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Periodic locked orbit in the Winfree Model with N oscillators

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Abstract

We give in this paper a sufficient condition for the existence of locked state in finite dimensional Winfree model independently of choice of natural frequencies and the number of oscillators. The main result consists to prove the existence of periodic orbit in a torus which is equivalent to the existence of rotation vectors, the proof in this paper can be applied to more generalized Winfree model.

Keywords: Winfree model, coupled oscillators, locked-state, synchronization, rotation vector

1 Introduction

The simplest synchronization model may be described by the behavior of two pendulums of equal mass coupled by an horizontal string. One notice that the two pendulums behave in the same way after some time and begin to oscillate with the same frequency. When the frequencies of the two oscillators are identical, they are said to be “locked”. This behavior seems to appears very often in biological complex systems. We will study a particular model, called the Winfree model, that may be described by N oscillators coupled uniformly.

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In 1967 Winfree proposed a mean field model which describes the synchronization of a population of organisms or units that interact simultaneously. We assume that the state of each unit is described by a point on a cycle. We call natural frequency, the frequency of a unit, if it were isolated from the others. The natural frequencies are supposed to be distributed inside an interval $[1 - \gamma, 1 + \gamma]$ for some constant γ called *spectrum width*. The interaction of the rest of the population on each unit is supposed to be independent of the unit and controlled by a single parameter called the *coupling strength* κ . There exist different states: the locking state where all the units possess the same frequency, the death state where all the states are frozen with zero frequency, the incoherence state where each unit oscillates at independent frequencies. There may also exist mixed states where part of the oscillators is synchronized and the other part is dead for instance. For small values of κ , the Winfree model may be reduced to the Kuramoto model. In both models the interaction of the outside world on each single unit is the same: we use the word “mean-field” to describe this kind of interaction.

The collective behavior of a population of oscillators has first been studied by Winfree in [12]: for a fixed coupling and small spectrum width, all the oscillators synchronize to a unique frequency. Kuramoto [7] extended the model by passing to the limit when the number of oscillators goes to infinity.

The Winfree model is given by the following differential equation

$$\dot{x}_i = \omega_i - \kappa \frac{1}{N} \sum_{j=1}^N P(x_j) R(x_i) \quad (1)$$

where P and R are two periodic functions, $X(t) = (x_1(t), \dots, x_N(t))$ is the state, and $x_i(t)$ is the phase of the i -th oscillator. Although $x_i(t)$ should represent a scalar in $[0, 2\pi]$, we actually consider its unique continuous lift in \mathbb{R} , that we continue to call $x_i(t)$. The parameter $\kappa \geq 0$ is the coupling strength; the vector of natural frequencies $\Omega := (\omega_1, \dots, \omega_N)$ satisfy

$$1 - \gamma \leq \omega_i \leq 1 + \gamma, \quad \forall i = 1, \dots, N, \quad (2)$$

where $\gamma \in [0, 1[$. We actually assume a more particular form of the mean-field interaction, as in Ariaratnam and Strogatz [1, 2], we assume

$$\dot{x}_i = \omega_i - \kappa \sigma(X) \sin(x_i) \quad \text{where} \quad \sigma(X) = \frac{1}{N} \sum_{j=1}^N [1 + \cos(x_j)]. \quad (3)$$

Notice that the mean-field interaction σ satisfies $\sigma(X) \in [0, 2]$ for every state $X = (x_1, \dots, x_N) \in \mathbb{R}^N$.

Since the vector field is uniformly bounded, the flow $\Phi^t(X)$, $X \in \mathbb{R}^d$, is defined for all time. Because of the presence of the coupling, the instantaneous frequency $\rho_i(t) := \frac{x_i(t)}{t}$ may not be equal to ω_i . A numerical study shows that, for large t , depending on (γ, κ) , three major cases occur: the *death state* where all the oscillators are frozen, the *locked state* where all $\rho_i(t) = \text{const} \neq 0$, and the *incoherence state* where $\rho_i(t)$ is strictly increasing in ω_i ; in addition there are two secondary cases: the *partial death state* where some of the oscillators are frozen and the others are incoherent, and the *partial locked state* where some are locked and the others are incoherent. Intermediate cases exist numerically but are more difficult to visualize. Ariaratnam and Strogatz [1, 2] have given a precise definition of these transitions in the case $N \rightarrow +\infty$. The partial locking is still not understood very well. Giannuzzi, Marinazzo, Nardulli, Pellicoro, and Stramaglia [4] have extended Ariaratnam and Strogatz result by putting a factor in front of the mean-field σ proportional to some power of the modulus of the average phase $\frac{1}{N} \sum_{k=1}^N \exp(ix_k)$.

Nevertheless the fact that the instantaneous frequency $\rho_i(t)$ admits a limit, in other words that the rotation vector exists, has never been addressed (except of course in the death state). Our main result is a partial result in that direction in the locked state when $\kappa \in]0, \kappa_*[$ and $\gamma \approx 0$ where κ_* is the locking bifurcation critical parameter for the Winfree model $\gamma = 0$ and $N = 1$ defined by

$$\kappa_* := \max\{\kappa > 0 : 1 - \kappa(1 + \cos x) \sin x > 0, \forall x \in \mathbb{R}\}. \quad (4)$$

Main result 1. *We consider the Winfree model given by (3) and satisfying the frequency condition (2). Then, there exists a open set U in the space of parameters $(\gamma, \kappa) \in [0, 1] \times [0, 1]$, independent of N , whose closure contains $\{0\} \times [0, \kappa_*]$ such that for every parameter $(\gamma, \kappa) \in U$, for every $N \geq 1$ and every choice of natural frequencies $(\omega_i)_{i=1}^N$ satisfying condition (2),*

1. *There exists an open set $C_{\gamma, \kappa}^N$ invariant by the flow Φ^t , of the form,*

$$C_{\gamma, \kappa}^N := \left\{ X = (x_i)_{i=1}^N \in \mathbb{R}^N : \max_{i,j} |x_j - x_i| < \Delta_{\gamma, \kappa} \left(\frac{1}{N} \sum_{i=1}^N x_i \right) \right\}$$

where $\Delta_{\gamma, \kappa} : \mathbb{R} \rightarrow]0, 1[$ is a 2π -periodic smooth function independent of N .

