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# Supplement to High-dimensional Bayesian inference via the Unadjusted Langevin Algorithm

Alain Durmus<sup>1</sup>      Éric Moulines<sup>2</sup>

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## 1 Postponed proofs

### 1.1 Proof of (23)

Consider the constant sequence  $\gamma_k = \gamma$  for all  $k \in \mathbb{N}^*$  with  $\gamma \in (0, 1/(m + L)]$ . By (20), we have for all  $n \in \mathbb{N}^*$  and  $x \in \mathbb{R}^d$

$$\vartheta_{n,0}^{(1)}(x) \leq \gamma D_1(\gamma, d) + \gamma^3 D_2(\gamma) \sum_{i=1}^n (1 - \kappa\gamma/2)^{n-i} \delta_{i,n,0}(x) ,$$

where

$$D_1(\gamma, d) = 2L^2 \kappa^{-1} (\kappa^{-1} + \gamma) (2d + L^2 \gamma^2 / 6) , \quad D_2(\gamma) = L^4 (\kappa^{-1} + \gamma) .$$

In addition, using that  $\kappa \geq 2m$  and for all  $t \geq 0$ ,  $1 - t \leq e^{-t}$ ,

$$\begin{aligned} \sum_{i=1}^n (1 - \kappa\gamma/2)^{n-i} \delta_{i,n,0}(x) &= \sum_{i=1}^n \left[ (1 - \kappa\gamma/2)^{n-i} \left\{ e^{-2m\gamma(i-1)} \|x - x^*\|^2 \right. \right. \\ &\quad \left. \left. + \left( 1 - e^{-2m\gamma(i-1)} \right) (d/m) \right\} \right] \\ &\leq n e^{-m\gamma(n-1)} \|x - x^*\|^2 + 2d(\kappa\gamma m)^{-1} . \end{aligned} \quad (\text{S1})$$

Therefore for all  $n \geq 1$  and  $x \in \mathbb{R}^d$  we get

$$\vartheta_{n,0}^{(1)}(x) \leq \gamma D_1(\gamma) + \gamma^3 D_2(\gamma) \left\{ n e^{-m\gamma(n-1)} \|x - x^*\|^2 + 2d(\kappa\gamma m)^{-1} \right\} . \quad (\text{S2})$$

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Let now  $\ell \in \mathbb{N}^*$ ,  $\ell \geq \lceil \gamma^{-1} \rceil + 1$  and  $n = \ell - \lceil \gamma^{-1} \rceil$ . Then,

$$\begin{aligned} & \sum_{k=n+1}^{\ell} \{(\gamma_k^3 L^2/3) \varrho_{1,k-1}(x) + d\gamma_k^2\} \\ & \leq (L^2 \gamma^3/3) \left\{ (1 - \kappa\gamma)^n (\ell - n - 1) \|x - x^*\|^2 + 2\kappa^{-1} \gamma d (\ell - n - 1) \right\} + d\gamma^2 (\ell - n - 1) \\ & \leq (L^2 \gamma^3/3) \left\{ (1 + \gamma^{-1})(1 - \kappa\gamma)^{\ell - \lceil \gamma^{-1} \rceil} \|x - x^*\|^2 + 2(1 + \gamma)\kappa^{-1} d \right\} + d(1 + \gamma). \end{aligned}$$

Combining this inequality and (S2) in the bound provided by Theorem 14 shows (23).

## 1.2 Proof of (61)

First for all  $n \geq 1$  and  $x \in \mathbb{R}^d$ , we have

$$\vartheta_{n,0}^{(2)}(x) \leq \gamma^2 \mathbf{E}_1(\gamma, d) + \gamma^3 \mathbf{E}_2(\gamma) \sum_{i=1}^n \prod_{k=i+1}^n (1 - \kappa\gamma_k/2) \delta_{i,n,0}(x),$$

where

$$\mathbf{E}_1(\gamma, d) = 2d\kappa^{-1} \left\{ 2L^2 + 4\kappa^{-1}(\tilde{L}^2/12 + \gamma L^4/4) + \gamma^2 L^4/6 \right\}, \mathbf{E}_2(\gamma) = L^4(4\kappa^{-1}/3 + \gamma).$$

$$\mathbf{E}_3(\gamma, d, x) = (\ell - \gamma^{-1}) e^{-m\gamma(\ell - 2\gamma^{-1} - 1)} \|x - x^*\|^2 + 2d/(\kappa\gamma m)$$

$$\mathbf{E}_4(\gamma, d, x) = e^{-m\gamma(\ell - 2\gamma^{-1} - 1)} \|x - x^*\|^2 + 2\kappa^{-1}d + d/m.$$

By (S1), we get for all  $n \geq 1$  and  $x \in \mathbb{R}^d$ ,

$$\vartheta_{n,0}^{(2)}(x) \leq \gamma^2 \mathbf{E}_1(\gamma, d) + \gamma^3 \mathbf{E}_2(\gamma) \left\{ n e^{-m\gamma(n-1)} \|x - x^*\|^2 + 2d(\kappa\gamma m)^{-1} \right\}. \quad (\text{S3})$$

On the other hand, for all  $x \in \mathbb{R}^d$ ,  $\ell, n \in \mathbb{N}$ ,  $n \geq 1$ ,  $\ell > n$  we have using that  $\kappa \geq 2m$  and for all  $t \geq 0$ ,  $1 - t \leq e^{-t}$ ,

$$\begin{aligned} \vartheta_{n,\ell}^{(2)}(x) & \leq \gamma^3 n \mathbf{E}_1(\gamma) + \gamma^3 n \mathbf{E}_2(\gamma) \left\{ e^{-m\gamma(n-1)} \varrho_{n,\ell}(x) + d/m \right\} \\ & \leq \gamma^3 n \mathbf{E}_1(\gamma) + \gamma^3 n \mathbf{E}_2(\gamma) \left\{ e^{-m\gamma(n-1)} \left( (1 - \kappa\gamma)^{\ell-n} \|x - x^*\|^2 + 2\kappa^{-1}d \right) + d/m \right\} \\ & \leq \gamma^3 n \mathbf{E}_1(\gamma) + \gamma^3 n \mathbf{E}_2(\gamma) \left\{ e^{-m\gamma(\ell-1)} \|x - x^*\|^2 + 2\kappa^{-1}d + d/m \right\}. \end{aligned} \quad (\text{S4})$$

Finally, for all  $\ell \in \mathbb{N}^*$  and  $x \in \mathbb{R}^d$ , we have

$$(\gamma^3 L^2/3) \varrho_{1,\ell-1}(x) + d\gamma^2 \leq (\gamma^3 L^2/3) \left\{ (1 - \kappa\gamma)^{\ell-1} \|x - x^*\|^2 + 2d\kappa^{-1} \right\} + d\gamma^2. \quad (\text{S5})$$

Combining (S3), (S4) and (S5) in the bound given by Lemma 31, and using that  $\gamma^{-1} \leq 2^{n(\gamma)} \leq 2\gamma^{-1}$  we have for all  $\ell \in \mathbb{N}^*$ ,  $\ell > 2^{n(\gamma)}$ ,

$$\begin{aligned}
\|\delta_x P_{\ell\gamma} - \delta_x R_{\gamma}^{\ell}\|_{\text{TV}} &\leq 2^{-3/2} L \left[ (\gamma^3 L^2 / 3) \left\{ (1 - \kappa\gamma)^{\ell-1} \|x - x^*\|^2 + 2d\kappa^{-1} \right\} + d\gamma^2 \right]^{1/2} \\
&+ (4\pi)^{-1/2} \left[ \gamma^2 \mathbf{E}_1(\gamma) + \gamma^3 \mathbf{E}_2(\gamma) \left\{ (\ell - \gamma^{-1}) e^{-m\gamma(\ell-2\gamma^{-1}-1)} \|x - x^*\|^2 + 2d(\kappa\gamma m)^{-1} \right\} \right]^{1/2} \\
&+ \sum_{k=1}^{n(\gamma)} \left[ \frac{\gamma^3 2^{k-1} \mathbf{E}_1(\gamma) + \gamma^3 2^{k-1} \mathbf{E}_2(\gamma) \left\{ e^{-m\gamma(\ell-2^{k-1})} \|x - x^*\|^2 + 2\kappa^{-1}d + d/m \right\}}{\pi 2^{k+1}\gamma} \right]^{1/2} \\
&\leq 2^{-3/2} L \left\{ (\gamma^3 L^2 / 3) \left\{ (1 - \kappa\gamma)^{\ell-1} \|x - x^*\|^2 + 2d\kappa^{-1} \right\} + d\gamma^2 \right\}^{1/2} \\
&+ (4\pi)^{-1/2} \left[ \gamma^2 \mathbf{E}_1(\gamma) + \gamma^3 \mathbf{E}_2(\gamma) \left\{ (\ell - \gamma^{-1}) e^{-m\gamma(\ell-2\gamma^{-1}-1)} \|x - x^*\|^2 + 2d(\kappa\gamma m)^{-1} \right\} \right]^{1/2} \\
&+ (4\pi)^{-1/2} n(\gamma) \left[ \gamma^2 \mathbf{E}_1(\gamma) + \gamma^2 \mathbf{E}_2(\gamma) \left\{ e^{-m\gamma(\ell-2\gamma^{-1}-1)} \|x - x^*\|^2 + 2\kappa^{-1}d + d/m \right\} \right]^{1/2}.
\end{aligned}$$

