilon \int_{\mathbb{R}^d} |f| m \quad \text{and} \quad T_3 = - \int_{\mathbb{R}^d} |f| m \frac{x \cdot \nabla m}{m}.$$

This implies that

$$\int_{\mathbb{R}^d} \mathcal{B}_\varepsilon f (\text{sign} f) m \leq \int_{\mathbb{R}^d} |f| m \left( C \langle x \rangle^{-2} - \frac{x \cdot \nabla m}{m} + M C_R \kappa_\varepsilon - M \chi_R \right)$$

and we conclude as in the proof of Lemma 2.4. We refer to [4, 6] for the proof in the case  $\varepsilon = 0$ .  $\square$

**Lemma 2.6.** *Let  $s \in \mathbb{N}$  and  $q > d/2 + s$ . For any  $a > d/2 - q + s$ , there exist  $\varepsilon_0 > 0$ ,  $M \geq 0$  and  $R \geq 0$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ ,  $\mathcal{B}_\varepsilon - a$  is hypodissipative in  $H^s(m)$ .*

*Proof.* The case  $s = 0$  is nothing but Lemma 2.4. We now deal with the case  $s = 1$ . We consider  $f_t$  a solution to

$$\partial_t f_t = \mathcal{B}_\varepsilon f_t.$$

From the previous lemma, we already know that

$$(2.12) \quad \frac{1}{2} \frac{d}{dt} \|f_t\|_{L^2(m)}^2 \leq \int_{\mathbb{R}^d} f_t^2 m^2 (\psi_R^\varepsilon - M \chi_R).$$

We now want to compute the evolution of the derivative of  $f_t$ :

$$\partial_t \partial_x f_t = \mathcal{B}(\partial_x f_t) + M \partial_x (\chi_R^c) (k_\varepsilon * f_t - f_t) + \partial_x f_t,$$

which in turn implies that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_x f_t\|_{L^2(m)}^2 &= \int_{\mathbb{R}^d} (\partial_x f_t) \partial_t (\partial_x f_t) m^2 \\ &= \int_{\mathbb{R}^d} (\partial_x f_t) \mathcal{B}(\partial_x f_t) m^2 + \int_{\mathbb{R}^d} M \partial_x (\chi_R^c) (k_\varepsilon * f_t) (\partial_x f_t) m^2 \\ &\quad - \int_{\mathbb{R}^d} M \partial_x (\chi_R^c) f_t (\partial_x f_t) m^2 + \int_{\mathbb{R}^d} (\partial_x f_t)^2 m^2 \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Concerning  $T_1$ , using the proof of Lemma 2.4, we obtain

$$(2.13) \quad T_1 \leq \int_{\mathbb{R}^d} (\partial_x f_t)^2 m^2 (\psi_R^\varepsilon - M \chi_R).$$

Then, to deal with  $T_2$ , we first notice that using Jensen inequality and (2.1), we have

$$\begin{aligned} \|k_\varepsilon * f\|_{L^2(m)}^2 &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k_\varepsilon(x-y) f(y) dy \right)^2 m^2(x) dx \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) m^2(x) dx f^2(y) dy \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} k(z) m^2(y + \varepsilon z) dz f^2(y) dy \\ &\leq C \int_{\mathbb{R}^d} k(z) m^2(z) dz \int_{\mathbb{R}^d} f^2 m^2. \end{aligned}$$

We thus obtain using that  $k \in L^1_{2q+3}$ :

$$\|k_\varepsilon * f\|_{L^2(m)} \leq C \|f\|_{L^2(m)}.$$

The term  $T_2$  is then treated using Cauchy-Schwarz inequality, Young inequality and the fact that  $|\partial_x(\chi_R^c)|$  is bounded by a constant depending only on  $R$ :

$$\begin{aligned} (2.14) \quad T_2 &\leq M C_R \|k_\varepsilon *_x f_t\|_{L^2(m)} \|\partial_x f_t\|_{L^2(m)} \\ &\leq M C_R \|f_t\|_{L^2(m)} \|\partial_x f_t\|_{L^2(m)} \\ &\leq M C_R K(\zeta) \|f_t\|_{L^2(m)}^2 + M C_R \zeta \|\partial_x f_t\|_{L^2(m)}^2 \end{aligned}$$

for any  $\zeta > 0$  as small as we want.

The term  $T_3$  is handled using an integration by parts and with the fact that  $|\partial_x^2(\chi_R^c)|$  is bounded with a constant which only depends on  $R$ :

$$(2.15) \quad T_3 = \frac{M}{2} \int_{\mathbb{R}^d} \partial_x^2(\chi_R^c) f_t^2 m^2 + \frac{M}{2} \int_{\mathbb{R}^d} \partial_x(\chi_R^c) f_t^2 \partial_x(m^2) \leq M C_R \|f_t\|_{L^2(m)}^2.$$

Finally, we have

$$(2.16) \quad T_4 = \|\partial_x f_t\|_{L^2(m)}^2.$$

Combining estimates (2.13), (2.14), (2.15) and (2.16), we deduce

$$(2.17) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_x f_t\|_{L^2(m)}^2 &\leq C_{R,M,\zeta} \int_{\mathbb{R}^d} f_t^2 m^2 \\ &\quad + \int_{\mathbb{R}^d} (\partial_x f_t)^2 m^2 (\psi_R^\varepsilon + M C_R \zeta + 1 - M \chi_R). \end{aligned}$$

To conclude the proof in the case  $s = 1$ , we introduce the norm

$$\|f\|_{H^1(m)}^2 := \|f\|_{L^2(m)}^2 + \eta \|\partial_x f\|_{L^2(m)}^2, \quad \eta > 0.$$

Combining (2.12) and (2.17), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f_t\|_{H^1(m)}^2 &\leq \int_{\mathbb{R}^d} f_t^2 m^2 (\psi_R^\varepsilon + \eta C_{R,M,\zeta} - M \chi_R) \\ &\quad + \eta \int_{\mathbb{R}^d} (\partial_x f_t)^2 m^2 (\psi_R^\varepsilon + M C_R \zeta + 1 - M \chi_R). \end{aligned}$$

Using the same strategy as in the proof of Lemma 2.4, if  $a > d/2 - q + 1$ , we can choose  $M, R$  large enough and  $\zeta, \varepsilon_0, \eta$  small enough such that we have on  $\mathbb{R}^d$

$$\psi_R^\varepsilon + \eta C_{R,M,\zeta} - M \chi_R \leq a \quad \text{and} \quad \psi_R^\varepsilon + M C_R \zeta + 1 - M \chi_R \leq a$$

for any  $\varepsilon \in (0, \varepsilon_0]$ , which implies that

$$\frac{1}{2} \frac{d}{dt} \|f_t\|_{H^1(m)}^2 \leq a \|f_t\|_{H^1(m)}^2.$$

The higher order derivatives are treated with the same method introducing the norm

$$(2.18) \quad \|f\|_{H^s(m)}^2 := \sum_{j=0}^s \eta^j \|\partial_x^j f\|_{L^2(m)}^2.$$

□

2.5. **Uniform regularization properties of  $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)$ .** We introduce the notation

$$(2.19) \quad I_\varepsilon(f) := \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))^2 k_\varepsilon(x - y) dx dy.$$

**Lemma 2.7.** *There exists a constant  $K > 0$  such that for any  $\varepsilon > 0$ , the following estimate holds:*

$$(2.20) \quad \|k_\varepsilon * f\|_{\dot{H}^1}^2 \leq K I_\varepsilon(f).$$

*Proof.* First, performing a change of variable, one can notice that

$$\forall \xi \in \mathbb{R}^d, \quad \widehat{k_\varepsilon}(\xi) = \widehat{k}(\varepsilon \xi).$$

Using that  $\int_{\mathbb{R}^d} k_\varepsilon = 1$ , we have

$$\begin{aligned} I_\varepsilon(f) &= \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f^2(x) k_\varepsilon(x - y) dx dy + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f^2(y) k_\varepsilon(x - y) dx dy \\ &\quad - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)f(y) k_\varepsilon(x - y) dx dy \\ &= \frac{1}{\varepsilon^2} \left( \int_{\mathbb{R}^d} f^2 - \int_{\mathbb{R}^d} (k_\varepsilon * f) f \right). \end{aligned}$$

As a consequence, using Plancherel formula, we get

$$I_\varepsilon(f) = \frac{1}{\varepsilon^2} \left( \int_{\mathbb{R}^d} \widehat{f}^2 - \int_{\mathbb{R}^d} \widehat{k_\varepsilon} \widehat{f}^2 \right) = \int_{\mathbb{R}^d} \widehat{f}^2(\xi) \frac{1 - \widehat{k}(\varepsilon \xi)}{\varepsilon^2} d\xi.$$

Then, we again use Plancherel formula to obtain

$$\begin{aligned} \|k_\varepsilon * f\|_{\dot{H}^1}^2 &= \|\partial_x(k_\varepsilon * f)\|_{L^2}^2 = \|\mathcal{F}(\partial_x(k_\varepsilon * f))\|_{L^2}^2 \\ &= \int_{\mathbb{R}^d} |\xi|^2 \widehat{k}(\varepsilon \xi)^2 \widehat{f}^2. \end{aligned}$$

We conclude to (2.20) by using (2.3).  $\square$

We now introduce the following notation  $\lambda := 1/(2K) > 0$ . Before going into the proof of regularization lemmas, we recall a result from [5] which is going to be useful.

**Lemma 2.8.** *Consider two Banach spaces  $X, Y$  and a function  $u : \mathbb{R}^+ \rightarrow \mathcal{B}(X) + \mathcal{B}(Y)$ . For  $a_0, b \in \mathbb{R}$ ,  $a_0 < b$ , we assume that*

- (1)  $ue^{-at} \in L^1(0, \infty; \mathcal{B}(X) \cap \mathcal{B}(Y))$  for any  $a > a_0$ ;
- (2)  $ue^{-bt} \in L^1(0, \infty; \mathcal{B}(X, Y))$ .

*Then, for any  $a > a_0$ , there exists  $n \in \mathbb{N}$  such that  $u^{(*n)}e^{-at} \in L^1(0, \infty; \mathcal{B}(X, Y))$ , with explicit constant uniquely depending on the two assumed bounds (1) and (2).*

**Lemma 2.9.** *Consider  $s_1 < s_2 \in \mathbb{N}$  and  $q > d/2 + s_2$ . Let  $M, R$  and  $\varepsilon_0$  so that the conclusion of Lemma 2.6 holds in both spaces  $H^{s_1}(m)$  and  $H^{s_2}(m)$ . Then, for any  $a \in (\max\{d/2 - q + s_2, -\lambda\}, 0)$ , there exists  $n \in \mathbb{N}$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ , we have the following estimate*

$$\int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{H^{s_1}(m) \rightarrow H^{s_2}(m)} e^{-at} dt \leq C_a$$

for some constant  $C_a > 0$ .

*Proof.* We first give the proof for the case  $(s_1, s_2) = (0, 1)$ . We consider  $a \in (\max\{d/2 - q + 1, -\lambda\}, 0)$ ,  $b \in (\max\{d/2 - q + 1, -\lambda\}, a)$  and  $f_t := S_{\mathcal{B}_\varepsilon}(t)f$ , i.e. that satisfies

$$\partial_t f_t = \mathcal{B}_\varepsilon f_t, \quad f_0 = f.$$

From the proof of Lemma 2.6, for any  $\varepsilon \in (0, \varepsilon_0]$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f_t\|_{L^2(m)}^2 &\leq -\frac{1}{2} \left( \frac{1}{\varepsilon^2} - M \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))^2 k_\varepsilon(x-y) m^2(x) dy dx + a \|f_t\|_{L^2(m)}^2 \\ &\leq -\frac{1}{4\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(y) - f(x))^2 k_\varepsilon(x-y) dy dx + a \|f_t\|_{L^2(m)}^2 \\ &\leq -\frac{1}{2} I_\varepsilon(f_t) + a \|f_t\|_{L^2(m)}^2 \end{aligned}$$

where we have used that  $M \leq 1/(2\varepsilon^2)$  for any  $\varepsilon \in (0, \varepsilon_0]$ . Using Lemma 2.7, we obtain

$$\begin{aligned} \frac{d}{dt} \|f_t\|_{L^2(m)}^2 &\leq -2\lambda \|k_\varepsilon * f_t\|_{\dot{H}^1}^2 + 2a \|f_t\|_{L^2(m)}^2 \\ &\leq 2a \|k_\varepsilon * f_t\|_{\dot{H}^1}^2 + 2a \|f_t\|_{L^2(m)}^2. \end{aligned}$$

Multiplying this inequality by  $e^{-2at}$ , it implies that

$$\frac{d}{dt} \left( \|f_t\|_{L^2(m)}^2 e^{-2at} \right) \leq 2a \|k_\varepsilon * f_t\|_{\dot{H}^1}^2 e^{-2at}$$

and thus, integrating in time

$$\|f_t\|_{L^2(m)}^2 e^{-2at} - 2a \int_0^t \|k_\varepsilon * f_s\|_{\dot{H}^1}^2 e^{-2as} ds \leq \|f\|_{L^2(m)}^2.$$

In particular, we obtain

$$(2.21) \quad \int_0^t \|k_\varepsilon * f_s\|_{\dot{H}^1}^2 e^{-2as} ds \leq -\frac{1}{2a} \|f\|_{L^2(m)}^2, \quad \forall t \geq 0.$$

We now want to estimate

$$\begin{aligned} &\int_0^t \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|_{H^1(m)}^2 e^{-2as} ds = \int_0^t \|\mathcal{A}_\varepsilon f_s\|_{H^1(m)}^2 e^{-2as} ds \\ &= \int_0^t \|\mathcal{A}_\varepsilon f_s\|_{L^2(m)}^2 e^{-2as} ds + \int_0^t \|\partial_x (\mathcal{A}_\varepsilon f_s)\|_{L^2(m)}^2 e^{-2as} ds \\ &\leq \int_0^t \|\mathcal{A}_\varepsilon f_s\|_{L^2(m)}^2 e^{-2as} ds + \int_0^t \|M \partial_x (\chi_R) k_\varepsilon * f_s\|_{L^2(m)}^2 e^{-2as} ds \\ &\quad + \int_0^t \|M \chi_R \partial_x (k_\varepsilon * f_s)\|_{L^2(m)}^2 e^{-2as} ds \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Using dissipativity properties of  $\mathcal{B}_\varepsilon$  and boundedness of  $\mathcal{A}_\varepsilon$ , we get

$$I_1 \leq \int_0^t e^{2bs} e^{-2as} ds \|f\|_{L^2(m)}^2 \leq C \|f\|_{L^2(m)}^2.$$

We deal with  $I_2$  using the fact that  $M \partial_x (\chi_R)$  is compactly supported, Young inequality and dissipativity properties of  $\mathcal{B}_\varepsilon$ :

$$I_2 \leq C \int_0^t \|k_\varepsilon * f_s\|_{L^2}^2 ds \leq C \int_0^t \|f_s\|_{L^2}^2 ds \leq C \int_0^t e^{2bs} ds \|f\|_{L^2(m)}^2 \leq C \|f\|_{L^2(m)}^2.$$

Finally, for  $I_3$ , we use (2.21) to obtain

$$I_3 \leq \int_0^t \|k_\varepsilon * f_s\|_{\dot{H}^1}^2 e^{-2as} ds \leq C \|f\|_{L^2(m)}^2.$$

Passing to the limit  $t \rightarrow \infty$ , we obtain

$$\int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s) f\|_{H^1(m)}^2 e^{-2as} ds \leq C \|f\|_{L^2(m)}^2.$$

Consequently, using Cauchy-Schwarz inequality, we have

$$(2.22) \quad \left( \int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^1(m)} e^{-as/2} ds \right)^2 = \left( \int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^1(m)} e^{-as} e^{as/2} ds \right)^2 \\ \leq \int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^1(m)}^2 e^{-2as} ds \int_0^\infty e^{as} ds \\ \leq C \|f\|_{L^2(m)}^2.$$

To conclude the proof in the case  $(s_1, s_2) = (0, 1)$ , we use Lemma 2.8 with  $X = L^1(m)$ ,  $Y = L^2(m)$  and  $u(t) = \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)$ . Lemmas 2.3 and 2.4-2.6 allow us to check that assumption (1) is satisfied and assumption (2) comes from (2.22).

