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Supplement to “High-dimensional Bayesian inference via the Unadjusted Langevin Algorithm”

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1 Discussions on Theorem 5

In this section, we give explicit bounds for ULA which are consequences of Theorem 5. First note that, if $(\gamma_k)_{k \geq 1}$ is a non-increasing sequence of step sizes with $\gamma_1 < (m + L)^{-1}$, we have by (9) that

$$u_n^{(2)}(\gamma) \leq \sum_{i=1}^n \{A_0 \gamma_i^2 + A_1 \gamma_i^3\} \prod_{k=i+1}^n (1 - \kappa \gamma_k / 2), \quad (\text{S1})$$

where κ is given by (42), and

$$A_0 = 2L^2 \kappa^{-1} d, \quad (\text{S2})$$

$$A_1 = 2L^2 d + dL^4 (\kappa^{-1} + (m + L)^{-1}) (m^{-1} + 6^{-1} (m + L)^{-1}). \quad (\text{S3})$$

1.1 Explicit bounds for fixed step size and fixed precision

If $(\gamma_k)_{k \geq 1}$ is a constant step size, $\gamma_k = \gamma$ for all $k \geq 1$, then a straightforward consequence of Theorem 5 and (S1) is the following result, which gives the minimal number of iterations n_ε and a step size γ_ε to get $W_2(\delta_{x^*} Q_\gamma^n, \pi)$ smaller than $\varepsilon > 0$.

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Corollary S1 (of Theorem 5). *Assume **H1** and **H2**. Let x^* be the unique minimizer of U . Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Set for all $k \in \mathbb{N}$, $\gamma_k = \gamma$ with*

$$\begin{aligned} \gamma &= \frac{-\mathbf{A}_0 + (\mathbf{A}_0^2 + \varepsilon^2 \kappa \mathbf{A}_1)^{1/2}}{2\mathbf{A}_1} \wedge (m+L)^{-1}, \\ n &= \lceil \log^{-1}(1 - \kappa\gamma/2) \{-\log(\varepsilon^2/2) + \log(2d/m)\} \rceil. \end{aligned} \quad (\text{S4})$$

Then $W_2(\delta_{x^*} R_\gamma^n, \pi) \leq \varepsilon$.

Note that if γ is given by (S4), and is different from $1/(m+L)$, then $\gamma \leq \varepsilon(4\mathbf{A}_1\kappa^{-1})^{-1/2}$ and $2\kappa^{-1}(\mathbf{A}_0\gamma + \mathbf{A}_1\gamma^2) = \varepsilon^2/2$. Therefore,

$$\gamma \geq (\varepsilon^2\kappa/4) \left\{ \mathbf{A}_0 + \varepsilon(\mathbf{A}_1\kappa/4)^{1/2} \right\}^{-1}.$$

It is shown in [1, Corollary 1, Proposition 2] that under **H2**, for constant step size for any $\varepsilon > 0$, we can choose γ and $n \geq 1$ such that if for all $k \geq 1$, $\gamma_k = \gamma$, then $\|\nu^* Q_n^\gamma - \pi\|_{\text{TV}} \leq \varepsilon$ where ν^* is the Gaussian measure on \mathbb{R}^d with mean x^* and covariance matrix $L^{-1} \mathbf{I}_d$ or a warm start. We stress that the results in [1, Corollary 1, Proposition 2] hold only for particular choices of the initial distribution ν^* , (which might seem a rather artificial assumption) whereas Theorem 5 holds for any initial distribution in $\mathcal{P}_2(\mathbb{R}^d)$.

We compare the optimal value of γ and n obtained from Corollary S1 with those given in [1, Corollary 1, Proposition 2] and [2, Table 2] for the total variation distance and established under the same conditions as Theorem 5. This comparison is summarized in Table 1; for simplicity, we provide only the dependencies of the minimal number of simulations n as a function of the dimension d , the precision ε and the constants m, L . It can be seen that the dependency on the dimension is significantly better than those in [1, Corollary 1]. The dependency on the dimension and the precision is same compared to [1, Proposition 2] (up to logarithmic terms), but this result only holds if the initial distribution is a warm start. In addition, the dependency on L and m is not explicit in [1, Proposition 2], and that is why we do indicate it in Table 1. On the other hand, we get the same dependency on ε and d as [2, Table 2]. Note that however the result of [2] holds for the total variation and not the Wasserstein distance.

Parameter	d	ε	L	m
Corollary S1	$d \log(d)$	$\varepsilon^{-2} \log(\varepsilon) $	L^2	$ \log(m) m^{-3}$
[1, Corollary 1] Gaussian start	d^3	$\varepsilon^{-2} \log(\varepsilon) $	L^3	$ \log(m) m^{-2}$
[1, Proposition 2] warm start	d	$\varepsilon^{-2} \log(\varepsilon) $	—	—
[2, Table 2]	$d \log(d)$	$\varepsilon^{-2} \log(\varepsilon) $	L^2	m^{-2}

Table 1: Dependencies of n

1.2 Explicit bounds for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0, 1]$

We give here a bound on the sequences $(u_n^{(1)}(\gamma))_{n \geq 0}$ and $(u_n^{(2)}(\gamma))_{n \geq 0}$ for $(\gamma_k)_{k \geq 1}$ defined by $\gamma_1 < 1/(m+L)$ and $\gamma_k = \gamma_1 k^{-\alpha}$ for $\alpha \in (0, 1]$. Also for that purpose we introduce

for $t \in \mathbb{R}_+^*$,

$$\psi_\beta(t) = \begin{cases} (t^\beta - 1)/\beta & \text{for } \beta \neq 0 \\ \log(t) & \text{for } \beta = 0. \end{cases} \quad (\text{S5})$$

We easily get for $a \geq 0$ that for all $n, p \geq 1$, $n \leq p$

$$\psi_{1-a}(p+1) - \psi_{1-a}(n) \leq \sum_{k=n}^p k^{-a} \leq \psi_{1-a}(p) - \psi_{1-a}(n) + 1, \quad (\text{S6})$$

and for $a \in \mathbb{R}$

$$\sum_{k=n}^p k^{-a} \leq \psi_{1-a}(p+1) - \psi_{1-a}(n) + 1. \quad (\text{S7})$$

1. For $\alpha = 1$, using that for all $t \in \mathbb{R}$, $(1+t) \leq e^t$ and by (S6) and (S7), we have

$$u_n^{(1)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2}, \quad u_n^{(2)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^1 \mathbf{A}_j(\psi_{\kappa\gamma_1/2-1-j}(n+1) + 1).$$

2. For $\alpha \in (0, 1)$, by (S6) and Lemma 23 applied with $\ell = \lceil n/2 \rceil$, where $\lceil \cdot \rceil$ is the ceiling function, we have

$$\begin{aligned} u_n^{(1)}(\gamma) &\leq \exp(-\kappa\gamma_1\psi_{1-\alpha}(n+1)/2) \\ u_n^{(2)}(\gamma) &\leq \sum_{j=0}^1 \mathbf{A}_j \left[2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} + \gamma_1^{j+2} \{ \psi_{1-\alpha(j+2)}(\lceil n/2 \rceil) + 1 \} \right] \\ &\quad \times \exp(-(\kappa\gamma_1/2) \{ \psi_{1-\alpha}(n+1) - \psi_{1-\alpha}(\lceil n/2 \rceil) \}). \end{aligned} \quad (\text{S8})$$

