Supplement to ”High-dimensional Bayesian inference via the Unadjusted Langevin Algorithm”

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1 Postponed proofs

1.1 Proof of (23)

Consider the constant sequence $\gamma_k = \gamma$ for all $k \in \mathbb{N}^*$ with $\gamma \in (0, 1/(m+L)]$. By (20), we have for all $n \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$

$$\vartheta^{(1)}_{n,0}(x) \leq \gamma \mathsf{D}_1(\gamma, d) + \gamma^3 \mathsf{D}_2(\gamma) \sum_{i=1}^{n} (1 - \kappa\gamma/2)^{n-i} \delta_{i,n,0}(x) \ ,$$

where

$$\mathsf{D}_1(\gamma, d) = 2L^2 \kappa^{-1} \left(\kappa^{-1} + \gamma\right) \left(2d + L^2\gamma^2/6\right) \ , \quad \mathsf{D}_2(\gamma) = L^4 \left(\kappa^{-1} + \gamma\right) \ .$$

In addition, using that $\kappa \geq 2m$ and for all $t \geq 0$, $1 - t \leq \mathrm{e}^{-t}$,

$$\begin{aligned}\sum_{i=1}^{n} (1 - \kappa\gamma/2)^{n-i} \delta_{i,n,0}(x) &= \sum_{i=1}^{n} \Big[(1 - \kappa\gamma/2)^{n-i} \Big\{ \mathrm{e}^{-2m\gamma(i-1)} \left\| x - x^\star \right\|^2 \\ &\qquad + \left(1 - \mathrm{e}^{-2m\gamma(i-1)}\right) (d/m) \Big\} \Big] \\ &\leq n \mathrm{e}^{-m\gamma(n-1)} \left\| x - x^\star \right\|^2 + 2d(\kappa\gamma m)^{-1} \ . \end{aligned} \tag{S1}$$

Therefore for all $n \geq 1$ and $x \in \mathbb{R}^d$ we get

$$\vartheta^{(1)}_{n,0}(x) \leq \gamma \mathsf{D}_1(\gamma) + \gamma^3 \mathsf{D}_2(\gamma) \left\{ n \mathrm{e}^{-m\gamma(n-1)} \left\| x - x^\star \right\|^2 + 2d(\kappa\gamma m)^{-1} \right\} \ . \tag{S2}$$

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Let now $\ell \in \mathbb{N}^*$, $\ell \geq \lceil \gamma^{-1} \rceil + 1$ and $n = \ell - \lceil \gamma^{-1} \rceil$. Then,

$$\begin{aligned}
&\sum_{k=n+1}^{\ell} \{(\gamma_k^3 L^2/3)\varrho_{1,k-1}(x) + d\gamma_k^2\} \\
&\leq (L^2\gamma^3/3)\left\{(1-\kappa\gamma)^n(\ell-n-1)\|x-x^\star\|^2 + 2\kappa^{-1}\gamma d(\ell-n-1)\right\} + d\gamma^2(\ell-n-1) \\
&\leq (L^2\gamma^3/3)\left\{(1+\gamma^{-1})(1-\kappa\gamma)^{\ell-\lceil\gamma^{-1}\rceil}\|x-x^\star\|^2 + 2(1+\gamma)\kappa^{-1}d\right\} + d(1+\gamma)\,.
\end{aligned}$$

Combining this inequality and (S2) in the bound provided by Theorem 14 shows (23).

1.2 Proof of (61)

First for all $n \geq 1$ and $x \in \mathbb{R}^d$, we have

$$\vartheta_{n,0}^{(2)}(x) \leq \gamma^2 \mathsf{E}_1(\gamma, d) + \gamma^3 \mathsf{E}_2(\gamma) \sum_{i=1}^{n} \prod_{k=i+1}^{n} (1-\kappa\gamma_k/2)\delta_{i,n,0}(x)\,,$$

where

$$\mathsf{E}_1(\gamma,d) = 2d\kappa^{-1}\left\{2L^2 + 4\kappa^{-1}(\tilde{L}^2/12 + \gamma L^4/4) + \gamma^2 L^4/6\right\}\,, \mathsf{E}_2(\gamma) = L^4(4\kappa^{-1}/3 + \gamma)\,.$$

$$\begin{aligned}
\mathsf{E}_3(\gamma,d,x) &= (\ell-\gamma^{-1})\mathrm{e}^{-m\gamma(\ell-2\gamma^{-1}-1)}\|x-x^\star\|^2 + 2d/(\kappa\gamma m) \\
\mathsf{E}_4(\gamma,d,x) &= \mathrm{e}^{-m\gamma(\ell-2\gamma^{-1}-1)}\|x-x^\star\|^2 + 2\kappa^{-1}d + d/m\,.
\end{aligned}$$

By (S1), we get for all $n \geq 1$ and $x \in \mathbb{R}^d$,

$$\vartheta_{n,0}^{(2)}(x) \leq \gamma^2\mathsf{E}_1(\gamma,d) + \gamma^3\mathsf{E}_2(\gamma)\left\{n\mathrm{e}^{-m\gamma(n-1)}\|x-x^\star\|^2 + 2d(\kappa\gamma m)^{-1}\right\}\,. \tag{S3}$$

On the other hand, for all $x \in \mathbb{R}^d$, $\ell, n \in \mathbb{N}$, $n \geq 1$, $\ell > n$ we have using that $\kappa \geq 2m$ and for all $t \geq 0$, $1 - t \leq \mathrm{e}^{-t}$,

$$\begin{aligned}
\vartheta_{n,\ell}^{(2)}(x) &\leq \gamma^3 n\mathsf{E}_1(\gamma) + \gamma^3 n\mathsf{E}_2(\gamma)\left\{\mathrm{e}^{-m\gamma(n-1)}\varrho_{n,\ell}(x) + d/m\right\} \\
&\leq \gamma^3 n\mathsf{E}_1(\gamma) + \gamma^3 n\mathsf{E}_2(\gamma)\left\{\mathrm{e}^{-m\gamma(n-1)}\left((1-\kappa\gamma)^{\ell-n}\|x-x^\star\|^2 + 2\kappa^{-1}d\right) + d/m\right\} \\
&\leq \gamma^3 n\mathsf{E}_1(\gamma) + \gamma^3 n\mathsf{E}_2(\gamma)\left\{\mathrm{e}^{-m\gamma(\ell-1)}\|x-x^\star\|^2 + 2\kappa^{-1}d + d/m\right\}\,.
\end{aligned} \tag{S4}$$

Finally, for all $\ell \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$, we have

$$(\gamma^3L^2/3)\varrho_{1,\ell-1}(x) + d\gamma^2 \leq (\gamma^3L^2/3)\left\{(1-\kappa\gamma)^{\ell-1}\|x-x^\star\|^2 + 2d\kappa^{-1}\right\} + d\gamma^2\,. \tag{S5}$$

Combining (S3), (S4) and (S5) in the bound given by Lemma 31, and using that $\gamma^{-1} \leq 2^{\mathsf{n}(\gamma)} \leq 2\gamma^{-1}$ we have for all $\ell \in \mathbb{N}^*$, $\ell > 2^{\mathsf{n}(\gamma)}$,

$$
\begin{aligned}
&\|\delta_x P_{\ell\gamma} - \delta_x R_\gamma^\ell\|_{\mathrm{TV}} \leq 2^{-3/2} L \left[(\gamma^3 L^2/3) \left\{ (1-\kappa\gamma)^{\ell-1} \|x - x^\star\|^2 + 2d\kappa^{-1} \right\} + d\gamma^2 \right]^{1/2} \\
&+ (4\pi)^{-1/2} \left[\gamma^2 \mathsf{E}_1(\gamma) + \gamma^3 \mathsf{E}_2(\gamma) \left\{ (\ell - \gamma^{-1}) \mathrm{e}^{-m\gamma(\ell - 2\gamma^{-1} - 1)} \|x - x^\star\|^2 + 2d(\kappa\gamma m)^{-1} \right\} \right]^{1/2} \\
&+ \sum_{k=1}^{\mathsf{n}(\gamma)} \left[\frac{\gamma^3 2^{k-1} \mathsf{E}_1(\gamma) + \gamma^3 2^{k-1} \mathsf{E}_2(\gamma) \left\{ \mathrm{e}^{-m\gamma(\ell - 2^k - 1)} \|x - x^\star\|^2 + 2\kappa^{-1} d + d/m \right\}}{\pi 2^{k+1} \gamma} \right]^{1/2} \\
&\qquad\qquad\leq 2^{-3/2} L \left\{ (\gamma^3 L^2/3) \left\{ (1-\kappa\gamma)^{\ell-1} \|x - x^\star\|^2 + 2d\kappa^{-1} \right\} + d\gamma^2 \right\}^{1/2} \\
&+ (4\pi)^{-1/2} \left[\gamma^2 \mathsf{E}_1(\gamma) + \gamma^3 \mathsf{E}_2(\gamma) \left\{ (\ell - \gamma^{-1}) \mathrm{e}^{-m\gamma(\ell - 2\gamma^{-1} - 1)} \|x - x^\star\|^2 + 2d(\kappa\gamma m)^{-1} \right\} \right]^{1/2} \\
&+ (4\pi)^{-1/2} \mathsf{n}(\gamma) \left[\gamma^2 \mathsf{E}_1(\gamma) + \gamma^2 \mathsf{E}_2(\gamma) \left\{ \mathrm{e}^{-m\gamma(\ell - 2\gamma^{-1} - 1)} \|x - x^\star\|^2 + 2\kappa^{-1} d + d/m \right\} \right]^{1/2} .
\end{aligned}
$$

