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STABILITY OF THE TANGENT BUNDLE OF G/P IN POSITIVE CHARACTERISTICS

INDRANIL BISWAS, PIERRE-EMMANUEL CHAPUT, AND CHRISTOPHE MOURougANE

ABSTRACT. Let G be an almost simple simply-connected affine algebraic group over an algebraically closed field k of characteristic $p > 0$. If G has type B_n , C_n or F_4 , we assume that $p > 2$, and if G has type G_2 , we assume that $p > 3$. Let $P < G$ be a parabolic subgroup. We prove that the tangent bundle of G/P is Frobenius stable with respect to the anticanonical polarization on G/P .

1. INTRODUCTION

Let G be an almost simple simply-connected affine algebraic group over an algebraically closed field k , and let $P < G$ be a parabolic subgroup. If the characteristic $\text{char}(k)$ is zero, then it is known that the tangent bundle of G/P is stable with respect to the anticanonical polarization on G/P . Throughout, (semi)stability means slope (semi)stability. In the complex case it was proved long ago that this bundle admits a Kähler-Einstein metric (see [Ko55] or [Be87, Chapter 8]), which implies polystability. Simplicity of this bundle was proved in [AB10], proving the stability; A. Boralevi proved stability of $T(G/P)$ when G is of type ADE [Bor12, Theorem C]. Our aim here is to address stability of $T(G/P)$ in the case where $\text{char}(k)$ is positive.

If G is of type B_n , C_n or F_4 , we assume that $\text{char}(k) > 2$; if G is of type G_2 , we assume that $\text{char}(k) > 3$.

The main Theorem of this note says that under the above assumption, the tangent bundle of G/P and all its iterated Frobenius pull-backs are stable with respect to the anticanonical polarization on G/P .

The method of proof of the main Theorem is as follows. We prove that the stability of $T(G/P)$ is equivalent to certain statement on the quotient $\text{Lie}(G)/\text{Lie}(P)$ considered as a P -module. The statement in question is shown to be independent of the characteristic of k (as long as the above assumptions hold). Finally, the main Theorem follows from the fact that $T(G/P)$ is stable if $\text{char}(k) = 0$.

A natural question to ask is whether $T(G/P)$ remains stable with respect to polarizations on G/P other than the anticanonical one. A. Boralevi gave a negative answer to this question. She constructed examples of G/P and polarization on them with respect to which $T(G/P)$ is not even semistable [Bor12, Theorem D].

2. TANGENT BUNDLE OF G/P

Let G be an almost simple simply-connected affine algebraic group defined over an algebraically closed field k . The Lie algebra of G will be denoted by \mathfrak{g} . Let $P \subsetneq G$ be a parabolic subgroup. We start with a result which is valid in all characteristics.

Proposition 2.1. *Let M_1, M_2 be two G -modules such that $H^0(G/P, T(G/P)) = M_1 \otimes M_2$ as G -modules. Then either $M_1 = k$ or $M_2 = k$.*

Proof. Let θ be the highest root of \mathfrak{g} . We claim that θ is a maximal weight of $H^0(G/P, T(G/P))$ in the sense that $\theta + \alpha$ is not a weight of $H^0(G/P, T(G/P))$ for any positive root α . To prove this, first note that if $H^0(G/P, T(G/P)) = \mathfrak{g}$, then this is in fact the definition of the highest root. By [De77, Théorème 1], there are only three cases where $H^0(G/P, T(G/P)) \neq \mathfrak{g}$:

- (1) $G = \text{Sp}(2n)$ of type C_n with $H^0(G/P, T(G/P)) = \mathfrak{sl}(2n)$,

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- (2) $G = \mathrm{SO}(n+2)$ of type B_n with $H^0(G/P, T(G/P)) = \mathfrak{so}(2n+2)$, and
- (3) $G = G_2$ with $H^0(G/P, T(G/P)) = \mathfrak{so}(7)$.