1.3 Optimal strategy with a fixed number of iterations

Corollary S2. *Let $n \in \mathbb{N}^*$ be a fixed number of iteration. Assume **H1**, **H2**, and $(\gamma_k)_{k \geq 1}$ is a constant sequence, $\gamma_k = \gamma$ for all $k \geq 1$. Set*

$$\begin{aligned} \gamma^+ &= 2(\kappa n)^{-1} \left[\log(\kappa n/2) + \log(2(\|x - x^*\|^2 + d/m)) - \log(2\kappa^{-1}\mathbf{A}_0) \right] \\ \gamma^- &= 2(\kappa n)^{-1} \left[\log(\kappa n/2) + \log(2(\|x - x^*\|^2 + d/m)) \right. \\ &\quad \left. - \log \{ 2\kappa^{-1}(\mathbf{A}_0 + 2\mathbf{A}_1(m+L)^{-1}) \} \right]. \end{aligned}$$

Assume $\gamma^+ \in (0, (m+L)^{-1})$. Then, the optimal choice of γ to minimize the bound on $W_2(\delta_x R_\gamma^n, \pi)$ given by Theorem 5 belongs to $[\gamma^-, \gamma^+]$. Moreover if $\gamma = \gamma_+$, then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x R_\gamma^n, \pi)$ given by Theorem 5 is equivalent to $Cdn^{-1} \log(n)$ as n goes to $+\infty$.

Similarly, we have the following result.

Corollary S3. Assume **H1** and **H2**. Let $(\gamma_k)_{k \geq 1}$ be the decreasing sequence, defined by $\gamma_k = \gamma_\alpha/k^\alpha$, with $\alpha \in (0, 1)$. Let $n \geq 1$ and set

$$\gamma_\alpha = 2(1 - \alpha)\kappa^{-1}(2/n)^{1-\alpha} \log(\kappa n/(2(1 - \alpha))) .$$

Assume $\gamma_\alpha \in (0, (m + L)^{-1})$. Then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x Q_\gamma^n, \pi)$ given by Theorem 5 is equivalent to $Cdn^{-1} \log(n)$ as n goes to $+\infty$.

Proof. Follows from (S1), (S8) and the choice of γ_α . □

2 Discussion on Theorem 8

Based on Theorem 8, we can follow the same discussion as for Theorem 5. Note that

$$u_n^{(3)}(\gamma) \leq \sum_{i=1}^n \{B_0 \gamma_i^3 + B_1 \gamma_i^4\} \prod_{k=i+1}^n (1 - \kappa \gamma_k/2) , \quad (\text{S9})$$

where κ is given by (42), and

$$B_0 = d \left[2L^2 + (4/(3\kappa)) \left\{ d\tilde{L}^2 + L^4/m \right\} \right] , \quad (\text{S10})$$

$$B_1 = d \left[\kappa^{-1}L^4 + L^4/(6(m + L)) + m^{-1} \right] . \quad (\text{S11})$$

2.1 Explicit bounds for fixed step size and fixed precision

The following result gives for a target precision $\varepsilon > 0$, the minimal number of iterations n_ε and a step size γ_ε to get $W_2(\delta_{x^*} Q_\gamma^n, \pi)$ smaller than ε , when $(\gamma_k)_{k \geq 1}$ is constant, $\gamma_k = \gamma_\varepsilon$ for all $k \geq 1$.

Corollary S4. Assume **H1**, **H2** and **H3**. Let x^* be the unique minimizer of U . Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Set for all $k \in \mathbb{N}$, $\gamma_k = \gamma$ with

$$\begin{aligned} \gamma &= \left[(\varepsilon/(2\kappa)) \left\{ B_0 + B_1(m + L)^{-1} \right\}^{-1/2} \right] \wedge (1/(m + L)) , \\ n &= \left\lceil \log^{-1}(1 - \kappa\gamma/2) \left\{ -\log(\varepsilon^2/2) + \log(2d/m) \right\} \right\rceil . \end{aligned}$$

Then $W_2(\delta_{x^*} R_\gamma^n, \pi) \leq \varepsilon$.

We provide the dependencies of the minimal number of simulations n_ε as a function of the dimension d , the precision ε and the constants m, L, \tilde{L} in Table 2.

Parameter	d	ε	L	m
Corollary S4	$d \log(d)$	$\varepsilon^{-1} \log(\varepsilon) $	L^2	$ \log(m) m^{-2}$

Table 2: Dependencies of n

2.2 Explicit bounds for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0, 1]$

We give here a bound on the sequence $(u_n^{(3)}(\gamma))_{n \geq 0}$ for $(\gamma_k)_{k \geq 1}$ defined by $\gamma_1 < 1/(m+L)$ and $\gamma_k = \gamma_1 k^{-\alpha}$ for $\alpha \in (0, 1]$. Bounds for $(u_n^{(1)}(\gamma))_{n \geq 0}$ have already been given in Section 1.2. Recall that the function $\boldsymbol{\psi}$ is defined by (S5). For $\alpha \in (0, 1]$, by (S6) and Lemma 23 applied with $\ell = \lceil n/2 \rceil$, where $\lceil \cdot \rceil$ is the ceiling function, we have

$$u_n^{(3)}(\gamma) \leq \sum_{j=1}^2 \mathbf{B}_{j-1} \left[2\kappa^{-1} \gamma_1^{j+1} (n/2)^{-\alpha(j+1)} + \gamma_1^{j+2} (\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2 \rceil) + 1) \right. \\ \left. \times \exp(-(\kappa\gamma_1/2) \{ \boldsymbol{\psi}_{1-\alpha}(n+1) - \boldsymbol{\psi}_{1-\alpha}(\lceil n/2 \rceil) \}) \right]. \quad (\text{S12})$$

2.3 Optimal strategy with a fixed number of iterations

Corollary S5. *Let $n \in \mathbb{N}^*$ be a fixed number of iteration. Assume **H1**, **H2**, **H3** and $(\gamma_k)_{k \geq 1}$ is a constant sequence, $\gamma_k = \gamma^*$ for all $k \geq 1$, with*

$$\gamma^* = 4(\kappa n)^{-1} \left\{ \log(\kappa n/2) + \log(2(\|x - x^*\|^2 + d/m)) \right\}.$$

Assume $\gamma^ \in (0, (m+L)^{-1})$. Then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x R_\gamma^n, \pi)$ is equivalent to $Cd^2 n^{-2} \log^2(n)$ as n goes to $+\infty$.*

Similarly, we have the following result.

Corollary S6. *Assume **H1**, **H2** and **H3**. Let $(\gamma_k)_{k \geq 1}$ be the decreasing sequence, defined by $\gamma_k = \gamma_\alpha/k^\alpha$, with $\alpha \in (0, 1)$. Let $n \geq 1$ and set*

$$\gamma_\alpha = 2(1-\alpha)\kappa^{-1}(2/n)^{1-\alpha} \log(\kappa n/(2(1-\alpha))).$$

Assume $\gamma_\alpha \in (0, (m+L)^{-1})$. Then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x R_\gamma^n, \pi)$ is equivalent to $Cd^2 n^{-2} \log^2(n)$ as n goes to $+\infty$.

Proof. Follows from (S9), (S12) and the choice of γ_α . \square

Note that in Corollary S5 and Corollary S6, we do not find the optimal convergence rates obtained for the sequence of step sizes $\gamma_k = \gamma_1/k$, for $k \geq 1$ and $\gamma_1 > 0$, up to a logarithmic factor $\log(n)$. This is most likely due to the fact that the bounds (for example (S12)) used to compute the optimal parameters γ^* and γ_α are not the most appropriate.

