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# Directed variations of the 1-2-3 Conjecture

Emma Barthe<sup>a</sup>, Julien Bensmail<sup>a</sup>

<sup>a</sup>LIP, UMR 5668 ENS Lyon, CNRS, UCBL, INRIA, Université de Lyon, France

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## Abstract

In this paper, we consider the following question, which stands as a directed analogue of the well-known 1-2-3 Conjecture: Given a digraph  $D$  with no arc  $\vec{uv}$  verifying  $d^+(u) = d^-(v) = 1$ , is it possible to weight the arcs of  $D$  with weights among  $\{1, 2, 3\}$  so that, for every arc  $\vec{uv}$  of  $D$ , the sum of incident weights out-going from  $u$  is different from the sum of incident weights in-coming to  $v$ ? Towards this question, we first verify it for  $D$  belonging to particular classes of digraphs, before then proving its weakening where 3 is replaced by some absolute constant, namely 17. In the same spirit, we investigate a total version of the same question inspired by the 1-2 Conjecture, and prove that even less weights are necessary in this context, namely 10.

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## 1. Introduction

We here focus on *vertex-distinguishing weightings*, a graph theory notion that attracted more and more attention in the last decade. Basically, given an undirected graph  $G$ , the goal is to weight some *elements* of  $G$  so that some *well-identified vertices* of  $G$  get distinguished relatively to some *aggregate* computed from the weighting. As emphasized in the previous sentence, the such problems of correctly weighting a graph are hence made of three main parameters. For any of these variants, the main goal is, given a graph, to deduce the smallest number of weights necessary to obtain a correct distinguishing weighting.

In this paper, we focus on those such problems where edges are part of the weighted elements, and the distinguishing aggregate is the *sum* of weights incident to the vertices. More formally, given an edge-weighting  $w$  of some graph  $G$ , for every vertex  $v$  one may compute<sup>1</sup>

$$\sigma_w(v) := \sum_{u \in N(v)} w(vu),$$

that is, the sum of the weights incident to  $v$ . In case  $w$  is a total-weighting, every vertex  $v$  also has its own weight, which must be involved when computing  $\sigma_w(v)$ , that is

$$\sigma_w(v) := w(v) + \sum_{u \in N(v)} w(vu)$$

in such a situation. In the setting where  $\sigma_w$  is the distinguishing parameter, three main notions have been studied in literature:

1. If *edge-weightings* are considered and *all vertices* of  $G$  must be distinguished by  $\sigma$ , the least number of necessary consecutive edge weights is denoted  $s(G)$  (and is called the *irregularity strength* of  $G$  in literature).

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<sup>1</sup>In case no ambiguity is possible, we will sometimes voluntarily omit the subscript  $w$  (e.g. write  $\sigma$  for  $\sigma_w$ ) to lighten the notations.

2. If *edge-weightings* are considered and only the *adjacent vertices* of  $G$  must be distinguished, the least number of necessary consecutive weights is denoted  $\chi_{\Sigma}^e(G)$ .
3. If *total-weightings* are considered and only the *adjacent vertices* of  $G$  must be distinguished, the least number of necessary consecutive weights is denoted  $\chi_{\Sigma}^t(G)$ .

As we only focus on Items (2) and (3) (that is, on *sum-colouring edge-weighting* and *sum-colouring total-weighting*) in this paper, we will below recall some of their associated backgrounds. For more general details on this wide area (and on the upcoming introductory details), we refer the interested reader to the recent survey by Seamone on this topic [10].

The  $\chi_{\Sigma}^e$  parameter is related to the well-know **1-2-3 Conjecture** raised by Karoński, Łuczak and Thomason [7], which reads as follows (where a *nice graph* refers to a graph with no component isomorphic to  $K_2$ ).

**1-2-3 Conjecture.** *For every nice graph  $G$ , we have  $\chi_{\Sigma}^e(G) \leq 3$ .*

Many constant upper bounds on  $\chi_{\Sigma}^e$  were given towards the 1-2-3 Conjecture, the best one of which being due to Kalkowski, Karoński and Pfender who proved that  $\chi_{\Sigma}^e(G) \leq 5$  whenever  $G$  is nice [5]. Concerning the  $\chi_{\Sigma}^t$  parameter, the following so-called **1-2 Conjecture** was raised by Przybyło and Woźniak [9].

