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Abstract

We study the long-time behavior of variants of the telegraph process with position-dependent jump-rates, which result in a monotone gradient-like drift toward the origin. We compute their invariant laws and obtain, via probabilistic couplings arguments, some quantitative estimates of the total variation distance to equilibrium. Our techniques extend ideas previously developed for a simplified piecewise deterministic Markov model of bacterial chemotaxis.

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1 Introduction

1.1 The model and main results

Piecewise Deterministic Markov Process (PDMP) have been extensively studied in the last two decades (see [6, 7, 15] for general background) and have recently received renewed attention, motivated by their natural application in areas such as biology [22, 8], communication networks [9] or reliability of complex systems, to name a few. Understanding the ergodic properties of these models, in particular the rate at which they stabilize towards equilibrium, has in turn increased the interest in the long-time behavior of PDMPs.

In this paper we pursue the study of these questions on PDMP models of bacterial chemotaxis, initiated in [10, 11] by means of analytic tools, and deepened in [13] and [20] on simplified versions that can be seen as variants of Kac's classic "telegraph process" [16].

We consider the simple PDMP of kinetic type $(Z_t)_{t\geq 0} = ((Y_t, W_t))_{t\geq 0}$ with values in $\mathbb{R} \times \{-1, +1\}$ and infinitesimal generator

$$Lf(y, w) = w\partial_y f(y, w) + \big(a(y)\mathbb{1}_{\{yw\leq 0\}} + b(y)\mathbb{1}_{\{yw>0\}}\big)(f(y, -w) - f(y, w)), \tag{1}$$

where a and b are nonnegative functions in $\mathbb{R}$. That is, the continuous component Y evolves according to $\frac{dY_t}{dt} = W_t$ and represents the position of a particle on the real line, whereas the discrete component W represents the velocity of the particle and jumps between $+1$ and -1, with instantaneous state-dependent rate given by $a(y)$ (resp. $b(y)$) if the particle at position y approaches (resp. goes away from) the origin. This process describes, in a naive way, the motion of flagellated bacteria as a sequence of linear "runs", the directions of which randomly change at rates that depend on the position of the bacterium. The emergence of macroscopical drift is expected when the response mechanism favors longer runs in specific directions (reflecting the propensity to move for instance towards a source of nutriments). We refer the reader to [22] for a scaling limit of the processes introduced in [10, 11] that leads to simplified models like (1).

In the particular case where the jump rates are constants such that $b > a > 0$, the convergence to equilibrium of the process (1) has been investigated in a previous work [13], where fully explicit and asymptotically sharp (in the natural diffusive scaling limit of the process) bounds were obtained. We also refer the reader to [20] for the study of related models in the circle, relying on a spectral decomposition, and to [21] for a general approach to some kinetic models including the above one, based on functional inequalities.

In the present work we will consider position dependent jump-rates which throughout will be assumed to satisfy:

Hypothesis 1.1. *Function b (resp. a) is measurable, even, non decreasing (resp. non increasing) on $[0, +\infty)$, bounded from below by $\underline{b} > 0$ (resp. $\underline{a} > 0$). Moreover we assume that $b(y) > a(y)$ for all $y \neq 0$.*

In the sequel, $\bar{b}$ stands for $\sup_{y>0} b(y) \in [\underline{b}, \infty]$ and $\operatorname{sgn} : \mathbb{R} \to \{-1, +1\}$ denotes de function

$$\operatorname{sgn}(y) = \mathbb{1}_{\{y\geq 0\}} - \mathbb{1}_{\{y<0\}}.$$

Let us denote by $\mu_t^{y,w}$ the law of $Z_t = (Y_t, W_t)$ when issued from $Z_0 = (y, w)$. The following is a our main result:

Theorem 1.2 (Convergence to equilibrium)**.** *There exists $\kappa > 0$, $K > 0$, and $\lambda > 0$ such that for any $y, \tilde{y} \in \mathbb{R}$ and $w, \tilde{w} \in \{-1, +1\}$,*

$$\left\| \mu_t^{y,w} - \mu_t^{\tilde{y},\tilde{w}} \right\|_{\mathrm{TV}} \leq K e^{\kappa |y| \vee |\tilde{y}|} e^{-\lambda t}. \tag{2}$$

The constants above can be expressed in terms of the functions a and b following the lines of the proof. We will try to provide as explicit as possible bounds in each of its steps.

The proof of Theorem 1.2 relies on a probabilistic coupling argument, reminiscent of Meyn-Tweedie-Foster-Lyapunov techniques, see [19, 17]. Variants of this type of methods have been developed in several previous works on specific instances of PDMP [4, 2, 1, 18]. The model under study in the present paper is harder to deal with, since the vector fields that drive the continuous part are not contractive.

Our approach will be based on extensions of some ideas and methods developed in [13]. However, due to the non constant jump-rates we will have to work with explicit couplings of jump-times, for two copies of the process found at different positions. Moreover, we will need to make use of some discrete time Markov process embedded in their trajectories (reminiscent of [3]), in order to obtain controls of the global coupling time. These additional technicalities prevent us from getting estimates as explicit as in [13].

Before delving into the proof of Theorem 1.2, we point out the explicit form of the equilibrium of the process (Y, W) and its relation to one dimensional diffusion processes in a convex potential.

Proposition 1.3 (Invariant distribution)**.** *The invariant distribution of* (Y, W) *on* $\mathbb{R} \times \{-1, +1\}$ *is given by*

$$\mu(dy, dw) = \frac{1}{C_F} e^{-F(y)} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw)$$

where $C_F := \int_{\mathbb{R}} e^{-F(y)}\, dy < \infty$ *and* F *is the convex function*

$$y \in \mathbb{R} \mapsto F(y) = \int_0^y \mathrm{sgn}(z)(b(z) - a(z))\, dz. \tag{3}$$

The domain of the Laplace transform of μ *is* $(-\bar{b} + \underline{a}, \bar{b} - \underline{a}) \times \{-1, +1\}$.

Example 1.4 (Laplace and Gaussian equilibria)**.** *If* a *and* b *are constant functions, then*

$$\mu(dy, dw) = \frac{b-a}{2} e^{-(b-a)|y|} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

If a *is a constant function and* b *is the map* $y \mapsto a + |y|$, *then*

$$\mu(dy, dw) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

Figure 1 compares in the latter case the empirical law of Y_t *to its invariant measure at increasing time instants.*

Proof of Proposition 1.3. We first note that the constant C_F is finite as soon as the functions a and b satisfy Hypothesis 1.1, since $z \mapsto b(z) - a(z)$ is non decreasing and positive on $(0, \infty)$. Furthermore,

$$\lim_{y \to +\infty} \frac{1}{y} \int_0^y (b(z) - a(z))\, dz = \bar{b} - \underline{a}.$$

This ensures that the Laplace transform of μ is finite on $(-\bar{b} + \underline{a}, \bar{b} - \underline{a}) \times \{-1, +1\}$ and infinite on the complement. For any function $f \in \mathcal{C}^1$ on $\mathbb{R} \times \{-1, +1\}$ with compact support one has, from the definition of F,

$$\begin{aligned} Lf(y,1) + Lf(y,-1) &= \partial_y(f(y,1) - f(y,-1)) - \mathrm{sgn}(y)(b(y) - a(y))(f(y,1) - f(y,-1)) \\ &= \partial_y(f(y,1) - f(y,-1)) - F'(x)(f(y,1) - f(y,-1)). \end{aligned}$$

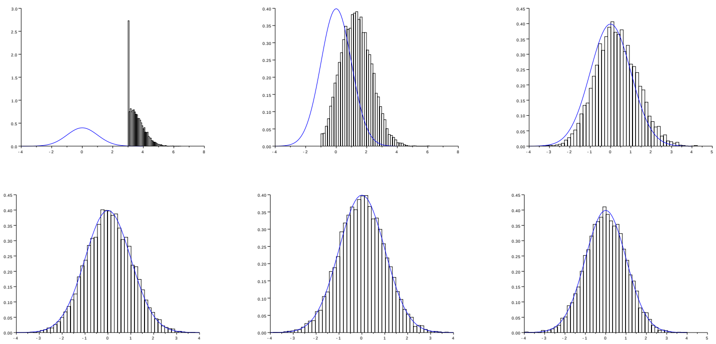

Figure 1: Empirical distribution of Y_t starting at $(5,-1)$ for $t \in \{2, 6, 10, 14, 18, 22\}$ with $a(y) = 1$ and $b(y) = 1 + |y|$.

An integration by parts ensures that

$$\int (\partial_y f(y,1) - \partial_y f(y,-1)) e^{-F(y)}\, dy = \int (f(y,1) - f(y,-1)) F'(y) e^{-F(y)}\, dy,$$

which yields $\int Lf(y,w)\,\mu(dy,dw) = 0$. In other words, μ is an invariant measure for L. $\square$

The next result is independent of the previous and can be seen as a generalization of the renormalisation of the telegraph process studied by Kac [16] (see also [14], [13]). It shows that, under the suitable scaling, process (1) behaves like the diffusion processes expected from Proposition 1.3:

Theorem 1.5 (Diffusive scaling)**.** *For each $N \geq 1$ let $a_N, b_N : \mathbb{R} \to \mathbb{R}_+$ be jump-rates satisfying Hypothesis 1.1 such that $y \mapsto a_N(y) + b_N(y)$ is of class $\mathcal{C}^1$ and*

i) $a_N(0) + b_N(0) \to \infty$,

ii) $b_N - a_N \to 2c_1$ and $\frac{a_N' + b_N'}{a_N + b_N} \to 2c_2$ locally uniformly for some functions $c_1, c_2 : \mathbb{R} \to \mathbb{R}$

when $N \to \infty$. Let $(Y_t^{(N)}, W_t^{(N)})_{t\geq 0}$ denote the process driven by (1) with $a = a_N, b = b_N$ and assume that $Y_0^{(N)} \to \xi_0$ in law as $N \to \infty$. Then, the sequence of processes

$$\left(\xi_\cdot^{(N)}\right)_{N\geq 1} := \left(Y_{\tau_N(\cdot)}^{(N)}\right)_{N\geq 1},$$

with τ_N the solution of $\tau_N'(t) = \frac{1}{2}(a_N(Y_{\tau_N(t)}^{(N)}) + b_N(Y_{\tau_N(t)}^{(N)}))$, $\tau_N(0) = 0$, weakly converges in $C([0,\infty), \mathbb{R})$ when $N \to \infty$ to the solution ξ_t of the stochastic differential equation

$$d\xi_t = dB_t - (\operatorname{sgn}(\xi_t) c_1(\xi_t) + c_2(\xi_t))\, dt\,, \tag{4}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion independent from ξ_0.