3 Generalization of Theorem 5 and Theorem 8

In this section, we weaken the assumption $\gamma_1 \leq 1/(m+L)$ of Theorem 5 and Theorem 8. We assume now:

G1. *The sequence $(\gamma_k)_{k \geq 1}$ is non-increasing, and there exists n_1 such that $\gamma_{n_1} \leq 1/(m+L)$.*

Under **G1**, we denote by

$$n_0 = \min \{k \in \mathbb{N} \mid \gamma_k \leq 2/(m+L)\} \quad (\text{S13})$$

We first give an extension of Proposition 2-(i). Denote in the sequel $(\cdot)_+ = \max(\cdot, 0)$. Recall that under **H2**, x^* is the unique minimizer of U , and κ is defined in (A)

Theorem S7. *Assume **H1**, **H2** and **G1**. Then for all $n, p \in \mathbb{N}^*$, $n \leq p$*

$$\int_{\mathbb{R}^d} \|x - x^*\|^2 \mu_0 Q_n^p(dx) \leq \mathbf{G}_{n,p}(\mu_0, \gamma),$$

where

$$\begin{aligned} \mathbf{G}_{n,p}(\mu_0, \gamma) &= \exp\left(-\sum_{k=n}^p \gamma_k \kappa + \sum_{k=n}^{n_0-1} L^2 \gamma_k^2\right) \int_{\mathbb{R}^d} \|x - x^*\|^2 \mu_0(dx) \\ &+ 2d\kappa^{-1} + 2d \left\{ \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} (1 + L^2 \gamma_k^2) \right\} \exp\left(-\sum_{k=n}^p \kappa \gamma_k + \sum_{k=n}^{n_0-1} \gamma_k^2 mL\right). \end{aligned} \quad (\text{S14})$$

Proof. For any $\gamma > 0$, we have for all $x \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \|y - x^*\|^2 R_\gamma(x, dy) = \|x - \gamma \nabla U(x) - x^*\|^2 + 2\gamma d.$$

Using that $\nabla U(x^*) = 0$, (42) and **H1**, we get from the previous inequality:

$$\begin{aligned} &\int_{\mathbb{R}^d} \|y - x^*\|^2 R_\gamma(x, dy) \\ &\leq (1 - \kappa\gamma) \|x - x^*\|^2 + \gamma \left(\gamma - \frac{2}{m+L}\right) \|\nabla U(x) - \nabla U(x^*)\|^2 + 2\gamma d \\ &\leq \eta(\gamma) \|x - x^*\|^2 + 2\gamma d, \end{aligned}$$

where $\eta(\gamma) = (1 - \kappa\gamma + \gamma L(\gamma - 2/(m+L)))_+$. Denote for all $k \geq 1$, $\eta_k = \eta(\gamma_k)$. By a straightforward induction, we have by definition of Q_n^p for $p, n \in \mathbb{N}$, $p \leq n$,

$$\int_{\mathbb{R}^d} \|x - x^*\|^2 \mu_0 Q_n^p(dx) \leq \prod_{k=n}^p \eta_k \int_{\mathbb{R}^d} \|x - x^*\| \mu_0(dx) + (2d) \sum_{i=n}^p \prod_{k=i+1}^p \eta_k \gamma_i. \quad (\text{S15})$$

For the first term of the right hand side, we simply use the bound, for all $x \in \mathbb{R}$, $(1+x) \leq e^x$, and we get by **G1**

$$\prod_{k=n}^p \eta_k \leq \exp\left(-\sum_{k=n}^p \kappa \gamma_k + \sum_{k=n}^{n_0-1} L^2 \gamma_k^2\right), \quad (\text{S16})$$

where n_0 is defined in (S13). Consider now the second term in the right hand side of (S15).

$$\begin{aligned}
\sum_{i=n}^p \prod_{k=i+1}^p \eta_k \gamma_i &\leq \sum_{i=n_0}^p \prod_{k=i+1}^p (1 - \kappa \gamma_k) \gamma_i + \sum_{i=n}^{n_0-1} \prod_{k=i+1}^p \eta_k \gamma_i \\
&\leq \kappa^{-1} \sum_{i=n_0}^p \left\{ \prod_{k=i+1}^p (1 - \kappa \gamma_k) - \prod_{k=i}^p (1 - \kappa \gamma_k) \right\} \\
&\quad + \left\{ \sum_{i=n}^{n_0-1} \prod_{k=i+1}^{n_0-1} (1 + L^2 \gamma_k^2) \gamma_i \right\} \prod_{k=n_0}^p (1 - \kappa \gamma_k) \quad (\text{S17})
\end{aligned}$$

Since $(\gamma_k)_{k \geq 1}$ is non-increasing, we have

$$\begin{aligned}
\sum_{i=n}^{n_0-1} \prod_{k=i+1}^{n_0-1} (1 + L^2 \gamma_k^2) \gamma_i &= \sum_{i=n}^{n_0-1} (\gamma_i L^2)^{-1} \left\{ \prod_{k=i}^{n_0-1} (1 + L^2 \gamma_k^2) - \prod_{k=i+1}^{n_0-1} (1 + L^2 \gamma_k^2) \right\} \\
&\leq \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} (1 + L^2 \gamma_k^2).
\end{aligned}$$

Furthermore for $k < n_0$ $\gamma_k > 2/(m + L)$. This implies with the bound $(1 + x) \leq e^x$ on \mathbb{R} :

$$\begin{aligned}
\prod_{k=n_0}^p (1 - \kappa \gamma_k) &\leq \exp \left(- \sum_{k=n_0}^p \kappa \gamma_k \right) \exp \left(\sum_{k=n_0}^{n_0-1} \kappa \gamma_k \right) \\
&\leq \exp \left(- \sum_{k=n_0}^p \kappa \gamma_k \right) \exp \left(\sum_{k=n_0}^{n_0-1} \gamma_k^2 m L \right).
\end{aligned}$$

Using the two previous inequalities in (S17), we get

$$\begin{aligned}
\sum_{i=n}^p \prod_{k=i+1}^p \eta_k \gamma_i &\leq \kappa^{-1} + \left\{ \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} (1 + L^2 \gamma_k^2) \right\} \exp \left(- \sum_{k=n_0}^p \kappa \gamma_k + \sum_{k=n_0}^{n_0-1} \gamma_k^2 m L \right). \quad (\text{S18})
\end{aligned}$$

Combining (S16) and (S18) in (S15) concluded the proof. \square

We now deal with bounds on $W_2(\mu_0 Q_\gamma^n, \pi)$ under **G1**. But before we preface our result by some technical lemmas.