## 2 Discussion on Theorem 5

Note that

$$u_n^{(2)}(\gamma) \leq \sum_{i=1}^n \{A_0 \gamma_i^2 + A_1 \gamma_i^3\} \prod_{k=i+1}^n (1 - \kappa\gamma_k/2), \quad (\text{S6})$$

where  $\kappa$  is given by (3), and

$$A_0 = 2L^2 \kappa^{-1} d, \quad (\text{S7})$$

$$A_1 = 2L^2 \kappa^{-1} d(m+L)^{-1} + dL^4 (\kappa^{-1} + (m+L)^{-1})(m^{-1} + 6^{-1}(m+L)^{-1}). \quad (\text{S8})$$

If  $(\gamma_k)_{k \geq 1}$  is a constant step size,  $\gamma_k = \gamma$  for all  $k \geq 1$ , then a straightforward consequence of Theorem 5 and (S6) is the following result, which gives the minimal number of iterations  $n_{\epsilon}$  and a step-size  $\gamma_{\epsilon}$  to get  $W_2(\delta_{x^*} Q_{\gamma}^n, \pi)$  smaller than  $\epsilon > 0$ .

**Corollary 1** (of Theorem 5). *Assume H1 and H2. Let  $x^*$  be the unique minimizer of  $U$ . Let  $x \in \mathbb{R}^d$  and  $\epsilon > 0$ . Set for all  $k \in \mathbb{N}$ ,  $\gamma_k = \gamma$  with*

$$\begin{aligned}
\gamma &= \frac{-A_0 + (A_0^2 + \epsilon^2 \kappa A_1)^{1/2}}{2A_1} \wedge (m+L)^{-1}, \\
n &= \lceil \log^{-1}(1 - \kappa\gamma/2) \{ -\log(\epsilon^2/2) + \log(2d/m) \} \rceil.
\end{aligned} \quad (\text{S9})$$

Then  $W_2(\delta_{x^*} R_{\gamma}^n, \pi) \leq \epsilon$ .

Note that if  $\gamma$  is given by (S9), and is different from  $1/(m+L)$ , then  $\gamma \leq \epsilon(4A_1\kappa^{-1})^{-1/2}$  and  $2\kappa^{-1}(A_0\gamma + A_1\gamma^2) = \epsilon^2/2$ . Therefore,

$$\gamma \geq (\epsilon^2 \kappa / 4) \left\{ A_0 + \epsilon(A_1/(4\kappa))^{1/2} \right\}^{-1}.$$

It is shown in [1, Corollary 1] that under **H2**, for constant step size for any  $\epsilon > 0$ , we can choose  $\gamma$  and  $n \geq 1$  such that if for all  $k \geq 1$ ,  $\gamma_k = \gamma$ , then  $\|\nu^\star Q_n^\gamma - \pi\|_{\text{TV}} \leq \epsilon$  where  $\nu^\star$  is the Gaussian measure on  $\mathbb{R}^d$  with mean  $x^\star$  and covariance matrix  $L^{-1} \mathbf{I}_d$ . We stress that the results in [1, corollary 1] hold only for a particular choice of the initial distribution  $\nu^\star$ , (which might seem a rather artificial assumption) whereas Theorem 5 holds for any initial distribution in  $\mathcal{P}_2(\mathbb{R}^d)$ .

We compare the optimal value of  $\gamma$  and  $n$  obtained from Corollary 1 with those given in [1, Corollary 1]. This comparison is summarized in Table 1 and Table 2; for simplicity, we provide only the dependencies of the optimal stepsize  $\gamma$  and minimal number of simulations  $n$  as a function of the dimension  $d$ , the precision  $\epsilon$  and the constants  $m, L$ . It can be seen that the dependency on the dimension is significantly better than those in [1, Corollary 1].

Parameter	$d$	$\epsilon$	$L$	$m$
Corollary 1	$d^{-1}$	$\epsilon^2$	$L^{-2}$	$m^2$
[1, Corollary 1]	$d^{-2}$	$\epsilon^2$	$L^{-2}$	$m$

Table 1: Dependencies of  $\gamma$

Parameter	$d$	$\epsilon$	$L$	$m$
Corollary 1	$d \log(d)$	$\epsilon^{-2}  \log(\epsilon) $	$L^2$	$ \log(m)  m^{-3}$
[1, Corollary 1]	$d^3$	$\epsilon^{-2}  \log(\epsilon) $	$L^3$	$ \log(m)  m^{-2}$

Table 2: Dependencies of  $n$

## 2.1 Explicit bounds for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0, 1]$

We give here a bound on the sequences  $(u_n^{(1)}(\gamma))_{n \geq 0}$  and  $(u_n^{(2)}(\gamma))_{n \geq 0}$  for  $(\gamma_k)_{k \geq 1}$  defined by  $\gamma_1 < 1/(m + L)$  and  $\gamma_k = \gamma_1 k^{-\alpha}$  for  $\alpha \in (0, 1]$ . Also for that purpose we introduce for  $t \in \mathbb{R}_+$ ,

$$\psi_\beta(t) = \begin{cases} (t^\beta - 1)/\beta & \text{for } \beta \neq 0 \\ \log(t) & \text{for } \beta = 0. \end{cases} \quad (\text{S10})$$

We easily get for  $a \geq 0$  that for all  $n, p \geq 1$ ,  $n \leq p$

$$\psi_{1-a}(p+1) - \psi_{1-a}(n) \leq \sum_{k=n}^p k^{-a} \leq \psi_{1-a}(p) - \psi_{1-a}(n) + 1, \quad (\text{S11})$$

and for  $a \in \mathbb{R}$

$$\sum_{k=n}^p k^{-a} \leq \psi_{1-a}(p+1) - \psi_{1-a}(n) + 1. \quad (\text{S12})$$

1. For  $\alpha = 1$ , using that for all  $t \in \mathbb{R}$ ,  $(1+t) \leq e^t$  and by (S11) and (S12), we have

$$u_n^{(1)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2}, \quad u_n^{(2)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^1 A_j(\psi_{\kappa\gamma_1/2-1-j}(n+1)+1).$$

2. For  $\alpha \in (0, 1)$ , by (S11) and Lemma 21 applied with  $\ell = \lceil n/2 \rceil$ , where  $\lceil \cdot \rceil$  is the ceiling function, we have

$$\begin{aligned} u_n^{(1)}(\gamma) &\leq \exp(-\kappa\gamma_1\psi_{1-\alpha}(n+1)/2) \\ u_n^{(2)}(\gamma) &\leq \sum_{j=0}^1 A_j \left[ 2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} + \gamma_1^{j+2} \{ \psi_{1-\alpha(j+2)}(\lceil n/2 \rceil) + 1 \} \right. \\ &\quad \left. \times \exp(-(\kappa\gamma_1/2) \{ \psi_{1-\alpha}(n+1) - \psi_{1-\alpha}(\lceil n/2 \rceil) \}) \right]. \quad (\text{S13}) \end{aligned}$$

## 2.2 Optimal strategy with a fixed number of iterations

**Corollary 2.** Let  $n \in \mathbb{N}^*$  be a fixed number of iteration. Assume **H1**, **H2**, and  $(\gamma_k)_{k \geq 1}$  is a constant sequence,  $\gamma_k = \gamma$  for all  $k \geq 1$ . Set

$$\begin{aligned} \gamma^+ &= 2(\kappa n)^{-1} \left[ \log(\kappa n/2) + \log(2(\|x - x^*\|^2 + d/m)) - \log(2\kappa^{-1}A_0) \right] \\ \gamma_- &= 2(\kappa n)^{-1} \left[ \log(\kappa n/2) + \log(2(\|x - x^*\|^2 + d/m)) \right. \\ &\quad \left. - \log \{ 2\kappa^{-1}(A_0 + 2A_1(m+L)^{-1}) \} \right]. \end{aligned}$$

Assume  $\gamma^+ \in (0, (m+L)^{-1})$ . Then, the optimal choice of  $\gamma$  to minimize the bound on  $W_2(\delta_x R_\gamma^n, \pi)$  given by Theorem 5 belongs to  $[\gamma_-, \gamma^+]$ . Moreover if  $\gamma = \gamma_+$ , then there exists  $C \geq 0$  independent of the dimension such that the bound on  $W_2^2(\delta_x R_\gamma^n, \pi)$  is equivalent to  $Cdn^{-1} \log(n)$  as  $n$  goes to  $+\infty$ .

