Supplement to ” Non-asymptotic convergence analysis for the Unadjusted Langevin Algorithm ”

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1. Discussion of Theorem 16

Note that

$$u_n^{(2)}(\gamma) \leq \sum_{i=1}^{n} \left\{ \mathsf{A}_0 \gamma_i^2 + \mathsf{A}_1 \gamma_i^3 \right\} \prod_{k=i+1}^{n} (1 - \kappa \gamma_k / 2) \,, \tag{S1}$$

where κ is given by (12), and

$$\mathsf{A}_0 \overset{\text{def}}{=} 2L^2 \kappa^{-1} d \,, \tag{S2}$$

$$\mathsf{A}_1 \overset{\text{def}}{=} 2L^2 \kappa^{-1} d (m+L)^{-1} + dL^4 (\kappa^{-1} + (m+L)^{-1})(m^{-1} + 6^{-1}(m+L)^{-1}) \,. \tag{S3}$$

If $(\gamma_k)_{k \geq 1}$ is a constant step size ,$\gamma_k = \gamma$ for all $k \geq 1$, then a straightforward consequence of Theorem 16 and (S1) is the following result, which gives the minimal number of iterations n and a step-size γ to get $W_2(\delta_{x^\star} Q_\gamma^n, \pi)$ smaller than $\epsilon > 0$.

Corollary S1 (of Theorem 16)**.** *Assume* ***H****1 and* ***H****3. Let $x^\star$ be the unique minimizer of U. Let $x \in \mathbb{R}^d$ and $\epsilon > 0$. Set for all $k \in \mathbb{N}$, $\gamma_k = \gamma$ with*

$$\gamma = \frac{-2\mathsf{A}_0 \kappa^{-1} + (4\kappa^{-1}(\mathsf{A}_0^2 \kappa^{-1} + \epsilon^2 \mathsf{A}_1))^{1/2}}{4\mathsf{A}_1 \kappa^{-1}} \wedge (m+L)^{-1} \,, \tag{S4}$$

$$n = \left\lceil 2(\kappa\gamma)^{-1} \left\{ -\log(\epsilon^2/4) + \log(2d/m) \right\} \right\rceil \,. \tag{S5}$$

Then $W_2(\delta_{x^\star} Q_n^\gamma, \pi) \leq \epsilon$.

Note that if γ is given by (S4), and is different from $1/(m+L)$, then $\gamma \leq \epsilon(4\mathsf{A}_1\kappa^{-1})^{-1/2}$ and $2\kappa^{-1}(\mathsf{A}_0\gamma + \mathsf{A}_1\gamma^2) = \epsilon^2/2$. Therefore,

$$\gamma \geq (\epsilon^2 \kappa/4) \left\{ \mathsf{A}_0 + \epsilon(\mathsf{A}_1/(4\kappa))^{1/2} \right\}^{-1} \,.$$

It is shown in [S1, Corollary 1] that under **H**3, for constant step size for any $\epsilon > 0$, we can choose γ and $n \geq 1$ such that if for all $k \geq 1$, $\gamma_k = \gamma$, then $\|\nu^\star Q_n^\gamma - \pi\|_{\mathrm{TV}} \leq \epsilon$ where $\nu^\star$ is the Gaussian measure on $\mathbb{R}^d$ with mean $x^\star$ and covariance matrix $L^{-1}\,\mathrm{I}_d$. We stress that the results in [S1, corollary 1] hold only for a particular choice of the initial distribution $\nu^\star$, (which might seem a rather artificial assumption) whereas Theorem 16 hold for any initial distribution in $\mathcal{P}_2(\mathbb{R}^d)$.

We compare the optimal value of γ and n obtained from Corollary S1 with those given in [S1, Corollary 1]. This comparison is summarized in Table 1 and Table 2; for simplicity, we provide only the dependencies of the optimal stepsize γ and minimal number of simulations n as a function of the dimension d, the precision ϵ and the constants m, L. It can be seen that the dependency on the dimension is significantly better than those in [S1, Corollary 1].

Parameter	d	ϵ	L	m
Theorem 16 and (30)	d^{-1}	ϵ^2	L^{-2}	m^2
[S1, Corollary 1]	d^{-2}	ϵ^2	L^{-2}	m

TABLE 1
Dependencies of γ

Parameter	d	ϵ	L	m
Theorem 16 and (30)	$d\log(d)$	$\epsilon^{-2}\left\|\log(\epsilon)\right\|$	L^2	$\left\|\log(m)\right\| m^{-3}$
[S1, Corollary 1]	d^3	$\epsilon^{-2}\left\|\log(\epsilon)\right\|$	L^3	$\left\|\log(m)\right\| m^{-2}$

TABLE 2
Dependencies of n

1.1. Explicit bounds for $\gamma_k = \gamma_1 k^{\alpha}$ with $\alpha \in (0, 1]$

We give here a bound on the sequences $(u_n^{(1)}(\gamma))_{n\geq 0}$ and $(u_n^{(2)}(\gamma))_{n\geq 0}$ for $(\gamma_k)_{k\geq 1}$ defined by $\gamma_1 < 1/(m + L)$ and $\gamma_k = \gamma_1 k^{-\alpha}$ for $\alpha \in (0, 1]$. Also for that purpose we introduce for $t \in \mathbb{R}_+^*$,

$$\boldsymbol{\psi}_\beta(t) = \begin{cases} (t^\beta - 1)/\beta & \text{for } \beta \neq 0 \\ \log(t) & \text{for } \beta = 0 \,. \end{cases} \tag{S6}$$

We easily get for $a \geq 0$ that for all $n, p \geq 1$, $n \leq p$

$$\boldsymbol{\psi}_{1-a}(p + 1) - \boldsymbol{\psi}_{1-a}(n) \leq \sum_{k=n}^{p} k^{-a} \leq \boldsymbol{\psi}_{1-a}(p) - \boldsymbol{\psi}_{1-a}(n) + 1 \,, \tag{S7}$$

and for $a \in \mathbb{R}$

$$\sum_{k=n}^{p} k^{-a} \leq \boldsymbol{\psi}_{1-a}(p + 1) - \boldsymbol{\psi}_{1-a}(n) + 1 \,. \tag{S8}$$

