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On a directed variation of the 1-2-3 and 1-2 Conjectures[☆]

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Abstract

In this paper, we consider the following question, which stands as a directed analogue of the well-known 1-2-3 Conjecture: Given any digraph D with no arc \vec{uv} verifying $d^+(u) = d^-(v) = 1$, is it possible to weight the arcs of D with weights among $\{1, 2, 3\}$ so that, for every arc \vec{uv} of D , the sum of incident weights out-going from u is different from the sum of incident weights in-coming to v ? We answer positively to this question, and investigate digraphs for which even the weights among $\{1, 2\}$ are sufficient. In relation with the so-called 1-2 Conjecture, we also consider a total version of the problem, which we prove to be false. Our investigations turn to have interesting relations with open questions related to the 1-2-3 Conjecture.

1. Introduction

We here focus on *vertex-distinguishing weightings*, a graph theory notion that attracted more and more attention in the last decade. Basically, given an undirected graph G , the goal is to weight some *elements* of G so that some *well-identified vertices* of G get distinguished relatively to some *aggregate* computed from the weighting. As emphasized in the previous sentence, the such problems of correctly weighting a graph are hence made of three main parameters. For any of these variants, the main goal is, given a graph, to deduce the smallest number of consecutive weights $1, \dots, k$ necessary to obtain a correct distinguishing weighting.

In this paper, we focus on those such problems where edges (among maybe other elements) have to be weighted, and the distinguishing aggregate is the *sum* of weights incident to the vertices. More formally, given an edge-weighting w of some graph G , for

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every vertex v one may compute¹

$$\sigma_w(v) := \sum_{u \in N(v)} w(vu),$$

that is, the sum of the weights incident to v . In case w is a total-weighting, every vertex v also has its own weight, which must be involved when computing $\sigma_w(v)$, that is

$$\sigma_w(v) := w(v) + \sum_{u \in N(v)} w(vu)$$

in such a situation. In the setting where σ_w is the distinguishing parameter, three main notions have been studied in literature:

- (1) If *edge-weightings* are considered and *all vertices* of G must be distinguished by σ , the least number of necessary consecutive edge weights is denoted $s(G)$ (and is called the *irregularity strength* of G in literature).
- (2) If *edge-weightings* are considered and only the *adjacent vertices* of G must be distinguished, the least number of necessary consecutive weights is denoted $\chi_{\Sigma}^e(G)$.
- (3) If *total-weightings* are considered and only the *adjacent vertices* of G must be distinguished, the least number of necessary consecutive weights is denoted $\chi_{\Sigma}^t(G)$.

As we only focus on Items (2) and (3) (that is, on *sum-colouring edge-weighting* and *sum-colouring total-weighting*) in this paper, we will below recall some of their associated backgrounds. For more general details on this wide area (and on the upcoming introductory details), we refer the interested reader to the recent survey by Seamone on this topic [14].

The parameter χ_{Σ}^e is related to the well-known **1-2-3 Conjecture** raised by Karoński, Łuczak and Thomason [10], which reads as follows (where a *nice graph* refers to a graph with no component isomorphic to K_2).

1-2-3 Conjecture (Karoński, Łuczak, Thomason [10]). *For every nice graph G , we have $\chi_{\Sigma}^e(G) \leq 3$.*

Several constant upper bounds on χ_{Σ}^e were given towards the 1-2-3 Conjecture, the best one of which being due to Kalkowski, Karoński and Pfender, who proved that $\chi_{\Sigma}^e(G) \leq 5$ whenever G is nice [8]. Concerning the parameter χ_{Σ}^t , the following so-called **1-2 Conjecture** was raised by Przybyło and Woźniak [13].

1-2 Conjecture (Przybyło, Woźniak [13]). *For every graph G , we have $\chi_{\Sigma}^t(G) \leq 2$.*

Towards the 1-2 Conjecture, the best known result so far is due to Kalkowski [7], who proved that every graph G verifies $\chi_{\Sigma}^t(G) \leq 3$.

There have been a few attempts for bringing the 1-2-3 and 1-2 Conjectures to directed graphs, see *e.g.* [1, 3, 5, 11]. Most of all these different directed versions of the 1-2-3 and 1-2 Conjectures were shown to hold, even under strong additional constraints such as list requirements. This results from the fact that these versions, though seemingly close to

¹In case no ambiguity is possible, we will sometimes voluntarily omit the subscript w (*e.g.* write σ for σ_w) to lighten the notations.

the 1-2-3 and 1-2 Conjectures in essence, were based on several behaviours that are not so comparable to the ones we have to deal with when considering the original conjectures. Notably, the definitions of some of these versions make the use of induction arguments possible, while such are generally not applicable in the undirected context. This makes us wonder what should be the directed analogues to the 1-2-3 and 1-2 Conjectures that would mimic their behaviours and inherent hardness the best, while fitting to the particularities of the directed context.

In that spirit, we introduce and study new directed analogues of the 1-2-3 and 1-2 Conjectures. Our directed analogue of the 1-2-3 Conjecture is introduced in Section 2, while our analogue of the 1-2 Conjecture is studied in Section 3. We more precisely show our directed analogue of the 1-2-3 Conjecture to be equivalent to solved cases of the 1-2-3 Conjecture, hence giving a positive answer to a question addressed by Łuczak [12]. Using that equivalence, we point out that our directed analogue of the 1-2 Conjecture, though true in specific contexts, is false in general. Unexpected implications of our investigations on the 1-2-3 Conjecture are discussed in Section 4.