Similarly, we have the following result.

**Corollary 3.** Assume **H1** and **H2**. Let  $(\gamma_k)_{k \geq 1}$  be the decreasing sequence, defined by  $\gamma_k = \gamma_\alpha k^{-\alpha}$ , with  $\alpha \in (0, 1)$ . Let  $n \geq 1$  and set

$$\gamma_\alpha = 2(1-\alpha)\kappa^{-1}(2/n)^{1-\alpha} \log(\kappa n/(2(1-\alpha))).$$

Assume  $\gamma_\alpha \in (0, (m+L)^{-1})$ . Then there exists  $C \geq 0$  independent of the dimension such that the bound on  $W_2^2(\delta_x Q_\gamma^n, \pi)$  is equivalent to  $Cdn^{-1} \log(n)$  as  $n$  goes to  $+\infty$ .

*Proof.* Follows from (S6), (S13) and the choice of  $\gamma_\alpha$ .  $\square$

## 3 Discussion on Theorem 8

Based on Theorem 8, we can follow the same discussion as for Theorem 5. Note that

$$u_n^{(3)}(\gamma) \leq \sum_{i=1}^n \{ B_0 \gamma_i^3 + B_1 \gamma_i^4 \} \prod_{k=i+1}^n (1 - \kappa\gamma_k/2), \quad (\text{S14})$$

where  $\kappa$  is given by (3), and

$$\mathbf{B}_0 = d \left[ 2L^2 + \kappa^{-1} \left\{ \tilde{L}^2/3 + 4L^4/(3m) \right\} \right] , \quad (\text{S15})$$

$$\mathbf{B}_1 = d \left[ \kappa^{-1} L^4 + L^4/(6(m+L)) + m^{-1} \right] . \quad (\text{S16})$$

The following result gives for a target precision  $\epsilon > 0$ , the minimal number of iterations  $n_\epsilon$  and a step-size  $\gamma_\epsilon$  to get  $W_2(\delta_{x^\star} Q_\gamma^n, \pi)$  smaller than  $\epsilon$ , when  $(\gamma_k)_{k \geq 1}$  is a constant step size,  $\gamma_k = \gamma_\epsilon$  for all  $k \geq 1$ .

**Corollary 4.** *Assume **H1**, **H2** and **H3**. Let  $x^\star$  be the unique minimizer of  $U$ . Let  $x \in \mathbb{R}^d$  and  $\epsilon > 0$ . Set for all  $k \in \mathbb{N}$ ,  $\gamma_k = \gamma$  with*

$$\begin{aligned} \gamma &= \left[ (\epsilon/2) \kappa^{-1} \left\{ \mathbf{B}_0 + \mathbf{B}_1(m+L)^{-1} \right\}^{-1/2} \right] \wedge (1/(m+L)) , \\ n &= \left\lceil \log^{-1}(1 - \kappa\gamma/2) \left\{ -\log(\epsilon^2/2) + \log(2d/m) \right\} \right\rceil . \end{aligned}$$

Then  $W_2(\delta_{x^\star} R_\gamma^n, \pi) \leq \epsilon$ .

We provide the dependencies of the optimal stepsize  $\gamma_\epsilon$  and minimal number of simulations  $n_\epsilon$  as a function of the dimension  $d$ , the precision  $\epsilon$  and the constants  $m, L, \tilde{L}$  in Table 3 and Table 4.

Parameter	$d$	$\epsilon$	$L$	$m$
Corollary 4	$d^{-1/2}$	$\epsilon$	$L^{-2}$	$m$

Table 3: Dependencies of  $\gamma$

Parameter	$d$	$\epsilon$	$L$	$m$
Corollary 4	$d^{1/2} \log(d)$	$\epsilon^{-1}  \log(\epsilon) $	$L^2$	$ \log(m)  m^{-2}$

Table 4: Dependencies of  $n$

### 3.1 Explicit bounds for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0, 1]$

We give here a bound on the sequence  $(u_n^{(3)}(\gamma))_{n \geq 0}$  for  $(\gamma_k)_{k \geq 1}$  defined by  $\gamma_1 < 1/(m+L)$  and  $\gamma_k = \gamma_1 k^{-\alpha}$  for  $\alpha \in (0, 1]$ . Bounds for  $(u_n^{(1)}(\gamma))_{n \geq 0}$  have already been given in Section 2.1. Recall that the function  $\psi$  is defined by (S10). For  $\alpha \in (0, 1]$ , by (S11) and Lemma 21 applied with  $\ell = \lceil n/2 \rceil$ , where  $\lceil \cdot \rceil$  is the ceiling function, we have

$$\begin{aligned} u_n^{(3)}(\gamma) &\leq \sum_{j=1}^2 \mathbf{B}_{j-1} \left[ 2\kappa^{-1} \gamma_1^{j+1} (n/2)^{-\alpha(j+1)} + \gamma_1^{j+2} (\psi_{1-\alpha(j+2)}(\lceil n/2 \rceil) + 1) \right. \\ &\quad \times \exp(-(\kappa\gamma_1/2) \{ \psi_{1-\alpha}(n+1) - \psi_{1-\alpha}(\lceil n/2 \rceil) \}) \left. \right] . \end{aligned} \quad (\text{S17})$$

### 3.2 Optimal strategy with a fixed number of iterations

**Corollary 5.** *Let  $n \in \mathbb{N}^*$  be a fixed number of iteration. Assume **H1**, **H2**, **H3** and  $(\gamma_k)_{k \geq 1}$  is a constant sequence,  $\gamma_k = \gamma^*$  for all  $k \geq 1$ , with*

$$\gamma^* = 4(\kappa n)^{-1} \left\{ \log(\kappa n/2) + \log(2(\|x - x^*\|^2 + d/m)) \right\}.$$

*Assume  $\gamma^* \in (0, (m+L)^{-1})$ . Then there exists  $C \geq 0$  independent of the dimension such that the bound on  $W_2^2(\delta_x R_\gamma^n, \pi)$  is equivalent to  $Cdn^{-2} \log^2(n)$  as  $n$  goes to  $+\infty$ .*

Similarly, we have the following result.

**Corollary 6.** *Assume **H1**, **H2** and **H3**. Let  $(\gamma_k)_{k \geq 1}$  be the decreasing sequence, defined by  $\gamma_k = \gamma_\alpha k^{-\alpha}$ , with  $\alpha \in (0, 1)$ . Let  $n \geq 1$  and set*

$$\gamma_\alpha = 2(1 - \alpha)\kappa^{-1}(2/n)^{1-\alpha} \log(\kappa n/(2(1 - \alpha))) .$$

*Assume  $\gamma_\alpha \in (0, (m+L)^{-1})$ . Then there exists  $C \geq 0$  independent of the dimension such that the bound on  $W_2^2(\delta_x R_\gamma^n, \pi)$  is equivalent to  $Cdn^{-2} \log^2(n)$  as  $n$  goes to  $+\infty$ .*

*Proof.* Follows from **(S14)**, **(S17)** and the choice of  $\gamma_\alpha$ .  $\square$

## 4 Generalization of Theorem 5

In this section, we weaken the assumption  $\gamma_1 \leq 1/(m+L)$  of Theorem 5. We assume now:

**G1.** *The sequence  $(\gamma_k)_{k \geq 1}$  is non-increasing, and there exists  $\rho > 0$  and  $n_1$  such that  $(1 + \rho)\gamma_{n_1} \leq 2/(m+L)$ .*

Under **G1**, we denote by

$$n_0 = \min \{k \in \mathbb{N} \mid \gamma_k \leq 2/(m+L)\} \quad (\text{S18})$$

We first give an extension of Theorem 3. Denote in the sequel  $(\cdot)_+ = \max(\cdot, 0)$ . Recall that under **H2**,  $x^*$  is the unique minimizer of  $U$ , and  $\kappa$  is defined in **(4)**

**Theorem S7.** *Assume **H1**, **H2** and **G1**. Then for all  $n, p \in \mathbb{N}^*$ ,  $n \leq p$*

$$\int_{\mathbb{R}^d} \|x - x^*\|^2 \mu_0 Q_n^p(dx) \leq G_{n,p}(\mu_0, \gamma) ,$$

where

$$\begin{aligned} G_{n,p}(\mu_0, \gamma) = & \exp \left( - \sum_{k=n}^p \gamma_k \kappa + \sum_{k=n}^{n_0-1} L^2 \gamma_k^2 \right) \int_{\mathbb{R}^d} \|x - x^*\|^2 \mu_0(dx) \\ & + 2d\kappa^{-1} + 2d \left\{ \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} (1 + L^2 \gamma_k^2) \right\} \exp \left( - \sum_{k=n}^p \kappa \gamma_k + \sum_{k=n}^{n_0-1} \gamma_k^2 m L \right) . \end{aligned} \quad (\text{S19})$$