2. *There exists a constant rotation number $\rho_{\gamma, \kappa} > 0$, and an initial condition $X_* \in C_{\gamma, \kappa}^N$ such that,*

$$\Phi_i^t(X_*) = \rho_{\gamma, \kappa} t + \Psi_{i, \gamma, \kappa}^N(t), \quad \forall i = 1, \dots, N, \quad \forall t \geq 0,$$

where $\Psi_{i,\gamma,\kappa}^N : \mathbb{R}^+ \rightarrow \mathbb{R}$ are C^∞ and $\frac{2\pi}{\rho_{\gamma,\kappa}}$ -periodic functions uniformly bounded with respect to N .

In particular, the set of parameters U corresponds to the weak locked state: whatever the initial condition $X(0) \in C_{\gamma,\kappa}^N$, the dispersion $|x_i(t) - x_j(t)|$ is uniformly bounded in t ; for some initial condition $X_*(0)$, the instantaneous frequency $\rho_i(t) := \frac{x_i(t)}{t}$ converges to a scalar $\rho_{\gamma,\kappa} > 0$ independently of the oscillator x_i .

It is common to study the Winfree model together with the Kuramoto model [7] which is obtained by passing to the limit $\epsilon \rightarrow 0$ in the Winfree model parametrize with $\omega_i := 1 + \epsilon\bar{\omega}_i$, $\kappa := \epsilon\bar{\kappa}$, and $x_i(t) := t + \epsilon\bar{x}_i(t)$. A first order approximation gives

$$\dot{\bar{x}}_i = \bar{\omega}_i - \frac{\kappa}{N} \sum_{j=1}^N H(\bar{x}_j - \bar{x}_i), \quad H(\bar{x}) := \frac{1}{2\pi} \int_0^{2\pi} P(\bar{x} + \theta) R(\theta) d\theta. \quad (5)$$

Kuramoto model is thus seen as an approximation of the Winfree model for small γ and κ . Although our open set of parameters is also close to the origin, we do not take any limit $\kappa \rightarrow 0$ and our result is valid for every N . The recent researches on the Winfree model may be found in Louca and Attay in [8], in Pazó and Montbrío in [10], or in Ha, Park and Ryoo in [5]. To be close to the physical and biological terminology, we define the locking state, as follows

Definition 2. We call rotation number of the i -th oscillator $x_i(t)$ the following limit, if it exists,

$$\rho_i := \lim_{t \rightarrow \infty} \frac{x_i(t)}{t}.$$

We call rotation vector of the system (3), the vector of rotation numbers of all oscillators, if they all exist.

Definition 3 (Weak Locking). Two oscillators $x_i(t)$ and $x_j(t)$ of the system (3) are said to be *weakly locked* if there exists a constant $M > 0$ such that

$$|x_i(t) - x_j(t)| \leq M, \quad \forall t \geq 0$$

Definition 4 (Locking). Two oscillators $x_i(t)$ and $x_j(t)$ of the system (3) are said to be *locked* if they are weakly locked and their rotation numbers exist and are positive.

Notice that if two oscillators $x_i(t)$ and $x_j(t)$ are weakly locked and one of them has a rotation number ρ_i , then ρ_j exists and $\rho_j = \rho_i$.

2 Reduction of the Winfree model

We want to show in this section that the Winfree model (3) can be analyzed by a linear differential equation. We note, throughout this paper, for any function $h : \mathbb{R} \rightarrow \mathbb{R}$ and any constant $c \in \mathbb{R}$, the linear vector field $\mathfrak{L}[c, h](z, t)$ defined for every function $z : \mathbb{R} \rightarrow \mathbb{R}$ and every $t \geq 0$ as follow

$$\mathfrak{L}[c, h](z, t) = c - h(t)z.$$

We denote by $\mathfrak{L}[c](z, t)$, the particular linear vector field $\mathfrak{L}[c, h](z, t)$ where

$$h(t) = \frac{\kappa C(t)}{1 - \kappa S(t)}$$

and

$$C(t) := [1 + \cos(t)] \cos(t), \quad S(t) := [1 + \cos(t)] \sin(t).$$

We recall that κ is the coupling strength of the Winfree model (3). Let us introduce for any vector $Y = (y_1, \dots, y_N) \in \mathbb{R}^N$, the dispersion of $\delta_{i,j}(Y)$, the global dispersion $\delta(Y)$ and the mean of Y

$$\delta_{i,j}(Y) := y_i - y_j, \quad \delta(Y) := \max_{i,j} |\delta_{i,j}(Y)| \quad \text{and} \quad \mu(Y) := \frac{1}{N} \sum_{j=1}^N y_j.$$

Recall that $X(t)$ denote the solution of the system (3). The mean $\mu(X)$ and the dispersion $\delta_{i,j}(X)$ of the Winfree model (3) satisfy the non-autonomous differential equation

$$\frac{d}{dt} \mu(X) = \mu(\Omega) - \kappa \sigma(X) \frac{1}{N} \sum_{j=1}^N \sin(x_j), \quad (6)$$

$$\frac{d}{dt} \delta_{i,j}(X) = \delta_{i,j}(\Omega) - 2\kappa \sigma(X) \cos \frac{x_i + x_j}{2} \sin \frac{\delta_{i,j}(X)}{2}. \quad (7)$$

Equation (7) is a non-autonomous Riccati equation which is difficult to solve since the non-constant coefficient is unknown. We use the comparison principle of solutions of differential equations to bound from above the dispersion function $\delta_{i,j}(X)$.

Proposition 5. *Let be $t_* > 0$ and $D \in]0, 1]$, and consider the Winfree model given by (3). For every parameter $(\gamma, \kappa) \in]0, 1[\times]0, 1[$, for every $N \in \mathbb{N}^*$ and for every choice of natural frequencies $(\omega_i)_{i=1}^N$ satisfying the conditions*

(2), suppose that $\delta(X(t)) < D$ on $[0, t_*]$, then for every $1 \leq i, j \leq N$ and $t \in [0, t_*]$:

$$\dot{\delta}_{i,j}(X) < \mathfrak{L}[c, h](\delta_{i,j}(X), t) \quad (8)$$

where $c = 2\gamma + 4\kappa D^2$ and $h(t) = \kappa C(\mu(X(t)))$.

