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# ON USUAL, VIRTUAL AND WELDED KNOTTED OBJECTS UP TO HOMOTOPY

BENJAMIN AUDOUX, PAOLO BELLINGERI, JEAN-BAPTISTE MEILHAN, AND EMMANUEL WAGNER

**ABSTRACT.** We consider several classes of knotted objects, namely usual, virtual and welded pure braids and string links, and two equivalence relations on those objects, induced by either self-crossing changes or self-virtualizations. We provide a number of results which point out the differences between these various notions. The proofs are mainly based on the techniques of Gauss diagram formulae.

## 1. INTRODUCTION

In this note, we study several variations of the notion of pure braids and string links. Recall that string links are pure tangles without closed components, which form a monoid that contains the pure braid group as the group of units. As usual in knot theory, these objects can be regarded as diagrams up to Reidemeister moves. When allowing virtual crossings in such diagrams, modulo a suitably extended set of Reidemeister moves, one defines the notions of virtual pure braids and virtual string links. Another related class of object is that of welded knotted objects. Welded knots are a natural quotient of virtual knots, by the so-called Overcrossings Commute relation, which is one of the two forbidden moves in virtual knot theory. What makes this Overcrossings Commute relation natural is that the virtual knot group, and hence any virtual knot invariant derived from it, factors through it. These welded knotted objects first appeared in a work of Fenn-Rimanyi-Rourke in the more algebraic context of braids [10].

The study of these three classes (usual, virtual and welded) of knotted objects is currently the subject of an ongoing project of Bar-Natan and Dancso [4, 5, 3], which aims at relating certain algebraic structures to the finite type theories for these objects.

Works of Habegger and Lin [14] show that, in the usual case, any string link is link-homotopic to a pure braid, and that string links are completely classified up to link-homotopy by their action on the reduced free group  $RF_n$ . Here, the link-homotopy is the equivalence relation on knotted object generated by self-crossing changes, and the reduced free group is the smallest quotient of the free group  $F_n$  where each generator commutes with all of its conjugates.

In [2], the authors gave welded analogues of these results, recalled in Theorems 4.1 and 4.2 below. There, it appears that the right analogue of link-homotopy in the virtual/welded setting is the notion of  $v$ -equivalence, which is the equivalence relation on virtual knotted objects generated by self-virtualization, i.e. replacement of a classical self crossing by a virtual one.

This note contains a series of results which analyse further the various quotients of usual, virtual and welded pure braids and string links up to link-homotopy and  $v$ -equivalence, and the relations between them. The summary of our results, stated and proved in Section 4, is given in Figure 1 below. Although all notation and definitions needed for this diagram will be given in Section 2, let us outline here that

- $P_n$  and  $SL_n$  stand for the (usual) sets of pure braids and string links on  $n$  strands, and the prefix  $v$  and  $w$  refer to their virtual and welded counterpart, respectively;
- the superscripts  $v$  and  $cc$  refer respectively to the equivalence relations generated by self-virtualization and self-crossing change.

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## 2. DEFINITIONS

Before we define below the main objects of this note, let us fix a few notation that will be used throughout. We set  $n$  to be a non negative integer, and we denote by  $\llbracket 1, n \rrbracket$  the set of integers between 1 and  $n$ .

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*Date:* July 4, 2015.

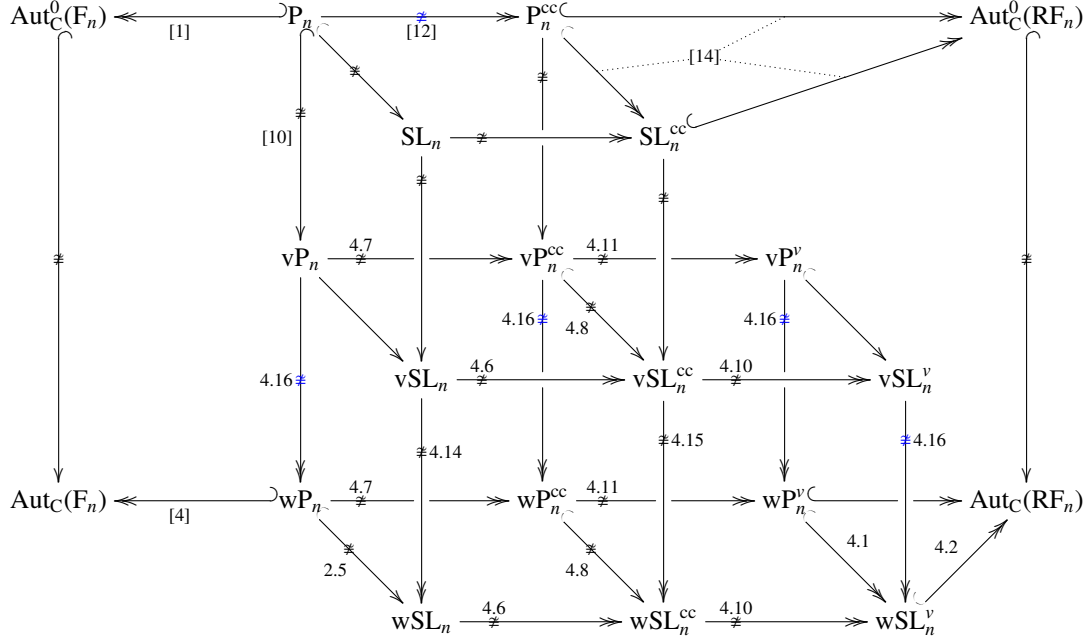


FIGURE 1. Connections between usual/virtual/welded pure braids and string links.

All statements hold for  $n \geq 2$  except for the blue “ $\cong$ ”, which become isomorphisms for  $n = 2$

Denote by  $I$  the unit closed interval. We fix  $n$  distinct points  $\{p_i\}_{i \in \llbracket 1, n \rrbracket}$  in  $I$ .

We will also use the following algebraic notation. Let  $F_n$  be the free group on  $n$  generators  $x_1, \dots, x_n$ . We denote by  $\text{RF}_n := F_n / \{[x_i; g^{-1}x_i g] \mid i \in \llbracket 1, n \rrbracket, g \in \text{RF}_n\}$  the *reduced free group* on  $n$  generators. Let  $G$  be either  $F_n$  or  $\text{RF}_n$  with its chosen system of generators; we define

- $\text{Aut}_C(G) := \{f \in \text{Aut}(G) \mid \forall i \in \llbracket 1, p \rrbracket, \exists g \in G, f(g_i) = g^{-1}g_i g\}$ , the group of *basis-conjugating automorphisms* of  $G$ ;
- $\text{Aut}_C^0(G) := \{f \in \text{Aut}_C(G) \mid f(x_1 \cdots x_n) = x_1 \cdots x_n\}$ .