Remark 1.6 (Diffusion in a convex potential)**.** *The drift term in (4) is odd since a_N and b_N are even. Notice from point ii) above that $\frac{a_N(y)+b_N(y)}{a_N(0)+b_N(0)} = e^{2\int_0^y c_2(s)ds + o(1)}$ uniformly on compact sets as $N \to \infty$. Hence $a_N(y) + b_N(y) \to \infty$ for all ≥ 0 by i), and $c_2 = 0$ if and only if $a_N(y) + b_N(y) \sim a_N(0) + b_N(0)$ for each $y \geq 0$. Thus, any diffusion with generator of the form $\frac{1}{2} f''(y) - U'(y) f'(y)$ for an even convex potential U can be obtained as a limit, taking for instance $a_N(y) = a_N(0) \longrightarrow \infty$ and $b_N(y) = a_N(0) + 2U'(y)$.*

The remainder of the paper is organized as follows. In the next subsection we briefly recall generalities on the coupling approach to long-time convergence in total variation distance. We also define therein the reflected version of process (1), a detailed study of which is crucial for proving Theorem 1.2. Section 2 is devoted to the study of jump and hitting times of the latter process. A coalescent coupling for it is then constructed in Section 3 and the corresponding convergence estimates are established. The proof of Theorem 1.2 is achieved in Section 4. Finally, Theorem 1.5 is proved in Section 5.

1.2 Preliminaries

In the sequel we will use the notation $\stackrel{\mathcal{L}}{=}$ meaning "equal in law to" and $E(\lambda)$ for an exponential random variable with mean $1/\lambda$.

Recall that the total variation distance between two probability measures η and $\tilde{\eta}$ on a measurable space $\mathcal{X}$ is given by

$$\|\eta-\tilde{\eta}\|_{\mathrm{TV}} = \inf\left\{\mathbb{P}(X\neq\tilde{X})\ :\ X,\tilde{X} \text{ random variables with } \mathcal{L}(X)=\eta,\ \mathcal{L}(\tilde{X})=\tilde{\eta}\right\}. \quad (5)$$

If η and $\tilde{\eta}$ are absolutely continuous with respect to a measure ν with respective densities f and $\tilde{f}$ then

$$\|\eta-\tilde{\eta}\|_{\mathrm{TV}} = \frac{1}{2}\int\left|f-\tilde{f}\right|d\nu = 1-\int f\wedge\tilde{f}\,d\nu.$$

See [17] for alternative definitions of this distance and its main properties. A pair of stochastic processes $(U_t,\tilde{U}_t)_{t\geq 0}$ constructed in the same probability space, for which an almost surely finite random time T satisfying $U_{t+T}=\tilde{U}_{t+T}$ for any $t\geq 0$ exists, is called a coalescent coupling. The random variable

$$T_*=\inf\left\{t\geq 0 : U_{t+s}=\tilde{U}_{t+s}\,\forall s\geq 0\right\}$$

is then called the *coupling time*. It follows in this case that

$$\left\|\mathcal{L}(U_t)-\mathcal{L}(\tilde{U}_t)\right\|_{\mathrm{TV}}\leq\mathbb{P}(T_*>t).$$

A helpful notion in obtaining an effective control of the distance is *stochastic domination* (see [17] for a complete introduction).

Definition 1.7 (Stochastic domination)**.** *Let S and T be two real random variables with respective cumulative distribution functions F and G. We say that S is stochastically smaller than T and we write $S\leq_{\text{sto.}}T$, if $F(t)\geq G(t)$ for any $t\in\mathbb{R}$.*

In particular, for a couple $(U_t,\tilde{U}_t)$ as above, Chernoff's inequality yields

$$\left\|\mathcal{L}(U_t)-\mathcal{L}(\tilde{U}_t)\right\|_{\mathrm{TV}}\leq\mathbb{P}(T>t)\leq\mathbb{E}\left(e^{\lambda T}\right)e^{-\lambda t} \quad (6)$$

for any non-negative random variable T such that $T_*\leq_{\text{sto.}}T$, and any $\lambda\geq 0$ in the domain of the Laplace transform $\lambda\mapsto\mathbb{E}\left(e^{\lambda T}\right)$ of T.

We will use these ideas to obtain the exponential convergence to equilibrium for (Y,W) in Theorem 1.2, and in Theorem 1.9 below for its reflected version (X,V) which we now introduce. The Markov process $((X_t,V_t))_{t\geq 0}$ is defined by its infinitesimal generator:

$$Af(x,v)=v\partial_x f(x,v)+\left(a(x)\mathbb{1}_{\{v=-1\}}+b(x)\mathbb{1}_{\{v=1\}}+\frac{\mathbb{1}_{\{x=0\}}}{\mathbb{1}_{\{x>0\}}}\right)(f(x,-v)-f(x,v)), \quad (7)$$

where the maps a and b satisfy Hypothesis 1.1. The term $\mathbb{1}_{\{x=0\}}(\mathbb{1}_{\{x>0\}})^{-1}$ means that V flips from -1 to $+1$ as soon as X hits zero. In other words, X is reflected at zero.

Remark 1.8. *Given a path $((Y_t, W_t))_{t\geq 0}$ driven by (1), a path of $((X_t, V_t))_{t\geq 0}$ can be constructed taking*

$$X_t = |Y_t|, \quad V_0 = \operatorname{sgn}(Y_0)W_0$$

and defining the set of jump times of V to be

$$\{t > 0 \,:\, \Delta V_t \neq 0\} = \{t > 0 \,:\, \Delta W_t \neq 0\} \cup \{t > 0 \,:\, Y_t = 0\}.$$

Since W does not jump with positive probability when Y hits the origin, one can also construct a path of $((Y_t, W_t))_{t\geq 0}$ from an initial value $y \in \mathbb{R}$ and a path $((X_t, V_t))_{t\geq 0}$ driven by (7): set $\sigma_0 = 0$ and $(\sigma_i)_{i\geq 1}$ for the successive hitting times of the origin and

$$(Y_t, W_t) = (-1)^i \operatorname{sgn}(y)(X_t, V_t) \quad \text{if } t \in [\sigma_i, \sigma_{i+1}).$$

Let us state our results about the long time behavior of (X, V).

Theorem 1.9. *The invariant measure of (X, V) is the product measure on $\mathbb{R}_+ \times \{-1, +1\}$ given by*

$$\nu(dx, dv) = \frac{2}{C_F} e^{-F(x)}\, dx \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dv).$$

where F and C_F are given by (3). Moreover, denoting by $\nu_t^{x,v}$ the law of (X_t, V_t) when $X_0 = x$ and $V_0 = v$, there exists $\lambda > 0$, $K > 0$ and $c > 0$ such that, for any $x, \tilde{x} \geq 0$ and $v, \tilde{v} \in \{-1, +1\}$,

$$\left\| \nu_t^{x,v} - \nu_t^{\tilde{x},\tilde{v}} \right\|_{\mathrm{TV}} \leq K e^{-\lambda t} e^{c(x \vee \tilde{x})}. \tag{8}$$

Following the lines of the proof, the constants λ, K and c can be expressed in terms of the jump rate functions a and b. Let us summarize some important random times involved in the proof of Theorem 1.9 (all related to the reflected process):

- $T_{(x,v)}$ stands for the first jump time starting at (x, v) and $\varphi_{(x,v)}$ stands for its Laplace transform,
- $Z_{(x,v)}$ stands for the first hitting time of $(0, +1)$ starting at (x, v),
- T_c stands for the first crossing time of the continuous components of two paths,
- T_* stands for the coupling time (i.e. from T_* on the two paths are equal forever).

Roughly, we will let evolve both paths until the first crossing time T_c (which is stochastically controlled by hitting times of 0) and then couple the whole processes by using explicit couplings of the jump-times. Notice that it is not obvious to deduce Theorem 1.2 from Theorem 1.9.

2 Basic properties of the reflected process

2.1 Distribution of the jump times

Let us denote by $T_{(x,v)}$ the first jump time of the stochastic process (X, V) starting from $(X_0, V_0) = (x, v)$ with infinitesimal generator defined by (7). This random time satisfies

$$T_{(x,v)} = \inf\left\{ t \geq 0 : \int_0^t c(X_s) \geq E \right\},$$

where E is an exponential variable with unit mean and c stands for the function b when $v = +1$ and, when $v = -1$,

$$c(x) = \begin{cases} a(x) & \text{for } x > 0, \\ +\infty & \text{for } x \leq 0. \end{cases}$$

The process (X, V) being deterministic between jump times, we have

$$T_{(x,v)} = \inf\left\{ t \geq 0 : \int_0^t c(x + vs) \geq E \right\}.$$

Consequently, if B is the primitive of b with $B(0) = 0$, we have

$$T_{(x,+1)} = B^{-1}(E + B(x)) - x, \tag{9}$$

and if A is the primitive of a with $A(0) = 0$,

$$T_{(x,-1)} = \begin{cases} x - A^{-1}(A(x) - E) & \text{if } E < A(x), \\ x & \text{otherwise.} \end{cases} \tag{10}$$

The functions A^{-1} and B^{-1} are well defined since a and b are positive functions.

Lemma 2.1 (Law of jump times). *Let $x \in \mathbb{R}_+$. The random variable $T_{(x,+1)}$ is absolutely continuous with density given by*

$$t \mapsto b(t + x)e^{-(B(t+x)-B(x))}\mathbb{1}_{(0,\infty)}(t).$$

The random variable $T_{(x,-1)}$ is a mixture of an absolutely continuous random variable and the constant variable x. Its distribution is

$$e^{-A(x)}\delta_x + a(x - t)e^{-(A(x)-A(x-t))}\mathbb{1}_{[0,x]}(t)dt,$$

where δ_x denotes the Dirac mass at x and dt the Lebesgue measure on $\mathbb{R}$.

Proof. We notice that $T_{(x,+1)}$ is almost surely finite since $\int_0^\infty b(x + s)ds = +\infty$. Let E be an exponential variable with unit mean and $t \geq 0$, then

$$\mathbb{P}(T_{(x,+1)} > t) = \mathbb{P}\left(\int_0^t b(x + s)ds < E \right) = e^{-\int_0^t b(x+s)ds}.$$

We obtain the density of $T_{(x,+1)}$ by derivation. The distribution of $T_{(x,-1)}$ is similarly obtained noting that $\mathbb{P}(E > A(x)) = e^{-A(x)}$. $\square$

Lemma 2.2 (Laplace transform of jump times). *Let $x \geq 0$ be fixed. The Laplace transform of $T_{(x,+1)}$ is finite on $(-\infty, \overline{b})$. Furthermore, if $\lambda < b(x)$,*

$$\mathbb{E}\left[e^{\lambda T_{(x,+1)}} \right] \leq \frac{b(x)}{b(x) - \lambda}.$$

Proof. Let $x \geq 0$ be fixed. Since b is non decreasing, we first we notice that $T_{(x,+1)} \leq_{\text{sto.}} E/b(x)$, where E is an exponential variable with mean 1. Then its Laplace transform is at least defined on $(-\infty, b(x))$ and is bounded from above by that of $E/b(x)$ on this interval.