Proof. Let be $D \in]0, 1[$ and assume $\delta(X(t)) < D$ on $[0, t_*]$, then for every $1 \leq i, j \leq N$ and $t \in [0, t_*]$,

$$|x_i - \mu(X)| = \left| \frac{1}{N} \sum_{j=1}^N \delta_{i,j}(X) \right| \leq \frac{1}{N} \sum_{j=1}^N |\delta_{i,j}(X)| \leq \delta(X) < D.$$

By the mean value theorem applied twice we have for every $1 \leq i, j, k \leq N$

$$\left| (1 + \cos(x_k)) \cos\left(\frac{x_i + x_j}{2}\right) - [1 + \cos(\mu(X))] \cos(\mu(X)) \right| \leq 3D,$$

which implies for every $1 \leq i, j \leq N$

$$\left| \sigma(X) \cos\left(\frac{x_i + x_j}{2}\right) - [1 + \cos(\mu(X))] \cos(\mu(X)) \right| \leq 3D.$$

We substitute in (7) using $|\sin \frac{\delta_{i,j}(X)}{2}| \leq \frac{\delta(X)}{2} < \frac{D}{2}$ and $\delta_{i,j}(\Omega) \leq 2\gamma$. We obtain for all $t \in [0, t_*]$ and for all $1 \leq i, j \leq N$,

$$\dot{\delta}_{i,j}(X) < 2\gamma + 3\kappa D^2 - 2\kappa[1 + \cos(\mu(X))] \cos(\mu(X)) \sin\left(\frac{\delta_{i,j}(X)}{2}\right).$$

We now use the fact that $|\sin(z) - z| \leq \frac{1}{2}z^2$ for all $z \in [0, \frac{1}{2}]$ to obtain finally

$$\dot{\delta}_{i,j}(X) < 2\gamma + 4\kappa D^2 - \kappa[1 + \cos(\mu(X))] \cos(\mu(X)) \delta_{i,j}(X), \quad (9)$$

for every $t \in [0, t_*]$ and every $1 \leq i, j \leq N$. \square

Our goal in the following is to show that for some parameters (γ, κ, D) the dispersion curve satisfies $|\delta_{i,j}(X(t))| < D$ for every $t \geq 0$. Our strategy consists in making the change of variable $s = \mu(X(t))$ in equation (8) and find a super-solution of a new linear equation with periodic coefficients. We will need the following lemma.

Lemma 6. Consider a linear scalar differential equation :

$$\dot{z}(s) = \mathfrak{L}[\alpha, \beta](z, s) \quad (10)$$

where $\alpha \in \mathbb{R}_+^*$ is a constant and $\beta(s) : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous 2π -periodic function satisfying

$$\int_0^{2\pi} \beta(\zeta) d\zeta \geq \beta_1 > 0 \quad \text{and} \quad - \int_s^t \beta(\zeta) d\zeta \leq \beta_0, \quad \forall \quad 0 \leq s \leq t \leq 2\pi \quad (11)$$

for some constant β_0, β_1 . Then equation (10) admits a C^1 , 2π -periodic, positive solution $z(s)$, such that

$$\max_{s \in [0, 2\pi]} z(s) \leq \alpha 2\pi \frac{\exp(\beta_0)}{1 - \exp(-\beta_1)}$$

Proof. Since $\beta(s)$ is 2π -periodic, by uniqueness of solutions of differential equations, a solution of (10) satisfying $z(0) = z(2\pi)$ is periodic $z(s) = z(2\pi + s)$ for all $s \in \mathbb{R}$. The general solution $y(s)$ of (10) of initial condition $y(0)$ is given by

$$y(s) = \exp\left(-\int_0^s \beta(\zeta) d\zeta\right) y(0) + \alpha \int_0^s \exp\left(-\int_t^s \beta(\zeta) d\zeta\right) dt. \quad (12)$$

For a solution $z(s)$ of (10) to be 2π -periodic and strictly positive, it is sufficient to solve

$$z(0) = z(2\pi) = \exp\left(-\int_0^{2\pi} \beta(\zeta) d\zeta\right) z(0) + \alpha \int_0^{2\pi} \exp\left(-\int_t^{2\pi} \beta(\zeta) d\zeta\right) dt$$

Since $\int_0^{2\pi} \beta(\zeta) d\zeta \geq \beta_1 > 0$, we finally obtain

$$z(s) = \alpha \frac{\int_s^{2\pi+s} \exp\left(-\int_t^{2\pi+s} \beta(\zeta) d\zeta\right) dt}{1 - \exp\left(-\int_0^{2\pi} \beta(\zeta) d\zeta\right)}$$

and $\max_{s \in [0, 2\pi]} z(s) \leq \alpha 2\pi \frac{\exp(\beta_0)}{1 - \exp(-\beta_1)}$. \square

Recall that κ_* is the locking bifurcation parameter for the Winfree model $\gamma = 0$ and $N = 1$ defined by

$$\kappa_* := \max\{\kappa > 0 : 1 - \kappa S(x) > 0, \forall x \in \mathbb{R}\}.$$

We find the numerical value $\kappa_* = [\sup_{x \in [0, 2\pi]} S(x)]^{-1} = [S(\frac{\pi}{3})]^{-1} = \frac{4}{3\sqrt{3}}$.

Proposition 7. *Let be $t_* > 0$ and $D \in]0, 1[$. Consider the Winfree model given by (3). For every parameter $\gamma \in]0, 1[$, for every $N \in \mathbb{N}^*$ and for every choice of natural frequencies $(\omega_i)_{i=1}^N$ satisfying condition (2), suppose that $\delta(X(t)) < D$ on $[0, t_*]$, then there exists $\kappa_{\gamma, D} \in]0, \kappa_*[$ such that for every $\kappa \in [0, \kappa_{\gamma, D}]$ the function $t \rightarrow \mu(X(t))$ is strictly increasing and in particular a diffeomorphism from $[0, t_*]$ onto $[\mu(X(0)), \mu(X(t_*))]$. More precisely,*

$$\frac{d}{dt}\mu(X(t)) \geq 1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1) > 0 \quad \text{where} \quad \kappa_{\gamma, D} = \kappa_* \frac{1 - \gamma}{3D\kappa_* + 1}.$$

Proof. By the Mean value theorem and by same arguments as in the proof of proposition (5), we find that for every $1 \leq i, j \leq N$ and $t \in [0, t_*]$

$$|(1 + \cos(x_j)) \sin(x_i) - (1 + \cos(\mu(X))) \sin(\mu(X))| < 3D,$$

which implies

$$\left| \sigma(X) \frac{1}{N} \sum_{i=1}^N \sin(x_i) - (1 + \cos(\mu(X))) \sin(\mu(X)) \right| < 3D. \quad (13)$$

We substitute in equation (6) and since $\frac{1}{N} \sum_{j=1}^N \omega_j > 1 - \gamma$ we find

$$\frac{d}{dt}\mu(X) \geq 1 - \gamma - 3\kappa D - \kappa(1 + \cos(\mu(X))) \sin(\mu(X))$$

