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A Model of Dynamics Driven by Repeated Harmonic Interactions

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ABSTRACT

We consider an exactly soluble W^*-dynamical system driven by repeated harmonic interactions. Although dynamics is Hamiltonian and quasi-free, it leads in the large-time limit to relaxation of initial states to a steady state. We found explicitly the rate of the entropy production which accompanies this relaxation. Besides, we study evolution of subsystems to elucidate their eventual correlations and convergence to equilibrium states. Finally we prove a universality of dynamics driven by repeated harmonic perturbations in a short-time interaction limit.

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1 Preliminaries and the Model

1.1 Setup

We consider a quantum system recently proposed in [TZ]. It is a harmonic system (one-mode quantum oscillator $\mathcal{S}$) successively perturbed by time-dependent stationary repeated harmonic interactions. This sequence of perturbation is switched on at the moment $t = 0$ and it acts successively on the interval $0 \leq t < \infty$. It is a common fashion to present this sequence as repeated interactions of the system $\mathcal{S}$ with an *infinite* time-equidistant *chain*: $\mathcal{C} = \mathcal{S}_1 + \mathcal{S}_2 + \ldots$, of subsystems $\{\mathcal{S}_k\}_{k\geq 1}$.

Note that there is a physical interpretation [NVZ], [BJM], behind of this mathematical setting. For the model [TZ], the system $\mathcal{C}_N$ is identified with a chain of N quantum particles ("atoms") with infinitely many *harmonic* internal degrees of freedom. They interact *one-by-one* with a one-mode quantum resonator (cavity) $\mathcal{S}$. This is a caricature of the one-atom maser system. In contract to [NVZ], but similar to the two-level Jaynes-Cummings atoms [BJM], the interaction with harmonic atoms is *inelastic*. This yields a very different evolution of the model [TZ] comparing to the model [NVZ], in which interaction is completely *elastic*, see Sections 4 and 5.

Below we suppose that the states of $\mathcal{S}$ and of every $\mathcal{S}_k$ are *normal*, i.e. defined by the *density matrices* ρ_0 and $\{\rho_k\}_{k=1}^{\infty}$ on the Hilbert spaces $\mathscr{H}_{\mathcal{S}}$ and $\{\mathscr{H}_{\mathcal{S}_k}\}_{k=1}^{\infty}$, respectively. The Hilbert space of the total system is then the tensor product $\mathscr{H}_{\mathcal{S}} \otimes \mathscr{H}_{\mathcal{C}}$. Here the infinite product $\mathscr{H}_{\mathcal{C}} = \otimes_{k\geq 1}\mathscr{H}_{\mathcal{S}_k}$ stays for the Hilbert space chain.

Since for any fixed moment $t \geq 0$, only a *finite* number $N(t)$ of repeated interactions are involved into the dynamics, the subsystems $\{\mathcal{S}_k\}_{k>N(t)}$ are still *independent* for different k, as well as they are independent of components $\mathcal{S}$ and $\{\mathcal{S}_k\}_{k=1}^{N(t)}$. On the other hand, the problem of correlations between components $\mathcal{S}$, $\mathcal{S}_k$ for $k \leq N(t)$ and between $\mathcal{S}_k$, $\mathcal{S}_{k'}$ for $1 \leq k < k' \leq N(t)$ is considered in Section 5. This peculiarity of the repeated interactions allows to reduce the analysis of dynamics to the finite tensor product: $\mathscr{H}_{\mathcal{C}_N} = \otimes_{k=1}^{N}\mathscr{H}_{\mathcal{S}_k}$. Then one recovers the above infinite chain $\mathcal{S}+\mathcal{C}$ as the limit $N \to \infty$, *a*

posteriori. Details of dynamics are presented in the next Section 2. We finish this section by formulation of some of our hypothesises.

Hypothesis (H1): (*Initial states*) For $t \leq 0$, *all* components of $\mathcal{S}$ and $\{\mathcal{S}_k\}_{k=1}^N$ are *independent*, i.e. the state of $\mathcal{S} + \mathcal{C}_N$ is described as a finite tensor product: $\omega_{\mathcal{S}+\mathcal{C}_N} := \omega_{\mathcal{S}} \otimes \bigotimes_{k=1}^N \omega_{\mathcal{S}_k}$. We suppose that each of the state in the product is *normal*.

Hypothesis (H2): (*Tuned interaction*) We consider repeated perturbations in the tuned regime: for any moment $t \geq 0$ exactly *one* subsystem ("atom") $\mathcal{S}_n$ is interacting with the system $\mathcal{S}$ (quantum resonator) during a fixed time $\tau > 0$. Here $n = [t/\tau] + 1$, where $[x]$ denotes the *integer* part of $x \geq 0$.

Let $\mathscr{H}_0$ be the Hilbert space for the system $\mathcal{S}$ and $\mathscr{H}_k$ be the Hilbert space for the the system $\mathcal{S}_k$ for $k = 1, \cdots, N$. Then for $k = 0, 1, \cdots, N$, the space $\mathscr{H}_k$ is a copy of the one-mode boson Fock space $\mathscr{F}$ with the vacuum vector $\Omega \in \mathscr{F}$ and with densely defined boson annihilation and (adjoint) creation operators: a and a^*, defined by $a\,\Omega = 0$. The total system $\mathcal{S} + \mathcal{C}_N$ lives in the Hilbert space

$$\mathscr{H}^{(N)} := \mathscr{H}_0 \otimes \bigotimes_{k=1}^{N} \mathscr{H}_k = \mathscr{F}^{\otimes(N+1)} \ . \tag{1.1}$$

Here $\mathbb{1}$ is the unit operator on $\mathscr{F}$. In the space (1.1) we define operators

$$b_k := \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \ , \ b_k^* := \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a^* \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \ , \tag{1.2}$$

where operator a, or a^*, is the $(k+1)$th factor in (1.2). Operators (1.2) formally satisfy the Canonical Commutation Relations (CCR)

$$[b_k, b_{k'}^*] = \delta_{k,k'}\mathbb{1}, \quad [b_k, b_{k'}] = [b_k^*, b_{k'}^*] = 0 \ , \ k, k' = 0, 1, 2, \cdots, N. \tag{1.3}$$

Hypothesis (H3): (*Harmonic interaction*) The time-dependent repeated interaction described by (H2) is a piecewise *constant* operator in (1.1). It is the sum over $n \geq 1$ of the *bilinear* forms in operators (1.2) in the space $\mathscr{H}_0 \otimes \mathscr{H}_n$:

$$K_n(t) := \chi_{[(n-1)\tau, n\tau)}(t)\, \eta\, (b_0^* b_n + \ b_n^* b_0) \ , \qquad \eta > 0. \tag{1.4}$$

Here $\chi_{\mathcal{I}}(x)$ is the characteristic function of the set $\mathcal{I}$.

1.2 The Model

For any $N \geq 1$ and $t < N\tau$, the self-adjoint Hamiltonian $H_N(t)$ of the non-autonomous system $\mathcal{S} + \mathcal{C}_N$ is defined in the space (1.1) as the sum of Hamiltonians corresponding the systems $\mathcal{S}$, $\mathcal{S}_k$ and interaction (1.4) [TZ]:

$$\begin{aligned} H_N(t) &:= H_{\mathcal{S}} + \sum_{k=1}^{N} (H_{\mathcal{S}_k} + K_k(t)) \qquad (1.5) \\ &= E b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta \sum_{k=1}^{N} \chi_{[(k-1)\tau, k\tau)}(t)\ (b_0^* b_k + \ b_k^* b_0)\,, \end{aligned}$$

Hypothesis (H4):(*Semi-boundedness*) To keep the self-adjoint Hamiltonian (1.5) semi-bounded from *below* we suppose that $E, \epsilon > 0$ and we impose the condition

$$\eta^2 \leq E\,\epsilon\ . \tag{1.6}$$

By virtue of (1.4), (1.5) only $\mathcal{S}_n$ interacts with $\mathcal{S}$ for $t \in [(n-1)\tau, n\tau)$, $n \geq 1$, i.e. the system $\mathcal{S} + \mathcal{C}_N$ is *autonomous* on this time-interval with self-adjoint Hamiltonian

$$H_n := E\, b_0^* b_0 + \epsilon \sum_{k=1}^{N} b_k^* b_k + \eta\,(b_0^* b_n +\ b_n^* b_0)\ ,\ n \leq N\ . \tag{1.7}$$

Note that there exists a CCR-preserving linear transformation, which diagonalise (1.7):

$$\widetilde{H}_n := \varepsilon_0\, c_0^* c_0 + \varepsilon_1\, c_1^* c_1 + \sum_{k=2}^{N} \varepsilon_k c_k^* c_k\ . \tag{1.8}$$

Here $\varepsilon_2 = \ldots = \varepsilon_N = \epsilon$, and $\varepsilon_{0,1} := \frac{1}{2}[(E+\epsilon) \pm \sqrt{(E-\epsilon)^2 + 4\eta^2}] > 0$ by (H4).

The next key lemma follows from the harmonic structure of the Hamiltonian (1.7).

Lemma 1.1 *For $j = 0, 1, 2, \ldots, N$ and $n = 1, 2, \ldots, N$, one gets*

$$e^{itH_n} b_j e^{-itH_n} = \sum_{k=0}^{N} (U_n^*(t))_{jk} b_k, \quad e^{itH_n} b_j^* e^{-itH_n} = \sum_{k=0}^{N} \overline{(U_n^*(t))_{jk}} b_k^*, \tag{1.9}$$

$$e^{-itH_n} b_j e^{itH_n} = \sum_{k=0}^{N} (U_n(t))_{jk} b_k, \quad e^{-itH_n} b_j^* e^{itH_n} = \sum_{k=0}^{N} \overline{(U_n(t))_{jk}} b_k^*\ , \tag{1.10}$$

for $t \geq 0$. Here $U_n(t)$ and $V_n(t)$ are $(N+1)\times(N+1)$ matrices related by $U_n(t) := e^{it\epsilon} V_n(t)$, where

$$(V_n(t))_{jk} := \begin{cases} g(t)z(t)\,\delta_{k0} + g(t)w(t)\,\delta_{kn} & (j=0) \\ g(t)w(t)\,\delta_{k0} + g(t)z(-t)\,\delta_{kn} & (j=n) \\ \quad \delta_{jk} & (\textit{otherwise})\ , \end{cases} \tag{1.11}$$

and

$$g(t) := e^{it(E-\epsilon)/2}\ ,\ w(t) := \frac{2i\eta}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}\ , \tag{1.12}$$

$$z(t) := \cos t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} + \frac{i(E-\epsilon)}{\sqrt{(E-\epsilon)^2 + 4\eta^2}} \sin t \sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}\ . \tag{1.13}$$

Remark 1.2 *Note that by definitions (1.12) and (1.13), we get $|z(t)|^2 + |w(t)|^2 = 1$, $z(-t) = \overline{z(t)}$ and $w(t) = -\overline{w(t)}$. Therefore, the matrix*

$$M(t) := \begin{pmatrix} z(t) & w(t) \\ w(t) & z(-t) \end{pmatrix}$$

is unitary. For $N = 1$, one gets $M(t) = \overline{g(t)} V_1(t)$, see (1.11). Moreover, (1.9) and (1.10) imply that $\{V_n(t)\}_{t\in\mathbb{R}}$ and $\{U_n(t)\}_{t\in\mathbb{R}}$ are in fact one-parameter groups of $(N+1)\times(N+1)$ unitary matrices.