Let us now fix $\lambda < \overline{b}$. Thus there exists $z \geq 0$ such that $\lambda < b(x+z)$. The distribution of $T_{(x,+1)}$ conditional on $T_{(x,+1)} > z$ is equal to the distribution of $z + T_{(x+z,+1)}$ and the Laplace transform of $T_{(x,+1)}$ can be split as follows

$$\begin{aligned}\mathbb{E}\Big[e^{\lambda T_{(x,+1)}}\Big] &= \mathbb{E}\Big[e^{\lambda T_{(x,+1)}} \mathbb{1}_{T_{(x,+1)} \leq z}\Big] + \mathbb{E}\Big[e^{\lambda T_{(x,+1)}} \mathbb{1}_{T_{(x,+1)} > z}\Big] \\ &= \int_0^z b(x+t) e^{\int_0^t (\lambda - b(x+u))du} dt + e^{\int_0^z (\lambda - b(x+u))du} \mathbb{E}\Big[e^{\lambda T_{(x+z,+1)}}\Big].\end{aligned}$$

which is finite from the definition of z. $\square$

Lemma 2.3 (Stochastic order for jump times)**.** *For $0 \leq \tilde{x} < x$, we have*

$$T_{(x,+1)} \leq_{\text{sto.}} T_{(\tilde{x},+1)} \quad \textit{and} \quad T_{(x,-1)} \geq_{\text{sto.}} T_{(\tilde{x},-1)}.$$

Proof. We first consider $v = +1$. Let E be an exponential variable with unit mean and

$$T_{(x,+1)} := B^{-1}(E + B(x)) - x \quad \text{and} \quad T_{(\tilde{x},-1)} := B^{-1}(E + B(\tilde{x})) - \tilde{x}$$

We have

$$\int_0^{T_{(x,+1)}} b(x+s)ds = E = \int_0^{T_{(\tilde{x},+1)}} b(\tilde{x}+s)ds$$

with b a non-decreasing function and $x > \tilde{x}$, then clearly $T_{(x,+1)} \leq T_{(\tilde{x},+1)}$ (inequalities are strict when the jump rates are strictly monotone functions). The proof is similar for $T_{(x,-1)}$. $\square$

We now give helpful results in order to construct a coalescent coupling of two processes starting from different initial data, which is in fact a generalization of a well known result on exponential random variables. Indeed, when $a < b$ are positive constants, the following equalities in distribution hold:

$$\begin{aligned}E(b) &\stackrel{\mathcal{L}}{=} E(a) \wedge E(b-a), \\ E(a) &\stackrel{\mathcal{L}}{=} E(b) + \varepsilon\, E'(a),\end{aligned}$$

where random variables on the left hand side are independent and ε is Bernoulli with parameter $(1 - a/b)$.

Lemma 2.4 (Decomposition of jump times, part I)**.** *For $x \geq \tilde{x} \geq 0$, the identity in law*

$$T_{(x,+1)} \stackrel{\mathcal{L}}{=} T_{(\tilde{x},+1)} \wedge Z_+$$

holds, with Z_+ a random variable with values in $(0,\infty]$ independent of $T_{(\tilde{x},+1)}$, such that

$$\mathbb{P}(Z_+ > t) = \exp\left(-B(x+t) + B(x) + B(\tilde{x}+t) - B(\tilde{x})\right) \textit{ for all } t \in [0, \tilde{x}). \tag{11}$$

Moreover, there exists a coupling $(T_{(x,+1)}, T_{(\tilde{x},+1)})$ such that, almost surely,

$$T_{(\tilde{x},+1)} \geq T_{(x,+1)}$$

and, conditionally on $\{T_{(x,+1)}=t\}$,

$$T_{(\tilde{x},+1)} \stackrel{\mathcal{L}}{=} t+\xi_t \hat{T}_{(\tilde{x}+t,+1)}$$

with $\hat{T}_{(\tilde{x}+t,+1)} \stackrel{\mathcal{L}}{=} T_{(\tilde{x}+t,+1)}$ *and* ξ_t *a Bernoulli r.v. independent of* $\hat{T}_{(x+t,+1)}$ *of parameter*

$$\beta_t := \frac{b(x+t)-b(\tilde{x}+t)}{b(x+t)} \in [0,1).$$

We observe that if $b(x)$ goes to $+\infty$ as $x\to\infty$, we have $Z_+<+\infty$ a.s., whereas $Z_+=+\infty$ a.s. if $b(x)=b$ is constant.

Proof. Since b is a non-decreasing function, we have for any $t\geq 0$,

$$B(t+x)-B(x) \geq B(t+\tilde{x})-B(\tilde{x}).$$

Using the representation (9) and the memoryless property of the exponential distribution, we thus have for all $t\geq 0$ that

$$\begin{aligned}\mathbb{P}\big(T_{(x,+1)}>t\big) &= \mathbb{P}(E>B(x+t)-B(x)) \\ &= \mathbb{P}\big(T_{(\tilde{x},+1)}>t\big)\mathbb{P}(E>B(x+t)-B(x)-B(\tilde{x}+t)+B(\tilde{x})).\end{aligned}$$

The first statement follows. We next check that $(T_{(x,+1)},T_{(\tilde{x},+1)}) := (T_{(\tilde{x},+1)}\wedge Z_+, T_{(\tilde{x},+1)})$, with $(T_{(\tilde{x},+1)},Z_+)$ as before, is the required coupling. Since

$$\{T_{(\tilde{x},+1)}>Z_+\} = \{T_{(\tilde{x},+1)}>T_{(x,+1)}\}, \tag{12}$$

we deduce that $T_{(\tilde{x},+1)} = T_{(x,+1)} + \big(T_{(\tilde{x},+1)}-T_{(x,+1)}\big)\,\mathbb{1}_{\{T_{(\tilde{x},+1)}>T_{(x,+1)}\}}$. Thus, we just need to check that, conditionally on $\{T_{x,+1}=t\}$,

$$(\mathbb{1}_{\{T_{(\tilde{x},+1)}>T_{(x,+1)}\}}, T_{(\tilde{x},+1)}-T_{(x,+1)}) \stackrel{\mathcal{L}}{=} (\xi_t,\hat{T}_{(\tilde{x}+t,+1)}).$$

Using (12), and the expression for the density of $T_{(\tilde{x},+1)}$ together with (11), we get

$$\begin{aligned}\mathbb{P}\left(T_{(\tilde{x},+1)}>T_{(x,+1)}+r, T_{(x,+1)}>s\right) &= \mathbb{P}\left(T_{(\tilde{x},+1)}>Z_++r, Z_+>s\right) \\ &= \int_s^\infty \left[\frac{b(x+t)-b(\tilde{x}+t)}{b(x+t)} e^{-(B(\tilde{x}+r+t)-B(\tilde{x}+t))}\right] e^{-(B(x+t)-B(x))} b(x+t)dt\end{aligned}$$

for all $s,r\geq 0$. Alternatively,

$$\begin{aligned}&\mathbb{P}\left(T_{(\tilde{x},+1)}>T_{(x,+1)}+r, T_{(x,+1)}>s\right) = \\ &\quad\int_s^\infty \mathbb{P}\left(T_{(\tilde{x},+1)}-T_{(x,+1)}>r, T_{(\tilde{x},+1)}>T_{(x,+1)} | T_{(x,+1)}=t\right) e^{-(B(x+t)-B(x))} b(x+t)dt.\end{aligned}$$

Taking derivative with respect to s in the two above integrals, one concludes by comparing the two different obtained expressions. $\square$

The function a being non-increasing, we obtain an analogous result for $T_{(x,-1)}$:

Lemma 2.5 (Decomposition of jump times, part II)**.** *For $x > \tilde{x} > 0$, the identity in law*

$$T_{(\tilde{x},-1)} \overset{\mathcal{L}}{=} T_{(x,-1)} \wedge Z_-$$

holds, with Z_- a random variable with values in $(0, \tilde{x}]$ independent of $T_{(x,+1)}$, such that

$$\mathbb{P}(Z_- > t) = \exp\left(-A(\tilde{x}) + A(\tilde{x} - t) + A(x) - A(x - t)\right) \text{ for all } t \in [0, \tilde{x}). \tag{13}$$

Moreover, there exists a coupling $(T_{(x,-1)}, T_{(\tilde{x},-1)})$ such that, almost surely,

$$T_{(x,-1)} \geq T_{(\tilde{x},-1)}$$

and, conditionally on $\{T_{(\tilde{x},-1)} = t\}$,

$$T_{(x,-1)} \overset{\mathcal{L}}{=} t + \chi_t \hat{T}_{(x-t,-1)}$$

with $\hat{T}_{(x-t,-1)} \overset{\mathcal{L}}{=} T_{(x-t,+1)}$ and χ_t a Bernoulli r.v. independent of $\hat{T}_{(x-t,+1)}$ of parameter

$$\alpha_t := \frac{a(\tilde{x} - t) - a(x - t)}{a(\tilde{x} - t)} \in [0, 1) \quad \text{if } t < \tilde{x} \quad \text{and} \quad \alpha_{\tilde{x}} := 1.$$

Example 2.6 (Explicit laws for jump times)**.** *In the case $b(x) = \underline{b} + x$ with $\underline{b} > 0$ (as in the TCP model studied in [9, 5]), $T_{(x,+1)}$ has the density*

$$f_{(x,+1)}(t) = (\underline{b} + x + t)e^{-\frac{(t+x+\underline{b})^2 - (x+\underline{b})^2}{2}} \mathbb{1}_{\{t>0\}}$$

and an everywhere finite Laplace transform given by $\mathbb{E}[e^{\lambda T_{(x,+1)}}] = 1 + \lambda\eta(x + \underline{b} - \lambda)$, with $\eta(u) = e^{\frac{u^2}{2}}\sqrt{2\pi}(1 - \Phi(u))$ and Φ the cumulative distribution function of a standard Gaussian variable. We also notice in this case that for $0 \leq \tilde{x} \leq x$,

$$T_{(x,+1)} \overset{\mathcal{L}}{=} T_{(\tilde{x},+1)} \wedge E(x - \tilde{x}),$$

for $E(x - \tilde{x})$ an exponential variable of mean $1/(x - \tilde{x})$ independent of $T_{(\tilde{x},+1)}$, and

$$\mathbb{P}\big(E(x - \tilde{x}) > T_{(\tilde{x},+1)}\big) = 1 - (x - \tilde{x})\eta(x + \underline{b}).$$

2.2 Hitting time of the origin

Let $(x, v) \in \mathbb{R}_+ \times \{-1, +1\}$. We notice that

$$Z_{(x,+1)} \overset{\mathcal{L}}{=} Z_{(x,-1)} + S_x,$$

where $Z_{(x,v)}$ is the first hitting time of $(0, +1)$ of a path starting from (x, v) and S_x is an excursion above x independent of $Z_{(x,-1)}$. Consequently $Z_{(x,-1)}$ is stochastically smaller than $Z_{(x,+1)}$.

The Laplace transform of the hitting time of zero starting from (x, v) was explicitly computed in [13] in the case where a and b are both constant. Let us recall this result in the following proposition.