In order to obtain $\mu(X)$ strictly increasing on $[0, t_*]$ it's sufficient to choose κ so that $1 - \gamma - 3\kappa D - \kappa(1 + \cos(z)) \sin(z) > 0$, $\forall z \in [0, 2\pi]$. As $\gamma < 1$ and $\max_{z \in [0, 2\pi]} (1 + \cos(z)) \sin(z) = \frac{1}{\kappa_*}$ then it's sufficient that

$$\frac{d}{dt}\mu(X) \geq 1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1) > 0.$$

Let be $\kappa_{\gamma, D} = \kappa_* \frac{1 - \gamma}{3D\kappa_* + 1}$. □

Under the hypothesis of proposition 7 above, note $s_0 := \mu(X(0))$ and $s_* := \mu(X(t_*))$ and consider the change of variable $s = \mu(X(t))$. We will use later the notation $\mu_X(t) := \mu(X(t))$. More precisely, let

$$\tau_X : \begin{cases} [s_0, s_*] & \rightarrow [0, t_*] \\ s & \mapsto \tau_X(s) \end{cases} \quad (14)$$

be the inverse function of μ_X , and define

$$x_i^*(s) = x_i \circ \tau_X(s), \quad X^*(s) = X \circ \tau_X(s).$$

With respect to the new variable s , equation (8) becomes:

Corollary 8. *Let be $t_* > 0$ and $D \in]0, 1[$, and consider the Winfree model given by (3). For every parameter $\gamma \in]0, 1[$, $\kappa \in]0, \kappa_{\gamma, D}[$, (where $\kappa_{\gamma, D}$ is given by proposition 7), for every $N \in \mathbb{N}^*$ and for every choice of natural frequencies $(\omega_i)_{i=1}^N$ satisfying condition (2), suppose $\delta(X(t)) < D$ on $[0, t_*]$, then for every $1 \leq i, j \leq N$, the sub-differential equation (8) admits the following (strict) sub-differential equation*

$$\frac{d}{ds} \delta_{i,j}(X^*(s)) < \mathfrak{L}[\alpha(\gamma, \kappa, D)](\delta_{i,j}(X^*(s), s), \quad \forall s \in [s_0, s_*], \quad (15)$$

where $\alpha(\gamma, \kappa, D) = 2\kappa_* \left[\frac{\gamma + 3\kappa D}{\kappa_* - \kappa} \frac{\gamma + \kappa D + 2\kappa D^2}{1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1)} + \frac{\gamma + 2\kappa D^2}{\kappa_* - \kappa} \right]$.

Proof. Let be $t_* > 0$, $D \in]0, 1[$. Suppose that $\delta(X(t)) < D$ on $[0, t_*]$ and $\kappa \in]0, \kappa_{\gamma, D}[$. Thanks to proposition 7, the function $\mu_X(t)$ is strictly increasing and in particular is a diffeomorphism from $[0, t_*]$ to $[s_0, s_*]$. We get

$$\frac{d}{ds} \delta_{i,j}(X^*(s)) \frac{d}{dt} \mu_X(t) = \frac{d}{dt} \delta_{i,j}(X(t)) \quad (16)$$

which is equivalent to

$$\frac{d}{ds} \delta_{i,j}(X^*(s))(1 - \kappa S(s)) = -\frac{d}{ds} \delta_{i,j}(X^*(s))[\dot{\mu}_X(t) - (1 - \kappa S(s))] + \frac{d}{dt} \delta_{i,j}(X(t)).$$

We first find an upper bound of $|\frac{d}{ds} \delta_{i,j}(X^*(s))[\dot{\mu}_X(t) - (1 - \kappa S(s))]|$.

On the one hand, by proposition 7, $\dot{\mu}_X(t) > 1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1)$ and by equation (16)

$$\left| \frac{d}{ds} \delta_{i,j}(X^*(s)) \right| (1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1)) \leq \left| \frac{d}{dt} \delta_{i,j}(X(t)) \right|.$$

Since $|\delta_{i,j}(X(t))| < D$ on $[0, t_*]$, by proposition 5 and equation (9), we get $|\frac{d}{dt} \delta_{i,j}(X(t))| < 2(\gamma + 2\kappa D^2 + \kappa D)$, and

$$\left| \frac{d}{ds} \delta_{i,j}(X^*(s)) \right| < 2 \frac{(\gamma + 2\kappa D^2 + \kappa D)}{1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1)}. \quad (17)$$

On the other hand, from equation (13) of the proof of proposition (7),

$$\begin{aligned} \dot{\mu}_X(t) &= \mu(\Omega) - \kappa \sigma(X) \frac{1}{N} \sum_{i=1}^N \sin(x_i), \\ \left| \sigma(X) \frac{1}{N} \sum_{i=1}^N \sin(x_i) - (1 + \cos(\mu_X)) \sin(\mu_X) \right| &\leq 3D, \\ |\dot{\mu}_X(t) - (1 - \kappa S(s))| &\leq \gamma + 3\kappa D, \quad \forall t \in [0, t_*], \end{aligned}$$

which implies

$$\frac{d}{ds}\delta_{i,j}(X^*)(1 - \kappa S(s)) < 2(\gamma + 3\kappa D)\frac{(\gamma + 2\kappa D^2 + \kappa D)}{1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1)} + \frac{d}{dt}\delta_{i,j}(X).$$

We next find an upper bound of the second expression $\frac{d}{dt}\delta_{i,j}(X)$. Since $1 - \kappa S(s) > 1 - \frac{\kappa}{\kappa_*}$, thanks to equation (8) of proposition 5, we have

$$\begin{aligned} \frac{d}{ds}\delta_{i,j}(X^*) &< 2\kappa_*\frac{(\gamma + 3\kappa D)(\gamma + 2\kappa D^2 + \kappa D)}{(\kappa_* - \kappa)(1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1))} + 2\kappa_*\frac{\gamma + 2\kappa D^2}{\kappa_* - \kappa} \\ &\quad - \frac{\kappa C(s)}{1 - \kappa S(s)}\delta_{i,j}(X). \end{aligned}$$

Let be $\alpha(\gamma, \kappa, D) = 2\kappa_*\left[\frac{(\gamma + 3\kappa D)(\gamma + 2\kappa D^2 + \kappa D)}{(\kappa_* - \kappa)(1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1))} + \frac{\gamma + 2\kappa D^2}{\kappa_* - \kappa}\right]$. We finally obtain

$$\begin{aligned} \frac{d}{ds}\delta_{i,j}(X^*) &< \alpha(\gamma, \kappa, D) - \frac{\kappa C(s)}{1 - \kappa S(s)}\delta_{i,j}(X^*) \\ &= \mathfrak{L}[\alpha(\gamma, \kappa, D)](\delta_{i,j}(X^*), s). \end{aligned} \quad \square$$

3 The dispersion curve

By the definition of $\alpha(\gamma, \kappa, D)$ given in corollary 8 and based on lemma 6, we discuss in this section the solution of the linear scalar periodic differential equation $\frac{d}{ds}z(s) = \mathfrak{L}[\alpha(\gamma, \kappa, D)](z, s)$, we show it admits a 2π -periodic and C^∞ scalar solution on \mathbb{R} that we note $\Delta_{\gamma, \kappa, D}$. This solution will play the role of a upper positive function of the solution of equation (15). We first begin by the following lemma.