**2.1. Usual, virtual and welded knotted objects.** In this section, we introduce the main objects of this note.

**Definition 2.1.** An  $n$ -component *virtual string link diagram* is an immersion  $L$  of  $n$  oriented intervals  $\bigsqcup_{i \in \llbracket 1, n \rrbracket} I_i$  in  $I \times I$ , called *strands*, such that

- each strand  $I_i$  has boundary  $\partial I_i = \{p_i\} \times \{0, 1\}$  and is oriented from  $\{p_i\} \times \{0\}$  to  $\{p_i\} \times \{1\}$  ( $i \in \llbracket 1, n \rrbracket$ );
- the singular set of  $L$  is a finite set of transverse double points.
- a decoration is added at each double point, and the decorated double point is called either a *classical crossing* or a *virtual crossing*, as indicated in Figure 2.

A classical crossing where the two preimages belong to the same component is called a *self-crossing*.

Up to isotopy, the set of virtual string link diagrams is naturally endowed with a monoidal structure by the stacking product, and with unit element the trivial diagram  $\bigcup_{i \in \llbracket 1, n \rrbracket} p_i \times I$ .

Two virtual string link diagrams are equivalent if they are related by a finite sequence of *generalized Reidemeister moves*, represented in Figure 3.

As is well known, virtual and mixed Reidemeister moves imply the more general *detour move*, which replaces an arc going through only virtual crossings by any other such arc, fixing the boundary [17].

We denote by  $\text{vSL}_n$  the quotient of  $n$ -component virtual string link diagrams up to isotopy and generalized Reidemeister moves, which is a monoid with composition induced by the stacking product. We call its elements  $n$ -component *virtual string links*.

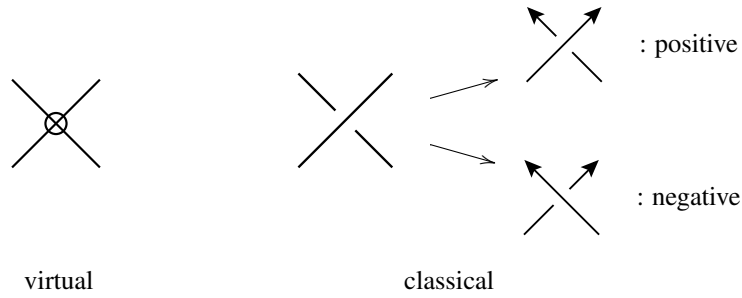


FIGURE 2. Virtual and classical crossings

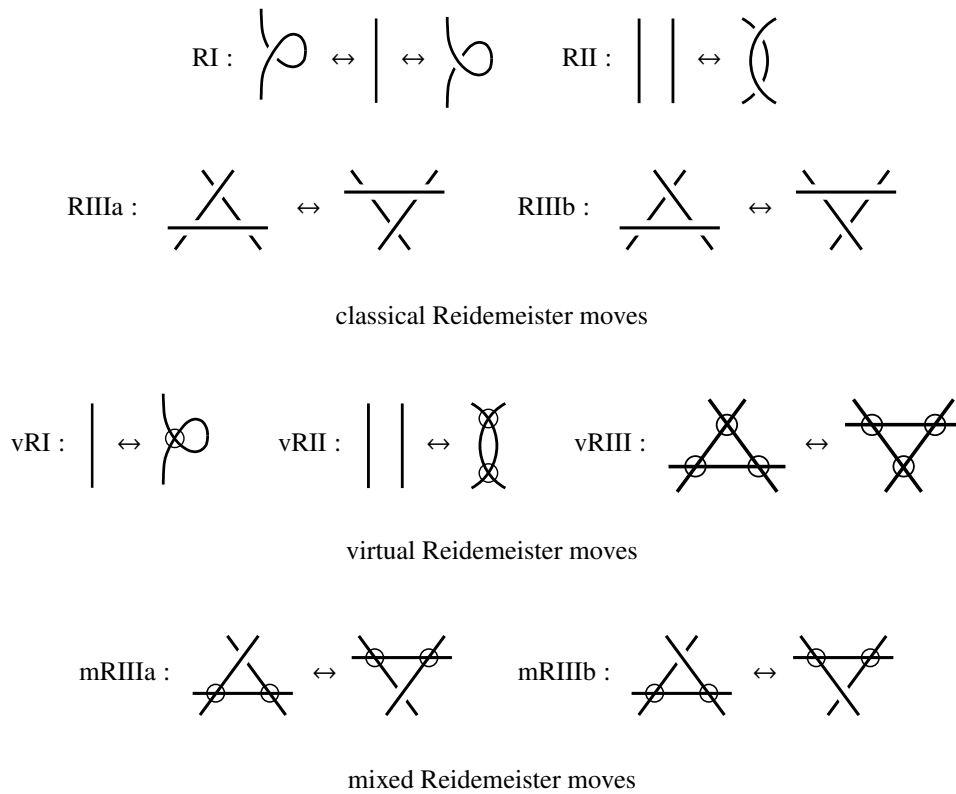


FIGURE 3. Generalized Reidemeister moves on diagrams; here, all lines are pieces of strands which may belong to the same strand or not, and can have any orientation.

**Definition 2.2.** An  $n$ -component virtual string link diagram is *monotone* if it intersects  $I \times \{t\}$  at exactly  $n$  points for all  $t \in I$ .

We denote by  $vP_n$  the submonoid of  $vSL_n$  of monotone elements, up to monotone transformations. This monoid is in fact the group of virtual pure braids studied in [6].

*Remark 2.3.* One could also consider  $vP_n$  to be the set of diagrams admitting a monotone representative (up to non-monotone transformations). The question of whether these two definitions agree is equivalent to that of the embedding of  $vP_n$  into  $vSL_n$ ; see Question 5.2.

As explained in the introduction, there is a natural quotient of virtual knot theory, where one of the forbidden move is allowed.

**Definition 2.4.** We define the Overcrossings Commute (OC) move as

$$\text{OC} : \begin{array}{c} \text{X} \\ \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array} \longleftrightarrow \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \text{X} \end{array} .$$

We denote by  $\text{wSL}_n := \text{vSL}_n / \text{OC}$  the quotient of  $\text{vSL}_n$  up to OC moves, which inherits a monoid structure from the stacking product. We call its elements  $n$ -component *welded string links*.

We denote by  $\text{wP}_n$  the submonoid of  $\text{wSL}_n$  of monotone elements up to monotone transformations.

This monoid is in fact the welded pure braid group studied, for instance, in [4]. Thus, we will freely call welded pure braids the elements of  $\text{wP}_n$ .

*Remark 2.5.* Unlike in the virtual case, the welded pure braid group is known to be isomorphic to the subset of  $\text{wSL}_n$  admitting a monotone representative, see [2, Rem. 3.7].