2. A directed 1-2-3 Conjecture

Let D be a simple digraph, and w be an arc-weighting of D . For every vertex v , one can compute two sums incident to v , namely

$$\sigma_w^-(v) := \sum_{u \in N^-(v)} w(\overrightarrow{uv}),$$

i.e. the incident in-coming sum, and

$$\sigma_w^+(v) := \sum_{u \in N^+(v)} w(\overrightarrow{vu}),$$

i.e. the incident out-going sum. We call w *sum-colouring* if, for every arc \overrightarrow{uv} of D , we have

$$\sigma_w^+(u) \neq \sigma_w^-(v).$$

The least number of weights in a sum-colouring k -arc-weighting (if any) of D is denoted $\chi_{\mathbb{L}}^e(D)$.

Before starting investigating the parameter $\chi_{\mathbb{L}}^e$, let us, reusing the notions and terminology above, describe the previously introduced directed versions of the 1-2-3 Conjecture mentioned in Section 1. In [5, 11] was introduced the variant where, for every arc \overrightarrow{uv} , we require the *relative sums* of u and v , *i.e.* $\sigma^-(u) - \sigma^+(u)$ and $\sigma^-(v) - \sigma^+(v)$, to be different. In [1, 3] was studied the variant where every two adjacent vertices must have different *out-sums* (or, equivalently, *in-sums*), that is we require $\sigma^+(u) \neq \sigma^+(v)$ (resp. $\sigma^-(u) \neq \sigma^-(v)$) for every arc \overrightarrow{uv} . The notion of sum-colouring arc-weighting above is hence different in the sense that, among the three directed variants, it is the only one where the distinction between two vertices connected by an arc depends on the direction of that arc.

As a very first observation, it is worth mentioning that not all digraphs admit a sum-colouring arc-weighting. To be convinced of this statement, just consider a digraph D having an arc \overrightarrow{uv} such that $d^+(u) = d^-(v) = 1$. Then, no matter what weight x is assigned to \overrightarrow{uv} , clearly we will get $\sigma^+(u) = \sigma^-(v) = x$; so there is no hope to find a sum-colouring arc-weighting. However, one can easily convince themselves that if D is *nice*, in the sense

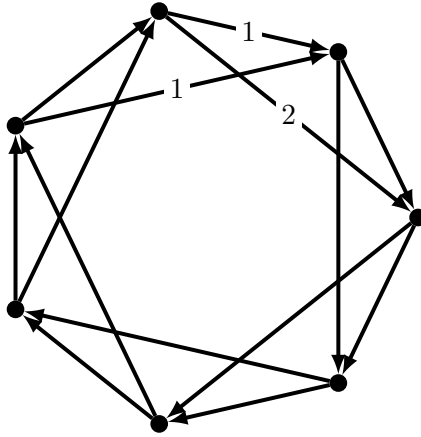


Figure 1: Illustration of the argument why \vec{C}_7 admits no sum-colouring 2-arc-weighting. Having a weight (here, 1) on some arc (here, the topmost arc) forces two incident arcs to be weighted differently.

that it does not admit this configuration, then D admits a sum-colouring arc-weighting (just consider sufficiently fast increasing weights).

Experimentations on some nice digraphs suggest that the following conjecture, which stands as a directed analogue of the 1-2-3 Conjecture, should be true.

Directed 1-2-3 Conjecture. *For every nice digraph D , we have $\chi_{\vec{L}}^e(D) \leq 3$.*

It is worth mentioning that the value 3 in our conjecture would be best possible as there exist nice digraphs admitting no sum-colouring 2-arc-weighting. One easy family of digraphs whose $\chi_{\vec{L}}^e$ is 3 is squares of odd cycles in which the two underlying cycles are directed to form two directed cycles (see Figure 1). Assume indeed we use weights 1 and 2 only on such a digraph. Such a digraph is 2-regular and weighting, say, 1 an arc, say, $\vec{v_1v_2}$ forces the weights on the second arc out-going from v_1 and the second arc in-coming to v_2 to be different (so that $\sigma^+(v_1) \neq \sigma^-(v_2)$). Repeating this argument until all arcs are weighted following successive deductions, eventually we easily reach a contradiction. So such a digraph can only be weighted with at least three weights. In upcoming Section 2.2, we will point out that actually many other such digraphs exist.

This section is organized as follows. We start by giving a direct proof of the Directed 1-2-3 Conjecture in Section 2.1. Our proof relies on an equivalence between the Directed 1-2-3 Conjecture and the 1-2-3 Conjecture for particular undirected graphs. Then we investigate, in Section 2.2, digraphs D verifying $\chi_{\vec{L}}^e(D) \leq 2$. In particular, we show that some families of digraphs have this property, and point out that some other do not (hence providing more examples of nice digraphs needing all weights among $\{1, 2, 3\}$).