Using the same strategy, we can easily obtain that

$$\int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^2(m)}^2 e^{-2as} ds \leq C \|f\|_{H^1(m)}^2.$$

We can thus deduce that

$$\int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*2)}(t)f\|_{H^2(m)}^2 e^{-2at} dt \\ \leq \int_0^\infty \int_0^t \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t-s)\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^2(m)}^2 e^{-2a(t-s)} e^{-2as} ds dt \\ \leq \int_0^\infty \int_s^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t-s)\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^2(m)}^2 e^{-2a(t-s)} dt e^{-2as} ds \\ \leq \int_0^\infty \int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^2(m)}^2 e^{-2at} dt e^{-2as} ds \\ \leq C \int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(s)f\|_{H^1(m)}^2 e^{-2as} ds \\ \leq C \|f\|_{L^2(m)}^2.$$

Reiterating the process, we can conclude the proof of the lemma. We refer to [4, 6] for the proof in the case  $\varepsilon = 0$ . □

**Lemma 2.10.** *Consider  $q > d/2$  and  $M, R, \varepsilon_0$  so that the conclusions of Lemmas 2.4 and 2.5 hold. Then, for any  $a \in (-q, 0)$ , there exists  $n \in \mathbb{N}$  such that the following estimate holds for any  $\varepsilon \in [0, \varepsilon_0]$ :*

$$\forall t \geq 0, \quad \int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{\mathcal{B}(L^1(m), L^2(m))} e^{-at} dt \leq C_a,$$

for some constant  $C_a > 0$ .

*Proof.* We first introduce the formal dual operators of  $\mathcal{A}_\varepsilon$  and  $\mathcal{B}_\varepsilon$ :

$$\mathcal{A}_\varepsilon^* \phi := k_\varepsilon * (M \chi_R \phi), \quad \mathcal{B}_\varepsilon^* \phi := \frac{1}{\varepsilon^2} (k_\varepsilon * \phi - \phi) - x \cdot \nabla \phi - k_\varepsilon * (M \chi_R \phi).$$

We use the same computation as the one used to deal with  $T_1$  in the proof of Lemma 2.4 and Cauchy-Schwarz inequality:

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^* \phi) \phi \leq -\frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) (\phi(y) - \phi(x))^2 dy dx \\ + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi^2(y) - \phi^2(x)) k_\varepsilon(x-y) dy dx \\ + \frac{d}{2} \int_{\mathbb{R}^d} \phi^2 + \|k_\varepsilon * (M \chi_R \phi)\|_{L^2} \|\phi\|_{L^2}.$$

We then notice that the second term equals 0 and we use Young inequality and the fact that  $\|k_\varepsilon\|_{L^1} = 1$  to get

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^* \phi) \phi &\leq -\frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) (\phi(y) - \phi(x))^2 dy dx + \frac{d}{2} \int_{\mathbb{R}^d} \phi^2 + \frac{1}{2} \|M \chi_R \phi\|_{L^2}^2 + \frac{1}{2} \|\phi\|_{L^2}^2 \\ &\leq -I_\varepsilon(\phi) + C \int_{\mathbb{R}^d} \phi^2 \end{aligned}$$

where  $I_\varepsilon$  is defined in (2.19). We also have the following inequality:

$$\begin{aligned} I_\varepsilon(\chi_R \phi) &\leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) \phi^2(x) (\chi_R(y) - \chi_R(x))^2 dy dx \\ &\quad + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) \chi_R^2(y) (\phi(y) - \phi(x))^2 dy dx \\ &\leq C \|\nabla \chi_R\|_\infty \int_{\mathbb{R}^d} \phi^2 + 2I_\varepsilon(\phi). \end{aligned}$$

If we denote  $\phi_t := S_{\mathcal{B}_\varepsilon^*}(t)\phi$ , we thus have

$$\frac{1}{2} \frac{d}{dt} \|\phi_t\|_{L^2}^2 \leq -\lambda \|k_\varepsilon * (\chi_R \phi_t)\|_{H^1}^2 + b \|\phi_t\|_{L^2}^2, \quad b > 0.$$

Multiplying this inequality by  $e^{-bt}$ , we obtain

$$\frac{d}{dt} (\|\phi_t\|_{L^2}^2 e^{-bt}) \leq -2\lambda \|k_\varepsilon * (\chi_R \phi_t)\|_{H^1}^2 e^{-bt}, \quad \forall t \geq 0,$$

and integrating in time, we get

$$(2.23) \quad \|\phi_t\|_{L^2}^2 e^{-bt} + 2\lambda \int_0^t \|k_\varepsilon * (\chi_R \phi_s)\|_{H^1}^2 e^{-bs} ds \leq \|\phi\|_{L^2(m)}^2, \quad \forall t \geq 0.$$

We now estimate

$$\begin{aligned} \int_0^t \|\mathcal{A}_\varepsilon^* S_{\mathcal{B}_\varepsilon^*}(s) \phi\|_{H^1}^2 e^{-2bs} ds &= \int_0^t \|\mathcal{A}_\varepsilon^* \phi_s\|_{H^1}^2 e^{-2bs} ds \\ &= \int_0^t \|k_\varepsilon * (M \chi_R \phi_s)\|_{L^2}^2 e^{-2bs} ds + \int_0^t \|k_\varepsilon * (M \chi_R \phi_s)\|_{H^1}^2 e^{-2bs} ds. \end{aligned}$$

Using Young inequality and (2.23), we conclude that

$$\int_0^\infty \|\mathcal{A}_\varepsilon^* S_{\mathcal{B}_\varepsilon^*}(t) \phi\|_{H^1}^2 e^{-2bs} ds \leq C \|\phi\|_{L^2}^2.$$

As in the proof of Lemma 2.9, we can obtain that for any  $s \in \mathbb{N}$ , there exists a constant  $C > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0]$ ,

$$\int_0^\infty \|(\mathcal{A}_\varepsilon^* S_{\mathcal{B}_\varepsilon^*})^{(*s)}(t)\|_{L^2 \rightarrow H^s}^2 e^{-2bt} dt \leq C.$$

From this, we deduce that for any  $\varepsilon \in (0, \varepsilon_0]$ ,

$$\int_0^\infty \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*s)}(t)\|_{H^{-s} \rightarrow L^2}^2 e^{-2bt} dt \leq C.$$

Taking  $\ell > d/2$  and using the continuous Sobolev embedding  $L^1(\mathbb{R}^d) \subset H^{-\ell}(\mathbb{R}^d)$ , we obtain

$$\int_0^\infty \|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*\ell)}(t)\|_{L^1 \rightarrow L^2}^2 e^{-2bt} dt \leq C.$$

The integer  $\ell$  is thus fixed such that  $\ell > d/2$ . Then noticing that

$$(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{*(\ell+1)} = \mathcal{A}_\varepsilon (S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*\ell)} *_t S_{\mathcal{B}_\varepsilon}$$



and using the fact that  $\mathcal{A}_\varepsilon$  is compactly supported combined with Lemma 2.5, we get

$$\begin{aligned}
& \int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{*(\ell+1)} f\|_{L^2(m)}^2 e^{-2bt} dt \\
& \leq \int_0^\infty \int_0^t \|(\mathcal{S}_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{*(\ell)}(s) S_{\mathcal{B}_\varepsilon}(t-s) f\|_{L^2}^2 e^{-2bt} dt \\
& \leq \int_0^\infty \int_s^\infty \|(\mathcal{S}_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{*(\ell)}(s)\|_{L^1 \rightarrow L^2}^2 e^{-2bs} \|S_{\mathcal{B}_\varepsilon}(t-s) f\|_{L^1(m)}^2 e^{-2b(t-s)} dt ds \\
& \leq \int_0^\infty \|(\mathcal{S}_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{*(\ell)}(s)\|_{L^1 \rightarrow L^2}^2 e^{-2bs} ds \int_0^\infty e^{2(a-b)t} dt \|f\|_{L^1(m)}^2 \\
& \leq C \|f\|_{L^1(m)}^2.
\end{aligned}$$

Consequently, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(2.24) \quad & \left( \int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{*(\ell+1)} f\|_{L^2(m)} e^{-2bt} dt \right)^2 \leq \int_0^\infty \|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{*(\ell+1)} f\|_{L^2(m)}^2 e^{-2bt} dt \int_0^\infty e^{-2bt} dt \\
& \leq C \|f\|_{L^1(m)}^2.
\end{aligned}$$

To conclude the proof, we use Lemma 2.8 with  $X = L^1(m)$ ,  $Y = L^2(m)$  and  $u(t) := (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{*(\ell+1)}(t)$ . We are able to check that assumption (1) is satisfied thanks to Lemmas 2.3 and 2.4-2.5. Assumption (2) is nothing but inequality (2.24). We refer to [4, 6] for the proof in the case  $\varepsilon = 0$ .  $\square$

## 2.6. Spectral analysis.

**Lemma 2.11.** *For any  $\varepsilon > 0$ ,  $\Lambda_\varepsilon$  satisfies Kato's inequalities:*

$$\forall f \in D(\Lambda_\varepsilon), \quad \Lambda_\varepsilon(\theta(f)) \geq \theta'(f) (\Lambda_\varepsilon f), \quad \theta(s) = |s| \quad \text{or} \quad \theta(s) = s_+.$$

*It follows that for any  $\varepsilon > 0$ , the semigroup associated to  $\Lambda_\varepsilon$  is positive in the following sense that if  $f \in L^1(m)$  and  $f \geq 0$ , then for any  $t \geq 0$ ,  $S_{\Lambda_\varepsilon}(t)f \geq 0$ .*

*Proof.* First, we have

$$\begin{aligned}
& \text{sign} f(x) \Lambda_\varepsilon f(x) \\
& = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} k_\varepsilon(x-y) (f(y) - f(x)) dy \text{sign} f(x) + d f(x) \text{sign} f(x) + x \cdot \nabla f(x) \text{sign} f(x) \\
& \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} k_\varepsilon(x-y) (|f|(y) - |f|(x)) dy + d |f|(x) + x \cdot \nabla |f|(x) = \Lambda_\varepsilon |f|(x),
\end{aligned}$$

which ends the proof of the Kato inequality in the case  $\theta(s) = |s|$ . Using that  $s_+ = (s + |s|)/2$ , we obtain the result in the case  $\theta(s) = s_+$ .

We consider  $f \leq 0$  and denote  $f(t) := S_{\Lambda_\varepsilon}(t)f$ . We define  $\beta(s) = s_+ = (|s| + s)/2$ . Using Kato's inequality, we have  $\partial_t \beta(f_t) \leq \Lambda_\varepsilon \beta(f_t)$ , and then

$$0 \leq \int_{\mathbb{R}^d} \beta(f_t) \leq \int_{\mathbb{R}^d} \beta(f) = 0, \quad \forall t \geq 0,$$

from which we deduce  $f_t \leq 0$  for any  $t \geq 0$ .  $\square$

The operator  $-\Lambda_\varepsilon$  satisfies the following form of the strong maximum principle.

**Lemma 2.12.** *Any nonnegative eigenfunction associated to the eigenvalue 0 is positive. In other words, we have*

$$f \in D(\Lambda_\varepsilon), \quad \Lambda_\varepsilon f = 0, \quad f \geq 0, \quad f \neq 0 \quad \text{implies} \quad f > 0.$$

*Proof.* We define

$$\mathcal{C}f = \frac{1}{\varepsilon^2} k_\varepsilon * f, \quad \mathcal{D}f = x \cdot \nabla_x f + \lambda f, \quad \lambda := d - \frac{1}{\varepsilon^2}$$

and the semigroup

$$S_{\mathcal{D}}(t)g := g(e^t x) e^{\lambda t}$$

with generator  $\mathcal{D}$ . Thanks to the Duhamel formula

$$S_{\Lambda_\varepsilon}(t) = S_{\mathcal{D}}(t) + \int_0^t S_{\mathcal{D}}(s) \mathcal{C} S_{\Lambda_\varepsilon}(t-s) ds,$$

the eigenfunction  $f$  satisfies

$$\begin{aligned} f &= S_{\Lambda_\varepsilon}(t)f = S_{\mathcal{D}}(t)f + \int_0^t S_{\mathcal{D}}(s) \mathcal{C} S_{\Lambda_\varepsilon}(t-s)f ds \\ &\geq \int_0^t S_{\mathcal{D}}(s) \mathcal{C} f ds \quad \forall t > 0. \end{aligned}$$

By assumption, there exists  $x_0 \in \mathbb{R}^d$  such that  $f \not\equiv 0$  on  $B(x_0, r/2)$ . As a consequence, denoting  $\rho := \|f\|_{L^1(B(x_0, r/2))} > 0$ , we have

$$\mathcal{C}f \geq \frac{\kappa \rho}{\varepsilon^2} \mathbb{1}_{B(x_0, r/2)},$$

and then

$$f \geq \frac{\kappa \rho}{\varepsilon^2} \sup_{t>0} \int_0^t e^{\lambda s} \mathbb{1}_{B(e^{-s}x_0, e^{-t}r/2)} ds \geq \kappa_1 \mathbb{1}_{B(x_0, r/4)}, \quad \kappa_1 > 0.$$

Using that lower bound, we obtain

$$\mathcal{C}f \geq \theta_d \frac{\kappa \kappa_{i-1}}{\varepsilon^2} \mathbb{1}_{B(x_0, u_i r)}, \quad \text{and then } f \geq \kappa_i \mathbb{1}_{B(x_0, v_i r)},$$

with  $i = 2$ ,  $u_2 = 1$ ,  $\kappa_2 > 0$ ,  $v_2 = 3/4$ . Repeating once more the argument, we get the same lower estimate with  $i = 3$ ,  $u_3 = 7/4$ ,  $\kappa_3 > 0$  and  $v_3 = 3/2$ . By an induction argument, we finally get  $f > 0$  on  $\mathbb{R}^d$ .  $\square$

We are now able to prove Theorem 2.1.

*Proof of part (1) in Theorem 2.1.* Using Lemmas 2.3-2.6-2.5, 2.11, 2.12 and the fact that  $\Lambda_\varepsilon^* 1 = 0$ , we can apply Krein-Rutman theorem which implies that for any  $\varepsilon > 0$ , there exists a unique  $G_\varepsilon > 0$  such that  $\|G_\varepsilon\|_{L^1} = 1$ ,  $\Lambda_\varepsilon G_\varepsilon = 0$  and  $\Pi_\varepsilon f = \langle f \rangle G_\varepsilon$  where  $\langle f \rangle = \int_{\mathbb{R}^d} f$ . It also implies that for any  $\varepsilon > 0$ , there exists  $a_\varepsilon < 0$  such that in  $X = L^1(m)$  or  $X = H^s(m)$  for any  $s \in \mathbb{N}$ , there holds

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_\varepsilon} = \{0\}$$

and

$$(2.25) \quad \forall t \geq 0, \quad \|S_{\Lambda_\varepsilon}(t)f - \langle f \rangle G_\varepsilon\|_X \leq e^{at} \|f - \langle f \rangle G_\varepsilon\|_X, \quad \forall a > a_\varepsilon.$$

*Proof of part (2) in Theorem 2.1.* We now have to establish that estimate (2.25) can be obtained uniformly in  $\varepsilon \in [0, \varepsilon_0]$ . In order to do so, we use a perturbation argument in the same line as in [7, 11] to prove that our operator  $\Lambda_\varepsilon$  has a spectral gap in  $H^3(m)$  which does not depend on  $\varepsilon$ .