1. For $\alpha = 1$, using that for all $t \in \mathbb{R}$, $(1 + t) \leq \mathrm{e}^t$ and by (S7) and (S8), we have

$$u_n^{(1)}(\gamma) \leq (n + 1)^{-\kappa\gamma_1/2} \;,\; u_n^{(2)}(\gamma) \leq (n + 1)^{-\kappa\gamma_1/2} \sum_{j=0}^{1} \mathsf{A}_j (\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n + 1) + 1) \,.$$

2. For $\alpha \in (0, 1)$, by (S7) and Lemma 39 applied with $\ell = \lceil n/2 \rceil$, where $\lceil\cdot\rceil$ is the ceiling function, we have

$$\begin{aligned} u_n^{(1)}(\gamma) &\leq \exp\left(-\kappa\gamma_1\boldsymbol{\psi}_{1-\alpha}(n + 1)/2\right) \\ u_n^{(2)}(\gamma) &\leq \sum_{j=0}^{1} \mathsf{A}_j \left(2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} + \gamma_1^{j+2}\left(\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2\rceil) + 1\right)\right. \\ &\quad \times \exp\left\{-(\kappa\gamma_1/2)(\boldsymbol{\psi}_{1-\alpha}(n + 1) - \boldsymbol{\psi}_{1-\alpha}(\lceil n/2\rceil))\right\}\,. \end{aligned} \tag{S9}$$

1.2. *Optimal strategy with a fixed number of iterations*

Corollary S2. *Assume* **H**1*,* **H**3*, and* $(\gamma_k)_{k\geq 1}$ *is a constant sequence,* $\gamma_k = \gamma$ *for all* $k \geq 1$*. Let* $n \geq 1$*, and set*

$$\gamma^{+} \stackrel{\text{def}}{=} 2(\kappa n)^{-1}(\log(\kappa n/2) + \log(2(\|x - x^\star\|^2 + d/m)) - \log(2\kappa^{-1}\mathsf{A}_0))$$
$$\gamma_{-} \stackrel{\text{def}}{=} 2(\kappa n)^{-1}(\log(\kappa n/2) + \log(2(\|x - x^\star\|^2 + d/m)) - \log(2\kappa^{-1}(\mathsf{A}_0 + 2\mathsf{A}_1(m+L)^{-1}))) \ .$$

Assume $\gamma^+ \in \left(0, (m+L)^{-1}\right)$*.Then, the optimal choice of* γ *to minimize the bound on* $W_2(\delta_x Q_n^\gamma, \pi)$ *given by Theorem 16 belongs to* $[\gamma_-, \gamma^+]$*. Moreover if* $\gamma = \gamma_+$*, then the bound on* $W_2^2(\delta_x Q_n^\gamma, \pi)$ *is equivalent to* $4\mathsf{A}_0(\kappa^2 n)^{-1}\log(\kappa n/2)$ *as* $n \to +\infty$*.*

Similarly, we have the following result.

Corollary S3. *Assume* **H**1 *and* **H**3*. Let* $(\gamma_k)_{k\geq 1}$ *be the decreasing sequence, defined by* $\gamma_k = \gamma_\alpha k^{-\alpha}$*, with* $\alpha \in (0,1)$*. Let* $n \geq 1$ *and set*

$$\gamma_\alpha \stackrel{\text{def}}{=} 2(1-\alpha)\kappa^{-1}(2/n)^{1-\alpha}\log(\kappa n/(2(1-\alpha))) \ .$$

Assume $\gamma_\alpha \in \left(0, (m+L)^{-1}\right)$*. Then the bound on* $W_2^2(\delta_x Q_n^\gamma, \pi)$ *given by Theorem 16 is smaller for large* n *than* $8(1-\alpha)\mathsf{A}_0(\kappa^2 n)^{-1}\log(\kappa n/(2(1-\alpha)))$*.*

Proof. Follows from (S1), (S9) and the choice of γ_α. □

2. Generalization of Theorem 16

In this section, we weaken the assumption $\gamma_1 \leq 1/(m+L)$ of Theorem 16. We assume now:

G1. *The sequence* $(\gamma_k)_{k\geq 1}$ *is non-increasing, and there exists* $\rho > 0$ *and* n_1 *such that* $(1+\rho)\gamma_{n_1} \leq 2/(m+L)$*.*

Under **G**1, we denote by

$$n_0 \stackrel{\text{def}}{=} \min\left\{k \in \mathbb{N} \mid \gamma_k \leq 2/(m+L)\right\} \tag{S10}$$

We first give an extension of Theorem 14. Denote in the sequel $(\cdot)_+ = \max(\cdot, 0)$. Recall that under **H**3, $x^\star$ is the unique minimizer of U, and κ is defined in (14)

Theorem S4. *Assume* **H**1*,* **H**3 *and* **G**1*. Then for all* $n, p \in \mathbb{N}^*$*,* $n \leq p$*,*

$$\int_{\mathbb{R}^d} \|x - x^\star\|^2 \mu_0 Q_n^p(\mathrm{d}x) \leq E_{n,p}(\mu_0, \gamma) \ ,$$

where

$$E_{n,p}(\mu_0,\gamma) \stackrel{\text{def}}{=} \exp\left(-\sum_{k=n}^{p}\gamma_k\kappa + \sum_{k=n}^{n_0-1}L^2\gamma_k^2\right)\int_{\mathbb{R}^d}\|x - x^\star\|^2\mu_0(\mathrm{d}x)$$
$$+ 2d\kappa^{-1} + 2d\left\{\prod_{k=n}^{n_0-1}(\gamma_{n_0-1}L^2)^{-1}\left(1 + L^2\gamma_k^2\right)\right\}\exp\left(-\sum_{k=n}^{p}\kappa\gamma_k + \sum_{k=n}^{n_0-1}\gamma_k^2 mL\right) \ . \tag{S11}$$

Proof. For any $\gamma > 0$, we have for all $x \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \|y - x^\star\|^2 R_\gamma(x, \mathrm{d}y) = \|x - \gamma \nabla U(x) - x^\star\|^2 + 2\gamma d \,.$$