2 Discussion on Theorem 5

Note that

$$
u_n^{(2)}(\gamma) \leq \sum_{i=1}^{n} \left\{ \mathsf{A}_0 \gamma_i^2 + \mathsf{A}_1 \gamma_i^3 \right\} \prod_{k=i+1}^{n} (1 - \kappa\gamma_k/2) , \tag{S6}
$$

where κ is given by (3), and

$$
\mathsf{A}_0 = 2L^2 \kappa^{-1} d , \tag{S7}
$$

$$
\mathsf{A}_1 = 2L^2 \kappa^{-1} d (m+L)^{-1} + dL^4 (\kappa^{-1} + (m+L)^{-1})(m^{-1} + 6^{-1}(m+L)^{-1}) . \tag{S8}
$$

If $(\gamma_k)_{k \geq 1}$ is a constant step size ,$\gamma_k = \gamma$ for all $k \geq 1$, then a straightforward consequence of Theorem 5 and (S6) is the following result, which gives the minimal number of iterations n_ϵ and a step-size γ_ϵ to get $W_2(\delta_{x^\star} Q_\gamma^n, \pi)$ smaller than $\epsilon > 0$.

Corollary 1 (of Theorem 5)**.** *Assume* ***H****1 and* ***H****2. Let $x^\star$ be the unique minimizer of U. Let $x \in \mathbb{R}^d$ and $\epsilon > 0$. Set for all $k \in \mathbb{N}$, $\gamma_k = \gamma$ with*

$$
\begin{aligned}
\gamma &= \frac{-\mathsf{A}_0 + (\mathsf{A}_0^2 + \epsilon^2 \kappa \mathsf{A}_1)^{1/2}}{2\mathsf{A}_1} \wedge (m+L)^{-1} , \\
n &= \left\lceil \log^{-1}(1 - \kappa\gamma/2) \left\{ -\log(\epsilon^2/2) + \log(2d/m) \right\} \right\rceil .
\end{aligned} \tag{S9}
$$

Then $W_2(\delta_{x^\star} R_\gamma^n, \pi) \leq \epsilon$.

Note that if γ is given by (S9), and is different from $1/(m+L)$, then $\gamma \leq \epsilon(4\mathsf{A}_1\kappa^{-1})^{-1/2}$ and $2\kappa^{-1}(\mathsf{A}_0\gamma + \mathsf{A}_1\gamma^2) = \epsilon^2/2$. Therefore,

$$
\gamma \geq (\epsilon^2 \kappa/4) \left\{ \mathsf{A}_0 + \epsilon(\mathsf{A}_1/(4\kappa))^{1/2} \right\}^{-1} .
$$

It is shown in [1, Corollary 1] that under **H**2, for constant step size for any $\epsilon > 0$, we can choose γ and $n \geq 1$ such that if for all $k \geq 1$, $\gamma_k = \gamma$, then $\|\nu^\star Q_n^\gamma - \pi\|_{\mathrm{TV}} \leq \epsilon$ where $\nu^\star$ is the Gaussian measure on $\mathbb{R}^d$ with mean $x^\star$ and covariance matrix $L^{-1}\,\mathrm{I}_d$. We stress that the results in [1, corollary 1] hold only for a particular choice of the initial distribution $\nu^\star$, (which might seem a rather artificial assumption) whereas Theorem 5 holds for any initial distribution in $\mathcal{P}_2(\mathbb{R}^d)$.

We compare the optimal value of γ and n obtained from Corollary 1 with those given in [1, Corollary 1]. This comparison is summarized in Table 1 and Table 2; for simplicity, we provide only the dependencies of the optimal stepsize γ and minimal number of simulations n as a function of the dimension d, the precision ϵ and the constants m, L. It can be seen that the dependency on the dimension is significantly better than those in [1, Corollary 1].

Parameter	d	ϵ	L	m
Corollary 1	d^{-1}	ϵ^2	L^{-2}	m^2
[1, Corollary 1]	d^{-2}	ϵ^2	L^{-2}	m

Table 1: Dependencies of γ

Parameter	d	ϵ	L	m
Corollary 1	$d\log(d)$	$\epsilon^{-2}\,\lvert\log(\epsilon)\rvert$	L^2	$\lvert\log(m)\rvert\, m^{-3}$
[1, Corollary 1]	d^3	$\epsilon^{-2}\,\lvert\log(\epsilon)\rvert$	L^3	$\lvert\log(m)\rvert\, m^{-2}$

Table 2: Dependencies of n

2.1 Explicit bounds for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0, 1]$

We give here a bound on the sequences $(u_n^{(1)}(\gamma))_{n\geq 0}$ and $(u_n^{(2)}(\gamma))_{n\geq 0}$ for $(\gamma_k)_{k\geq 1}$ defined by $\gamma_1 < 1/(m+L)$ and $\gamma_k = \gamma_1 k^{-\alpha}$ for $\alpha \in (0,1]$. Also for that purpose we introduce for $t \in \mathbb{R}_+^*$,

$$\boldsymbol{\psi}_\beta(t) = \begin{cases} (t^\beta - 1)/\beta & \text{for } \beta \neq 0 \\ \log(t) & \text{for } \beta = 0\;. \end{cases} \tag{S10}$$

We easily get for $a \geq 0$ that for all $n, p \geq 1$, $n \leq p$

$$\boldsymbol{\psi}_{1-a}(p+1) - \boldsymbol{\psi}_{1-a}(n) \leq \sum_{k=n}^{p} k^{-a} \leq \boldsymbol{\psi}_{1-a}(p) - \boldsymbol{\psi}_{1-a}(n) + 1\;, \tag{S11}$$

and for $a \in \mathbb{R}$

$$\sum_{k=n}^{p} k^{-a} \leq \boldsymbol{\psi}_{1-a}(p+1) - \boldsymbol{\psi}_{1-a}(n) + 1\;. \tag{S12}$$

1. For $\alpha = 1$, using that for all $t \in \mathbb{R}$, $(1+t) \leq \mathrm{e}^t$ and by (S11) and (S12), we have

$$u_n^{(1)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \;, \; u_n^{(2)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^{1} \mathsf{A}_j (\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n+1)+1) \;.$$

2. For $\alpha \in (0,1)$, by (S11) and Lemma 21 applied with $\ell = \lceil n/2 \rceil$, where $\lceil \cdot \rceil$ is the ceiling function, we have

$$u_n^{(1)}(\gamma) \leq \exp\left(-\kappa\gamma_1 \boldsymbol{\psi}_{1-\alpha}(n+1)/2\right)$$
$$u_n^{(2)}(\gamma) \leq \sum_{j=0}^{1} \mathsf{A}_j \left[2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} + \gamma_1^{j+2}\left\{\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2 \rceil)+1\right\}\right.$$
$$\left. \times \exp\left(-(\kappa\gamma_1/2)\left\{\boldsymbol{\psi}_{1-\alpha}(n+1) - \boldsymbol{\psi}_{1-\alpha}(\lceil n/2 \rceil)\right\}\right)\right] \;. \tag{S13}$$

2.2 Optimal strategy with a fixed number of iterations

Corollary 2. *Let $n \in \mathbb{N}^*$ be a fixed number of iteration. Assume **H**1, **H**2, and $(\gamma_k)_{k\geq 1}$ is a constant sequence, $\gamma_k = \gamma$ for all $k \geq 1$. Set*

$$\gamma^+ = 2(\kappa n)^{-1}\left[\log(\kappa n/2) + \log(2(\|x - x^\star\|^2 + d/m)) - \log(2\kappa^{-1}\mathsf{A}_0)\right]$$
$$\gamma_- = 2(\kappa n)^{-1}\left[\log(\kappa n/2) + \log(2(\|x - x^\star\|^2 + d/m)) \right.$$
$$\left. - \log\left\{2\kappa^{-1}(\mathsf{A}_0 + 2\mathsf{A}_1(m+L)^{-1})\right\}\right] \;.$$

Assume $\gamma^+ \in \left(0, (m+L)^{-1}\right)$. Then, the optimal choice of γ to minimize the bound on $W_2(\delta_x R_\gamma^n, \pi)$ given by Theorem 5 belongs to $[\gamma_-, \gamma^+]$. Moreover if $\gamma = \gamma_+$, then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x R_\gamma^n, \pi)$ is equivalent to $Cdn^{-1}\log(n)$ as n goes to $+\infty$.