Proof (of Lemma 1.1): Let $\{J_n\}_{n=1}^N$ and $\{X_n\}_{n=1}^N$ be $(N+1)\times(N+1)$ Hermitian matrices given by

$$(J_n)_{jk} := \begin{cases} 1 & (j=k=0 \ \text{ or } \ j=k=n) \\ 0 & \text{otherwise} \end{cases} , \tag{1.14}$$

$$(X_n)_{jk} := \begin{cases} (E-\epsilon)/2 & (j,k)=(0,0) \\ -(E-\epsilon)/2 & (j,k)=(n,n) \\ \eta & (j,k)=(0,n) \\ \eta & (j,k)=(n,0) \\ 0 & \text{otherwise} \end{cases} . \tag{1.15}$$

We define the matrices

$$Y_n := \epsilon I + \frac{E-\epsilon}{2} J_n + X_n \quad (n=1,\ldots,N) \ , \tag{1.16}$$

where I is the $(N+1)\times(N+1)$ identity matrix. Then Hamiltonian (1.7) takes the form

$$H_n = \sum_{j,k=0}^{N} (Y_n)_{jk} b_j^* b_k \ . \tag{1.17}$$

Since Y_n is Hermitian, there exists a *diagonal* matrix Λ and unitary mapping $\mathcal{U}_n : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, such that $Y_n = \mathcal{U}_n^* \, \Lambda \, \mathcal{U}_n$ holds. Recall that after canonical transformation the matrix $\Lambda := \{\Lambda_{ij}\}_{i,j=0}^N = \{\delta_{ij}\, \varepsilon_j\}_{i,j=0}^N$ (1.8) is universal and independent of n. The new operators:

$$c_j = \sum_{k=0}^{N} (\mathcal{U}_n)_{jk} \ b_k, \quad c_j^* = \sum_{k=0}^{N} \overline{(\mathcal{U}_n)_{jk}} \ b_k^* \quad (j=0,1,\ldots,N) \ , \tag{1.18}$$

satisfy CCR in the space $\mathscr{H}^{(N)}$ (1.1) and diagonalise (1.17): $\widetilde{H}_n = \sum_{j=0}^N \Lambda_{jj} c_j^* c_j$, where $\Lambda_{jj} = \varepsilon_j$ (1.8). Therefore, the set of all eigenvectors of $\widetilde{H}_n$ is

$$\Big\{ \prod_{j=0}^{N} \frac{(c_j^*)^{n_j}}{\sqrt{n_j!}} \, \Omega \otimes \ldots \otimes \Omega \, \Big| \, n_j \in \mathbb{Z}_+ \quad (j=0,1,\ldots,N) \Big\} \ . \tag{1.19}$$

Note that it forms a complete orthonormal basis in $\mathscr{H}^{(N)}$. The linear envelope $\mathscr{H}_0^{(N)}$ of the set (1.19) is invariant subspace for transformations $e^{it\widetilde{H}_n}$ and its norm-closure coincides with $\mathscr{H}^{(N)}$. Then by (1.18) one gets on vectors (1.19):

$$e^{it\widetilde{H}_n} c_j e^{-it\widetilde{H}_n} = e^{-it\Lambda_{jj}} c_j, \quad e^{it\widetilde{H}_n} c_j^* e^{-it\widetilde{H}_n} = e^{it\Lambda_{jj}} c_j^* \ .$$

Now taking into account canonical transformation (1.18), we obtain

$$e^{itH_n} b_j e^{-itH_n} = \sum_{k=0}^{N} (\mathcal{U}_n^*)_{jk} \, e^{it\widetilde{H}_n} c_k e^{-it\widetilde{H}_n}$$

$$= \sum_{k,l=0}^{N} (\mathcal{U}_n^*)_{jk} e^{-it\Lambda_{kk}} (\mathcal{U}_n)_{kl} b_l = \sum_{l=0}^{N} \left(e^{-it\mathcal{U}_n^*\Lambda\mathcal{U}_n}\right)_{jl} b_l = \sum_{l=0}^{N} \left(e^{-itY_n}\right)_{jl} b_l \ . \tag{1.20}$$

Similarly we obtain $e^{itH_n} b_j^* e^{-itH_n} = \sum_{l=0}^{N} \overline{\left(e^{-itY_n}\right)_{jl}} b_l^*$.

Note that by virtue of (1.14), (1.15), one has identities

$$X_n^2 = \left(\frac{(E-\epsilon)^2}{4} + \eta^2\right) J_n \quad \text{and} \quad J_n X_n = X_n \ .$$

Together with definition (1.16) and (1.11), they yield

$$e^{itY_n} = e^{it\epsilon}\Bigg(I - J_n + e^{it(E-\epsilon)/2}\Bigg\{J_n \cos t\sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2} \tag{1.21}$$

$$+ iX_n \left[\frac{(E-\epsilon)^2}{4} + \eta^2\right]^{-1/2} \sin t\sqrt{\frac{(E-\epsilon)^2}{4} + \eta^2}\ \Bigg\}\Bigg) = e^{it\epsilon} V_n(t) = U_n(t).$$

Inserting now (1.21) into (1.20), we prove (1.9). Since $U_n(t)^* = U_n(-t)$, one can similarly establish (1.10). □

Remark 1.3 *Hereafter, we are going to use the short-hand notations:*

$$g := g(\tau),\ w := w(\tau),\ z := z(\tau) \ \text{ and } \ V_n := V_n(\tau),\ U_n := U_n(\tau) \ . \tag{1.22}$$

In Section 2, we give explicit description of the Hamiltonian dynamics for the non-autonomous system $\mathcal{S} + \mathcal{C}$ driven by harmonic repeated interactions (H3). We show that our model of bosons (1.5) is a quasi-free W^*-dynamical system. In Section 3 we recall formulae for the entropy of the CCR quasi-free states. We use them in Section 4 for calculations of the entropy production. Section 5 is dedicated to analysis of reduced dynamics of subsystems, of their correlations and of convergence to equilibrium. We prove a universality of the short-time interaction limit of this dynamics for the subsystem $\mathcal{S}$.

2 Hamiltonian Dynamics

A well-known way to avoid the problem of evolution of unbounded creation-annihilation operators is to construct dynamics of the subsystem $\mathcal{S}$ on the unital Weyl CCR C^*-algebra $\mathscr{A}(\mathscr{F})$, see e.g. [AJP1] (Lectures 4 and 5), [BR2]. Here $\mathscr{A}(\mathscr{F})$ is generated on the Fock space $\mathscr{F}$ as the operator-norm closure of the linear span $\mathscr{A}_w$ of the Weyl operator system:

$$\{\widehat{w}(\alpha) = e^{i\Phi(\alpha)/\sqrt{2}}\}_{\alpha\in\mathbb{C}} \ . \tag{2.1}$$

Here $\Phi(\alpha) := \bar{\alpha}a + \alpha a^*$ is a self-adjoint operator with domain in $\mathscr{F}$ and the CCR take then the Weyl form:

$$\widehat{w}(\alpha_1)\widehat{w}(\alpha_2) = e^{-i\,\mathrm{Im}(\bar{\alpha}_1\alpha_2)/2}\ \widehat{w}(\alpha_1+\alpha_2)\ , \quad \alpha_1, \alpha_2 \in \mathbb{C}\ . \tag{2.2}$$

Note that $\mathscr{A}(\mathscr{F})$ is a minimal C^*-algebra, which contains the linear span $\mathscr{A}_w$ of the Weyl operator system (2.1). Algebra $\mathscr{A}(\mathscr{F})$ is contained in the unital C^*-algebra $\mathcal{L}(\mathscr{F})$ of all bounded operators on $\mathscr{F}$.

Similarly we define the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}) \subset \mathcal{L}(\mathscr{H})$ over $\mathscr{H} := \mathscr{H}^{(N)}$ (1.1). It is appropriate for description the system $\mathcal{S} + \mathcal{C}$. This algebra is generated by operators

$$W(\zeta) = \bigotimes_{k=0}^{N} \widehat{w}(\zeta_k), \qquad \zeta = \{\zeta_k\}_{k=0}^{N} \in \mathbb{C}^{N+1} \ , \ N \geq 1. \tag{2.3}$$

Using definitions of the boson operators $\{b_k, b_k^*\}_{k=1}^N$ and of the sesquilinear forms

$$\langle \zeta, b \rangle := \sum_{j=0}^{N} \bar{\zeta}_j b_j, \qquad \langle b, \zeta \rangle := \sum_{j=0}^{N} \zeta_j b_j^* \ , \tag{2.4}$$

the Weyl operators (2.3) can be rewritten as

$$W(\zeta) = \exp[i\big(\langle \zeta, b \rangle + \langle b, \zeta \rangle\big)/\sqrt{2}] \ . \tag{2.5}$$

We denote by $\mathfrak{C}_1(\mathscr{F}) \subset \mathcal{L}(\mathscr{F})$, the set of all trace-class operators on $\mathscr{F}$. A self-adjoint, non-negative operator $\rho \in \mathfrak{C}_1(\mathscr{F})$ with *unit* trace is called *density matrix*. The state $\omega_\rho(\cdot)$ generated on the C^*-algebra of bounded operators $\mathcal{L}(\mathscr{F})$ by ρ :

$$\omega_\rho(A) := \mathrm{Tr}_{\mathscr{F}}(\rho\, A) \ , \quad A \in \mathcal{L}(\mathscr{F}) \ , \tag{2.6}$$

is a *normal* state. Let $\{\rho_k\}_{k=0}^N$ be density matrices on $\mathscr{F}$. Then the normal *product-state* on the C^*-algebra $\mathscr{A}(\mathscr{H})$ (isometrically isomorphic to the tensor product $\otimes_{k=0}^N \mathscr{A}(\mathscr{F})$) is

$$\omega_{\rho^\otimes}(\cdot) := \mathrm{Tr}_{\mathscr{H}}(\rho^\otimes \ \cdot \ ) \ , \quad \rho^\otimes := \otimes_{k=0}^N \rho_k \ . \tag{2.7}$$

If we put $C_k(\alpha) := \mathrm{Tr}_{\mathscr{F}}[\rho_k\, \widehat{w}(\alpha)]$, $\alpha \in \mathbb{C}$, then by (2.3) one obtains for $\rho^\otimes$ (2.7) the representation:

$$\omega_{\rho^\otimes}(W(\zeta)) := \mathrm{Tr}_{\mathscr{H}}[\rho^\otimes\, W(\zeta)] = \prod_{k=0}^{N} C_k(\zeta_k) \ . \tag{2.8}$$

Let $\varrho \in \mathfrak{C}_1(\mathscr{H})$ be a density matrix on $\mathscr{H}$. Then for the system $\mathcal{S}+\mathcal{C}$, the Hamiltonian evolution $T_t : \varrho \mapsto \varrho(t)$ of initial density matrix $\varrho(0) := \varrho$ is defined as a solution of the Cauchy problem for the *non-autonomous* Liouville equation

$$\partial_t \varrho(t) = L(t)(\varrho(t)) \ , \ \varrho(t)\big|_{t=0} = \varrho \ . \tag{2.9}$$

By virtue of (1.7) the equation (2.9) is *autonomous* for each of the interval $[(n-1)\tau, n\tau)$:

$$L(t)(\cdot) = L_n(\cdot) = -i[H_n \ , \ \cdot \ ] \ , \qquad t \in [(n-1)\tau, n\tau) \ , \ n \geqslant 1 \ . \tag{2.10}$$

Since any $t \geqslant 0$ has the representation:

$$t := n(t)\tau + \nu(t) \ , \ n(t) := [t/\tau] \ \text{ and } \ \nu(t) \in [0, \tau) \ , \tag{2.11}$$

by the Markovian independence of generators (2.10), the trace-norm ($\|\cdot\|_1$)-continuous solution of the Cauchy problem (2.9) [Za] takes the iterative form:

$$\varrho(t) = T_t(\varrho) := T_{\nu(t),n}(T_{\tau,n-1}(\ldots T_{\tau,1}(\varrho)\ldots)) = \\ e^{-i\nu(t)H_n} e^{-i\tau H_{n-1}} \ldots e^{-i\tau H_1} \varrho\, e^{i\tau H_1} \ldots e^{i\tau H_{n-1}} e^{i\nu(t)H_n} \ . \tag{2.12}$$

Here $t \in [(n-1)\tau, n\tau)$, $n = n(t) < N$. By the $\|\cdot\|_1$-continuity we obtain from (2.12) that

$$\varrho(N\tau - 0) = \varrho(N\tau) = T_{N\tau}(\varrho) = e^{-i\tau H_N} \ldots e^{-i\tau H_1} \varrho\, e^{i\tau H_1} \ldots e^{i\tau H_N}. \tag{2.13}$$