First, we introduce the following spaces:

$$X_1 := H_1^6(m) \subset X_0 := H^3(m) \subset X_{-1}(m) := L^2(m)$$

where  $m = \langle x \rangle^q$  with  $q > d/2 + 5$  so that the conclusion of Lemma 2.6 is satisfied in the three spaces  $X_i$ ,  $i = -1, 0, 1$ .

One can notice that we also have the following embedding

$$X_1 \subset H_1^5(m) \subset D(\Lambda_\varepsilon) = D(\mathcal{B}_\varepsilon) \subset D(\mathcal{A}_\varepsilon) \subset X_0.$$

We now summarize the necessary results to apply a perturbative argument (obtained thanks to Lemmas 2.2, 2.3, 2.4, 2.6 and 2.9 and from [4, 6]).

There exist  $a_0 < 0$  and  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ :

- (i) For any  $i = -1, 0, 1$ ,  $\mathcal{A}_\varepsilon \in \mathcal{B}(X_i)$  uniformly in  $\varepsilon$ .
- (ii) For any  $a > a_0$  and  $\ell \geq 0$ , there exists  $C_{\ell,a} > 0$  such that

$$\forall i = -1, 0, 1, \quad \forall t \geq 0, \quad \|\mathcal{S}_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon \mathcal{S}_{\mathcal{B}_\varepsilon})^{(*\ell)}(t)\|_{X_i \rightarrow X_i} \leq C_{\ell,a} e^{at}.$$

- (iii) For any  $a > a_0$ , there exist  $n \geq 1$  and  $C_{n,a} > 0$  such that

$$\forall i = -1, 0, \quad \int_0^\infty \|(\mathcal{A}_\varepsilon \mathcal{S}_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{X_i \rightarrow X_{i+1}} e^{-at} dt \leq C_{n,a}.$$

- (iv) There exists a function  $\eta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$  such that

$$\forall i = -1, 0, \quad \|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{X_i \rightarrow X_i} \leq \eta(\varepsilon) \quad \text{and} \quad \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_i \rightarrow X_{i-1}} \leq \eta(\varepsilon).$$

- (v)  $\Sigma(\Lambda_0) \cap \Delta_{a_0} = \{0\}$  in spaces  $X_i$ ,  $i = -1, 0, 1$ , where 0 is a one dimensional eigenvalue.

Using a perturbative argument as in [11], from the facts (i)–(v), we can deduce the following proposition:

**Proposition 2.13.** *There exist  $a_0 < 0$  and  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ , the following properties hold in  $X_0 = H^3(m)$ :*

- (1)  $\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_0} = \{0\}$ ;
- (2) for any  $f \in X_0$  and any  $a > a_0$ ,

$$\|\mathcal{S}_{\Lambda_\varepsilon}(t)f - G_\varepsilon \langle f \rangle\|_{X_0} \leq C_a e^{at} \|f - G_\varepsilon \langle f \rangle\|_{X_0}, \quad \forall t \geq 0$$

for some explicit constant  $C_a > 0$ .

To end the proof of Theorem 2.1, we enlarge the space where the previous estimates hold. To do that, we use an extension argument (see [4, 7]) and Lemmas 2.3, 2.5-2.6 and 2.9-2.10. Our “small space” is  $H^3(m)$  and our “large” space is  $L^1(m)$ .

### 3. FROM FRACTIONAL TO CLASSICAL FOKKER-PLANCK EQUATION

In this part, we denote  $\alpha := 2 - \varepsilon \in (0, 2]$  and we deal with the equations

$$(3.26) \quad \begin{cases} \partial_t f = -(-\Delta)^{\alpha/2} f + \operatorname{div}(xf) = \Lambda_{2-\alpha} f =: \mathcal{L}_\alpha f, & \alpha \in (0, 2) \\ \partial_t f = \Delta f + \operatorname{div}(xf) = \Lambda_0 f =: \mathcal{L}_2 f. \end{cases}$$

We here recall (see (1.2)) that for  $\alpha \in (0, 2)$ , the fractional Laplacian of Schwartz function is defined using an integral formulation as follows:

$$(3.27) \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \quad (-\Delta)^{\alpha/2} f(x) := c_\alpha \int_{\mathbb{R}^d} \frac{f(x) - f(y) + \chi(x-y)(x-y) \cdot \nabla f(x)}{|x-y|^{d+\alpha}} dy,$$

where  $\chi \in \mathcal{D}(\mathbb{R}^d)$  and  $\mathbf{1}_{B(0,1)} \leq \chi \leq \mathbf{1}_{B(0,2)}$ . Moreover,  $c_\alpha$  is a constant depending on  $\alpha$  satisfying

$$\frac{c_\alpha}{2} \int_{|z| \leq i} \frac{z_i^2}{|z|^{d+\alpha}} = 1, \quad \forall i = 1, \dots, d,$$

which implies that  $c_\alpha \approx (2 - \alpha)$ . Also, notice that by duality, we can extend the definition of the fractional Laplacian to the following class of functions:

$$\left\{ f \in L^1_{\text{loc}}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |f(x)| \langle x \rangle^{-d-\alpha} dx < \infty \right\}.$$

Consequently, one can define  $(-\Delta)^{\alpha/2} m$  when  $q < \alpha$ .

We recall that the equation  $\partial_t f = \mathcal{L}_\alpha f$  admits a unique equilibrium of mass 1 that we denote  $G_\alpha$  (see [3] for the case  $\alpha < 2$ ). Moreover, if  $\alpha < 2$ , one can prove that  $G_\alpha(x) \approx \langle x \rangle^{-d-\alpha}$  (see [12]) and for  $\alpha = 2$ , we have  $G_2(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$ . The main result of this section reads:

**Theorem 3.14.** *Assume  $\alpha_0 \in (0, 2)$  and  $q < \alpha_0$ . There exists an explicit constant  $a_0 < 0$  such that for any  $\alpha \in [\alpha_0, 2]$ , the semigroup  $S_{\mathcal{L}_\alpha}(t)$  associated to the fractional Fokker-Planck equation (3.26) satisfies: for any  $f \in L^1_q$ , any  $a > a_0$  and any  $\alpha \in [\alpha_0, 2]$ ,*

$$\|S_{\mathcal{L}_\alpha}(t)f - G_\alpha\langle f \rangle\|_{L^1_q} \leq C_a e^{at} \|f - G_\alpha\langle f \rangle\|_{L^1_q}$$

for some explicit constant  $C_a \geq 1$ . In particular, the spectrum  $\Sigma(\mathcal{L}_\alpha)$  of  $\mathcal{L}_\alpha$  satisfies the separation property  $\Sigma(\mathcal{L}_\alpha) \cap \Delta_{a_0} = \{a_0\}$  in  $L^1_q$  for any  $\alpha \in [\alpha_0, 2]$ .

**3.1. Exponential decay in  $L^2(G_\alpha^{-1/2})$ .** We recall a result from [3] which establishes an exponential decay to equilibrium for the semigroup  $S_{\mathcal{L}_\alpha}(t)$ .

**Theorem 3.15.** *There exists a constant  $a_0 < 0$  such that for any  $\alpha \in (0, 2)$ ,*

- (1) *in  $L^2(G_\alpha^{-1/2})$ , there holds  $\Sigma(\mathcal{L}_\alpha) \cap \Delta_{a_0} = \{0\}$ ;*
- (2) *we have the following estimate for any  $a > a_0$ ,*

$$\|S_{\mathcal{L}_\alpha}(t)f - G_\alpha\langle f \rangle\|_{L^2(G_\alpha^{-1/2})} \leq e^{at} \|f - G_\alpha\langle f \rangle\|_{L^2(G_\alpha^{-1/2})}, \quad \forall t \geq 0.$$

*Proof.* The proof is immediate going back to the proof of the exponential decay in the space  $L^2(G_\alpha^{-1/2})$  from [3]. Indeed, we can notice that the rate of decrease is uniform in  $\alpha$ .  $\square$

**3.2. Splitting of  $\mathcal{L}_\alpha$ .** We define  $\mathcal{A}_\alpha := M \chi_R$  and  $\mathcal{B}_\alpha := \mathcal{L}_\alpha - \mathcal{A}_\alpha$  for some  $M, R > 0$  to be chosen later.

**3.3. Uniform boundedness of  $\mathcal{A}_\alpha$ .**

**Lemma 3.16.** *Consider  $s \in \mathbb{N}$  and  $p \geq 1$ . The operator is uniformly bounded in  $\alpha$  from  $W^{s,p}(\nu)$  to  $W^{s,p}$  with  $\nu = m$  or  $\nu = G_\alpha^{-1/2}$ .*

*Proof.* The proof is immediate using that  $M \chi_R$  and all its derivatives are compactly supported.  $\square$

**3.4. Uniform dissipativity properties of  $\mathcal{B}_\alpha$ .**

**Lemma 3.17.** *For any  $a > -q$ , there exist  $M > 0$  and  $R > 0$  such that for any  $\alpha \in [\alpha_0, 2]$ ,  $\mathcal{B}_\alpha - a$  is dissipative in  $L^1(m)$ .*

*Proof.* We just have to adapt the proof Lemma 5.1 from [12] taking into account the constant  $c_\alpha$ . Indeed, we have

$$\int_{\mathbb{R}^d} (\mathcal{L}_\alpha f) \operatorname{sign} f m \leq \int_{\mathbb{R}^d} |f| m \left( \frac{I_\alpha(m)}{m} - \frac{x \cdot \nabla m}{m} \right).$$

We can then show that thanks to the rescaling constant  $c_\alpha$ ,  $I_\alpha(m)/m$  goes to 0 at infinity uniformly in  $\alpha \in [\alpha_0, 2)$ . As a consequence, if  $a > -q$ , since  $(x \cdot \nabla m)/m$  goes to  $-q$  at infinity, one may choose  $M$  and  $R$  such that for any  $\alpha \in [\alpha_0, 2)$ ,

$$\frac{I_\alpha(m)}{m} - \frac{x \cdot \nabla m}{m} - M \chi_R \leq a, \quad \text{on } \mathbb{R}^d,$$

which gives the result.  $\square$

**Lemma 3.18.** *For any  $a > a_0$  where  $a_0$  is defined in Theorem 3.15,  $\mathcal{B}_\alpha - a$  is dissipative in  $L^2(G_\alpha^{-1/2})$ .*

*Proof.* The proof also comes from [12, Lemma 5.1].  $\square$

### 3.5. Uniform regularization properties of $\mathcal{A}_\alpha S_{\mathcal{B}_\alpha}(t)$ .

**Lemma 3.19.** *There exist some constants  $b \in \mathbb{R}$  and  $C > 0$  such that for any  $\alpha \in [\alpha_0, 2]$ , the following estimates hold:*

$$\forall t \geq 0, \quad \|S_{\mathcal{B}_\alpha}(t)\|_{\mathcal{B}(L^1, L^2)} \leq C \frac{e^{bt}}{t^{d/2\alpha_0}}.$$

As a consequence, we can prove that for any  $a > \max(-q, a_0)$  and any  $\alpha \in [\alpha_0, 2]$ ,

$$(3.28) \quad \forall t \geq 0, \quad \|(\mathcal{A}_\alpha S_{\mathcal{B}_\alpha})^{(*n)}(t)\|_{\mathcal{B}(L^1(m), L^2(G_\alpha^{-1/2}))} \leq C e^{at}.$$

*Proof.* We do not write the proof for the case  $\alpha = 2$  and refer to [4, 6].

*Step 1.* The key argument to prove this regularization property of  $S_{\mathcal{B}_\alpha}(t)$  is the Nash inequality. For  $\alpha \in [\alpha_0, 2)$ , from the proof of [12, Lemma 5.3], we obtain that there exist  $b \geq 0$  and  $C > 0$  such that for any  $\alpha \in [\alpha_0, 2)$ ,

$$\forall t \geq 0, \quad \|S_{\mathcal{B}_\alpha}(t)f\|_{L^2} \leq C \frac{e^{bt}}{t^{d/(2\alpha_0)}} \|f\|_{L^1}.$$

*Step 2.* Then, using that  $\mathcal{A}_\alpha$  is compactly supported, we can write

$$\|\mathcal{A}_\alpha S_{\mathcal{B}_\alpha}(t)f\|_{L^2(m)} \leq C \|S_{\mathcal{B}_\alpha}(t)f\|_{L^2} \leq C \frac{e^{bt}}{t^{d/(2\alpha_0)}} \|f\|_{L^1}.$$

Using the same method as in [4], we can first deduce that there exists  $\ell_0 \in \mathbb{N}$ ,  $\gamma \in [0, 1)$  and  $K \in \mathbb{R}$  such that for any  $\alpha \in [\alpha_0, 2]$ ,

$$\|(\mathcal{A}_\alpha S_{\mathcal{B}_\alpha})^{(*\ell_0)}(t)f\|_{L^2(G_\alpha^{-1/2})} \leq C \frac{e^{bt}}{t^\gamma} \|f\|_{L^1(m)}.$$

We can then conclude that (3.28) holds using [4, Lemma 2.17] combined with Lemmas 3.17 and 3.16.  $\square$

**3.6. Spectral analysis.** Before going into the proof of Theorem 3.14, let us notice that we can make explicit the projection  $\Pi_\alpha$  onto the null space  $\mathcal{N}(\mathcal{L}_\alpha)$  through the following formula:  $\Pi_\alpha f = \langle f \rangle G_\alpha$ . Moreover, since the mass is preserved by the equation  $\partial_t f = \mathcal{L}_\alpha f$ , we can deduce that  $\Pi_\alpha(S_{\mathcal{L}_\alpha}(t)f) = \Pi_\alpha f$  for any  $t \geq 0$ .

*Proof of Theorem 3.14.* We can apply [4, Theorem 2.13] for each  $\alpha \in [\alpha_0, 2]$  because combining Theorem 3.15 with Lemmas 3.16, 3.17, 3.18 and 3.19, we can check the assumptions of the theorem are satisfied.  $\square$

## 4. FROM DISCRETE TO FRACTIONAL FOKKER-PLANCK EQUATION

Let us fix  $\alpha \in (0, 2)$ . We consider the equations

$$(4.29) \quad \begin{cases} \partial_t f = k_\varepsilon * f - \|k_\varepsilon\|_{L^1} f + \operatorname{div}_x(xf) =: \Lambda_\varepsilon f, & \varepsilon > 0 \\ \partial_t f = -(-\Delta)^{\alpha/2} f + \operatorname{div}_x(xf) =: \Lambda_0 f \end{cases}$$

where

$$k_\varepsilon(x) := \mathbf{1}_{\varepsilon \leq |x| \leq 1/\varepsilon} k_0(x) + \mathbf{1}_{|x| < \varepsilon} k_0(\varepsilon), \quad k_0(x) := |x|^{-d-\alpha}.$$

Notice that

$$(4.30) \quad \forall x \in \mathbb{R}^d \setminus \{0\}, \quad k_\varepsilon(x) \nearrow k_0(x) \quad \text{as } \varepsilon \rightarrow 0.$$

We here recall that for  $\alpha \in (0, 2)$ , the fractional Laplacian on Schwartz functions is defined through the formula (3.27). Since  $\alpha$  is fixed in this part, we can get rid of the constant  $c_\alpha$  and consider that it equals 1. The main theorem of this section reads:

**Theorem 4.20.** *Assume  $0 < q < \alpha/2$ .*

(1) *For any  $\varepsilon > 0$ , there exists a positive and unit mass normalized steady state  $G_\varepsilon \in L^1_q(\mathbb{R}^d)$  to the discrete fractional Fokker-Planck equation (4.29).*

(2) *There exist an explicit constant  $a_0 < 0$  and a constant  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in [0, \varepsilon_0]$ , the semigroup  $S_{\Lambda_\varepsilon}(t)$  associated to the discrete and fractional Fokker-Planck equations (4.29) satisfies: for any  $f \in L^1_q$  and any  $a > a_0$ ,*

$$\|S_{\Lambda_\varepsilon}(t)f - G_\varepsilon\langle f \rangle\|_{L^1_q} \leq C_a e^{at} \|f - G_\varepsilon\langle f \rangle\|_{L^1_q} \quad \forall t \geq 0,$$

for some explicit constant  $C_a \geq 1$ . In particular, the spectrum  $\Sigma(\Lambda_\varepsilon)$  of  $\Lambda_\varepsilon$  satisfies the separation property  $\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_0} = \{0\}$  in  $L^1_q$ .