Lemma 9. *For all $D \in]0, 1[$ and $(\gamma, \kappa) \in]0, 1[\times]0, \kappa_{\gamma, D}[$ the linear scalar differential equation*

$$\frac{d}{ds}z(s) = \mathfrak{L}[\alpha(\gamma, \kappa, D)]z(s) \quad (18)$$

admits a positive solution denoted $\Delta_{\gamma, \kappa, D}(s)$, 2π -periodic and C^∞ on \mathbb{R} .

Proof. Recall that for all $D \in]0, 1[$ and for all $(\gamma, \kappa) \in]0, 1[\times]0, \kappa_{\gamma, D}[$ the constant $\alpha(\gamma, \kappa, D)$ is defined and positive, moreover $1 - \kappa S(\zeta) > 0$. To be

able to use lemma (6), we need to estimate the integral from below

$$\begin{aligned} \int_0^{2\pi} \frac{\kappa C(\zeta)}{1 - \kappa S(\zeta)} d\zeta &= -\ln \left[\frac{1 - \kappa S(2\pi)}{1 - \kappa S(0)} \right] + \int_0^{2\pi} \frac{\kappa \sin^2(\zeta)}{1 - \kappa S(\zeta)} d\zeta \\ &= \int_0^{2\pi} \frac{\kappa \sin^2(\zeta)}{1 - \kappa S(\zeta)} d\zeta \\ &\geq \min_{t \in [0, 2\pi]} \frac{\kappa}{1 - \kappa S(t)} \int_0^{2\pi} \sin^2(\zeta) d\zeta = \frac{\pi \kappa}{1 + \frac{\kappa}{\kappa_*}}. \end{aligned}$$

Then lemma (6) implies that the linear periodic equation (18) admits a 2π -periodic solution $\Delta_{\gamma, \kappa, D}(s)$, C^∞ and positive. \square

Our goal is to choose γ and κ so that there exists $D \in]0, 1[$ such that the corresponding function $\Delta_{\gamma, \kappa, D}$ is bounded from above by D as we show for example in figure 1b. The following lemma gives a sufficient condition of existence of such a function.

Lemma 10. *There exists an open set U of parameter (γ, κ) whose closure contains $\{0\} \times [0, \kappa_*]$, defined in the following way*

$$U := \{(\gamma, \kappa) \in]0, 1[\times]0, \kappa_*[: 0 < \gamma < \kappa D^2(\kappa)\}$$

where $D(\kappa) = L_* (1 - \frac{\kappa}{\kappa_*})^2$ and $L_* := \frac{1}{280} \exp(-\frac{\pi}{4} \frac{\kappa_*}{1 - \kappa_*} - \frac{\pi}{2} \kappa_*)$, such that

$$\max_{s \in [0, 2\pi]} \Delta_{\gamma, \kappa, D(\kappa)}(s) < D(\kappa).$$

Definition 11. We call *dispersion curve* the periodic function

$$\Delta_{\gamma, \kappa} := \Delta_{\gamma, \kappa, D(\kappa)}.$$

Proof of the lemma 10. Part1. We first estimate the constants β_0 and β_1 in lemma 6 for the periodic function $h(s) = \kappa C(s)/(1 - \kappa S(s))$. We already proved in lemma 9

$$\int_0^{2\pi} \frac{\kappa C(\zeta)}{1 - \kappa S(\zeta)} d\zeta \geq \frac{\pi \kappa}{1 + \frac{\kappa}{\kappa_*}} \geq \frac{\pi}{2} \kappa := \beta_1.$$

Moreover $C(s) = (1 + \cos(s)) \cos(s)$ is non negative for $s \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$. By discarding negative terms we have for every $0 \leq s \leq t \leq 2\pi$

$$-\int_s^t \frac{\kappa C(\zeta)}{1 - \kappa S(\zeta)} d\zeta \leq \int_{\max(s, \pi/2)}^{\min(t, 3\pi/2)} \frac{\kappa(-C(\zeta))}{1 - \kappa S(\zeta)} d\zeta \leq \int_{\pi/2}^{3\pi/2} \frac{\kappa(-C(\zeta))}{1 - \kappa S(\zeta)} d\zeta.$$

Using $\max_{s \in [\pi/2, 3\pi/2]} (-C(s)) \leq \frac{1}{4}$ and $\min_{s \in [\pi/2, 3\pi/2]} (1 - \kappa S(s)) \geq 1 - \kappa$, we have for every $0 \leq s \leq t \leq 2\pi$

$$-\int_s^t \frac{\kappa C(\zeta)}{1 - \kappa S(\zeta)} d\zeta \leq \frac{\pi}{4} \frac{\kappa}{1 - \kappa} \leq \frac{\pi}{4} \frac{\kappa_*}{1 - \kappa_*} := \beta_0.$$

Part 2. We show that $\max_{s \in [0, 2\pi]} \Delta_{\gamma, \kappa, D}(s) < D$ is implied by

$$\frac{\gamma + 3\kappa D}{1 - \frac{\kappa}{\kappa_*}} \frac{\gamma + \kappa D + 2\kappa D^2}{1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1)} + \frac{\gamma + 2\kappa D^2}{1 - \frac{\kappa}{\kappa_*}} < L(\kappa_*)\kappa D \quad (19)$$

where $L(\kappa_*) = \frac{1}{8} \exp(-\frac{\pi}{4} \frac{\kappa_*}{1 - \kappa_*} - \frac{\pi}{2} \kappa_*)$. Indeed, it is implied by

$$\max_{s \in [0, 2\pi]} \Delta_{\gamma, \kappa, D}(s) \leq \alpha(\gamma, \kappa, D) 2\pi \frac{\exp(\beta_0)}{1 - \exp(-\beta_1)} < D, \quad (20)$$

which is implied by equation (19) using the estimate

$$1 - \exp(-\beta_1) = 1 - \exp(-\frac{\pi}{2} \kappa) = \frac{\pi}{2} \kappa \exp(-\zeta) \geq \frac{\pi}{2} \kappa \exp(-\frac{\pi}{2} \kappa_*).$$