Note that equivalent and often more convenient description of density matrices evolution (2.12) is the *dual* dynamics $T_t^* : \mathcal{L}(\mathscr{H}) \to \mathcal{L}(\mathscr{H})$:

$$\omega_{T_t(\varrho)}(A) = \mathrm{Tr}_{\mathscr{H}}(T_t(\varrho)\, A) =: \mathrm{Tr}_{\mathscr{H}}(\varrho\, T_t^*(A)), \text{ for } (\varrho, A) \in \mathfrak{C}_1(\mathscr{F}) \times \mathcal{L}(\mathscr{H}). \tag{2.14}$$

Since $t \mapsto T_t(\varrho)$ is $\|\cdot\|_1$-continuous and since $\mathcal{L}(\mathscr{H})$ is topologically dual of $\mathfrak{C}_1(\mathscr{H})$, one gets that $t \mapsto T_t^*(A)$ in (2.14) is a one-parameter $*$-automorphism of the unital C^*-algebra of bounded operators $\mathcal{L}(\mathscr{H})$. The automorphism of the C^*-dynamical system $(\mathcal{L}(\mathscr{H}), T_t^*)$ is not time-continuous for bosons. To ensure the continuity of T_t^* one considers instead of the C^*-algebra $\mathscr{A}(\mathscr{H}) \subset \mathcal{L}(\mathscr{H})$, the von Neumann algebra $\mathfrak{M}(\mathscr{H})$, which is closure of the Weyl linear span $\mathscr{A}_w$ generated by (2.1), (2.3) in the weak*-topology. Since it is weaker than C^*-algebra topology, $\mathfrak{M}(\mathscr{H})$ is $*$-isomorphic to $\mathcal{L}(\mathscr{H})$. Then $\|\cdot\|_1$-continuity of $T_t(\varrho)$ implies continuity of the dual mapping $t \mapsto T_t^*(A)$ in the weak*-topology on $\mathfrak{M}(\mathscr{H})$ and defines a W^*-dynamical system $(\mathfrak{M}(\mathscr{H}), T_t^*)$, see e.g. [AJP1] (Lectures 2 and 4).

Remark 2.1 *Below we show that T_t^* maps $\mathscr{A}(\mathscr{H})$ into itself, and that the action of T_t^* on Weyl operators can be calculated in the explicit form. Since $\mathscr{A}(\mathscr{H})$ is $*$-weakly dense in $\mathcal{L}(\mathscr{H})$, these allow to deduce properties of evolution $\rho(t)$.*

Using (2.13) and dual representation (2.14), we prove the main result of this section.

Lemma 2.2 *For $t = N\tau$, the expectation (2.8) of the Weyl operator (2.5) with respect to the evolved state has the form*

$$\omega_{\rho(N\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \ldots U_N\ \zeta)) = \prod_{k=0}^{N} C_k((U_1 \ldots U_N\ \zeta)_k)\,. \tag{2.15}$$

Here

$$(U_1 \ldots U_N\ \zeta)_0 = e^{iN\tau\epsilon}\big((gz)^N \zeta_0 + \sum_{j=1}^{N} gw(gz)^{j-1}\ \zeta_j\big)\,, \tag{2.16}$$

whereas

$$(U_1 \ldots U_N\ \zeta)_k = e^{iN\tau\epsilon}\big(gw(gz)^{N-k}\zeta_0 + g\bar{z}\zeta_k + \sum_{j=k+1}^{N} g^2 w^2 (gz)^{j-k-1}\zeta_j\big)\,, \tag{2.17}$$

for $0 < k < N$, and

$$(U_1 \ldots U_N\ \zeta)_N = e^{iN\tau\epsilon}\big(gw\zeta_0 + g\bar{z}\zeta_N\big)\,, \tag{2.18}$$

see definitions (1.12) and (1.13).

Proof : Note that (2.8), (2.13) and duality (2.14) yield

$$\omega_{\rho(N\tau)}(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[\rho\ T^*_{N\tau}(W(\zeta))] = \mathrm{Tr}_{\mathscr{H}}[\rho\ e^{i\tau H_1}\dots e^{i\tau H_N} W(\zeta) e^{-i\tau H_N}\dots e^{-i\tau H_1}]$$

$$= \mathrm{Tr}_{\mathscr{H}}[\rho\ W(U_1\dots U_N\ \zeta)] = \prod_{k=0}^{N} C_k((U_1\dots U_N\ \zeta)_k). \tag{2.19}$$

To generate the mapping $\zeta \mapsto U_1\dots U_N\,\zeta$ in (2.19), we use Lemma 1.1 and sesquilinear forms (2.4) to obtain

$$\begin{aligned} e^{i\tau H_1}\dots e^{i\tau H_N}\langle\zeta, b\rangle e^{-i\tau H_N}\dots e^{-i\tau H_1} &= \langle\zeta, U_N^*\dots U_1^* b\rangle \\ &= \langle U_1\dots U_N\ \zeta, b\rangle \end{aligned} \tag{2.20}$$

and the similar expression for its conjugate, which we then insert into (2.5).

Moreover, by the same Lemma 1.1, we get that $U_1\dots U_N\ \zeta = e^{iN\tau\epsilon}V_1\dots V_N\,\zeta$, where

$$(V_1\dots V_N)_{0j} = \begin{cases} (V_1)_{00}\dots(V_N)_{00} = (gz)^N & (j=0) \\ (V_1)_{00}\dots(V_{j-1})_{00}(V_j)_{0j}(V_{j+1})_{jj}\dots(V_N)_{jj} = (gz)^{j-1}gw & (0<j\leqslant N), \end{cases}$$

and for $0<k\leqslant N$:

$$(V_1\dots V_N)_{kj} = \begin{cases} (V_1\dots V_{k-1})_{kk}(V_k)_{k0}(V_{k+1}\dots V_N)_{00} = gw(gz)^{N-k} & (j=0) \\ 0 & (0<j<k) \\ (V_1\dots V_{k-1})_{kk}(V_k)_{kk}(V_{k+1}\dots V_N)_{kk} = g\bar{z} & (j=k) \\ (V_1\dots V_{k-1})_{kk}(V_k)_{k0}(V_{k+1}\dots V_{j-1})_{00}(V_j)_{0j}(V_{j+1}\dots V_N)_{jj} & \\ \qquad = gw(gz)^{j-k-1}gw & (k<j\leqslant N). \end{cases}$$

Collecting these formulae, one obtains explicit expressions for components (2.16) and (2.17) of the vector $U_1\dots U_N\,\zeta$. □

Remark 2.3 *Note that for a fixed N and for any $t = m\tau$, $1\le m\le N$, the arguments of Lemma 2.2 give a general formula*

$$\begin{aligned} \omega_{\rho(m\tau)}(W(\zeta)) &= \omega_\rho(T^*_{m\tau}(W(\zeta))) = \omega_\rho(W(U_1\dots U_m\ \zeta)) \\ &= \prod_{k=0}^{N} C_k((U_1\dots U_m\ \zeta)_k)\ . \end{aligned} \tag{2.21}$$

Following the same line of reasoning as for (2.17) one obtains explicit formulae for the components $\{(U_1\dots U_m\ \zeta)_k\}_{k=0}^N$:

$$(U_1\dots U_m\ \zeta)_k =$$

$$\begin{cases} e^{im\tau\epsilon}\big((gz)^m\zeta_0 + \sum_{j=1}^m gw(gz)^{j-1}\ \zeta_j\big) & (k=0) \\ e^{im\tau\epsilon}\big(gw(gz)^{m-k}\zeta_0 + g\bar{z}\zeta_k + \sum_{j=k+1}^m g^2w^2(gz)^{j-k-1}\zeta_j\big) & (1\leqslant k<m) \\ e^{im\tau\epsilon}\big(gw\zeta_0 + g\bar{z}\zeta_m\big) & (k=m) \\ e^{im\tau\epsilon}\ \zeta_k & (m<k\leqslant N) \end{cases}$$

Note that for $m = N$, these formulae coincide with (2.16)-(2.18), except the last line, which is void in this case.

Remark 2.4 *Recall that unity preserving $*$-dynamics $t \mapsto T_t^*$ on the von Neumann algebra $\mathfrak{M}(\mathscr{H})$ generated by $\{W(\zeta)\}_{\zeta\in\mathbb{C}}$ (2.5) is quasi-free, if there exist a mapping $U_t : \zeta \mapsto U_t\zeta$ and a complex-valued function $\Omega_t : \zeta \mapsto \Omega_t(\zeta)$, such that*

$$T_t^*(W(\zeta)) = \Omega_t(\zeta)W(U_t\zeta) \ , \ \Omega_0 = 1 \ , \ U_0 = I \ , \tag{2.22}$$

see e.g., [AJP1], [BR2] or [Ve]. Then by Remark 2.3, the step-wise dynamics

$$T_{m\tau}^*(W(\zeta)) = W(U_1 \dots U_m \ \zeta) \ , \qquad m = 0, 1, \dots, N$$

is quasi-free, with $\Omega_t(\zeta) = 1$ and the matrices $\{U_j\}_{j=1}^N$ on $\mathbb{C}^{N+1}$ defined by Lemma 1.1.

3 Entropy of Quasi-Free States on CCR C^*-Algebras

In this section, we establish some useful formulae relating expectations of the Weyl operators (Weyl characteristic function) and the entropy of boson quasi-free states. We formulate them in a way that is restricted but sufficient for our purposes. For general settings see, e.g. [Fa], [AJP1], [BR2], [Ve] and references therein.

Definition 3.1 *A state ω on the CCR C^*-algebra $\mathscr{A}(\mathscr{F})$ (2.1) is called quasi-free, if its characteristic function has the form*

$$\omega(\widehat{w}(\alpha)) := e^{-\frac{1}{4}|\alpha|^2 - \frac{1}{2}h(\alpha)} \ , \tag{3.1}$$

where $h : \alpha \mapsto \widehat{h}(\alpha, \alpha)$ is a (closable) non-negative sesquilinear form on $\mathbb{C}\times\mathbb{C}$. A quasi-free state ω is gauge-invariant if $\omega(\widehat{w}(\alpha)) = \omega(\widehat{w}(e^{i\varphi}\alpha))$ for $\varphi \in [0, 2\pi)$.

Let ω_β denote the Gibbs state with parameter β (dimensionless inverse temperature) given by the density matrix $\rho(\beta) = e^{-\beta a^* a}/Z(\beta)$, where $Z(\beta) = (1 - e^{-\beta})^{-1}$. Since

$$\omega_\beta(\widehat{w}(\alpha)) = e^{-\frac{1}{4}|\alpha|^2 - \frac{1}{2}h_\beta(\alpha)} \ , \quad h_\beta(\alpha) = \frac{|\alpha|^2}{e^\beta - 1} \ , \ \alpha \in \mathbb{C} \ , \tag{3.2}$$

this state is quasi-free and gauge-invariant. Note that the entropy of ω_β is given by

$$s(\beta) := -\mathrm{Tr}_{\mathscr{F}}[\rho(\beta)\ln\rho(\beta)] = \beta\omega_\beta(a^*a) - \ln(1 - e^{-\beta}) \ \text{ and } \ \omega_\beta(a^*a) = \frac{1}{e^\beta - 1} \ . \tag{3.3}$$

In terms of the variable $x := (1 + e^{-\beta})/(1 - e^{-\beta})$ the entropy (3.3) is

$$s(\beta) = \sigma(x) := \frac{x+1}{2}\ln\frac{x+1}{2} - \frac{x-1}{2}\ln\frac{x-1}{2} \ . \tag{3.4}$$

Here $\sigma : (1, \infty) \to (0, \infty)$ and $\sigma'(x) > 0$.