Using that $\nabla U(x^\star) = 0$, (12) and **H**1, we get from the previous inequality:

$$\begin{aligned}\int_{\mathbb{R}^d} \|y - x^\star\|^2 R_\gamma(x, \mathrm{d}y) &\leq (1-\kappa\gamma)\|x - x^\star\|^2 + \gamma\left(\gamma - \frac{2}{m+L}\right)\|\nabla U(x) - \nabla U(x^\star)\|^2 + 2\gamma d \\ &\leq \eta(\gamma)\|x - x^\star\|^2 + 2\gamma d \,,\end{aligned}$$

where $\eta(\gamma) = (1 - \kappa\gamma + \gamma L(\gamma - 2/(m+L))_+)$. Denote for all $k \geq 1$, $\eta_k = \eta(\gamma_k)$. By a straightforward induction, we have by definition of Q_n^p for $p, n \in \mathbb{N}$, $p \leq n$,

$$\int_{\mathbb{R}^d} \|x - x^\star\|^2 \mu_0 Q_n^p(\mathrm{d}x) \leq \prod_{k=n}^{p} \eta_k \int_{\mathbb{R}^d} \|x - x^\star\| \mu_0(\mathrm{d}x) + (2d) \sum_{i=n}^{p} \prod_{k=i+1}^{p} \eta_k \gamma_i \,, \tag{S12}$$

with the convention that for $n, p \in \mathbb{N}$, $n < p$, $\prod_p^n = 1$. For the first term of the right hand side, we simply use the bound, for all $x \in \mathbb{R}$, $(1 + x) \leq \mathrm{e}^x$, and we get by **G**1

$$\prod_{k=n}^{p} \eta_k \leq \exp\left(-\sum_{k=n}^{p} \kappa\gamma_k + \sum_{k=n}^{n_0-1} L^2\gamma_k^2\right) \,, \tag{S13}$$

where n_0 is defined in (S10). Consider now the second term in the right hand side of (S12).

$$\begin{aligned}\sum_{i=n}^{p}\prod_{k=i+1}^{p}\eta_k\gamma_i &\leq \sum_{i=n_0}^{p}\prod_{k=i+1}^{p}(1-\kappa\gamma_k)\gamma_i + \sum_{i=n}^{n_0-1}\prod_{k=i+1}^{p}\eta_k\gamma_i \\ &\leq \kappa^{-1}\sum_{i=n_0}^{p}\left\{\prod_{k=i+1}^{p}(1-\kappa\gamma_k) - \prod_{k=i}^{p}(1-\kappa\gamma_k)\right\} + \left\{\sum_{i=n}^{n_0-1}\prod_{k=i+1}^{n_0-1}\left(1+L^2\gamma_k^2\right)\gamma_i\right\}\prod_{k=n_0}^{p}(1-\kappa\gamma_k)\end{aligned} \tag{S14}$$

Since $(\gamma_k)_{k\geq 1}$ is nonincreasing, we have

$$\begin{aligned}\sum_{i=n}^{n_0-1}\prod_{k=i+1}^{n_0-1}\left(1+L^2\gamma_k^2\right)\gamma_i &= \sum_{i=n}^{n_0-1}(\gamma_i L^2)^{-1}\left\{\prod_{k=i}^{n_0-1}\left(1+L^2\gamma_k^2\right) - \prod_{k=i+1}^{n_0-1}\left(1+L^2\gamma_k^2\right)\right\} \\ &\leq \prod_{k=n}^{n_0-1}(\gamma_{n_0-1}L^2)^{-1}\left(1+L^2\gamma_k^2\right) \,.\end{aligned}$$

Furthermore for $k < n_0$ $\gamma_k > 2/(m+L)$. This implies with the bound $(1+x) \leq \mathrm{e}^x$ on $\mathbb{R}$:

$$\prod_{k=n_0}^{p}(1-\kappa\gamma_k) \leq \exp\left(-\sum_{k=n}^{p}\kappa\gamma_k\right)\exp\left(\sum_{k=n}^{n_0-1}\kappa\gamma_k\right) \leq \exp\left(-\sum_{k=n}^{p}\kappa\gamma_k\right)\exp\left(\sum_{k=n}^{n_0-1}\gamma_k^2 mL\right) \,.$$

Using the two previous inequalities in (S14), we get

$$\sum_{i=n}^{p}\prod_{k=i+1}^{p}\eta_k\gamma_i \leq \kappa^{-1} + \left\{\prod_{k=n}^{n_0-1}(\gamma_{n_0-1}L^2)^{-1}\left(1+L^2\gamma_k^2\right)\right\}\exp\left(-\sum_{k=n}^{p}\kappa\gamma_k + \sum_{k=n}^{n_0-1}\gamma_k^2 mL\right) \,. \tag{S15}$$

Combining (S13) and (S15) in (S12) concluded the proof. □

We now deal with bounds on $W_2(\mu_0 Q_\gamma^n, \pi)$ under **G**1. But before we preface our result by some techinical lemmas.

Lemma S5. *Assume **H**1 and **H**3. Let $\zeta_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, $(Y_t, \overline{Y}_t)_{t \geq 0}$ such that $(Y_0, \overline{Y}_0)$ is distributed according to ζ_0 and given by (21). Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration associated with $(B_t)_{t\geq 0}$ with $\mathcal{F}_0$, the σ-field generated by $(Y_0, \overline{Y}_0)$. Then for all $n \geq 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$,*

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|Y_{\Gamma_{n+1}} - \overline{Y}_{\Gamma_{n+1}}\right\|^2\right] \\
&\qquad \leq \left\{1 - \gamma_{n+1}\left(\kappa - 2\epsilon_1\right) + \gamma_{n+1} L((1+\epsilon_2)\gamma_{n+1} - 2/(m+L))_+\right\} \left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2 \\
&\qquad\quad + \gamma_{n+1}^2 (1/(2\epsilon_1) + (1+\epsilon_2^{-1})\gamma_{n+1}) \left(dL^2 + (L^4\gamma_{n+1}/2)\left\|Y_{\Gamma_n} - x^\star\right\|^2 + dL^4 \gamma_{n+1}^2/12\right) .
\end{aligned}
$$

Proof. Let $n \geq 0$ and $\epsilon_1 > 0$, and set $\Delta_n = Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}$ by definition we have:

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\Delta_{n+1}\right\|^2\right] &= \left\|\Delta_n\right\|^2 + \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \left\{\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\right\} \mathrm{d}s\right\|^2\right] \\
&- 2\gamma_{n+1}\left\langle \Delta_n, \nabla U(Y_{\Gamma_n}) - \nabla U(\overline{Y}_{\Gamma_n})\right\rangle - 2\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\langle \Delta_n, \left\{\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\right\}\right\rangle \mathrm{d}s\right] .
\end{aligned}
$$