**1-2 Conjecture.** *For every graph  $G$ , we have  $\chi_{\Sigma}^t(G) \leq 2$ .*

Towards the 1-2 Conjecture, the best known result so far is due to Kalkowski [4], who proved that every graph  $G$  verifies  $\chi_{\Sigma}^t(G) \leq 3$ .

There have been a few attempts for bringing the 1-2-3 and 1-2 Conjectures to directed graphs, see *e.g.* [1, 2, 3, 8]. Most of all these different directed versions of the 1-2-3 and 1-2 Conjectures were shown to hold, even under strong additional constraints such as list requirements. This results from the fact that these versions, though seemingly close to the 1-2-3 and 1-2 Conjectures in essence, were based on several behaviours that are not so comparable to the ones we have to deal with when considering the original conjectures. This makes us wonder what should be the directed analogues to the 1-2-3 and 1-2 Conjectures that would mimic their behaviours and inherent hardness the most.

In that spirit, we introduce and study new directed analogues of the 1-2-3 and 1-2 Conjectures, which, as desired, seem seemingly harder than the previously considered directed variants. Our analogue to the 1-2-3 Conjecture is introduced in Section 2, while our analogue to the 1-2 Conjecture is studied in Section 3. We mainly verify these two conjectures for several classes of digraphs, before proving weakenings of these two conjectures. Our conclusions and perspectives for future work are gathered in Section 4.

## 2. A directed 1-2-3 Conjecture

Let  $D$  be a simple digraph, and  $w$  be an arc-weighting of  $D$ . For every vertex  $v$ , one can compute two sums incident to  $v$ , namely

$$\sigma_w^-(v) := \sum_{u \in N^-(v)} w(\overrightarrow{uv}),$$

*i.e.* the incident in-coming sum, and

$$\sigma_w^+(v) := \sum_{u \in N^+(v)} w(\overrightarrow{vu}),$$

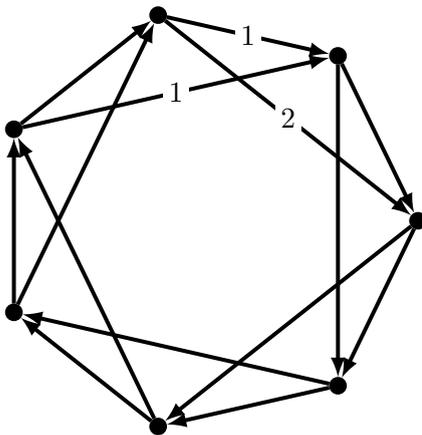


Figure 1: Illustration of the argument why  $\vec{C}_7^2$  admits no sum-colouring 2-arc-weighting. Having a weight (here, 1) on some arc (here, the top-most arc) forces two incident arcs to be weighted differently.

*i.e.* the incident out-going sum. We call  $w$  *sum-colouring* if, for every arc  $\vec{uv}$  of  $D$ , we have

$$\sigma_w^+(u) \neq \sigma_w^-(v).$$

The least number of weights in a sum-colouring  $k$ -arc-weighting (if any) of  $D$  is denoted  $\chi_{\Sigma^\pm}^e(D)$ .

As a very first observation, it is worth mentioning that not all digraphs admit a sum-colouring arc-weighting. To be convinced of this statement, just consider a digraph  $D$  having an arc  $\vec{uv}$  such that  $d^+(u) = d^-(v) = 1$ . Then, in such a situation, no matter what weight  $x$  is assigned to  $\vec{uv}$ , clearly we will get  $\sigma^+(u) = \sigma^-(v) = x$  – so there is no hope to find a sum-colouring arc-weighting. However, one can easily convince himself that if  $D$  is *nice*, in the sense that it does not admit this configuration, then  $D$  admits a sum-colouring arc-weighting (just consider large enough arc weights).