To extend (3.4) to the space (1.1) we note that a general *gauge-invariant* quasi-free states on the CCR C^*-algebra $\mathscr{A}(\mathscr{H})$ are defined by density matrices of the form [Ve]:

$$\rho_L = \frac{1}{Z_L}\, e^{-\langle b, Lb\rangle} \ , \ Z_L = \det\big[I - e^{-L}\big]^{-1} \ . \tag{3.5}$$

Here sesquilinear operator-valued forms $\langle b, Lb\rangle = \sum_{n,m=0}^{N} \ell_{nm} b_n^* b_m$ are parameterised by $(N+1)\times(N+1)$ *positive-definite* Hermitian matrix $L = \{\ell_{nm}\}_{0\leqslant n,m\leqslant N}$. Note that the $*$-automorphism G_φ on $\mathscr{A}(\mathscr{H})$ (the gauge transformation) :

$$G_\varphi : b_n^* \mapsto b_n^* e^{i\varphi},\ b_m \mapsto b_m e^{-i\varphi} \qquad (\varphi\in\mathbb{R}\ ,\ n,m=0,1,\dots N)\ , \tag{3.6}$$

leaves the state (3.5) invariant. Then characteristic function of the Weyl operators $W(\zeta)$ takes the form

$$\omega_{\rho_L}(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[\rho_L W(\zeta)] = \exp\big[-\frac{1}{4}\langle\zeta,\zeta\rangle - \frac{1}{2}\langle\zeta, \frac{I}{e^L - I}\,\zeta\rangle\big]\ . \tag{3.7}$$

Here the (*transposed*) vector in the argument is $\zeta^{\mathrm{tr}} = (\zeta_0,\zeta_1,\dots\zeta_N)\in\mathbb{C}^{N+1}$. Note that the entropy of the state ω_{ρ_L} is given by

$$S(\rho_L) = -\mathrm{Tr}_{\mathscr{H}}[\rho_L \ln\rho_L] = \mathrm{tr}_{\mathbb{C}^{N+1}}[L(e^L - I)^{-1} - \ln(I - e^{-L})]\ . \tag{3.8}$$

If we define the matrix $X := (I + e^{-L})(I - e^{-L})^{-1}$, then the characteristic function (3.7) takes the form:

$$\omega_{\rho_L}(W(\zeta)) = \exp\big[-\frac{1}{4}\langle\zeta, X\zeta\rangle\big]\ , \tag{3.9}$$

and for the entropy (3.8) we obtain

$$S(\rho_L) = \mathrm{tr}\Big[\frac{X+I}{2}\ln\frac{X+I}{2} - \frac{X-I}{2}\ln\frac{X-I}{2}\Big]. \tag{3.10}$$

Below we need a bit more specified set up than (3.9), (3.10). Let $\rho(\beta,\delta;\xi)$ be density matrix of a quasi-free state (3.5) corresponding to the operator-valued sesquilinear form

$$\langle b, L(\beta,\delta;\xi)b\rangle := \beta\sum_{n=0}^{N} b_n^* b_n + \delta\langle b,\xi\rangle\langle\xi, b\rangle. \tag{3.11}$$

on $\mathbb{C}^{N+1}\times\mathbb{C}^{N+1}$. Here $\beta>0,\ \delta>-\beta$, and the vector $\xi^{\mathrm{tr}} = (\xi_0,\xi_1,\dots,\xi_N)\in\mathbb{C}^{N+1}$.

Lemma 3.2 *The partition function of the state*

$$\rho(\beta,\delta;\xi) = \frac{1}{Z(\beta,\delta;\xi)}\ \exp\big[-\langle b, L(\beta,\delta;\xi)b\rangle\big]\ ,$$

is given by

$$Z(\beta,\delta;\xi) = \mathrm{Tr}_{\mathscr{H}}[e^{-\langle b, L(\beta,\delta;\xi)b\rangle}] = (1-e^{-\beta})^{-N}(1 - e^{-(\beta+\delta\langle\xi,\xi\rangle)})^{-1}\ . \tag{3.12}$$

The characteristic function and the entropy of this state are respectively:

$$\mathrm{Tr}_{\mathscr{H}}[\rho(\beta,\delta;\xi)W(\zeta)] = \exp\big[-\frac{1}{4}\,\frac{1+e^{-\beta}}{1-e^{-\beta}}\,\langle\zeta,\zeta\rangle\big]$$

$$\times\exp\big[-\frac{1}{4}\Big(\frac{1+e^{-\beta-\delta\langle\xi,\xi\rangle}}{1-e^{-\beta-\delta\langle\xi,\xi\rangle}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big)|\langle\xi,\zeta\rangle|^2/\langle\xi,\xi\rangle\big]\ , \tag{3.13}$$

and

$$S(\rho(\beta,\delta;\xi)) = -\mathrm{Tr}_{\mathscr{H}}[\rho(\beta,\delta;\xi)\ln\rho(\beta,\delta;\xi)] = Ns(\beta) + \ s(\beta+\delta\langle\xi,\xi\rangle)\ . \tag{3.14}$$

Proof : Proof of (3.12) follows from (3.5) and (3.11). Indeed, since by (3.5) any orthogonal transformation $\mathcal{O}$ on $\mathbb{C}^{N+1}$ leaves the partition function invariant: $Z_{\mathcal{O}^T L \mathcal{O}} = Z_L$, one can calculate it with $O\xi$ (instead of ξ), where $\mathcal{O}\xi$ has only one non-zero component equals to the vector norm $\langle \xi, \xi \rangle^{1/2}$. Then the right-hand side of (3.12) follows straightforwardly from the calculation of the left-hand side for this choice of $\mathcal{O}\xi$.

Since this transformation $\mathcal{O}$ also diagonalise the matrix $L := L(\beta, \delta; \xi)$, one uses it to simplify (3.9) and then to return back to ξ at the last step. To this aim we note that

$$\omega_{\rho_L}(W(\zeta)) = \exp\big[-\frac{1}{4} \langle \mathcal{O}\zeta, \mathcal{O}X\mathcal{O}^*\mathcal{O}\zeta \rangle \big] = \tag{3.15}$$
$$\exp\big[-\frac{1}{4}\, \frac{1+e^{-\beta}}{1-e^{-\beta}}\, \langle \mathcal{O}\zeta, \mathcal{O}\zeta \rangle' \big] \exp\big[-\frac{1}{4}\, \frac{1+e^{-\beta-\delta\langle \xi,\xi \rangle}}{1-e^{-\beta-\delta\langle \xi,\xi \rangle}} |(\mathcal{O}\zeta)_0|^2 \big] \ .$$

Here $\langle \mathcal{O}\zeta, \mathcal{O}\zeta \rangle' := \sum_{k=1}^N |(\mathcal{O}\zeta)_k|^2$ and we choose transformation $\mathcal{O}$ in such a way that $(\mathcal{O}\xi)_j = \delta_{0,j} \|\xi\|$. Since

$$|(\mathcal{O}\zeta)_0|^2 = \frac{1}{\langle \xi, \xi \rangle} \langle \mathcal{O}\zeta, \mathcal{O}\xi \rangle \langle \mathcal{O}\xi, \mathcal{O}\zeta \rangle \ , \tag{3.16}$$

the identities (3.15) prove (3.13). The same method is valid for entropy (3.8). Calculation of the trace in diagonal representation for $L = L(\beta, \delta; \xi)$ gives formula (3.14). □

Recall that the state ω on the CCR C^*-algebra $\mathscr{A}(\mathscr{H})$ is *regular*, if the map $s \mapsto \omega(W(s\,\zeta))$ is a continuous function of $s \in \mathbb{R}$ for any $\zeta \in \mathbb{C}^{N+1}$. This property follows from the explicit expression (3.13). Since by the Araki-Segal theorem, see e.g. [AJP1](Lecture 5), a regular state is completely defined by its characteristic function, (3.13) and (3.14) yield the following statement.

Lemma 3.3 *The entropy $S(\rho)$ of the quasi-free state ω_ρ on the CCR C^*-algebra $\mathscr{A}(\mathscr{H})$ with characteristic function*

$$\omega_\rho(W(\zeta)) = \exp\Big[-\frac{1}{4} \big(x\langle \zeta, \zeta \rangle + x_0 |\langle \xi, \zeta \rangle|^2 \big) \Big] \tag{3.17}$$

is uniquely determined by the parameters (ξ, x, x_0), where $\xi \in \mathbb{C}^{N+1}$, $x > 1,\ x_0 > 1 - x$ and it has the form

$$S(\rho) = N\sigma(x) + \sigma(x + x_0 \langle \xi, \xi \rangle) \ , \tag{3.18}$$

where $\sigma(\cdot)$ is defined by (3.4).

Proof : The proof follows directly from definition (3.4), if one puts

$$x_0\, \langle \xi, \xi \rangle = \frac{1+e^{-\beta-\delta\langle \xi,\xi \rangle}}{1-e^{-\beta-\delta\langle \xi,\xi \rangle}} - \frac{1+e^{-\beta}}{1-e^{-\beta}} \ ,$$

in (3.13) and uses (3.4) in (3.14). □

4 Repeated Perturbations and Entropy Production

We consider evolution (2.12) of the system $\mathcal{S}+\mathcal{C}$, when initial density matrix (2.7) corresponds to the product of gauge-invariant Gibbs quasi-free states with parameter $\beta_0 \geq 0$ for $\mathcal{S}$ and with parameter $\beta \geq 0$ for $\mathcal{C}$:

$$\rho = \rho_0 \otimes \bigotimes_{k=1}^{N} \rho_k \ , \qquad \rho_0 = e^{-\beta_0 a^* a}/Z(\beta_0) \ , \quad \rho_k = e^{-\beta a^* a}/Z(\beta) \ , \ k = 1, 2, \ldots, N \ . \tag{4.1}$$

This case corresponds to ρ_L in (3.5) with diagonal matrix $L = \mathrm{diag}(\beta_0, \beta, \ldots, \beta)$ and to $\rho(\beta, \delta; \xi)$ in representation (3.11) with $(\beta, \delta; \xi) = (\beta, \beta_0 - \beta; e)$, i.e.,

$$\rho = \rho(\beta, \beta_0 - \beta; e) = \exp\big[- \beta_0 b_0^* b_0 - \beta \sum_{j=1}^{N} b_j^* b_j \big] / Z(\beta, \beta_0 - \beta) \ . \tag{4.2}$$

Here $e^{\mathrm{tr}} = (1, 0, \ldots, 0) \in \mathbb{C}^{N+1}$ and

$$Z(\beta, \beta_0 - \beta) = Z(\beta_0) Z(\beta)^N = \frac{1}{(1 - e^{-\beta_0})(1 - e^{-\beta})^N} .$$

A straightforward application of formulae (3.13), (3.14) and Lemma 3.2 for $\xi = e$ (i.e. for $\langle \xi, \xi \rangle = 1$, $\langle \xi, \zeta \rangle = \zeta_0$) to the state (4.1) (or (4.2)), yields the following statement:

Lemma 4.1 *The characteristic function of (4.1) (or (4.2)) is*

$$\begin{aligned} &\omega_\rho(W(\zeta)) = \mathrm{Tr}_{\mathscr{H}}[\rho \ W(\zeta)] = \\ &\exp\Big[- \frac{|\zeta_0|^2}{4} \Big(\frac{1 + e^{-\beta_0}}{1 - e^{-\beta_0}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big) - \frac{\langle \zeta, \zeta \rangle}{4} \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big] \ , \end{aligned} \tag{4.3}$$

and the entropy is equal to

$$S(\rho) = N s(\beta) + s(\beta_0) \, . \tag{4.4}$$

Since by (2.12) the density matrix $\rho(t)$ of the total system $\mathcal{S}+\mathcal{C}$ for $t = N\tau$ is

$$\rho(N\tau) = e^{-i\tau H_N} \ldots e^{-i\tau H_1} \rho \, e^{i\tau H_1} \ldots e^{i\tau H_N} \ , \tag{4.5}$$

we obtain for evolution of the characteristic function and the entropy the statement:

Lemma 4.2 *Characteristic function of the state with density matrix (4.5) is equal to*

$$\begin{aligned} &\omega_{\rho(N\tau)}(W(\zeta)) = \\ &\exp\Big[- \frac{|(U_1 \ldots U_N \zeta)_0|^2}{4} \Big(\frac{1 + e^{-\beta_0}}{1 - e^{-\beta_0}} - \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big) - \frac{\langle \zeta, \zeta \rangle}{4} \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \Big] \ , \end{aligned} \tag{4.6}$$

whereas the total entropy rests invariant:

$$S(\rho(N\tau)) = S(\rho) = N s(\beta) + s(\beta_0) \ .$$

Here the mapping $U_1 \ldots U_N : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ is given by (2.16) and (2.17).