Similarly, we have the following result.

Corollary 3. *Assume **H**1 and **H**2. Let $(\gamma_k)_{k\geq 1}$ be the decreasing sequence, defined by $\gamma_k = \gamma_\alpha k^{-\alpha}$, with $\alpha \in (0,1)$. Let $n \geq 1$ and set*

$$\gamma_\alpha = 2(1-\alpha)\kappa^{-1}(2/n)^{1-\alpha}\log(\kappa n/(2(1-\alpha))) \;.$$

Assume $\gamma_\alpha \in \left(0, (m+L)^{-1}\right)$. Then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x Q_\gamma^n, \pi)$ is equivalent to $Cdn^{-1}\log(n)$ as n goes to $+\infty$.

Proof. Follows from (S6), (S13) and the choice of γ_α. □

3 Discussion on Theorem 8

Based on Theorem 8, we can follow the same discussion as for Theorem 5. Note that

$$u_n^{(3)}(\gamma) \leq \sum_{i=1}^{n}\left\{\mathsf{B}_0\gamma_i^3 + \mathsf{B}_1\gamma_i^4\right\}\prod_{k=i+1}^{n}(1-\kappa\gamma_k/2) \;, \tag{S14}$$

where κ is given by (3), and

$$\mathsf{B}_0 = d\left[2L^2 + \kappa^{-1}\left\{\tilde{L}^2/3 + 4L^4/(3m)\right\}\right] , \tag{S15}$$

$$\mathsf{B}_1 = d\left[\kappa^{-1}L^4 + L^4/(6(m+L)) + m^{-1}\right] . \tag{S16}$$

The following result gives for a target precision $\epsilon > 0$, the minimal number of iterations n_ϵ and a step-size γ_ϵ to get $W_2(\delta_{x^\star} Q_\gamma^n, \pi)$ smaller than ϵ, when $(\gamma_k)_{k\geq 1}$ is a constant step size, $\gamma_k = \gamma_\epsilon$ for all $k \geq 1$.

Corollary 4. *Assume* ***H*** *1,* ***H*** *2 and* ***H*** *3. Let $x^\star$ be the unique minimizer of U. Let $x \in \mathbb{R}^d$ and $\epsilon > 0$. Set for all $k \in \mathbb{N}$, $\gamma_k = \gamma$ with*

$$\gamma = \left[(\epsilon/2)\kappa^{-1}\left\{\mathsf{B}_0 + \mathsf{B}_1(m+L)^{-1}\right\}^{-1/2}\right] \wedge (1/(m+L)) ,$$

$$n = \left\lceil \log^{-1}(1-\kappa\gamma/2)\left\{-\log(\epsilon^2/2) + \log(2d/m)\right\}\right\rceil .$$

Then $W_2(\delta_{x^\star} R_\gamma^n, \pi) \leq \epsilon$.

We provide the dependencies of the optimal stepsize γ_ϵ and minimal number of simulations n_ϵ as a function of the dimension d, the precision ϵ and the constants $m, L, \tilde{L}$ in Table 3 and Table 4.

Parameter	d	ϵ	L	m
Corollary 4	$d^{-1/2}$	ϵ	L^{-2}	m

Table 3: Dependencies of γ

Parameter	d	ϵ	L	m
Corollary 4	$d^{1/2}\log(d)$	$\epsilon^{-1}\lvert\log(\epsilon)\rvert$	L^2	$\lvert\log(m)\rvert\, m^{-2}$

Table 4: Dependencies of n

3.1 Explicit bounds for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0,1]$

We give here a bound on the sequence $(u_n^{(3)}(\gamma))_{n\geq 0}$ for $(\gamma_k)_{k\geq 1}$ defined by $\gamma_1 < 1/(m+L)$ and $\gamma_k = \gamma_1 k^{-\alpha}$ for $\alpha \in (0,1]$. Bounds for $(u_n^{(1)}(\gamma))_{n\geq 0}$ have already been given in Section 2.1. Recall that the function $\boldsymbol{\psi}$ is defined by (S10). For $\alpha \in (0,1]$, by (S11) and Lemma 21 applied with $\ell = \lceil n/2 \rceil$, where $\lceil\cdot\rceil$ is the ceiling function, we have

$$u_n^{(3)}(\gamma) \leq \sum_{j=1}^{2} \mathsf{B}_{j-1}\left[2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} + \gamma_1^{j+2}\left(\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2\rceil) + 1\right)\right.$$
$$\left.\times \exp\left(-(\kappa\gamma_1/2)\left\{\boldsymbol{\psi}_{1-\alpha}(n+1) - \boldsymbol{\psi}_{1-\alpha}(\lceil n/2\rceil)\right\}\right)\right] . \tag{S17}$$

3.2 Optimal strategy with a fixed number of iterations

Corollary 5. *Let $n \in \mathbb{N}^*$ be a fixed number of iteration. Assume* **H1**, **H2**, **H3** *and $(\gamma_k)_{k\geq 1}$ is a constant sequence, $\gamma_k = \gamma^\star$ for all $k \geq 1$, with*

$$\gamma^\star = 4(\kappa n)^{-1} \left\{ \log(\kappa n/2) + \log(2(\|x - x^\star\|^2 + d/m)) \right\} .$$

Assume $\gamma^\star \in \left(0, (m+L)^{-1}\right)$. Then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x R_\gamma^n, \pi)$ is equivalent to $Cdn^{-2}\log^2(n)$ as n goes to $+\infty$.

Similarly, we have the following result.

Corollary 6. *Assume* **H1**, **H2** *and* **H3**. *Let $(\gamma_k)_{k\geq 1}$ be the decreasing sequence, defined by $\gamma_k = \gamma_\alpha k^{-\alpha}$, with $\alpha \in (0,1)$. Let $n \geq 1$ and set*

$$\gamma_\alpha = 2(1-\alpha)\kappa^{-1}(2/n)^{1-\alpha}\log(\kappa n/(2(1-\alpha))) .$$

Assume $\gamma_\alpha \in \left(0, (m+L)^{-1}\right)$. Then there exists $C \geq 0$ independent of the dimension such that the bound on $W_2^2(\delta_x R_\gamma^n, \pi)$ is equivalent to $Cdn^{-2}\log^2(n)$ as n goes to $+\infty$.

Proof. Follows from (S14), (S17) and the choice of γ_α. □

4 Generalization of Theorem 5

In this section, we weaken the assumption $\gamma_1 \leq 1/(m+L)$ of Theorem 5. We assume now:

G1. *The sequence $(\gamma_k)_{k\geq 1}$ is non-increasing, and there exists $\rho > 0$ and n_1 such that $(1+\rho)\gamma_{n_1} \leq 2/(m+L)$.*

Under **G1**, we denote by

$$n_0 = \min\left\{k \in \mathbb{N} \mid \gamma_k \leq 2/(m+L)\right\} \tag{S18}$$

We first give an extension of Theorem 3. Denote in the sequel $(\cdot)_+ = \max(\cdot, 0)$. Recall that under **H2**, $x^\star$ is the unique minimizer of U, and κ is defined in (4)

Theorem S7. *Assume* **H1**, **H2** *and* **G1**. *Then for all $n, p \in \mathbb{N}^*$, $n \leq p$*

$$\int_{\mathbb{R}^d} \|x - x^\star\|^2 \mu_0 Q_n^p(\mathrm{d}x) \leq \mathsf{G}_{n,p}(\mu_0, \gamma) ,$$

where

$$\mathsf{G}_{n,p}(\mu_0,\gamma) = \exp\left(-\sum_{k=n}^{p}\gamma_k\kappa + \sum_{k=n}^{n_0-1} L^2\gamma_k^2\right)\int_{\mathbb{R}^d}\|x - x^\star\|^2 \mu_0(\mathrm{d}x)$$
$$+ 2d\kappa^{-1} + 2d\left\{\prod_{k=n}^{n_0-1}(\gamma_{n_0-1}L^2)^{-1}\left(1 + L^2\gamma_k^2\right)\right\}\exp\left(-\sum_{k=n}^{p}\kappa\gamma_k + \sum_{k=n}^{n_0-1}\gamma_k^2 mL\right) . \tag{S19}$$