Using the two inequalities $|\langle a, b\rangle| \leq \epsilon_1 \|a\|^2 + (4\epsilon_1)^{-1}\|b\|^2$ and (12), we get

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\Delta_{n+1}\right\|^2\right] \leq \left\{1 - \gamma_{n+1}(\kappa - 2\epsilon_1)\right\}\left\|\Delta_n\right\|^2 - 2\gamma_{n+1}/(m+L)\left\|\nabla U(Y_{\Gamma_n}) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2 \\
&+ \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \left\{\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\right\}\mathrm{d}s\right\|^2\right] + \frac{1}{2\epsilon_1}\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\right\|^2\right]\mathrm{d}s .
\end{aligned}
$$

Using $\|a+b\|^2 \leq (1+\epsilon_2)\|a\|^2 + (1+\epsilon_2^{-1})\|b\|^2$ and the Jensen's inequality, we have

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\int_{\Gamma_n}^{\Gamma_{n+1}} \left\{\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\right\}\mathrm{d}s\right\|^2\right] &\leq (1+\epsilon_2)\gamma_{n+1}^2 \left\|\nabla U(Y_{\Gamma_n}) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2 \\
&+ \gamma_{n+1}(1+\epsilon_2^{-1})\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\int_{\Gamma_n}^{\Gamma_{n+1}} \left\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\right\|^2 \mathrm{d}s\right] .
\end{aligned}
$$

This result and **H**1 imply,

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\Delta_{n+1}\right\|^2\right] &\leq \left\{1 - \gamma_{n+1}(\kappa - 2\epsilon_1) + \gamma_{n+1}L((1+\epsilon_2)\gamma_{n+1} - 2/(m+L))_+\right\}\left\|\Delta_n\right\|^2 \\
&+ \left((1+\epsilon_2^{-1})\gamma_{n+1} + (2\epsilon_1)^{-1}\right)\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\right\|^2\right]\mathrm{d}s . \qquad \text{(S16)}
\end{aligned}
$$

By **H**1, the Markov property of $(Y_t)_{t\geq 0}$ and (66), we have

$$
\int_{\Gamma_n}^{\Gamma_{n+1}} \mathbb{E}^{\mathcal{F}_{\Gamma_n}}\left[\left\|\nabla U(Y_s) - \nabla U(Y_{\Gamma_n})\right\|^2\right]\mathrm{d}s \leq L^2\left(d\gamma_{n+1}^2 + dL^2\gamma_{n+1}^4/12 + (L^2\gamma_{n+1}^3/2)\left\|Y_{\Gamma_n} - x^\star\right\|^2\right) .
$$

Plugging this bound in (S16) concludes the proof. □

Lemma S6. *Let $(\gamma_k)_{k\geq 1}$ be a nonincreasing sequence of positive numbers. Let $\varpi, \beta > 0$ be positive constants satisfying $\varpi^2 \leq 4\beta$ and $\tau > 0$. Assume there exists $N \geq 1$, $\gamma_N \leq \tau$ and $\gamma_N \varpi \leq 1$. Then for all $n \geq 0$, $j \geq 2$*

(i)

$$\sum_{i=1}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi + \gamma_k \beta (\gamma_k - \tau)_+\right) \gamma_i^j \leq \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi\right) \gamma_i^j + \left\{ \beta^{-1} \gamma_1^{j-2} \prod_{k=1}^{N-1} \left(1 + \gamma_k^2 \beta\right) \right\} \prod_{k=N}^{n+1} \left(1 - \varpi \gamma_k\right) \;.$$

(ii) For all $\ell \in \{N, \ldots, n\}$,

$$\sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi\right) \gamma_i^j \leq \exp\left(- \sum_{k=\ell}^{n+1} \varpi \gamma_k \right) \sum_{i=N}^{\ell-1} \gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi} \;.$$

Proof. By definition of N we have

$$\begin{aligned}
&\sum_{i=1}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi + \gamma_k \beta (\gamma_k - \tau)_+\right) \gamma_i^j \\
&\leq \sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi\right) \gamma_i^j + \left\{ \sum_{i=1}^{N-1} \prod_{k=i+1}^{N-1} \left(1 + \gamma_k^2 \beta\right) \gamma_i^j \right\} \prod_{k=N}^{n+1} \left(1 - \gamma_k \varpi\right) \;.
\end{aligned} \tag{S17}$$

Using that $(\gamma_k)_{k\geq 1}$ is nonincreasing, we have

$$\begin{aligned}
\sum_{i=1}^{N-1} \prod_{k=i+1}^{N-1} \left(1 + \gamma_k^2 \beta\right) \gamma_i^j &\leq \sum_{i=1}^{N-1} \frac{\gamma_i^{j-2}}{\beta} \left\{ \prod_{k=i}^{N-1} \left(1 + \gamma_k^2 \beta\right) - \prod_{k=i+1}^{N-1} \left(1 + \gamma_k^2 \beta\right) \right\} \\
&\leq \beta^{-1} \gamma_1^{j-2} \prod_{k=1}^{N-1} \left(1 + \gamma_k^2 \beta\right) \;.
\end{aligned}$$

Plugging this inequality in (S17) concludes the proof of (i). Let $\ell \in \{N, \ldots, n+1\}$. Since $(\gamma_k)_{k\geq 1}$ is nonincreasing and for every $x \in \mathbb{R}$, $(1+x) \leq \mathrm{e}^x$, we get

$$\begin{aligned}
\sum_{i=N}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi\right) \gamma_i^j &= \sum_{i=N}^{\ell-1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi\right) \gamma_i^j + \sum_{i=\ell}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi\right) \gamma_i^j \\
&\leq \sum_{i=N}^{\ell-1} \exp\left(- \sum_{k=i+1}^{n+1} \varpi \gamma_k \right) \gamma_i^j + \gamma_\ell^{j-1} \sum_{i=\ell}^{n+1} \prod_{k=i+1}^{n+1} \left(1 - \gamma_k \varpi\right) \gamma_i \\
&\leq \exp\left(- \sum_{k=\ell}^{n+1} \varpi \gamma_k \right) \sum_{i=N}^{\ell-1} \gamma_i^j + \frac{\gamma_\ell^{j-1}}{\varpi} \;.
\end{aligned}$$