In these three cases, we have exceptional automorphisms that account for additional vector fields and we have $H^0(G/P, T(G/P)) = \mathfrak{g} \oplus V$, where V has a unique highest weight which is not higher than θ . For example, if $G = \mathrm{Sp}_{2n}$, then $G/P = \mathrm{SL}(2n)/P_{\mathrm{SL}(2n)}$ is a projective space of dimension $2n-1$, so that $H^0(G/P, T(G/P))$ is $\mathfrak{sl}(2n)$. Then V is a module with unique highest weight $\epsilon_1 + \epsilon_2$, whereas $\theta = 2\epsilon_1$ (in the notation of [Bou05, Chap VI, Planches]). So the claim is proved.

As θ is a maximal weight of $H^0(G/P, T(G/P)) = M_1 \otimes M_2$, there are maximal weights ω_1 and ω_2 of M_1 and M_2 respectively, such that

$$\theta = \omega_1 + \omega_2. \quad (1)$$

Since ω_1 and ω_2 are maximal, they are dominant. In all types except A_n and C_n , we have θ to be a fundamental weight. Therefore, from the equality in (1) it follows that either $\omega_1 = 0$ or $\omega_2 = 0$, hence the proposition is proved in these cases.

For the remaining cases of A_n and C_n , assume that $\omega_1 \neq 0$ and $\omega_2 \neq 0$. Let ϖ_i denote the i -th fundamental weight. In case of A_n , we have $\theta = \varpi_1 + \varpi_n$, so up to a permutation, $\omega_1 = \varpi_1$ and $\omega_2 = \varpi_n$. Since the Weyl group orbits of both ϖ_1 and ϖ_n have $n+1$ elements, it follows that $\dim M_1 \geq n+1$ and $\dim M_2 \geq n+1$. This implies that $\dim H^0(G/P, T(G/P)) \geq (n+1)^2$ which is a contradiction. In case of C_n , we have $\theta = 2\varpi_1$, so similarly we get $\omega_1 = \omega_2 = \varpi_1$, and $\dim H^0(G/P, T(G/P)) \geq (2n)^2$. This is again a contradiction. \square

3. THE MAIN RESULT

We now impose the following assumptions on the characteristic of k :

Working assumption.

- The characteristic $\mathrm{char}(k)$ of k is positive, and
- $\mathrm{char}(k)$ is bigger than all the coefficients $\langle \alpha^\vee, \beta \rangle$ for all roots α, β of G with $\alpha \neq \beta$.

In other words, if the root system of G is simply-laced, then $\mathrm{char}(k)$ is only assumed to be positive; if G is any of B_n, C_n and F_4 , we assume that $\mathrm{char}(k) > 2$; if $G = G_2$, we assume that $\mathrm{char}(k) > 3$.

Main Theorem. *Under the previous assumption, the tangent bundle $T(G/P)$ is Frobenius stable with respect to the anticanonical polarization on G/P .*

We will divide the proof into several steps. The question of stability will be reduced to characteristic zero. The reduction to characteristic zero is achieved using the following construction: Let $G_{\mathbb{Z}}$ be the split simply-connected Chevalley group scheme over \mathbb{Z} having the same root system as G . By the theory of reductive algebraic group schemes, as the root system characterizes simply-connected groups up to isomorphism, we have $G \simeq G_{\mathbb{Z}} \otimes \mathrm{Spec} k$. On the other hand, we denote $G_{\mathbb{Z}} \otimes \mathrm{Spec} \mathbb{C}$ by $G_{\mathbb{C}}$. There exists a parabolic group $P_{\mathbb{Z}} \subset G_{\mathbb{Z}}$ such that $P_{\mathbb{Z}} \otimes \mathrm{Spec} k$ is conjugate to P . The parabolic subgroup $P \otimes \mathrm{Spec} \mathbb{C}$ of $G_{\mathbb{C}}$ will be denoted by $P_{\mathbb{C}}$.

Fix a maximal torus $T \subset G$ and a Borel subgroup B . Assume $T \subset B \subset P$. Let R denote the set of roots of \mathfrak{g} . The set of positive (respectively, negative) roots of \mathfrak{g} will be denoted by R^+ (respectively, R^-). The eigenspace corresponding to any $\alpha \in R$ will be denoted by \mathfrak{g}^α .

A subsheaf $E \subset T(G/P)$ is called G -stable if it is preserved by the left action of G on $T(G/P)$. Since the left translation action of G on G/P is transitive, any G -stable subsheaf of $T(G/P)$ is a subbundle.