Part 3. We now chose (γ, κ, D) as in lemma 10 : $\kappa \in]0, \kappa_*[$, $D = D(\kappa)$ and $\gamma \in]0, \kappa D^2[$. We show that the inequality (19) is satisfied. We have

$$\begin{aligned} \gamma < \kappa D^2 < \kappa D, \quad \gamma + 3\kappa D < 4\kappa D < \frac{1}{2} \left(1 - \frac{\kappa}{\kappa_*}\right), \\ \gamma + \kappa D + 2\kappa D^2 < 4\kappa D, \quad \gamma + 2\kappa D^2 < 3\kappa D^2, \\ 1 - \gamma - \frac{\kappa}{\kappa_*}(3D\kappa_* + 1) &= \left(1 - \frac{\kappa}{\kappa_*}\right) - (\gamma + 3\kappa D) > \frac{1}{2} \left(1 - \frac{\kappa}{\kappa_*}\right). \end{aligned}$$

The first term of the left hand side of (19) is bounded from above by

$$\frac{4\kappa D}{1 - \frac{\kappa}{\kappa_*}} \frac{8\kappa D}{1 - \frac{\kappa}{\kappa_*}} < \frac{32\kappa D^2}{\left(1 - \frac{\kappa}{\kappa_*}\right)^2}.$$

The second term of the left hand side is bounded from above by

$$\frac{3\kappa D^2}{1 - \frac{\kappa}{\kappa_*}} < \frac{3\kappa D^2}{\left(1 - \frac{\kappa}{\kappa_*}\right)^2}.$$

By adding the two estimates, inequality (19) is satisfied since

$$\frac{35\kappa D^2}{\left(1 - \frac{\kappa}{\kappa_*}\right)^2} = L(\kappa_*)\kappa D. \quad \square$$

By an explicit resolution of equation (20) we find a curve that define a larger domain of parameters U that we described in figure 1a.

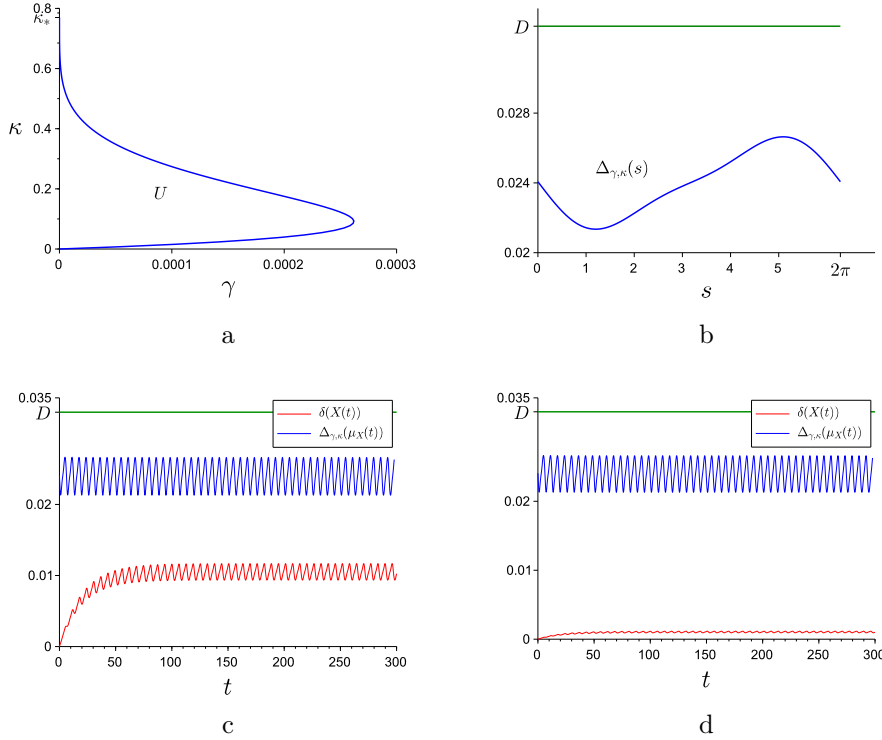


Figure 1: (a) we show the open set U of parameters (γ, κ) , for which the Winfree model presents a locked state, this domain is calculated by solving explicitly equation (20). (b) shows for $(\gamma = 0.0002610, \kappa = 0.1) \in U$ the 2π -periodic function $\Delta_{\gamma, \kappa}$ on the period $[0, 2\pi]$ which is bounded by the dispersion constant $D = 0.0329804$. (c) shows the phase dispersion $\delta(X(t))$ for the solution $X(t)$ of the Winfree model with a uniform distribution of natural frequencies $(\gamma = 0.0002610, \kappa = 0.1) \in U$ and $N = 100$; the initial condition of $X(t)$ satisfies $X(0) = (0, \dots, 0) \in C_{\gamma, \kappa}$. The phase dispersion is bounded from above by the dispersion curve $\Delta_{\gamma, \kappa}(\mu_X(t))$ and the constant $D = 0.0329804$. (d) shows the phase dispersion $\delta(X(t))$ for the solution $X(t)$ of the Winfree model with binomial distribution $B(99, 0.8)$ of natural frequencies and with identical parameters as in figure 1a.

4 Weak Locking

We are now able to prove the existence of a Φ^t -invariant open set C in \mathbb{R}^N as we show in figure 1c and 1d. This set has the shape of a cylinder of bounded and convex base. We recall that U denotes the set of parameters (γ, κ) defined in lemma 10.

Proposition 12. *For every parameter $(\gamma, \kappa) \in U$, for every $N \in \mathbb{N}^*$ and for every choice of natural frequencies $(\omega_i)_{i=1}^N$ satisfying condition (2), let $\Delta_{\gamma, \kappa}$ be the dispersion curve defined in 11, and define*

$$C_{\gamma, \kappa}^N := \left\{ X = (x_i)_{i=1}^N \in \mathbb{R}^N : \max_{i,j} |x_j - x_i| < \Delta_{\gamma, \kappa} \left(\frac{1}{N} \sum_{i=1}^N x_i \right) \right\}.$$

Then $C_{\gamma, \kappa}^N$ is invariant by the flow Φ^t . In particular the oscillators $x_i(t)$ are weakly locked altogether.