□

Lemma S7. *Let $(\gamma_k)_{k\geq 1}$ be a nonincreasing sequence of positive numbers, $\varpi, \beta, \tau > 0$ be positive real numbers, and $N \geq 1$ satisfying the assumptions of Lemma S6. Let $\mathsf{P} \in \mathbb{N}^*$, $C_i \geq 0$, $i = 0, \ldots, \mathsf{P}$ be positive constants and $(u_n)_{n\geq 0}$ be a sequence of real numbers with $u_0 \geq 0$ satisfying for all $n \geq 0$*

$$u_{n+1} \leq (1 - \gamma_{n+1}\varpi + \beta\gamma_{n+1}(\gamma_{n+1} - \tau)_+)u_n + \sum_{i=0}^{\mathsf{P}} C_j \gamma_{n+1}^{j+2} .$$

Then for all $n \geq 1$,

$$u_n \leq \left\{\prod_{k=1}^{N-1}\left(1 + \beta\gamma_k^2\right)\right\}\prod_{k=N}^{n}\left(1 - \gamma_k\varpi\right)u_0 + \sum_{j=0}^{\mathsf{P}} C_j \sum_{i=N}^{n}\prod_{k=i+1}^{n}\left(1 - \gamma_k\varpi\right)\gamma_i^{j+2}$$
$$+ \left\{\sum_{j=0}^{\mathsf{P}} C_j\beta^{-1}\gamma_1^j \prod_{k=1}^{N-1}\left(1 + \gamma_k^2\beta\right)\right\}\prod_{k=N}^{n}\left(1 - \varpi\gamma_k\right) .$$

Proof. This is a consequence of a straightforward induction and Lemma S6-(i). □

Proposition S8. *Assume* ***H** 1*, ***H** 3* *and* ***G** 1*. Let $x^\star$ be the unique minimizer of U. Let $\zeta_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, $(Y_t, \overline{Y}_t)_{t\geq 0}$ such that $(Y_0, \overline{Y}_0)$ is distributed according to ζ_0 and given by (21). Then for all $n \geq 0$ and $t \in [\Gamma_n, \Gamma_{n+1}]$:*

$$\mathbb{E}\left[\left\|Y_t - \overline{Y}_t\right\|^2\right] \leq \tilde{u}_n^{(1)}\mathbb{E}\left[\left\|Y_0 - \overline{Y}_0\right\|^2\right] + \tilde{u}_n^{(4)} + \tilde{u}_{t,n}^{(5)} ,$$

where

$$\tilde{u}_n^{(1)}(\gamma) \stackrel{\text{def}}{=} \left\{\prod_{k=1}^{n_1-1}\left(1 + L^2(1+\rho)\gamma_k^2\right)\right\}\prod_{k=n_1}^{n}\left(1 - \kappa\gamma_k/2\right) , \tag{S18}$$

$$\tilde{u}_n^{(4)}(\gamma) \stackrel{\text{def}}{=} \sum_{i=n_1}^{n}\gamma_i^2 \mathsf{f}(\gamma_i)\prod_{k=i+1}^{n}\left(1 - \kappa\gamma_k/2\right)$$
$$+ \mathsf{f}(\gamma_1)(L^2(1+\rho))^{-1}\left\{\prod_{k=1}^{n_1-1}\left(1 + \gamma_k^2(1+\rho)L^2\right)\right\}\prod_{k=n_1}^{n}\left(1 - \kappa\gamma_k/2\right) ,$$

where for $\gamma > 0$,

$$\mathsf{f}(\gamma) = \left\{2\kappa^{-1} + (1+\rho^{-1})\gamma\right\}\left(dL^2 + \delta L^4\gamma/2 + dL^4\gamma^2/12\right) ,$$
$$\delta = \max_{i\geq 1}\left\{\mathrm{e}^{-2m\Gamma_{i-1}}\mathbb{E}\left[\|Y_0 - x^\star\|^2\right] + (1 - \mathrm{e}^{-2m\Gamma_{i-1}})(d/m)\right\} ,$$

and

$$\tilde{u}_{t,n}^{(5)}(\gamma) \stackrel{\text{def}}{=} \frac{m+L}{2}\left(\frac{(t-\Gamma_n)^3 L^2}{3}E_{1,n}(\mu_0, \gamma) + (t-\Gamma_n)^2 d\right) ,$$

where $E_{1,n}(\mu_0, \gamma)$ is given by (S11) and μ_0 is the initial distribution of $\overline{Y}_0$.

Proof. Lemma S5 with $\epsilon_1 = \kappa/4$ and $\epsilon_2 = \rho$, **G**1, Lemma S6, (65) imply for all $n \geq 0$

$$\mathbb{E}\left[\left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2\right] \leq \tilde{u}_n^{(1)}(\gamma)\mathbb{E}\left[\left\|Y_0 - \overline{Y}_0\right\|^2\right] + \tilde{u}_n^{(4)}(\gamma) \;. \tag{S19}$$

Now let $n \geq 0$ and $t \in [\Gamma_n, \Gamma_{n+1}]$. By (21),

$$\left\|Y_t - \overline{Y}_t\right\|^2 = \left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2 - 2\int_{\Gamma_n}^t \left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_s \right\rangle \mathrm{d}s \;. \tag{S20}$$

Moreover for all $s \in [\Gamma_n, \Gamma_{n+1}]$, by (12) we get

$$\begin{aligned}\left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_s \right\rangle &= \left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_{\Gamma_n} + \overline{Y}_{\Gamma_n} - \overline{Y}_s \right\rangle \\ &\geq (m+L)^{-1}\left\|\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2 + \left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), \overline{Y}_{\Gamma_n} - \overline{Y}_s \right\rangle \;. \end{aligned} \tag{S21}$$

Since $|\langle a, b\rangle| \leq (m+L)^{-1}\|a\|^2 + (m+L)\|b\|^2/4$, we have

$$\begin{aligned}&\left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), \overline{Y}_{\Gamma_n} - \overline{Y}_s \right\rangle \geq \\ &\qquad - (m+L)^{-1}\left\|\nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n})\right\|^2 - (m+L)\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2/4 \;.\end{aligned}$$