The anticanonical line bundles of G/P and $G_{\mathbb{C}}/P_{\mathbb{C}}$ are ample. Fix the anticanonical polarization on G/P and also on $G_{\mathbb{C}}/P_{\mathbb{C}}$.

Proposition 3.1. *Let $E \subset T(G/P)$ be a G -stable subbundle of $T(G/P)$. There exists a subbundle $E_{\mathbb{C}} \subset T(G_{\mathbb{C}}/P_{\mathbb{C}})$ such that $\mathrm{rk}(E_{\mathbb{C}}) = \mathrm{rk}(E)$ and $\mathrm{deg}(E_{\mathbb{C}}) = \mathrm{deg}(E)$.*

Proof. Let $x_0 = eP/P \in G/P$ be the base point. The set of roots α such that $\mathfrak{g}^\alpha \subset \mathfrak{p}$ will be denoted by $I(P)$. We have

$$T_{x_0}(G/P) \simeq \mathfrak{g}/\mathfrak{p} \simeq \bigoplus_{\alpha \in R \setminus I(P)} \mathfrak{g}^\alpha.$$

Sending a G -stable subbundle $V \subset T(G/P)$ to the P -module V_{x_0} an equivalence between G -stable subbundles of $T(G/P)$ and P -submodules of $T_{x_0}(G/P)$ is obtained. Let M be the P -submodule of $T_{x_0}(G/P)$ corresponding to E . Since M is a T -stable subspace of $\bigoplus_{\alpha \notin I(P)} \mathfrak{g}^\alpha$, there is a subset $I(M) \subset R \setminus I(P)$ such that $M = \bigoplus_{\alpha \in I(M)} \mathfrak{g}^\alpha$. By the following Lemma 3.2, we have

$$\forall \beta \in I(P), \forall \alpha \in I(M), \alpha + \beta \in R \setminus I(P) \implies \alpha + \beta \in I(M).$$

Thus, $M_{\mathbb{C}} := \bigoplus_{\alpha \in I(M)} \mathfrak{g}_{\mathbb{C}}^\alpha$ is a $P_{\mathbb{C}}$ -submodule of $T_{x_0}(G_{\mathbb{C}}/P_{\mathbb{C}})$ and the subbundle $E_{\mathbb{C}} \subset T(G_{\mathbb{C}}/P_{\mathbb{C}})$ corresponding to $M_{\mathbb{C}}$ satisfies the conditions in the proposition. \square

In the following Lemma, we consider the vector space $\bigoplus_{\alpha \in R \setminus I(P)} \mathfrak{g}^\alpha$. This is isomorphic as a vector space to $\mathfrak{g}/\mathfrak{p}$, and therefore has a natural P -module structure.

Lemma 3.2. *Let $I \subset R \setminus I(P)$ be a set of negative roots. Then the sum $M(I) := \bigoplus_{\alpha \in I} \mathfrak{g}^\alpha$ is a P -stable submodule of $\bigoplus_{\alpha \in R \setminus I(P)} \mathfrak{g}^\alpha$ if, and only if,*

$$\forall \beta \in I(P), \forall \alpha \in I, \alpha + \beta \in R \setminus I(P) \implies \alpha + \beta \in I. \quad (2)$$

Proof. Take $\alpha \in I$ and $\beta \in I(P)$ such that $\alpha + \beta \in R \setminus I(P)$. In particular, we have $\beta \neq \pm\alpha$. Since G is simply-connected, \mathfrak{g} is the Lie algebra defined by Serre's relations (this is explained for example in [CR10, Remark 2.2.3]), so we can choose a basis of \mathfrak{g} such that the coefficients of the Lie bracket are those of the Chevalley basis [Ca72]. Consider the biggest integer p such that $\alpha - p\beta \in R$. This p is smaller than the length of the β -string of roots through α minus 1 (since $\alpha + \beta \in R$), and thus, by the working Assumption, we have $p \leq \text{char}(k) - 2$. This implies that $p + 1 < \text{char}(k)$. It now follows from [Ca72, Theorem 4.2.1] that $[\mathfrak{g}^\beta, \mathfrak{g}^\alpha] = \mathfrak{g}^{\alpha+\beta}$. Assuming that $M(I)$ is P -stable, we have it to be \mathfrak{p} -stable, and therefore $\alpha + \beta \in I$.