Experimentations on some nice digraphs suggest that the following conjecture, which stands as a direct directed analogue of the 1-2-3 Conjecture, could be true.

**Directed 1-2-3 Conjecture.** *For every nice digraph  $D$ , we have  $\chi_{\Sigma^\pm}^e(D) \leq 3$ .*

It is worth mentioning that the value 3 in our conjecture would be best possible as there exist nice digraphs admitting no sum-colouring 2-arc-weighting. One easy family of digraphs whose  $\chi_{\Sigma^\pm}^e$  is 3 is squares of odd cycles in which the two underlying cycles are directed to form two circuits (see Figure 1). Assume indeed we use weights 1 and 2 only on such a digraph. Such a digraph is 2-regular and weighting, say, 1 an arc, say,  $\vec{v_1v_2}$  forces the weights on the second arc out-going from  $v_1$  and the second arc in-coming to  $v_2$  to be different (so that  $\sigma^+(v_1) \neq \sigma^-(v_2)$ ). Repeating this argument until all arcs are weighted following successive deductions, eventually we easily reach a contradiction – so such a digraph can only be weighted with at least three weights. In upcoming Section 2.1, we will point out that actually many other such digraphs exist.

This section is organized as follows. First, we give support to the Directed 1-2-3 Conjecture by showing, in Section 2.1, that many digraphs verify it. These digraphs range from acyclic digraphs and digraphs with particular colouring properties. Then, in Section 2.2, we make a first step towards the Directed 1-2-3 Conjecture by showing that there is a constant number of weights, namely 17, that suffice to weight all nice digraphs.

### 2.1. Support to the Directed 1-2-3 Conjecture

We start by showing that the Directed 1-2-3 Conjecture holds for acyclic digraphs, and digraphs with particular colouring properties.

#### Acyclic digraphs

We here consider *acyclic digraphs*, i.e. digraphs with no directed circuit. More precisely, we prove that acyclic digraphs agree with the Directed 1-2-3 Conjecture, see upcoming Theorem 2. Using three weights on nice acyclic digraphs is actually best possible. To illustrate this, and as going to be explained in the next section, just take any nice bipartite graph  $G$  with  $\chi_{\Sigma}^e(G) = 3$  and direct all arcs towards the same part. In doing so, we obtain an acyclic oriented bipartite graph  $\vec{G}$  with  $\chi_{\Sigma_{\pm}}^e(\vec{G}) = 3$ . We however show that, in particular situations, acyclic digraphs may have their  $\chi_{\Sigma_{\pm}}^e$  being at most 2.

**Theorem 1.** *For every nice acyclic tournament  $\vec{T}$ , we have  $\chi_{\Sigma_{\pm}}^e(\vec{T}) \leq 2$ .*

*Proof.* Since  $\vec{T}$  is acyclic, it admits a vertex ordering  $v_1, v_2, \dots, v_n$  such that for all  $i$  and  $j$  with  $i < j$ , the arc between  $v_i$  and  $v_j$  is directed “to the right”, that is towards  $j$ . We produce a 2-arc-weighting  $w$  of  $\vec{T}$  with (perhaps almost) the following properties:

1. For all  $i \in \{1, 2, \dots, n-1\}$ , we have  $\sigma_w^+(v_i)$  odd.
2. For all  $i \in \{2, 3, \dots, n\}$ , we have  $\sigma_w^-(v_i)$  even.

Assuming  $w$  satisfies Properties (1) and (2) above, clearly it is sum-colouring.

The main idea to obtain such a  $w$  consists in picking pairs of vertices  $v_i$  and  $v_j$ , one of their common out-neighbours  $v_k$ , set  $w(\overrightarrow{v_i v_j}) = w(\overrightarrow{v_j v_k}) = 1$ , and set all other arcs out-going from  $v_i$  and  $v_j$  to 2. In doing so, note that Property (1) will be met for  $v_i$  and  $v_j$ , while the parity of  $\sigma_w^-(v_k)$  will remain unchanged.