Proof : From (2.15), one gets $\omega_{\rho(N\tau)}(W(\zeta)) = \omega_\rho(W(U_1 \dots U_N\ \zeta))$. Since the mappings $U_j : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$, $j = 1, \dots, N$ are unitary (Lemma 2.2), (4.3) yields (4.6). Finally, we obtain that the mapping (4.5) leaves the total entropy (4.4) invariant, see (3.3). □

Let ω and ω_0 be two normal states on the Weyl CCR algebra $\mathscr{A}(\mathscr{H})$ with density matrices ϱ and ϱ_0. Following Araki [Ar1], we introduce the *relative entropy* of the state ω with respect to ω_0:

$$\mathrm{Ent}(\varrho|\varrho_0) := \mathrm{Tr}_{\mathscr{H}}[\varrho(\ln \varrho - \ln \varrho_0)] \geq 0 \ , \tag{4.7}$$

see also [AJP3].

Lemma 4.3 *The relative entropy of $\omega_{\rho(N\tau)}$ with respect to ω_ρ is*

$$\mathrm{Ent}(\rho(N\tau)|\rho) \ = \ \frac{(\beta_0 - \beta)(e^{\beta_0} - e^{\beta})}{(e^{\beta_0} - 1)(e^{\beta} - 1)}(1 - |z|^{2N}) \ , \tag{4.8}$$

where $z := z(\tau)$ is defined by (1.13) and (1.22).

Proof : The trace cyclicity yields

$$\mathrm{Ent}(\rho(N\tau)|\rho) = \mathrm{Tr}_{\mathscr{H}}[\rho(N\tau)(\ln \rho(N\tau) - \ln \rho)] \tag{4.9}$$

$$= \mathrm{Tr}_{\mathscr{H}}[\rho(\ln \rho - e^{i\tau H_1} \dots e^{i\tau H_N} \ln \rho e^{-i\tau H_N} \dots e^{-i\tau H_1})]$$

$$= \frac{\beta - \beta_0}{Z(\beta, \beta_0 - \beta)} \mathrm{Tr}_{\mathscr{H}}\big[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j} \big(b_0^* b_0 - e^{i\tau H_1} \dots e^{i\tau H_N} b_0^* b_0 e^{-i\tau H_N} \dots e^{-i\tau H_1})\big] \ .$$

Note that one gets $b_0^* b_0 = \langle b, e\rangle\langle e, b\rangle$ by (2.4). Hence, (2.20) implies

$$e^{i\tau H_1} \cdots e^{i\tau H_N} b_0^* b_0 e^{-i\tau H_N} \cdots e^{-i\tau H_1} = \sum_{k=0}^N (U_1 \dots U_N\ e)_k b_k^* \ \sum_{k'=0}^N \overline{(U_1 \dots U_N\ e)}_{k'}\, b_{k'} \tag{4.10}$$

Note also that the gauge invariance of the state ρ implies the selection rule:

$$\frac{1}{Z(\beta, \beta_0 - \beta)} \mathrm{Tr}_{\mathscr{H}}\big[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j}\ b_k^* b_{k'}\ \big] = 0 \ \text{ for } \ k \neq k' \ . \tag{4.11}$$

By this rule after injection of (4.10) into (4.9) only diagonal terms with $k = k'$ survive in the expectation:

$$\mathrm{Ent}(\rho(N\tau)|\rho) =$$

$$\frac{\beta - \beta_0}{Z(\beta, \beta_0 - \beta)} \mathrm{Tr}_{\mathscr{H}}\big[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j} \big(b_0^* b_0 - \sum_{k=0}^N |(U_1 \dots U_N e)_k|^2 b_k^* b_k\big)\big]$$

Finally, by Lemma 2.2, (2.16), (2.17), and by (3.3), we obtain

$$\mathrm{Ent}(\rho(N\tau)|\rho) =$$

$$\frac{\beta - \beta_0}{Z(\beta, \beta_0 - \beta)} \mathrm{Tr}_{\mathscr{H}}\big[e^{-\beta_0 b_0^* b_0 - \beta \sum_{j=1}^N b_j^* b_j} \big((1 - |z|^{2N}) b_0^* b_0 - \sum_{k=1}^N |w|^2 |z|^{2N-2k} b_k^* b_k)\big)\big]$$

$$= \frac{(\beta_0 - \beta)(e^{\beta_0} - e^{\beta})}{(e^{\beta_0} - 1)(e^{\beta} - 1)}(1 - |z|^{2N}) \ ,$$

that proves (4.8). □

Remark 4.4 *The relative entropy defined by (4.7) is non-negative. In contrast to invariant total entropy (Lemma 4.2), the relative entropy (4.8) is increasing monotonously with $N \to \infty$ for $|z| < 1$ (see Lemma 1.1, Remark 1.2). It converges to the limit:*

$$\lim_{N\to\infty} \mathrm{Ent}(\rho(N\tau)|\rho) \;=\; (\beta - \beta_0)\left[\frac{1}{e^{\beta_0}-1} - \frac{1}{e^{\beta}-1}\right] \geq 0 \;, \tag{4.12}$$

which is positive for $\beta_0 \neq \beta$. The limit (4.12) gives asymptotic amount of the entropy production, when one starts with the initial state corresponding to (4.1) and then consider $N\tau \to \infty$, see [BJM].

5 Evolution of Subsystems

5.1 Convergence to Equilibrium

Subsystem $\mathcal{S}$. We start with the simplest subsystem $\mathcal{S}$. Let the initial state of the total system $\mathcal{S} + \mathcal{C}$ in (1.1) be a tensor-product of the corresponding density matrices $\rho = \rho_S \otimes \rho_C$, see (H1). Then for $t \geq 0$ the state $\omega^t_{\mathcal{S}}(\cdot)$ of the subsystem $\mathcal{S}$ is given on the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_0)$ by

$$\omega^t_{\mathcal{S}}(\cdot) := \omega_{\rho(t)}(\cdot \otimes \mathbb{1}) \;. \tag{5.1}$$

For $\zeta = (\alpha, 0, \ldots, 0) \in \mathbb{C}^{N+1}$, let us consider the Weyl operator $W(\zeta) = \widehat{w}(\alpha) \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$ (2.3). By virtue of (2.8), (2.21) and (5.1), we obtain for $t = m\tau$ ($1 \leq m \leq N$):

$$\omega^{m\tau}_{\mathcal{S}}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta)) = \omega_{\rho}(W(U_1 \ldots U_m\, \zeta)) \;. \tag{5.2}$$

Then for components $\{(U_1 \ldots U_m\, \zeta)_k\}_{k=0}^N$ of the vector $U_1 \ldots U_m \zeta$ in (5.2), one obtains the expression:

$$(U_1 \ldots U_m\, \zeta)_k = \begin{cases} e^{im\tau\epsilon}(gz)^m \alpha & (k = 0) \\ e^{im\tau\epsilon} g w (gz)^{m-k} \alpha & (1 \leqslant k < m) \\ e^{im\tau\epsilon} g w \alpha & (k = m) \\ 0 & (m < k \leqslant N) \;, \end{cases} \tag{5.3}$$

which follows from Remark 2.3.

If the initial density matrices: $\rho = \rho_S \otimes \rho_C$ corresponds to the product of Gibbs quasi-free states for different temperatures as in (4.1), then (5.2) and Lemma4.1 yield

$$\omega^{m\tau}_{\mathcal{S}}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|z^m\alpha|^2}{4}\left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\right)\Big] \tag{5.4}$$

Note that for any moment $t = m\tau$ the state $\omega^{m\tau}_{\mathcal{S}}(\cdot)$ is a quasi-free Gibbs equilibrium state with parameter $\beta^*(m\tau)$ which satisfies the equation

$$\frac{1+e^{-\beta^*(m\tau)}}{1-e^{-\beta^*(m\tau)}} = |z|^{2m}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|z|^{2m})\frac{1+e^{-\beta}}{1-e^{-\beta}} \;. \tag{5.5}$$

This equation yields that either $\beta \leq \beta^*(m\tau) \leq \beta_0$, or $\beta_0 \leq \beta^*(m\tau) \leq \beta$.

For $m \to \infty$ ($N \to \infty$) the Weyl characteristic function (5.4) has the limit

$$\lim_{m\to\infty} \omega_{\mathcal{S}}^{m\tau}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ . \tag{5.6}$$

Hence, in the limit $t \to \infty$ the subsystem $\mathcal{S}$ evolves from the Gibbs equilibrium state with parameter β_0 to another equilibrium state with parameter β *imposed* by the chain $\mathcal{C}$.

Subsystem $\mathcal{S}_1$. The initial state $\omega_{\mathcal{S}_1}^0(\cdot) = \omega_{\mathcal{S}_1}^t(\cdot)|_{t=0}$ of this subsystem corresponds to a one-point reduced density matrix or to the partial trace on the CCR Weyl algebra $\mathscr{A}(\mathscr{H}_1)$:

$$\omega_{\mathcal{S}_1}^0(\widehat{w}(\alpha)) = \omega_\rho(\mathbb{1} \otimes \widehat{w}(\alpha) \otimes \bigotimes_{k=2}^{N} \mathbb{1}) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ . \tag{5.7}$$

Now we choose vector $(\zeta^1)^{\mathrm{tr}} := (0, \alpha, 0, \ldots, 0) \in \mathbb{C}^{N+1}$. Then

$$\omega_{\mathcal{S}_1}^{m\tau}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta^{(1)})) = \omega_{\rho_S\otimes\rho_C}(W(U_1 \ldots U_m\ \zeta^{(1)})) \tag{5.8}$$

for $1 < m \leq N$. By Remark 2.3, the components $\{(U_1 \ldots U_m\ \zeta^{(1)})_k\}_{k=0}^N$ are:

$$(U_1 \ldots U_m\ \zeta)_k = \begin{cases} e^{im\tau\epsilon} g w\, \alpha & (k=0) \\ e^{im\tau\epsilon}\, \delta_{k,1}\, g\overline{z}\, \alpha & (1 \leqslant k < m) \\ 0 & (m \leqslant k \leqslant N). \end{cases} \tag{5.9}$$

Then, we have

$$\omega_{\mathcal{S}_1}^{m\tau}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|w\alpha|^2}{4}\left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\right)\Big] \tag{5.10}$$

for any $1 < m \leq N$. Therefore, the initial state (5.7) changes to (5.10) after the first act of interaction on the interval $[0, \tau)$ and there is no further evolution of this state for $t > \tau$.

Note that (5.10) is characteristic function of a quasi-free Gibbs equilibrium state with parameter β^*, which satisfies the equation

$$\frac{1+e^{-\beta^*}}{1-e^{-\beta^*}} = |w|^2\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|w|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}} \ .$$

Again, this equation implies that either $\beta \leq \beta^* \leq \beta_0$, or $\beta_0 \leq \beta^* \leq \beta$.

Evolution of $\mathcal{S}_1$ has a transparent interpretation: after the one act of interaction during the time $t \in [0, \tau)$, subsystem $\mathcal{S}_1$ relaxes to an *intermediate* equilibrium with the subsystem $\mathcal{S}$. This results in a shift of initial parameter β to β^*, which rests unchangeable since there is no perturbations of subsystem $\mathcal{S}_1$ for $t > \tau$.