2.1. A proof of the Directed 1-2-3 Conjecture

Let G be a bipartite graph with bipartition $A \cup B$. In the following, we say that G is *anti-matchable* if G is *balanced*, i.e. $|A| = |B|$, and the complement of G admits a perfect matching joining A and B . Said differently, G is anti-matchable if it is balanced and has a set of disjoint non-edges between A and B covering all its vertices. Assuming the vertices in A and B are explicitly ordered, i.e. from first to last, we call G *anti-matched* if, for every $i \in \{1, \dots, |A|\}$, the i th vertex of A is not adjacent to the i th vertex of B . Note that a perfect matching in the complement of G can be directly deduced when G is anti-matched.

We below prove the Directed 1-2-3 Conjecture by essentially proving an equivalence between this conjecture and the 1-2-3 Conjecture for nice bipartite graphs.

Theorem 1. *The following two problems are equivalent:*

- (1) *The Directed 1-2-3 Conjecture for nice digraphs.*
- (2) *The 1-2-3 Conjecture for nice bipartite graphs.*

Proof. (2) \Rightarrow (1) Let D be a nice digraph. We describe below how to deduce a sum-colouring 3-arc-weighting w' of D . Let v_1, v_2, \dots, v_n denote the vertices of D following an arbitrary ordering. Now consider the bipartite graph $G(D)$ with bipartition $V^+ \cup V^-$ constructed from D as follows:

- For every vertex v_i of D , add a vertex v_i^+ to V^+ , as well as a vertex v_i^- to V^- .
- For every arc $\overrightarrow{v_i v_j}$ of D , add the edge $v_i^+ v_j^-$ to $G(D)$.

Clearly $G(D)$ is nice since otherwise D would not be nice. Furthermore, $G(D)$ is anti-matched. Assume now we give some edge-weighting w of $G(D)$, and let w' be the arc-weighting of D where, for every arc $\overrightarrow{v_i v_j}$ of D , we put $w'(\overrightarrow{v_i v_j}) = w(v_i^+ v_j^-)$. Note that w' is well-defined since every arc of D is associated with exactly one edge in $G(D)$. Furthermore, by the way $G(D)$ was constructed and w' was obtained, for every vertex v_i we have $\sigma_{w'}^+(v_i) = \sigma_w(v_i^+)$ and $\sigma_{w'}^-(v_i) = \sigma_w(v_i^-)$. So if w is sum-colouring in $G(D)$, then in particular w' is sum-colouring in D . The result then follows immediately since every nice bipartite graph admits a sum-colouring 3-edge-weighting according to our initial hypothesis.

(1) \Rightarrow (2) Let G be a nice bipartite graph with bipartition $A \cup B$. In case G is not balanced or anti-matchable, just add some isolated vertices to A and B until G fulfils these two properties. Note that this operation preserves niceness of G . Assuming $|A| = |B| = n$, relabel the vertices of A and B so that $A = \{v_1^+, v_2^+, \dots, v_n^+\}$ and $B = \{v_1^-, v_2^-, \dots, v_n^-\}$, and every two vertices of the form v_i^+ and v_i^- are not adjacent, which is possible since G was made anti-matchable. So G is now anti-matched.

Now just perform the construction converse to the one described in the proof of (2) \Rightarrow (1) to get a digraph $D(G)$. More precisely, for every pair $\{v_i^+, v_i^-\}$ of vertices of G , add a vertex v_i to $D(G)$. Now, for every edge $v_i^+ v_j^-$ of G , add the arc $\overrightarrow{v_i v_j}$ to $D(G)$. Clearly this operation is valid since $i \neq j$ by the labelling of the vertices of G . Besides, $D(G)$ is nice since otherwise G would not be nice. It should be now clear that, similarly as in the previous case, from a sum-colouring 3-arc-weighting of $D(G)$, which exists by the initial hypothesis, we can just copy the arc weights onto the edges of G to get a sum-colouring 3-edge-weighting of G . This concludes the proof. \square

The Directed 1-2-3 Conjecture now follows directly from Theorem 1 since the 1-2-3 Conjecture holds for all nice bipartite graphs (as proved by Karoński, Łuczak and Thomason in [10]).

Theorem 2 (Karoński, Łuczak, Thomason [10]). *For every nice bipartite graph G , we have $\chi_\Sigma^e(G) \leq 3$.*

Corollary 3. *The Directed 1-2-3 Conjecture is true.*

Theorem 1 in particular gives another explanation why the directed squares of odd cycles mentioned in Section 2 have their value of χ_L^e being 3. Note indeed that for every such digraph D , the graph $G(D)$ we obtain is a cycle of the form C_{4k+2} , which is known to be non sum-colouring 2-edge-weightable.

2.2. Using weights 1 and 2 only

We have seen in the previous section that every nice digraph can be arc-weighted in a sum-colouring way with weights among $\{1, 2, 3\}$. In this section, we study situations in which all weights among $\{1, 2, 3\}$ are necessary, and situations where only the weights among $\{1, 2\}$ are needed.

Bipartite digraphs

Our upcoming arguments and remarks rely on the following alternative proof of the Directed 1-2-3 Conjecture for nice bipartite digraphs.