Assume first  $n$  is odd. Starting from  $v_1$ , consider disjoint pairs of consecutive vertices until  $v_{n-1}$  is reached (that is,  $\{v_1, v_2\}$ , then  $\{v_3, v_4\}$ , and so on). For every such pair  $\{v_i, v_{i+1}\}$ , set  $w(\overrightarrow{v_i v_n}) = w(\overrightarrow{v_{i+1} v_n}) = 1$ . At the end of the process, assign 2 to all non-weighted arcs. Clearly  $w$  directly respects Properties (1) and (2) above, so it is sum-colouring.

If  $n$  is even, then we proceed as follows. We repeat the same procedure as in the previous case but with the pairs  $\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{i-3}, v_{i-2}\}$  only, and with choosing the arcs towards  $v_{n-1}$  to be weighted 1. Once the 1’s are attributed, put 2 on all remaining arcs, including  $\overrightarrow{v_{n-1} v_n}$ . Note that Property (1) is violated by  $\sigma_w^+(v_{n-1})$  only, which is equal exactly to 2. But since  $n > 2$  (since otherwise  $\vec{T}$  would not be nice), we have  $\sigma_w^-(v_n) > 2 = \sigma_w^+(v_{n-1})$ . So  $w$  is sum-colouring.  $\square$

Modifying the proof of Theorem 1, we now prove that acyclic digraphs all agree with the Directed 1-2-3 Conjecture.

**Theorem 2.** *For every nice acyclic digraph  $D$ , we have  $\chi_{\Sigma_{\pm}}^e(D) \leq 3$ .*

*Proof.* We describe below how to deduce a sum-colouring 3-arc-weighting  $w$  of  $D$ . The process to obtain  $w$  is somewhat similar to that described in the proof of Theorem 1. Start from some ordering  $v_1, v_2, \dots, v_n$  of the vertices of  $D$  such that all arcs are directed to the right. Now iteratively construct one subset  $X$  of vertices and one subset  $F$  of arcs as follows. As long as  $D - X$  has two vertices  $v_i$  and  $v_j$  having a common out-neighbour  $v_k$  in the original  $D$ , add  $\overrightarrow{v_i v_k}$  and  $\overrightarrow{v_j v_k}$  to  $F$ , and add  $v_i$  and  $v_j$  to  $X$ . When the process ends, clearly the following properties are fulfilled by  $X$  and  $F$ :

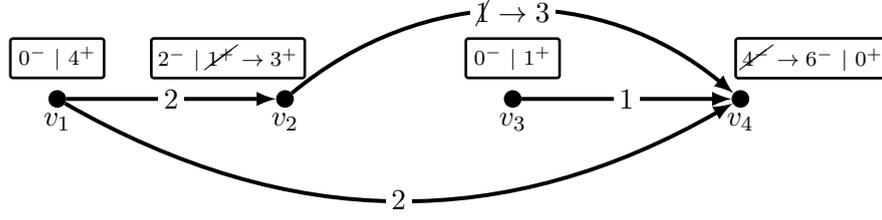


Figure 2: Illustration of the modification scheme described in the proof of Theorem 2. The original arc-weighting  $w$  is not sum-colouring since  $\sigma_w^+(v_1) = \sigma_w^-(v_4)$ . This conflict is solved by changing the weight on  $\overrightarrow{v_2v_4}$  to 3.

1. Every vertex of  $X$  is incident to exactly one forward arc in  $F$ .
2. Every vertex of  $D$  has an even number of incident in-coming arcs in  $F$ .
3. Every two vertices in  $V(D) \setminus X$  have no common out-neighbours.

Now start with  $w$  assigning 1 to all arcs of  $F$ , and 2 to all other arcs of  $D$ . By Property (2), so far  $\sigma_w^-(v_i)$  is even for every  $v_i$  of  $D$ . By Property (1), clearly  $\sigma_w^+(v_i)$  is odd for every  $v_i$  in  $X$ . However  $\sigma_w^+(v_i)$  is even for every  $v_i$  in  $V(D) \setminus X$ , so if this set is not empty, then  $w$  may not be sum-colouring – but these are the only possible conflicts.