Using this inequality in (S21), we get

$$\left\langle \nabla U(Y_s) - \nabla U(\overline{Y}_{\Gamma_n}), Y_s - \overline{Y}_s \right\rangle \geq -(m+L)\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2/4 \;,$$

and (S20) becomes

$$\left\|Y_t - \overline{Y}_t\right\|^2 \leq \left\|Y_{\Gamma_n} - \overline{Y}_{\Gamma_n}\right\|^2 + ((m+L)/2)\int_{\Gamma_n}^t \left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2 \mathrm{d}s \tag{S22}$$

Therefore, by the previous inequality, it remains to bound the expectation of $\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\|^2$. By (21) and using $\nabla U(x^\star) = 0$,

$$\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2 = \left\|-(s-\Gamma_n)(\nabla U(\overline{Y}_{\Gamma_n}) - \nabla U(x^\star)) + \sqrt{2}(B_s - B_{\Gamma_n})\right\|^2 \;.$$

Then taking the expectation, using the Markov property of $(B_t)_{t\geq 0}$ and **H**1, we have

$$\mathbb{E}\left[\left\|\overline{Y}_s - \overline{Y}_{\Gamma_n}\right\|^2\right] \leq (s-\Gamma_n)^2 L^2 \mathbb{E}\left[\left\|\overline{Y}_{\Gamma_n} - x^\star\right\|^2\right] + 2(s-\Gamma_n)d \;. \tag{S23}$$

The proof follows from taking the expectation in (S22), combining (S19)-(S23), and using Theorem S4. ☐

Theorem S9. *Assume **H**1, **H**3 and **G**1. Then for all $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $n \geq 1$,*

$$W_2^2(\mu_0 Q_\gamma^n, \pi) \leq \tilde{u}_n^{(1)}(\gamma) W_2^2(\mu_0, \pi) + \tilde{u}_n^{(2)}(\gamma) \;, \tag{S24}$$

where $(\tilde{u}_n^{(1)})_{n\geq 0}$ is given by (S18) and

$$\begin{aligned}\tilde{u}_n^{(2)}(\gamma) &\stackrel{\text{def}}{=} \sum_{i=n_1}^n \gamma_i^2 \phi(\gamma_i) \prod_{k=i+1}^n (1 - \kappa\gamma_k/2) \\ &\quad + \phi(\gamma_1)(L^2(1+\rho))^{-1}\left\{\prod_{k=1}^{n_1-1}(1+\gamma_k^2(1+\rho)L^2)\right\}\prod_{k=n_1}^n (1-\kappa\gamma_k/2) \;,\end{aligned} \tag{S25}$$

where

$$\phi(\gamma) = \left\{2\kappa^{-1} + (1+\rho^{-1})\gamma\right\}\left(dL^2 + dL^4\gamma/(2m) + dL^4\gamma^2/12\right) .$$

Proof. Let ζ_0 be an optimal transference plan of μ_0 and π. Let $(Y_t, \overline{Y}_t)_{t\geq 0}$ with $(Y_0, \overline{Y}_0)$ distributed according to ζ_0 and defined by (21). By definition of W_2 and since for all $t \geq 0$, π is invariant for P_t, $W_2^2(\mu_0 Q^n, \pi) \leq \mathbb{E}[\|Y_{\Gamma_n} - X_{\Gamma_n}\|^2]$. Then the proof follows from Proposition S8 since Y_0 is distributed according to π and by (16), which shows that $\delta \leq d/m$. □

2.1. Explicit bound based on Theorem S9 for $\gamma_k = \gamma_1 k^{\alpha}$ with $\alpha \in (0,1]$

We give here a bound on the sequences $(\tilde{u}_n^{(1)}(\gamma))_{n\geq 1}$ and $(\tilde{u}_n^{(2)}(\gamma))_{n\geq 1}$ for $(\gamma_k)_{k\geq 1}$ defined by $\gamma_1 > 0$ and $\gamma_k = \gamma_1 k^{-\alpha}$ for $\alpha \in (0,1]$. Recall that $\boldsymbol{\psi}_\beta$ is given by (S6). First note, since $(\gamma_k)_{k\geq 1}$ is nonincreasing, for all $n \geq 1$, we have

$$\tilde{u}_n^{(2)}(\gamma) \leq \sum_{j=0}^{1} \mathsf{B}_j \sum_{i=n_1}^{n} \gamma_i^{j+2} \prod_{k=i+1}^{n} (1-\kappa\gamma_k/2) + \sum_{j=0}^{1} \mathsf{B}_j (L^2(1+\rho))^{-1}\gamma_1^j \left\{\prod_{k=1}^{n_1-1}(1+\gamma_k^2(1+\rho)L^2)\right\} \prod_{k=n_1}^{n}(1-\kappa\gamma_k/2) , \quad \text{(S26)}$$

where

$$\mathsf{B}_1 = \mathsf{b} dL^2 , \mathsf{B}_2 = \mathsf{b}(dL^4/(2m) + \gamma_1 dL^4/12) , \mathsf{b} = 2\kappa^{-1} + (1+\rho^{-1})\gamma_1 .$$

1. For $\alpha = 1$ and $n_1 = 1$, by (S7) and (S8), we have

$$\tilde{u}_n^{(1)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2}$$
$$\tilde{u}_n^{(2)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^{1} \mathsf{B}_j \left\{\gamma_1^{j+2}(\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n+1)+1) + (L^2(1+\rho))^{-1}\gamma_1^j\right\} .$$

   For $n_1 > 1$, since $(\gamma_k)_{k\geq 0}$ is non increasing, using again (S7), (S8), and the bound for $t \in \mathbb{R}$, $(1+t) \leq \mathrm{e}^t$

$$\tilde{u}_n^{(1)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \exp\left\{\kappa\gamma_1\boldsymbol{\psi}_0(n_1)/2 + L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{-1}(n_1-1)+1)\right\}$$
$$\tilde{u}_n^{(2)}(\gamma) \leq (n+1)^{-\kappa\gamma_1/2} \sum_{j=0}^{1} \mathsf{B}_j \left(\gamma_1^{j+2}(\boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n+1) - \boldsymbol{\psi}_{\kappa\gamma_1/2-1-j}(n_1)+1) + (\gamma_1^j/(L^2(1+\rho)) \exp\left\{\kappa\gamma_1\boldsymbol{\psi}_0(n_1)/2 + L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{-1}(n_1-1)+1)\right\}\right) .$$