Proof. For any $\gamma > 0$, we have for all $x \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \|y - x^\star\|^2 R_\gamma(x, \mathrm{d}y) = \|x - \gamma \nabla U(x) - x^\star\|^2 + 2\gamma d \; .$$

Using that $\nabla U(x^\star) = 0$, (3) and **H**1, we get from the previous inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \|y - x^\star\|^2 R_\gamma(x, \mathrm{d}y) \\
&\leq (1 - \kappa\gamma) \|x - x^\star\|^2 + \gamma \left(\gamma - \frac{2}{m+L}\right) \|\nabla U(x) - \nabla U(x^\star)\|^2 + 2\gamma d \\
&\leq \eta(\gamma) \|x - x^\star\|^2 + 2\gamma d \; ,
\end{aligned}$$

where $\eta(\gamma) = (1 - \kappa\gamma + \gamma L(\gamma - 2/(m+L))_+)$. Denote for all $k \geq 1$, $\eta_k = \eta(\gamma_k)$. By a straightforward induction, we have by definition of Q_n^p for $p, n \in \mathbb{N}$, $p \leq n$,

$$\int_{\mathbb{R}^d} \|x - x^\star\|^2 \mu_0 Q_n^p(\mathrm{d}x) \leq \prod_{k=n}^{p} \eta_k \int_{\mathbb{R}^d} \|x - x^\star\| \mu_0(\mathrm{d}x) + (2d) \sum_{i=n}^{p} \prod_{k=i+1}^{p} \eta_k \gamma_i \; , \tag{S20}$$

with the convention that for $n, p \in \mathbb{N}$, $n < p$, $\prod_p^n = 1$. For the first term of the right hand side, we simply use the bound, for all $x \in \mathbb{R}$, $(1 + x) \leq \mathrm{e}^x$, and we get by **G**1

$$\prod_{k=n}^{p} \eta_k \leq \exp\left(-\sum_{k=n}^{p} \kappa\gamma_k + \sum_{k=n}^{n_0 - 1} L^2\gamma_k^2\right) \; , \tag{S21}$$

where n_0 is defined in (S18). Consider now the second term in the right hand side of (S20).

$$\begin{aligned}
\sum_{i=n}^{p} \prod_{k=i+1}^{p} \eta_k \gamma_i &\leq \sum_{i=n_0}^{p} \prod_{k=i+1}^{p} (1 - \kappa\gamma_k) \gamma_i + \sum_{i=n}^{n_0-1} \prod_{k=i+1}^{p} \eta_k \gamma_i \\
&\leq \kappa^{-1} \sum_{i=n_0}^{p} \left\{ \prod_{k=i+1}^{p} (1 - \kappa\gamma_k) - \prod_{k=i}^{p} (1 - \kappa\gamma_k) \right\} \\
&+ \left\{ \sum_{i=n}^{n_0-1} \prod_{k=i+1}^{n_0-1} (1 + L^2\gamma_k^2) \gamma_i \right\} \prod_{k=n_0}^{p} (1 - \kappa\gamma_k)
\end{aligned} \tag{S22}$$

Since $(\gamma_k)_{k \geq 1}$ is nonincreasing, we have

$$\begin{aligned}
\sum_{i=n}^{n_0-1} \prod_{k=i+1}^{n_0-1} (1 + L^2\gamma_k^2) \gamma_i &= \sum_{i=n}^{n_0-1} (\gamma_i L^2)^{-1} \left\{ \prod_{k=i}^{n_0-1} (1 + L^2\gamma_k^2) - \prod_{k=i+1}^{n_0-1} (1 + L^2\gamma_k^2) \right\} \\
&\leq \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} (1 + L^2\gamma_k^2) \; .
\end{aligned}$$

Furthermore for $k < n_0$ $\gamma_k > 2/(m+L)$. This implies with the bound $(1+x) \le \mathrm{e}^x$ on $\mathbb{R}$:

$$\prod_{k=n_0}^{p} (1-\kappa\gamma_k) \le \exp\left(-\sum_{k=n}^{p} \kappa\gamma_k\right) \exp\left(\sum_{k=n}^{n_0-1} \kappa\gamma_k\right)$$
$$\le \exp\left(-\sum_{k=n}^{p} \kappa\gamma_k\right) \exp\left(\sum_{k=n}^{n_0-1} \gamma_k^2 mL\right) .$$

Using the two previous inequalities in (S22), we get

$$\sum_{i=n}^{p} \prod_{k=i+1}^{p} \eta_k \gamma_i$$
$$\le \kappa^{-1} + \left\{ \prod_{k=n}^{n_0-1} (\gamma_{n_0-1} L^2)^{-1} \left(1 + L^2\gamma_k^2\right) \right\} \exp\left(-\sum_{k=n}^{p} \kappa\gamma_k + \sum_{k=n}^{n_0-1} \gamma_k^2 mL\right) . \quad \text{(S23)}$$

Combining (S21) and (S23) in (S20) concluded the proof. □

We now deal with bounds on $W_2(\mu_0 Q_\gamma^n, \pi)$ under **G** 1. But before we preface our result by some techincal lemmas.

Lemma S8. *Assume **H** 1 and **H** 2. Let $\zeta_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, $(Y_t, \overline{Y}_t)_{t\ge 0}$ such that $(Y_0, \overline{Y}_0)$ is distributed according to ζ_0 and given by (9). Let $(\mathcal{F}_t)_{t\ge0}$ be the filtration associated with $(B_t)_{t\ge0}$ with $\mathcal{F}_0$, the σ-field generated by $(Y_0, \overline{Y}_0)$. Then for all $n \ge 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$,*

$$\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|Y_{\Gamma_{n+1}} - \overline{Y}_{\Gamma_{n+1}}\right\|^2\right]$$
$$\le \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1} L((1+\epsilon_2)\gamma_{n+1} - 2/(m+L))_+\} \left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2$$
$$+ \gamma_{n+1}^2 (1/(2\epsilon_1) + (1+\epsilon_2^{-1})\gamma_{n+1}) \left(dL^2 + (L^4\gamma_{n+1}/2) \left\|Y_{\Gamma_n} - x^\star\right\|^2 + dL^4\gamma_{n+1}^2/12\right) .$$

Proof. Let $n \ge 0$ and $\epsilon_1 > 0$, and set $\Delta_n = Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}$ by definition we have:

$$\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\|\Delta_{n+1}\|^2\right] = \|\Delta_n\|^2 + \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \{\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\}\,\mathrm{d}s\right\|^2\right]$$
$$- 2\gamma_{n+1}\left\langle \Delta_n, \nabla U(Y_{\Gamma_n}) - \nabla U(\overline{Y}_{\Gamma_n})\right\rangle - 2\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\langle \Delta_n, \{\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\}\rangle\,\mathrm{d}s\right] .$$

Using the two inequalities $|\langle a, b\rangle| \le \epsilon_1\|a\|^2 + (4\epsilon_1)^{-1}\|b\|^2$ and (3), we get

$$\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\|\Delta_{n+1}\|^2\right] \le \{1 - \gamma_{n+1}(\kappa - 2\epsilon_1)\}\|\Delta_n\|^2 - 2\gamma_{n+1}/(m+L)\left\|\nabla U(Y_{\Gamma_n}) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2$$
$$+ \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \{\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\}\mathrm{d}s\right\|^2\right]$$
$$+ (2\epsilon_1)^{-1}\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2\right]\mathrm{d}s . \quad \text{(S24)}$$

Using $\|a+b\|^2 \le (1+\epsilon_2)\|a\|^2 + (1+\epsilon_2^{-1})\|b\|^2$ and the Jensen's inequality, we have

$$\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \{\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\}\,\mathrm{d}s\right\|^2\right] \le (1+\epsilon_2)\gamma_{n+1}^2 \left\|\nabla U(Y_{\Gamma_n}) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2 \\ + \gamma_{n+1}\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\int_{\Gamma_n}^{\Gamma_{n+1}} \|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2\,\mathrm{d}s\right] .$$