Lemma S8. *Assume **H1** and **H2**. Let $\zeta_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, $(Y_t, \bar{Y}_t)_{t \geq 0}$ such that (Y_0, \bar{Y}_0) is distributed according to ζ_0 and given by (50). Let $(\mathcal{F}'_t)_{t \geq 0}$ be the filtration associated with $(B_t)_{t \geq 0}$ with \mathcal{F}'_0 , the σ -field generated by (Y_0, \bar{Y}_0) .*

(i) For all $n \geq 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| Y_{\Gamma_{n+1}} - \bar{Y}_{\Gamma_{n+1}} \right\|^2 \right] \\ & \leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1}L((1 + \epsilon_2)\gamma_{n+1} - 2/(m + L))_+\} \left\| Y_{\Gamma_n} - \bar{Y}_{\Gamma_n} \right\|^2 \\ & + \gamma_{n+1}^2(1/(2\epsilon_1) + (1 + \epsilon_2^{-1})\gamma_{n+1}) \left(dL^2 + (L^4\gamma_{n+1}/2) \left\| Y_{\Gamma_n} - x^* \right\|^2 + dL^4\gamma_{n+1}^2/12 \right). \end{aligned}$$

(ii) If in addition **H3** holds then for all $n \geq 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| Y_{\Gamma_{n+1}} - \bar{Y}_{\Gamma_{n+1}} \right\|^2 \right] \\ & \leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1}L((1 + \epsilon_2)\gamma_{n+1} - 2/(m + L))_+\} \left\| Y_{\Gamma_n} - \bar{Y}_{\Gamma_n} \right\|^2 \\ & + (2\epsilon_1)^{-1}\gamma_{n+1}^3 \left\{ (2L^4/3) \left\| Y_{\Gamma_n} - x^* \right\|^2 + L^4d\gamma_{n+1}/2 + 2d^2\tilde{L}^2/3 \right\} \\ & + \gamma_{n+1}^3(1 + \epsilon_2^{-1}) \left(dL^2 + (L^4\gamma_{n+1}/2) \left\| Y_{\Gamma_n} - x^* \right\|^2 + dL^4\gamma_{n+1}^2/12 \right). \end{aligned}$$

Proof. (i) Let $n \geq 0$ and $\epsilon_1 > 0$, and set $\Delta_n = Y_{\Gamma_n} - \bar{Y}_{\Gamma_n}$ by definition we have:

$$\begin{aligned} \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| \Delta_{n+1} \right\|^2 \right] & = \left\| \Delta_n \right\|^2 + \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| \int_{\Gamma_n}^{\Gamma_{n+1}} \{ \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}) \} ds \right\|^2 \right] \quad (\text{S19}) \\ & - 2\gamma_{n+1} \langle \Delta_n, \nabla U(Y_{\Gamma_n}) - \nabla U(\bar{Y}_{\Gamma_n}) \rangle - 2 \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} [\langle \Delta_n, \{ \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) \} \rangle ds]. \end{aligned}$$

Using the two inequalities $|\langle a, b \rangle| \leq \epsilon_1 \|a\|^2 + (4\epsilon_1)^{-1} \|b\|^2$ and (42), we get

$$\begin{aligned} \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| \Delta_{n+1} \right\|^2 \right] & \leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1)\} \left\| \Delta_n \right\|^2 \\ & - 2\gamma_{n+1}/(m + L) \left\| \nabla U(Y_{\Gamma_n}) - \nabla U(\bar{Y}_{\Gamma_n}) \right\|^2 \quad (\text{S20}) \end{aligned}$$

$$\begin{aligned} & + \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| \int_{\Gamma_n}^{\Gamma_{n+1}} \{ \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}) \} ds \right\|^2 \right] \\ & + (2\epsilon_1)^{-1} \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) \right\|^2 \right] ds. \quad (\text{S21}) \end{aligned}$$

Using $\|a + b\|^2 \leq (1 + \epsilon_2) \|a\|^2 + (1 + \epsilon_2^{-1}) \|b\|^2$ and the Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\left\| \int_{\Gamma_n}^{\Gamma_{n+1}} \{ \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}) \} ds \right\|^2 \right] & \leq (1 + \epsilon_2)\gamma_{n+1}^2 \left\| \nabla U(Y_{\Gamma_n}) - \nabla U(\bar{Y}_{\Gamma_n}) \right\|^2 \\ & + (1 + \epsilon_2^{-1})\gamma_{n+1} \mathbb{E}^{\mathcal{F}'_{\Gamma_n}} \left[\int_{\Gamma_n}^{\Gamma_{n+1}} \left\| \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) \right\|^2 ds \right]. \quad (\text{S22}) \end{aligned}$$

This result and **H1** imply,

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|\Delta_{n+1}\|^2 \right] &\leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1}L((1 + \epsilon_2)\gamma_{n+1} - 2/(m + L))_+\} \|\Delta_n\|^2 \\ &\quad + ((1 + \epsilon_2^{-1})\gamma_{n+1} + (2\epsilon_1)^{-1}) \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2 \right] ds. \end{aligned} \quad (\text{S23})$$

By **H1**, the Markov property of $(Y_t)_{t \geq 0}$ and Lemma **21**, we have

$$\begin{aligned} \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2 \right] ds \\ \leq L^2 \left(d\gamma_{n+1}^2 + dL^2\gamma_{n+1}^4/12 + (L^2\gamma_{n+1}^3/2) \|Y_{\Gamma_n} - x^*\|^2 \right). \end{aligned} \quad (\text{S24})$$

Plugging this bound in **(S23)** concludes the proof.

(ii) Let $n \geq 0$ and $\epsilon > 0$, and set $\Theta_n = Y_{\Gamma_n} - \bar{Y}_{\Gamma_n}$. Using Itô's formula, we have for all $s \in [\Gamma_n, \Gamma_{n+1})$,

$$\begin{aligned} \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) &= \int_{\Gamma_n}^s \left\{ \nabla^2 U(Y_u) \nabla U(Y_u) + \vec{\Delta}(\nabla U)(Y_u) \right\} du \\ &\quad + \sqrt{2} \int_{\Gamma_n}^s \nabla^2 U(Y_u) dB_u. \end{aligned} \quad (\text{S25})$$

Since Θ_n is \mathcal{F}_{Γ_n} -measurable and $(\int_0^s \nabla^2 U(Y_u) dB_u)_{s \in [0, \Gamma_{n+1}]}$ is a $(\mathcal{F}_s)_{s \in [0, \Gamma_{n+1}]}$ -martingale under **H1**, by **(S25)** we have:

$$\begin{aligned} &|\mathbb{E}^{\mathcal{F}_{\Gamma_n}} [\langle \Theta_n, \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) \rangle]| \\ &= \left| \left\langle \Theta_n, \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\int_{\Gamma_n}^s \left\{ \nabla^2 U(Y_u) \nabla U(Y_u) + \vec{\Delta}(\nabla U)(Y_u) \right\} du \right] \right\rangle \right| \end{aligned}$$

Combining this equality, **(S22)** and $|\langle a, b \rangle| \leq \epsilon_1 \|a\|^2 + (4\epsilon_1)^{-1} \|b\|^2$ in we have

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|\Theta_{n+1}\|^2 \right] &\leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1}L((1 + \epsilon_2)\gamma_{n+1} - 2/(m + L))_+\} \|\Theta_n\|^2 \\ &\quad + (2\epsilon_1)^{-1} A + (1 + \epsilon_2^{-1})\gamma_{n+1} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\int_{\Gamma_n}^{\Gamma_{n+1}} \|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2 ds \right], \end{aligned} \quad (\text{S26})$$

where

$$A = \int_{\Gamma_n}^{\Gamma_{n+1}} \left\| \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\int_{\Gamma_n}^s \nabla^2 U(Y_u) \nabla U(Y_u) + \vec{\Delta}(\nabla U)(Y_u) du \right] \right\|^2 ds.$$

We now separately bound the two last terms of the right hand side. By **H1**, the Markov property of $(Y_t)_{t \geq 0}$ and Lemma **21**, we have

$$\begin{aligned} \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2 \right] ds \\ \leq L^2 \left(d\gamma_{n+1}^2 + dL^2\gamma_{n+1}^4/12 + (1/2)L^2\gamma_{n+1}^3 \|Y_{\Gamma_n} - x^*\|^2 \right). \end{aligned} \quad (\text{S27})$$