*Proof.* For any  $\gamma > 0$ , we have for all  $x \in \mathbb{R}^d$ :

$$\int_{\mathbb{R}^d} \|y - x^\star\|^2 R_\gamma(x, dy) = \|x - \gamma \nabla U(x) - x^\star\|^2 + 2\gamma d.$$

Using that  $\nabla U(x^\star) = 0$ , (3) and **H1**, we get from the previous inequality:

$$\begin{aligned} & \int_{\mathbb{R}^d} \|y - x^\star\|^2 R_\gamma(x, dy) \\ & \leq (1 - \kappa\gamma) \|x - x^\star\|^2 + \gamma \left( \gamma - \frac{2}{m+L} \right) \|\nabla U(x) - \nabla U(x^\star)\|^2 + 2\gamma d \\ & \leq \eta(\gamma) \|x - x^\star\|^2 + 2\gamma d, \end{aligned}$$

where  $\eta(\gamma) = (1 - \kappa\gamma + \gamma L(\gamma - 2/(m+L))_+)$ . Denote for all  $k \geq 1$ ,  $\eta_k = \eta(\gamma_k)$ . By a straightforward induction, we have by definition of  $Q_n^p$  for  $p, n \in \mathbb{N}$ ,  $p \leq n$ ,

$$\int_{\mathbb{R}^d} \|x - x^\star\|^2 \mu_0 Q_n^p(dx) \leq \prod_{k=n}^p \eta_k \int_{\mathbb{R}^d} \|x - x^\star\| \mu_0(dx) + (2d) \sum_{i=n}^p \prod_{k=i+1}^p \eta_k \gamma_i, \quad (\text{S20})$$

with the convention that for  $n, p \in \mathbb{N}$ ,  $n < p$ ,  $\prod_p^n = 1$ . For the first term of the right hand side, we simply use the bound, for all  $x \in \mathbb{R}$ ,  $(1+x) \leq e^x$ , and we get by **G1**

$$\prod_{k=n}^p \eta_k \leq \exp \left( - \sum_{k=n}^p \kappa \gamma_k + \sum_{k=n}^{n_0-1} L^2 \gamma_k^2 \right), \quad (\text{S21})$$

where  $n_0$  is defined in (S18). Consider now the second term in the right hand side of (S20).

$$\begin{aligned} \sum_{i=n}^p \prod_{k=i+1}^p \eta_k \gamma_i & \leq \sum_{i=n_0}^p \prod_{k=i+1}^p (1 - \kappa \gamma_k) \gamma_i + \sum_{i=n}^{n_0-1} \prod_{k=i+1}^p \eta_k \gamma_i \\ & \leq \kappa^{-1} \sum_{i=n_0}^p \left\{ \prod_{k=i+1}^p (1 - \kappa \gamma_k) - \prod_{k=i}^p (1 - \kappa \gamma_k) \right\} \\ & \quad + \left\{ \sum_{i=n}^{n_0-1} \prod_{k=i+1}^{n_0-1} (1 + L^2 \gamma_k^2) \gamma_i \right\} \prod_{k=n_0}^p (1 - \kappa \gamma_k) \end{aligned} \quad (\text{S22})$$

Since  $(\gamma_k)_{k \geq 1}$  is nonincreasing, we have

$$\begin{aligned} \sum_{i=n}^{n_0-1} \prod_{k=i+1}^{n_0-1} (1 + L^2 \gamma_k^2) \gamma_i & = \sum_{i=n}^{n_0-1} (\gamma_i L^2)^{-1} \left\{ \prod_{k=i}^{n_0-1} (1 + L^2 \gamma_k^2) - \prod_{k=i+1}^{n_0-1} (1 + L^2 \gamma_k^2) \right\} \\ & \leq \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} (1 + L^2 \gamma_k^2). \end{aligned}$$

Furthermore for  $k < n_0$   $\gamma_k > 2/(m+L)$ . This implies with the bound  $(1+x) \leq e^x$  on  $\mathbb{R}$ :

$$\begin{aligned} \prod_{k=n_0}^p (1 - \kappa\gamma_k) &\leq \exp\left(-\sum_{k=n}^p \kappa\gamma_k\right) \exp\left(\sum_{k=n}^{n_0-1} \kappa\gamma_k\right) \\ &\leq \exp\left(-\sum_{k=n}^p \kappa\gamma_k\right) \exp\left(\sum_{k=n}^{n_0-1} \gamma_k^2 mL\right). \end{aligned}$$

Using the two previous inequalities in (S22), we get

$$\begin{aligned} \sum_{i=n}^p \prod_{k=i+1}^p \eta_k \gamma_i \\ \leq \kappa^{-1} + \left\{ \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} (1 + L^2 \gamma_k^2) \right\} \exp\left(-\sum_{k=n}^p \kappa\gamma_k + \sum_{k=n}^{n_0-1} \gamma_k^2 mL\right). \end{aligned} \quad (\text{S23})$$

Combining (S21) and (S23) in (S20) concluded the proof.  $\square$

We now deal with bounds on  $W_2(\mu_0 Q_\gamma^n, \pi)$  under **G 1**. But before we preface our result by some technical lemmas.

**Lemma S8.** *Assume **H1** and **H2**. Let  $\zeta_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ ,  $(Y_t, \bar{Y}_t)_{t \geq 0}$  such that  $(Y_0, \bar{Y}_0)$  is distributed according to  $\zeta_0$  and given by (9). Let  $(\mathcal{F}_t)_{t \geq 0}$  be the filtration associated with  $(B_t)_{t \geq 0}$  with  $\mathcal{F}_0$ , the  $\sigma$ -field generated by  $(Y_0, \bar{Y}_0)$ . Then for all  $n \geq 0$ ,  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$ ,*

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \|Y_{\Gamma_{n+1}} - \bar{Y}_{\Gamma_{n+1}}\|^2 \right] \\ \leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1}L((1 + \epsilon_2)\gamma_{n+1} - 2/(m+L))_+\} \|Y_{\Gamma_n} - \bar{Y}_{\Gamma_n}\|^2 \\ + \gamma_{n+1}^2 (1/(2\epsilon_1) + (1 + \epsilon_2^{-1})\gamma_{n+1}) \left( dL^2 + (L^4 \gamma_{n+1}/2) \|Y_{\Gamma_n} - x^*\|^2 + dL^4 \gamma_{n+1}^2 / 12 \right). \end{aligned}$$

*Proof.* Let  $n \geq 0$  and  $\epsilon_1 > 0$ , and set  $\Delta_n = Y_{\Gamma_n} - \bar{Y}_{\Gamma_n}$  by definition we have:

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \|\Delta_{n+1}\|^2 \right] &= \|\Delta_n\|^2 + \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \left\| \int_{\Gamma_n}^{\Gamma_{n+1}} \{\nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n})\} ds \right\|^2 \right] \\ &\quad - 2\gamma_{n+1} \langle \Delta_n, \nabla U(Y_{\Gamma_n}) - \nabla U(\bar{Y}_{\Gamma_n}) \rangle - 2 \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} [\langle \Delta_n, \{\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\} \rangle ds]. \end{aligned}$$

Using the two inequalities  $|\langle a, b \rangle| \leq \epsilon_1 \|a\|^2 + (4\epsilon_1)^{-1} \|b\|^2$  and (3), we get

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \|\Delta_{n+1}\|^2 \right] &\leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1)\} \|\Delta_n\|^2 - 2\gamma_{n+1}/(m+L) \|\nabla U(Y_{\Gamma_n}) - \nabla U(\bar{Y}_{\Gamma_n})\|^2 \\ &\quad + \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \left\| \int_{\Gamma_n}^{\Gamma_{n+1}} \{\nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n})\} ds \right\|^2 \right] \\ &\quad + (2\epsilon_1)^{-1} \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2 \right] ds. \end{aligned} \quad (\text{S24})$$

Using  $\|a + b\|^2 \leq (1 + \epsilon_2) \|a\|^2 + (1 + \epsilon_2^{-1}) \|b\|^2$  and the Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \left\| \int_{\Gamma_n}^{\Gamma_{n+1}} \{ \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}) \} ds \right\|^2 \right] &\leq (1 + \epsilon_2) \gamma_{n+1}^2 \left\| \nabla U(Y_{\Gamma_n}) - \nabla U(\bar{Y}_{\Gamma_n}) \right\|^2 \\ &\quad + \gamma_{n+1} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \int_{\Gamma_n}^{\Gamma_{n+1}} \left\| \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) \right\|^2 ds \right]. \end{aligned}$$