This result and **H**1 imply,

$$\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\|\Delta_{n+1}\|^2\right] \le \{1-\gamma_{n+1}(\kappa-2\epsilon_1) + \gamma_{n+1}L((1+\epsilon_2)\gamma_{n+1} - 2/(m+L))_+\}\|\Delta_n\|^2 \\ + ((1+\epsilon_2^{-1})\gamma_{n+1} + (2\epsilon_1)^{-1})\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2\right]\mathrm{d}s\,. \quad \text{(S25)}$$

By **H**1, the Markov property of $(Y_t)_{t\ge 0}$ and Lemma 19, we have

$$\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\|^2\right]\mathrm{d}s \\ \le L^2\left(d\gamma_{n+1}^2 + dL^2\gamma_{n+1}^4/12 + (L^2\gamma_{n+1}^3/2)\|Y_{\Gamma_n} - x^\star\|^2\right) .$$

Plugging this bound in (S25) concludes the proof. □

Lemma S9. *Let $(\gamma_k)_{k\ge 1}$ be a nonincreasing sequence of positive numbers. Let $\varpi, \beta > 0$ be positive constants satisfying $\varpi^2 \le 4\beta$ and $\tau > 0$. Assume there exists $N \ge 1$, $\gamma_N \le \tau$ and $\gamma_N\varpi \le 1$. Then for all $n \ge 0$, $j \ge 2$*

(i)

$$\sum_{i=1}^{n+1}\prod_{k=i+1}^{n+1}(1-\gamma_k\varpi + \gamma_k\beta(\gamma_k-\tau)_+)\gamma_i^j \le \sum_{i=N}^{n+1}\prod_{k=i+1}^{n+1}(1-\gamma_k\varpi)\gamma_i^j \\ + \left\{\beta^{-1}\gamma_1^{j-2}\prod_{k=1}^{N-1}(1+\gamma_k^2\beta)\right\}\prod_{k=N}^{n+1}(1-\varpi\gamma_k) .$$

(ii) For all $\ell \in \{N,\dots,n\}$,

$$\sum_{i=N}^{n+1}\prod_{k=i+1}^{n+1}(1-\gamma_k\varpi)\gamma_i^j \le \exp\left(-\sum_{k=\ell}^{n+1}\varpi\gamma_k\right)\sum_{i=N}^{\ell-1}\gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi} .$$

Proof. By definition of N we have

$$\sum_{i=1}^{n+1}\prod_{k=i+1}^{n+1}(1-\gamma_k\varpi + \gamma_k\beta(\gamma_k-\tau)_+)\gamma_i^j \\ \le \sum_{i=N}^{n+1}\prod_{k=i+1}^{n+1}(1-\gamma_k\varpi)\gamma_i^j + \left\{\sum_{i=1}^{N-1}\prod_{k=i+1}^{N-1}(1+\gamma_k^2\beta)\gamma_i^j\right\}\prod_{k=N}^{n+1}(1-\gamma_k\varpi) . \quad \text{(S26)}$$

Using that $(\gamma_k)_{k\geq 1}$ is nonincreasing, we have

$$
\begin{aligned}
\sum_{i=1}^{N-1}\prod_{k=i+1}^{N-1}\left(1+\gamma_k^2\beta\right)\gamma_i^j &\leq \sum_{i=1}^{N-1}\frac{\gamma_i^{j-2}}{\beta}\left\{\prod_{k=i}^{N-1}\left(1+\gamma_k^2\beta\right)-\prod_{k=i+1}^{N-1}\left(1+\gamma_k^2\beta\right)\right\} \\
&\leq \beta^{-1}\gamma_1^{j-2}\prod_{k=1}^{N-1}\left(1+\gamma_k^2\beta\right) \; .
\end{aligned}
$$

Plugging this inequality in (S26) concludes the proof of (i). Let $\ell \in \{N,\ldots,n+1\}$. Since $(\gamma_k)_{k\geq 1}$ is nonincreasing and for every $x\in\mathbb{R}$, $(1+x)\leq \mathrm{e}^x$, we get

$$
\begin{aligned}
\sum_{i=N}^{n+1}\prod_{k=i+1}^{n+1}\left(1-\gamma_k\varpi\right)\gamma_i^j &= \sum_{i=N}^{\ell-1}\prod_{k=i+1}^{n+1}\left(1-\gamma_k\varpi\right)\gamma_i^j + \sum_{i=\ell}^{n+1}\prod_{k=i+1}^{n+1}\left(1-\gamma_k\varpi\right)\gamma_i^j \\
&\leq \sum_{i=N}^{\ell-1}\exp\left(-\sum_{k=i+1}^{n+1}\varpi\gamma_k\right)\gamma_i^j + \gamma_\ell^{j-1}\sum_{i=\ell}^{n+1}\prod_{k=i+1}^{n+1}\left(1-\gamma_k\varpi\right)\gamma_i \\
&\leq \exp\left(-\sum_{k=\ell}^{n+1}\varpi\gamma_k\right)\sum_{i=N}^{\ell-1}\gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi} \; .
\end{aligned}
$$

$\square$

Lemma S10. *Let $(\gamma_k)_{k\geq 1}$ be a nonincreasing sequence of positive numbers, $\varpi,\beta,\tau>0$ be positive real numbers, and $N\geq 1$ satisfying the assumptions of Lemma S9. Let $\mathsf{P}\in\mathbb{N}^*$, $C_i\geq 0$, $i=0,\ldots,\mathsf{P}$ be positive constants and $(u_n)_{n\geq 0}$ be a sequence of real numbers with $u_0\geq 0$ satisfying for all $n\geq 0$*

$$
u_{n+1}\leq (1-\gamma_{n+1}\varpi+\beta\gamma_{n+1}(\gamma_{n+1}-\tau)_+)u_n+\sum_{i=0}^{\mathsf{P}}C_j\gamma_{n+1}^{j+2} \; .
$$

Then for all $n\geq 1$,

$$
\begin{aligned}
u_n \leq \left\{\prod_{k=1}^{N-1}\left(1+\beta\gamma_k^2\right)\right\}\prod_{k=N}^{n}\left(1-\gamma_k\varpi\right)u_0 + \sum_{j=0}^{\mathsf{P}}C_j\sum_{i=N}^{n}\prod_{k=i+1}^{n}\left(1-\gamma_k\varpi\right)\gamma_i^{j+2} \\
+\left\{\sum_{j=0}^{\mathsf{P}}C_j\beta^{-1}\gamma_1^j\prod_{k=1}^{N-1}\left(1+\gamma_k^2\beta\right)\right\}\prod_{k=N}^{n}\left(1-\varpi\gamma_k\right) \; .
\end{aligned}
$$

Proof. This is a consequence of a straightforward induction and Lemma S9-(i). $\square$

Proposition S11. *Assume **H**1, **H**2 and **G**1. Let $x^\star$ be the unique minimizer of U. Let $\zeta_0\in\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$, $(Y_t,\overline{Y}_t)_{t\geq 0}$ such that $(Y_0,\overline{Y}_0)$ is distributed according to ζ_0 and given by (9). Then for all $n\geq 0$ and $t\in[\Gamma_n,\Gamma_{n+1}]$:*

$$
\mathbb{E}\left[\left\|Y_t-\overline{Y}_t\right\|^2\right]\leq \tilde{u}_n^{(1)}\mathbb{E}\left[\left\|Y_0-\overline{Y}_0\right\|^2\right]+\tilde{u}_n^{(4)}+\tilde{u}_{t,n}^{(5)} \; ,
$$

where

$$\tilde{u}_n^{(1)}(\gamma) = \left\{ \prod_{k=1}^{n_1-1} \left(1 + L^2(1+\rho)\gamma_k^2\right) \right\} \prod_{k=n_1}^{n} (1 - \kappa\gamma_k/2) \ , \tag{S27}$$

$$\tilde{u}_n^{(4)}(\gamma) = \sum_{i=n_1}^{n} \gamma_i^2 \mathsf{a}(\gamma_i) \prod_{k=i+1}^{n} (1 - \kappa\gamma_k/2) \\ + \mathsf{a}(\gamma_1)(L^2(1+\rho))^{-1} \left\{ \prod_{k=1}^{n_1-1} (1 + \gamma_k^2(1+\rho)L^2) \right\} \prod_{k=n_1}^{n} (1 - \kappa\gamma_k/2) \ ,$$

where for $\gamma > 0$,

$$\mathsf{a}(\gamma) = \left\{2\kappa^{-1} + (1+\rho^{-1})\gamma\right\} (dL^2 + \delta L^4\gamma/2 + dL^4\gamma^2/12) \ , \\ \delta = \max_{i\geq 1} \left\{ \mathrm{e}^{-2m\Gamma_{i-1}} \mathbb{E}\left[\|Y_0 - x^\star\|^2\right] + (1 - \mathrm{e}^{-2m\Gamma_{i-1}})(d/m) \right\} \ ,$$

and

$$\tilde{u}_{t,n}^{(5)}(\gamma) = \frac{m+L}{2} \left(\frac{(t-\Gamma_n)^3 L^2}{3} \mathsf{G}_{1,n}(\mu_0, \gamma) + (t-\Gamma_n)^2 d \right) \ ,$$

where $\mathsf{G}_{1,n}(\mu_0,\gamma)$ *is given by* (S19) *and* μ_0 *is the initial distribution of* $\overline{Y}_0$.