*Warnings 2.6.*

- The following Undercrossings Commute (UC) move

$$\text{UC} : \begin{array}{c} \text{X} \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array} \longleftrightarrow \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{X} \end{array} ,$$

was forbidden in the virtual context and is still forbidden in the welded context.

- Virtual and welded notions do not coincide, even for  $n = 1$ , where we get respectively the notion of virtual and welded long knots (see [4], and also Section 4, for a summary of results).

Let us now turn to the classical versions of these objects.

**Definition 2.7.** An  $n$ -component *string link* is an embedding  $L = \sqcup_{i \in \llbracket 1, n \rrbracket} I_i$  of  $n$  disjoint copies of the oriented interval  $I$  in the standard cube  $I^3$  such that  $I_i$  runs from  $p_i \times \{0\}$  to  $p_i \times \{1\}$  for all  $i \in \llbracket 1, n \rrbracket$ . We denote by  $\text{SL}_n$  the set of  $n$ -component classical string link up to isotopy. It is naturally endowed with a monoidal structure by the stacking product, with unit  $\cup_{i \in \llbracket 1, n \rrbracket} \{p_i\} \times I$ . Borrowing the terminology of [4, 5], we shall call such string links *usual* in order to distinguish them from the virtual and welded ones.

Any usual string link can be generically projected in  $I^3$  onto a virtual string link diagram with classical crossings only. Hence  $\text{SL}_n$  can be described as the set of virtual string link diagrams with no virtual crossing, modulo the classical Reidemeister moves only. Note that the question of whether the map from  $\text{SL}_n$  to  $\text{vSL}_n$  and/or to  $\text{wSL}_n$  is injective remains open.

We also denote by  $\text{P}_n$  the (usual) pure braid group on  $n$  strands, which can likewise be seen as the set of monotone virtual string links with no virtual crossing. Recall from [10] that this map is well-defined and is an embedding of  $\text{P}_n$  in  $\text{vP}_n$  and in  $\text{wP}_n$ .

*Remark 2.8.* Unlike the class of usual knotted objects, which is intrinsically topological, virtual and welded objects are diagrammatical in nature. However, both theories enjoy nice topological interpretations. It is now well known and understood [8, 18] that virtual knot theory can be realized topologically as the theory of knots in thickened surfaces modulo handle stabilization. Note that the Overcrossings Commute relation is not satisfied in this topological setting.

A topological realization of welded theory is given by considering a certain class of surfaces embedded in 4-space. In particular, welded string links map surjectively onto the monoid of ribbon tubes studied in [2]. This had first been pointed out by Satoh for the case of welded knots [24], but some key ideas already appeared in early works of Yajima [25].

**2.2. Self-local moves and homotopy relations.** In this note, we consider two types of equivalence relations on the above usual/virtual/welded objects, both generated by self-local moves.

**Definition 2.9.** Two virtual string link diagrams are related by a *self-virtualization* if one can be obtained from the other by turning a classical self-crossing into a virtual one. We call *v-equivalence* the equivalence relation on  $vSL_n$  and  $wSL_n$  generated by self-virtualization.

We denote respectively by  $vSL_n^v$  and  $wSL_n^v$  the quotient of  $vSL_n$  and  $wSL_n$  under *v-equivalence*, which are monoids with composition induced by the stacking product. We also denote by  $vP_n^v \subset vSL_n^v$  and  $wP_n^v \subset wSL_n^v$  the respective subsets of elements having a monotone representative.

In the context of usual string links, there is also a natural notion of crossing change, which is a local move that switches a positive classical crossing to a negative one, and vice-versa. If we further require that the two strands involved belong to the same component, we define a *self-crossing change*.

The classical notion of link-homotopy is the equivalence relation on usual (string) links generated by self-crossing changes. It was introduced for links by Milnor in [20], and later used by Habegger and Lin for string links [14], in order to “study (string) links modulo knot theory”, and focus on the interactions between distinct components. More precisely, link-homotopy on usual string links allows not only to unknot each component individually, but also simultaneously, since every usual string link is link-homotopic to a pure braid.

Thanks to the interpretation in terms of local crossing changes, the notion of link-homotopy can be extended to the whole monoids  $vSL_n$  and  $wSL_n$ .

**Definition 2.10.** We call *cc-equivalence* the equivalence relation on virtual knotted objects generated by self-crossing changes.

We denote the quotient of  $vSL_n$  under *cc-equivalence* by  $vSL_n^{cc}$ . In addition, we denote by  $vP_n^{cc}$  the image of  $vP_n$  in  $vSL_n^{cc}$ . We shall use similar notation in the usual and welded cases.

Since a crossing change can be realized by a sequence of two (de)virtualization moves, the *cc-equivalence* is clearly sharper than the *v-equivalence*. It is also *a priori* a more natural extension of the classical situation. However, as already noted in [2], it appears not to be the relevant notion for the study of welded string links “modulo knot theory”; this is recalled in further details in Section 4.1.

### 3. GAUSS DIAGRAM FORMULAE FOR VIRTUAL AND WELDED STRING LINKS

In this section, we recall the main tools for proving the results of this paper, namely Gauss diagram formulae [11, 13, 23].

**3.1. Gauss diagrams.** We first roughly review the notion of Gauss diagrams.

**Definition 3.1.** A *Gauss diagram* is a set of signed and oriented (thin) arrows on  $n$  ordered and oriented vertical (thick) strands, up to isotopy of the underlying strands. Endpoints of arrows are called *ends* and are divided in two parts, *heads* and *tails*, defined by the orientation of the arrow (which goes by convention from the tail to the head). An arrow having both ends on the same strand is called a *self-arrow*.

Examples of Gauss diagrams can be found in Figures 5 to 10 in the next section. As these figures also illustrate, Gauss diagrams serve as a combinatorial tool for faithfully encoding virtual/welded knotted objects. Indeed, it is well known that to any virtual string link diagram  $L$ , there is a unique associated Gauss diagram  $G_L$ , where the set of classical crossings in  $L$  is in one-to-one correspondence with the set of arrows in  $G_L$ , and that this correspondence induces a bijection between  $vSL_n$  and the set of Gauss diagrams up to the natural analogues of classical Reidemeister moves<sup>1</sup> of Figure 3. See [11, 13] for the knot case.

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<sup>1</sup>Note that there are no Gauss diagram analogues of the mixed and virtual Reidemeister moves, since virtual crossings are simply not materialized in Gauss diagrams.

Likewise, welded diagrams are faithfully encoded by equivalence classes of Gauss diagram up to the following Tails Commute (TC) move, which is the Gauss diagram analogue of Overcrossings Commute:

$$\text{TC : } \begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \leftarrow \varepsilon \quad \quad \quad \rightarrow \eta \\ \left| \quad \left| \quad \left| \right. \right. \end{array} \longleftrightarrow \begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \leftarrow \varepsilon \quad \quad \quad \rightarrow \eta \\ \left| \quad \left| \quad \left| \right. \right. \end{array} ,$$

where the signs  $\varepsilon$  and  $\eta$  are arbitrary.