The method of the proof is similar to the one of Section 2. We introduce a suitable splitting  $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ , establish some dissipativity and regularity properties on  $\mathcal{B}_\varepsilon$  and  $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$  and apply the Krein-Rutman theory revisited in [9, 5]. However, let us emphasize that we introduce a new splitting for the fractional operator (a different one from Section 3 and from [12]) and we also develop a new perturbative argument in the same line as [7, 11, 5] but with some less restrictive assumptions on the operators  $\mathcal{A}_\varepsilon$  and  $\mathcal{B}_\varepsilon$ , requiring that they are fulfilled only on the limit operator (i.e. for  $\varepsilon = 0$ ).

**4.1. Splittings of  $\Lambda_\varepsilon$ .** For any  $0 < \beta < \beta'$ , as previously, we denote  $\chi_\beta(x) := \chi(x/\beta)$ ,  $\chi_\beta^c := 1 - \chi_\beta$ ; we also define  $\chi_{\beta, \beta'} := \chi_{\beta'} - \chi_\beta$  and introduce the function  $\xi_\beta$  defined on  $\mathbb{R}^d \times \mathbb{R}^d$  by  $\xi_\beta(x, y) := \chi_\beta(x) + \chi_\beta(y) - \chi_\beta(x)\chi_\beta(y)$  and  $\xi_\beta^c := 1 - \xi_\beta$ . We denote  $I_0(f) := -(-\Delta)^{\alpha/2}f$  and  $I_\varepsilon(f) := k_\varepsilon * f - \|k_\varepsilon\|_{L^1}f$  for  $\varepsilon > 0$ . We split these operators into several parts: for any  $\varepsilon \geq 0$ ,

$$\begin{aligned} I_\varepsilon(f)(x) &= \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_\eta(x-y) (f(y) - f(x) - \chi(x-y)(y-x) \cdot \nabla f(x)) dy \\ &\quad + \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_\eta^c(x-y) (f(y) - f(x)) dy \\ &= \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_\eta(x-y) (f(y) - f(x) - \chi(x-y)(y-x) \cdot \nabla f(x)) dy \\ &\quad + \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_L^c(x-y) (f(y) - f(x)) dy \\ (4.31) \quad &\quad + \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_{\eta, L}(x-y) (f(y) - f(x)) \xi_R^c(x, y) dy \\ &\quad - \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_{\eta, L}(x-y) \xi_R(x, y) dy f(x) \\ &\quad + \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_{\eta, L}(x-y) \xi_R(x, y) f(y) dy \\ &=: \mathcal{B}_\varepsilon^1 f + \mathcal{B}_\varepsilon^2 f + \mathcal{B}_\varepsilon^3 f + \mathcal{B}_\varepsilon^4 f + \mathcal{A}_\varepsilon f. \end{aligned}$$

where the constants  $\eta \in [\varepsilon, 1]$ ,  $R > 0$  and  $0 < L \leq 1/\varepsilon$  will be chosen later. One can notice that given the facts that  $\eta \geq \varepsilon$  and  $L \leq 1/\varepsilon$ , we have for any  $\varepsilon > 0$ ,  $\mathcal{A}_\varepsilon = \mathcal{A}_0 =: \mathcal{A}$ . Finally, we denote for any  $\varepsilon \geq 0$ ,

$$\mathcal{B}_\varepsilon^5 f = \operatorname{div}(xf) \quad \text{and} \quad \mathcal{B}_\varepsilon f = \mathcal{B}_\varepsilon^1 f + \mathcal{B}_\varepsilon^2 f + \mathcal{B}_\varepsilon^3 f + \mathcal{B}_\varepsilon^4 f + \mathcal{B}_\varepsilon^5 f.$$

**4.2. Convergence  $\mathcal{B}_\varepsilon \rightarrow \mathcal{B}_0$ .**

**Lemma 4.21.** *Consider  $p \in (1, \infty)$  and  $q \in (0, \alpha/p)$ . The following convergence holds:*

$$\|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathcal{B}(W^{s+2, p(m)}, W^{s, p(m)})} \leq \eta_1(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad s = -2, 0.$$

*Proof.* Let us notice that  $\mathcal{B}_\varepsilon - \mathcal{B}_0 = \Lambda_\varepsilon - \Lambda_0$ .

*Step 1.* We first deal with the case  $s = 0$  and we introduce the notation  $k_{0,\varepsilon} := k_0 - k_\varepsilon$ . We then compute

$$\begin{aligned} \|\Lambda_\varepsilon f - \Lambda_0 f\|_{L^p(m)}^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} k_{0,\varepsilon}(z) (f(x+z) - f(x) - \chi(z)z \cdot \nabla f(x)) dz \right|^p m^p(x) dx \\ &\leq C \int_{\mathbb{R}^d} \left| \int_{|z| \leq 1} k_{0,\varepsilon}(z) (f(x+z) - f(x) - \chi(z)z \cdot \nabla f(x)) dz \right|^p m^p(x) dx \\ &\quad + C \int_{\mathbb{R}^d} \left| \int_{|z| \geq 1} k_{0,\varepsilon}(z) (f(x+z) - f(x) - \chi(z)z \cdot \nabla f(x)) dz \right|^p m^p(x) dx \\ &=: T_1 + T_2. \end{aligned}$$

To deal with  $T_1$ , we perform a Taylor expansion of  $f$ :

$$f(x+z) = f(x) + z \cdot \nabla f(x) + \int_0^1 (1-s) D^2 f(x+sz)(z, z) ds.$$

Since  $\chi(z) = 1$  if  $|z| \leq 1$ , we thus obtain

$$T_1 \leq C \int_{\mathbb{R}^d} \left( \int_{|z| \leq 1} k_{0,\varepsilon}(z) |z|^2 \int_0^1 (1-s) |D^2 f(x+sz)| ds dz \right)^p m^p(x) dx.$$

Then, from Holder inequality applied with the measure  $\mu_\varepsilon(dz) := \mathbf{1}_{|z| \leq 1} k_{0,\varepsilon}(z) |z|^2 dz$ , we have

$$\begin{aligned} T_1 &\leq C \left( \int_{\mathbb{R}^d} \mu_\varepsilon(dz) \right)^{p/p'} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \int_0^1 (1-s) |D^2 f(x+sz)| ds \right)^p \mu_\varepsilon(dz) m^p(x) dx \\ &\leq C \left( \int_{\mathbb{R}^d} \mu_\varepsilon(dz) \right)^{p/p'} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \int_0^1 |D^2 f(x+sz)| ds \right)^p \mu_\varepsilon(dz) m^p(x) dx \end{aligned}$$

where  $p' = p/(p-1)$  is the Holder conjugate of  $p$ . Using now Jensen inequality with the probability measure  $\mathbf{1}_{[0,1]}(s) ds$ , we get

$$\begin{aligned} T_1 &\leq C \left( \int_{\mathbb{R}^d} \mu_\varepsilon(dz) \right)^{p/p'} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^1 |D^2 f(x+sz)|^p ds \mu_\varepsilon(dz) m^p(x) dx \\ &\leq C \left( \int_{\mathbb{R}^d} \mu_\varepsilon(dz) \right)^{p/p'} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^1 |D^2 f(x+sz)|^p m^p(x+sz) m^p(s) ds \mu_\varepsilon(dz) dx \\ &\leq C \left( \int_{\mathbb{R}^d} \mu_\varepsilon(dz) \right)^{p/p'} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^1 |D^2 f(x)|^p m^p(x) ds \mu_\varepsilon(dz) dx \end{aligned}$$

where we have used that for  $|z| \leq 1$  and  $s \in [0, 1]$ ,  $m^p(s) \leq C$  and have performed a change of variable. We then deduce that

$$T_1 \leq C \left( \int_{\mathbb{R}^d} \mu_\varepsilon(dz) \right)^p \int_{\mathbb{R}^d} |D^2 f(x)|^p m^p(x) dx$$

with

$$\int_{\mathbb{R}^d} \mu_\varepsilon(dz) = \int_{|z| \leq 1} k_{0,\varepsilon}(z) |z|^2 dz \xrightarrow{\varepsilon \rightarrow 0} 0$$

by dominated convergence since for any  $\varepsilon \geq 0$ ,

$$|k_{0,\varepsilon}(z)| |z|^2 \mathbf{1}_{|z| \leq 1} \leq 2k_0(z) |z|^2 \mathbf{1}_{|z| \leq 1} \in L^1(\mathbb{R}^d).$$

To treat  $T_2$ , we first notice that the term involving  $\nabla f(x)$  gives no contribution so that

$$T_2 \leq C \int_{\mathbb{R}^d} \left| \int_{|z| \geq 1} k_{0,\varepsilon}(z) (f(x+z) - f(x)) dz \right|^p m^p(x) dx.$$

Then, using again Holder inequality with the measure  $\mathbb{1}_{|z|\geq 1} k_{0,\varepsilon}(z) dz$ , we get doing similar computations as for  $T_1$

$$\begin{aligned} T_2 &\leq C \left( \int_{|z|\geq 1} k_{0,\varepsilon}(z) dz \right)^{p/p'} \int_{\mathbb{R}^d} \int_{|z|\geq 1} |k_{0,\varepsilon}(z)| (|f|^p(x+z) + |f|^p(x)) dz m^p(x) dx \\ &\leq C \left( \int_{|z|\geq 1} k_{0,\varepsilon}(z) dz \right)^{p/p'} \left( \int_{\mathbb{R}^d} \int_{|z|\geq 1} |k_{0,\varepsilon}(z)| |f|^p(x+z) m^p(x+z) m^p(z) dz dx \right. \\ &\quad \left. + \int_{|z|\geq 1} k_{0,\varepsilon}(z) dz \int_{\mathbb{R}^d} |f|^p(x) m^p(x) dx \right) \\ &\leq C \left( \int_{|z|\geq 1} k_{0,\varepsilon}(z) m^p(z) dz \right)^p \int_{\mathbb{R}^d} |f|^p(x) m^p(x) dx, \end{aligned}$$

with

$$\int_{|z|\geq 1} k_{0,\varepsilon}(z) m^p(z) dz \xrightarrow{\varepsilon \rightarrow 0} 0$$

by dominated convergence since for any  $\varepsilon \geq 0$ ,

$$k_{0,\varepsilon}(z) m^p(z) \mathbb{1}_{|z|\geq 1} \leq 2 k_0(z) m^p(z) \mathbb{1}_{|z|\geq 1} \in L^1(\mathbb{R}^d).$$

As a consequence, we obtain

$$\|(\Lambda_\varepsilon - \Lambda_0)(f)\|_{L^p(m)} \leq \eta(\varepsilon) \|f\|_{W^{2,p}(m)}, \quad \eta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

*Step 2.* We now consider the case  $s = -2$ , and we recall that by definition

$$\|\Lambda_\varepsilon f - \Lambda_0 f\|_{W^{-2,p}(m)} = \sup_{\|\phi\|_{W^{2,p'}} \leq 1} \int_{\mathbb{R}^d} f (\Lambda_\varepsilon - \Lambda_0)^*(\phi m) = \sup_{\|\phi\|_{W^{2,p'}} \leq 1} \int_{\mathbb{R}^d} f (\Lambda_\varepsilon - \Lambda_0)(\phi m)$$

where  $p' = p/(p-1)$  and because  $(\Lambda_\varepsilon - \Lambda_0)^* = \Lambda_\varepsilon - \Lambda_0$  (where  $\Lambda^*$  stands for the formal dual operator of  $\Lambda$ ). We then estimate the integral in the right hand side of the previous equality:

$$\int_{\mathbb{R}^d} f (\Lambda_\varepsilon - \Lambda_0)(\phi m) = \int_{\mathbb{R}^d} \frac{(\Lambda_\varepsilon - \Lambda_0)(\phi m)}{m} f m \leq \|(\Lambda_\varepsilon - \Lambda_0)(\phi m)/m\|_{L^{p'}} \|f\|_{L^p(m)}.$$

Moreover,

$$\begin{aligned} (\Lambda_\varepsilon - \Lambda_0)(\phi m)(x) &= (I_\varepsilon - I_0)(\phi m)(x) \\ (4.32) \quad &= (I_\varepsilon - I_0)(\phi)(x) m(x) + \int_{\mathbb{R}^d} k_{0,\varepsilon}(z) \phi(x+z) (m(x+z) - m(x) - \nabla m(x) \cdot z \chi(z)) dz \\ &\quad + \int_{\mathbb{R}^d} k_{0,\varepsilon}(z) \chi(z) z \cdot \nabla m(x) (\phi(x+z) - \phi(x)) dz. \end{aligned}$$

We deduce that

$$\begin{aligned} \|(\Lambda_\varepsilon - \Lambda_0)(\phi m)/m\|_{L^{p'}} &\leq \|(I_\varepsilon - I_0)(\phi)\|_{L^{p'}} \\ &\quad + \left( \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left| \int_{\mathbb{R}^d} k_{0,\varepsilon}(z) \phi(x+z) (m(x+z) - m(x) - \nabla m(x) \cdot z \chi(z)) dz \right|^{p'} dx \right)^{1/p'} \\ &\quad + \left( \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left| \int_{\mathbb{R}^d} k_{0,\varepsilon}(z) \chi(z) z \cdot \nabla m(x) (\phi(x+z) - \phi(x)) dz \right|^{p'} dx \right)^{1/p'} \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

To deal with  $J_1$ , we use the step 1 of the proof which gives us

$$\|(I_\varepsilon - I_0)(\phi)\|_{L^{p'}} \leq \eta(\varepsilon) \|\phi\|_{W^{2,p'}}, \quad \eta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$



The term  $J_2$  is split into two parts:

$$\begin{aligned} J_2^{p'} &\leq C \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left| \int_{|z| \leq 1} k_{0,\varepsilon}(z) \phi(x+z) (m(x+z) - m(x) - \nabla m(x) \cdot z \chi(z)) dz \right|^{p'} dx \\ &\quad + C \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left| \int_{|z| \geq 1} k_{0,\varepsilon}(z) \phi(x+z) (m(x+z) - m(x) - \nabla m(x) \cdot z \chi(z)) dz \right|^{p'} dx \\ &=: J_{21} + J_{22}. \end{aligned}$$

We first notice that for  $|z| \leq 1$ ,

$$m(x+z) - m(x) - \nabla m(x) \cdot z \chi(z) = m(x+z) - m(x) - \nabla m(x) \cdot z = \frac{1}{2} D^2 m(x + \theta z)(z, z)$$

for some  $\theta \in (0, 1)$ , which implies that

$$J_{21} \leq C \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left( \int_{|z| \leq 1} k_{0,\varepsilon}(z) |z|^2 |D^2 m(x + \theta z)| |\phi|(x+z) dz \right)^{p'} dx.$$

Since  $0 < q < 2$ ,  $|D^2 m| \leq C$  and  $1/m^{p'} \leq C$  in  $\mathbb{R}^d$ , we thus deduce using Holder inequality and a change of variable,

$$\begin{aligned} J_{21} &\leq C \int_{\mathbb{R}^d} \left( \int_{|z| \leq 1} k_{0,\varepsilon}(z) |z|^2 |\phi|(x+z) dz \right)^{p'} dx \\ &\leq C \left( \int_{|z| \leq 1} k_{0,\varepsilon}(z) |z|^2 dz \right)^{p'} \|\phi\|_{L^{p'}}^{p'} \quad \text{with} \quad \int_{|z| \leq 1} k_{0,\varepsilon}(z) |z|^2 dz \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