On the other hand, let $U_\beta \subset G$ be the one-parameter additive subgroup corresponding to the root β . Since $U_\beta \cdot \mathfrak{g}^\alpha \subset \bigoplus_{k \geq 0} \mathfrak{g}^{\alpha+k\beta}$, from (2) it follows that $M(I)$ is U_β -stable for any root $\beta \in I(P)$, and thus $M(I)$ is P -stable. \square

Lemma 3.3. *The tangent bundle $T(G/P)$ is polystable.*

Proof. Let E be the first term of the Harder-Narasimhan filtration of $T(G/P)$. First assume $E \neq T(G/P)$, so

$$\mu(E) > \mu(T(G/P)), \quad (3)$$

where μ denotes the slope, namely the quotient of the degree by the rank. Since the anticanonical polarization of G/P is fixed by G , from the uniqueness of the Harder-Narasimhan filtration it follows that E is G -stable. By Proposition 3.1 and stability of $T(G_{\mathbb{C}}/P_{\mathbb{C}})$ in characteristic 0 [AB10, Theorem 2.1], we thus have $\mu(E) < \mu(T(G/P))$ which contradicts (3). So $T(G/P)$ is semistable.

We can then similarly argue with the polystable socle (cf. [HL97, page 23, Lemma 1.5.5]) of $T(G/P)$ to deduce that $T(G/P)$ is polystable. \square

Since $T(G/P)$ is polystable there are non-isomorphic stable vector bundles E_1, \dots, E_r of same slope such that the natural map

$$\bigoplus_{i=1}^r \text{Hom}(E_i, T(G/P)) \otimes E_i \longrightarrow T(G/P) \quad (4)$$

is an isomorphism. We note that E_1, \dots, E_r are unique up to permutations of $\{1, \dots, r\}$.

Lemma 3.4. *Take any $g \in G$ and integer $1 \leq j \leq r$. Then $g^* E_j \simeq E_j$ as vector bundles on G/P .*

Proof. Let $\phi : G \times (G/P) \rightarrow G/P$ be the left-translation action. Let $p_2 : G \times (G/P) \rightarrow G/P$ be the projection to the second factor. The action ϕ produces an isomorphism of vector bundles

$$\Phi : \bigoplus_{i=1}^r \text{Hom}(E_i, T(G/P)) \otimes \phi^* E_i = \phi^* T(G/P) \rightarrow p_2^* T(G/P) = \bigoplus_{i=1}^r \text{Hom}(E_i, T(G/P)) \otimes p_2^* E_i. \quad (5)$$

For $i \neq \ell$, as E_i and E_ℓ are stable of the same slope, we have

$$\text{Hom}((\phi^* E_i)|_{\{e\} \times G/P}, (p_2^* E_\ell)|_{\{e\} \times G/P}) = \text{Hom}(E_i, E_\ell) = 0.$$

Hence, using semi-continuity,

$$\text{Hom}(\phi^* E_i, p_2^* E_\ell) = 0. \quad (6)$$

From (6) it follows immediately that Φ in (5) takes $\text{Hom}(E_i, T(G/P)) \otimes \phi^* E_i$ to itself for every $1 \leq i \leq r$. In particular, we have $\text{Hom}(E_j, T(G/P)) \otimes \phi^* E_j \simeq \text{Hom}(E_j, T(G/P)) \otimes p_2^* E_j$. Fix $g \in G$: restricting to $\{g\} \times G/P$, we get

$$\text{Hom}(E_j, T(G/P)) \otimes g^* E_j \simeq \text{Hom}(E_j, T(G/P)) \otimes E_j. \quad (7)$$

Since E_j is stable, we know that $g^* E_j$ is indecomposable. Now in view of the uniqueness of the decomposition into a direct sum of indecomposable vector bundles (see [At56, p. 315, Theorem 2]), from (7) we conclude that $g^* E_j \simeq E_j$. \square