Proposition 2.7 (Hitting time of 0 for constant jump rates [13]). *Let us define* $\lambda_c = \frac{1}{2}(\sqrt{b}-\sqrt{a})^2$ *and, for any* $\lambda \leq \lambda_c$,

$$c(\lambda) = \frac{b-a-\sqrt{(a+b-2\lambda)^2-4ab}}{2} \quad \text{and} \quad \psi(\lambda) = \frac{a+b-2\lambda-\sqrt{(a+b-2\lambda)^2-4ab}}{2a}.$$

Then, for any $\lambda \in (-\infty, \lambda_c]$,

$$\mathbb{E}\Big(e^{\lambda Z_{(x,-1)}}\Big) = e^{xc(\lambda)} \quad \text{and} \quad \mathbb{E}\Big(e^{\lambda Z_{(x,+1)}}\Big) = \psi(\lambda)e^{xc(\lambda)}.$$

Moreover, these Laplace transforms are infinite on (λ_c, ∞).

If the jump rates a and b are not constant, the evolution away from the origin is no longer invariant by translation. Consequently, we have to consider a new way to estimate the distribution of the hitting time of zero.

Proposition 2.8 (Hitting time of 0 for general jump rates). *Let* $M > 0$ *such that*

$$\sqrt{\frac{b(M)}{a(M)}}e^{M\left(\sqrt{b(M)}-\sqrt{a(M)}\right)^2}\Big(1-e^{-A(M)}\Big) < 1.$$

Then, the Laplace transform of the first hitting time $Z_{(x,v)}$ *of* $(0,+1)$ *starting from* $(x,v) \in \mathbb{R}\times\{-1,+1\}$ *satisfies*

$$\mathbb{E}\Big[e^{\lambda Z_{(x,v)}}\Big] \leq Ce^{\frac{(x\vee M)(b(M)-a(M))}{2}} \quad \text{for all } \lambda \leq \frac{1}{2}(\sqrt{b(M)}-\sqrt{a(M)})^2,$$

where $C > 0$ *is an explicit constant depending on* M, a *and* b.

Proof. We first notice that $f(x,v) = e^{\alpha x+\beta v}$ with $\alpha, \beta > 0$, is a Lyapunov function for the infinitesimal generator A of $((X_t,V_t))_{t\geq 0}$ defined by (7). More precisely, we have

$$\begin{aligned} Af(x,+1) &= f(x,+1)\Big[\alpha - b(x)\Big(1-e^{-2\beta}\Big)\Big], \\ Af(x,-1) &= f(x,-1)\Big[-\alpha + a(x)\Big(e^{2\beta}-1\Big)\Big] \end{aligned}$$

for all $x > 0$. If we choose $\alpha, \beta > 0$ and a compact set $K = [0,M]\times\{-1,1\}$ such that

$$-\alpha + a(M)\Big(e^{2\beta}-1\Big) < 0 \quad \text{and} \quad \alpha - b(M)\Big(1-e^{-2\beta}\Big) < 0, \tag{14}$$

by monotony of a and b there are $\rho = \rho(\alpha,\beta,M,a) > 0$ to be specified and $c > 0$ such that

$$Af(x,v) \leq -\rho f(x,v) + c\mathbb{1}_K(x,v).$$

The Laplace transform of the first hitting time of $[0,M)\times\{-1,1\}$ starting from (x,v), denoted $\tau_{(x,v)} := \inf\{t > 0 : (X_t,V_t) \in [0,M)\times\{-1,1\}\}$, can then classically be controlled as follows:

$$\begin{aligned} f(X_{t\wedge\tau_{(x,v)}}, V_{t\wedge\tau_{(x,v)}})e^{\rho t} &= f(x,v) + \int_0^{t\wedge\tau_{(x,v)}} [Af(X_s,V_s) + \rho f(X_s,V_s)]\, e^{\rho t} ds + N_{t\wedge\tau_{(x,v)}} \\ &\leq f(x,v) + N_{t\wedge\tau_{(x,v)}} \end{aligned}$$

where $(N_t)_{t\geq 0}$ is a martingale with respect to the filtration generated by $((X_t,V_t))_{t\geq 0}$. Taking the expectation in the previous inequality we deduce that, for $x > M$,

$$\mathbb{E}[e^{\rho\tau_{(x,v)}}] \leq e^{\alpha(x-M)+\beta(v+1)}. \tag{15}$$

We next choose $\alpha,\beta > 0$ in order to optimize ρ. For $\alpha,\beta,M > 0$ satisfying (14), we set

$$\rho = \Big[b(M)(1-e^{-2\beta})-\alpha\Big] \wedge \Big[\alpha - a(M)(e^{2\beta}-1)\Big].$$

First we choose $\alpha > 0$ such that

$$2\alpha = b(M)(1-e^{-2\beta}) + a(M)(e^{2\beta}-1)$$

and then

$$\beta = \frac{1}{4}[\log(b(M)) - \log(a(M))]. \tag{16}$$

With this choice, we have

$$2\alpha = b(M) - a(M) \quad \text{and} \quad \rho(M) = \frac{1}{2}\Big(\sqrt{b(M)} - \sqrt{a(M)}\Big)^2. \tag{17}$$

Condition (14) is satisfied for any $M > 0$ since

$$-\alpha + a(M)\Big(e^{2\beta}-1\Big) = \alpha - b(M)\Big(1-e^{-2\beta}\Big) = -\frac{1}{2}\Big(\sqrt{b(M)} - \sqrt{a(M)}\Big)^2.$$

We now obtain an estimate for the Laplace transform of $Z_{(x,+1)}$. Let $M > 0$ be arbitrarily fixed for the moment and $\lambda > 0$ such that $\lambda \leq \rho(M)$. For $x \geq M$, we have

$$Z_{(x,+1)} \stackrel{\mathcal{L}}{=} \tau_{(x,+1)} + Z_{(M,-1)} \tag{18}$$

where $\tau_{(x,+1)}$ and $Z_{(M,-1)}$ are independent.
Let us denote by T_1 and T_2 the first two jumps of V. From Figure 2.2 we see that, before a path starting from $(M,-1)$ hits $(0,+1)$, either

1. $T_1 = M$, in which case we have $Z_{(M,-1)} = M$;
2. $T_1 < M$ and $T_2 > T_1$, in which case we have
$$Z_{(M,-1)} = 2T_1 + Z_{(M,+1)} \leq_{\text{sto.}} 2M + \tau_{(M,+1)} + \tilde{Z}_{(M,-1)}; \text{ or}$$
3. $T_1 < M$ and $T_2 \leq T_1$, and then
$$Z_{(M,-1)} = T_1 + T_2 + Z_{(M-T_1+T_2,-1)} \leq_{\text{sto.}} 2M + \tilde{Z}_{(M,-1)},$$
with $\tilde{Z}_{(M,-1)}$ an independent hitting time of zero starting from $(M,-1)$.

Lemma 2.1 ensures that $\mathbb{P}_{(M,-1)}(T_1 = M) = e^{-A(M)}$. As a consequence, if φ is the Laplace transform of $Z_{(M,-1)}$, one has

$$\varphi(\lambda) \leq e^{-A(M)}e^{\lambda M} + \Big(1-e^{-A(M)}\Big)e^{2\lambda M}\mathbb{E}\Big[e^{\lambda\tau_{(M,+1)}}\Big]\varphi(\lambda).$$

For any $\lambda < \rho(M)$ we get, thanks to Hölder inequality and (15), that

$$\mathbb{E}\Big[e^{\lambda\tau_{(M,+1)}}\Big] \leq \Big(\mathbb{E}\Big[e^{\rho(M)\tau_{(M,+1)}}\Big]\Big)^{\lambda/\rho(M)} \leq e^{2\lambda\beta(M)/\rho(M)}.$$

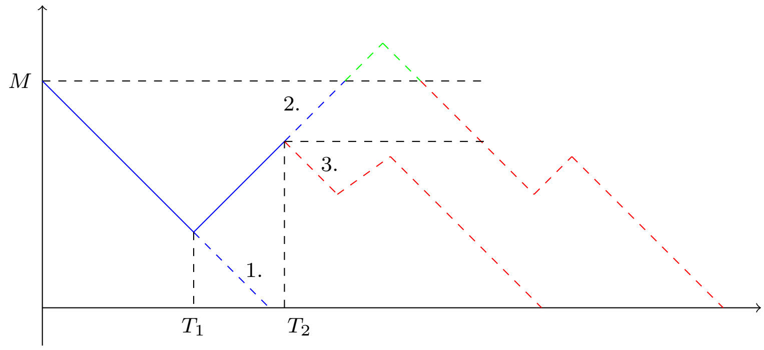

Figure 2: The three different types of paths from $(M,-1)$ to $(0,+1)$.

Thanks to (16), if M is chosen in order that

$$\sqrt{\frac{b(M)}{a(M)}}e^{M\left(\sqrt{b(M)}-\sqrt{a(M)}\right)^2}\left(1-e^{-A(M)}\right)<1$$

then for any $\lambda \leq \rho(M)$, φ is finite and

$$\varphi(\lambda) \leq \frac{e^{-A(M)}e^{\lambda M}}{1-\left(1-e^{-A(M)}\right)e^{2\lambda(M+\beta(M)/\rho(M))}}.$$

Combining (18) and (15) with the previous estimate completes the proof in the main case $x \geq M$. If $x \leq M$ then

$$Z_{(x,-1)} \leq_{\text{sto.}} Z_{(M,-1)} \quad \text{ and } \quad Z_{(x,+1)} \leq_{\text{sto.}} M+\tau_{(M,+1)}+\tilde{Z}_{(M,-1)},$$

and one can conclude as in the previous case. □

3 The coupling time for the reflected process

This section is dedicated to the construction of a coalescent coupling of two paths of the reflected process driven by (7) starting from two different initial conditions.

3.1 The first crossing time

Let us consider (x,v) and $(\tilde{x},\tilde{v})$ two initial data with $\tilde{x}<x$. The first crossing time of two paths (X,V) and $(\tilde{X},\tilde{V})$ starting respectively from (x,v) and $(\tilde{x},\tilde{v})$ is defined by

$$T_c = T_c(x,v,\tilde{x},\tilde{v}) = \inf\left\{t \geq 0 \;:\; X_t = \tilde{X}_t\right\}.$$

Since $(X_t)_{t\geq 0}$ is continuous, T_c is stochastically smaller than the hitting time of zero $Z_{(x,v)}$ of the initially upper path, whatever the joint law of the pair. The first crossing point X_{T_c} is such that

$$X_{T_c} \leq \sup_{t\in[0,Z_{(x,v)}]} X_t - (x-\tilde{x}) \leq \frac{1}{2}\left(Z_{(x,v)}+\tilde{x}-x\right). \tag{19}$$

Notice that at time T_c the two velocities are opposites.

3.2 A way to stick the two paths

In what follows we assume that $(X_0, V_0) = (x, +1)$ and $(\tilde{X}_0, \tilde{V}_0) = (x, -1)$ and construct two paths which are equal after a coalescent time $T_*(x)$. The successful coupling consists in producing a jump of (exactly) one of the two velocities V or $\tilde{V}$ at a crossing time of their position components X and $\tilde{X}$. We will use the coupled jump-times studied in Lemmas 2.4 and 2.5 in order to minimize the time required to do so, in the spirit of [13].