We now bound A . We get using Jensen's inequality, Fubini's theorem, $\nabla U(x^*) = 0$ and (10)

$$\begin{aligned} A &\leq 2 \int_{\Gamma_n}^{\Gamma_{n+1}} (s - \Gamma_n) \int_{\Gamma_n}^s \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|\nabla^2 U(Y_u) \nabla U(Y_u)\|^2 \right] du ds \\ &\quad + 2 \int_{\Gamma_n}^{\Gamma_{n+1}} (s - \Gamma_n) \int_{\Gamma_n}^s \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|\vec{\Delta}(\nabla U)(Y_u)\|^2 \right] du ds \\ &\leq 2 \int_{\Gamma_n}^{\Gamma_{n+1}} (s - \Gamma_n) L^4 \int_{\Gamma_n}^s \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[\|Y_u - x^*\|^2 \right] du ds + 2\gamma_{n+1}^3 d^2 \tilde{L}^2 / 3. \end{aligned} \quad (\text{S28})$$

By Lemma 21-(i), the Markov property and for all $t \geq 0$, $1 - e^{-t} \leq t$, we have for all $s \in [\Gamma_n, \Gamma_{n+1}]$,

$$\int_{\Gamma_n}^s \mathbb{E}^{\mathcal{F}_{\Gamma_n}} c \gamma_{n+1}^3 \left[\|Y_u - x^*\|^2 \right] du \leq (2m)^{-1} (1 - e^{-2m(s-\Gamma_n)}) \|Y_{\Gamma_n} - x^*\|^2 + d(s - \Gamma_n)^2.$$

Using this inequality in (S28) and for all $t \geq 0$, $1 - e^{-t} \leq t$, we get

$$A \leq (2L^4 \gamma_{n+1}^3 / 3) \|Y_{\Gamma_n} - x^*\|^2 + L^4 d \gamma_{n+1}^4 / 2 + 2\gamma_{n+1}^3 d^2 \tilde{L}^2 / 3.$$

Combining this bound and (S27) in (S26) concludes the proof. \square

Lemma S9. Let $(\gamma_k)_{k \geq 1}$ be a non-increasing sequence of positive numbers. Let $\varpi, \beta > 0$ be positive constants satisfying $\varpi^2 \leq 4\beta$ and $\tau > 0$. Assume there exists $N \geq 1$, $\gamma_N \leq \tau$ and $\gamma_N \varpi \leq 1$. Then for all $n \geq 0$, $j \geq 2$

(i)

$$\begin{aligned} \sum_{i=1}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi + \gamma_k \beta (\gamma_k - \tau)_+) \gamma_i^j &\leq \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j \\ &\quad + \left\{ \beta^{-1} \gamma_1^{j-2} \prod_{k=1}^{N-1} (1 + \gamma_k^2 \beta) \right\} \prod_{k=N}^{n+1} (1 - \varpi \gamma_k). \end{aligned}$$

(ii) For all $\ell \in \{N, \dots, n\}$,

$$\sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j \leq \exp \left(- \sum_{k=\ell}^{n+1} \varpi \gamma_k \right) \sum_{i=N}^{\ell-1} \gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi}.$$

Proof. (i) By definition of N we have

$$\begin{aligned} &\sum_{i=1}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi + \gamma_k \beta (\gamma_k - \tau)_+) \gamma_i^j \\ &\leq \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j + \left\{ \sum_{i=1}^{N-1} \prod_{k=i+1}^{N-1} (1 + \gamma_k^2 \beta) \right\} \prod_{k=N}^{n+1} (1 - \gamma_k \varpi). \end{aligned} \quad (\text{S29})$$

Using that $(\gamma_k)_{k \geq 1}$ is non-increasing, we have

$$\begin{aligned} \sum_{i=1}^{N-1} \prod_{k=i+1}^{N-1} (1 + \gamma_k^2 \beta) \gamma_i^j &\leq \sum_{i=1}^{N-1} \frac{\gamma_i^{j-2}}{\beta} \left\{ \prod_{k=i}^{N-1} (1 + \gamma_k^2 \beta) - \prod_{k=i+1}^{N-1} (1 + \gamma_k^2 \beta) \right\} \\ &\leq \beta^{-1} \gamma_1^{j-2} \prod_{k=1}^{N-1} (1 + \gamma_k^2 \beta) . \end{aligned}$$

Plugging this inequality in (S29) concludes the proof of (i).

(ii) Let $\ell \in \{N, \dots, n+1\}$. Since $(\gamma_k)_{k \geq 1}$ is non-increasing and for every $x \in \mathbb{R}$, $(1+x) \leq e^x$, we get

$$\begin{aligned} \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j &= \sum_{i=N}^{\ell-1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j + \sum_{i=\ell}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j \\ &\leq \sum_{i=N}^{\ell-1} \exp\left(-\sum_{k=i+1}^{n+1} \varpi \gamma_k\right) \gamma_i^j + \gamma_\ell^{j-1} \sum_{i=\ell}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i \\ &\leq \exp\left(-\sum_{k=\ell}^{n+1} \varpi \gamma_k\right) \sum_{i=N}^{\ell-1} \gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi} . \end{aligned}$$

(iii)

□

Lemma S10. Let $(\gamma_k)_{k \geq 1}$ be a non-increasing sequence of positive numbers, $\varpi, \beta, \tau > 0$ be positive real numbers, and $N \geq 1$ satisfying the assumptions of Lemma S9. Let $P \in \mathbb{N}^*$, $C_i \geq 0$, $i = 0, \dots, P$ be positive constants and $(u_n)_{n \geq 0}$ be a sequence of real numbers with $u_0 \geq 0$ satisfying for all $n \geq 0$

$$u_{n+1} \leq (1 - \gamma_{n+1} \varpi + \beta \gamma_{n+1} (\gamma_{n+1} - \tau)_+) u_n + \sum_{i=0}^P C_j \gamma_{n+1}^{j+2} .$$

Then for all $n \geq 1$,

$$\begin{aligned} u_n &\leq \left\{ \prod_{k=1}^{N-1} (1 + \beta \gamma_k^2) \right\} \prod_{k=N}^n (1 - \gamma_k \varpi) u_0 + \sum_{j=0}^P C_j \sum_{i=N}^n \prod_{k=i+1}^n (1 - \gamma_k \varpi) \gamma_i^{j+2} \\ &\quad + \left\{ \sum_{j=0}^P C_j \beta^{-1} \gamma_1^j \prod_{k=1}^{N-1} (1 + \gamma_k^2 \beta) \right\} \prod_{k=N}^n (1 - \varpi \gamma_k) . \end{aligned}$$

Proof. This is a consequence of a straightforward induction and Lemma S9-(i). □

Theorem S11. Assume **H1**, **H2** and **G1**.