Finally, the next observation allows to study the two homotopy relations introduced in Section 2.2. At the level of Gauss diagrams, the cc-equivalence is generated by the local move which switches both the orientation and the sign of a self-arrow; the v-equivalence, on the other hand, is simply generated by the removal of a self-arrow.

**3.2. Gauss diagrams formulae.** We now review Gauss diagrams formulae. First, let us define an *arrow diagram* to be an unsigned Gauss diagram i.e. an arrow diagram on  $n$  strands of  $n$  intervals with unlabeled arrows (see [22]).

Given a Gauss diagram  $G$ , there is an associated formal linear combination of arrow diagrams

$$i(G) := \sum_{G' \subseteq G} \sigma(G') A_{G'},$$

where the sum runs over all subdiagrams of  $G$ ,  $\sigma(G')$  denotes the product of the signs in the subdiagram  $G'$  and  $A_{G'}$  is the arrow diagram obtained from  $G'$  by forgetting the signs.

The  $\mathbb{Z}$ -module  $\mathring{A}_n$  generated by arrow diagrams on  $n$  strands comes equipped with a natural scalar product  $\langle -, - \rangle$ , defined by  $\langle A, A' \rangle = \delta_{A, A'}$  for any two arrow diagrams  $A$  and  $A'$ . So given any formal linear combination  $F$  of arrow diagrams in  $\mathring{A}_n$ , one can define a map on the set of Gauss diagrams on  $n$  strands  $\langle F; - \rangle$  by setting

$$\langle F; G \rangle := \langle F, i(G) \rangle$$

for any Gauss diagram  $G$ . Roughly speaking, this map counts with signs subdiagrams of  $G$ .

We can define in this way a map on the set of virtual diagrams by setting  $\langle F; L \rangle := \langle F; G_L \rangle$ , for any virtual string link diagram  $L$  with associated Gauss diagram  $G_L$ . We say that  $F$  is the *defining linear combination* of the map  $\langle F; - \rangle$ . Now, such a map does not, in general, factor through the generalized Reidemeister moves. We recall below a simple criterion, due to Mortier [21], which gives a sufficient condition for getting a virtual string link invariant in this way. To state this criterion, we need a few more definitions.

A *degenerate arrow diagram* is an arrow diagram where two arrow ends are allowed to coincide. We denote by  $\mathcal{D}_n$  the abelian group freely generated by degenerate arrow diagrams on  $n$  strands, modulo the relations

$$\begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \nearrow \quad \nwarrow \\ \left| \quad \left| \quad \left| \right. \right. \end{array} = \begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \nearrow \quad \nearrow \\ \left| \quad \left| \quad \left| \right. \right. \end{array} + \begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \nwarrow \quad \nwarrow \\ \left| \quad \left| \quad \left| \right. \right. \end{array} ; \quad \begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \nwarrow \quad \nearrow \\ \left| \quad \left| \quad \left| \right. \right. \end{array} = \begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \nwarrow \quad \nwarrow \\ \left| \quad \left| \quad \left| \right. \right. \end{array} + \begin{array}{c} \left| \quad \left| \quad \left| \right. \right. \\ \nearrow \quad \nearrow \\ \left| \quad \left| \quad \left| \right. \right. \end{array} .$$

Two arrow ends are called *adjacent* if they are met consecutively when running along some strand. An *internal edge* of an arrow diagram is a portion of a strand cobounded by two *distinct* adjacent arrow ends. We define a linear map  $d : \mathring{A}_n \rightarrow \mathcal{D}_n$  by sending any arrow diagram  $A$  on  $n$  strands to

$$d(A) := \sum_{\text{internal edges } e \text{ of } A} (-1)^{\uparrow_e} \cdot \eta(e) \cdot A_e,$$

where  $A_e \in \mathcal{D}_n$  denote the degenerate arrow diagram obtained by shrinking  $e$  to a point,  $\uparrow_e \in \{0; 1; 2\}$  is the number of arrow heads bounding  $e$ , and  $\eta(e) \in \{\pm 1\}$  is given by

$$\eta(e) = \begin{cases} -1 & \text{if the two arrows cobounding } e \text{ do not cross in } A, \\ +1 & \text{otherwise,} \end{cases}$$

with the convention that two arrows *do not cross* in an arrow diagram if, when running along the  $n$  strands  $I_1$  to  $I_n$ , in order and following the orientations, we meet the two ends of one of these arrows consecutively.

**Theorem 3.2.** *Let  $F \in \mathring{A}_n$  be a linear combination of arrow diagrams on  $n$  strands:*

- (1) *if  $F$  does not contain any arrow with adjacent ends, then  $\langle F; - \rangle$  is invariant under move RI;*

- (2) if  $F$  does not contain two arrows with adjacent heads and adjacent tails, then  $\langle F; - \rangle$  is invariant under move RII;
- (3) if  $\langle F; - \rangle$  is invariant under move RII and if  $d(F)$  is zero in  $\mathcal{D}_n$ , then  $\langle F; - \rangle$  is invariant under moves RIII;
- (4) if  $F$  does not contain any pair of adjacent arrow tails, then  $\langle F; - \rangle$  is invariant under OC;
- (5) if, for each diagram of  $F$  which has a self-arrow  $\vec{a}$ , the diagram obtained by reversing the orientation of  $\vec{a}$  also appears in  $F$  with opposite sign, then  $\langle F; - \rangle$  is invariant under cc-equivalence;
- (6) if  $F$  does not contain any self-arrow, then  $\langle F; - \rangle$  is invariant under  $v$ -equivalence.

The only difficult part of the statement is the invariance under RIII and it is due to A. Mortier. It can be found in [21] where it is stated as an equivalence in the context of virtual knots. However, the arguments adapt verbatim to the case of virtual string links. Actually, Mortier pointed out the fact that each point in Theorem 3.2 is also an equivalence in the string link case (this fact can be proved using a suitable Polyak algebra).