Concerning  $J_{22}$ , we use that  $|z \chi(z)| \leq C$  for any  $|z| \geq 1$  and that  $|\nabla m| \leq C m$  in  $\mathbb{R}^d$ , we obtain

$$\begin{aligned} J_{22} &\leq C \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left( \int_{|z| \geq 1} k_{0,\varepsilon}(z) |\phi|(x+z) (m(x+z) + m(x) + |\nabla m(x)|) dz \right)^{p'} dx \\ &\leq C \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left( \int_{|z| \geq 1} k_{0,\varepsilon}(z) |\phi|(x+z) (m(x)m(z) + m(x)) dz \right)^{p'} dx \\ &\leq C \int_{\mathbb{R}^d} \frac{1}{m^{p'}(x)} \left( \int_{|z| \geq 1} k_{0,\varepsilon}(z) |\phi|(x+z) m(x) m(z) dz \right)^{p'} dx \\ &\leq C \int_{\mathbb{R}^d} \left( \int_{|z| \geq 1} k_{0,\varepsilon}(z) |\phi|(x+z) m(z) dz \right)^{p'} dx, \end{aligned}$$

which implies, using Holder inequality and a change of variable,

$$J_{22} \leq C \left( \int_{|z| \geq 1} k_{0,\varepsilon}(z) m^p(z) dz \right)^{p'} \|\phi\|_{L^{p'}}^{p'} \quad \text{with} \quad \int_{|z| \geq 1} k_{0,\varepsilon}(z) m^p(z) dz \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Finally, we handle  $J_3$  performing a Taylor expansion of  $\phi$ :

$$\phi(x+z) - \phi(x) = \int_0^1 (1-s) \nabla \phi(x+sz) \cdot z ds$$

which implies, using that  $|\nabla m|^{p'}/m^{p'} \in L^\infty(\mathbb{R}^d)$ , Holder inequality and a change of variable,

$$\begin{aligned} J_3 &\leq \left( \int_{\mathbb{R}^d} \frac{|\nabla m|^{p'}(x)}{m^{p'}(x)} \left( \int_{|z|\leq 2} k_{0,\varepsilon}(z) |z|^2 \int_0^1 |\nabla \phi|(x + sz) ds dz \right)^{p'} dx \right)^{1/p'} \\ &\leq C \int_{|z|\leq 2} k_{0,\varepsilon}(z) |z|^2 dz \|\nabla \phi\|_{L^{p'}} \quad \text{with} \quad \int_{|z|\leq 2} k_{0,\varepsilon}(z) |z|^2 dz \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

As a consequence, we obtain that

$$\|(\Lambda_\varepsilon - \Lambda_0)(\phi m)/m\|_{L^{p'}} \leq \eta(\varepsilon) \|\phi\|_{W^{2,p'}}, \quad \eta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0,$$

which concludes the proof.  $\square$

#### 4.3. Regularization properties of $\mathcal{A}_\varepsilon$ .

**Lemma 4.22.** *For any  $p \in (1, \infty)$ ,  $(s, t) = (-2, 0)$  or  $(0, 2)$ , the operator  $\mathcal{A}_\varepsilon = \mathcal{A}_0 = \mathcal{A}$  defined in (4.31) by*

$$\mathcal{A}f = \int_{\mathbb{R}^d} k_0(x-y) \chi_{\eta,L}(x-y) \xi_R(x,y) f(y) dy$$

is bounded from  $W^{s,p}$  to  $W^{t,p}(\nu)$  for any weight function  $\nu$ .

*Proof.* First, one can notice that

$$\begin{aligned} (4.33) \quad \xi_R(x,y) \chi_{\eta,L}(x-y) &\leq (\chi_R(x) + \chi_R(y)) \chi_{\eta,L}(x-y) \\ &\leq (\mathbf{1}_{|x|\leq 2R} + \mathbf{1}_{|y|\leq 2R}) \mathbf{1}_{\eta \leq |x-y| \leq 2L} \\ &\leq 2 \mathbf{1}_{\eta \leq |x-y| \leq 2L} \mathbf{1}_{|x|\leq 2(R+L)} \mathbf{1}_{|y|\leq 2(R+L)}, \end{aligned}$$

the proof is hence immediate in the case  $s = t = 0$  using Young inequality:

$$\|\mathcal{A}f\|_{L^p(\nu)} \leq C \|\mathcal{A}f\|_{L^p} \leq \|k_0 \mathbf{1}_{\eta \leq |\cdot| \leq 2L}\|_{L^1} \|f\|_{L^p}.$$

We now deal with the case  $(s, t) = (0, 2)$ . First, we have for  $\ell = 1, 2$

$$\partial_x^\ell(\mathcal{A}f)(x) = \sum_{i+j+k=\ell} \int_{\mathbb{R}^d} \partial_x^i(k_0(x-y)) \partial_x^j(\chi_{\eta,L}(x-y)) \partial_x^k(\xi_R(x,y)) f(y) dy,$$

and for any  $(i, j, k)$  such that  $i + j + k = \ell$ ,

$$|\partial_x^i(k_0(x-y)) \partial_x^j(\chi_{\eta,L}(x-y)) \partial_x^k(\xi_R(x,y))| \leq C |\partial_x^i(k_0(x-y))| \mathbf{1}_{\eta \leq |x-y| \leq 2L} \mathbf{1}_{|x|\leq 2(R+L)}.$$

As a consequence, for  $\ell = 0, 1, 2$ ,

$$\|\partial_x^\ell(\mathcal{A}f)\|_{L^p(\nu)} \leq \sum_{i=0}^2 \|\partial_x^i k_0 \mathbf{1}_{\eta \leq |\cdot| \leq 2L}\|_{L^1} \|f\|_{L^p},$$

which concludes the proof in the case  $(s, t) = (0, 2)$ .

Finally, we argue by duality to prove the last part corresponding to the case  $(s, t) = (-2, 0)$ , we use the previous case:

$$\begin{aligned} \|\mathcal{A}f\|_{L^p(\nu)} &\leq C \|\mathcal{A}f\|_{L^p} = C \sup_{\|\phi\|_{L^{p'}} \leq 1} \int_{\mathbb{R}^d} (\mathcal{A}f) \phi \\ &= C \sup_{\|\phi\|_{L^{p'}} \leq 1} \int_{\mathbb{R}^d} (\mathcal{A}\phi) f \\ &\leq C \sup_{\|\phi\|_{L^{p'}} \leq 1} \|f\|_{W^{-2,p}} \|\mathcal{A}\phi\|_{W^{2,p'}} \\ &\leq C \sup_{\|\phi\|_{L^{p'}} \leq 1} \|f\|_{W^{-2,p}} \|\phi\|_{L^{p'}} \leq C \|f\|_{W^{-2,p}}. \end{aligned}$$

$\square$

#### 4.4. Dissipativity properties of $\mathcal{B}_\varepsilon$ and $\mathcal{B}_0$ .

**Lemma 4.23.** *Consider  $p \in [1, 2]$  and  $q \in (0, \alpha/p)$ . For any  $a > d(1 - 1/p) - q$ , there exist  $\varepsilon_1 > 0$ ,  $\eta > 0$ ,  $L > 0$  and  $R > 0$  such that for any  $\varepsilon \in [0, \varepsilon_1]$ ,  $\mathcal{B}_\varepsilon - a$  is dissipative in  $L^p(m)$ .*

*Proof.* We consider  $a > d(1 - 1/p) - q$  and we estimate the integral  $\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^i f) (\text{sign} f) |f|^{p-1} m^p$ , for  $i = 1, \dots, 5$ .

We first deal with  $\mathcal{B}_\varepsilon^1$  in both cases  $\varepsilon > 0$  and  $\varepsilon = 0$  simultaneously noticing that for any  $\varepsilon \geq 0$ ,

$$\mathcal{B}_\varepsilon^1 f(x) = \int_{\mathbb{R}^d} k_\varepsilon(x-y) \chi_\eta(x-y) (f(y) - f(x) - (y-x) \cdot \nabla f(x)) dy.$$

Then, using that  $\Phi : s \mapsto |s|^p/p$  is convex, we have

$$\begin{aligned} & (f(y) - f(x)) \text{sign}(f(x)) |f|^{p-1}(x) \\ (4.34) \quad &= ((f(y) - f(x))\Phi'(f(x)) + \Phi(f(x)) - \Phi(f(y))) + (\Phi(f(y)) - \Phi(f(x))) \\ &\leq \frac{1}{p} (|f|^p(y) - |f|^p(x)). \end{aligned}$$

Consequently,

$$\begin{aligned} & \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^1 f) (\text{sign} f) |f|^{p-1} m^p \\ &\leq \frac{1}{p} \int_{\mathbb{R}^d \times \mathbb{R}^d} (|f|^p(y) - |f|^p(x) - (y-x) \cdot \nabla |f|^p(x)) k_\varepsilon(x-y) \chi_\eta(x-y) dy m^p(x) dx \\ &= \frac{1}{p} \int_{\mathbb{R}^d \times \mathbb{R}^d} (m^p(y) - m^p(x) - (y-x) \cdot \nabla m^p(x)) k_\varepsilon(x-y) \chi_\eta(x-y) dy |f|^p(x) dx. \end{aligned}$$

We estimate the last term for any  $\varepsilon \geq 0$  thanks to a Taylor expansion:

$$m^p(y) - m^p(x) - (y-x) \cdot \nabla m^p(x) = \frac{1}{2} D^2 m^p(x + \theta(y-x))(y-x, y-x)$$

for some  $\theta \in (0, 1)$ . Using that  $pq < \alpha < 2$ , we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^d} (m^p(y) - m^p(x) - (y-x) \cdot \nabla m^p(x)) k_\varepsilon(x-y) \chi_\eta(x-y) dy \\ &\leq C \int_{|z| \leq 2\eta} |z|^2 k_0(z) dz \end{aligned}$$

and thus

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^1 f) (\text{sign} f) |f|^{p-1} m^p \leq \kappa_\eta \int_{\mathbb{R}^d} |f|^p m^p \quad \text{with} \quad \kappa_\eta \approx \int_{|z| \leq 2\eta} k_0(z) |z|^2 dz \xrightarrow{\eta \rightarrow 0} 0.$$

Concerning  $\mathcal{B}_\varepsilon^2$ , we also treat the case  $\varepsilon > 0$  and  $\varepsilon = 0$  in a same time using (4.34):

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^2 f) (\text{sign} f) |f|^{p-1} m^p &\leq \frac{1}{p} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) (|f|^p(y) - |f|^p(x)) \chi_L^\varepsilon(x-y) m^p(x) dy dx \\ &= \frac{1}{p} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) (m^p(y) - m^p(x)) \chi_L^\varepsilon(x-y) |f|^p(x) dy dx. \end{aligned}$$

We now use the fact that the function  $s \mapsto s^{pq/2}$  is  $pq/2$ -Holder continuous since  $pq/2 < \alpha/2 \leq 1$  to obtain

$$\begin{aligned} (4.35) \quad |m^p(x) - m^p(y)| &\leq C ||x| - |y||^{pq/2} (|x| + |y|)^{pq/2} \\ &\leq C |x-y|^{pq/2} \min \left( (|x| + |x-y| + |x|)^{pq/2}, (|y| + |x-y| + |y|)^{pq/2} \right) \\ &\leq C \left( \min \left( |x-y|^{pq/2} |x|^{pq/2}, |x-y|^{pq/2} |y|^{pq/2} \right) + |x-y|^{pq} \right) \\ &\leq C \langle x-y \rangle^{pq} \min \left( \langle x \rangle^{pq/2}, \langle y \rangle^{pq/2} \right). \end{aligned}$$

We deduce that

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^2 f) (\text{sign} f) |f|^{p-1} m^p &\leq C \int_{|z| \geq L} k_0(z) m^p(z) dz \int_{\mathbb{R}^d} |f|^p(x) \langle x \rangle^{pq/2} dx \\ &\leq \kappa_L \int_{\mathbb{R}^d} |f|^p m^p, \quad \text{with} \quad \kappa_L \approx \int_{|z| \geq L} k_0(z) m^p(z) dz \xrightarrow{L \rightarrow +\infty} 0. \end{aligned}$$

We now handle the third term  $\mathcal{B}_\varepsilon^3$  first using inequality (4.34):

$$\begin{aligned} &\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^3 f) (\text{sign} f) |f|^{p-1} m^p \\ &\leq \frac{1}{p} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(x-y) \chi_{\eta,L}(x-y) \xi_R^c(x,y) (|f|^p(y) - |f|^p(x)) m^p(x) dy dx \\ &= \frac{1}{p} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(z) \chi_{\eta,L}(z) \xi_R^c(y+z,y) |f|^p(y) (m^p(y+z) - m^p(y)) dy dz. \end{aligned}$$

We then use the Taylor-Lagrange formula which gives us the existence of  $\theta \in (0, 1)$  such that

$$m^p(y+z) = m^p(y) + z \cdot \nabla m^p(y + \theta z).$$

Notice that there exists a constant  $C_L > 0$  depending on  $L$  such that  $|\nabla m^p(y + \theta z)| \leq C_L \langle y \rangle^{pq-1}$  for any  $y \in \mathbb{R}^d$ ,  $|z| \leq 2L$ . We hence obtain

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^3 f) (\text{sign} f) |f|^{p-1} m^p &\leq C_L \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(z) |z| \chi_{\eta,L}(z) \xi_R^c(y+z,y) |f|^p(y) \langle y \rangle^{pq-1} dy dz \\ &\leq C_L \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\varepsilon(z) |z| \chi_{\eta,L}(z) \chi_R^c(y) |f|^p(y) \frac{m^p(y)}{\langle y \rangle} dy dz \\ &\leq C_L \int_{\eta \leq |z| \leq 2L} k_0(z) |z| dz \int_{|y| \geq 2R} |f|^p(y) \frac{m^p(y)}{\langle y \rangle} dy. \end{aligned}$$

which leads to

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^3 f) (\text{sign} f) |f|^{p-1} m^p \leq C_{\eta,L} \int_{\mathbb{R}^d} |f|^p(y) \frac{m^p(y)}{R} dy.$$

As a consequence, we obtain

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^3 f) (\text{sign} f) |f|^{p-1} m^p \leq \kappa_R C_{\eta,L} \int_{\mathbb{R}^d} |f| m \quad \text{with} \quad \kappa_R \approx \frac{1}{R} \xrightarrow{R \rightarrow +\infty} 0.$$

We just estimate the term involving  $\mathcal{B}_\varepsilon^4$  using that  $\xi_R(x,y) \geq \chi_R(x)$ , we get

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^4 f) (\text{sign} f) |f|^{p-1} m^p \leq - \int_{2\eta \leq |z| \leq L} k_\varepsilon(z) dz \int_{|x| \leq R} |f|^p m^p.$$

Finally, using integration by parts, we have

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon^5 f) (\text{sign} f) |f|^{p-1} m^p &= \int_{\mathbb{R}^d} |f|^p(x) m^p(x) \left( d \left( 1 - \frac{1}{p} \right) - \frac{x \cdot \nabla m^p(x)}{p m^p(x)} \right) dx \\ &\leq \int_{\mathbb{R}^d} |f|^p(x) m(x) \left( d \left( 1 - \frac{1}{p} \right) - \frac{x \cdot \nabla m^p(x)}{p m^p(x)} \right) \mathbb{1}_{|x| \geq R} dx. \end{aligned}$$

If we gather all the previous estimates and we denote

$$\psi_{\eta,L,R}^\varepsilon(x) := \kappa_\eta + \kappa_L + \kappa_R C_{\eta,L} - \int_{2\eta \leq |z| \leq L} k_\varepsilon(z) dz \mathbb{1}_{|x| \leq R} - \left( d \left( 1 - \frac{1}{p} \right) - \frac{x \cdot \nabla m^p(x)}{p m^p(x)} \right) \mathbb{1}_{|x| \geq R},$$

we obtain

$$\int_{\mathbb{R}^d} (\mathcal{B}_\varepsilon f) (\text{sign} f) |f|^{p-1} m^p \leq \int_{\mathbb{R}^d} \psi_{\eta,L,R}^\varepsilon(x) |f|^p(x) m^p(x) dx.$$