Lemma 3.5. *For all $j \in [1, r]$, the vector bundle E_j is G -equivariant.*

Proof. Fix an integer $1 \leq j \leq r$. We now introduce the group of symmetries of the vector bundle E_j : Let \tilde{G} denote the set of pairs (g, h) , where $g \in G$ and $h \in \text{Aut}(E_j)$, such that the diagram

$$\begin{array}{ccc} E_j & \xrightarrow{h} & E_j \\ \downarrow & & \downarrow \\ G/P & \xrightarrow{g} & G/P \end{array}$$

commutes. Since E_j is simple, $\text{Aut}_{G/P}(E_j) \simeq \mathbb{G}_m$, and therefore we get a central extension

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{G} \xrightarrow{pr_1} G \rightarrow 1.$$

By Lemma 3.4, the above homomorphism pr_1 is surjective. This \tilde{G} is an algebraic group. To see this, consider the direct image $p_{2*} \mathcal{I}so(\phi^* E_j, p_2^* E_j)$, where ϕ and p_2 are the projections in the proof of Lemma 3.4, and $\mathcal{I}so(\phi^* E_j, p_2^* E_j)$ is the sheaf of isomorphisms between the two vector bundles $\phi^* E_j$ and $p_2^* E_j$. This direct image is a principal \mathbb{G}_m -bundle over G/P . The total space of this principal \mathbb{G}_m -bundle is identified with \tilde{G} .

We consider the derived subgroup $[\tilde{G}, \tilde{G}]$. Since G is simple and not abelian, we have $[G, G] = G$, so $\pi([\tilde{G}, \tilde{G}]) = G$. The unipotent radical of \tilde{G} is trivial. Indeed, the unipotent radical is mapped to the trivial subgroup of G since G is simple. Therefore it is included in \mathbb{G}_m and so the unipotent radical is trivial. Since \tilde{G} is reductive, $[\tilde{G}, \tilde{G}]$ is semi-simple, hence a proper subgroup of \tilde{G} (the radical of \tilde{G} contains \mathbb{G}_m hence \tilde{G} is not semi-simple). Thus the restriction of pr_1 to $[\tilde{G}, \tilde{G}]$ is an isogeny. Since G is simply-connected, the restriction of pr_1 to $[\tilde{G}, \tilde{G}]$ is an isomorphism. Consequently, the tautological action of $[\tilde{G}, \tilde{G}]$ on E_j makes it a G -equivariant bundle. \square

Lemma 3.6. *The integer r in (4) is 1.*

Proof. Since $\text{Hom}(E_1, T(G/P)) \otimes E_1$ is a direct summand of $T(G/P)$ (see (4)), from Lemma 3.3 we know that the slope of $\text{Hom}(E_1, T(G/P)) \otimes E_1$ coincides with the slope of $T(G/P)$. In the proof of Lemma 3.5 we saw that $\text{Hom}(E_1, T(G/P)) \otimes E_1$ is a G -equivariant direct summand of $T(G/P)$. As $T(G_C/P_C)$ is stable, [AB10, Theorem 2.1], from Proposition 3.1 it now follows that $\text{Hom}(E_1, T(G/P)) \otimes E_1 = T(G/P)$. \square

Lemma 3.7. $\dim \text{Hom}(E_1, T(G/P)) = 1$.

Proof. From Lemma 3.6 we have $H^0(G/P, T(G/P)) = \text{Hom}(E_1, T(G/P)) \otimes H^0(G/P, E_1)$. Since $T(G/P)$ is globally generated, so is E_1 and thus $\dim H^0(G/P, E_1) > 1$. Thus, as E_1 is G -equivariant, the lemma follows from Proposition 2.1. \square

From Lemma 3.3, Lemma 3.6 and Lemma 3.7 it follows that $T(G/P)$ is stable. The following lemma completes the proof of the main Theorem.

Lemma 3.8. *Let E be a semi-stable (respectively, stable) G -equivariant vector bundle on G/P . Then E is Frobenius semi-stable (respectively, Frobenius stable).*

Proof. The absolute Frobenius morphism on G/P will be denoted by F . First assume that E is semi-stable. Let again W be the first term of the Harder-Narasimhan filtration of F^*E . We use the correspondence between vector bundles on G/P and P -modules. Thus W corresponds to a P -stable subspace of $(F^*E)_{x_0}$, the fiber of F^*E at the base point in G/P . This is the same as an F^*P -stable subspace S of E_{x_0} . Since $F : P \rightarrow P$ is bijective, this S is also a P -submodule of E_{x_0} . Thus, there exists a subbundle $W' \subset E$ of slope $\frac{\mu(W)}{p} \geq \frac{\mu(F^*E)}{p} = \mu(E)$ such that $W = F^*W'$. By semi-stability of E , we have $W' = E$. Thus we get that $W = F^*E$.