*Example 3.3.* As an example, let us consider the two invariants  $v_{2,1}$  and  $v_{2,2}$ , defined in [13] by

$$v_{2,1} = \left\langle \begin{array}{c} \uparrow \\ \textcircled{\uparrow} \\ \downarrow \end{array}, - \right\rangle \quad \text{and} \quad v_{2,2} = \left\langle \begin{array}{c} \uparrow \\ \textcircled{\uparrow} \\ \textcircled{\uparrow} \\ \downarrow \end{array}, - \right\rangle.$$

They are easily seen to be invariants of virtual 1-string links. Indeed, for, say, the latter one, we have

$$d \left( \begin{array}{c} \uparrow \\ \textcircled{\uparrow} \\ \textcircled{\uparrow} \\ \downarrow \end{array} \right) = - \begin{array}{c} \uparrow \\ \textcircled{\uparrow} \\ \downarrow \end{array} + \begin{array}{c} \uparrow \\ \textcircled{\uparrow} \\ \downarrow \end{array} - \begin{array}{c} \uparrow \\ \textcircled{\uparrow} \\ \downarrow \end{array} = 0 \in \mathcal{D}_1.$$

The invariant  $v_{2,1}$  is moreover a welded 1-string links invariant, while  $v_{2,2}$  is not, since the defining diagram of the latter contains two adjacent arrow tails. As a matter of fact, the virtual string link  $K''$  of Figure 5 is trivial in  $wSL_1$ , but we have  $v_{2,2}(K'') = -1$ , while  $v_{2,2}$  clearly vanishes on the trivial 1-string link.

In the rest of the paper, we leave it as an exercise to the reader to check using Theorem 3.2 that each invariant defined via an arrow diagram formula has the desired invariance properties.

#### 4. RESULTS ON USUAL, VIRTUAL AND WELDED BRAID-LIKE OBJECTS

In this section, we recall some comparative results on usual, virtual and welded knotted objects, and we provide further results comparing various notions of homotopy for these objects. They are roughly summarized in Figure 1.

**4.1. Some analogies between the usual and welded theories.** Let us start by recalling a couple of results from [2], on the  $v$ -equivalence for welded string links. On one hand, we have the following

**Theorem 4.1** ([2]). *Every welded string link is monotone up to self-virtualization.*

This is a welded analogue of a result of Habegger and Lin [14], which states that any usual string link is link-homotopic to a pure braid.

On the other hand, we have a classification result, analogous to [14, Thm. 1.7] in the usual case.

**Theorem 4.2** ([2]). *The monoids  $wSL_n^v$ ,  $wP_n^v$  and  $AutC(RF_n)$  are all isomorphic.*

Note, moreover, that this classification of welded string links up to  $v$ -equivalence is achieved by a virtual extension of Milnor invariants.

The next theorem illustrates the fact, suggested by the above results, that the  $v$ -equivalence can indeed be seen as a natural extension of the usual link-homotopy to the welded case.

**Theorem 4.3.** *Let  $L_1, L_2 \in SL_n$ , and let  $\iota_w : SL_n \rightarrow wSL_n$  be the natural map induced by the inclusion at the level of diagrams. If  $\iota_w(L_1)$  and  $\iota_w(L_2)$  are  $v$ -equivalent, then  $L_1$  and  $L_2$  are cc-equivalent.*

In other words, the notion of self-virtualization restricted to welded string links with only classical crossings coincides with the usual self-crossing change.



*Proof.* Recall that  $\text{Aut}_C^0(\text{RF}_n)$  is the set of automorphisms in  $\text{Aut}_C(\text{RF}_n)$  which leave the product  $x_1 x_2 \cdots x_n$  invariant. Theorem 1.7 of [14] states that  $\text{SL}_n^{\text{cc}} \cong \text{Aut}_C^0(\text{RF}_n)$  and it is easily checked at the diagram level that this isomorphism is compatible, through the map  $\iota_w$ , with the one of Theorem 4.2. Therefore if  $L_1, L_2 \in \text{SL}_n$  are  $v$ -equivalent, then they represent the same automorphism in  $\text{Aut}_C(\text{RF}_n)$  which is actually in  $\text{Aut}_C^0(\text{RF}_n)$  since it corresponds to some usual string links. According to Theorem 1.7 of [14], this implies that  $\iota_w(L_1)$  and  $\iota_w(L_2)$  are cc-equivalent.  $\square$

We will see below that the cc-equivalence, on the other hand, does not allow such generalizations, and hence appears not to be the right notion to be considered in this context.

**4.2. cc-equivalence for virtual and welded string links.** In this section, we compare virtual and welded pure braids and string links up to cc-equivalence.

In the case  $n = 1$ , the situation is rather simple and well-known.

Obviously, we have  $vP_1^{\text{cc}} \cong wP_1^{\text{cc}} \cong \{1\}$ , since the virtual and welded braid groups on one strand themselves are trivial. In the string link case, however, virtual and welded objects differ:

**Lemma 4.4.** *Self-crossing change is an unknotting operation for welded 1-string links, but isn't for virtual 1-string links. In other words, we have  $wSL_1^{\text{cc}} \cong \{1\}$ , whereas  $vSL_1^{\text{cc}} \neq \{1\}$ .*

*Proof.* Let us prove the first assertion. A welded string link on one strand has a Gauss diagram consisting of a single vertical strand and several signed self-arrows, and a crossing change on this welded long knot corresponds to switching both the sign and orientation of one arrow. So, for any two arrow ends that are adjacent on the vertical strand, we may safely assume up to crossing changes that these are two arrow tails, hence we may freely exchange their relative positions on the strand using TC. This implies that, up to crossing change and TC, any Gauss diagram of a long knot can be turned into a diagram consisting of only isolated arrows. By R1, such a Gauss diagram is clearly trivial.

We now turn to the virtual case. Consider the virtual long knot  $K_0$  shown in Figure 4, which is a string link version of the Kishino knot. As pointed out in [9], the closure of  $K_0$  is a virtual knot which cannot be unknotted by crossing changes. This proves that  $K_0$  is not cc-equivalent to the trivial long knot.  $\square$

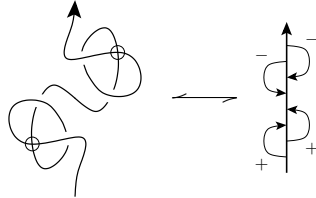


FIGURE 4. The virtual long knot  $K_0$ , and its Gauss diagram.

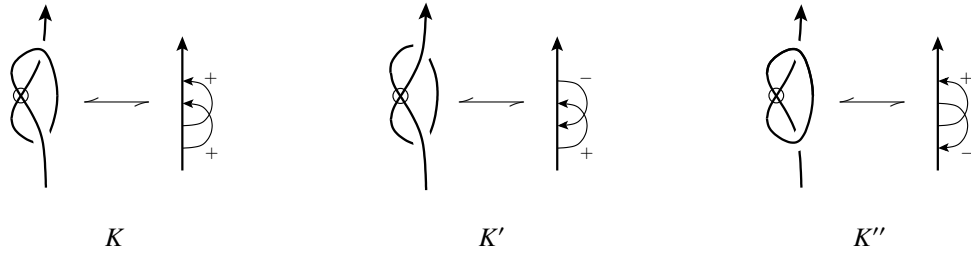
Similar results in the knot case can be found in [9, Sec. 1]; note however that, unlike in the usual case, the closure map from virtual/welded long knots to virtual/welded knots is not an isomorphism [17].