This result and **H1** imply,

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \left\| \Delta_{n+1} \right\|^2 \right] &\leq \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1}L((1 + \epsilon_2)\gamma_{n+1} - 2/(m + L))_+\} \left\| \Delta_n \right\|^2 \\ &\quad + ((1 + \epsilon_2^{-1})\gamma_{n+1} + (2\epsilon_1)^{-1}) \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \left\| \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) \right\|^2 \right] ds. \quad (\text{S25}) \end{aligned}$$

By **H1**, the Markov property of  $(Y_t)_{t \geq 0}$  and Lemma 19, we have

$$\begin{aligned} \int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}} \left[ \left\| \nabla U(Y_s) - \nabla U(Y_{\Gamma_n}) \right\|^2 \right] ds \\ \leq L^2 \left( d\gamma_{n+1}^2 + dL^2\gamma_{n+1}^4/12 + (L^2\gamma_{n+1}^3/2) \|Y_{\Gamma_n} - x^*\|^2 \right). \end{aligned}$$

Plugging this bound in (S25) concludes the proof.  $\square$

**Lemma S9.** *Let  $(\gamma_k)_{k \geq 1}$  be a nonincreasing sequence of positive numbers. Let  $\varpi, \beta > 0$  be positive constants satisfying  $\varpi^2 \leq 4\beta$  and  $\tau > 0$ . Assume there exists  $N \geq 1$ ,  $\gamma_N \leq \tau$  and  $\gamma_N \varpi \leq 1$ . Then for all  $n \geq 0$ ,  $j \geq 2$*

(i)

$$\begin{aligned} \sum_{i=1}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi + \gamma_k \beta (\gamma_k - \tau)_+) \gamma_i^j &\leq \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j \\ &\quad + \left\{ \beta^{-1} \gamma_1^{j-2} \prod_{k=1}^{N-1} (1 + \gamma_k^2 \beta) \right\} \prod_{k=N}^{n+1} (1 - \varpi \gamma_k). \end{aligned}$$

(ii) For all  $\ell \in \{N, \dots, n\}$ ,

$$\sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j \leq \exp \left( - \sum_{k=\ell}^{n+1} \varpi \gamma_k \right) \sum_{i=N}^{\ell-1} \gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi}.$$

*Proof.* By definition of  $N$  we have

$$\begin{aligned} \sum_{i=1}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi + \gamma_k \beta (\gamma_k - \tau)_+) \gamma_i^j \\ \leq \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j + \left\{ \sum_{i=1}^{N-1} \prod_{k=i+1}^{N-1} (1 + \gamma_k^2 \beta) \gamma_i^j \right\} \prod_{k=N}^{n+1} (1 - \gamma_k \varpi). \quad (\text{S26}) \end{aligned}$$

Using that  $(\gamma_k)_{k \geq 1}$  is nonincreasing, we have

$$\begin{aligned} \sum_{i=1}^{N-1} \prod_{k=i+1}^{N-1} (1 + \gamma_k^2 \beta) \gamma_i^j &\leq \sum_{i=1}^{N-1} \frac{\gamma_i^{j-2}}{\beta} \left\{ \prod_{k=i}^{N-1} (1 + \gamma_k^2 \beta) - \prod_{k=i+1}^{N-1} (1 + \gamma_k^2 \beta) \right\} \\ &\leq \beta^{-1} \gamma_1^{j-2} \prod_{k=1}^{N-1} (1 + \gamma_k^2 \beta) . \end{aligned}$$

Plugging this inequality in (S26) concludes the proof of (i). Let  $\ell \in \{N, \dots, n+1\}$ . Since  $(\gamma_k)_{k \geq 1}$  is nonincreasing and for every  $x \in \mathbb{R}$ ,  $(1+x) \leq e^x$ , we get

$$\begin{aligned} \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j &= \sum_{i=N}^{\ell-1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j + \sum_{i=\ell}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i^j \\ &\leq \sum_{i=N}^{\ell-1} \exp \left( - \sum_{k=i+1}^{n+1} \varpi \gamma_k \right) \gamma_i^j + \gamma_\ell^{j-1} \sum_{i=\ell}^{n+1} \prod_{k=i+1}^{n+1} (1 - \gamma_k \varpi) \gamma_i \\ &\leq \exp \left( - \sum_{k=\ell}^{n+1} \varpi \gamma_k \right) \sum_{i=N}^{\ell-1} \gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi} . \end{aligned}$$

□

**Lemma S10.** Let  $(\gamma_k)_{k \geq 1}$  be a nonincreasing sequence of positive numbers,  $\varpi, \beta, \tau > 0$  be positive real numbers, and  $N \geq 1$  satisfying the assumptions of Lemma S9. Let  $P \in \mathbb{N}^*$ ,  $C_i \geq 0$ ,  $i = 0, \dots, P$  be positive constants and  $(u_n)_{n \geq 0}$  be a sequence of real numbers with  $u_0 \geq 0$  satisfying for all  $n \geq 0$

$$u_{n+1} \leq (1 - \gamma_{n+1} \varpi + \beta \gamma_{n+1} (\gamma_{n+1} - \tau)_+) u_n + \sum_{i=0}^P C_j \gamma_{n+1}^{j+2} .$$

Then for all  $n \geq 1$ ,

$$\begin{aligned} u_n &\leq \left\{ \prod_{k=1}^{N-1} (1 + \beta \gamma_k^2) \right\} \prod_{k=N}^n (1 - \gamma_k \varpi) u_0 + \sum_{j=0}^P C_j \sum_{i=N}^n \prod_{k=i+1}^n (1 - \gamma_k \varpi) \gamma_i^{j+2} \\ &\quad + \left\{ \sum_{j=0}^P C_j \beta^{-1} \gamma_1^j \prod_{k=1}^{N-1} (1 + \gamma_k^2 \beta) \right\} \prod_{k=N}^n (1 - \varpi \gamma_k) . \end{aligned}$$

*Proof.* This is a consequence of a straightforward induction and Lemma S9-(i). □

**Proposition S11.** Assume **H1**, **H2** and **G1**. Let  $x^*$  be the unique minimizer of  $U$ . Let  $\zeta_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ ,  $(Y_t, \bar{Y}_t)_{t \geq 0}$  such that  $(Y_0, \bar{Y}_0)$  is distributed according to  $\zeta_0$  and given by (9). Then for all  $n \geq 0$  and  $t \in [\Gamma_n, \Gamma_{n+1}]$ :

$$\mathbb{E} \left[ \|Y_t - \bar{Y}_t\|^2 \right] \leq \tilde{u}_n^{(1)} \mathbb{E} \left[ \|Y_0 - \bar{Y}_0\|^2 \right] + \tilde{u}_n^{(4)} + \tilde{u}_{t,n}^{(5)} ,$$

where

$$\tilde{u}_n^{(1)}(\gamma) = \left\{ \prod_{k=1}^{n_1-1} (1 + L^2(1 + \rho)\gamma_k^2) \right\} \prod_{k=n_1}^n (1 - \kappa\gamma_k/2) , \quad (\text{S27})$$

$$\begin{aligned} \tilde{u}_n^{(4)}(\gamma) &= \sum_{i=n_1}^n \gamma_i^2 \mathbf{a}(\gamma_i) \prod_{k=i+1}^n (1 - \kappa\gamma_k/2) \\ &\quad + \mathbf{a}(\gamma_1)(L^2(1 + \rho))^{-1} \left\{ \prod_{k=1}^{n_1-1} (1 + \gamma_k^2(1 + \rho)L^2) \right\} \prod_{k=n_1}^n (1 - \kappa\gamma_k/2) , \end{aligned}$$

where for  $\gamma > 0$ ,

$$\begin{aligned} \mathbf{a}(\gamma) &= \{2\kappa^{-1} + (1 + \rho^{-1})\gamma\} (dL^2 + \delta L^4\gamma/2 + dL^4\gamma^2/12) , \\ \delta &= \max_{i \geq 1} \left\{ e^{-2m\Gamma_{i-1}} \mathbb{E} \left[ \|Y_0 - x^*\|^2 \right] + (1 - e^{-2m\Gamma_{i-1}})(d/m) \right\} , \end{aligned}$$

and

$$\tilde{u}_{t,n}^{(5)}(\gamma) = \frac{m + L}{2} \left( \frac{(t - \Gamma_n)^3 L^2}{3} \mathbf{G}_{1,n}(\mu_0, \gamma) + (t - \Gamma_n)^2 d \right) ,$$

where  $\mathbf{G}_{1,n}(\mu_0, \gamma)$  is given by (S19) and  $\mu_0$  is the initial distribution of  $\bar{Y}_0$ .