Subsystem $\mathcal{S}_m$. For $1 < m \leq N$ the initial state $\omega_{\mathcal{S}_m}^0(\cdot) = \omega_{\mathcal{S}_m}^t(\cdot)|_{t=0}$ of this subsystem is defined on the CCR Weyl algebra $\mathscr{A}(\mathscr{H}_m)$ by the partial trace :

$$\omega_{\mathcal{S}_m}^0(\widehat{w}(\alpha)) = \omega_\rho(\bigotimes_{k=0}^{m-1} \mathbb{1} \otimes \widehat{w}(\alpha) \otimes \bigotimes_{k=m+1}^{N} \mathbb{1}) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \ . \tag{5.11}$$

Now we choose vector $(\zeta^m)^{\rm tr} := (0,\dots,0,\alpha,0,\dots,0) \in \mathbb{C}^{N+1}$, where α occupies the $m+1$ position. Consequently

$$\omega^{m\tau}_{\mathcal{S}_m}(\widehat{w}(\alpha)) = \omega_{\rho(m\tau)}(W(\zeta^{(m)})) = \omega_{\rho_S\otimes\rho_C}(W(U_1\dots U_m\,\zeta^{(m)}))\ . \tag{5.12}$$

The components $\{(U_1\dots U_m\,\zeta^{(m)})_k\}_{k=0}^N$ are:

$$(U_1\dots U_m\,\zeta^{(m)})_k = \begin{cases} e^{im\tau\epsilon}\,gw(gz)^{m-1}\,\alpha & (k=0)\\ e^{im\tau\epsilon}\,g^2w^2\,(gz)^{m-k-1}\,\alpha & (1\leqslant k<m)\\ e^{im\tau\epsilon}g\overline{z}\,\alpha\ , & (k=m)\\ 0 & (m<k\leqslant N). \end{cases} \tag{5.13}$$

which again follows from Remark 2.3. Then evolution of the state of subsystem $\mathcal{S}_m$ is:

$$\begin{aligned}&\omega^{m\tau}_{\mathcal{S}_m}(\widehat{w}(\alpha)) = \\ &\exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}} - \frac{|w\alpha|^2}{4}|z|^{2(m-1)}\left(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\right)\Big]\ .\end{aligned} \tag{5.14}$$

Note that interaction for $t\in[(m-1)\tau, m\tau)$ push out the subsystem $\mathcal{S}_m$ from the Gibbs equilibrium state (5.11), but its effect attenuates for large m:

$$\lim_{m\to\infty}\omega^{m\tau}_{\mathcal{S}_m}(\widehat{w}(\alpha)) = \exp\Big[-\frac{|\alpha|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]\ . \tag{5.15}$$

Again, this is evolution of a quasi-free Gibbs equilibrium state with time-dependent inverse temperature parameter $\beta^{**}(m\tau)$, which satisfies the equation

$$\frac{1+e^{-\beta^{**}(m\tau)}}{1-e^{-\beta^{**}(m\tau)}} = |w|^2|z|^{2(m-1)}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} + (1-|w|^2|z|^{2(m-1)})\frac{1+e^{-\beta}}{1-e^{-\beta}}\ . \tag{5.16}$$

As above, the value of the parameter $\beta^{**}(m\tau)$ is always between β_0 and β.

To interpret the evolution of $\mathcal{S}_m$ and the coincidence between (5.15) and (5.6) note that the state of the subsystem $\mathcal{S}$ relaxes to that of initial state of the chain $\mathcal{C}$, see (5.6). Therefore, after interaction of the subsystem $\mathcal{S}_m$, i.e. at the moment $t=m\tau$, its parameter $\beta^{**}(m\tau)$ has a value between β and $\beta^*((m-1)\tau)$ since (5.5) and (5.16) yield

$$\frac{1+e^{-\beta^{**}(m\tau)}}{1-e^{-\beta^{**}(m\tau)}} = |w|^2\frac{1+e^{-\beta^*((m-1)\tau)}}{1-e^{-\beta^*((m-1)\tau)}} + (1-|w|^2)\frac{1+e^{-\beta}}{1-e^{-\beta}}\ .$$

As in the case $m=1$, there is no further evolution: $\omega^{n\tau}_{\mathcal{S}_m} = \omega^{m\tau}_{\mathcal{S}_m}$ for $n\geqslant m$.

Next, we consider the composed subsystems $\mathcal{S}+\mathcal{S}_m$ and $\mathcal{S}_{m-n}+\mathcal{S}_m$. Our aim is to study the eventual *correlations* imposed by repeated perturbations due to $\mathcal{S}$.

Subsystem $\mathcal{S}+\mathcal{S}_m$. For $1<m\leqslant N$ the initial state $\omega^0_{\mathcal{S}+\mathcal{S}_m}(\cdot) = \omega^t_{\mathcal{S}+\mathcal{S}_m}(\cdot)|_{t=0}$ of this *composed* subsystem is defined by the partial trace on the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_0\otimes\mathscr{H}_m)\approx\mathscr{A}(\mathscr{H}_0)\otimes\mathscr{A}(\mathscr{H}_m)$ by:

$$\begin{aligned}\omega^0_{\mathcal{S}+\mathcal{S}_m}(\widehat{w}(\alpha_0)\otimes\widehat{w}(\alpha_1)) &:= \omega_\rho(\widehat{w}(\alpha_0)\otimes\bigotimes_{k=1}^{m-1}\mathbb{1}\otimes\widehat{w}(\alpha_1)\otimes\bigotimes_{k=m+1}^{N}\mathbb{1})\\ &= \exp\Big[-\frac{|\alpha_0|^2}{4}\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}}\Big]\exp\Big[-\frac{|\alpha_1|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]\ .\end{aligned} \tag{5.17}$$

This is the characteristic function of the product state corresponding to two isolated systems with different temperatures. If one defines vector $(\zeta^{(0,m)})^{\rm tr} := (\alpha_0, 0, \dots, 0, \alpha_1, 0, \dots, 0) \in \mathbb{C}^{N+1}$, where α_1 occupies the $m+1$ position, then

$$\omega^{m\tau}_{\mathcal{S}+\mathcal{S}_m}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) = \omega_{\rho(m\tau)}(W(\zeta^{(0,m)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \dots U_m\ \zeta^{(0,m)}))\ . \tag{5.18}$$

The components $\{(U_1 \dots U_m\ \zeta^{(0,m)})_k\}_{k=0}^N$ are deduced from Remark 2.3:

$$(U_1 \dots U_m\ \zeta^{(0,m)})_k = \begin{cases} e^{im\tau\epsilon}\ (gz)^{m-1}\ [gz\,\alpha_0 + gw\,\alpha_1], & (k=0) \\ e^{im\tau\epsilon}(gz)^{m-k-1} g^2 [wz\,\alpha_0 + w^2\,\alpha_1], & (1 \leqslant k < m) \\ e^{im\tau\epsilon}\ [gw\,\alpha_0 + g\overline{z}\,\alpha_1], & (k=m) \\ 0 & (m < k \leqslant N). \end{cases} \tag{5.19}$$

Together with (2.8), one gets for $m \to \infty$:

$$\begin{aligned} &\omega^{m\tau}_{\mathcal{S}+\mathcal{S}_m}(\widehat{w}(\alpha_0) \otimes \widehat{w}(\alpha_1)) \\ &= \exp\Big[-\frac{1}{4}|z\alpha_0 + w\alpha_1|^2 |z|^{2(m-1)} \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}}\Big] \\ &\times \exp\Big[-\frac{1}{4}|z\alpha_0 + w\alpha_1|^2 (1-|z|^{2(m-1)}) \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \exp\Big[-\frac{1}{4}|w\alpha_0 + \overline{z}\alpha_1|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \\ &\longrightarrow \exp\Big[-\frac{1}{4}(|\alpha_0|^2 + |\alpha_1|^2) \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]\ . \end{aligned} \tag{5.20}$$

Hence, in this limit the composed subsystem $\mathcal{S}+\mathcal{S}_m$ evolves from the product of two quasi-free equilibrium states (5.17) with different parameters β_0 and β to the product of quasi-free equilibrium states for the same parameter β imposed by repeated interaction with the chain $\mathcal{C}$, when $m \to \infty$. Interpretation is similar to the case *Subsystem* $\mathcal{S}_m$.

Subsystem $\mathcal{S}_{m-n}+\mathcal{S}_m$. We suppose that $1 < (m-n) < m \leqslant N$. Then the initial state $\omega^t_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\cdot)|_{t=0}$ of this *composed* subsystem is the partial trace over the Weyl C^*-algebra $\mathscr{A}(\mathscr{H}_{m-n} \otimes \mathscr{H}_m) \approx \mathscr{A}(\mathscr{H}_{m-n}) \otimes \mathscr{A}(\mathscr{H}_m)$:

$$\begin{aligned} &\omega^0_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) := \\ &\omega_\rho\Big(\bigotimes_{k=0}^{m-n-1} \mathbb{1} \otimes \widehat{w}(\alpha_1) \otimes \bigotimes_{k=m-n+1}^{m-1} \mathbb{1} \otimes \widehat{w}(\alpha_2) \otimes \bigotimes_{k=m+1}^{N} \mathbb{1}\Big) = \\ &= \exp\Big[-\frac{|\alpha_1|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] \exp\Big[-\frac{|\alpha_2|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]\ . \end{aligned} \tag{5.21}$$

This is the characteristic function of the product state corresponding to two isolated systems with the same temperatures.

We define vector $(\zeta^{(m-n,m)})^{\rm tr} := (0, 0, \dots, 0, \alpha_1, 0, \dots, 0, \alpha_2, 0, \dots, 0) \in \mathbb{C}^{N+1}$, where α_1 occupies the $m-n+1$ position, and α_2 occupies the $m+1$ position, then

$$\begin{aligned} &\omega^{m\tau}_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) = \\ &\omega_{\rho(m\tau)}(W(\zeta^{(m-n,m)})) = \omega_{\rho_S \otimes \rho_C}(W(U_1 \dots U_m\ \zeta^{(m-n,m)}))\ . \end{aligned} \tag{5.22}$$

With help of Remark 2.3 we can calculate the values of components $\{(U_1 \dots U_m\, \zeta^{(m-n,m)})_k\}_{k=0}^N$:

$$(U_1 \dots U_m\, \zeta^{(m-n,m)})_k = \tag{5.23}$$

$$= \begin{cases} e^{im\tau\epsilon}\, (gz)^{m-n-1}\, gw[\alpha_1 + (gz)^n \alpha_2] & (k=0) \\ e^{im\tau\epsilon}\, [g^2 w^2 (gz)^{m-n-k-1}\, \alpha_1 + g^2 w^2 (gz)^{m-k-1}\, \alpha_2] & (1 \leqslant k < m-n) \\ e^{im\tau\epsilon}\, [g\overline{z}\, \alpha_1 + g^2 w^2\, (gz)^{m-k-1}\, \alpha_2] & (k = m-n) \\ e^{im\tau\epsilon}\, g^2 w^2\, (gz)^{m-k-1}\, \alpha_2 & (m-n < k < m) \\ e^{im\tau\epsilon}\, g\overline{z}\, \alpha_2 & (k=m) \\ 0 & (m < k \leqslant N) \end{cases} .$$

When $m \to \infty$, then for any fixed n we obtain for (5.22):

$$\begin{aligned} &\omega^{m\tau}_{\mathcal{S}_{m-n}+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) \\ &= \exp\Big[-\frac{1}{4}|w|^2 |\alpha_1 + (gz)^{n+1}\alpha_2|^2 |z|^{2(m-n-1)} \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} \Big] \\ &\times \exp\Big[-\frac{1}{4}\{|w|^2(1-|z|^{2(m-n-1)}) + |z|^2\}|\alpha_1|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}} \Big] \\ &\times \exp\Big[-\frac{1}{4}(1-|w|^2|z|^{2(m-1)})|\alpha_2|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}} \Big] . \\ &\longrightarrow \exp\Big[-\frac{1}{4}(|\alpha_1|^2 + |\alpha_2|^2) \frac{1+e^{-\beta}}{1-e^{-\beta}} \Big] . \end{aligned} \tag{5.24}$$

Therefore, in this limit, the composed subsystem $\mathcal{S}_{m-n} + \mathcal{S}_m$ evolves from the initial product of two quasi-free equilibrium states (5.21) to the *same* final state, although for a finite m the evolution (5.24) is nontrivial. This again easily understandable taking into account our analysis of *Subsystem* $\mathcal{S}_m$ and *Subsystem* $\mathcal{S} + \mathcal{S}_m$.