We notice that  $A_{\eta,L}^\varepsilon := \int_{2\eta \leq |z| \leq L} k_\varepsilon(z) dz \rightarrow \infty$  as  $\varepsilon \rightarrow 0$  and  $\eta \rightarrow 0$ . We can thus choose  $\varepsilon_1, \eta, L$ , and  $R$  such that for any  $\varepsilon \in [0, \varepsilon_1]$ ,

$$\kappa_\eta + \kappa_L + \kappa_R C_{\eta,L} - A_{\eta,L}^\varepsilon \leq a.$$

Then, using that  $-(x \cdot \nabla m^p(x))/(p m^p(x))$  goes to  $-q$  at infinity and that  $a > d(1 - 1/p) - q$ , up to change the value of  $R$ , we have

$$|x| \geq 2R \Rightarrow \kappa_\eta + \kappa_L + \kappa_R C_L - \left( d \left( 1 - \frac{1}{p} \right) - \frac{x \cdot \nabla m^p(x)}{p m^p(x)} \right) \leq a.$$

As a conclusion, for any  $x \in \mathbb{R}^d$ , we have  $\psi_{\eta, L, R}^\varepsilon(x) \leq a$ , which yields the result.  $\square$

**Lemma 4.24.** *Consider  $q \in (0, \alpha/2)$ . There exists  $b \in \mathbb{R}$  such that for any  $s \in \mathbb{N}$ ,  $\mathcal{B}_0 - b$  is hypodissipative in  $H^s(m)$ .*

*Proof. Step 1.* We first treat the case  $s = 0$ . We write that  $\mathcal{B}_0 = \Lambda_0 - \mathcal{A}_0$  and compute  $\int_{\mathbb{R}^d} (\mathcal{B}_0 f) f m^2$ :

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_0 f) f m^2 &= \int_{\mathbb{R}^d} (\Lambda_0 f) f m^2 - \int_{\mathbb{R}^d} (\mathcal{A}_0 f) f m^2 \\ &= \int_{\mathbb{R}^d} I_0(f) f m^2 + \int_{\mathbb{R}^d} \operatorname{div}(xf) f m^2 - \int_{\mathbb{R}^d} (\mathcal{A}_0 f) f m^2 =: T_1 + T_2 + T_3. \end{aligned}$$

Concerning  $T_1$ , we have

$$\begin{aligned} T_1 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (f(y) - f(x) - \chi(x-y)(y-x) \cdot \nabla f(x)) f(x) m^2(x) dy dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (f(y) - f(x))^2 dy m^2(x) dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (f^2(y) - f^2(x) - \chi(x-y)(y-x) \cdot \nabla f^2(x)) m^2(x) dy dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (f(y) - f(x))^2 dy m^2(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} I_0(f^2) m^2 \\ &= -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (f(y) - f(x))^2 dy m^2(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} f^2 I_0(m^2). \end{aligned}$$

Since one can prove that  $I_0(m^2)/m^2$  goes to 0 at infinity (cf Lemma 5.1 from [12]) and is thus bounded in  $\mathbb{R}^d$ , we can deduce that there exists  $C \in \mathbb{R}$  such that

$$T_1 \leq -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (f(y) - f(x))^2 dy m^2(x) dx + C \int_{\mathbb{R}^d} f^2 m^2.$$

We can notice that

$$\begin{aligned} &-\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (f(y) - f(x))^2 dy m^2(x) dx \\ &\leq -\frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) ((fm)(y) - (fm)(x))^2 dy dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (m(y) - m(x))^2 dx f^2(y) dy. \end{aligned}$$

Moreover, there exists  $\theta \in (0, 1)$  such that

$$\begin{aligned} &\int_{|x-y| \leq 1} k_0(x-y) (m(y) - m(x))^2 dx f^2(y) dy \\ &\leq \int_{|x-y| \leq 1} k_0(x-y) |x-y|^2 |\nabla m(x + \theta(y-x))|^2 dx f^2(y) dy \\ &\leq C \int_{|z| \leq 1} k_0(z) |z|^2 dz \int_{\mathbb{R}^d} f^2 m^2 \\ &\leq C \int_{\mathbb{R}^d} f^2 m^2 \end{aligned}$$

and

$$\begin{aligned}
& \int_{|x-y|\geq 1} k_0(x-y) (m(y) - m(x))^2 dx f^2(y) dy \\
& \leq C \int_{|x-y|\geq 1} k_0(x-y) (m^2(y) + m^2(x)) dx f^2(y) dy \\
& \leq C \int_{|z|\geq 1} k_0(z) m^p(z) dz \int_{\mathbb{R}^d} f^2 m^2 \\
& \leq C \int_{\mathbb{R}^d} f^2 m^2.
\end{aligned}$$

We here recall that the homogeneous Sobolev space  $\dot{H}^s$  for  $s$  a real number is the set of tempered distributions  $u$  such that  $\widehat{u}$  belongs to  $L^1_{loc}$  and

$$\|u\|_{\dot{H}^s}^2 := \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

Moreover, if  $s \in (0, 1)$ , one can prove that there exists a constant  $c_0 > 0$  such that

$$\|u\|_{\dot{H}^s}^2 = c_0^{-1} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+2s}} dx dy$$

from which we deduce the following important identity:

$$(4.36) \quad c_0 \|u\|_{\dot{H}^{\alpha/2}}^2 = \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k_0(x-y) dx dy \quad \forall \alpha \in (0, 2).$$

As a consequence, up to change the value of  $C$ , we have

$$T_1 \leq -\frac{c_0}{4} \|f m\|_{\dot{H}^{\alpha/2}}^2 + C \int_{\mathbb{R}^d} f^2 m^2.$$

Then, we compute

$$T_2 = \int_{\mathbb{R}^d} f^2 m^2 \left( \frac{d}{2} - \frac{x \cdot \nabla m^2}{2 m^2} \right) \leq \frac{d}{2} \int_{\mathbb{R}^d} f^2 m^2.$$

Concerning  $T_3$ , we use Lemma 4.22 and Cauchy-Schwarz inequality:

$$T_3 \leq \|\mathcal{A}_0 f\|_{L^2(m)} \|f\|_{L^2(m)} \leq C \|f\|_{L^2(m)}^2.$$

As a consequence, gathering the three previous inequalities, we have

$$\int_{\mathbb{R}^d} (\mathcal{B}_0 f) f m^2 \leq -\frac{c_0}{4} \|f m\|_{\dot{H}^{\alpha/2}}^2 + b_0 \int_{\mathbb{R}^d} f^2 m^2, \quad b_0 \in \mathbb{R}.$$

*Step 2.* We now consider  $b > b_0$  and we prove that for any  $s \in \mathbb{N}$ ,  $\mathcal{B}_0 - b$  is hypodissipative in  $H^s(m)$ . For  $s \in \mathbb{N}^*$ , we recall the definition of the triple norm introduce in (2.18):

$$\|f\|_{H^s(m)}^2 = \sum_{j=0}^s \eta^j \|\partial_x^j f\|_{L^2(m)}^2, \quad \eta > 0$$

which is equivalent to the classical  $H^s(m)$  norm. We use again that  $\mathcal{B}_0 = \Lambda_0 - \mathcal{A}_0$  and we only deal with the case  $s = 1$ , the higher order derivatives being treated in the same way. First, we have

$$\partial_x(\mathcal{B}_0 f) = \Lambda_0(\partial_x f) + \partial_x f - \partial_x(\mathcal{A}_0 f).$$

Then, we can notice that

$$\mathcal{A}_0 f(x) = \int_{\mathbb{R}^d} k_0(z) \chi_{\eta,L}(z) \xi_R(x, x+z) f(x+z) dz$$

so that

$$\partial_x(\mathcal{A}_0 f)(x) = \mathcal{A}_0(\partial_x f)(x) + \widetilde{\mathcal{A}}_0 f(x), \quad \text{with } \|\widetilde{\mathcal{A}}_0 f\|_{L^2(m)} \leq C \|f\|_{L^2},$$

where the last inequality is obtained thanks to inequality (4.33) as in the proof of Lemma 4.22. We deduce that

$$\partial_x(\mathcal{B}_0 f) = \mathcal{B}_0(\partial_x f) + \partial_x f - \widetilde{\mathcal{A}}_0 f.$$

Then, doing the same computations as in the case  $s = 0$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \partial_x(\mathcal{B}_0 f) (\partial_x f) m^2 &= \int_{\mathbb{R}^d} \mathcal{B}_0(\partial_x f) (\partial_x f) m^2 + \int_{\mathbb{R}^d} (\partial_x f)^2 m^2 - \int_{\mathbb{R}^d} \widetilde{\mathcal{A}}_0 f (\partial_x f) m^2 \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

with

$$\begin{aligned} J_1 &\leq -\frac{c_0}{4} \|(\partial_x f) m\|_{\dot{H}^{\alpha/2}}^2 + b_0 \int_{\mathbb{R}^d} (\partial_x f)^2 m^2 \\ &\leq -\frac{c_0}{8} \|f m\|_{\dot{H}^{1+\alpha/2}}^2 + \frac{c_0}{4} \|f \partial_x m\|_{\dot{H}^{\alpha/2}}^2 + b_0 \int_{\mathbb{R}^d} (\partial_x f)^2 m^2 \\ &\leq -\frac{c_0}{8} \|f m\|_{\dot{H}^{1+\alpha/2}}^2 + C \left( \|f\|_{L^2(m)}^2 + \|f m\|_{\dot{H}^1}^2 \right), \end{aligned}$$

and also

$$J_2 \leq \frac{1}{2} \left( \|f\|_{L^2(m)}^2 + \|f m\|_{\dot{H}^1}^2 \right)$$

and finally using Cauchy-Schwarz inequality, we have

$$J_3 \leq \|\widetilde{\mathcal{A}}_0 f\|_{L^2(m)} \|\partial_x f\|_{L^2(m)} \leq C \left( \|f\|_{L^2(m)}^2 + \|f m\|_{\dot{H}^1}^2 \right).$$

As a consequence, we have

$$\int_{\mathbb{R}^d} \partial_x(\mathcal{B}_0 f) (\partial_x f) m^2 \leq -\frac{c_0}{8} \|f m\|_{\dot{H}^{1+\alpha/2}}^2 + b_1 \left( \|f\|_{L^2(m)}^2 + \|f m\|_{\dot{H}^1}^2 \right), \quad b_1 \in \mathbb{R}.$$

We now introduce  $f_t$  solution of

$$\partial_t f_t = \mathcal{B}_0 f_t, \quad f_0 = f$$

and we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f_t\|_{\dot{H}^1(m)}^2 &= \int_{\mathbb{R}^d} (\mathcal{B}_0 f_t) f_t m^2 + \eta \int_{\mathbb{R}^d} \partial_x(\mathcal{B}_0 f_t) (\partial_x f_t) m^2 \\ &\leq -\frac{c_0}{4} \|f_t m\|_{\dot{H}^{\alpha/2}}^2 - \eta \frac{c_0}{8} \|f_t m\|_{\dot{H}^{1+\alpha/2}}^2 \\ &\quad + \|f_t\|_{L^2(m)}^2 (b_0 + \eta b_1) + \eta b_1 \|f_t m\|_{\dot{H}^1}^2. \end{aligned}$$

We now use the following interpolation inequality:

$$\|h\|_{\dot{H}^1} \leq \|h\|_{\dot{H}^{\alpha/2}}^{\alpha/2} \|h\|_{\dot{H}^{1+\alpha/2}}^{1-\alpha/2},$$

which implies

$$(4.37) \quad \|h\|_{\dot{H}^1}^2 \leq K(\zeta) \|h\|_{\dot{H}^{\alpha/2}}^2 + \zeta \|h\|_{\dot{H}^{1+\alpha/2}}^2, \quad \zeta > 0.$$

We obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|f_t\|_{\dot{H}^1(m)}^2 \\ &\leq \left( -\frac{c_0}{4} + \eta b_1 K(\zeta) \right) \|f_t m\|_{\dot{H}^{\alpha/2}}^2 + \eta \left( -\frac{c_0}{8} + \zeta b_1 \right) \|f_t m\|_{\dot{H}^{1+\alpha/2}}^2 + \|f_t\|_{L^2(m)}^2 (b_0 + \eta b_1). \end{aligned}$$

Choosing  $\zeta$  small enough so that  $-c_0/8 + \zeta b_1 < 0$  and then  $\eta$  small enough so that  $-c_0/4 + \eta b_1 K(\zeta) < 0$  and  $b_0 + \eta b_1 < b$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|f_t\|_{\dot{H}^1(m)}^2 \leq b \|f_t\|_{\dot{H}^1(m)}^2$$

which concludes the proof in the case  $s = 1$ .  $\square$

We now introduce the “renormalized” operator  $\mathcal{B}_{0,m}$  defined by

$$(4.38) \quad \mathcal{B}_{0,m}(h) = m \mathcal{B}_0(m^{-1}h).$$

**Corollary 4.25.** *Consider  $q$  such that  $2q < \alpha$ . There exists  $b \in \mathbb{R}$  such that for any  $s \in \mathbb{N}$ ,  $\mathcal{B}_{0,m} - b$  is hypodissipative in  $H^s$ .*

*Proof.* The proof comes from Lemma 4.24 and is immediate noticing that the norms defined on  $H^s(m)$  by

$$\|f\|_1^2 = \sum_{j=0}^s \|\partial_x^j f\|_{L^2(m)}^2 \quad \text{and} \quad \|f\|_2^2 := \|f m\|_{H^s}^2$$

are equivalent.  $\square$

**Lemma 4.26.** *Consider  $q$  such that  $2q < \alpha$ . There exists  $b \in \mathbb{R}$  such that for any  $s \in \mathbb{N}$ ,  $\mathcal{B}_{0,m} - b$  is hypodissipative in  $H^{-s}$ , (or equivalently,  $\mathcal{B}_0 - b$  is hypodissipative in  $H^{-s}(m)$ ).*

*Proof.* We introduce the dual operator of  $\mathcal{B}_{0,m}$  defined by:

$$\mathcal{B}_{0,m}^* \phi = \omega I_0(m \phi) - x \cdot \nabla \phi - \frac{x \cdot \nabla m}{m} \phi - \omega \mathcal{A}_0(m \phi)$$

where  $\omega := m^{-1}$ . We now want to prove that  $\mathcal{B}_{0,m}^*$  is hypodissipative in  $H^s$ .

*Step 1.* We consider first the case  $s = 0$  and we estimate  $\int_{\mathbb{R}^d} (\mathcal{B}_{0,m}^* \phi) \phi$ :

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}_{0,m}^* \phi) \phi &= \int_{\mathbb{R}^d} I_0(m \phi) \omega \phi - \int_{\mathbb{R}^d} x \cdot (\nabla \phi) \phi - \int_{\mathbb{R}^d} \frac{x \cdot \nabla m}{m} \phi^2 - \int_{\mathbb{R}^d} \omega \mathcal{A}_0(m \phi) \phi \\ &=: T_1 + \dots + T_4. \end{aligned}$$

We have

$$T_2 = \frac{d}{2} \int_{\mathbb{R}^d} \phi^2 \quad \text{and} \quad T_3 \leq 0.$$

Next, using (4.33), we have  $\|\mathcal{A}_0(m \phi)\|_{L^2} \leq C \|\mathcal{A}_0(|\phi|)\|_{L^2}$  and thus

$$T_4 \leq C (\|\mathcal{A}_0(|\phi|)\|_{L^2}^2 + \|\phi\|_{L^2}^2) \leq C \|\phi\|_{L^2}^2$$

from Lemma 4.22. Let us now estimate  $T_1$ .