*Proof.* Lemma S8 with  $\epsilon_1 = \kappa/4$  and  $\epsilon_2 = \rho$ , G1, Lemma S9, (i) imply for all  $n \geq 0$

$$\mathbb{E} \left[ \|Y_{\Gamma_n} - \bar{Y}_{\Gamma_n}\|^2 \right] \leq \tilde{u}_n^{(1)}(\gamma) \mathbb{E} \left[ \|Y_0 - \bar{Y}_0\|^2 \right] + \tilde{u}_n^{(4)}(\gamma) . \quad (\text{S28})$$

Now let  $n \geq 0$  and  $t \in [\Gamma_n, \Gamma_{n+1}]$ . By (9),

$$\|Y_t - \bar{Y}_t\|^2 = \|Y_{\Gamma_n} - \bar{Y}_{\Gamma_n}\|^2 - 2 \int_{\Gamma_n}^t \langle \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}), Y_s - \bar{Y}_s \rangle ds . \quad (\text{S29})$$

Moreover for all  $s \in [\Gamma_n, \Gamma_{n+1}]$ , by (3) we get

$$\begin{aligned} \langle \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}), Y_s - \bar{Y}_s \rangle &= \langle \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}), Y_s - \bar{Y}_{\Gamma_n} + \bar{Y}_{\Gamma_n} - \bar{Y}_s \rangle \\ &\geq (m + L)^{-1} \|\nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n})\|^2 + \langle \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}), \bar{Y}_{\Gamma_n} - \bar{Y}_s \rangle . \end{aligned} \quad (\text{S30})$$

Since  $|\langle a, b \rangle| \leq (m + L)^{-1} \|a\|^2 + (m + L) \|b\|^2 / 4$ , we have

$$\begin{aligned} \langle \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}), \bar{Y}_{\Gamma_n} - \bar{Y}_s \rangle \\ \geq -(m + L)^{-1} \|\nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n})\|^2 - (m + L) \|\bar{Y}_s - \bar{Y}_{\Gamma_n}\|^2 / 4 . \end{aligned}$$

Using this inequality in (S30), we get

$$\langle \nabla U(Y_s) - \nabla U(\bar{Y}_{\Gamma_n}), Y_s - \bar{Y}_s \rangle \geq -(m + L) \|\bar{Y}_s - \bar{Y}_{\Gamma_n}\|^2 / 4 ,$$

and (S29) becomes

$$\|Y_t - \bar{Y}_t\|^2 \leq \|Y_{\Gamma_n} - \bar{Y}_{\Gamma_n}\|^2 + ((m+L)/2) \int_{\Gamma_n}^t \|\bar{Y}_s - \bar{Y}_{\Gamma_n}\|^2 ds \quad (\text{S31})$$

Therefore, by the previous inequality, it remains to bound the expectation of  $\|\bar{Y}_s - \bar{Y}_{\Gamma_n}\|^2$ . By (9) and using  $\nabla U(x^*) = 0$ ,

$$\|\bar{Y}_s - \bar{Y}_{\Gamma_n}\|^2 = \left\| -(s - \Gamma_n)(\nabla U(\bar{Y}_{\Gamma_n}) - \nabla U(x^*)) + \sqrt{2}(B_s - B_{\Gamma_n}) \right\|^2.$$

Then taking the expectation, using the Markov property of  $(B_t)_{t \geq 0}$  and **H1**, we have

$$\mathbb{E} \left[ \|\bar{Y}_s - \bar{Y}_{\Gamma_n}\|^2 \right] \leq (s - \Gamma_n)^2 L^2 \mathbb{E} \left[ \|\bar{Y}_{\Gamma_n} - x^*\|^2 \right] + 2(s - \Gamma_n)d. \quad (\text{S32})$$

The proof follows from taking the expectation in (S31), combining (S28)-(S32), and using Theorem S7.  $\square$

**Theorem S12.** Assume **H1**, **H2** and **G1**. Then for all  $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$  and  $n \geq 1$ ,

$$W_2^2(\mu_0 Q_\gamma^n, \pi) \leq \tilde{u}_n^{(1)}(\gamma) W_2^2(\mu_0, \pi) + \tilde{u}_n^{(2)}(\gamma), \quad (\text{S33})$$

where  $(\tilde{u}_n^{(1)})_{n \geq 0}$  is given by (S27) and

$$\begin{aligned} \tilde{u}_n^{(2)}(\gamma) = & \sum_{i=n_1}^n \gamma_i^2 \mathbf{b}(\gamma_i) \prod_{k=i+1}^n (1 - \kappa \gamma_k / 2) \\ & + \mathbf{b}(\gamma_1) (L^2(1 + \rho))^{-1} \left\{ \prod_{k=1}^{n_1-1} (1 + \gamma_k^2(1 + \rho)L^2) \right\} \prod_{k=n_1}^n (1 - \kappa \gamma_k / 2), \end{aligned} \quad (\text{S34})$$

where

$$\mathbf{b}(\gamma) = \{2\kappa^{-1} + (1 + \rho^{-1})\gamma\} (dL^2 + dL^4\gamma/(2m) + dL^4\gamma^2/12).$$

*Proof.* Let  $\zeta_0$  be an optimal transference plan of  $\mu_0$  and  $\pi$ . Let  $(Y_t, \bar{Y}_t)_{t \geq 0}$  with  $(Y_0, \bar{Y}_0)$  distributed according to  $\zeta_0$  and defined by (9). By definition of  $W_2$  and since for all  $t \geq 0$ ,  $\pi$  is invariant for  $P_t$ ,  $W_2^2(\mu_0 Q_\gamma^n, \pi) \leq \mathbb{E}[\|Y_{\Gamma_n} - X_{\Gamma_n}\|^2]$ . Then the proof follows from Proposition S11 since  $Y_0$  is distributed according to  $\pi$  and by Theorem 1-(ii), which shows that  $\delta \leq d/m$ .  $\square$

#### 4.1 Explicit bound based on Theorem S12 for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0, 1]$

We give here a bound on the sequences  $(\tilde{u}_n^{(1)}(\gamma))_{n \geq 1}$  and  $(\tilde{u}_n^{(2)}(\gamma))_{n \geq 1}$  for  $(\gamma_k)_{k \geq 1}$  defined by  $\gamma_1 > 0$  and  $\gamma_k = \gamma_1 k^{-\alpha}$  for  $\alpha \in (0, 1]$ . Recall that  $\psi_\beta$  is given by (S10). First note,

since  $(\gamma_k)_{k \geq 1}$  is nonincreasing, for all  $n \geq 1$ , we have

$$\begin{aligned} \tilde{u}_n^{(2)}(\gamma) &\leq \sum_{j=0}^1 C_j \sum_{i=n_1}^n \gamma_i^{j+2} \prod_{k=i+1}^n (1 - \kappa \gamma_k / 2) \\ &\quad + \sum_{j=0}^1 C_j (L^2(1 + \rho))^{-1} \gamma_1^j \left\{ \prod_{k=1}^{n_1-1} (1 + \gamma_k^2(1 + \rho)L^2) \right\} \prod_{k=n_1}^n (1 - \kappa \gamma_k / 2), \quad (\text{S35}) \end{aligned}$$

where

$$C_1 = \mathbf{b} d L^2, C_2 = \mathbf{b} (d L^4 / (2m) + \gamma_1 d L^4 / 12), \mathbf{b} = 2\kappa^{-1} + (1 + \rho^{-1})\gamma_1.$$

1. For  $\alpha = 1$  and  $n_1 = 1$ , by (S11) and (S12), we have

$$\begin{aligned} \tilde{u}_n^{(1)}(\gamma) &\leq (n+1)^{-\kappa \gamma_1 / 2} \\ \tilde{u}_n^{(2)}(\gamma) &\leq (n+1)^{-\kappa \gamma_1 / 2} \sum_{j=0}^1 C_j \left\{ \gamma_1^{j+2} (\psi_{\kappa \gamma_1 / 2 - 1 - j}(n+1) + 1) + (L^2(1 + \rho))^{-1} \gamma_1^j \right\}. \end{aligned}$$

For  $n_1 > 1$ , since  $(\gamma_k)_{k \geq 0}$  is non increasing, using again (S11), (S12), and the bound for  $t \in \mathbb{R}$ ,  $(1 + t) \leq e^t$

$$\begin{aligned} \tilde{u}_n^{(1)}(\gamma) &\leq (n+1)^{-\kappa \gamma_1 / 2} \exp \left\{ \kappa \gamma_1 \psi_0(n_1) / 2 + L^2(1 + \rho) \gamma_1^2 (\psi_{-1}(n_1 - 1) + 1) \right\} \\ \tilde{u}_n^{(2)}(\gamma) &\leq (n+1)^{-\kappa \gamma_1 / 2} \sum_{j=0}^1 C_j \left( \gamma_1^{j+2} (\psi_{\kappa \gamma_1 / 2 - 1 - j}(n+1) - \psi_{\kappa \gamma_1 / 2 - 1 - j}(n_1) + 1) \right. \\ &\quad \left. + (\gamma_1^j / (L^2(1 + \rho))) \exp \left\{ \kappa \gamma_1 \psi_0(n_1) / 2 + L^2(1 + \rho) \gamma_1^2 (\psi_{-1}(n_1 - 1) + 1) \right\} \right). \end{aligned}$$

Thus, for  $\gamma_1 > \kappa/2$ , we get a bound in  $\mathcal{O}(n^{-1})$ .