*Remark 4.5.* It is still unknown whether the closure map used in the latter part of the proof, from cc-equivalence classes of virtual long knots to virtual knots up to crossing changes, has a non trivial kernel.

**Lemma 4.6.** *For  $n \geq 1$ , there are distinct virtual and welded string links which are cc-equivalent, i.e. the canonical projections  $vSL_n \twoheadrightarrow vSL_n^{\text{cc}}$  and  $wSL_n \twoheadrightarrow wSL_n^{\text{cc}}$  are not injective.*

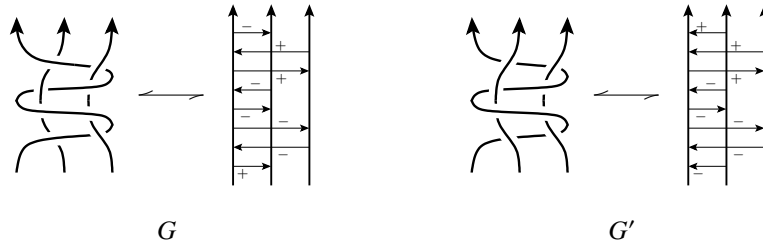
*Proof.* Recall from Section 3 that the invariant  $v_{2,1}$  from [13] is an invariant of welded (and in particular, virtual) 1-string links. Clearly, the virtual string link  $K$  of Figure 5 is cc-equivalent to  $K'$ , shown in the same figure. However, we have  $v_{2,1}(K') = -1$ , whereas  $v_{2,1}$  doesn't detect  $K$ . This proves the statement in both the virtual and welded settings.  $\square$

Recall from [12] that the canonical projection from the pure braid group to  $P_n^{\text{cc}}$  is not injective. The proof of Goldsmith actually applies to the virtual and welded context:


 FIGURE 5. The virtual string links  $K$ ,  $K'$  and  $K''$ , and their Gauss diagrams.

**Lemma 4.7.** *For  $n > 2$ , there are distinct virtual and welded pure braids which are  $cc$ -equivalent, i.e. the canonical projections  $vP_n \twoheadrightarrow vP_n^{cc}$  and  $wP_n \twoheadrightarrow wP_n^{cc}$  are not injective.*

*Proof.* Consider the pure braids  $G$  and  $G'$  shown in Figure 6, which (implicitly) appear in Figure 2 of [12]. As shown there, these two pure braids are  $cc$ -equivalent. The result then follows by noting that


 FIGURE 6. The virtual pure braids  $G$  and  $G'$ , and their Gauss diagrams.

usual braids embed injectively in  $vP_n$  and in  $wP_n$ . Indeed, the latter inclusion follows immediately from the interpretations of  $P_n$  and  $wP_n$  in terms of automorphism groups of  $F_n$  (see the left rectangle in Figure 1), and clearly implies the former.  $\square$

More strikingly, although any 1-component welded string link can be unknotted using crossing changes, this cannot always be achieved simultaneously for all strands of a welded string link with two or more components:

**Lemma 4.8.** *For all  $n > 1$ , there are virtual and welded string links which are not  $cc$ -equivalent to any welded pure braid, i.e. the inclusions  $vP_n^{cc} \hookrightarrow vSL_n^{cc}$  and  $wP_n^{cc} \hookrightarrow wSL_n^{cc}$  are not surjective.*

*Proof.* Consider the welded 2-string link  $L'$  of Figure 7 and the invariant  $S_2 : vSL_2 \rightarrow \mathbb{Z}$  defined by

$$S_2 = \langle \begin{array}{c} \uparrow \\ \diagdown \\ \downarrow \end{array} - \begin{array}{c} \uparrow \\ \diagup \\ \downarrow \end{array} - \begin{array}{c} \uparrow \\ \downarrow \\ \downarrow \end{array} + \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \end{array} - \begin{array}{c} \uparrow \\ \downarrow \\ \downarrow \end{array} + \begin{array}{c} \uparrow \\ \downarrow \\ \downarrow \end{array}, - \rangle.$$

Notice that  $S_2$  is a welded 2-string link invariant, and is an invariant of  $cc$ -equivalence. Note also that  $S_2$  does detect  $L'$ , since  $S_2(L') = 1$ . Now, if  $L'$  were  $cc$ -equivalent to a pure braid  $\tilde{L}$ , then it would admit a representative whose Gauss diagram has only horizontal arrows. But then we would have  $S_2(\tilde{L}) = 0$ , since the defining formula for  $S_2$  contains no such diagram with only horizontal arrows. This proves that  $L'$  is not  $cc$ -equivalent to a pure braid, hence completes the proof in the welded case.

The result in the virtual case follows, by simply noting that the above argument applies equally well when regarding  $L'$  as a virtual string link.  $\square$

*Remark 4.9.* The above proof shows in particular that the monoids  $vSL_n^{cc}$  and  $wSL_n^{cc}$  are both non trivial (this was already clear from Lemma 4.4 in the virtual case). It is easily checked that this is already the case

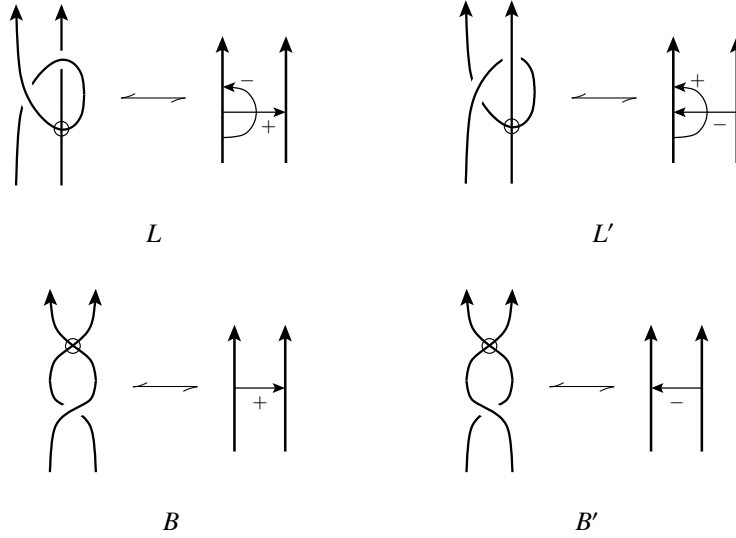


FIGURE 7. The virtual 2-string links  $L$ ,  $L'$ ,  $B$  and  $B'$ , and their Gauss diagrams.

for  $vP_n^{cc}$  and  $wP_n^{cc}$ ; for example, the virtual pure braid  $B$  of Figure 7 has a non trivial virtual linking number  $lk_{1/2}$  (see e.g. [13, Sec. 1.7]), which is a welded invariant.

**4.3. Comparing cc and v-equivalences.** We now compare the cc-equivalence and the v-equivalence for welded knotted objects.