*Case  $\alpha < 1$ .* We can write that

$$\begin{aligned} T_1 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) ((m\phi)(y) - (m\phi)(x)) \omega(x) \phi(x) dy dx \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (\phi(y) - \phi(x)) \phi(x) dy dx \\ &\quad + \int_{|x-y| \leq 1} k_0(x-y) (m(y) - m(x)) \omega(x) \phi(y) \phi(x) dy dx \\ &\quad + \int_{|x-y| \geq 1} k_0(x-y) (m(y) - m(x)) \omega(x) \phi(y) \phi(x) dy dx \\ &=: T_{11} + T_{12} + T_{13}. \end{aligned}$$

Let us point out here that from (4.36), we have

$$\begin{aligned} T_{11} &= \int_{\mathbb{R}^d} I_0(\phi) \phi \\ &= -\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (\phi(y) - \phi(x))^2 dy dx + \frac{1}{2} \int_{\mathbb{R}^d} I_0(\phi^2) \\ &= -\frac{c_0}{2} \|\phi\|_{\dot{H}^{\alpha/2}}^2. \end{aligned}$$

Then, using a Taylor expansion, there exists  $\theta \in (0, 1)$  such that

$$\begin{aligned} (4.39) \quad T_{12} &= \int_{|x-y| \leq 1} k_0(x-y) (m(y) - m(x)) \omega(x) \phi(y) \phi(x) dy dx \\ &= \int_{|x-y| \leq 1} k_0(x-y) \nabla m(x + \theta(y-x)) \cdot (y-x) \omega(x) \phi(y) \phi(x) dy dx \\ &\leq C \int_{|x-y| \leq 1} k_0(x-y) |x-y| |\nabla m(x + \theta(y-x))| \omega(x) (\phi^2(y) + \phi^2(x)) dy dx. \end{aligned}$$



Using then that for any  $x, y \in \mathbb{R}^d$ ,  $|x - y| \leq 1$ , we have  $|\nabla m(x + \theta(y - x))| \omega(x) \leq C$  and that  $\alpha < 1$ , we deduce that

$$T_{12} \leq C \int_{\mathbb{R}^d} \phi^2.$$

Concerning  $T_{13}$ , we have from (4.35)

$$|m(y) - m(x)| \leq C \langle x - y \rangle^q \min(\langle x \rangle^{q/2}, \langle y \rangle^{q/2}),$$

from which we deduce that we also have

$$T_{13} \leq C \int_{\mathbb{R}^d} \phi^2.$$

We thus obtain

$$T_1 \leq -\frac{c_0}{2} \|\phi\|_{\dot{H}^{\alpha/2}}^2 + C \int_{\mathbb{R}^d} \phi^2.$$

*Case  $\alpha \in [1, 2)$ .* We have:

$$\begin{aligned} T_1 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x - y) ((m\phi)(y) - (m\phi)(x) - \nabla(m\phi)(x) \cdot (y - x) \chi(x - y)) \omega(x) \phi(x) dy dx \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x - y) (\phi(y) - \phi(x) - \nabla\phi(x) \cdot (y - x) \chi(x - y)) \phi(x) dy dx \\ &\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x - y) [(m(y) - m(x)) \phi(y) - \nabla m(x) \cdot (y - x) \chi(x - y)] \omega(x) \phi(x) dy dx \\ &= \int_{\mathbb{R}^d} I_0(\phi) \phi \\ &\quad + \int_{|x-y| \leq 1} k_0(x - y) (m(y) - m(x) - (y - x) \cdot \nabla m(x) \chi(x - y)) \omega(x) \phi(y) \phi(x) dy dx \\ &\quad + \int_{|x-y| \geq 1} k_0(x - y) (m(y) - m(x) - (y - x) \cdot \nabla m(x) \chi(x - y)) \omega(x) \phi(y) \phi(x) dy dx \\ &\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x - y) (\phi(y) - \phi(x)) \phi(x) \omega(x) \nabla m(x) \cdot (y - x) \chi(y - x) dy dx \\ &=: T_{11} + T_{12} + T_{13} + T_{14}. \end{aligned}$$

We still have

$$T_{11} = -\frac{c_0}{2} \|\phi\|_{\dot{H}^{\alpha/2}}^2.$$

Arguing similarly as for  $T_{12}$  in (4.39) i.e. using a Taylor expansion (at order 2 instead of 1), we obtain

$$T_{12} \leq C \int_{\mathbb{R}^d} \phi^2.$$

Next, we split  $T_{13}$  into two parts:

$$\begin{aligned} T_{13} &\leq C \int_{|x-y| \geq 1} k_0(x - y) |m(y) - m(x)| \omega(x) (\phi^2(x) + \phi^2(y)) dx dy \\ &\quad + C \int_{1 \leq |x-y| \leq 2} k_0(x - y) |x - y| |\nabla m(x)| \omega(x) (\phi^2(x) + \phi^2(y)) dx dy \\ &\leq C \int_{|x-y| \geq 1} k_0(x - y) \langle x - y \rangle^q \langle x \rangle^{-q/2} (\phi^2(x) + \phi^2(y)) dx dy \\ &\quad + C \int_{1 \leq |x-y| \leq 2} k_0(x - y) (\phi^2(x) + \phi^2(y)) dx dy \end{aligned}$$

where we have used (4.35), we thus obtain:

$$T_{13} \leq C \int_{\mathbb{R}^d} \phi^2.$$

Concerning  $T_{14}$ , we use Young inequality which implies that for any  $\zeta > 0$ ,

$$\begin{aligned} T_{14} &\leq \zeta \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) (\phi(y) - \phi(x))^2 dy dx \\ &\quad + K(\zeta) \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) \phi^2(x) \frac{|\nabla m(x)|^2}{m^2(x)} |y-x|^2 \chi^2(x-y) dy dx \\ &\leq \zeta c_0 \|\phi\|_{\dot{H}^{\alpha/2}}^2 + K(\zeta) \int_{|z| \leq 2} k(z) |z|^2 dz \int_{\mathbb{R}^d} \phi^2. \end{aligned}$$

Consequently, taking  $\zeta > 0$  small enough, we have

$$T_1 \leq -\frac{c_0}{4} \|\phi\|_{\dot{H}^{\alpha/2}}^2 + C \int_{\mathbb{R}^d} \phi^2.$$

We hence conclude that

$$\int_{\mathbb{R}^d} (\mathcal{B}_{0,m}^* \phi) \phi \leq -\frac{c_0}{4} \|\phi\|_{\dot{H}^{\alpha/2}}^2 + b_0 \int_{\mathbb{R}^d} \phi^2, \quad b_0 \in \mathbb{R}.$$

*Step 2.* We now consider  $b > b_0$  and we prove that for any  $s \in \mathbb{N}$ ,  $\mathcal{B}_{0,m}^* - b$  is hypodissipative in  $H^s$ . As in (2.18), for  $s \in \mathbb{N}^*$ , we introduce the norm

$$\|\phi\|_{H^s}^2 := \sum_{j=0}^s \eta^j \|\partial_x^j \phi\|_{L^2}^2, \quad \eta > 0$$

which is equivalent to the classical  $H^s$  norm. We only deal with the case  $s = 1$ , the higher order derivatives are treated in the same way. First, using the identity (4.32) (with  $k_0$  instead of  $k_{0,\varepsilon}$ ), we notice that

$$\mathcal{B}_{0,m}^* \phi = I_0(\phi) + \omega \mathcal{C}_m^1(\phi) + \omega \mathcal{C}_m^2(\phi) - x \cdot \nabla \phi - \frac{x \cdot \nabla m}{m} \phi - \omega \mathcal{A}_0(m \phi)$$

where

$$\begin{aligned} \mathcal{C}_m^1(\phi)(x) &= \int_{\mathbb{R}^d} k_0(x-y) \phi(y) (m(y) - m(x) - (y-x) \cdot \nabla m(x) \chi(x-y)) dy \\ &= \int_{\mathbb{R}^d} k_0(z) \phi(x+z) (m(x+z) - m(x) - z \cdot \nabla m(x) \chi(z)) dz \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_m^2(\phi)(x) &= \int_{\mathbb{R}^d} k_0(x-y) (\phi(y) - \phi(x)) \nabla m(x) \cdot (y-x) \chi(x-y) dy \\ &= \int_{\mathbb{R}^d} k_0(z) (\phi(x+z) - \phi(x)) \nabla m(x) \cdot z \chi(z) dz. \end{aligned}$$

Before going into the computation of  $\partial_x(\mathcal{B}_{0,m}^* \phi)$ , we also notice that

$$\partial_x(\omega \mathcal{A}_0(m \phi)) = \omega \mathcal{A}_0(m \partial_x \phi) + \widehat{\mathcal{A}_{0,m}}(\phi)$$

where  $\widehat{\mathcal{A}_{0,m}}$  satisfies

$$\|\widehat{\mathcal{A}_{0,m}}(\phi)\|_{L^2} \leq C \|\phi\|_{L^2}$$

thanks to (4.33). Consequently, we have

$$\begin{aligned} \partial_x(\mathcal{B}_{0,m}^* \phi) &= \mathcal{B}_{0,m}^*(\partial_x \phi) + \omega \mathcal{C}_{\partial_x m}^1(\phi) + \omega \mathcal{C}_{\partial_x m}^2(\phi) + \partial_x \omega \mathcal{C}_m^1(\phi) + \partial_x \omega \mathcal{C}_m^2(\phi) \\ &\quad - \partial_x \phi - \partial_x \left( \frac{x \cdot \nabla m}{m} \right) \phi - \widehat{\mathcal{A}_{0,m}}(\phi) \end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^d} \partial_x (\mathcal{B}_{0,m}^* \phi) \partial_x \phi &= \int_{\mathbb{R}^d} \mathcal{B}_{0,m}^* (\partial_x \phi) (\partial_x \phi) + \int_{\mathbb{R}^d} \omega \mathcal{C}_{\partial_x m}^1 (\phi) (\partial_x \phi) + \int_{\mathbb{R}^d} \omega \mathcal{C}_{\partial_x m}^2 (\phi) (\partial_x \phi) \\
&\quad + \int_{\mathbb{R}^d} \partial_x \omega \mathcal{C}_m^1 (\phi) (\partial_x \phi) + \int_{\mathbb{R}^d} \partial_x \omega \mathcal{C}_m^2 (\phi) (\partial_x \phi) - \int_{\mathbb{R}^d} (\partial_x \phi)^2 \\
&\quad - \int_{\mathbb{R}^d} \partial_x \left( \frac{x \cdot \nabla m}{m} \right) \phi (\partial_x \phi) - \int_{\mathbb{R}^d} \widehat{\mathcal{A}_{0,m}} (\phi) (\partial_x \phi) \\
&=: J_1 + \dots + J_8.
\end{aligned}$$

We have from the step 1 of the proof

$$J_1 \leq -\frac{c_0}{4} \|\phi\|_{\dot{H}^{1+\alpha/2}}^2 + b_0 \int_{\mathbb{R}^d} (\partial_x \phi)^2.$$

Moreover, we easily obtain that

$$J_6 + J_7 + J_8 \leq C \left( \int_{\mathbb{R}^d} \phi^2 + \int_{\mathbb{R}^d} (\partial_x \phi)^2 \right).$$

The term  $J_2$  is first separated into two parts:

$$\begin{aligned}
J_2 &= \\
&\int_{|x-y| \leq 1} k_0(x-y) \phi(y) ((\partial_x m)(y) - (\partial_x m)(x) - (y-x) \cdot \nabla (\partial_x m)(x) \chi(x-y)) \omega(x) \partial_x \phi(x) dy dx \\
&+ \int_{|x-y| \geq 1} k_0(x-y) \phi(y) ((\partial_x m)(y) - (\partial_x m)(x) - (y-x) \cdot \nabla (\partial_x m)(x) \chi(x-y)) \omega(x) \partial_x \phi(x) dy dx \\
&=: J_{21} + J_{22}.
\end{aligned}$$

The term  $J_{21}$  is treated as  $T_{12}$  is the step 1 of the proof. Concerning  $J_{22}$ , as for  $T_{13}$ , we divide it into two parts:

$$\begin{aligned}
J_{22} &\leq \int_{|x-y| \geq 1} k_0(x-y) |(\partial_x m)(y) - (\partial_x m)(x)| \omega(x) (\phi^2(y) + (\partial_x \phi)^2(x)) dx dy \\
&\quad + \int_{1 \leq |x-y| \leq 2} k_0(x-y) |x-y| |\nabla (\partial_x m)(x)| \omega(x) (\phi^2(y) + (\partial_x \phi)^2(y)) dx dy \\
&\leq C \int_{|x-y| \geq 1} k_0(x-y) (\phi^2(y) + (\partial_x \phi)^2(x)) dx dy \\
&\quad + C \int_{1 \leq |x-y| \leq 2} k_0(x-y) (\phi^2(y) + (\partial_x \phi)^2(y)) dx dy,
\end{aligned}$$

where the second inequality comes from the fact that

$$|(\partial_x m)(y) - (\partial_x m)(x)| \omega(x) \leq C \quad \text{and} \quad |\nabla (\partial_x m)(x)| \omega(x) \leq C \quad \forall x, y \in \mathbb{R}^d$$

because  $q < \alpha/2 < 1$ . We hence deduce that

$$J_2 \leq C \left( \int_{\mathbb{R}^d} \phi^2 + \int_{\mathbb{R}^d} (\partial_x \phi)^2 \right).$$

Concerning  $J_3$ , we perform a Taylor expansion of  $\phi$  and use that  $|\nabla (\partial_x m)| \omega \in L^\infty(\mathbb{R}^d)$ :

$$\begin{aligned}
(4.40) \quad J_3 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} k_0(x-y) \int_0^1 (1-t) \nabla \phi(y + t(x-y)) \cdot (y-x) dt \\
&\quad \nabla (\partial_x m)(x) \cdot (y-x) \chi(x-y) \omega(x) \partial_x \phi(x) dy dx \\
&\leq C \int_{|x-y| \leq 2} k_0(x-y) |x-y|^2 \int_0^1 |\nabla \phi(x + t(y-x))| dt |\partial_x \phi(x)| dy dx \\
&\leq C \int_{|z| \leq 2} k_0(z) |z|^2 \int_0^1 |\nabla \phi(x + tz)|^2 dt dz dx + \int_{|z| \leq 2} k_0(z) |z|^2 |\partial_x \phi(x)|^2 dz dx
\end{aligned}$$

where we have used Jensen inequality with the probability measure  $\mathbb{1}_{[0,1]}(t) dt$  and Young inequality. We use a change of variable for the first term of the RHS of (4.40), it implies that

$$J_3 \leq C \|\phi\|_{\dot{H}^1}^2.$$

We deal with  $J_4$  splitting it into two parts ( $|x-y| \leq 1$  and  $|x-y| \geq 1$ ) and using the same method as for  $T_{12}$  and  $T_{13}$  in the step 1 of the proof, we obtain

$$J_4 \leq C \left( \int_{\mathbb{R}^d} \phi^2 + \int_{\mathbb{R}^d} (\partial_x \phi)^2 \right).$$

To deal with  $J_5$ , we proceed exactly as for  $J_3$  and obtain

$$J_5 \leq C \|\phi\|_{\dot{H}^1}^2.$$

Summarizing the previous inequalities and using (4.37), we obtain that for any  $\zeta > 0$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} \partial_x (\mathcal{B}_{0,m}^* \phi) \partial_x \phi &\leq -\frac{c_0}{4} \|\phi\|_{\dot{H}^{1+\alpha/2}}^2 + b_1 (\|\phi\|_{L^2}^2 + \|\phi\|_{\dot{H}^1}^2) \\ &\leq -\frac{c_0}{4} \|\phi\|_{\dot{H}^{1+\alpha/2}}^2 + b_1 (\|\phi\|_{L^2}^2 + K(\zeta) \|\phi\|_{\dot{H}^{\alpha/2}}^2 + \zeta \|\phi\|_{\dot{H}^{1+\alpha/2}}^2), \quad b_1 \in \mathbb{R}. \end{aligned}$$

This implies that if  $\phi_t$  is the solution of

$$\partial_t \phi_t = \mathcal{B}_{0,m}^* \phi_t, \quad \phi_0 = \phi$$

then

$$\frac{1}{2} \frac{d}{dt} \|\phi_t\|_{\dot{H}^1}^2 \leq \left( -\frac{c_0}{4} + \eta b_1 K(\zeta) \right) \|\phi_t\|_{\dot{H}^{\alpha/2}}^2 + \eta \left( -\frac{c_0}{4} + \zeta b_1 \right) \|\phi_t\|_{\dot{H}^{1+\alpha/2}}^2 + (b_0 + \eta b_1) \|\phi_t\|_{L^2}^2.$$

Taking  $\zeta$  and  $\eta$  small enough, we deduce that

$$\frac{1}{2} \frac{d}{dt} \|\phi_t\|_{\dot{H}^1}^2 \leq b \|\phi_t\|_{\dot{H}^1}^2,$$

this concludes the proof in the case  $s = 1$ .  $\square$

We now fix  $0 < q < \alpha/2$ . From Lemma 4.23 applied with  $p = 1$ , there exists  $a < 0$  such that  $\mathcal{B}_\varepsilon - a$  is dissipative in  $L^1(m)$  for any  $\varepsilon \in [0, \varepsilon_1]$  (or equivalently,  $\mathcal{B}_{\varepsilon,m} - a$  is dissipative in  $L^1$  where  $\mathcal{B}_{\varepsilon,m}$  is defined as  $\mathcal{B}_{0,m}$  in (4.38)). From Lemma 4.23 applied with  $p = 2$ , Corollary 4.25 and Lemma 4.26, there exists  $b \in \mathbb{R}$  such that  $\mathcal{B}_\varepsilon - b$  is dissipative in  $L^2(m)$  for any  $\varepsilon \in [0, \varepsilon_1]$  (or equivalently,  $\mathcal{B}_{\varepsilon,m} - b$  is dissipative in  $L^2$ ),  $\mathcal{B}_{0,m} - b$  is hypodissipative in  $H^s$  and  $H^{-s}$  for any  $s \in \mathbb{N}^*$ .