   Thus, for $\gamma_1 > \kappa/2$, we get a bound in $\mathcal{O}(n^{-1})$.
2. For $\alpha \in (0,1)$ and $n_1 = 1$, by (S7) and Lemma S6-(ii) applied with $\ell = \lceil n/2 \rceil$, we have

$$\tilde{u}_n^{(1)}(\gamma) \leq \exp\left(-(\kappa\gamma_1/2)\boldsymbol{\psi}_{\alpha-1}(n+1)\right)$$
$$\tilde{u}_n^{(2)}(\gamma) \leq \sum_{j=0}^{1} \mathsf{B}_j \left\{\gamma_1^{j+2}\left(\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2\rceil)+1\right)\exp\left\{-(\kappa\gamma_1/2)(\boldsymbol{\psi}_{1-\alpha}(n+1) - \boldsymbol{\psi}_{1-\alpha}(\lceil n/2\rceil))\right\} + 2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} + \gamma_1^j \exp\left\{-(\kappa\gamma_1/2)\boldsymbol{\psi}_{\alpha-1}(n+1)\right\}\right\} .$$

For $n_1 > 1$ and $\lceil n/2 \rceil \geq n_1$, since $(\gamma_k)_{k\geq 0}$ is non increasing, using again (S7), and Lemma S6-(ii) applied with $\ell = \lceil n/2 \rceil$, and the bound for $t \in \mathbb{R}$, $(1+t) \leq \mathrm{e}^t$, we get

$$\tilde{u}_n^{(1)}(\gamma) \leq \exp\left\{-\kappa\gamma_1(\boldsymbol{\psi}_{\alpha-1}(n+1) - \boldsymbol{\psi}_{1-\alpha}(n_1))/2 + L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{1-2\alpha}(n_1-1)+1)\right\}$$

$$\tilde{u}_n^{(2)}(\gamma) \leq \sum_{j=0}^{1} \mathsf{B}_j \left\{ 2\kappa^{-1}\gamma_1^{j+1}(n/2)^{-\alpha(j+1)} \right.$$
$$+ \gamma_1^{j+2}\left(\boldsymbol{\psi}_{1-\alpha(j+2)}(\lceil n/2 \rceil) - \boldsymbol{\psi}_{1-\alpha(j+2)}(n_1) + 1\right) \exp\left\{-(\kappa\gamma_1/2)(\boldsymbol{\psi}_{1-\alpha}(n+1) - \boldsymbol{\psi}_{1-\alpha}(\lceil n/2 \rceil))\right\}$$
$$\left. +(\gamma_1^j/(L^2(1+\rho))\exp\left\{-\kappa\gamma_1(\boldsymbol{\psi}_{\alpha-1}(n+1) - \boldsymbol{\psi}_{1-\alpha}(n_1))/2 + L^2(1+\rho)\gamma_1^2(\boldsymbol{\psi}_{1-2\alpha}(n_1-1)+1)\right\} \right\} .$$

3. Explicit bound on the MSE under strongly convex assumptions for $\gamma_k = \gamma_1 k^{-\alpha}$ with $\alpha \in [0,1]$

Without loss of generality, assume that $\|f\|_{\mathrm{Lip}} = 1$. In the following, denote by $\Omega(x) \overset{\text{def}}{=} \|x - x^\star\|^2 + d/m$ and C a constant (which may take different values upon each appearance), which does not depend on m, L, γ_1, α and $\|x - x^\star\|$.

1. First for $\alpha = 0$, recall by (S1), (29) and (30) we have for all $p \geq 1$,

$$u_p^{(1)}(\gamma)W_2^2(\delta_x, \pi) + u_p^{(2)}(\gamma) \leq 2\Omega(x)(1-\kappa\gamma_1/2)^p + 2\kappa^{-1}(\mathsf{A}_0\gamma_1 + \mathsf{A}_1\gamma_1^2) ,$$

where A_0 and A_1 are given by (S2) and (S3) respectively. So by Proposition 18 and Lemma 39, we have the following bound for the bias

$$\left\{\mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f)\right\}^2 \leq C\left(\frac{\kappa^{-1}\exp(-\kappa N\gamma_1/2)\Omega(x)}{\gamma_1 n} + \kappa^{-1}\mathsf{A}_0\gamma_1\right) .$$

Therefore plugging this inequality and the one given by Theorem 24 in (32) implies:

$$\mathrm{MSE}_f(N,n) \leq C\left(\kappa^{-1}\mathsf{A}_0\gamma_1 + \frac{\kappa^{-2} + \kappa^{-1}\exp(-\kappa N\gamma_1/2)\Omega(x)}{n\gamma_1}\right) . \tag{S27}$$

So with fixed γ_1 this bound is of order γ_1. If we fix the number of iterations n, we can optimize the choice of γ_1. Set

$$\gamma_{\star,0}(n) = (\kappa^{-1}\mathsf{A}_0)^{-1}(C_{\mathrm{MSE},0}/n)^{1/2} \ , \ \text{where } C_{\mathrm{MSE},0} \overset{\text{def}}{=} \kappa^{-3}\mathsf{A}_0 \ ,$$

and (S27) becomes if $\gamma_1 \leftarrow \gamma_{\star,0}(n)$,

$$\mathrm{MSE}_f(N,n) \leq C(C_{\mathrm{MSE},0}n)^{-1/2}\left(\kappa^{-1}\exp(-\kappa N\gamma_{\star,0}(n)/2)\Omega(x) + C_{\mathrm{MSE},0}\right) .$$

Setting $N_0(n) = 2(\kappa\gamma_{\star,0}(n))^{-1}\log(\Omega(x))$, we end up with

$$\mathrm{MSE}_f(N_0(n), n) \leq C(C_{\mathrm{MSE},0}/n)^{1/2} .$$

Note that $N_0(n)$ is of order $n^{1/2}$.