Proof. Lemma S8 with $\epsilon_1 = \kappa/4$ and $\epsilon_2 = \rho$, **G**1, Lemma S9, ((i)) imply for all $n \geq 0$

$$\mathbb{E}\left[\left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2\right] \leq \tilde{u}_n^{(1)}(\gamma)\mathbb{E}\left[\left\|Y_0 - \overline{Y}_0\right\|^2\right] + \tilde{u}_n^{(4)}(\gamma) \ . \tag{S28}$$

Now let $n \geq 0$ and $t \in [\Gamma_n, \Gamma_{n+1}]$. By (9),

$$\left\|Y_t - \overline{Y}_t\right\|^2 = \left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2 - 2\int_{\Gamma_n}^{t} \left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_s \right\rangle \mathrm{d}s \ . \tag{S29}$$

Moreover for all $s \in [\Gamma_n, \Gamma_{n+1}]$, by (3) we get

$$\left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_s \right\rangle = \left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_{\Gamma_n} + \overline{Y}_{\Gamma_n} - \overline{Y}_s \right\rangle \\ \geq (m+L)^{-1} \left\|\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2 + \left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), \overline{Y}_{\Gamma_n} - \overline{Y}_s \right\rangle \ . \tag{S30}$$

Since $|\langle a,b\rangle| \leq (m+L)^{-1}\|a\|^2 + (m+L)\|b\|^2/4$, we have

$$\left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), \overline{Y}_{\Gamma_n} - \overline{Y}_s \right\rangle \\ \geq -(m+L)^{-1} \left\|\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2 - (m+L)\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2/4 \ .$$

Using this inequality in (S30), we get

$$\left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_s \right\rangle \geq -(m+L)\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2/4 \ ,$$

and (S29) becomes

$$\left\|Y_t - \overline{Y}_t\right\|^2 \leq \left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2 + ((m+L)/2) \int_{\Gamma_n}^t \left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2 \mathrm{d}s \tag{S31}$$

Therefore, by the previous inequality, it remains to bound the expectation of $\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2$. By (9) and using $\nabla U(x^\star) = 0$,

$$\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2 = \left\|-(s-\Gamma_n)(\nabla U(\overline{Y}_{\Gamma_n}) - \nabla U(x^\star)) + \sqrt{2}(B_s - B_{\Gamma_n})\right\|^2 .$$

Then taking the expectation, using the Markov property of $(B_t)_{t\geq 0}$ and **H**1, we have

$$\mathbb{E}\left[\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2\right] \leq (s-\Gamma_n)^2 L^2 \mathbb{E}\left[\left\|\overline{Y}_{\Gamma_n} - x^\star\right\|^2\right] + 2(s-\Gamma_n)d . \tag{S32}$$

The proof follows from taking the expectation in (S31), combining (S28)-(S32), and using Theorem S7. □

Theorem S12. *Assume **H**1, **H**2 and **G**1. Then for all $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $n \geq 1$,*

$$W_2^2(\mu_0 Q_\gamma^n, \pi) \leq \tilde{u}_n^{(1)}(\gamma) W_2^2(\mu_0, \pi) + \tilde{u}_n^{(2)}(\gamma) , \tag{S33}$$

where $(\tilde{u}_n^{(1)})_{n\geq 0}$ is given by (S27) and

$$\tilde{u}_n^{(2)}(\gamma) = \sum_{i=n_1}^n \gamma_i^2 \mathsf{b}(\gamma_i) \prod_{k=i+1}^n (1-\kappa\gamma_k/2) \\ + \mathsf{b}(\gamma_1)(L^2(1+\rho))^{-1} \left\{\prod_{k=1}^{n_1-1}(1+\gamma_k^2(1+\rho)L^2)\right\} \prod_{k=n_1}^n (1-\kappa\gamma_k/2) , \tag{S34}$$

where

$$\mathsf{b}(\gamma) = \left\{2\kappa^{-1} + (1+\rho^{-1})\gamma\right\}(dL^2 + dL^4\gamma/(2m) + dL^4\gamma^2/12) .$$

Proof. Let ζ_0 be an optimal transference plan of μ_0 and π. Let $(Y_t, \overline{Y}_t)_{t\geq 0}$ with $(Y_0, \overline{Y}_0)$ distributed according to ζ_0 and defined by (9). By definition of W_2 and since for all $t \geq 0$, π is invariant for P_t, $W_2^2(\mu_0 Q^n, \pi) \leq \mathbb{E}[\|Y_{\Gamma_n} - X_{\Gamma_n}\|^2]$. Then the proof follows from Proposition S11 since Y_0 is distributed according to π and by Theorem 1-(ii), which shows that $\delta \leq d/m$. □

4.1 Explicit bound based on Theorem S12 for $\gamma_k = \gamma_1 k^\alpha$ with $\alpha \in (0,1]$

We give here a bound on the sequences $(\tilde{u}_n^{(1)}(\gamma))_{n\geq 1}$ and $(\tilde{u}_n^{(2)}(\gamma))_{n\geq 1}$ for $(\gamma_k)_{k\geq 1}$ defined by $\gamma_1 > 0$ and $\gamma_k = \gamma_1 k^{-\alpha}$ for $\alpha \in (0,1]$. Recall that $\boldsymbol{\psi}_\beta$ is given by (S10). First note, since $(\gamma_k)_{k\geq 1}$ is nonincreasing, for all $n \geq 1$, we have

$$\tilde{u}_n^{(2)}(\gamma) \leq \sum_{j=0}^{1} \mathsf{C}_j \sum_{i=n_1}^{n} \gamma_i^{j+2} \prod_{k=i+1}^{n} (1-\kappa\gamma_k/2) + \sum_{j=0}^{1} \mathsf{C}_j (L^2(1+\rho))^{-1} \gamma_1^j \left\{ \prod_{k=1}^{n_1-1} (1+\gamma_k^2(1+\rho)L^2) \right\} \prod_{k=n_1}^{n} (1-\kappa\gamma_k/2) \ , \quad \text{(S35)}$$

where

$$\mathsf{C}_1 = \mathsf{b} d L^2 \ , \mathsf{C}_2 = \mathsf{b}(dL^4/(2m) + \gamma_1 d L^4/12) \ , \mathsf{b} = 2\kappa^{-1} + (1+\rho^{-1})\gamma_1 \ .$$

1. For $\alpha = 1$ and $n_1 = 1$, by (S11) and (S12), we have

$$\tilde{u}_n^{(1)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2}$$
$$\tilde{u}_n^{(2)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^{1} \mathsf{C}_j \left\{ \gamma_1^{j+2}(\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n+1)+1) + (L^2(1+\rho))^{-1}\gamma_1^j \right\} \ .$$

For $n_1 > 1$, since $(\gamma_k)_{k\geq 0}$ is non increasing, using again (S11), (S12), and the bound for $t \in \mathbb{R}$, $(1+t) \leq \mathrm{e}^t$

$$\tilde{u}_n^{(1)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \exp\left\{ \kappa\gamma_1 \boldsymbol{\psi}_0(n_1)/2 + L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{-1}(n_1-1)+1) \right\}$$
$$\tilde{u}_n^{(2)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^{1} \mathsf{C}_j \Big(\gamma_1^{j+2}(\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n+1) - \boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n_1)+1)$$
$$+ (\gamma_1^j/(L^2(1+\rho)) \exp\left\{ \kappa\gamma_1 \boldsymbol{\psi}_0(n_1)/2 + L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{-1}(n_1-1)+1) \right\} \Big) \ .$$

Thus, for $\gamma_1 > \kappa/2$, we get a bound in $\mathcal{O}(n^{-1})$.