The 1-component case is again rather trivial. As seen in Lemma 4.4, the cc-equivalence yields different quotients on  $vSL_1$  and  $wSL_1$ . The v-equivalence, on the other hand, trivializes both:  $vSL_1^v \cong wSL_1^v \cong \{1\}$ . Indeed, virtualizing all crossings of a welded (or virtual) long knot always yields the trivial element.

For  $n > 1$ , the situation is different:

**Lemma 4.10.** *For all  $n > 1$ , there are virtual and welded string links which are v-equivalent but not cc-equivalent, i.e. the canonical projections  $vSL_n^{cc} \twoheadrightarrow vSL_n^v$  and  $wSL_n^{cc} \twoheadrightarrow wSL_n^v$  are not injective.*

*Proof.* Consider again the welded 2-string links  $L$  and  $L'$  in Figure 7. As shown in the proof of Lemma 4.8,  $L$  is not cc-equivalent to a pure braid. However,  $L'$  is equivalent, up to self-virtualization to the pure braid  $B$  shown in Figure 7. As for Lemma 4.8, the argument applies to both the virtual and welded context.  $\square$

This remains true when restricting to pure braid groups:

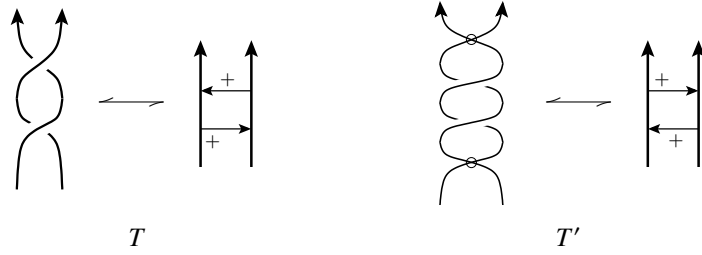
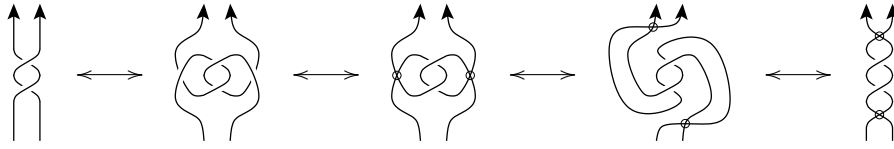
**Lemma 4.11.** *For all  $n > 1$ , there are virtual and welded pure braids which are v-equivalent but not cc-equivalent, i.e. the surjective maps  $vP_n^{cc} \twoheadrightarrow vP_n^v$  and  $wP_n^{cc} \twoheadrightarrow wP_n^v$  are not injective.*

*Proof.* Let  $T$  and  $T'$  be the welded pure braids shown in Figure 8. On one hand, we have that  $T$  and  $T'$  are v-equivalent. This is shown in Figure 9 below; in this figure, the first move is achieved by a sequence of classical Reidemeister moves, the second is a pair of self-virtualizations, the third is a pair of detour moves, and the final move is a planar isotopy. Note that, at the Gauss diagram level, this is merely an instance of a more general result on commutation of arrows supported by two strands, stated in [2, Prop. 4.11].

On the other hand,  $T$  and  $T'$  are not cc-equivalent. This can be checked using the invariant  $Q_2$  defined by the formula

$$Q_2 = \langle \begin{array}{c} \uparrow \uparrow \\ \leftarrow \rightarrow \end{array} - \begin{array}{c} \uparrow \uparrow \\ \rightarrow \leftarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \circlearrowleft \end{array} - \begin{array}{c} \uparrow \uparrow \\ \circlearrowright \end{array} + \begin{array}{c} \uparrow \uparrow \\ \leftarrow \circlearrowleft \end{array} - \begin{array}{c} \uparrow \uparrow \\ \leftarrow \circlearrowright \end{array}, - \rangle,$$

which is an invariant of welded 2-string links up to cc-equivalence. Indeed, it is straightforwardly checked that  $Q_2(T) = 1$ , while  $Q_2(T') = -1$ .  $\square$


 FIGURE 8. The welded pure braids  $T$  and  $T'$ , and their Gauss diagrams.

 FIGURE 9. The welded pure braids  $T$  and  $T'$  are v-equivalent.

Lemma 4.6 readily implies that the canonical projections  $vSL_n \rightarrow vSL_n^v$  and  $wSL_n \rightarrow wSL_n^v$  are not injective for  $n \geq 1$ , and the same observation holds for  $vP_n \rightarrow vP_n^v$  and  $wP_n \rightarrow wP_n^v$  by Lemma 4.7, for  $n > 2$ . Actually, in the welded pure braid case, this remains true for  $n = 2$ :

**Lemma 4.12.** *We have  $wP_2 \not\cong wP_2^v$ .*

*Proof.* One can easily prove that any automorphism in  $\text{Aut}_C(\text{RF}_2)$  can be written as  $\xi_{\eta_1, \eta_2}$  for some  $\eta_1, \eta_2 \in \mathbb{N}$ , where  $\xi_{\eta_1, \eta_2}(x_1) = x_2^{\eta_1} x_1 x_2^{-\eta_1}$  and  $\xi_{\eta_1, \eta_2}(x_2) = x_1^{\eta_2} x_2 x_1^{-\eta_2}$ , and that  $\xi_{\eta_1, \eta_2} \xi_{\eta_3, \eta_4} = \xi_{\eta_1 + \eta_3, \eta_2 + \eta_4}$ . This implies that  $wP_2^v \cong \text{Aut}_C(\text{RF}_2) \cong \mathbb{Z}^2$ , while it is well-known that  $wP_2 = F_2$  (see for instance [10]).  $\square$

*Remark 4.13.* Lemma 4.3 can also be proved using the invariant  $Q_2$  used for Lemma 4.11. Indeed, it already follows from [2, Prop. 4.11] that  $wP_2^v$  is abelian, so it suffices to show that this is not the case for  $wP_2$ . This is a consequence of the fact that  $Q_2$  distinguishes the welded pure braids  $T$  and  $T'$  of Figure 8. More generally, since  $Q_2$  is an invariant of cc-equivalence, we have we have from this observation that none of  $vP_2$ ,  $vP_2^{\text{cc}}$ ,  $wP_2$  and  $wP_2^{\text{cc}}$  is abelian.