Proof of proposition 12. From lemmas 9 and 6, we have

$$\frac{d}{ds} \Delta_{\gamma, \kappa}(s) = \mathfrak{L}[\alpha(\gamma, \kappa, D(\kappa))](\Delta_{\gamma, \kappa}(s), s) \quad \text{and} \quad \max_{s \in \mathbb{R}} \Delta_{\gamma, \kappa}(s) < D(\kappa).$$

Let be $X(0) \in C_{\gamma, \kappa}^N$ and

$$t_* := \sup\{t \geq 0 : \forall 0 < t' < t, \delta(X(t')) < \Delta_{\gamma, \kappa}(\mu_X(t'))\}.$$

The proposition is proved if we show $t_* = +\infty$. Using the change of variable $s = \mu_X(t)$, corollary 8 implies, for every $s \in [s_0, s_*]$, $s_0 = \mu_X(0)$ and $s_* = \mu_X(t_*)$,

$$\frac{d}{ds} \delta_{i,j}(X^*(s)) < \mathfrak{L}[\alpha(\gamma, \kappa, D(\kappa))](\delta_{i,j}(X^*(s)), s).$$

Assume by contradiction that $t_* < +\infty$. Then there exist $1 \leq i_0, j_0 \leq N$ such that $\delta_{i_0, j_0}(X^*(s_*)) = \Delta_{\gamma, \kappa}(s_*)$,

$$\begin{aligned} \frac{d}{ds} \delta_{i_0, j_0}(X^*(s_*)) &< \mathfrak{L}[\alpha(\gamma, \kappa, D(\kappa))](\delta_{i_0, j_0}(X^*(s_*)), s_*) \\ &= \mathfrak{L}[\alpha(\gamma, \kappa, D(\kappa))](\Delta_{\gamma, \kappa}(s_*), s_*) = \frac{d}{ds} \Delta_{\gamma, \kappa}(s_*). \end{aligned}$$

There exists $s < s_*$ close enough to s_* such that $\delta_{i_0, j_0}(X^*(s)) > \Delta_{\gamma, \kappa}(s)$ or in other words there exists $t < t_*$ close enough to t_* such that $\delta_{i_0, j_0}(X(t)) > \Delta_{\gamma, \kappa}(\mu_X(t))$. We have obtained a contradiction. \square

5 Locking

We show in this section there exists a solution $X(t) = (x_1(t), \dots, x_N(t))$ of the system (3) with some initial condition in $C_{\gamma, \kappa}^N$ such that any two components $x_i(t)$ and $x_j(t)$ are locked, or more precisely, there exists a common rotation number $\rho_{\gamma, \kappa} > 0$ such that $x_i(t) = \rho_{\gamma, \kappa} t + \Psi_{i, \gamma, \kappa}^N(t)$, $\forall i = 1, \dots, N$ where $\Psi_{i, \gamma, \kappa}^N(t)$ are periodic function of period $2\pi/\rho_{\gamma, \kappa}$. Our strategy consists in constructing, by fixing the mean of $X(0)$, a compact and convex transverse section $\Sigma_{\gamma, \kappa}$ to the closure $\bar{C}_{\gamma, \kappa}^N$, and a continuous Poincaré map $P_{\gamma, \kappa} : \Sigma_{\gamma, \kappa} \rightarrow \Sigma_{\gamma, \kappa}$ by waiting the first time the mean of $X(t)$ return to 0. We then use Brouwer fixed point theorem to prove the existence of a fixed point of $P_{\gamma, \kappa}$.

Lemma 13. *Let be $(\gamma, \kappa) \in U$ where U is defined in lemma 10. Define*

$$\Sigma_{\gamma, \kappa} = \{X \in C_{\gamma, \kappa}^N : \mu(X) = 0\}.$$

Then there exist a C^∞ map (the Poincaré map) $P_{\gamma, \kappa} : \Sigma_{\gamma, \kappa} \rightarrow \Sigma_{\gamma, \kappa}$ and a C^∞ function (the return map) $\theta_{\gamma, \kappa} : \Sigma_{\gamma, \kappa} \rightarrow \mathbb{R}^+$ such that

$$\begin{aligned} \Phi^{\theta(X)}(X) &= P_{\gamma, \kappa}(X) + 2\pi\mathbb{1}, \quad \mathbb{1} = (1, \dots, 1) \in \mathbb{R}^N, \\ \frac{2\pi}{1 + \gamma + 2\kappa} &< \theta(X) < \frac{2\pi}{1 - \gamma - \frac{\kappa}{\kappa_*} - 3\kappa D(\kappa)}. \end{aligned}$$

Proof. Let be $X \in C_{\gamma, \kappa}^N$ such that $\mu(X) = 0$. Let be $\mu_X(t) := \mu(\Phi^t(X))$ and τ_X be the inverse function of μ_X as it has been defined in (14). Thanks to proposition 7, we obtain

$$1 - \gamma - \frac{\kappa}{\kappa_*} - 3\kappa D(\kappa) < \dot{\mu}_X(t) < 1 + \gamma + 2\kappa.$$

Define $\theta(X) := \tau_X(2\pi)$. Then $\int_0^{\tau_X(2\pi)} \dot{\mu}_X(t) dt = 2\pi$ implies the second estimate of the lemma. Define $P_{\gamma, \kappa}(X) := \Phi^{\theta(X)}(X) - 2\pi\mathbb{1}$. Then

$$\begin{aligned} \mu(P_{\gamma, \kappa}(X)) &= \mu_X(\theta(X)) - 2\pi = \mu_X \circ \tau_X(2\pi) - 2\pi = 0, \\ \delta(P_{\gamma, \kappa}(X)) &= \delta(\Phi^{\theta(X)}(X)) < \Delta_{\gamma, \kappa}(2\pi) = \Delta_{\gamma, \kappa}(0). \end{aligned}$$

We have shown that $P_{\gamma, \kappa}$ is a map from $\Sigma_{\gamma, \kappa}$ into itself. □

Corollary 14. *The Poincaré map $P_{\gamma, \kappa}$ defined in lemma 13 admits a fixed point $X_* \in \Sigma_{\gamma, \kappa}$.*