2. For  $\alpha \in (0, 1)$  and  $n_1 = 1$ , by (S11) and Lemma S9-(ii) applied with  $\ell = \lceil n/2 \rceil$ , we have

$$\begin{aligned} \tilde{u}_n^{(1)}(\gamma) &\leq \exp(-(\kappa \gamma_1 / 2) \psi_{\alpha-1}(n+1)) \\ \tilde{u}_n^{(2)}(\gamma) &\leq \sum_{j=0}^1 C_j \left\{ \gamma_1^{j+2} (\psi_{1-\alpha(j+2)}(\lceil n/2 \rceil) + 1) e^{-(\kappa \gamma_1 / 2) (\psi_{1-\alpha}(n+1) - \psi_{1-\alpha}(\lceil n/2 \rceil))} \right. \\ &\quad \left. + 2\kappa^{-1} \gamma_1^{j+1} (n/2)^{-\alpha(j+1)} + \gamma_1^j e^{-(\kappa \gamma_1 / 2) \psi_{\alpha-1}(n+1)} \right\}. \end{aligned}$$

For  $n_1 > 1$  and  $\lceil n/2 \rceil \geq n_1$ , since  $(\gamma_k)_{k \geq 0}$  is non increasing, using again (S11), and Lemma S9-(ii) applied with  $\ell = \lceil n/2 \rceil$ , and the bound for  $t \in \mathbb{R}$ ,  $(1 + t) \leq e^t$ , we

get

$$\begin{aligned}
\tilde{u}_n^{(1)}(\gamma) &\leq e^{-\kappa\gamma_1(\psi_{\alpha-1}(n+1)-\psi_{1-\alpha}(n_1))/2+L^2(1+\rho)\gamma_1^2(\psi_{1-2\alpha}(n_1-1)+1)} \\
\tilde{u}_n^{(2)}(\gamma) &\leq \sum_{j=0}^1 \mathsf{C}_j \left\{ 2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} \right. \\
&\quad + \gamma_1^{j+2} (\psi_{1-\alpha(j+2)}(\lceil n/2 \rceil) - \psi_{1-\alpha(j+2)}(n_1) + 1) e^{-(\kappa\gamma_1/2)(\psi_{1-\alpha}(n+1)-\psi_{1-\alpha}(\lceil n/2 \rceil))} \\
&\quad \left. + (\gamma_1^j/(L^2(1+\rho)))e^{-\kappa\gamma_1(\psi_{\alpha-1}(n+1)-\psi_{1-\alpha}(n_1))/2+L^2(1+\rho)\gamma_1^2(\psi_{1-2\alpha}(n_1-1)+1)} \right\}.
\end{aligned}$$

## 5 Explicit bounds on the MSE

Without loss of generality, assume that  $\|f\|_{\text{Lip}} = 1$ . In the following, denote by  $\Omega(x) = \|x - x^*\|^2 + d/m$  and  $C$  a constant (which may take different values upon each appearance), which does not depend on  $m, L, \gamma_1, \alpha$  and  $\|x - x^*\|$ .

### 5.1 Explicit bounds based on Theorem 5

1. First for  $\alpha = 0$ , recall by Theorem 5 and (S6) we have for all  $p \geq 1$ ,

$$W_2^2(\delta_x R_\gamma^p, \pi) \leq 2\Omega(x)(1 - \kappa\gamma_1/2)^p + 2\kappa^{-1}(\mathsf{A}_0\gamma_1 + \mathsf{A}_1\gamma_1^2),$$

where  $\mathsf{A}_0$  and  $\mathsf{A}_1$  are given by (S7) and (S8) respectively. So by (15) and Lemma 21, we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{\gamma_1 n} + \kappa^{-1} \mathsf{A}_0 \gamma_1 \right).$$

Therefore plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \kappa^{-1} \mathsf{A}_0 \gamma_1 + \frac{\kappa^{-2} + \kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{n \gamma_1} \right). \quad (\text{S36})$$

So with fixed  $\gamma_1$  this bound is of order  $\gamma_1$ . If we fix the number of iterations  $n$ , we can optimize the choice of  $\gamma_1$ . Set

$$\gamma_{\star,0}(n) = (\kappa^{-1} \mathsf{A}_0)^{-1} (C_{\text{MSE},0}/n)^{1/2}, \text{ where } C_{\text{MSE},0} = \kappa^{-3} \mathsf{A}_0,$$

and (S36) becomes if  $\gamma_1 \leftarrow \gamma_{\star,0}(n)$ ,

$$\text{MSE}_f^{N,n} \leq C (C_{\text{MSE},0} n)^{-1/2} (\kappa^{-1} \exp(-\kappa N \gamma_{\star,0}(n)/2) \Omega(x) + C_{\text{MSE},0}).$$

Setting  $N_0(n) = 2(\kappa \gamma_{\star,0}(n))^{-1} \log(\Omega(x))$ , we end up with

$$\text{MSE}_f^{N_0(n),n} \leq C (C_{\text{MSE},0}/n)^{1/2}.$$

Note that  $N_0(n)$  is of order  $n^{1/2}$ .



2. For  $\alpha \in (0, 1/2)$  by Theorem 5, Lemma 21, (S11) and (S13), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1}{(1-2\alpha)n^\alpha} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1}{(1-2\alpha)n^\alpha} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right). \quad (\text{S37})$$

At fixed  $\gamma_1$ , this bound is of order  $n^{-\alpha}$ , and is better than (S36) for  $(\gamma_k)_{k \geq 1}$  constant. If we fix the number of iterations  $n$ , we can optimize the choice of  $\gamma_1$  again. Set

$$\gamma_{*,\alpha}(n) = (\kappa^{-1} \mathbf{A}_0 / (1-2\alpha))^{-1} (C_{\text{MSE},\alpha} / n^{1-2\alpha})^{1/2}, \text{ where } C_{\text{MSE},\alpha} = \kappa^{-3} \mathbf{A}_0 / (1-2\alpha),$$

(S37) becomes with  $\gamma_1 \leftarrow \gamma_{*,\alpha}(n)$ ,

$$\begin{aligned} \text{MSE}_f^{N,n} \\ \leq C (C_{\text{MSE},\alpha} n)^{-1/2} (\kappa^{-1} \exp\{-\kappa N^{1-\alpha} \gamma_{*,\alpha}(n) / (2(1-\alpha))\} \Omega(x) + C_{\text{MSE},\alpha}) . \end{aligned}$$

Setting  $N_\alpha(n) = \{2(1-\alpha)(\kappa \gamma_{*,\alpha}(n))^{-1} \log(\Omega(x))\}^{1/(1-\alpha)}$ , we end up with

$$\text{MSE}_f^{N_\alpha(n),n} \leq C (C_{\text{MSE},\alpha} / n)^{1/2}.$$

It is worthwhile to note that the order of  $N_\alpha(n)$  in  $n$  is  $n^{(1-2\alpha)/(2(1-\alpha))}$ , and  $C_{\text{MSE},\alpha}$  goes to infinity as  $\alpha \rightarrow 1/2$ .