Theorem 4. *For every nice bipartite digraph D , we have $\chi_{\mathbb{L}}^e(D) \leq 3$.*

Proof. Assume $A \cup B$ is a bipartition of $V(\text{und}(D))$, where $\text{und}(D)$ denotes the underlying undirected graph of D . Let $D_{\uparrow\uparrow}$ and $D_{\downarrow\downarrow}$ be the subdigraphs of D induced by the arcs going from, say, A to B , and from B to A , respectively. We note that both $D_{\uparrow\uparrow}$ and $D_{\downarrow\downarrow}$ are nice, since otherwise D itself would not be nice. Of course, since D is bipartite, so are $D_{\uparrow\uparrow}$ and $D_{\downarrow\downarrow}$.

Let w be an arc-weighting of D obtained as follows. Consider $D_{\uparrow\uparrow}$ first. Since $\text{und}(D_{\uparrow\uparrow})$ is nice and bipartite, it admits a sum-colouring 3-edge-weighting w' (according to Theorem 2) which we directly transfer to $D_{\uparrow\uparrow}$ (i.e. if uv and \overrightarrow{uv} are corresponding edge and arc of $\text{und}(D_{\uparrow\uparrow})$ and $D_{\uparrow\uparrow}$, we give them the same weight, that is $w'(uv)$). Note that if \overrightarrow{uv} is an arc of $D_{\uparrow\uparrow}$ then, because $\sigma_{w'}(u) \neq \sigma_{w'}(v)$, and u has in-degree 0 and v has out-degree 0 in $D_{\uparrow\uparrow}$, clearly we have $\sigma_{w'}^+(u) \neq \sigma_{w'}^-(v)$ in $D_{\uparrow\uparrow}$. So w' is sum-colouring in $D_{\uparrow\uparrow}$. Analogously, $D_{\downarrow\downarrow}$ admits such an arc-weighting w'' as well, according to the same arguments.

To obtain w , it now suffices to directly copy the weights by w' and w'' of $D_{\uparrow\uparrow}$ and $D_{\downarrow\downarrow}$, respectively, to D (that is, if \overrightarrow{uv} belongs to $D_{\uparrow\uparrow}$, just set $w(\overrightarrow{uv}) = w'(\overrightarrow{uv})$, or $w(\overrightarrow{uv}) = w''(\overrightarrow{uv})$ otherwise). Then w retains the property of being sum-colouring, since, for every vertex v in A (resp. B), the in-degree of v in D is exactly its in-degree in $D_{\downarrow\downarrow}$ (resp. $D_{\uparrow\uparrow}$), while the out-degree of v in D is exactly its out-degree in $D_{\uparrow\uparrow}$ (resp. $D_{\downarrow\downarrow}$). So if w were not sum-colouring, then one of w' and w'' would not be sum-colouring. \square

Following the proof of Theorem 4, we get that, for any nice bipartite digraph D , we have

$$\chi_{\mathbb{L}}^e(D) = \max \{ \chi_{\Sigma}^e(\text{und}(D_{\downarrow\downarrow})), \chi_{\Sigma}^e(\text{und}(D_{\uparrow\uparrow})) \},$$

where $D_{\uparrow\uparrow}$ and $D_{\downarrow\downarrow}$ have the same meaning as in the proof above. This in particular means that if one of $D_{\uparrow\uparrow}$ and $D_{\downarrow\downarrow}$ has its value of χ_{Σ}^e being equal to 3, then $\chi_{\mathbb{L}}^e(D) = 3$. Since many bipartite graphs have their value of χ_{Σ}^e being exactly 3 (refer to [14] for a summary of all known such families), we get that a wide bunch of nice bipartite digraphs have their $\chi_{\mathbb{L}}^e$ being exactly 3. On the other hand, we get that if a nice bipartite digraph with bipartition $A \cup B$ is the combination of two nice bipartite graphs G_1 and G_2 such that:

- $\chi_{\Sigma}^e(G_1) \leq 2$ and $\chi_{\Sigma}^e(G_2) \leq 2$,
- all edges of G_1 are directed from, say, A to B , in D , and
- all edges of G_2 are directed from B to A in D ,

then $\chi_{\mathbb{L}}^e(D) \leq 2$.

We also note that the proof of Theorem 4 gives something more general. Namely, if a nice digraph D has a partition V_1, \dots, V_k of its vertex set such that:

1. every V_i is a stable set,
2. all vertices in a given V_i have all of their in-neighbours in a same V_j , and
3. all vertices in a given V_i have all of their out-neighbours in a same V_j ,

then a sum-colouring arc-weighting of D can be obtained by independently edge-weighting all of its underlying nice bipartite graphs of the form $\text{und}(D[V_i \cup V_j])$ for every $i \neq j$ (some of which may have no edge). In the next result, we call such a partition of $V(D)$ a *circular vertex-colouring* of D .

Corollary 5. *For every nice digraph D with a circular vertex-colouring V_1, \dots, V_k , we have*

$$\chi_{\mathbb{L}}^e(D) = \max \{ \chi_{\Sigma}^e(\text{und}(D[V_i \cup V_j])) : i, j \in \{1, 2, \dots, k\} \}.$$

From Corollary 5, we can deduce many digraphs that need all weights among $\{1, 2, 3\}$ to be weighted, and, conversely, many digraphs for which the weights among $\{1, 2\}$ suffice.