We choose  $\theta \in (0, 1)$  such that  $a_\theta := a\theta + b(1-\theta) < 0$  and  $2/(1-\theta) \in \mathbb{N}$ . We introduce  $p_\theta := 2/(1-\theta)$  and we denote

$$X_1 := W^{2,p_\theta}(m) \subset X_0 := L^{p_\theta}(m) \subset X_{-1} := W^{-2,p_\theta}(m).$$

**Lemma 4.27.** *The operator  $\mathcal{B}_0 - a_\theta$  is hypodissipative in  $X_i$ ,  $i = -1, 0, 1$  and the operator  $\mathcal{B}_\varepsilon - a_\theta$  is dissipative in  $X_0$  for any  $\varepsilon \in (0, \varepsilon_1]$ .*

*Proof.* We prove that  $\mathcal{B}_{0,m} - a_\theta$  is hypodissipative in  $W^{-2,p_\theta}$ ,  $L^{p_\theta}$  and  $W^{2,p_\theta}$  by interpolation. To conclude for  $X_0$ , we just have to interpolate the results coming from Lemma 4.23 with  $p = 1$  and Lemma 4.24 with  $s = 0$  and use the fact that  $[L^1, L^2]_\theta = L^{p_\theta}$  with  $1/p_\theta = \theta + (1-\theta)/2$  i.e.  $p_\theta = 2/(1-\theta)$ . Then, for  $X_1$  and  $X_{-1}$ , we first choose  $s_0$  large enough so that  $s_0(1-\theta) = 2$ . We then have  $[L^1, H^{s_0}]_\theta = W^{2,p_\theta}$ ,  $[L^1, H^{-s_0}]_\theta = W^{-2,p_\theta}$  and we conclude thanks to Lemma 4.23 with  $p = 1$  and Lemma 4.24 with  $s = s_0$ .

We prove that  $\mathcal{B}_\varepsilon - a_\theta$  is dissipative in  $X_0$  exactly in the same way as we proved that  $\mathcal{B}_0 - a_\theta$  is dissipative in  $X_0$ .  $\square$

**4.5. Spectral analysis.** We here divide the proof of Theorem 4.20 into two parts, using Krein Rutman theory for the first part and using both perturbative and enlargement arguments for the second part.

*Proof of part (1) of Theorem 4.20.* First, we notice that as in Section 2 (Lemmas 2.11 and 2.12), we can prove that the operator  $\Lambda_\varepsilon$  satisfies Kato's inequalities,  $S_{\Lambda_\varepsilon}$  is a positive semigroup and  $-\Lambda_\varepsilon$  satisfies a strong maximum principle. Using Krein-Rutman theory, this gives the first part of Theorem 4.20 i.e. that there exists a unique  $G_\varepsilon > 0$  such that  $\|G_\varepsilon\|_{L^1} = 1$ ,  $\Lambda_\varepsilon G_\varepsilon = 0$ . Moreover, it also implies that  $\Pi_\varepsilon f = \langle f \rangle G_\varepsilon$ .

*Proof of part (2) of Theorem 4.20.* We first develop a perturbative argument which is detailed in what follows, improving a bit similar results presented in [6, 11]. We then ends the proof using an enlargement argument.

**Lemma 4.28.** *For any  $z \in \Omega := \Delta_{a_\theta} \setminus \{0\}$  we define the family of operators*

$$K_\varepsilon(z) := -(\Lambda_\varepsilon - \Lambda_0) \mathcal{R}_{\Lambda_0}(z) (\mathcal{A} \mathcal{R}_{\mathcal{B}_\varepsilon}(z)).$$

*There exists a function  $\eta_2(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$  such that*

$$(4.41) \quad \|K_\varepsilon(z)\|_{\mathcal{B}(X_0)} \leq \eta_2(\varepsilon) \quad \forall z \in \Omega_\varepsilon := \Delta_a \setminus \bar{B}_\varepsilon, \quad B_\varepsilon := B(0, \eta_2(\varepsilon)).$$

*Moreover, there exists  $\varepsilon_2 \in (0, \varepsilon_1)$  such that for any  $\varepsilon \in (0, \varepsilon_2)$  the operators  $I + K_\varepsilon(z)$  and  $\Lambda_\varepsilon - z$  are invertible for any  $z \in \Omega_\varepsilon$  and*

$$\forall z \in \Omega_\varepsilon, \quad \mathcal{R}_{\Lambda_\varepsilon}(z) = \mathcal{U}_\varepsilon(z) (I + K_\varepsilon(z))^{-1}$$

*with*

$$\mathcal{U}_\varepsilon := \mathcal{R}_{\mathcal{B}_\varepsilon} - \mathcal{R}_{\Lambda_0} (\mathcal{A} \mathcal{R}_{\mathcal{B}_\varepsilon}).$$

*As an immediate consequence, there holds*

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_\theta} \subset \bar{B}_\varepsilon.$$

*Proof.* We know that the operators  $\mathcal{A} \mathcal{R}_{\mathcal{B}_\varepsilon}(z) : X_0 \rightarrow X_1$  (from Lemmas 4.22 and 4.27) and  $\mathcal{R}_{\Lambda_0}(z) : X_1 \rightarrow X_1$  (previous works from [4, 6]) are bounded for any  $z \in \Omega$  and that the operators  $\Lambda_\varepsilon - \Lambda_0 : X_1 \rightarrow X_0$  are small as  $\varepsilon \rightarrow 0$  uniformly in  $z \in \Omega$  (Lemma 4.21). Because 0 is a simple eigenvalue, we have

$$\|\mathcal{R}_{\Lambda_0}(z)\|_{\mathcal{B}(X_1)} \leq C |z|^{-1} \quad \forall z \in \Omega.$$

for some  $C > 0$ . We introduce the constant  $C_{a_\theta} > 0$  (coming from Lemmas 4.22 and 4.27) such that

$$\|\mathcal{A} \mathcal{S}_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{B}(X_0, X_1)} \leq C_{a_\theta} e^{a_\theta t}.$$

Defining  $\eta_2(\varepsilon) := (C C_{a_\theta} \eta_1(\varepsilon))^{1/2}$ , we deduce that for any  $z \in \Omega_\varepsilon$ ,

$$(4.42) \quad \|K_\varepsilon(z)\|_{\mathcal{B}(X_0)} \leq \eta_1(\varepsilon) \frac{C}{\eta_2(\varepsilon)} C_{a_\theta} = \eta_2(\varepsilon).$$

We choose  $\varepsilon_2 > 0$  such that  $\eta_2(\varepsilon) < 1$  for any  $\varepsilon \in (0, \varepsilon_2)$ , we thus obtain that  $\|K_\varepsilon(z)\| < 1$  for any  $\varepsilon \in (0, \varepsilon_2)$  and  $z \in \Omega_\varepsilon$ , which implies that  $I + K_\varepsilon(z)$  is invertible.

We compute

$$\begin{aligned} (\Lambda_\varepsilon - z) \mathcal{U}_\varepsilon &= (\mathcal{B}_\varepsilon - z + \mathcal{A}) \mathcal{R}_{\mathcal{B}_\varepsilon} - (\Lambda_\varepsilon - \Lambda_0 + \Lambda_0 - z) \mathcal{R}_{\Lambda_0} \mathcal{A} \mathcal{R}_{\mathcal{B}_\varepsilon} \\ &= Id + K_\varepsilon. \end{aligned}$$

For  $z \in \Omega_\varepsilon$ ,  $\varepsilon \in (0, \varepsilon_2)$ , we denote  $\mathcal{J}_\varepsilon(z) := \mathcal{U}_\varepsilon(z) (I + K_\varepsilon(z))^{-1}$ , so that

$$(\Lambda_\varepsilon - z) \mathcal{J}_\varepsilon(z) = Id,$$

which implies that  $\Lambda_\varepsilon - z$  has a right-inverse  $\mathcal{J}_\varepsilon(z)$ .

Since  $\Lambda_\varepsilon - z$  is invertible for  $\Re z$  large enough and  $\mathcal{J}_\varepsilon(z)$  is uniformly locally bounded in  $\Omega_\varepsilon$ , we deduce that  $\Lambda_\varepsilon - z$  is invertible in  $\Omega_\varepsilon$ , and its inverse is its right-inverse  $\mathcal{J}_\varepsilon(z)$ .  $\square$

**Lemma 4.29.** *Let us denote*

$$\Pi_\varepsilon := \frac{i}{2\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\Lambda_\varepsilon}(z) dz, \quad \Gamma_\varepsilon := \{z \in \mathbb{C} : |z| = \eta_2(\varepsilon)\}$$

*the spectral projector onto eigenspaces associated to eigenvalues contained in  $\bar{B}_\varepsilon$ . There exists  $\eta_3(\varepsilon)$  such that*

$$\|\Pi_\varepsilon - \Pi_0\|_{\mathcal{B}(X_0)} \leq \eta_3(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

*Proof.* First, we have

$$\begin{aligned} \Pi_\varepsilon &= \frac{i}{2\pi} \int_{\Gamma_\varepsilon} \{\mathcal{R}_{\mathcal{B}_\varepsilon}(z) - \mathcal{R}_{\Lambda_0}(z)(\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z))\} (I + K_\varepsilon(z))^{-1} dz \\ &= \frac{i}{2\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\mathcal{B}_\varepsilon}(z) \{I - K_\varepsilon(z)(I + K_\varepsilon(z))^{-1}\} dz \\ &\quad - \frac{i}{2\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\Lambda_0}(z)(\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)) \{I - K_\varepsilon(z)(I + K_\varepsilon(z))^{-1}\} dz \\ &= \frac{1}{2i\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\mathcal{B}_\varepsilon}(z) K_\varepsilon(z) (I + K_\varepsilon(z))^{-1} dz \\ &\quad - \frac{i}{2\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\Lambda_0}(z)(\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)) \{I - K_\varepsilon(z)(I + K_\varepsilon(z))^{-1}\} dz \end{aligned}$$

and similarly,

$$\begin{aligned} \Pi_0 &= \frac{i}{2\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\Lambda_0}(z) dz \\ &= \frac{i}{2\pi} \int_{\Gamma_\varepsilon} \{\mathcal{R}_{\mathcal{B}_0}(z) - \mathcal{R}_{\Lambda_0}(z)(\mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z))\} dz \\ &= \frac{1}{2i\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\Lambda_0}(z)(\mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z)) dz. \end{aligned}$$

Consequently,

$$\begin{aligned} \Pi_0 - \Pi_\varepsilon &= \frac{1}{2i\pi} \int_{\Gamma_\varepsilon} \mathcal{R}_{\Lambda_0}(z) \{\mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z) - \mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)\} dz \\ &\quad - \frac{1}{2i\pi} \int_{\Gamma_\varepsilon} \{\mathcal{R}_{\mathcal{B}_\varepsilon}(z) - \mathcal{R}_{\Lambda_0}(z)\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)\} K_\varepsilon(z) (I + K_\varepsilon(z))^{-1} dz \\ &=: T_1 + T_2. \end{aligned}$$

Concerning  $T_1$ , we use the identity

$$\mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z) - \mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z) = \mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z)(\mathcal{B}_\varepsilon - \mathcal{B}_0)\mathcal{R}_{\mathcal{B}_\varepsilon}(z)$$

with Lemmas 4.21, 4.22 and 4.27 which imply that

$$\mathcal{R}_{\mathcal{B}_\varepsilon}(z) \in \mathcal{B}(X_0), \quad \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_0 \rightarrow X_{-1}} \leq \eta_1(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z) \in \mathcal{B}(X_{-1}, X_0).$$

To treat  $T_2$ , we use estimate (4.41) on  $K_\varepsilon(z)$  and the facts that  $\mathcal{R}_{\mathcal{B}_\varepsilon}(z) \in \mathcal{B}(X_0)$  and that we also have  $\mathcal{R}_{\Lambda_0}(z)\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z) \in \mathcal{B}(X_0)$ . It concludes the proof.  $\square$

**Proposition 4.30.** *There exists  $\varepsilon_0 \in (0, \varepsilon_2)$  such that for any  $\varepsilon \in (0, \varepsilon_0)$ , the following properties hold in  $X_0$ :*

- (1)  $\Sigma(\Lambda_\varepsilon) \cap \Delta_{a_\theta} = \{0\}$ ;
- (2) for any  $f \in X_0$  and any  $a > a_\theta$ ,

$$\|S_{\Lambda_\varepsilon}(t)f - G_\varepsilon\langle f \rangle\|_{X_0} \leq C_a e^{at} \|f - G_\varepsilon\langle f \rangle\|_{X_0}, \quad \forall t \geq 0$$

for some explicit constant  $C_a \geq 1$ .

*Proof.* We know that if  $P$  and  $Q$  are two projectors such that  $\|P - Q\|_{\mathcal{B}(X_0)} < 1$ , then their ranges are isomorphic. Lemma 4.29 thus implies that there exists  $\varepsilon_0 \in (0, \varepsilon_1)$  such that for any  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\dim R(\Pi_\varepsilon) = \dim R(\Pi_0) = 1.$$

We also know that 0 is an eigenvalue for  $\Lambda_\varepsilon$  (cf. part (1) of Theorem 4.20). This concludes the proof of the first part of the proposition.

To get the estimate on the semigroup, we use a spectral mapping theorem coming from [9, Theorem 2.1]. The hypothesis of the theorem are satisfied because  $\mathcal{B}_\varepsilon - a$  is hypodissipative in  $X_0$  (and thus in  $D(\Lambda_{\varepsilon|X_0}) = D(\mathcal{B}_{\varepsilon|X_0})$ ) and  $\mathcal{A} \in \mathcal{B}(X_0, W_1^{2,p_\theta}(m))$  (and thus  $\mathcal{A} \in \mathcal{B}(X_0, D(\Lambda_{\varepsilon|X_0}))$ ).  $\square$

To conclude the proof of part (2) of Theorem 4.20, we use the previous Proposition 4.30 combined with an enlargement argument (see [4, 6]): our “small space” is  $E = L^{p_\theta}(m)$  and our “large” space is  $\mathcal{E} = L^1(m)$ . We then use Lemmas 4.22 and 4.23-4.27 and the fact that we clearly have  $\mathcal{A} \in \mathcal{B}(\mathcal{E}, E)$ .

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