(i) For all $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $n \geq 1$,

$$W_2^2(\mu_0 Q_\gamma^n, \pi) \leq \tilde{u}_n^{(1)}(\gamma) W_2^2(\mu_0, \pi) + \tilde{u}_n^{(2)}(\gamma), \quad (\text{S30})$$

where

$$\tilde{u}_n^{(1)}(\gamma) = \left\{ \prod_{k=1}^{n_1-1} (1 + 2L^2 \gamma_k^2) \right\} \prod_{k=n_1}^n (1 - \kappa \gamma_k / 2), \quad (\text{S31})$$

$$\begin{aligned} \tilde{u}_n^{(2)}(\gamma) &= \sum_{i=n_1}^n \gamma_i^2 \mathbf{b}(\gamma_i) \prod_{k=i+1}^n (1 - \kappa \gamma_k / 2) \\ &+ \mathbf{b}(\gamma_1) (2L^2)^{-1} \left\{ \prod_{k=1}^{n_1-1} (1 + 2\gamma_k^2 L^2) \right\} \prod_{k=n_1}^n (1 - \kappa \gamma_k / 2), \end{aligned} \quad (\text{S32})$$

with

$$\mathbf{b}(\gamma) = L^2 d \{ \kappa^{-1} + \gamma \} (2 + L^2 \gamma / m + L^2 \gamma^2 / 6).$$

(ii) If in addition **H3** holds, for all $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $n \geq 1$,

$$W_2^2(\mu_0 Q_\gamma^n, \pi) \leq \tilde{u}_n^{(1)}(\gamma) W_2^2(\mu_0, \pi) + \tilde{u}_n^{(3)}(\gamma), \quad (\text{S33})$$

where

$$\begin{aligned} \tilde{u}_n^{(3)}(\gamma) &= \sum_{i=n_1}^n \gamma_i^3 \mathbf{c}(\gamma_i) \prod_{k=i+1}^n (1 - \kappa \gamma_k / 2) \\ &+ \gamma_1 \mathbf{c}(\gamma_1) (2L^2)^{-1} \left\{ \prod_{k=1}^{n_1-1} (1 + 2\gamma_k^2 L^2) \right\} \prod_{k=n_1}^n (1 - \kappa \gamma_k / 2), \end{aligned} \quad (\text{S34})$$

with

$$\mathbf{c}(\gamma) = d \left\{ 2L^2 + \gamma_i L^4 \left(\frac{\gamma_i}{6} + m^{-1} \right) + \kappa^{-1} \left(\frac{4d\tilde{L}^2}{3} + \gamma_i L^4 + \frac{4L^4}{3m} \right) \right\}.$$

Proof. (i) Let ζ_0 be an optimal transference plan of μ_0 and π . Let $(Y_t, \bar{Y}_t)_{t \geq 0}$ with (Y_0, \bar{Y}_0) distributed according to ζ_0 and defined by (50). By definition of W_2 and since for all $t \geq 0$, π is invariant for P_t , $W_2^2(\mu_0 Q_\gamma^n, \pi) \leq \mathbb{E}[\|Y_{\Gamma_n} - X_{\Gamma_n}\|^2]$. Then the proof follows from Lemma S8-(i) and Lemma S10 using that for all $k \in \mathbb{N}$, $\mathbb{E}[\|Y_{\Gamma_k} - x^*\|] \leq d/m$ by since Y_0 is distributed according to π .

(ii) The proof follows the same line as the first statement using Lemma S8-(ii) instead of Lemma S8-(i). □

3.1 Explicit bound based on Theorem S11 for $\gamma_k = \gamma_1/k$

We give here a bound on the sequences $(\tilde{u}_n^{(1)}(\gamma))_{n \geq 1}$ and $(\tilde{u}_n^{(2)}(\gamma))_{n \geq 1}$, $(\tilde{u}_n^{(3)}(\gamma))_{n \geq 1}$ for $(\gamma_k)_{k \geq 1}$ defined by $\gamma_1 > 0$ and $\gamma_k = \gamma_1/k$. Recall that $\boldsymbol{\psi}_\beta$ is given by (S5). First note, since $(\gamma_k)_{k \geq 1}$ is non-increasing, for all $n \geq 1$, we have

$$\begin{aligned} \tilde{u}_n^{(2)}(\gamma) &\leq \sum_{j=0}^1 C_j \sum_{i=n_1}^n \gamma_i^{j+2} \prod_{k=i+1}^n (1 - \kappa\gamma_k/2) \\ &\quad + \sum_{j=0}^1 C_j (2L^2)^{-1} \gamma_1^j \left\{ \prod_{k=1}^{n_1-1} (1 + 2\gamma_k^2 L^2) \right\} \prod_{k=n_1}^n (1 - \kappa\gamma_k/2), \end{aligned} \quad (\text{S35})$$

where

$$C_1 = 2bdL^2, C_2 = b(dL^4/m + \gamma_1 dL^4/6), \mathbf{b} = \kappa^{-1} + \gamma_1.$$

and

$$\begin{aligned} \tilde{u}_n^{(3)}(\gamma) &\leq \sum_{j=0}^1 D_j \sum_{i=n_1}^n \gamma_i^{j+3} \prod_{k=i+1}^n (1 - \kappa\gamma_k/2) \\ &\quad + \sum_{j=0}^1 D_j (2L^2)^{-1} \gamma_1^{j+1} \left\{ \prod_{k=1}^{n_1-1} (1 + 2\gamma_k^2 L^2) \right\} \prod_{k=n_1}^n (1 - \kappa\gamma_k/2), \end{aligned} \quad (\text{S36})$$

where

$$D_1 = d \left[2L^2 + (4/(3\kappa)) \left\{ d\tilde{L}^2 + L^4/m \right\} \right], D_2 = d \left[\kappa^{-1} L^4 + L^4 \gamma_1 / (m + L) + m^{-1} \right].$$

1. We first give explicit bound based on Theorem S11-(i). For $n_1 = 1$, by (S6) and (S7), we have

$$\begin{aligned} \tilde{u}_n^{(1)}(\gamma) &\leq (n+1)^{-\kappa\gamma_1/2} \\ \tilde{u}_n^{(2)}(\gamma) &\leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^1 C_j \left\{ \gamma_1^{j+2} (\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n+1) + 1) + (2L^2)^{-1} \gamma_1^j \right\}. \end{aligned}$$

For $n_1 > 1$, since $(\gamma_k)_{k \geq 0}$ is non increasing, using again (S6), (S7), and the bound for $t \in \mathbb{R}$, $(1+t) \leq e^t$, we get

$$\begin{aligned} \tilde{u}_n^{(1)}(\gamma) &\leq (n+1)^{-\kappa\gamma_1/2} \exp \left\{ \kappa\gamma_1 \boldsymbol{\psi}_0(n_1)/2 + 2L^2 \gamma_1^2 (\boldsymbol{\psi}_{-1}(n_1-1) + 1) \right\} \\ \tilde{u}_n^{(2)}(\gamma) &\leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^1 C_j \left(\gamma_1^{j+2} (\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n+1) - \boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n_1) + 1) \right. \\ &\quad \left. + (\gamma_1^j / (2L^2)) \exp \left\{ \kappa\gamma_1 \boldsymbol{\psi}_0(n_1)/2 + 2L^2 \gamma_1^2 (\boldsymbol{\psi}_{-1}(n_1-1) + 1) \right\} \right). \end{aligned}$$

Thus, for $\gamma_1 > 2\kappa^{-1}$, the bound given by Theorem S11-(i) is of order $\mathcal{O}(n^{-1})$.