2. For $\alpha \in (0,1)$ and $n_1 = 1$, by (S11) and Lemma S9-(ii) applied with $\ell = \lceil n/2 \rceil$, we have

$$\tilde{u}_n^{(1)}(\gamma) \leq \exp\left(-(\kappa\gamma_1/2)\boldsymbol{\psi}_{\alpha-1}(n+1)\right)$$
$$\tilde{u}_n^{(2)}(\gamma) \leq \sum_{j=0}^{1} \mathsf{C}_j \left\{ \gamma_1^{j+2} \left(\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2 \rceil)+1 \right) \mathrm{e}^{-(\kappa\gamma_1/2)(\boldsymbol{\psi}_{1-\alpha}(n+1)-\boldsymbol{\psi}_{1-\alpha}(\lceil n/2 \rceil))} \right.$$
$$\left. + 2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} + \gamma_1^j \mathrm{e}^{-(\kappa\gamma_1/2)\boldsymbol{\psi}_{\alpha-1}(n+1)} \right\} \ .$$

For $n_1 > 1$ and $\lceil n/2 \rceil \geq n_1$, since $(\gamma_k)_{k\geq 0}$ is non increasing, using again (S11), and Lemma S9-(ii) applied with $\ell = \lceil n/2 \rceil$, and the bound for $t \in \mathbb{R}$, $(1+t) \leq \mathrm{e}^t$, we get

$$\tilde{u}_n^{(1)}(\gamma) \le \mathrm{e}^{-\kappa\gamma_1(\boldsymbol{\psi}_{\alpha-1}(n+1)-\boldsymbol{\psi}_{1-\alpha}(n_1))/2+L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{1-2\alpha}(n_1-1)+1)}$$

$$\tilde{u}_n^{(2)}(\gamma) \le \sum_{j=0}^{1} \mathsf{C}_j \Big\{2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)}$$

$$+ \gamma_1^{j+2}\left(\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2\rceil) - \boldsymbol{\psi}_{1-\alpha(j+2)}(n_1) + 1\right) \mathrm{e}^{-(\kappa\gamma_1/2)(\boldsymbol{\psi}_{1-\alpha}(n+1)-\boldsymbol{\psi}_{1-\alpha}(\lceil n/2\rceil))}$$

$$+(\gamma_1^j/(L^2(1+\rho))\mathrm{e}^{-\kappa\gamma_1(\boldsymbol{\psi}_{\alpha-1}(n+1)-\boldsymbol{\psi}_{1-\alpha}(n_1))/2+L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{1-2\alpha}(n_1-1)+1)}\Big\} .$$

5 Explicit bounds on the MSE

Without loss of generality, assume that $\|f\|_{\mathrm{Lip}} = 1$. In the following, denote by $\Omega(x) = \|x - x^\star\|^2 + d/m$ and C a constant (which may take different values upon each appearance), which does not depend on m, L, γ_1, α and $\|x - x^\star\|$.

5.1 Explicit bounds based on Theorem 5

1. First for $\alpha = 0$, recall by Theorem 5 and (S6) we have for all $p \ge 1$,

   $$W_2^2(\delta_x R_\gamma^p, \pi) \le 2\Omega(x)(1-\kappa\gamma_1/2)^p + 2\kappa^{-1}(\mathsf{A}_0\gamma_1 + \mathsf{A}_1\gamma_1^2) ,$$

   where A_0 and A_1 are given by (S7) and (S8) respectively. So by (15) and Lemma 21, we have the following bound for the bias

   $$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \le C\left(\frac{\kappa^{-1}\exp(-\kappa N\gamma_1/2)\Omega(x)}{\gamma_1 n} + \kappa^{-1}\mathsf{A}_0\gamma_1\right) .$$

   Therefore plugging this inequality and the one given by Theorem 10 implies:

   $$\mathrm{MSE}_f^{N,n} \le C\left(\kappa^{-1}\mathsf{A}_0\gamma_1 + \frac{\kappa^{-2} + \kappa^{-1}\exp(-\kappa N\gamma_1/2)\Omega(x)}{n\gamma_1}\right) . \tag{S36}$$

   So with fixed γ_1 this bound is of order γ_1. If we fix the number of iterations n, we can optimize the choice of γ_1. Set

   $$\gamma_{\star,0}(n) = (\kappa^{-1}\mathsf{A}_0)^{-1}(C_{\mathrm{MSE},0}/n)^{1/2} , \text{ where } C_{\mathrm{MSE},0} = \kappa^{-3}\mathsf{A}_0 ,$$

   and (S36) becomes if $\gamma_1 \leftarrow \gamma_{\star,0}(n)$,

   $$\mathrm{MSE}_f^{N,n} \le C(C_{\mathrm{MSE},0}n)^{-1/2}\left(\kappa^{-1}\exp(-\kappa N\gamma_{\star,0}(n)/2)\Omega(x) + C_{\mathrm{MSE},0}\right) .$$

   Setting $N_0(n) = 2(\kappa\gamma_{\star,0}(n))^{-1}\log(\Omega(x))$, we end up with

   $$\mathrm{MSE}_f^{N_0(n),n} \le C(C_{\mathrm{MSE},0}/n)^{1/2} .$$

   Note that $N_0(n)$ is of order $n^{1/2}$.

2. For $\alpha \in (0, 1/2)$ by Theorem 5, Lemma 21, (S11) and (S13), we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C \left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1}{(1-2\alpha)n^\alpha} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x)}{\gamma_1 n^{1-\alpha}} \right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1}{(1-2\alpha)n^\alpha} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right) . \tag{S37}$$

At fixed γ_1, this bound is of order $n^{-\alpha}$, and is better than (S36) for $(\gamma_k)_{k\geq 1}$ constant. If we fix the number of iterations n, we can optimize the choice of γ_1 again. Set

$$\gamma_{\star,\alpha}(n) = (\kappa^{-1}\mathsf{A}_0/(1-2\alpha))^{-1}(C_{\mathrm{MSE},\alpha}/n^{1-2\alpha})^{1/2} \ , \ \text{where } C_{\mathrm{MSE},\alpha} = \kappa^{-3}\mathsf{A}_0/(1-2\alpha) \ ,$$

(S37) becomes with $\gamma_1 \leftarrow \gamma_{\star,\alpha}(n)$,

$$\begin{aligned} &\mathrm{MSE}_f^{N,n} \\ &\quad \leq C(C_{\mathrm{MSE},\alpha} n)^{-1/2} \left(\kappa^{-1}\exp\left\{-\kappa N^{1-\alpha}\gamma_{\star,\alpha}(n)/(2(1-\alpha))\right\}\Omega(x) + C_{\mathrm{MSE},\alpha} \right) . \end{aligned}$$

Setting $N_\alpha(n) = \{2(1-\alpha)(\kappa\gamma_{\star,\alpha}(n))^{-1}\log(\Omega(x))\}^{1/(1-\alpha)}$, we end up with

$$\mathrm{MSE}_f^{N_\alpha(n),n} \leq C(C_{\mathrm{MSE},\alpha}/n)^{1/2} .$$

It is worthwhile to note that the order of $N_\alpha(n)$ in n is $n^{(1-2\alpha)/(2(1-\alpha))}$, and $C_{\mathrm{MSE},\alpha}$ goes to infinity as $\alpha \to 1/2$.

3. If $\alpha = 1/2$, by Theorem 5, Lemma 21, (S11) and (S13), we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C \left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1\log(n)}{n^{1/2}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1/2}/4\right\}\Omega(x)}{\gamma_1 n^{1/2}} \right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1\log(n)}{n^{1/2}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1/2}/4\right\}\Omega(x) + \kappa^{-2}}{\gamma_1 n^{1/2}} \right) . \tag{S38}$$

At fixed γ_1, the order of this bound is $\log(n)n^{-1/2}$, and is the best bound for the MSE. Fix the number of iterations n, and we now optimize the choice of γ_1. Set

$$\gamma_{\star,1/2}(n) = (\kappa^{-1}\mathsf{A}_0)^{-1}(C_{\mathrm{MSE},1/2}/\log(n))^{1/2} \ , \ \text{where } C_{\mathrm{MSE},1/2} = \kappa^{-3}\mathsf{A}_0 \ ,$$

and (S38) becomes with $\gamma_1 \leftarrow \gamma_{\star,1/2}(n)$,

$$\begin{aligned}&\mathrm{MSE}_f^{N,n}\\ &\quad\leq C\left(\frac{\log(n)}{nC_{\mathrm{MSE},1/2}}\right)^{1/2}\left(\kappa^{-1}\exp\left\{-\kappa N^{1/2}\gamma_{\star,1/2}(n)/4\right\}\Omega(x)+\frac{C_{\mathrm{MSE},1/2}}{\log(n)}\right) .\end{aligned}$$

Setting $N_{1/2}(n) = (4(\kappa\gamma_{\star,1/2}(n))^{-1}\log(\Omega(x)))^2$, we end up with

$$\mathrm{MSE}_f^{N_{1/2}(n),n} \leq C\left(\frac{\log(n)C_{\mathrm{MSE},1/2}}{n}\right)^{1/2} .$$

4. For $\alpha \in (1/2, 1]$, by Theorem 5, Lemma 21, (S11) and (S13), we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C\left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x)}{\gamma_1 n^{1-\alpha}}\right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C\left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}}\right) .$$

For fixed γ_1, the MSE is of order $n^{1-\alpha}$, and is worse than for $\alpha \in [0, 1/2]$. For a fixed number of iteration n, optimizing γ_1 would imply to choose $\gamma_1 \to +\infty$ as $n \to +\infty$. Therefore, in that case, the best choice of γ_1 is the largest possible value $1/(m+L)$.