Assume now that E is stable. So F^*E is semistable. Let $W \subset F^*E$ be a subbundle with $\mu(W) = \mu(F^*E)$. We consider the Cartier connection $F^*E \rightarrow F^*E \otimes \Omega_{G/P}^1$. The subbundle W is a Frobenius pull-back if and only if its image under the composition

$$W \rightarrow F^*E \rightarrow F^*E \otimes \Omega_{G/P}^1$$

is contained in $W \otimes \Omega_{G/P}^1$. Since both E and $\Omega_{G/P}^1$ are Frobenius semistable, the tensor product $E \otimes \Omega_{G/P}^1$ is again semi-stable [RR84, p. 285, Theorem 3.18]. But $\mu(F^*E \otimes \Omega_{G/P}^1) < \mu(F^*E) = \mu(W)$, so this composition vanishes. Therefore, let $W' \subset E$ be such that $W = F^*W'$. We have $\mu(W') = \mu(E)$. By stability of E , we get that $W' = E$ and hence $W = F^*E$. \square

4. AN EXAMPLE IN SMALL CHARACTERISTIC

We give an example of a tangent bundle which is semi-stable but not stable. We do not know if there are some tangent bundles to homogeneous spaces which are not semi-stable.

The example is that of $X = G/P = \mathbb{G}_\omega(n, 2n)$, the Grassmannian of Lagrangian spaces in a symplectic space of dimension $2n$, and we assume that k has characteristic 2. Namely, G is Sp_{2n} and P corresponds to the long simple root. Let U denote the universal bundle on X , of rank n and degree -1 . Then TX is a subbundle of $U^* \otimes U^*$; in fact if S^2U denotes the symmetric quotient of $U \otimes U$, then $TX \simeq (S^2U)^*$.

We will implicitly use the correspondence between P -modules and G -linearized homogeneous bundles on X . Note that the reductive quotient of P is $GL(U)$. Since there is an injection $F^*U \rightarrow S^2U$ of $GL(U)$ -modules (F denotes the Frobenius morphism), this defines an exact sequence of bundles on X :

$$0 \rightarrow F^*U \rightarrow S^2U \rightarrow K \rightarrow 0 \tag{8}$$

It follows that there is a subbundle $K^* \subset TX$. Since $\mu(F^*U) = \mu(S^2U) = 2\mu(U)$, we get $\mu(K^*) = \mu(TX)$ and TX is not stable. However since F^*U is the only $GL(U)$ -invariant subspace in S^2U , K^* is the only equivariant subbundle in TX . Thus the semi-stability inequality holds for this subbundle. Arguing as in the proof of Lemma 3.3, we deduce that TX is semi-stable.

For general homogeneous spaces G/P , we face two difficulties:

- There are equivariant subbundles in TX which do not lift to characteristic 0, and contrary to the above example, they are numerous in general.
- The stability of TX for characteristic 0 says nothing about $\mu(E)$ of such a subbundle $E \subset TX$. It is difficult to compute the $(\dim(G/P) - 1)$ -th power of the anticanonical polarization to be able to show the semi-stability inequality for E .

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA
E-mail address: indranil@math.tifr.res.in

DOMAINE SCIENTIFIQUE VICTOR GRIGNARD, 239, BOULEVARD DES AIGUILLETES, UNIVERSITÉ HENRI POINCARÉ NANCY 1,
B.P. 70239, F-54506 VANDOEUVRE-LÈS-NANCY CEDEX, FRANCE
E-mail address: pierre-emmanuel.chaput@univ-lorraine.fr

DÉPARTEMENT DE MATHÉMATIQUES, CAMPUS DE BEAULIEU, BÂT. 22-23, UNIVERSITÉ DE RENNES 1, 263 AVENUE DU GÉNÉRAL
LECLERC, CS 74205, 35042 RENNES CÉDEX, FRANCE
E-mail address: christophe.mourougane@univ-rennes1.fr