Consider now the case of a *fixed* $s := m - n \geqslant 1$. Then the limit in (5.24) is

$$\begin{aligned} &\lim_{m\to\infty} \omega^{m\tau}_{\mathcal{S}_s+\mathcal{S}_m}(\widehat{w}(\alpha_1) \otimes \widehat{w}(\alpha_2)) = \\ &= \exp\Big[-\frac{1}{4}|w|^2|z|^{2(s-1)}|\alpha_1|^2 \left\{ \frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}} \right\} \Big] \times \\ &\times \exp\Big[-\frac{1}{4}|\alpha_1|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}} \Big] \exp\Big[-\frac{1}{4}|\alpha_2|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}} \Big] \\ &= \exp\Big[-\frac{1}{4}|\alpha_1|^2 \frac{1+e^{-\beta^{**}(s\tau)}}{1-e^{-\beta^{**}(s\tau)}} \Big] \exp\Big[-\frac{1}{4}|\alpha_2|^2 \frac{1+e^{-\beta}}{1-e^{-\beta}} \Big] , \end{aligned} \tag{5.25}$$

where $\beta^{**}(s\tau)$ verifies equation (5.16). Hence, in this case the limit state (5.25) is the product of quasi-free Gibbs states with *different* parameters $\beta^{**}(s\tau)$ and β. This means that subsystem $\mathcal{S}_s$ keeps a *memory* about perturbation at the moment $t = s\tau$, when the parameter $\beta^*(s\tau)$ (5.5) of subsystem $\mathcal{S}$ was still different from β.

Note that (5.25) coincides with the product state (5.21) when $s \to \infty$.

Subsystem $\mathcal{S}_{\sim n}$. To define $\mathcal{S}_{\sim n}$ for $0 \leqslant n \leqslant k \leqslant N$, we divide the total system at the moment $t = k\tau$ into two subsystems: $\mathcal{S}_{n,k} + \mathcal{C}_{n,k}$. Here

$$\mathcal{S}_{n,k} := \mathcal{S} + \mathcal{S}_k + \mathcal{S}_{k-1} + \dots + \mathcal{S}_{k-n+1} \ , \ (\mathcal{S}_{0,k} := \mathcal{S}) \ , \tag{5.26}$$

whereas (see definitions in Section 1.1)

$$\mathcal{C}_{n,k} := \mathcal{S}_N + \cdots + \mathcal{S}_{k+1} + \mathcal{S}_{k-n} + \cdots + \mathcal{S}_1 \ . \tag{5.27}$$

We mean that $\mathcal{S}_{\sim n}$ is an entire "object" whose entity is $\mathcal{S}_{n,k}$ at the moment $t = k\tau$ ($k = n, n+1, \cdots, N$). As time is running, the elementary subsystems $\mathcal{S}_k$ in $\mathcal{S}_{\sim n}$ are replacing. We study the behaviour of $\mathcal{S}_{\sim n}$ for large $t = k\tau$, i.e., the k-dependence of the "state" of $\mathcal{S}_{n,k}$ at $t = k\tau$.

For any fixed $t = k\tau$ we can decompose the Hilbert space $\mathscr{H}$ into tensor product $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_c$. Here $\mathscr{H}_s$ is the Hilbert space of subsystem (5.26) and $\mathscr{H}_c$ corresponds to subsystem (5.27):

$$\mathscr{H}_s := \mathscr{H}_0 \otimes \bigotimes_{j=1}^{n} \mathscr{H}_{k-j+1}, \qquad \mathscr{H}_c := \bigotimes_{j=1}^{k-n} \mathscr{H}_j \otimes \bigotimes_{j=k+1}^{N} \mathscr{H}_j \ . \tag{5.28}$$

For a density matrix ϱ on $\mathscr{H}$, we introduce the *reduced* density matrix ϱ_s on $\mathscr{H}_s$ as the partial trace over $\mathscr{H}_c$:

$$\varrho_s := \mathrm{Tr}_{\mathscr{H}_c} \varrho \ . \tag{5.29}$$

To avoid a possible confusion causing by the fact that all $\mathscr{H}_j, j = 0, 1, \ldots$ are identical to $\mathscr{F}$ and by the change of components with time, we treat the Weyl algebra on the subsystem and the corresponding reduced density matrix of $\rho \in \mathfrak{C}_1(\mathscr{H})$ in the following way. For $n \leqslant N$ on the Fock space $\mathcal{F}^{\otimes(n+1)}$ we consider the Weyl operator

$$W_n(\zeta) = \exp\Big[i\frac{\langle \zeta, \tilde{b}, \rangle + \langle \tilde{b}, \zeta \rangle}{\sqrt{2}}\Big], \tag{5.30}$$

where $\zeta \in \mathbb{C}^{n+1}$, $\tilde{b}_0, \cdots, \tilde{b}_n$ and $\tilde{b}_0^*, \cdots, \tilde{b}_n^*$ are the annihilation and the creation operators in $\mathcal{F}^{\otimes(n+1)}$ satisfying the corresponding CCR, and

$$\langle \zeta, \tilde{b} \rangle = \sum_{j=0}^{n} \bar{\zeta}_j \tilde{b}_j, \qquad \langle \tilde{b}, \zeta \rangle = \sum_{j=0}^{n} \zeta_j \tilde{b}_j^*.$$

By $\mathscr{A}(\mathscr{F}^{\otimes(n+1)})$, we denote the C^* algebra generated by the Weyl operators (5.30). For any subset $J \subset \{1, 2, \cdots, N\}$, we define the operation of taking the partial trace

$$R^J : \mathfrak{C}_1(\mathscr{F}^{\otimes(N+1)}) \ni \rho \longmapsto R^J \rho \in \mathfrak{C}_1(\mathscr{F}^{\otimes(N+1-|J|)})$$

by

$$\omega_{R^J \rho}\big(W_{N-|J|}(\zeta)\big) = \omega_\rho\big(W_N(r_J \zeta)\big) \ .$$

Here the mapping

$$r_J : \mathbb{C}^{N+1-|J|} \ni \zeta \longmapsto r_J \zeta \in \mathbb{C}^{N+1}$$

is defined by

$$(r_J \zeta)_j := \begin{cases} \zeta_0 & (j = 0) \\ 0 & (j \in J) \\ \zeta_{j-|\{i \in J \,|\, i<j\}|} & (\text{otherwise}) \end{cases} ,$$

where $|A|$ denotes the cardinality of the set A.

Since all $\mathscr{H}_1, \mathscr{H}_2, \cdots$ are identical to $\mathscr{F}$, we do not care to distinguish the spaces

$$\bigotimes_{j\in\{0,1,\cdots,N\}\setminus J} \mathscr{H}_j \qquad \text{and} \qquad \bigotimes_{j\in\{0,1,\cdots,N\}\setminus J'} \mathscr{H}_j$$

when $J \neq J'$, but $|J| = |J'|$, and consider them as the same space $\mathscr{F}^{\otimes(N+1-|J|)}$. Instead, we pay attention to distinguishing projections

$$\bigotimes_{j=0}^{N} \mathscr{H}_j \longrightarrow \bigotimes_{j\in\{0,1,\cdots,N\}\setminus J} \mathscr{H}_j$$

for different subsets $J \subset \{1, 2, \cdots, N\}$ with same $|J|$.

Since we treat $\mathcal{S}_{n,k}$ at time $t = k\tau$ for $k = n, n+1, \cdots$ as the result of the time evolution of a *single* subsystem $\mathcal{S}_{\sim n}$, we define its state at the moment $t = k\tau$ by the reduced density matrix $\{\rho_s(k\tau)\}_{k\geqslant n}$ of this subsystem as follows:

$$\rho_s(k\tau) := R^{\{1,\cdots,k-n,k+1,\cdots,N\}}\big(\rho(k\tau)\big) = R^{\{1,\cdots,k-n,k+1,\cdots,N\}} T_{k\tau}(\rho), \tag{5.31}$$

see (2.12). Taking into account Lemma 4.2 and identity $\langle r_J\zeta, r_J\zeta\rangle_{\mathbb{C}^{N+1}} = \langle \zeta, \zeta\rangle_{\mathbb{C}^{N+1-|J|}}$, one readily obtains the following result.

Lemma 5.1 *For the initial density matrix (4.1),*

$$\omega_{\rho_s(k\tau)}(W_n(\zeta)) = \omega_{R^{J_{n,k}}\rho(k\tau)}(W_n(\zeta))$$

$$= \exp\Big[-\frac{|(U_1\ldots U_k\, r_{J_{n,k}}\zeta)_0|^2}{4}\Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) - \frac{\langle\zeta,\zeta\rangle}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big]$$

holds, where $J_{n,k} = \{1, 2, \cdots, k-n, k+1, \cdots, N\}$.

To study the limit $k \to \infty$ and $N \to \infty$ ($k \leqslant N$) for a fixed n, we note that $(U_1\ldots U_k\, r_{J_{n,k}}\zeta)_0 \to 0$ follows from (2.16) and $|z| < 1$. Lemma 5.1 implies that

$$\lim_{k\to\infty} \omega_{\rho_s(k\tau)}(W_n(\zeta)) = \exp\Big[-\frac{\langle\zeta,\zeta\rangle}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big] = \omega_{\rho_n^{(\beta)}}(W_n(\zeta)) \ , \tag{5.32}$$

where by the Araki-Segal theorem and irreducibility of the CCR algebra $\mathscr{A}(\mathscr{F}^{\otimes(n+1)})$ one has

$$\rho_n^{(\beta)} = \exp\big[-\beta\sum_{j=0}^{n} \tilde{b}_j^*\tilde{b}_j\big]/Z(\beta)^{n+1} \ , \ Z(\beta) = (1-e^{-\beta})^{-1} \ . \tag{5.33}$$

Therefore, we proved the following statement:

Theorem 5.2 *Let the initial state of the total system* $\mathcal{S}+\mathcal{C}$ *is defined by the density matrix (4.2):* $\rho = \rho(\beta, \beta_0 - \beta; e)$. *Then for any fixed* n, *the state* $\omega_{\rho_s(k\tau)}(\cdot)$ *of subsystem* $\mathcal{S}_{n,k}$ *converges to the equilibrium Gibbs state* $\omega_{\rho_n^{(\beta)}}(\cdot)$ *as* $k \to \infty$ *in the weak*-topology for the states on* $\mathscr{A}(\mathscr{F}^{\otimes(n+1)})$.

Theorem 5.3 *Under the same conditions as in Theorem 5.2, we obtain*

$$\lim_{k\to\infty} S(\rho_s(k\tau)) = S(\rho_n^{(\beta)}) \ .$$

Proof : Let the vector $\xi_{n,k} \in \mathbb{C}^{n+1}$ be defined by $(U_1 \dots U_k r_{J_{n,k}} \zeta)_0 =: \langle \xi_{n,k}, \zeta \rangle$. Then $\langle \xi_{n,k}, \xi_{n,k} \rangle \to 0$ as $k \to \infty$ for a fixed n and by Lemma 3.3, Lemma 5.1, we obtain that

$$S(\rho_s(k\tau)) = n\sigma\Big(\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) + \sigma\Big(\frac{1+e^{-\beta}}{1-e^{-\beta}} + \langle \xi_{n,k}, \xi_{n,k} \rangle \Big(\frac{1+e^{-\beta_0}}{1-e^{-\beta_0}} - \frac{1+e^{-\beta}}{1-e^{-\beta}}\Big)\Big)$$

$$\longrightarrow (n+1)\sigma\Big(\frac{1+e^{-\beta}}{1-e^{-\beta}}\Big) = S(\rho_n^{(\beta)}) \ .$$

□

Remark 5.4 *The local entropy decreases or increases according to $\beta > \beta_0$ or $\beta < \beta_0$, respectively.*

5.2 A Short-Time Limit for Repeated Perturbations

The results in the Subsection 5.1 are essentially due explicit knowledge of the initial density matrix (4.1) of the total system $\mathcal{S} + \mathcal{C}$. In this subsection, we show that the lack of this information is not decisive for certain results concerning the convergence to equilibrium if one considers the repeated interactions in a *short-time limit*.