3. If  $\alpha = 1/2$ , by Theorem 5, Lemma 21, (S11) and (S13), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1 \log(n)}{n^{1/2}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1/2}/4\} \Omega(x)}{\gamma_1 n^{1/2}} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1 \log(n)}{n^{1/2}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1/2}/4\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1/2}} \right). \quad (\text{S38})$$

At fixed  $\gamma_1$ , the order of this bound is  $\log(n)n^{-1/2}$ , and is the best bound for the MSE. Fix the number of iterations  $n$ , and we now optimize the choice of  $\gamma_1$ . Set

$$\gamma_{*,1/2}(n) = (\kappa^{-1} \mathbf{A}_0)^{-1} (C_{\text{MSE},1/2} / \log(n))^{1/2}, \text{ where } C_{\text{MSE},1/2} = \kappa^{-3} \mathbf{A}_0,$$

and (S38) becomes with  $\gamma_1 \leftarrow \gamma_{\star,1/2}(n)$ ,

$$\begin{aligned} \text{MSE}_f^{N,n} &\leq C \left( \frac{\log(n)}{nC_{\text{MSE},1/2}} \right)^{1/2} \left( \kappa^{-1} \exp \left\{ -\kappa N^{1/2} \gamma_{\star,1/2}(n)/4 \right\} \Omega(x) + \frac{C_{\text{MSE},1/2}}{\log(n)} \right). \end{aligned}$$

Setting  $N_{1/2}(n) = (4(\kappa\gamma_{\star,1/2}(n))^{-1} \log(\Omega(x)))^2$ , we end up with

$$\text{MSE}_f^{N_{1/2}(n),n} \leq C \left( \frac{\log(n) C_{\text{MSE},1/2}}{n} \right)^{1/2}.$$

4. For  $\alpha \in (1/2, 1]$ , by Theorem 5, Lemma 21, (S11) and (S13), we have the following bound for the bias

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp \left\{ -\kappa \gamma_1 N^{1-\alpha} / (2(1-\alpha)) \right\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp \left\{ -\kappa \gamma_1 N^{1-\alpha} / (2(1-\alpha)) \right\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right).$$

For fixed  $\gamma_1$ , the MSE is of order  $n^{1-\alpha}$ , and is worse than for  $\alpha \in [0, 1/2]$ . For a fixed number of iteration  $n$ , optimizing  $\gamma_1$  would imply to choose  $\gamma_1 \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Therefore, in that case, the best choice of  $\gamma_1$  is the largest possible value  $1/(m+L)$ .

5. For  $\alpha = 1$ , by Theorem 5, Lemma 21, (S11) and (S13), the bias is upper bounded by

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa \gamma_1/2} \Omega(x)}{\gamma_1 \log(n)} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \frac{\kappa^{-1} \mathbf{A}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa \gamma_1/2} \Omega(x) + \kappa^{-2}}{\gamma_1 \log(n)} \right).$$

For fixed  $\gamma_1$ , the order of the MSE is  $(\log(n))^{-1}$ . For a fixed number of iterations, the conclusions are the same than for  $\alpha \in (1/2, 1)$ .

## 5.2 Explicit bound based on Theorem 8

1. First for  $\alpha = 0$ , recall by Theorem 8 and (S14) we have for all  $p \geq 1$ ,

$$W_2^2(\delta_x R_\gamma^p, \pi) \leq 2\Omega(x)(1 - \kappa\gamma_1/2)^p + 2\kappa^{-1}(\mathbf{B}_0\gamma_1 + \mathbf{B}_1\gamma_1^2),$$

where  $\mathbf{B}_0$  and  $\mathbf{B}_1$  are given by (S15) and (S16) respectively. So by and Lemma 21, we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{\gamma_1 n} + \kappa^{-1} \mathbf{B}_0 \gamma_1^2 \right).$$

Therefore plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \kappa^{-1} \mathbf{B}_0 \gamma_1^2 + \frac{\kappa^{-2} + \kappa^{-1} \exp(-\kappa N \gamma_1/2) \Omega(x)}{n \gamma_1} \right). \quad (\text{S39})$$

So with fixed  $\gamma_1$  this bound is of order  $\gamma_1$ . If we fix the number of iterations  $n$ , we can optimize the choice of  $\gamma_1$ . Set

$$\gamma_{\star,0}(n) = (\kappa \mathbf{B}_0 n)^{-1/3},$$

and (S36) becomes if  $\gamma_1 \leftarrow \gamma_{\star,0}(n)$ ,

$$\text{MSE}_f^{N,n} \leq C(\mathbf{B}_0^{-1/2} n)^{-2/3} \left( \kappa^{-4/3} \exp(-\kappa N \gamma_{\star,0}(n)/2) \Omega(x) + \kappa^{-5/3} \right).$$

Setting  $N_0(n) = 2(\kappa \gamma_{\star,0}(n))^{-1} \log(\Omega(x))$ , we end up with

$$\text{MSE}_f^{N_0(n),n} \leq C(\mathbf{B}_0^{-1/2} \kappa^{5/2} n)^{-2/3}.$$

Note that  $N_0(n)$  is of order  $n^{1/3}$ .

2. For  $\alpha \in (0, 1/3)$  by Theorem 8, Lemma 21, (S11) and (S17), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1^2}{(1 - 3\alpha)n^{2\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1^2}{(1 - 3\alpha)n^{2\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right). \quad (\text{S40})$$

If we fix the number of iterations  $n$ , we can optimize the choice of  $\gamma_1$  again. Set

$$\gamma_{\star,\alpha}(n) = (n^{1-3\alpha} \kappa \mathbf{B}_0 / (1 - 3\alpha))^{-1/3},$$

(S40) becomes with  $\gamma_1 \leftarrow \gamma_{\star, \alpha}(n)$ ,

$$\text{MSE}_f^{N, n} \leq C(\mathbf{B}_0^{-1/2} n)^{-2/3} \left( \kappa^{-4/3} \exp(-\kappa N \gamma_{\star, 0}(n)/(2(1-\alpha))) \Omega(x) + \kappa^{-5/3} (1-3\alpha)^{-1/3} \right).$$

Setting  $N_\alpha(n) = \{(\kappa \gamma_{\star, \alpha}(n))^{-1} \log(\Omega(x))\}^{1/(1-\alpha)}$ , we end up with

$$\text{MSE}_f^{N_\alpha(n), n} \leq C(\mathbf{B}_0^{-1/2} \kappa^{5/2} n)^{-2/3}.$$

It is worthwhile to note that the order of  $N_\alpha(n)$  in  $n$  is  $n^{(1-3\alpha)/(3(1-\alpha))}$ .

3. If  $\alpha = 1/3$ , by Theorem 8, Lemma 21, (S11) and (S17), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1^2 \log(n)}{n^{2/3}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{2/3}/4\} \Omega(x)}{\gamma_1 n^{2/3}} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N, n} \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1^2 \log(n)}{n^{2/3}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{2/3}/4\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{2/3}} \right). \quad (\text{S41})$$

At fixed  $\gamma_1$ , the order of this bound is  $\log(n)n^{-2/3}$ , and is the best bound for the MSE. Fix the number of iterations  $n$ , and we now optimize the choice of  $\gamma_1$ . Set

$$\gamma_{\star, 1/2}(n) = (\kappa \mathbf{B}_0 \log(n))^{-1/3},$$

and (S41) becomes with  $\gamma_1 \leftarrow \gamma_{\star, 1/2}(n)$ ,

$$\text{MSE}_f^{N, n} \leq C \left( \frac{\log(n) \mathbf{B}_0}{n^2} \right)^{1/3} \left( \kappa^{-4/3} \exp\left\{-\kappa N^{1/2} \gamma_{\star, 1/2}(n)/4\right\} \Omega(x) + \kappa^{-5/3} \right).$$

Setting  $N_{1/2}(n) = (4(\kappa \gamma_{\star, 1/2}(n))^{-1} \log(\Omega(x)))^{3/2}$ , we end up with

$$\text{MSE}_f^{N_{1/2}(n), n} \leq C \left( \frac{\log(n) \mathbf{B}_0}{\kappa^5 n^2} \right)^{1/3}.$$

We can see that we obtain a worse bound than for  $\alpha = 0$  and  $\alpha \in (0, 1/3)$ .

4. For  $\alpha \in (1/3, 1]$ , by Theorem 8, Lemma 21, (S11) and (S17), we have the following bound for the bias

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N, n} \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp\{-\kappa \gamma_1 N^{1-\alpha}/(2(1-\alpha))\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right).$$

For fixed  $\gamma_1$ , the MSE is of order  $n^{1-\alpha}$ , and is worse than for  $\alpha = 1/2$ . For a fixed number of iteration  $n$ , optimizing  $\gamma_1$  would imply to choose  $\gamma_1 \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Therefore, in that case, the best choice of  $\gamma_1$  is the largest possible value  $1/(m+L)$ .

5. For  $\alpha = 1$ , by Theorem 8, Lemma 21, (S11) and (S13), the bias is upper bounded by

$$\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\}^2 \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa \gamma_1/2} \Omega(x)}{\gamma_1 \log(n)} \right).$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\text{MSE}_f^{N,n} \leq C \left( \frac{\kappa^{-1} \mathbf{B}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa \gamma_1/2} \Omega(x) + \kappa^{-2}}{\gamma_1 \log(n)} \right).$$

For fixed  $\gamma_1$ , the order of the MSE is  $(\log(n))^{-1}$ . For a fixed number of iterations, the conclusions are the same than for  $\alpha \in (1/2, 1)$ .

## References

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