Acyclic tournaments

We here consider *acyclic digraphs*, *i.e.* digraphs with no directed cycles. Using three weights on nice acyclic digraphs is actually best possible. To illustrate this, just take any nice bipartite graph G with $\chi_{\Sigma}^e(G) = 3$ and direct all arcs towards the same part. In doing so, as explained in previous Section 2.2, we obtain an acyclic oriented bipartite graph \vec{G} with $\chi_{\mathbb{L}}^e(\vec{G}) = 3$. We can however show that, in particular situations, such as for acyclic tournaments, acyclic digraphs may have their $\chi_{\mathbb{L}}^e$ being at most 2.

Theorem 6. *For every nice acyclic tournament \vec{T} , we have $\chi_{\mathbb{L}}^e(\vec{T}) \leq 2$.*

Proof. Since \vec{T} is acyclic, it admits a vertex ordering v_1, v_2, \dots, v_n such that for all i and j with $i < j$, the arc between v_i and v_j is directed “to the right”, that is towards j . Ideally, we would like to produce a 2-arc-weighting w of \vec{T} with the following properties:

- (1) For all $i \in \{1, 2, \dots, n-1\}$, we have $\sigma_w^+(v_i)$ odd.
- (2) For all $i \in \{2, 3, \dots, n\}$, we have $\sigma_w^-(v_i)$ even.

Assuming w satisfies Properties (1) and (2) above, clearly it is sum-colouring. Unfortunately, such a w cannot always be obtained but, in that case, we can nevertheless make sure that w is sum-colouring according to other arguments.

The main idea to obtain such a w consists in picking pairs of vertices v_i and v_j , and one of their common out-neighbours v_k , then setting $w(\overrightarrow{v_i v_k}) = w(\overrightarrow{v_j v_k}) = 1$, and setting all other arcs out-going from v_i and v_j to 2. In doing so, note that Property (1) will be met for v_i and v_j , while the parity of $\sigma_w^-(v_k)$ will remain unchanged.

Assume first n is odd. Starting from v_1 , consider disjoint pairs of consecutive vertices until v_{n-1} is reached (that is, $\{v_1, v_2\}$, then $\{v_3, v_4\}$, and so on). For every such pair $\{v_i, v_{i+1}\}$, set $w(\overrightarrow{v_i v_n}) = w(\overrightarrow{v_{i+1} v_n}) = 1$. At the end of the process, assign 2 to all non-weighted arcs. Clearly w directly respects Properties (1) and (2) above, so it is sum-colouring.

If n is even, then we proceed as follows. We repeat the same procedure as in the previous case but with the pairs $\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{n-3}, v_{n-2}\}$ only, choosing the arcs from their members towards v_{n-1} to be weighted 1. Once the 1’s are attributed, put 2 on all remaining arcs, including $\overrightarrow{v_{n-1} v_n}$. Note that Property (1) is violated by $\sigma_w^+(v_{n-1})$ only, which is equal exactly to 2. But since $n > 2$ (since otherwise \vec{T} would not be nice), we have $\sigma_w^-(v_n) > 2 = \sigma_w^+(v_{n-1})$. So w is sum-colouring. \square

3. A directed 1-2 Conjecture

We now investigate how helpful it is to be granted the possibility to locally modify σ^- and σ^+ at every vertex of a given digraph. Let us define this formally. Let w be a total-weighting of some digraph D . For every vertex v of D , we define:

$$\sigma_w^-(v) := w(v) + \sum_{u \in N^-(v)} w(\vec{uv}) \quad \text{and} \quad \sigma_w^+(v) := w(v) + \sum_{u \in N^+(v)} w(\vec{vu})$$

that is, the local weight by w is counted in both $\sigma_w^-(v)$ and $\sigma_w^+(v)$ (as if we were weighting a loop at v). Once more, we call w *sum-colouring* if $\sigma_w^+(u) \neq \sigma_w^-(v)$ for every arc \vec{uv} of D . This time, it should be clear that all digraphs admit a sum-colouring total-weighting (if D is not nice, in the sense defined for the arc version, just use vertex weights to “destroy” its bad configurations). For every digraph D , the chromatic parameter $\chi_{\mathbb{L}}^t(D)$, denoting the least number of consecutive weights in a sum-colouring k -total-weighting of D , is hence well-defined.

As a consequence of a remark above, we have the following.

Observation 7. *For every nice digraph D , we have $\chi_{\mathbb{L}}^t(D) \leq \chi_{\mathbb{L}}^e(D)$.*

Corollary 3 then implies that $\chi_{\mathbb{L}}^t(D) \leq 3$ holds for every nice digraph D . It can also be proved that this inequality holds when D is not nice.

Theorem 8. *For every digraph D , we have $\chi_{\mathbb{L}}^t(D) \leq 3$.*

Proof. Let D' be the digraph obtained from D by adding a loop at every vertex, and let $G(D')$ be the bipartite graph constructed from D' as described in the proof of Theorem 1. Note that, by construction, every loop $\vec{v_i v_i}$ of D' becomes an edge $v_i^+ v_i^-$ in $G(D')$. Since $G(D')$ is clearly nice, it admits a sum-colouring 3-edge-weighting according to Theorem 2. Now just transfer the edge weights from $G(D')$ to D' as described in the proof of Theorem 1 to obtain a 3-arc-weighting of D' which is sum-colouring (unless $\sigma^+(v_i) = \sigma^-(v_i)$ for some vertex v_i , but this is not a problem). To now get a sum-colouring 3-total-weighting of D , just transfer the arc weights from D' to D , with the exception that the weight of every loop $\vec{v_i v_i}$ in D' becomes the weight of v_i in D . \square

Since the possibility of locally modifying both σ^- and σ^+ is very handy, one could conjecture that the following, which is a direct analogue of the 1-2 Conjecture, should be true.