We study this limit for the subsystem $\mathcal{S}$. We keep to consider the initial state of the system $\mathcal{S} + \mathcal{C}$ to be a product state with the density matrix

$$\rho = \rho_0 \otimes \bigotimes_{k=1}^{N} \rho_k \in \mathfrak{C}_1(\mathscr{H}) \ , \tag{5.34}$$

see (2.7), but we essentially relax the conditions on ρ_0 and on $\{\rho_k\}_{k=1}^N$:

$$\begin{array}{ll} \text{(h1)} & \rho_1 = \rho_2 = \dots = \rho_N \in \mathfrak{C}_1(\mathscr{F}) \ ; \\ \text{(h2)} & \mathrm{Tr}_{\mathscr{F}}(\rho_1 a) = \mathrm{Tr}_{\mathscr{F}}(\rho_1 a^2) = \mathrm{Tr}_{\mathscr{F}}(\rho_1 a^*) = \mathrm{Tr}_{\mathscr{F}}(\rho_1 a^{*2}) = 0 \ ; \\ \text{(h3)} & \mathrm{Tr}_{\mathscr{F}}[\rho_1 (a^* a)^2] < \infty \ . \end{array}$$

Remark 5.5 *Note that hypothesis (h1)-(h3) are satisfied when the density matrices $\{\rho_k\}_{k=0}^N$ correspond to the gauge-invariant quasi-free states with parameter β_0 for $k = 0$ and β for $k = 1, 2, \dots, N$, see (4.1). Then (h2) is due to the gauge invariance and one gets for (h3):*

$$\mathrm{Tr}_{\mathscr{F}}[\rho_k (a^* a)^2] = (2n_\beta^2 + n_\beta) \ , \tag{5.35}$$

where $n_\beta = \mathrm{Tr}_{\mathscr{F}}(\rho_k\, a^* a) = (e^\beta - 1)^{-1}$, $k = 1, \dots, N$.

Below we denote by $|ya^* + \bar{y}a|$ the operator originated from the *polar decomposition* of the operator $ya^* + \bar{y}a = U\,|ya^* + \bar{y}a|$, where U is the partial isometry on $\mathscr{F}$.

Lemma 5.6 *Under hypothesis (h1)-(h3), the following bounds hold:*

$$\begin{aligned}
&\text{(i)} && \mathrm{Tr}_{\mathscr{F}}(\rho_k a^* a) < \infty, \\
&\text{(ii)} && \mathrm{Tr}_{\mathscr{F}}(\rho_k |y a^* + \bar{y} a|^2) \leqslant C|y|^2, \\
&\text{(iii)} && \mathrm{Tr}_{\mathscr{F}}(\rho_k |y a^* + \bar{y} a|^3) \leqslant C'|y|^3, \\
&\text{(iv)} && \mathrm{Tr}_{\mathscr{F}}(\rho_k |y a^* + \bar{y} a|^4) \leqslant C''|y|^4,
\end{aligned}$$

for all $k = 1, \ldots, N$. Here C, C', C'' are positive constants, which depend only on $\mathrm{Tr}[\rho_1 (a^* a)^2]$.

Proof : The first bound (i) is a consequence of the Cauchy-Schwarz inequality and (h3). Applying the inequalities

$$\begin{aligned}
|A + A^*|^2 &\leqslant |A + A^*|^2 + |A - A^*|^2 = 2(AA^* + A^* A), \\
|A + A^*|^4 &\leqslant |A + A^*|^4 + |A - A^*|^4 + |A + iA^*|^4 + |A - iA^*|^4 \\
&= 4(AA^* + A^* A)^2 + 4(A^2 A^{*2} + A^{*2} A^2),
\end{aligned}$$

to $A = \bar{y} a$, we obtain (ii) and (iv). Finally, a combination of (ii), (iv) with the Cauchy-Schwarz inequality yields (iii). □

Theorem 5.7 *Let $\tau \to 0$, $N \to \infty$ be short-time perturbation limit subject to demands: $\tau^2 N \to \infty$ and $\tau^3 N \to 0$. Then for any initial condition (5.34) verifying (h1)-(h3), the characteristic function $\omega_{\mathcal{S}}^{N\tau}(\widehat{w}(\theta))$ of the state for subsystem $\mathcal{S}$ at $t = N\tau$, converges to:*

$$\begin{aligned}
\omega_{\mathcal{S}}(\widehat{w}(\theta)) &:= \lim_{\tau \to 0, N \to \infty} \omega_{\mathcal{S}}^{N\tau}(\widehat{w}(\theta)) = \\
&\lim_{\tau \to 0, N \to \infty} \omega_{\rho(N\tau)}(W(\zeta_\theta)) = e^{-|\theta|^2 \mathrm{Tr}_{\mathscr{F}}[\rho_1 (a^* a + a a^*)]/4} \ .
\end{aligned} \tag{5.36}$$

Here $\theta \in \mathbb{C}$ and the $(N+1)$-component vector is $(\zeta_\theta)^{\mathrm{tr}} := (\theta, 0, 0, \ldots, 0) \in \mathbb{C}^{N+1}$.

Remark 5.8 *By (5.36) the state $\omega_{\mathcal{S}}^{N\tau}$ converges to $\omega_{\mathcal{S}}$ in the weak*-topology. From the right-hand side of (5.36) and Definition 3.1 we deduce that the limit state is gauge-invariant and quasi-free with $h(\theta) := |\theta|^2 \, \mathrm{Tr}_{\mathscr{F}}(\rho_1 \, a^* a)$.*

Remark 5.9 *Recall that the state ω over the Weyl algebra $\mathscr{A}(\mathscr{F}) = \overline{\mathscr{A}_w(\mathscr{F})}$ is regular, C^n-smooth or analytic, if the function (see (2.1))*

$$s \mapsto \omega(\widehat{w}(s\theta)) = \omega(e^{i\, s \Phi(\theta)/\sqrt{2}}) \tag{5.37}$$

is respectively continuous, C^n-smooth or analytic in the vicinity of $s = 0$. In the last case the characteristic function $\omega(\widehat{w}(s\theta))$ (and therefore the state) is completely determined by

$$\omega(\widehat{w}(s\theta)) = \exp\left\{ \sum_{m=1}^{\infty} \frac{i^m s^m}{m!} \, 2^{-m/2} \, \omega^T(\Phi^m(\theta)) \right\} \ . \tag{5.38}$$

Here $\{\omega^T(\Phi^m(\theta))\}_{m=0}^{\infty}$ are truncated correlation functions defined recursively by relations

$$\begin{aligned}
&\omega^T(\Phi(\theta)) := \omega(\Phi(\theta)) \ , \\
&\omega^T(\Phi^2(\theta)) := \omega(\Phi^2(\theta)) - \omega(\Phi(\theta))^2 \ , \\
&\omega^T(\Phi^3(\theta)) := \omega(\Phi^3(\theta)) - 3\omega(\Phi^2(\theta))\omega(\Phi(\theta)) + 2\omega(\Phi(\theta))^3 \ , \ \text{etc}
\end{aligned}$$

Lemma 5.6 implies that the states for density matrices $\rho_1 = \rho_2 = \ldots$ are C^4-smooth.

Proof (of Theorem 5.7): By (h2) and by Lemma 5.6 (i)-(iii) together with Remark 5.9, we obtain for the states $\omega(\cdot) = \omega_{\rho_k}(\cdot)$ the representation of (5.38) in the form:

$$C_k(\theta) = \omega_{\rho_k}(\widehat{w}(\theta)) = \exp[-\frac{1}{4}\ \omega^T_{\rho_k}(\Phi^2(\theta)) + R(\theta)]\ ,\ k = 1, 2, \ldots, N, \tag{5.39}$$

where $R(\theta) = O(|\theta|^3)$ in the vicinity of $\theta = 0$. For the self-adjoint operator $\Phi(\theta) = \bar{\theta}a + \theta a^*$, the hypothesis (h2) and Lemma 5.6 (i) imply

$$\omega^T_{\rho_k}(\Phi^2(\theta)) = |\theta|^2\ \mathrm{Tr}_{\mathscr{F}}[\rho_k\,(a^*a + aa^*)]\ . \tag{5.40}$$

Now, taking into account Lemma 2.2 for the vector ζ_θ, as well as (5.39) and (5.40), we obtain the representation:

$$\omega^{N\tau}_{\mathcal{S}}(\widehat{w}(\theta)) = \omega_{\rho(N\tau)}(W(\zeta_\theta)) = C_0(e^{i\epsilon\tau N}(gz)^N\theta)\prod_{k=1}^N C_k(e^{i\epsilon\tau N}gw\ (gz)^{N-k}\theta)$$

$$= C_0(e^{i\epsilon\tau N}(gz)^N\theta)\exp\Big(-\sum_{k=1}^N \frac{|\theta_k|^2}{4}\mathrm{Tr}_{\mathscr{F}}[(a^*a + aa^*)\rho_k] + \widehat{R}\Big)\ . \tag{5.41}$$

Here by (2.17) and by (5.39) one has

$$\theta_k := e^{i\epsilon N\tau}gw\ (gz)^{N-k}\theta\ ,\ \sum_{k=1}^N |\theta_k|^2 = |\theta|^2|w|^2\frac{1-|z|^{2N}}{1-|z|^2}\ ,\ \widehat{R} = \sum_{k=1}^N O(|\theta_k|^3)\ .$$

By virtue of (1.12) and (1.13), we get $|g(\tau)| = 1$, $|w(\tau)|^2 + |z(\tau)|^2 = 1$ and also

$$w(\tau) = i\eta\tau + O(\tau^3)\ ,\ |z(\tau)| = 1 - \frac{|\eta|^2\tau^2}{2} + O(\tau^4)\ ,$$

for small τ. This yields for small $\tau > 0$ and large N, the estimates $|(gz)^N| \leq O(e^{-|\eta|^2\tau^2 N/2})$, $|\theta_k| \leq O(\tau)$, and $\widehat{R} = O(\tau^3 N)$ by virtue of (h1). Then taking into account the conditions $\tau^2 N \to \infty$ and $\tau^3 N \to 0$, we get the limits:

$$\lim_{\tau\to 0, N\to\infty} C_0(e^{i\epsilon\tau N}(gz)^N\theta) = 1\ ,\ \lim_{\tau\to 0, N\to\infty}\sum_{k=1}^N |\theta_k|^2 = |\theta|^2\ ,\ \lim_{\tau\to 0, N\to\infty}\widehat{R} = 0\ .$$

C_0 is a continuous function since it is defined by a normal state with density matrix ρ_0. Inserting all these limits into (5.41), we obtain what is claimed as the limit (5.36). □

Corollary 5.10 *Suppose that density matrices $\{\rho_k\}_{k=1}^N$ correspond to the gauge-invariant quasi-free Gibbs state with parameter β (4.1). These states satisfy (h1)-(h3). The statement of Theorem 5.7 is valid with the limit*

$$\lim_{\tau\to 0, N\to\infty}\omega^{N\tau}_{\mathcal{S}}(\widehat{w}(\theta)) = \exp\left\{-\frac{|\theta|^2}{4}\frac{1+e^{-\beta}}{1-e^{-\beta}}\right\}\ . \tag{5.42}$$

It coincides with the result for equilibrium state (5.6) of the subsystem $\mathcal{S}$.

Hence, the short-time perturbation limit $\tau \to 0$, $N \to \infty$ subjected to $\tau^2 N \to \infty$ and $\tau^3 N \to 0$ gives a *universal* gauge-invariant quasi-free limiting state under hypothesis (h1)-(h3). The hypotheses (h2),(h3) control only first *two* moments of the initial states of the subsystem $\mathcal{C}$. Then stationarity and independence of repeated perturbation due to (h1), correspond to conditions for the non-commutative Central Limit Theorem [Ve]. Note also that the state ω_{ρ_0} of the subsystem $\mathcal{S}$ may be replaced by any *regular* state.

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