5. For $\alpha = 1$, by Theorem 5, Lemma 21, (S11) and (S13), the bias is upper bounded by

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C\left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1}{\log(n)} + \frac{\kappa^{-1}N^{-\kappa\gamma_1/2}\Omega(x)}{\gamma_1\log(n)}\right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C\left(\frac{\kappa^{-1}\mathsf{A}_0\gamma_1}{\log(n)} + \frac{\kappa^{-1}N^{-\kappa\gamma_1/2}\Omega(x) + \kappa^{-2}}{\gamma_1\log(n)}\right) .$$

For fixed γ_1, the order of the MSE is $(\log(n))^{-1}$. For a fixed number of iterations, the conclusions are the same than for $\alpha \in (1/2, 1)$.

5.2 Explicit bound based on Theorem 8

1. First for $\alpha = 0$, recall by Theorem 8 and (S14) we have for all $p \geq 1$,

$$W_2^2(\delta_x R_\gamma^p, \pi) \leq 2\Omega(x)(1 - \kappa\gamma_1/2)^p + 2\kappa^{-1}(\mathsf{B}_0\gamma_1 + \mathsf{B}_1\gamma_1^2) ,$$

where B_0 and B_1 are given by (S15) and (S16) respectively. So by and Lemma 21, we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C\left(\frac{\kappa^{-1}\exp(-\kappa N\gamma_1/2)\Omega(x)}{\gamma_1 n} + \kappa^{-1}\mathsf{B}_0\gamma_1^2\right) .$$

Therefore plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C\left(\kappa^{-1}\mathsf{B}_0\gamma_1^2 + \frac{\kappa^{-2} + \kappa^{-1}\exp(-\kappa N\gamma_1/2)\Omega(x)}{n\gamma_1}\right) . \tag{S39}$$

So with fixed γ_1 this bound is of order γ_1. If we fix the number of iterations n, we can optimize the choice of γ_1. Set

$$\gamma_{\star,0}(n) = (\kappa\mathsf{B}_0 n)^{-1/3} ,$$

and (S36) becomes if $\gamma_1 \leftarrow \gamma_{\star,0}(n)$,

$$\mathrm{MSE}_f^{N,n} \leq C(\mathsf{B}_0^{-1/2}n)^{-2/3}\left(\kappa^{-4/3}\exp(-\kappa N\gamma_{\star,0}(n)/2)\Omega(x) + \kappa^{-5/3}\right) .$$

Setting $N_0(n) = 2(\kappa\gamma_{\star,0}(n))^{-1}\log(\Omega(x))$, we end up with

$$\mathrm{MSE}_f^{N_0(n),n} \leq C(\mathsf{B}_0^{-1/2}\kappa^{5/2}n)^{-2/3} .$$

Note that $N_0(n)$ is of order $n^{1/3}$.

2. For $\alpha \in (0, 1/3)$ by Theorem 8, Lemma 21, (S11) and (S17), we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C\left(\frac{\kappa^{-1}\mathsf{B}_0\gamma_1^2}{(1-3\alpha)n^{2\alpha}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x)}{\gamma_1 n^{1-\alpha}}\right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C\left(\frac{\kappa^{-1}\mathsf{B}_0\gamma_1^2}{(1-3\alpha)n^{2\alpha}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}}\right) . \tag{S40}$$

If we fix the number of iterations n, we can optimize the choice of γ_1 again. Set

$$\gamma_{\star,\alpha}(n) = (n^{1-3\alpha}\kappa\mathsf{B}_0/(1-3\alpha))^{-1/3} ,$$

(S40) becomes with $\gamma_1 \leftarrow \gamma_{\star,\alpha}(n)$,

$$\mathrm{MSE}_f^{N,n} \leq C(\mathsf{B}_0^{-1/2} n)^{-2/3} \left(\kappa^{-4/3} \exp(-\kappa N \gamma_{\star,0}(n)/(2(1-\alpha)))\Omega(x) + \kappa^{-5/3}(1-3\alpha)^{-1/3}\right) .$$

Setting $N_\alpha(n) = \{(\kappa\gamma_{\star,\alpha}(n))^{-1}\log(\Omega(x))\}^{1/(1-\alpha)}$, we end up with

$$\mathrm{MSE}_f^{N_\alpha(n),n} \leq C(\mathsf{B}_0^{-1/2}\kappa^{5/2} n)^{-2/3} .$$

It is worthwhile to note that the order of $N_\alpha(n)$ in n is $n^{(1-3\alpha)/(3(1-\alpha))}$.

3. If $\alpha = 1/3$, by Theorem 8, Lemma 21, (S11) and (S17), we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C\left(\frac{\kappa^{-1}\mathsf{B}_0\gamma_1^2\log(n)}{n^{2/3}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{2/3}/4\right\}\Omega(x)}{\gamma_1 n^{2/3}}\right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C\left(\frac{\kappa^{-1}\mathsf{B}_0\gamma_1^2\log(n)}{n^{2/3}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{2/3}/4\right\}\Omega(x) + \kappa^{-2}}{\gamma_1 n^{2/3}}\right) . \tag{S41}$$

At fixed γ_1, the order of this bound is $\log(n)n^{-2/3}$, and is the best bound for the MSE. Fix the number of iterations n, and we now optimize the choice of γ_1. Set

$$\gamma_{\star,1/2}(n) = (\kappa\mathsf{B}_0\log(n))^{-1/3} ,$$

and (S41) becomes with $\gamma_1 \leftarrow \gamma_{\star,1/2}(n)$,

$$\mathrm{MSE}_f^{N,n} \leq C\left(\frac{\log(n)\mathsf{B}_0}{n^2}\right)^{1/3}\left(\kappa^{-4/3}\exp\left\{-\kappa N^{1/2}\gamma_{\star,1/2}(n)/4\right\}\Omega(x) + \kappa^{-5/3}\right) .$$

Setting $N_{1/2}(n) = (4(\kappa\gamma_{\star,1/2}(n))^{-1}\log(\Omega(x)))^{3/2}$, we end up with

$$\mathrm{MSE}_f^{N_{1/2}(n),n} \leq C\left(\frac{\log(n)\mathsf{B}_0}{\kappa^5 n^2}\right)^{1/3} .$$

We can see that we obtain a worse bound than for $\alpha = 0$ and $\alpha \in (0, 1/3)$.

4. For $\alpha \in (1/3, 1]$, by Theorem 8, Lemma 21, (S11) and (S17), we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C\left(\frac{\kappa^{-1}\mathsf{B}_0\gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x)}{\gamma_1 n^{1-\alpha}}\right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C\left(\frac{\kappa^{-1}\mathsf{B}_0\gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1}\exp\left\{-\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha))\right\}\Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}}\right) .$$

For fixed γ_1, the MSE is of order $n^{1-\alpha}$, and is worse than for $\alpha = 1/2$. For a fixed number of iteration n, optimizing γ_1 would imply to choose $\gamma_1 \to +\infty$ as $n \to +\infty$. Therefore, in that case, the best choice of γ_1 is the largest possible value $1/(m+L)$.

5. For $\alpha = 1$, by Theorem 8, Lemma 21, (S11) and (S13), the bias is upper bounded by

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C \left(\frac{\kappa^{-1} \mathsf{B}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa\gamma_1/2} \Omega(x)}{\gamma_1 \log(n)} \right) .$$

Plugging this inequality and the one given by Theorem 10 implies:

$$\mathrm{MSE}_f^{N,n} \leq C \left(\frac{\kappa^{-1} \mathsf{B}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa\gamma_1/2} \Omega(x) + \kappa^{-2}}{\gamma_1 \log(n)} \right) .$$

For fixed γ_1, the order of the MSE is $(\log(n))^{-1}$. For a fixed number of iterations, the conclusions are the same than for $\alpha \in (1/2, 1)$.

References

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