Directed 1-2 Conjecture. *For every digraph D , we have $\chi_{\mathbb{L}}^t(D) \leq 2$.*

This section is organized as follows. We first show, in Section 3.1, that the Directed 1-2 Conjecture, as currently stated, is actually false. So we propose, in the same section, a refined conjecture. We then prove in Section 3.2 that both versions of the Directed 1-2 Conjecture are true in some contexts.

3.1. Counterexamples to the Directed 1-2 Conjecture

As seen in the proof of Theorem 1, there is an equivalence between edge-weighting nice bipartite graphs and arc-weighting nice digraphs in a sum-colouring way. Some kind of similar relation can also be pointed out for our definition of sum-colouring total-weighting.

Let us introduce some more terminology. Given a digraph D , the balanced anti-matched bipartite graph $G(D)$ obtained from D as described in the first part of the proof of Theorem 1 is called the *bipartite anti-matched-representation* of D . Conversely, assuming G is

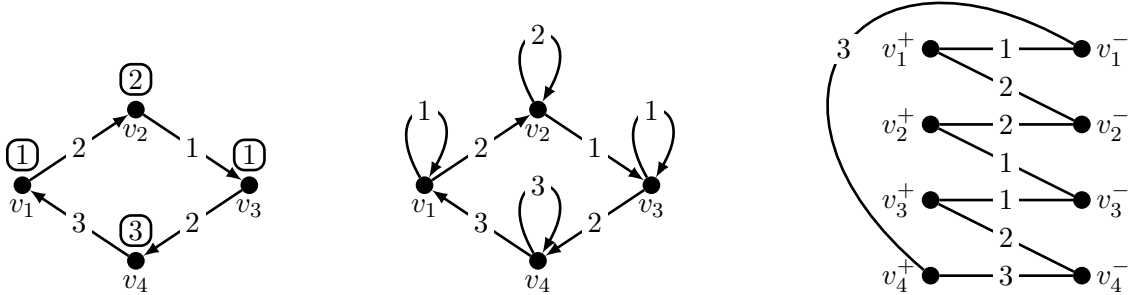


Figure 2: A digraph D (left), the same digraph D' with a loop at every vertex (middle), and the bipartite matched-representation $G(D)$ of D (right). From a sum-colouring edge-weighting of $G(D)$, one can deduce a sum-colouring arc-weighting of D' , from which is deduced a sum-colouring total-weighting of D .

an anti-matched bipartite graph, we call the digraph $D(G)$, as obtained in the second part of the proof of Theorem 1, the *directed representation* of G .

As seen in the proof of Theorem 8, note that, in the bipartite representation $G(D)$ of a digraph D , having an edge $v_i^+ v_i^-$ in $G(D)$ corresponds, in D , to an arc from v_i to itself. Furthermore, as within an arc-weighting, in D , a loop at v_i contributes to both $\sigma^-(v_i)$ and $\sigma^+(v_i)$, then weighting the edge $v_i^+ v_i^-$ in $G(D)$ can actually be seen as attributing the personal weight to v_i in the corresponding total-weighting of D . So an edge-weighting of $G(D)$ under the assumption that $G(D)$ is *matched*, *i.e.* its two partite sets can be ordered so that we have a perfect matching joining its every pair of i th vertices, is quite similar to a total-weighting of D (basically, for every vertex v_i , the weight of v_i in D is represented by the weight on $v_i^+ v_i^-$ in $G(D)$). Adding edges joining the first vertices, the second vertices, and so on, of the bipartite anti-matched representation of D , we obtain a bipartite graph which we call the *bipartite matched-representation* of D (see Figure 2 for an illustration). Assuming G is a matched bipartite graph with bipartition $\{a_1, a_2, \dots, a_n\} \cup \{b_1, b_2, \dots, b_n\}$, we call every edge $a_i b_i$ a *matched edge* of G .

There is an important point one should be careful with. By the remarks above, the following is true.

Observation 9. *Let D be a digraph, and $G(D)$ be the bipartite matched-representation of D . Then, we have $\chi_{\Sigma}^t(D) \leq \chi_{\Sigma}^e(G(D))$.*

However, the converse of Observation 9 is not true, in the sense that a sum-colouring total-weighting of D does not necessarily yield a sum-colouring edge-weighting of $G(D)$. This comes from the fact that, for every matched edge $v_i^+ v_i^-$ of $G(D)$, we require $\sigma(v_i^+) \neq \sigma(v_i^-)$, while the equivalent requirement in D would be to have $\sigma^+(v_i) \neq \sigma^-(v_i)$, which we do not impose.

Due to this remark, if we have a matched bipartite graph G which is not sum-colouring 2-edge-weightable, then we cannot directly deduce that $D(G)$ is not sum-colouring 2-total-weightable. We can nevertheless obtain non sum-colouring 2-total-weightable digraphs via the following observation.