2. For $\alpha \in (0, 1/2)$, recall by (S1), (29) and (30) we have for all $p \geq 1$,

$$u_p^{(1)}(\gamma) W_2^2(\delta_x, \pi) \leq 2\Omega(x) \prod_{i=1}^{p} (1 - \kappa\gamma_i/2) \; . \tag{S28}$$

So by Proposition 18, Lemma 39, (S7) and (S9), we have the following bound for the bias

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1}{(1-2\alpha) n^{\alpha}} + \frac{\kappa^{-1} \exp\left\{ -\kappa\gamma_1 N^{1-\alpha} / (2(1-\alpha)) \right\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right) \; .$$

Plugging this inequality and the one given by Theorem 24 in (32) implies:

$$\mathrm{MSE}_f(N, n) \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1}{(1-2\alpha) n^{\alpha}} + \frac{\kappa^{-1} \exp\left\{ -\kappa\gamma_1 N^{1-\alpha} / (2(1-\alpha)) \right\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right) \; . \tag{S29}$$

At fixed γ_1, this bound is of order $n^{-\alpha}$, and is better than (S27) for $(\gamma_k)_{k \geq 1}$ constant. If we fix the number of iterations n, we can optimize the choice of γ_1 again. Set

$$\gamma_{\star,\alpha}(n) = (\kappa^{-1} \mathsf{A}_0 / (1 - 2\alpha))^{-1} (C_{\mathrm{MSE},\alpha} / n^{1-2\alpha})^{1/2} \; , \text{ where } C_{\mathrm{MSE},\alpha} \stackrel{\mathrm{def}}{=} \kappa^{-3} \mathsf{A}_0 / (1 - 2\alpha) \; ,$$

(S29) becomes with $\gamma_1 \leftarrow \gamma_{\star,\alpha}(n)$,

$$\begin{aligned} &\mathrm{MSE}_f(N, n) \\ &\qquad \leq C (C_{\mathrm{MSE},\alpha} n)^{-1/2} \left(\kappa^{-1} \exp\left\{ -\kappa N^{1-\alpha} \gamma_{\star,\alpha}(n) / (2(1-\alpha)) \right\} \Omega(x) + C_{\mathrm{MSE},\alpha} \right) \; . \end{aligned}$$

Setting $N_\alpha(n) = \{2(1-\alpha)\kappa^{-1} \gamma_{\star,\alpha}(n) \log(\Omega(x))\}^{1/(1-\alpha)}$, we end up with

$$\mathrm{MSE}_f(N_\alpha(n), n) \leq C (C_{\mathrm{MSE},\alpha} / n)^{1/2} \; .$$

It is worthwhile to note that the order of $N_\alpha(n)$ in n is $n^{(1-2\alpha)/(2(1-\alpha))}$, and $C_{\mathrm{MSE},\alpha}$ goes to infinity as $\alpha \to 1/2$.

3. If $\alpha = 1/2$, (S28) still holds. By Lemma 39, (S7) and (S9), we have the following bound for the bias

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1 \log(n)}{n^{1/2}} + \frac{\kappa^{-1} \exp\left\{ -\kappa\gamma_1 N^{1/2} / 4 \right\} \Omega(x)}{\gamma_1 n^{1/2}} \right) \; .$$

Plugging this inequality and the one given by Theorem 24 in (32) implies:

$$\mathrm{MSE}_f(N, n) \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1 \log(n)}{n^{1/2}} + \frac{\kappa^{-1} \exp\left\{ -\kappa\gamma_1 N^{1/2} / 4 \right\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1/2}} \right) \; . \tag{S30}$$

At fixed γ_1, the order of this bound is $\log(n) n^{-1/2}$, and is the best bound for the MSE. Fix the number of iterations n, and we now optimize the choice of γ_1. Set

$$\gamma_{\star,1/2}(n) = (\kappa^{-1} \mathsf{A}_0)^{-1} (C_{\mathrm{MSE},1/2} / \log(n))^{1/2} \; , \text{ where } C_{\mathrm{MSE},1/2} \stackrel{\mathrm{def}}{=} \kappa^{-3} \mathsf{A}_0 \; ,$$

and (S30) becomes with $\gamma_1 \leftarrow \gamma_{\star,1/2}(n)$,

$$\mathrm{MSE}_f(N,n) \leq C \left(\frac{\log(n)}{n C_{\mathrm{MSE},1/2}} \right)^{1/2} \left(\kappa^{-1} \exp\left\{ -\kappa N^{1/2} \gamma_{\star,1/2}(n)/4 \right\} \Omega(x) + \frac{C_{\mathrm{MSE},1/2}}{\log(n)} \right) .$$

Setting $N_{1/2}(n) = (4\kappa^{-1}\gamma_{\star,1/2}(n)\log(\Omega(x)))^2$, we end up with

$$\mathrm{MSE}_f(N_{1/2}(n),n) \leq C \left(\frac{\log(n) C_{\mathrm{MSE},1/2}}{n} \right)^{1/2} .$$

We can see that we obtain a worse bound than for $\alpha = 0$ and $\alpha \in (0,1/2)$.

4. For $\alpha \in (1/2,1]$, (S28) still holds. By Lemma 39, (S7) and (S9), we have the following bound for the bias

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp\left\{ -\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha)) \right\} \Omega(x)}{\gamma_1 n^{1-\alpha}} \right) .$$

Plugging this inequality and the one given by Theorem 24 in (32) implies:

$$\mathrm{MSE}_f(N,n) \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1}{n^{1-\alpha}} + \frac{\kappa^{-1} \exp\left\{ -\kappa\gamma_1 N^{1-\alpha}/(2(1-\alpha)) \right\} \Omega(x) + \kappa^{-2}}{\gamma_1 n^{1-\alpha}} \right) .$$

For fixed γ_1, the MSE is of order $n^{1-\alpha}$, and is worse than for $\alpha = 1/2$. For a fixed number of iteration n, optimizing γ_1 would imply to choose $\gamma_1 \to +\infty$ as $n \to +\infty$. Therefore, in that case, the best choice of γ_1 is the largest possible value $1/(m+L)$.

5. For $\alpha = 1$, (S28) still holds. By Lemma 39, (S7) and (S9), the bias is upper bounded by

$$\left\{ \mathbb{E}_x[\hat{\pi}_n^N(f)] - \pi(f) \right\}^2 \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa\gamma_1/2} \Omega(x)}{\gamma_1 \log(n)} \right) .$$

Plugging this inequality and the one given by Theorem 24 in (32) implies:

$$\mathrm{MSE}_f(N,n) \leq C \left(\frac{\kappa^{-1} \mathsf{A}_0 \gamma_1}{\log(n)} + \frac{\kappa^{-1} N^{-\kappa\gamma_1/2} \Omega(x) + \kappa^{-2}}{\gamma_1 \log(n)} \right) .$$

For fixed γ_1, the order of the MSE is $(\log(n))^{-1}$. For a fixed number of iterations, the conclusions are the same than for $\alpha \in (1/2,1)$.

References

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