To be more precise, given $x > 0$ fixed, let us denote by U_+ and U_- (respectively L_- and L_+) the first and the second inter-jump time lapses of the path starting from $(x, +1)$ (resp. starting from $(x, -1)$). These random variables are constructed as follows. We first choose U_+ with distribution $T_{(x,+1)}$ and L_- with distribution $T_{(x,-1)}$ independently. We then define U_- and L_+ in such a way that (U_+, L_+) and (U_-, L_-) have the laws of the couplings defined in Lemmas 2.4 and 2.5 respectively and that $L_+ - U_+$ and $U_- - L_-$ are independent conditionally on (U_+, L_-). More precisely, conditionally on U_+ and L_-, we introduce two independent Bernoulli variables ξ and χ with

$$\mathbb{P}(\xi = 1 | U_+, L_-) = \frac{b(x + U_+) - b(x - L_- + U_+)}{b(x + U_+)} \quad \text{and}$$
$$\mathbb{P}(\chi = 1 | U_+, L_-) = \frac{a(x - L_-) - a(x + U_+ - L_-)}{a(x - L_-)} \mathbb{1}_{\{L_- < x\}} + \mathbb{1}_{\{L_- = x\}}$$

and two independent random variables $L_+ - U_+$ and $U_- - L_-$ with the same law as $T_{x+U_+-L_-,+1}$ and $T_{x+U_+-L_-,-1}$ respectively. Then we set

$$L_+ := U_+ + \xi(L_+ - U_+) \quad \text{and} \quad U_- := L_- + \chi(U_- - L_-).$$

Figure 3 shows the four possible outcomes. Those where exactly one of the Bernoulli variables is equal to 1 allow us to stick the paths at time $U_+ + L_-$ (i.e. on the rightmost corner of the rectangle): the velocities of the two paths are the same right after that instant, and the overshot length (beyond the rectangle's corner) determined by the previous coupling is compatible with the law of the two marginal processes from that moment on (because of their Markov property). We then say that the coupling attempt succeeded, and this happens conditionally on (U_+, L_-) with probability

$$\begin{aligned}
&\mathbb{P}(\xi = 0, \chi = 1 | U_+, L_-) + \mathbb{P}(\xi = 1, \chi = 0 | U_+, L_-) = \\
&\quad \left[\frac{b(x - L_- + U_+)}{b(x + U_+)} \left(1 - \frac{a(x - L_- + U_+)}{a(x - L_-)}\right) + \left(1 - \frac{b(x - L_- + U_+)}{b(x + U_+)}\right) \frac{a(x - L_- + U_+)}{a(x - L_-)} \right] \mathbb{1}_{\{L_- < x\}} \\
&\quad + \frac{b(U_+)}{b(x + U_+)} \mathbb{1}_{\{L_- = x\}}.
\end{aligned} \tag{20}$$

Observe that the success or failure of the coupling attempt is determined by the Bernoulli random variables ξ and χ. If the coupling attempt fails, the two trajectories cross or bounce off of each other at time $U_+ + L_-$ and by similar reason as before the (already determined) lengths $(L_+ - U_+)$ and $(U_- - L_-)$ can be used to restart two (upward and downward) trajectories from $x - L_- + U_+$, independently of each other conditionally on the past and consistently with the pathwise laws of each of the two processes.

The coupling construction is now obvious: we repeat this scheme starting from the new crossing or bouncing point, until we succeed in sticking the two paths. Notice that this iterative algorithm is more efficient than the general procedure of the Meyn-Tweedie

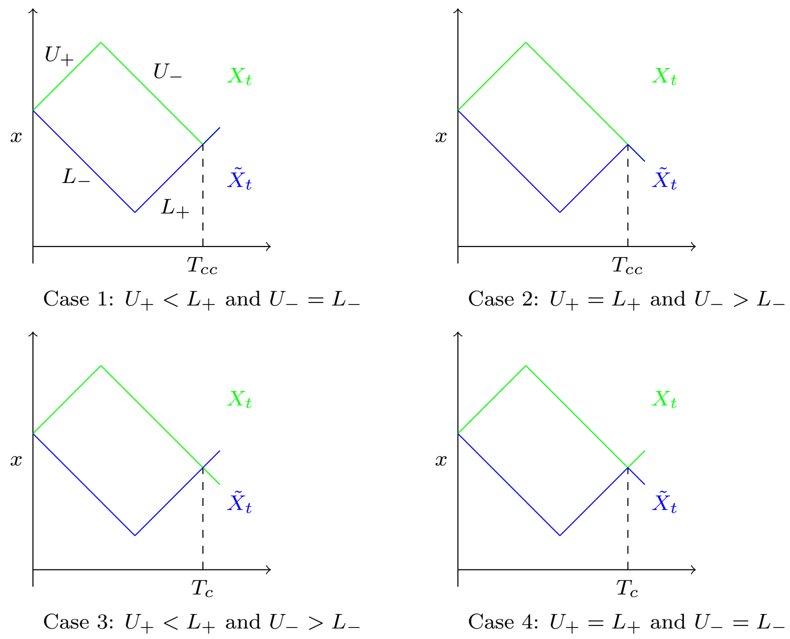

Figure 3: Position of both paths after one step.

method [19] since, after a fail, the two processes have already the same position (and still opposite velocities).

We point out that the coupling scheme implemented for the reflected process in [13] in the constant jump rates case is a particular case of the above described scheme. Notice however that here, in general, the upper path starting from $(x, +1)$ does not necessarily remain above the other path until the coupling time. We then cannot control the coupling time by the hitting time of 0 for the process starting at $(x, +1)$ as it was done in [13]. On the other hand, contrary to the constant rates case where the coupling could only succeed right after the lower process hit 0, the coupling can now succeed at an arbitrary step of the scheme, though not with a probability bounded from below uniformly in x (this can be easily seen from formula (20) e.g. in the case case when a is constant and $b(x) = a + x$). Therefore, a new approach to estimate the coupling time is needed, which is developed in next subsection.

3.3 Coupling time from a crossing point

In this section we will use the notation $\mathbb{P}_x$ (resp. $\mathbb{E}_x$) for the distribution (resp. the expectation) of a random variable associated with the coupling scheme given that the two copies started at position $x > 0$.
We first observe that for fixed $R > 0$, the probability of success in one step can be bounded from below (considering the last term in (20) and taking expectation) uniformly over $x \in [0, R]$ by some number $p_R \in (0, 1)$ satisfying

$$p_R \geq e^{-A(R)} \int_0^\infty b(u) e^{-\int_0^u b(R+s)ds} du. \tag{21}$$

This suggests that the number of trials below a fixed height $R > 0$ required in order to get a successful coupling can be stochastically dominated by some geometric random variable.

Notice that we do not expect a successful coupling to occur only below level R. We will rather use the above remark in order to construct the scheme in such a way that the coalescent time will be always smaller than or equal to some real random variable that we can control in terms of geometric number of positive time lapses.

First, we define a sequence of "rectangles" of potential trajectories of the two copies in the coupling scheme, on which the two copies will live at all times, irrespective of whether the coupling attempt has already been successful or not (of course once it has been so, their positions and velocities coincide from that moment on). More precisely, we define a discrete time Markov chain $(\Phi_n)_{n\geq 0}$ starting at x by

$$\begin{aligned} \Phi_0 &= x, \\ \Phi_{n+1} &= \Phi_n + T^{n+1}_{(\Phi_n,+1)} - T^{n+1}_{(\Phi_n,-1)}, \end{aligned} \tag{22}$$

where conditionally on all the past up to (and including) time n, $T^{n+1}_{(\Phi_n,+1)}$ and $T^{n+1}_{(\Phi_n,-1)}$ are independent and respectively equal in law to $T_{(y,+1)}$ and $T_{(y,-1)}$ on the event $\{\Phi_n = y\}$. Plainly, $(\Phi_n)_{n\geq 1}$ describes the height of the right-most corner of the $n-$th rectangle obtained by iterating the construction of Figure 3. Consider also the sequence of positive random variables (real time lengths) $(\sigma_n)_{n\geq 0}$ defined by

$$\begin{aligned} \sigma_0 &= 0, \\ \sigma_{n+1} &= T^{n+1}_{(\Phi_n,+1)} + T^{n+1}_{(\Phi_n,-1)}, \end{aligned} \tag{23}$$

which give the (real) time-position of the rectangles' right-most corners, and finally set $\Sigma_n = \sum_{i=1}^n \sigma_i$, with the convention $\Sigma_0 = 0$. Following Lemmas 2.4 and 2.5 and in order to determine at which attempt the coupling is successful, we introduce two sequences $(\xi_n)_{n\geq 1}$ and $(\chi_n)_{n\geq 1}$ of Bernoulli random variables, conditionally independent of each other given $(\sigma_k, \Phi_k)_{k\geq 0}$ and such that for $n \geq 0$, on $\left\{\Phi_n = y, T^{n+1}_{(\Phi_n,+1)} = t, T^{n+1}_{(\Phi_n,-1)} = s\right\}$,

$$\begin{aligned} \mathbb{P}(\xi_{n+1} = 1 | (\sigma_k, \Phi_k)_{k\leq n+1}) &= \frac{b(y+t) - b(y-s+t)}{b(y+t)}, \qquad \text{and} \\ \mathbb{P}(\chi_{n+1} = 1 | (\sigma_k, \Phi_k)_{k\leq n+1}) &= \frac{a(y-s) - a(y+t-s)}{a(y-s)} \mathbb{1}_{\{s<y\}} + \mathbb{1}_{\{s=y\}}. \end{aligned}$$

We also set $\xi_0 = \chi_0 = 1$ for notational simplicity. Observe that $(\sigma_n, \Phi_n, \xi_n, \chi_n)_{n\geq 0}$ is Markov process. We denote by $(\mathcal{F}_n)_{n\geq 0}$ the filtration it generates. We then define a sequence of random variables (κ_n) by

$$\kappa_n = \mathbb{1}_{\{\xi_n=1, \chi_n=0\}} + \mathbb{1}_{\{\xi_n=0, \chi_n=1\}}.$$

According to the discussion at the end of the previous subsection, the discrete time instant (or rectangle number) at which the coupling succeeds is

$$\rho := \inf\{n \geq 1 : \kappa_n = 1\}$$

and the real time spent in order that this happens is

$$T_* := \Sigma_\rho = \sum_{i=1}^{\rho} \sigma_i.$$

The trajectories of the two copies can be easily constructed from the previous objects, but we actually do not need to work with them.
Let us now introduce the discrete random variable

$$\rho_R := \inf\{n \geq 1 : \kappa_n = 1 \text{ and } \Phi_{n-1} < R\}.$$

Both ρ and ρ_R are stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Since $\rho \leq \rho_R$ a.s., we clearly have

$$T_* \leq T_*^R := \sum_{i=1}^{\rho_R} \sigma_i. \tag{24}$$

Our goal now is to exhibit an upper bound for the Laplace transform of the random time T_*^R under $\mathbb{P}_x$. We need to introduce the stopping time (with respect to $(\mathcal{F}_n)_{n\geq 0}$)

$$\tau_R(x) := \inf\{n \geq 0 \,:\, \Phi_n \in [0, R]\}, \tag{25}$$

and the real time

$$\Sigma_{\tau_R}(x) := \sum_{i=0}^{\tau_R(x)} \sigma_i \tag{26}$$

accumulated when the sequence $(\Phi_n)_{n\geq 0}$ reaches $[0, R]$ for the first time. Let $\varphi_{(x,v)}$ denote the Laplace transform of $T_{(x,v)}$ for $(x, v) \in \mathbb{R}^+ \times \{-1, +1\}$. We have

Lemma 3.1. *Assume there exist positive real numbers R, λ, β such that $\lambda < \beta$, $\lambda + \beta < \bar{b}$ and $\varphi_{(R,+1)}(\beta + \lambda)\varphi_{(R,-1)}(\lambda - \beta) < 1$. Then, $\tau_R < \infty$ a.s. and*

$$\mathbb{E}_x\left[e^{\beta\Phi_{\tau_R} + \lambda\sum_{i=0}^{\tau_R}\sigma_i}\eta^{\tau_R}\right] \leq e^{\beta x \mathbb{1}_{x>R}}$$

where $\eta := (\varphi_{(R,+1)}(\beta + \lambda)\varphi_{(R,-1)}(\lambda - \beta))^{-1} > 1$.