Observation 10. *Let D be a non-nice digraph having a directed cycle $\overrightarrow{v_1 v_2 \dots v_k v_1}$ such that all arcs $\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \dots, \overrightarrow{v_k v_1}$ are bad, *i.e.* we have $d^+(v_i) = d^-(v_{i+1}) = 1$ for every such arc $\overrightarrow{v_i v_{i+1}}$. If k is odd, then $\chi_{\Sigma}^t(D) > 2$.*

Observation 10 notably shows that $\chi_{\mathbb{L}}^t(D) > 2$ whenever D is a directed odd cycle. It is not so surprising that directed odd cycles are counterexamples to the Directed 1-2 Conjecture, as, since they are not nice, the vertex weights are really necessary here. This makes us wonder whether the following refined conjecture is true.

Refined Directed 1-2 Conjecture. *For every nice digraph D , we have $\chi_{\mathbb{L}}^t(D) \leq 2$.*

Towards that refined conjecture, recall that already Theorem 8 implies that we have $\chi_{\mathbb{L}}^t(D) \leq 3$ for every nice digraph D . Using Observation 9, we can also confirm the Refined Directed 1-2-3 Conjecture for digraphs D whose bipartite matched-representation $G(D)$ verifies $\chi_{\Sigma}^e(G(D)) \leq 2$. Many nice bipartite graphs are known to have this property, see [14], one of the most interesting results for our purpose being the following.

Theorem 11 (Chang, Lu, Wu, Yu [6]). *Let G be a nice connected bipartite graph with bipartition $A \cup B$. If at least one of $|A|$ and $|B|$ is even, then $\chi_{\Sigma}^e(G) \leq 2$.*

We say that a digraph is *connected* if its underlying undirected graph is connected.

Corollary 12. *Every connected digraph with even order agrees with the Refined Directed 1-2 Conjecture.*

Proof. Let D be a connected digraph with even order, and let $G(D)$ be the bipartite matched-representation of D . Since D has even order, both parts of $G(D)$ are of even size. Besides, $G(D)$ is necessarily nice since otherwise, as $G(D)$ is matched, it would mean that D has isolated vertices. So that we can apply Theorem 11 directly on $G(D)$ and get our conclusion, we just have to make sure that $G(D)$ is connected.

Since D is connected, for every two vertices u and v , there exists a path P from u to v in $\text{und}(D)$. We need to prove that there is a path from u^+ (or u^- , but this is equivalent as u^+u^- is an edge of $G(D)$) to v^+ (or v^- , for the same reason) in $G(D)$. Assuming u' is the vertex succeeding u in P , we actually just need to prove that u^+ or u^- is adjacent to u'^+ or u'^- in $G(D)$. But this is necessarily the case: since uu' is an edge of $\text{und}(D)$, either the arc $\overrightarrow{uu'}$ or $\overrightarrow{u'u}$ belongs to D . So $u^+u'^-$ is an edge of $G(D)$ in the first situation, while u'^+u^- is an edge in the second situation. Repeating this argument for every two subsequent vertices of P , we get that $G(D)$ indeed has the claimed path.

Since the arguments above apply whatever u and v are, we get that $G(D)$ is connected. So $G(D)$ admits a sum-colouring 2-edge-weighting according to Theorem 11, which we can directly turn into a sum-colouring 2-total-weighting of D . \square

As going to be discussed in concluding Section 4, proving the counterpart of Corollary 12 for connected digraphs with odd order does not seem as easy. This problem is in particular connected to other problems concerning nice bipartite graphs and the 1-2-3 Conjecture.

3.2. Digraphs verifying the (Refined) Directed 1-2 Conjecture

In this section, we prove that some classes of digraphs agree with the (Refined) Directed 1-2 Conjecture. We in particular consider acyclic digraphs, and bipartite digraphs.

Theorem 13. *For every acyclic digraph D , we have $\chi_{\mathbb{L}}^t(D) \leq 2$.*

Proof. Start by mimicking the beginning (*i.e.* the pairing part) of the proof of Theorem 6 as long as possible to get a 2-arc-weighting w of D such that all $\sigma_w^-(v_i)$'s are even and some $\sigma_w^+(v_i)$'s (but maybe not all) are odd. Put $w(v_i) = 2$ for every vertex v_i of D . From now on we will use notation w to deal with the resulting 2-total-weighting. In case there is no arc $\overrightarrow{v_i v_j}$ such that $\sigma_w^+(v_i) = \sigma_w^-(v_j)$, we are done. Otherwise, there are some such arcs $\overrightarrow{v_i v_j}$

such that $\sigma_w^+(v_i) = \sigma_w^-(v_j)$ with $\sigma_w^+(v_i)$ being even while it should be odd. From the point of view of $\overrightarrow{v_i v_j}$, we call v_i *bad*, while we call v_j *good*. Note that all sinks are necessarily good for all arcs.

To fix the conflicts, we prove that, starting from the rightmost conflicting arc (*i.e.* the one $\overrightarrow{v_i v_j}$ with v_i having the largest index), we can make v_i good by considering another arc $\overrightarrow{v_k v_i}$ and possibly making v_k bad. This means that we can basically “push” the conflicts towards the left, *i.e.* towards the sources, which we can handle easily (since we do not care about their value of σ_w^-).