**4.4. Comparing virtual and welded objects.** As pointed out in [4, Sec. 3.1], welded long knots are strictly weaker than virtual long knots, in the sense that there exist non trivial virtual long knots which are trivial up to OC. For example, as already noted at the end of Section 3, the virtual long knot  $K''$  shown on the right-hand side of Figure 5 is trivial up to OC, yet it is non trivial as virtual long knot, since it is detected by the invariant  $v_{2,2}$  defined in Section 3. This implies that, more generally, we have

**Lemma 4.14.** *For all  $n \geq 1$ , there are distinct virtual string links which are equal as welded string links, i.e.  $vSL_n \not\cong wSL_n$ .*

*Remark 4.15.* It may be worth mentioning that knottedness of each individual component is not the only obstruction to having  $vSL_n$  isomorphic to  $wSL_n$ . Actually, the following example shows that this also can't be valid among string links with trivial components. Let  $L$  be the virtual 2-string link shown in Figure 7, which is equivalent in  $wSL_2$  to the virtual pure braid  $B$  represented in the same figure. Now, let  $V_2 : vSL_2 \rightarrow \mathbb{Z}$  be the invariant introduced in [19] and defined by the Gauss diagram formula

$$V_2 = \langle \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) - \left( \begin{array}{c} \uparrow \\ \uparrow \end{array} \right) - \left( \begin{array}{c} \downarrow \\ \downarrow \end{array} \right), - \rangle.$$

We have that  $V_2(L) = -1$ , whereas  $V_2(B)$  vanishes on  $B$ .

The invariant  $V_2$  can moreover be symmetrized into

$$V_2^* = \langle \begin{array}{c} \uparrow \\ \diagdown \\ \uparrow \\ \diagup \\ \uparrow \end{array} - \begin{array}{c} \uparrow \\ \diagup \\ \uparrow \\ \diagdown \\ \uparrow \end{array} - \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} + \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} - \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} + \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array}, - \rangle.$$

which is invariant under self-crossing change. The same example proves then that  $vSL_n^{cc} \not\cong wSL_n^{cc}$  for  $n \geq 2$ .

By comparing the corresponding group presentations, given for instance in [4, Sec. 2], it can be seen that the group  $vP_2$  is isomorphic to  $wP_2$ . This remains true up to cc-equivalence, i.e. we have  $vP_2^{cc} \cong wP_2^{cc}$ . Up to v-equivalence, the isomorphism even holds for string links, i.e. we have  $vSL_2^v \cong wSL_2^v$ , as a consequence of [2, Prop. 4.11].

However, as soon as the number of strands is greater than 3, it is a general fact that, even up to cc or v-equivalence, the virtual and welded quotients are actually distinct:

**Lemma 4.16.** *For all  $n > 2$ , there are virtual pure braids which are distinct, even up to cc or v-equivalence, but are equivalent as welded pure braids.*

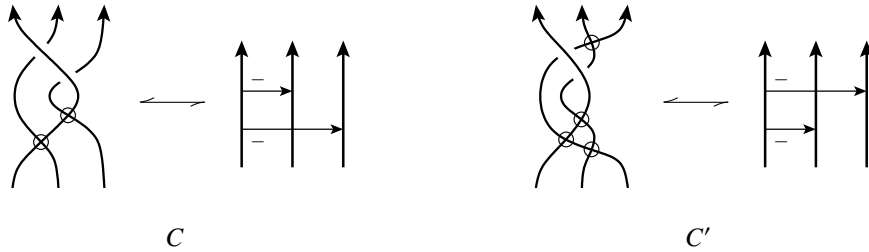


FIGURE 10. The virtual pure braids  $C$  and  $C'$ , and their Gauss diagrams.

It follows that the welded projections of  $vP_n$ ,  $vP_n^{cc}$ ,  $vP_n^v$ ,  $vSL_n$ ,  $vSL_n^{cc}$  and  $vSL_n^v$  are all non injective.

*Proof.* Consider the virtual pure braids  $C$  and  $C'$  shown in Figure 10. As obvious from the Gauss diagram point of view, they are equivalent in  $wP_3$ , but they are distinct in  $vP_3$ ,  $vP_3^{cc}$  and  $vP_3^v$ . Indeed, the virtual string link invariant

$$(1) \quad M_2 = \langle \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} - \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} - \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array}, - \rangle,$$

which is invariant under self-crossing change and self-virtualization, satisfies  $M_2(C) = 1$  but  $M_2(C') = 0$ .  $\square$

## 5. SOME OPEN QUESTIONS

All the connections between the different notions and quotients of string links are summarized in Figure 1. However, several questions remain open, and some are listed below.

*Question 5.1.* Do usual string links embed in their virtual or welded counterparts ?

This should be compared with the fact, used in the proof of Lemma 4.7, that  $P_n$  embeds into both  $vP_n$  and  $wP_n$ . Note that showing that the map  $SL_n \rightarrow wSL_n$  is injective would also give an affirmative answer in the virtual case.

*Question 5.2.* Is the inclusion map  $vP_n \rightarrow vSL_n$  injective ?

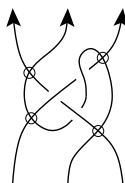
As noted in Remark 2.3, this is equivalent to showing that the virtual pure braid group  $vP_n$  is isomorphic to its quotient under non-monotone transformations. Recall that the analogous maps  $P_n \rightarrow SL_n$  and  $wP_n \rightarrow wSL_n$  for usual and welded objects are both injective. It is also known that  $P_n$  embeds in  $wSL_n$  (see e.g. [2]), which implies that  $P_n$  embeds in  $vSL_n$  as well.

*Question 5.3.* Are welded pure braids the only invertibles in  $wSL_n$  ? Are they in  $wSL_n^{cc}$  ? Likewise, are virtual pure braids the only invertibles in  $vSL_n$ ,  $vSL_n^{cc}$  and  $vSL_n^v$  ?

This is a natural question in view of the usual case, where pure braids form the group of units in  $SL_n$ , as shown in [15].

*Question 5.4.* Is the map  $vP_n^v \hookrightarrow vSL_n^v$  surjective? In other words, is any virtual string link monotone up to self-virtualization?

By Theorem 4.1, the answer is affirmative in the welded case. Also, as stated above, we have  $vSL_2^v \cong wSL_2^v \cong \mathbb{Z}^2$  by [2, Prop. 4.11], hence an affirmative answer in the 2-strand case. In the general case, however, the answer seems likely to be negative; for example, the virtual string link depicted below is a good candidate for a counter-example, although the techniques used in the present paper do not apply:



*Question 5.5.* Is the cc-equivalence trivial on  $vP_2$  and/or  $wP_2$ ?

In the usual case,  $P_2$  is known to coincide with  $P_2^{cc}$ . This follows from the fact that the automorphism of  $RF_2$  associated to the generator of  $P_2^{cc}$  (which is the image of the generator of  $P_2 = \mathbb{Z}$ ) has no finite order.

Finally, recall from the introduction that one of the main features of welded knotted object is that they are realized topologically as ribbon 2-knotted objects in 4-space, via Satoh's Tube map. Although less central in the present note, the following question seems worth adding.

*Question 5.6.* Is the Tube map, from welded string links to ribbon tubes [2], injective?

This question is in general open, see [4, 5]. It is true when restricting to welded braids by [7] and to string links up to self-virtualization [2], but fails in the case of welded knots [16].

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