Proof. $\bar{\Sigma}_{\gamma,\kappa}$ is compact and convex; $P_{\gamma,\kappa} : \bar{\Sigma}_{\gamma,\kappa} \rightarrow \bar{\Sigma}_{\gamma,\kappa}$ is continuous. By Brouwer fixed point theorem, $P_{\gamma,\kappa}$ admits a fixed point in $X_* \in \bar{\Sigma}_{\gamma,\kappa}$. We claim that $X_* \notin \partial\bar{\Sigma}_{\gamma,\kappa}$. Suppose by contradiction $X_* \in \partial\bar{\Sigma}_{\gamma,\kappa}$, then there would exist $1 \leq i_0, j_0 \leq N$, such that $\delta_{i_0, j_0}(X_*) = \Delta_{\gamma,\kappa}(0)$; as in the proof of proposition 12, there would exist $t_0 > 0$ small enough such that $\delta_{i_0, j_0}(X(t)) < \Delta_{\gamma,\kappa}(\mu(X(t)))$ for every $0 < t < t_0$; by repeating this argument for every $1 \leq i_1, j_1 \leq N$ satisfying the equality $\delta_{i_1, j_1}(X(t)) = \Delta_{\gamma,\kappa}(\mu(X(t)))$, we would obtain $\delta(X(t')) < \Delta_{\gamma,\kappa}(\mu(X(t')))$ for some $t' > 0$. But proposition 12 would imply $\delta(X(t)) < \Delta_{\gamma,\kappa}(\mu(X(t)))$ for every $t > t'$: which is the contradiction with the fact that $\delta(\Phi^{\theta(X_*)}(X_*)) = \Delta_{\gamma,\kappa}(2\pi) = \Delta_{\gamma,\kappa}(0)$ and $\mu(\Phi^{\theta(X_*)}(X_*)) = 2\pi$. \square

The following theorem is a consequence of the previous corollary, it states that if $(\gamma, \kappa) \in U$, the rotation vector of $\Phi^t(X_*) = (\Phi_1^t(X_*), \dots, \Phi_N^t(X_*))$ exists and is positive.

Theorem 15. *For any $(\gamma, \kappa) \in U$, there exists a constant rotation number $\rho_{\gamma,\kappa} > 0$ and an initial condition $X_* \in C_{\gamma,\kappa}^N$ such that,*

$$\Phi_i^t(X_*) = \rho_{\gamma,\kappa} t + \Psi_{i,\gamma,\kappa}^N(t), \quad \forall i = 1, \dots, N, \quad \forall t \geq 0,$$

where $\Psi_{i,\gamma,\kappa}^N : \mathbb{R}^+ \rightarrow \mathbb{R}$ is $\frac{2\pi}{\rho_{\gamma,\kappa}}$ -periodic, C^∞ and uniformly bounded with respect to N .

Proof. Corollary 14 implies the existence of a point $X_* \in C_{\gamma,\kappa}^N$ and a time $\theta_* > 0$ such that

$$\Phi^{\theta_*}(X_*) = X_* + 2\pi\mathbb{1}.$$

By the uniqueness property of the solutions of an ordinary differential equation, we have

$$\Phi^{\theta_*+t}(X_*) = \Phi^t(X_*) + 2\pi\mathbb{1}, \quad \forall t \geq 0.$$

Define

$$\Psi_{\gamma,\kappa}^N(s) := \Phi^s(X_*) - \frac{2\pi s}{\theta_*}\mathbb{1} = (\Psi_{1,\gamma,\kappa}^N(s), \dots, \Psi_{N,\gamma,\kappa}^N(s)), \quad \forall s \geq 0.$$

We show that $\Psi_{\gamma,\kappa}^N$ is periodic of period θ_* . We have

$$\begin{aligned} \Psi_{\gamma,\kappa}^N(s + \theta_*) &= \Phi^{s+\theta_*}(X_*) - 2\pi \frac{s + \theta_*}{\theta_*}\mathbb{1} \\ &= \Phi^s(X_*) + 2\pi\mathbb{1} - 2\pi \frac{s + \theta_*}{\theta_*}\mathbb{1} = \Psi_{\gamma,\kappa}^N(s). \end{aligned}$$

Moreover the return time θ_* is uniformly bounded from above as in lemma 13. \square

6 Conclusion

We found a particular initial condition with a periodic trajectory in a N -dimensional torus. The phase dispersion between any two oscillators is bounded by a 2π -periodic function, in particular the oscillators possess a common rotation number of the form $\rho_{\gamma,\kappa} = \frac{2\pi}{\theta^*}$ where θ^* is the time needed for the mean (or barycenter) of the oscillators to execute a complete revolution. We will investigate in the future the two following questions. Question 1: does there exist a set of parameters similar to the one described in the main result 1 for which the Winfree model is stable for an open set of initial conditions ? Question 2: is it possible to pass to the limit $N \rightarrow +\infty$ and obtain the stability of the continuous system ?

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References

- [1] J.T. Ariaratnam and S.H. Strogatz 2001 Phase Diagram for the Winfree Model of Coupled Nonlinear Oscillators *Phys. Rev. Lett.* **86** 4278.
- [2] J.T. Ariaratnam 2002 Collective Dynamics of the Winfree Model of Coupled Oscillators *Cornell University* Not published thesis.
- [3] N. Chopra and M. W. Spong 2009 On Exponential Synchronization of Kuramoto Oscillators, *IEEE Transactions on Automatic Control* **54** pp. 353-357
- [4] F. Giannuzzi, D. Marinazzo, G. Nardulli, M. Pellicoro, S. Stramaghi 2007 Phase Diagram of a Generalized Winfree Model *Phys. Rev E* **75** 051104.
- [5] Seung-Yeal Ha, Jinyeong Park, Sang Woo Ryou 2015 Emergence of Phase-locked States for the Winfree Model in a Large Coupling Regime *Discrete and Continuous Dynamical Systems* **35** 3417-3436.

- [6] A. Jadbabaie, N. Motee, and M. Barahona 2004 *On the Stability of the Kuramoto Model of Coupled Nonlinear Oscillators* (Proceedings of the American Control Conference) 4296-4301.
- [7] Y. Kuramoto 1975 International Symposium on Mathematical Problems in Theoretical Physics *Lecture Notes in Physics* Vol. **39** Springer New York p. 420.
- [8] S. Louca, F. M. Atay 2014 Spatially Structured Networks of Pulse-coupled Phase Oscillators on Metric Spaces. *AIMS Journal of Discrete and Continuous Dynamical Systems-A* vol. **34**
- [9] G. Nardulli, D. Mrinazzo, M. Pellicoro and S. Stramaglia 2004 Phase Shifts Between Synchronized Oscillators in the Winfree and Kuramoto Models *arxiv.org*
- [10] Diego Pazó and Ernest Montbrió 2014 Low-Dimensional Dynamics of Populations of Pulse-Coupled Oscillators *Phys. Rev. X* **4** 011009.
- [11] D.D. Quinn, R.H. Rand, and S.H. Strogatz 2007 Singular Unlocking Transition in the Winfree Model of Coupled Oscillators *Physical Review E* **75** 036218.
- [12] A. T. Winfree 1967 Biological rhythms and the behavior of populations of coupled oscillators *J. Theor. Biol.* **16** 15-42..