Consider a conflicting arc $\overrightarrow{v_i v_j}$. By definition, we have:

- $\sigma_w^-(v_j)$ even and $\sigma_w^+(v_j)$ odd or null (good), and
- $\sigma_w^-(v_i)$ even and $\sigma_w^+(v_i)$ even (bad).

If v_i is a source, then we can solve the problem by just switching $w(v_i)$, *i.e.* set this weight to $3 - w(v_i)$. Now, if v_i is not a source, then there is an arc $\overrightarrow{v_k v_i}$ with $k < i$. Then by just switching the weight $w(v_i)$, as well as the weight $w(\overrightarrow{v_k v_i})$, note that v_i becomes good.

By just repeating this argument until only sources of D are bad, we eventually can solve all conflicts and make w sum-colouring. \square

Theorem 14. *For every bipartite digraph D , we have $\chi_{\mathbb{L}}^t(D) \leq 2$.*

Proof. Let $A \cup B$ denote the bipartition of D . Start with assigning weight 2 to all arcs of D so that σ^- and σ^+ are even for all vertices. Now assign weight 1 on all vertices of A , and assign weight 2 on all vertices in B . The result is that,

- for every $a \in A$, both $\sigma^-(a)$ and $\sigma^+(a)$ are odd, and
- for every $b \in B$, both $\sigma^-(b)$ and $\sigma^+(b)$ are even.

It then follows that, for every arc \overrightarrow{uv} of D , the parities of $\sigma^+(u)$ and $\sigma^-(v)$ are different, so the two values are different. \square

4. Conclusion

In this paper, we have introduced new directed analogues of the 1-2 and 1-2-3 Conjectures. Although the Directed 1-2-3 Conjecture admits an easy proof, the unexpected equivalence between the Directed 1-2-3 (and 1-2) Conjecture and the 1-2-3 Conjecture we have exhibited in Theorem 1 is of interest. This is mainly because the status of the 1-2-3 Conjecture for nice bipartite graphs is not entirely understood. Indeed, apart from completely proving the 1-2-3 Conjecture, perhaps the most important open question related to the 1-2-3 Conjecture is the existence of an easy characterization of nice bipartite graphs G verifying $\chi_{\Sigma}^e(G) \leq 2$.

Question 15. *Which nice bipartite graphs G verify $\chi_{\Sigma}^e(G) \leq 2$?*

As summarized in [14], only a few classes of nice bipartite graphs G having $\chi_{\Sigma}^e(G) = 3$ are known at the moment; so studying the Directed 1-2-3 and 1-2 Conjectures may be a new way to attack Question 15. In particular, let us mention the following potential directions for future work.

1. When checking what is the value of $\chi_{\Sigma}^e(G)$ for a nice bipartite graph G , one may equivalently check the value of $\chi_{\mathbb{L}}^e(D(G))$ of a directed representation $D(G)$ of G . For that purpose, we recall that G should be anti-matched, and hence balanced. In case G is not balanced, we can get an equivalent nice balanced bipartite graph by adding isolated vertices to G . However, if G is already balanced, we note that the number of vertices of $D(G)$ will be $|V(G)|/2$ (while the size is preserved). So considering $D(G)$ may be simpler in some situations.
2. If we have $\chi_{\mathbb{L}}^e(D) = 3$ for some nice digraph D , by Theorem 1 we get an equivalent nice bipartite graph $G(D)$ verifying $\chi_{\Sigma}^e(G(D)) = 3$. More generally, if we could prove that it is NP-complete to decide whether $\chi_{\mathbb{L}}^e(D) \leq 2$ for a given nice digraph D , then we would prove that no easy characterization answering Question 15 exists. This is of interest as the complexity of deciding whether $\chi_{\Sigma}^e(G) \leq 2$ is still unknown under the restriction that G is a given nice bipartite graph.
3. Towards the Refined Directed 1-2-3 Conjecture, if we could prove that some nice digraph D refute it, then we would get another example of nice bipartite graph $G(D)$ not in the class mentioned in Question 15. Such a graph would more likely be a new graph not mentioned in the summary [14], as it can be easily checked that all such known graphs, when matched (*i.e.* have their vertices ordered as explained in Section 3.1), form a representation of digraphs which are not nice.
4. Conversely, to prove the counterpart of Corollary 12 for connected digraphs with odd order, it would be sufficient to prove that we have $\chi_{\Sigma}^e(G) \leq 2$ for every nice matched bipartite graph G . Actually it would be sufficient to prove that this result holds if we allow the two ends of every matched edge to have the same value of σ .
5. During previous investigations, it was noted that, in particular contexts, finding a correct weighting of some graph is similar to finding a decomposition into *irregular subgraphs*. This was notably considered in [2], where the authors study decompositions into *locally irregular subgraphs*, *i.e.* graphs with no adjacent vertices with the same degree, which are related to sum-colouring edge-weightings in regular graphs. This approach was further considered in [4], where, in the context of another directed variant of the 1-2-3 Conjecture, the authors consider decompositions of digraphs into locally irregular subgraphs (for some definition of irregularity in digraphs). We could consider this question as well in the context of our current investigations. Namely, one could define a *locally irregular digraph* as a digraph in which, for every arc \vec{uv} , we have $d^+(u) \neq d^-(v)$, and then study decompositions of digraphs into locally irregular subdigraphs.

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