Proof. For each $x > R$ and $\beta > \lambda$, from the stochastic monotonicity of the jump times (see Lemma 2.3) we get

$$\begin{aligned} \mathbb{E}_x\left[e^{\beta\Phi_1 + \lambda\sigma_1}\right] =& e^{\beta x}\mathbb{E}\left[e^{(\beta+\lambda)T_{x,+1}}\right]\mathbb{E}\left[e^{(\lambda-\beta)T_{x,-1}}\right] \\ \leq& e^{\beta x}\varphi_{(R,+1)}(\beta + \lambda)\varphi_{(R,-1)}(\lambda - \beta). \end{aligned} \tag{27}$$

If $\varphi_{(R,+1)}(\beta + \lambda)\varphi_{(R,-1)}(\lambda - \beta) < 1$, we deduce from (27) that $e^{\beta\Phi_{\tau_R\wedge n} + \lambda\sum_{i=1}^{n}\sigma_i}\eta^{\tau_R\wedge n}$ is a positive supermartingale with respect to $(\mathcal{F}_n)_{n\geq 0}$, hence

$$e^{\beta x \mathbb{1}_{x>R}} \geq \mathbb{E}_x\left[e^{\beta\Phi_{\tau_R\wedge n} + \lambda\sum_{i=0}^{\tau_R\wedge n}\sigma_i}\eta^{\tau_R\wedge n}\right] \geq \mathbb{E}_x\left[\eta^{\tau_R\wedge n}\right].$$

Letting $n \to \infty$ in the last expectation we get by monotone convergence $\mathbb{E}_x\left[\eta^{\tau_R}\right] < \infty$, hence $\tau_R < \infty$ a.s.. Letting then $n \to \infty$ in the first expectation and using Fatou's Lemma the statement follows. □

For each $\gamma > 0$ and $R > 0$, we now set

$$\mathcal{E}_R(\gamma) := \sup_{y\in[0,R]} \mathbb{E}_y\left[e^{\gamma T_{(y,+1)}}\mathbb{1}_{\kappa_1=0}\right].$$

Proposition 3.2 (Laplace transform of the coupling time starting at a crossing point)**.** *Assume that* (R,λ,β) *satisfy the conditions of Lemma 3.1 and moreover that* $\mathcal{E}_R(\lambda+\beta)<1$. *Then, the Laplace transform of* T_* *satisfies*

$$\mathbb{E}_x\left[e^{\lambda T_*}\right] \le \frac{e^{\beta x \mathbb{1}_{x>R}}\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)}{1-\mathcal{E}_R(\lambda+\beta)}. \tag{28}$$

Proof. Fix $x\in\mathbb{R}$ and R,λ,β satisfying conditions of Lemma 3.1. From (24), we just need to estimate T_*^R. To this aim, we consider the process $(\hat{\Phi}_n,\hat{\sigma}_n)_{n\ge 0}$ defined in terms of $(\Phi_n,\sigma_n)_{n\ge0}$ in the following way: first set $\hat{\tau}_0=0$ and $\hat{\tau}_1=\tau_R(x)+1$ and for all $n\ge 1$,

$$\hat{\tau}_{n+1}=\hat{\tau}_n+\tau_R^n+1, \quad \text{with} \quad \tau_R^n := \inf\{n\ge\hat{\tau}_n\ :\ \Phi_n\in[0,R]\}-\hat{\tau}_n.$$

In other words, $\hat{\tau}_{n+1}$ is the index of the first attempt to couple the paths that follows the first (discrete) return time $\hat{\tau}_n+\tau_R^n$ of $(\Phi_k)_{k\ge0}$ into $[0,R]$ after $\hat{\tau}_n$. Then, we set $\hat{\Phi}_0=x$, $\hat{\sigma}_0=0$, $\hat{\Phi}_1=\Phi_{\hat{\tau}_1}$, $\hat{\sigma}_1=\sum_{i=1}^{\hat{\tau}_1}\sigma_i$ and

$$\hat{\Phi}_{n+1}=\Phi_{\hat{\tau}_{n+1}}, \quad \hat{\sigma}_{n+1}=\sum_{i=\hat{\tau}_n+1}^{\hat{\tau}_{n+1}}\sigma_i.$$

Thus, $\hat{\sigma}_{n+1}$ is the sum of the real time

$$\Sigma^n_{\tau_R^n} := \sum_{i=\hat{\tau}_n+1}^{\hat{\tau}_{n+1}-1}\sigma_i$$

needed after $\hat{\sigma}_n$ in order to observe again a “rectangle corner” in $[0,R]$, plus the time $\sigma_{\hat{\tau}_{n+1}}$ spent in one coupling attempt right thereafter. Then, $\hat{\Phi}_{n+1}\in\mathbb{R}_+$ is the position of the discrete chain (or rectangle corner) at the time instant $\hat{\sigma}_{n+1}$. Notice that for each $i\ge 0$, $\hat{\tau}_i$ is a stopping time with respect to the filtration $(\mathcal{F}_n)_{n\ge0}$ and that, conditionally on $\mathcal{F}_{\hat{\tau}_n}\cap\{\hat{\Phi}_n=x\}$, $(\tau_R^n,\Sigma^n_{\tau_R^n})$ has the same law as the pair $(\tau_R(x),\Sigma_{\tau_R(x)})$ defined in (25) and (26). We can now write

$$T_*^R=\sum_{i=0}^{\hat{\rho}_R}\hat{\sigma}_i,$$

where $\hat{\rho}_R:=\inf\{n\ge1:\kappa_{\hat{\tau}_n}=1\}$ is a stopping time with respect to the filtration $(\mathcal{F}_{\hat{\tau}_n})_{n\ge0}$.

We notice that $\hat{\rho}_R<\infty$ a.s. since the probability of fail in one step starting from a position $x\in[0,R]$ is uniformly bounded on $[0,R]$ by $1-p_R$. We then can write

$$\mathbb{E}_x[e^{\lambda T_*^R}]=\sum_{n=1}^{\infty}\mathbb{E}_x\left[e^{\lambda\sum_{i=0}^n\hat{\sigma}_i}\mathbb{1}_{\hat{\rho}_R=n}\right]=\sum_{n=1}^{\infty}\mathbb{E}_x\left[e^{\lambda\sum_{i=0}^n\hat{\sigma}_i}\mathbb{1}_{\kappa_{\hat{\tau}_n}=0,\kappa_{\hat{\tau}_{n-1}}=0,\dots,\kappa_{\hat{\tau}_0}=1}\right]. \tag{29}$$

On one hand, we have

$$\begin{aligned}\mathbb{E}_x[e^{\lambda\hat{\sigma}_1}] &= \mathbb{E}\left[e^{\lambda\Sigma_{\tau_R(x)}}e^{\lambda\sigma_{\tau_R(x)+1}}\right]=\mathbb{E}\left[\mathbb{E}\left[e^{\lambda\sigma_{\tau_R(x)+1}}|\mathcal{F}_{\tau_R(x)}\right]e^{\lambda\Sigma_{\tau_R(x)}}\right]\\ &\le \mathbb{E}\left[\mathbb{E}_{\Phi_{\tau_R(x)}}\left[e^{\lambda\sigma_1}\right]e^{\lambda\Sigma_{\tau_R(x)}}\right].\end{aligned}$$

By Lemma 2.3, $\mathbb{E}_y\left[e^{\lambda\sigma_1}\right]=\varphi_{(y,+1)}(\lambda)\varphi_{(y,-1)}(\lambda)\le\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)$ for any point $y\in[0,R]$. Thus, using also Lemma 3.1 we get

$$\mathbb{E}_x[e^{\lambda\hat{\sigma}_1}]\le\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)e^{\beta x\mathbb{1}_{x>R}},$$

and

$$\mathbb{E}\Big[e^{\lambda\hat{\sigma}_n}\mathbb{1}_{\kappa_{\hat{\tau}_n}=1}|\mathcal{G}_{n-1}\Big] = \mathbb{E}_{\hat{\phi}_{n-1}}\Big[e^{\lambda\hat{\sigma}_1}\mathbb{1}_{\kappa_{\hat{\tau}_1}=1}\Big] \le \varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)e^{\beta\hat{\phi}_{n-1}\mathbb{1}_{\hat{\phi}_{n-1}>R}}.$$

On the other hand, we have

$$\mathbb{E}_x\Big[e^{\lambda\hat{\sigma}_1}e^{\beta\hat{\phi}_1\mathbb{1}_{\hat{\phi}_1>R}}\mathbb{1}_{\kappa_{\hat{\tau}_1}=0}\Big] = \mathbb{E}\Big[e^{\lambda\Sigma_{\tau_R}(x)}\mathbb{E}_{\Phi_{\tau_R}(x)}\Big[e^{\lambda\sigma_1+\beta\phi_1\mathbb{1}_{\phi_1>R}}\mathbb{1}_{\kappa_1=0}\Big]\Big]$$

and for all $y \in [0,R]$

$$\mathbb{E}_y\Big[e^{\lambda\sigma_1+\beta\phi_1\mathbb{1}_{\phi_1>R}}\mathbb{1}_{\kappa_1=0}\Big] \le \mathcal{E}_R(\lambda+\beta)e^{\beta y}.$$