2. We first give explicit bound based on Theorem S11-(ii). Note that bounds on $(\tilde{u}_n^{(1)}(\gamma))_{n \geq 1}$ have been obtained below. We just need to give some bounds on $(\tilde{u}_n^{(3)}(\gamma))_{n \geq 1}$. For $n_1 = 1$, by (S6), (S7), we have

$$\tilde{u}_n^{(3)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^1 D_j \left\{ \gamma_1^{j+3} (\boldsymbol{\psi}_{\kappa\gamma_1/2-2-j}(n+1) + 1) + (2L^2)^{-1} \gamma_1^{j+1} \right\} .$$

For $n_1 > 1$, since $(\gamma_k)_{k \geq 0}$ is non increasing, using again (S6), (S7), and the bound for $t \in \mathbb{R}$, $(1+t) \leq e^t$, we get

$$\begin{aligned} \tilde{u}_n^{(3)}(\gamma) &\leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^1 D_j \left(\gamma_1^{j+3} (\boldsymbol{\psi}_{\kappa\gamma_1/2-2-j}(n+1) - \boldsymbol{\psi}_{\kappa\gamma_1/2-2-j}(n_1) + 1) \right. \\ &\quad \left. + (\gamma_1^{j+1}/(2L^2)) \exp \left\{ \kappa\gamma_1 \boldsymbol{\psi}_0(n_1)/2 + 2L^2 \gamma_1^2 (\boldsymbol{\psi}_{-1}(n_1-1) + 1) \right\} \right) . \end{aligned}$$

Thus, for $\gamma_1 > 4\kappa^{-1}$, the bound given by Theorem S11-(i) is of order $\mathcal{O}(n^{-1})$.

4 Explicit bounds on the MSE

Without loss of generality, assume that $\|f\|_{\text{Lip}} = 1$. In the following, denote by $\Omega(x) = \|x - x^*\|^2 + d/m$ and C a constant (which may take different values upon each appearance), which does not depend on m, L, γ_1, α and $\|x - x^*\|$.

4.1 Explicit bounds based on Theorem 5

1. First for $\alpha = 0$, recall by Theorem 5 and (S1) we have for all $p \geq 1$,

$$W_2^2(\delta_x R_\gamma^p, \pi) \leq 2\Omega(x)(1 - \kappa\gamma_1/2)^p + 2\kappa^{-1}(\mathbf{A}_0\gamma_1 + \mathbf{A}_1\gamma_1^2) ,$$

where \mathbf{A} and \mathbf{A}_1 are given by (S2) and (S3) respectively. Set

$$\mathbf{A} = \mathbf{A} \vee (\mathbf{A}_1/(m+L))$$

So by (27) and Lemma 23, we have the following bound for the bias

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 \leq C \left(\frac{\kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{\gamma_1 n} + \kappa^{-1} \mathbf{A} \gamma_1 \right) .$$

Therefore plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\kappa^{-1} \mathbf{A} \gamma_1 + \frac{\kappa^{-2} + \kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{n \gamma_1} \right) . \quad (\text{S37})$$

So with fixed γ_1 this bound is of order γ_1 . If we fix the number of iterations n , we can optimize the choice of γ_1 . Set

$$\gamma_{\star,0}(n) = (\kappa^{-1}\mathbf{A})^{-1}(C_{\text{MSE},0}/n)^{1/2}, \text{ where } C_{\text{MSE},0} = \kappa^{-3}\mathbf{A},$$

and (S37) becomes if $\gamma_1 \leftarrow \gamma_{\star,0}(n)$,

$$\text{MSE}_f^{N,n} \leq C(C_{\text{MSE},0}n)^{-1/2} (\kappa^{-1} \exp(-\kappa N \gamma_{\star,0}(n)/2) \Omega(x) + C_{\text{MSE},0}) .$$

Setting $N_0(n) = 2(\kappa \gamma_{\star,0}(n))^{-1} \log(\Omega(x))$, we end up with

$$\text{MSE}_f^{N_0(n),n} \leq C(C_{\text{MSE},0}/n)^{1/2} .$$

Note that $N_0(n)$ is of order $n^{1/2}$.

- For $\alpha \in (0, 1/2)$ by Theorem 5, Lemma 23, (S6) and (S8), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left(\frac{\kappa^{-1}\mathbf{A}\gamma_1}{(1-2\alpha)n^\alpha} + \frac{\kappa^{-1} \exp\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right) .$$

Plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1}\mathbf{A}\gamma_1}{(1-2\alpha)n^\alpha} + \frac{\kappa^{-1} \exp\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right) . \tag{S38}$$

At fixed γ_1 , this bound is of order $n^{-\alpha}$, and is better than (S37) for $(\gamma_k)_{k \geq 1}$ constant. If we fix the number of iterations n , we can optimize the choice of γ_1 again. Set

$$\gamma_{\star,\alpha}(n) = (\kappa^{-1}\mathbf{A}/(1-2\alpha))^{-1}(C_{\text{MSE},\alpha}/n^{1-2\alpha})^{1/2}, \text{ where } C_{\text{MSE},\alpha} = \kappa^{-3}\mathbf{A}/(1-2\alpha),$$

(S38) becomes with $\gamma_1 \leftarrow \gamma_{\star,\alpha}(n)$,

$$\begin{aligned} \text{MSE}_f^{N,n} \\ \leq C(C_{\text{MSE},\alpha}n)^{-1/2} (\kappa^{-1} \exp\{-\kappa N^{1-\alpha} \gamma_{\star,\alpha}(n)/(2(1-\alpha))\} \Omega(x) + C_{\text{MSE},\alpha}) . \end{aligned}$$

Setting $N_\alpha(n) = \{2(1-\alpha)(\kappa \gamma_{\star,\alpha}(n))^{-1} \log(\Omega(x))\}^{1/(1-\alpha)}$, we end up with

$$\text{MSE}_f^{N_\alpha(n),n} \leq C(C_{\text{MSE},\alpha}/n)^{1/2} .$$

It is worthwhile to note that the order of $N_\alpha(n)$ in n is $n^{(1-2\alpha)/(2(1-\alpha))}$, and $C_{\text{MSE},\alpha}$ goes to infinity as $\alpha \rightarrow 1/2$.

3. If $\alpha = 1/2$, by Theorem 5, Lemma 23, (S6) and (S8), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left(\frac{\kappa^{-1} \mathbf{A} \gamma_1 \log(n)}{n^{1/2}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1/2}/4\} \Omega(x)}{\gamma_1 n^{1/2}} \right).$$

Plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1} \mathbf{A} \gamma_1 \log(n)}{n^{1/2}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1/2}/4\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1/2}} \right). \quad (\text{S39})$$

At fixed γ_1 , the order of this bound is $\log(n)n^{-1/2}$, and is the best bound for the MSE. Fix the number of iterations n , and we now optimize the choice of γ_1 . Set

$$\gamma_{\star,1/2}(n) = (\kappa^{-1} \mathbf{A})^{-1} (C_{\text{MSE},1/2} / \log(n))^{1/2}, \text{ where } C_{\text{MSE},1/2} = \kappa^{-3} \mathbf{A},$$

and (S39) becomes with $\gamma_1 \leftarrow \gamma_{\star,1/2}(n)$,

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\log(n)}{n C_{\text{MSE},1/2}} \right)^{1/2} \left(\kappa^{-1} \exp\{-\kappa N^{1/2} \gamma_{\star,1/2}(n)/4\} \Omega(x) + \frac{C_{\text{MSE},1/2}}{\log(n)} \right).$$

Setting $N_{1/2}(n) = (4(\kappa \gamma_{\star,1/2}(n))^{-1} \log(\Omega(x)))^2$, we end up with

$$\text{MSE}_f^{N_{1/2}(n),n} \leq C \left(\frac{\log(n) C_{\text{MSE},1/2}}{n} \right)^{1/2}.$$

4. For $\alpha \in (1/2, 1]$, by Theorem 5, Lemma 23, (S6) and (S8), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left(\frac{\kappa^{-1} \mathbf{A} \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right).$$

Plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1} \mathbf{A} \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right).$$

For fixed γ_1 , the MSE is of order $n^{1-\alpha}$, and is worse than for $\alpha \in [0, 1/2]$. For a fixed number of iteration n , optimizing γ_1 would imply to choose $\gamma_1 \rightarrow +\infty$ as $n \rightarrow +\infty$. Therefore, in that case, the best choice of γ_1 is the largest possible value $1/(m+L)$.