Then, again from Lemma 3.1 we get

$$\mathbb{E}_x\Big[e^{\lambda\hat{\sigma}_1}e^{\beta\hat{\phi}_1\mathbb{1}_{\hat{\phi}_1>R}}\mathbb{1}_{\kappa_{\hat{\tau}_1}=0}\Big] \le \mathcal{E}_R(\lambda+\beta)e^{\beta x\mathbb{1}_{x>R}}$$

and then, for all $k = 1,\dots,n-1$,

$$\begin{aligned}\mathbb{E}\Big[e^{\lambda\hat{\sigma}_k}e^{\beta\hat{\Phi}_k\mathbb{1}_{\hat{\phi}_k>R}}\mathbb{1}_{\kappa_{\hat{\tau}_k}=0}|\mathcal{G}_{k-1}\Big] &= \mathbb{E}_{\hat{\phi}_{k-1}}\Big[e^{\lambda\hat{\sigma}_1}e^{\beta\hat{\phi}_1\mathbb{1}_{\hat{\phi}_1>R}}\mathbb{1}_{\kappa_{\hat{\tau}_1}=0}\Big]\\ &\le \mathcal{E}_R(\lambda+\beta)e^{\beta\hat{\phi}_{k-1}\mathbb{1}_{\hat{\phi}_{k-1}>R}}.\end{aligned}$$

By successively conditioning in (29), we finally have

$$\begin{aligned}\mathbb{E}_x[e^{\lambda T_*^R}] &\le e^{\beta x\mathbb{1}_{x>R}}\sum_{n=1}^{\infty}\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)(\mathcal{E}_R(\lambda+\beta))^n\\ &= \frac{e^{\beta x\mathbb{1}_{x>R}}\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)}{1-\mathcal{E}_R(\lambda+\beta)}\end{aligned}$$

for parameters as required. □

Let us now verify the existence of (R,λ,β) such that all the assumptions of Proposition 3.2 hold. Notice first that for all $\beta > \lambda > 0$ and each $R > 0$ we have

$$\varphi_{(R,-1)}(\lambda-\beta) \le \frac{\bar{a}+(\beta-\lambda)e^{-(\bar{a}+\beta-\lambda)R}}{\bar{a}+\beta-\lambda}, \tag{30}$$

thanks to the fact that $T_{(R,-1)} \ge_{\text{sto.}} E(\bar{a}) \wedge R$. Since also $T_{(R,+1)} \le_{\text{sto.}} E(b(R))$, we furthermore have

$$\varphi_{(R,+1)}(\lambda+\beta) \le \frac{b(R)}{b(R)-(\lambda+\beta)} \tag{31}$$

for all $\lambda+\beta < b(R)$.

Given $\lambda > 0$, we take $\beta > \lambda$ of the form $\beta = \alpha\lambda$ for $\alpha > \frac{\bar{b}+\bar{a}}{\bar{b}-\bar{a}}$ (or simply $\alpha > 1$ if $\bar{b} = \infty$). Then, we have $\bar{b} > \frac{\alpha+1}{\alpha-1}\bar{a}$, hence we find R large enough such that

$$b(R)\big(1-e^{-\bar{a}R}\big) > \frac{\alpha+1}{\alpha-1}\bar{a}.$$

Thanks to (30) and (31), the assumptions of Lemma 3.1 are satisfied for all $\lambda \in (0, \lambda_c)$, where

$$\lambda_c := \inf \left\{ \lambda > 0 : (\alpha+1)\lambda \geq b(R)\left(1 - e^{-(\bar{a}+(\alpha-1)\lambda)R}\right) - \frac{\alpha+1}{\alpha-1}\bar{a} \right\}.$$

Indeed, from (30) and (31), condition $\varphi_{(R,+1)}(\beta+\lambda)\varphi_{(R,-1)}(\lambda-\beta) < 1$ holds as soon as

$$\begin{aligned}
& b(R)\left(\bar{a} + (\alpha-1)\lambda e^{-(\bar{a}+(\alpha-1)\lambda)R}\right) < (\bar{a}+(\alpha-1)\lambda)(b(R)-(\alpha+1)\lambda) \\
\Leftrightarrow \quad & b(R)(\alpha-1)\lambda e^{-(\bar{a}+(\alpha-1)\lambda)R} < (\alpha-1)\lambda b(R) - (\bar{a}+(\alpha-1)\lambda)(\alpha+1)\lambda \\
\Leftrightarrow \quad & (\alpha+1)\lambda < b(R)\left(1 - e^{-(\bar{a}+(\alpha-1)\lambda)R}\right) - \frac{\alpha+1}{\alpha-1}\bar{a}.
\end{aligned}$$

Since the previous inequality is satisfied for $\lambda = 0$, by continuity we have $\lambda_c > 0$; the function of λ on the r.h.s. being strictly concave, we also have $\lambda_c < \infty$.

Finally, notice that by Lemma 2.3 and Holder's inequality, for any $q > 1$,

$$\mathcal{E}_R(\gamma) \leq (1-p_R)^{1-1/q}\varphi^{1/q}_{(0,+1)}(q\gamma),$$

with $p_R \in (0,1)$ a quantity as in (21). Taking $q = q(\gamma) = \gamma^{-1}$, this in turn yields, for each fixed $R > 0$, $\limsup_{\gamma\to 0}\mathcal{E}_R(\gamma) \leq (1-p_R) < 1$. Therefore, there exists $\lambda_c' \in (0,\lambda_c)$ small enough such that $\mathcal{E}_R((\alpha+1)\lambda) < 1$ for all $\lambda \in (0,\lambda_c')$.

3.4 The coupling time for the reflected process

Let us consider two initial data (x,v) and $(\tilde{x},\tilde{v})$, with $x \geq \tilde{x}$. The coalescent time $T_*(x,\tilde{x})$ of a path (X,V) starting from (x,v) and a path $\left(\tilde{X},\tilde{V}\right)$ starting from $(\tilde{x},\tilde{v})$ is equal to the first crossing time $T_c(x,\tilde{x})$ of both paths plus the time spent to stick them using the coupling described in Section 3.2. Consequently, the coupling time is stochastically smaller than the hitting time $Z_{(x,v)}$ of the origin of the upper path (X,V), plus some remainder term.

For any (R,λ,β) satisfying assumptions of Proposition 3.2, the Laplace transform of the coupling time $T_*(x,\tilde{x})$ is bounded by

$$\mathbb{E}\left[e^{\lambda T_*(x,\tilde{x})}\right] \leq \frac{\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)}{1-\mathcal{E}_R(\lambda+\beta)}\mathbb{E}\left[e^{\lambda T_c(x,\tilde{x})}e^{\beta X_c \mathbb{1}_{X_c>R}}\right].$$

where the first crossing time $T_c(x,\tilde{x})$ is smaller than $Z_{(x,v)}$ and the first crossing point X_c is bounded from above by $\frac{1}{2}\left(Z_{(x,v)}+\tilde{x}-x\right)$. Consequently,

$$\mathbb{E}\left[e^{\lambda T_*(x,\tilde{x})}\right] \leq \frac{\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)}{1-\mathcal{E}_R(\lambda+\beta)}e^{\beta(\tilde{x}-x)/2}\mathbb{E}\left[e^{(\lambda+\beta/2)Z_{(x,v)}}\right].$$

Using now Proposition 2.8 , for $0 < \lambda < \beta$ satisfying conditions of Proposition 3.2 with $\lambda + \beta/2 < \frac{1}{2}(\sqrt{b(M_c)} - \sqrt{a(M_c)})^2$, we finally get

$$\mathbb{E}\left[e^{\lambda T_*(x,\tilde{x})}\right] \leq C\frac{\varphi_{(0,+1)}(\lambda)\varphi_{(R,-1)}(\lambda)}{1-\mathcal{E}_R(\lambda+\beta)}e^{\frac{\beta(\tilde{x}-x)}{2}}e^{\frac{x(b(M_c)-a(M_c))}{2}} \tag{32}$$

with C is given in Proposition 2.8 and

$$M_c = \sup\left\{ M > 0 \;:\; \sqrt{\frac{b(M)}{a(M)}}e^{M\left(\sqrt{b(M)}-\sqrt{a(M)}\right)^2}\left(1-e^{-A(M)}\right) < 1 \right\}.$$

This concludes the proof of Theorem 1.9.

4 The unreflected process

Let us construct a coalescent coupling of two unreflected processes starting from (y, w) and $(\tilde{y}, \tilde{w})$ respectively. For a given time $t_0 > 0$, the coupling algorithm is the following:

1. Define $(x, v) = (|y|, w\,\mathrm{sgn}(y))$ and $(\tilde{x}, \tilde{v}) = (|\tilde{y}|, \tilde{w}\,\mathrm{sgn}(\tilde{y}))$.
2. Couple two reflected processes starting at (x, v) and $(\tilde{x}, \tilde{v})$ as in Section 3.
3. Let them evolve until their common hitting time of 0.
4. Construct until that time the two associated unreflected processes starting at (y, w) and $(\tilde{y}, \tilde{w})$ as explained in Remark 1.8. The algorithm stops if, when at the origin, the two copies have the same velocities. Otherwise, go to Step 5.
5. Try to couple the unreflected processes starting from $(0, +1)$ and $(0, -1)$ before a fixed time t_0.
6. In case of failure, return to step 1 for two initial conditions in $[-t_0, t_0] \times \{-1, +1\}$.

The only remaining task is to analyze Step 5 of this algorithm. To that end, one has to study the law of (Y_t, W_t) when $Y_0 = 0$ and $W_0 = \pm 1$. Let us denote by $(T_n)_{n\geq 0}$ (with $T_0 = 0$) the successive jump times of the unreflected process. The variable $S_n = T_n - T_{n-1}$ stands for the n^{th} inter-jump time. In order to lighten the computation, we restrict ourselves to the law after 1 or 2 jumps.

Remark 4.1 (Jump times of the unreflected process)**.** *We can explicitly compute the law of the jump-times of the unreflected process. For $y \in \mathbb{R}$, set $A(y) = A(|y|)$ and $B(y) = B(|y|)$ where A and B were defined on $\mathbb{R}_+$ in Lemma 2.1. For $y > 0$, the law of the first jump time starting from $(y, -1)$ has the density $f_{(y,-1)}$ given by*

$$f_{(y,-1)}(t) = \begin{cases} a(y-t)e^{-(A(y)-A(y-t))} & \text{if } t < y, \\ e^{-A(y)}b(t-y)e^{-B(t-y)} & \text{if } t \geq y. \end{cases}$$

Moreover, the survival function $\bar{F}_{(y,+1)}(t) := \mathbb{P}_{(y,+1)}(T_1 > t)$ is given for $y \geq 0$ by

$$\bar{F}_{(y,+1)}(t) = e^{-(B(y+t)-B(y))},$$

and for $y < 0$ by

$$\bar{F}_{(y,+1)}(t) = \begin{cases} e^{-(A(y)-A(y+t))} & \text{if } y+t < 0, \\ e^{-A(y)}e^{-B(t+y)} & \text{if } y+t \geq 0. \end{cases}$$

For any bounded measurable function g on $\mathbb{R} \times \{-1, +1\}$, one then has

$$\begin{aligned} \mathbb{E}_{(0,-1)}\left[g(Y_t, W_t)\mathbb{1}_{\{T_1<t, T_2>t\}}\right] &= \mathbb{E}_{(0,-1)}\left[g(t-2S_1, +1)\mathbb{1}_{\{S_1<t, S_1+S_2>t\}}\right] \\ &= \int_0^t g(t-2s, +1) f_{(0,-1)}(-s)\bar{F}_{(-s,+1)}(t-s)\, ds \\ &= \int_{-t}^t g(u, +1) h_{-1}(u)\, du, \end{aligned}$$

where

$$h_{-1}(u) = \frac{1}{2} f_{(0,+1)}\left(\frac{t-u}{2}\right)\bar{F}_{\left(\frac{t-u}{2},+1\right)}\left(\frac{t+u}{2}\right).$$

Similarly,

$$\begin{aligned}\mathbb{E}_{(0,+1)}[g(Y_t, W_t)\mathbb{1}_{T_2<t,T_3>t}] &= \mathbb{E}_{(0,+1)}[g(t-2S_2,+1)\mathbb{1}_{S_1+S_2<t,T_3>t}]\\ &= \int_0^t\int_0^{t-s_2} g(t-2s_2,+1)f_{(0,+1)}(s_1)f_{(s_1,-1)}(s_2)\bar{F}_{(s_1-s_2,+1)}(t-s_1-s_2)\,ds_1\,ds_2\\ &= \int_{-t}^t g(u,+1)h_1(u)\,du,\end{aligned}$$

where

$$h_1(u) = \int_0^{\frac{t+u}{2}} \frac{1}{2}f_{(0,+1)}(s_1)f_{(s_1,-1)}\left(\frac{t-u}{2}\right)\bar{F}_{(s_1-\frac{t-u}{2},+1)}\left(\frac{t+u}{2}-s_1\right)ds_1.$$

Since $\mathcal{L}((Y_t,W_t)|Y_0=0,W_0=w)=\mathcal{L}((-Y_t,W_t)|Y_0=0,W_0=-w)$, two copies as in step 5 of the algorithm couple before time $t\geq 0$ with probability larger than

$$\varepsilon_t = 2\int_{-t}^t h_{-1}(u)\wedge h_1(u)\,du>0.$$

Remark 4.2 (Explicit lower bound)**.** *A lower bound of ε can be derived from the fact that $y\mapsto a(y)$ and $y\mapsto b(y)$ respectively belong to $[a(t),a(0)]$ and $[b(0),b(t)]$ on the interval $[-t,t]$.*

Let us now control the total duration of the algorithm. Notice that the estimates on the reflected process in Section 3 do no longer depend on the initial conditions, after the first crossing time of the reflected copies in the compact set $[-R,R]\times\{-1,+1\}$. This implies that, after the first iteration of Step 2 in the above algorithm, the duration of each step can be controlled independently of the initial data, and of the previous steps.

Moreover, the algorithm succeeds after at most a random number of iterations with geometric law of parameter ε_{t_0}. Since the duration of each step in the algorithm has a finite exponential moment, this is thus true for the coupling time as well. The upper bound of Theorem 1.2 can then be deduced. As a conclusion the bounds in Theorems 1.2 and 1.9 depends in the same way on initial data but the rate of convergence is smaller for the unreflected process.

5 Diffusive scaling

We finally prove Theorem 1.5. Omitting for a moment the sub and superscripts for notational simplicity, and writing

$$j_t := W_t+\kappa'(Y_t)-2\big(a(Y_t)\mathbb{1}_{\{Y_tW_t\leq 0\}}+b(Y_t)\mathbb{1}_{\{Y_tW_t>0\}}\big)\kappa(Y_t)W_t,$$

$J_t:=\int_0^t j_s ds$ and $\hat{Y}_t := Y_t+\kappa(Y_t)W_t$ for a given positive function κ of class $\mathcal{C}^1$, we see by Dynkin's theorem that the processes

$$M_t := \hat{Y}_t - J_t = Y_t+\kappa(Y_t)W_t - J_t$$

and

$$N_t := \hat{Y}_t^2 - 2\int_0^t \kappa(Y_s)ds - 2\int_0^t Y_s j_s ds - 2\int_0^t \kappa'(Y_s)\kappa(Y_s)W_s ds$$

are local martingales with respect to the filtration generated by (Y_t, W_t).

In fact, using $f(y,w) = y + \kappa(y)w$ and $g(y,w) = (y+\kappa(y)w)^2$, since $w^2 = 1$, we have

$$\begin{aligned}
Lf(y,w) &= w + \kappa'(y) - 2\big(a(y)\mathbb{1}_{\{yw\le 0\}} + b(y)\mathbb{1}_{\{yw>0\}}\big)\kappa(y)w,\\
Lg(y,w) &= 2w(1+\kappa'(y)w)(y+\kappa(y)w) - 4\big(a(y)\mathbb{1}_{\{yw\le 0\}} + b(y)\mathbb{1}_{\{yw>0\}}\big)y\kappa(y)w\\
&= 2\kappa(y) + 2y\big(w + \kappa'(y) - 2\big(a(y)\mathbb{1}_{\{yw\le 0\}} + b(y)\mathbb{1}_{\{yw>0\}}\big)k(y)w\big) + 2\kappa'(y)\kappa(y)w.
\end{aligned}$$

Integrating by parts we then get that

$$\begin{aligned}
M_t^2 &= \hat{Y}_t^2 - 2\hat{Y}_t J_t + J_t^2\\
&= \hat{Y}_t^2 - 2\int_0^t (Y_s + \kappa(Y_s)W_s)j_s ds - 2\int_0^t J_s d\hat{Y}_s + 2\int_0^t J_s j_s ds\\
&= N_t + 2\int_0^t \kappa(Y_s)ds - 2\int_0^t \kappa(Y_s)W_s j_s ds + 2\int_0^t \kappa'(Y_s)\kappa(Y_s)W_s ds\\
&\qquad - 2\int_0^t J_s d\hat{Y}_s + 2\int_0^t J_s j_s ds\\
&= N_t + 2\int_0^t \kappa(Y_s)ds - 2\int_0^t \kappa(Y_s)W_s[j_s - \kappa'(Y_s)]ds - 2\int_0^t J_s dM_s.
\end{aligned}$$

Thus, noting that

$$j_s = \kappa'(Y_s) + \operatorname{sgn}(Y_s)\left([2a(Y_s)\kappa(Y_s) - 1] + 2\mathbb{1}_{\{Y_sW_s>0\}}\left[1 - \kappa(Y_s)(a(Y_s)+b(Y_s))\right]\right)$$

we see for $\kappa(Y_s) = (a(Y_s)+b(Y_s))^{-1}$ that

$$\begin{aligned}
&M_t = Y_t - \left[-\int_0^t \left[\frac{a'(Y_s)+b'(Y_s)}{(a(Y_s)+b(Y_s))^2} + \operatorname{sgn}(Y_s)\left(\frac{b(Y_s)-a(Y_s)}{a(Y_s)+b(Y_s)}\right)\right]ds - \frac{W_t}{a(Y_t)+b(Y_t)}\right],\\
&M_t^2 - 2\int_0^t \left[\frac{1}{a(Y_s)+b(Y_s)} + W_s \operatorname{sgn}(Y_s)\frac{b(Y_s)-a(Y_s)}{(a(Y_s)+b(Y_s))^2}\right]ds
\end{aligned}$$

are local martingales.

The function τ_N of the statement is well defined by the Cauchy-Lipschitz Theorem, thanks to the assumptions on the coefficients and the fact that Y_t has Lipschitz trajectories. Moreover, τ_N is strictly increasing, the coefficients a and b being positive functions. Recalling the dependence on N of the coefficients, and defining for each $N \in \mathbb{N}$,

$$\begin{aligned}
\beta_t^{(N)} := &-\frac{1}{2}\int_0^t \left[\frac{a_N'(\xi_s^{(N)}) + b_N'(\xi_s^{(N)})}{a_N(\xi_s^{(N)}) + b_N(\xi_s^{(N)})} + \operatorname{sgn}(\xi_s^{(N)})\Big(b_N(\xi_s^{(N)}) - a_N(\xi_s^{(N)})\Big)\right]ds\\
&- \frac{W_{\tau_N(t)}^{(N)}}{a_N(\xi_t^{(N)}) + b_N(\xi_t^{(N)})}
\end{aligned}$$

and

$$\alpha_t^{(N)} := t + \frac{1}{2}\int_0^t W_{\tau_N(s)}^{(N)} \operatorname{sgn}(\xi_s^{(N)})\frac{b_N(\xi_s^{(N)}) - a_N(\xi_s^{(N)})}{a_N(\xi_s^{(N)}) + b_N(\xi_s^{(N)})}ds,$$

we see from the previous and Doob's optional stopping theorem that the processes

$$\left(\xi_\cdot^{(N)} - \beta_t^{(N)}\right)_{t\ge 0} \quad \text{and} \quad \left((\xi_t^{(N)} - \beta_t^{(N)})^2 - \alpha_t^{(N)}\right)_{t\ge 0}$$

are local martingales with respect to the filtration of the process $(\xi_t^{(N)}, W^{(N)}_{\tau_N(t)})_{t\geq 0}$. Defining for each $R>0$ the stopping time $\sigma_R^N = \inf\{t\geq 0 : |\xi_t^{(N)}| \geq R \text{ of } |\xi_{t-}^{(N)}| \geq R\}$, we notice that the hypotheses imply that

$$\mathbb{P}\left(\sup_{t\leq \sigma_R^N}\left|\alpha_t^{(N)} - t\right| \geq \varepsilon\right) + \mathbb{P}\left(\sup_{t\leq \sigma_R^N}\left|\beta_t^{(N)} + \int_0^t \operatorname{sgn}(\xi_s)c_1(\xi_s) + c_2(\xi_s)\,ds\right| \geq \varepsilon\right) \xrightarrow[N\to\infty]{} 0$$

for all $\varepsilon > 0$ and

$$\mathbb{E}\left(\sup_{t\leq \sigma_R^N}\left|\xi_t^{(N)} - \xi_{t-}^{(N)}\right|^2\right) + \mathbb{E}\left(\sup_{t\leq \sigma_R^N}\left|\beta_t^{(N)} - \beta_{t-}^{(N)}\right|^2\right) \xrightarrow[N\to\infty]{} 0.$$

The processes $\left(\xi_t^{(N)}, \alpha_t^{(N)}\right)_{t\geq 0}$ and $\left(\beta_t^{(N)}\right)_{t\geq 0}$ thus satisfy the hypotheses of Theorem 4.1 in [12, p. 354] (in the respective roles of the processes $X_n(\cdot), A_n(\cdot)$ and $B_n(\cdot)$ therein), which ensures that $\mathcal{L}((\xi_t^{(N)}, t\geq 0))$ converges weakly to the unique solution of the martingale problem with generator given for $f\in\mathcal{C}_c^\infty(\mathbb{R})$ by

$$Gf(x) := \frac{1}{2}f''(x) - (\operatorname{sgn}(x)c_1(x) + c_2(x))f'(x)$$

and initial law $\mathcal{L}(\xi_0)$.

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