5. For $\alpha = 1$, by Section 3.1, for $\gamma_1 > 2\kappa^{-1}$ there exists $\tilde{C}_1 \geq 0$, independent of d and n such that the bias is upper bounded by

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 = \tilde{C}_1 / \log(n).$$

Plugging this inequality and the one given by Theorem 15 implies there exists $\tilde{C} \geq 0$, independent of d and n such that the bias is upper bounded by

$$\text{MSE}_f^{N,n} = \tilde{C}_2 / \log(n).$$

4.2 Explicit bound based on Theorem 8

1. First for $\alpha = 0$, recall by Theorem 8 and (S9) we have for all $p \geq 1$,

$$W_2^2(\delta_x R_\gamma^p, \pi) \leq 2\Omega(x)(1 - \kappa\gamma_1/2)^p + 2\kappa^{-1}(\mathbf{B}_0\gamma_1 + \mathbf{B}_1\gamma_1^2),$$

where \mathbf{B}_0 and \mathbf{B}_1 are given by (S10) and (S11) respectively. Set

$$\mathbf{B} = \mathbf{B}_0 \vee (\mathbf{B}_1/(m + L))$$

So by and Lemma 23, we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left(\frac{\kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{\gamma_1 n} + \kappa^{-1} \mathbf{B} \gamma_1^2 \right).$$

Therefore plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\kappa^{-1} \mathbf{B} \gamma_1^2 + \frac{\kappa^{-2} + \kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{n \gamma_1} \right). \quad (\text{S40})$$

So with fixed γ_1 this bound is of order γ_1 . If we fix the number of iterations n , we can optimize the choice of γ_1 . Set

$$\gamma_{*,0}(n) = (\kappa \mathbf{B} n)^{-1/3},$$

and (S37) becomes if $\gamma_1 \leftarrow \gamma_{*,0}(n)$,

$$\text{MSE}_f^{N,n} \leq C (\mathbf{B}^{-1/2} n)^{-2/3} \left(\kappa^{-4/3} \exp(-\kappa N \gamma_{*,0}(n)/2) \Omega(x) + \kappa^{-5/3} \right).$$

Setting $N_0(n) = 2(\kappa \gamma_{*,0}(n))^{-1} \log(\Omega(x))$, we end up with

$$\text{MSE}_f^{N_0(n),n} \leq C (\mathbf{B}^{-1/2} \kappa^{5/2} n)^{-2/3}.$$

Note that $N_0(n)$ is of order $n^{1/3}$.

2. For $\alpha \in (0, 1/3)$ by Theorem 8, Lemma 23, (S6) and (S12), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left(\frac{\kappa^{-1} \mathbf{B} \gamma_1^2}{(1 - 3\alpha)n^{2\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right).$$

Plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1} \mathbf{B} \gamma_1^2}{(1 - 3\alpha)n^{2\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right). \quad (\text{S41})$$

If we fix the number of iterations n , we can optimize the choice of γ_1 again. Set

$$\gamma_{\star,\alpha}(n) = (n^{1-3\alpha}\kappa\mathbf{B}/(1-3\alpha))^{-1/3},$$

(S41) becomes with $\gamma_1 \leftarrow \gamma_{\star,\alpha}(n)$,

$$\text{MSE}_f^{N,n} \leq C(\mathbf{B}^{-1/2}n)^{-2/3} \left(\kappa^{-4/3} \exp(-\kappa N\gamma_{\star,0}(n)/(2(1-\alpha)))\Omega(x) + \kappa^{-5/3}(1-3\alpha)^{-1/3} \right).$$

Setting $N_\alpha(n) = \{(\kappa\gamma_{\star,\alpha}(n))^{-1} \log(\Omega(x))\}^{1/(1-\alpha)}$, we end up with

$$\text{MSE}_f^{N_\alpha(n),n} \leq C(\mathbf{B}^{-1/2}\kappa^{5/2}n)^{-2/3}.$$

It is worthwhile to note that the order of $N_\alpha(n)$ in n is $n^{(1-3\alpha)/(3(1-\alpha))}$.

3. If $\alpha = 1/3$, by Theorem 8, Lemma 23, (S6) and (S12), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left(\frac{\kappa^{-1}\mathbf{B}\gamma_1^2 \log(n)}{n^{2/3}} + \frac{\kappa^{-1} \exp\{-\kappa\gamma_1 N^{2/3}/4\} \Omega(x)}{\gamma_1 n^{2/3}} \right).$$

Plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1}\mathbf{B}\gamma_1^2 \log(n)}{n^{2/3}} + \frac{\kappa^{-1} \exp\{-\kappa\gamma_1 N^{2/3}/4\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{2/3}} \right). \quad (\text{S42})$$

At fixed γ_1 , the order of this bound is $\log(n)n^{-2/3}$, and is the best bound for the MSE. Fix the number of iterations n , and we now optimize the choice of γ_1 . Set

$$\gamma_{\star,1/2}(n) = (\kappa\mathbf{B} \log(n))^{-1/3},$$

and (S42) becomes with $\gamma_1 \leftarrow \gamma_{\star,1/2}(n)$,

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\log(n)\mathbf{B}}{n^2} \right)^{1/3} \left(\kappa^{-4/3} \exp\{-\kappa N^{1/2}\gamma_{\star,1/2}(n)/4\} \Omega(x) + \kappa^{-5/3} \right).$$

Setting $N_{1/2}(n) = (4(\kappa\gamma_{\star,1/2}(n))^{-1} \log(\Omega(x)))^{3/2}$, we end up with

$$\text{MSE}_f^{N_{1/2}(n),n} \leq C \left(\frac{\log(n)\mathbf{B}}{\kappa^5 n^2} \right)^{1/3}.$$

We can see that we obtain a worse bound than for $\alpha = 0$ and $\alpha \in (0, 1/3)$.

4. For $\alpha \in (1/3, 1]$, by Theorem 8, Lemma 23, (S6) and (S12), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left(\frac{\kappa^{-1}\mathbf{B}\gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right).$$

Plugging this inequality and the one given by Theorem 15 implies:

$$\text{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1} \mathbf{B} \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp \left\{ -\kappa \gamma_1 N^{1-\alpha} / (2(1-\alpha)) \right\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right).$$

For fixed γ_1 , the MSE is of order $n^{1-\alpha}$, and is worse than for $\alpha = 1/2$. For a fixed number of iteration n , optimizing γ_1 would imply to choose $\gamma_1 \rightarrow +\infty$ as $n \rightarrow +\infty$. Therefore, in that case, the best choice of γ_1 is the largest possible value $1/(m+L)$.

5. For $\alpha = 1$, by Section 3.1, for $\gamma_1 > 2\kappa^{-1}$ there exists $\tilde{C}_1 \geq 0$, independent of d and n such that the bias is upper bounded by

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 = \tilde{C}_1 / \log(n).$$

Plugging this inequality and the one given by Theorem 15 implies there exists $\tilde{C} \geq 0$, independent of d and n such that the bias is upper bounded by

$$\text{MSE}_f^{N,n} = \tilde{C}_2 / \log(n).$$

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