In order to fix these conflicts, we consider the vertices of  $D$  in increasing order of their indexes, and possibly modify  $w$  locally (see Figure 2). Assume  $v_i$  is, according to the ordering, the first vertex having an out-neighbour  $v_j$  such that  $\sigma_w^+(v_i) = \sigma_w^-(v_j)$ . Recall that  $v_i$  belongs to  $V(D) \setminus X$ , and hence  $w(\overrightarrow{v_iv_j}) = 2$ . If  $v_j$  is the only out-neighbour of  $v_i$ , then, because  $D$  is nice,  $v_j$  necessarily has in-neighbours different from  $v_i$  – so  $\sigma_w^-(v_j) > 2 = \sigma_w^+(v_i)$  and there would actually be no problem with  $\overrightarrow{v_iv_j}$ . So assume  $v_i$  has  $d$  out-neighbours  $u_1, u_2, \dots, u_d$  to the right, one of which is  $v_j$ . Since Property (1) is violated at  $v_i$ , all arcs from  $v_i$  to these  $d$  out-neighbours are weighted 2 by  $w$ . Since  $d \geq 2$ , we note that if  $\overrightarrow{v_iv_j}$  is a conflicting arc, meaning that  $\sigma_w^+(v_i) = \sigma_w^-(v_j)$ , then  $v_j$  necessarily has other incident in-coming arcs. Clearly all these arcs are weighted 1 by  $w$  since otherwise Property (3) would be violated. Now just turning one of these weights from 1 to 3 solves the conflict between  $v_i$  and  $v_j$  since only  $\sigma_w^-(v_j)$  is altered. Besides, it should be clear that none of Properties (1), (2) and (3) can be violated after the modification. Repeating the same argument for all  $u_j$ 's which are in conflict with  $v_i$ , we eventually fix all conflicts involving  $v_i$ .

The important thing to point out is that locally modifying  $w$  in this way cannot mess up with vertices which were considered before  $v_i$ . If this were the case, rephrased differently, this would mean that we *e.g.* locally modify, say,  $w(\overrightarrow{v_iv_j})$  to 3, making  $\sigma_w^-(v_j)$  rise by 2, but there is an arc  $\overrightarrow{v_kv_j}$  with  $k < i$  for which  $\sigma_w^+(v_k)$  is now equal to  $\sigma_w^-(v_j)$ . But this is impossible, since it would mean that  $v_i, v_k \in V(D) \setminus X$ , and  $v_i$  and  $v_k$  have  $v_j$  as a common out-neighbour, which would violate Property (3). Another important thing to keep in mind is that, by the described procedure, only the incident forward arcs may have their weight risen by 2. This implies that a situation where we cannot add 2 to some arc weight because it was already set to 3 before cannot occur. So the modification procedure is always applicable and, hence repeating it, we can make sure that  $w$  is sum-colouring.  $\square$

#### *Digraphs with particular colouring properties*

During previous investigations on related edge-weighting notions, it was often noted that starting from a particular vertex-partition is a good starting point in order to get

a distinguishing edge-weighting. In particular, having a *proper vertex-colouring*  $V_1, \dots, V_k$  of some graph  $G$ , that is a partition of  $V(G)$  into stable sets, is quite convenient, as it defines classes of vertices which can be assigned the same distinguishing aggregate when designing a particular edge-weighting. So knowing that  $G$  is  $k$ -colourable, *i.e.* admits a proper  $k$ -vertex-colouring, is a good hint to deduce an edge-weighting. It then appears natural to wonder how this strategy applies in our context.

Unfortunately things appear to be fairly less handleable for sum-colouring arc-weighting of digraphs, mainly because it seems generally hard to guarantee both the incident incoming and out-going sums of some vertex to have some specific values. But this can however be done in some contexts.

We start off by considering *bipartite digraphs*, *i.e.* those digraphs whose underlying graph is 2-colourable. We first need to recall the following result, due to Karoński, Łuczak and Thomason [7].

**Theorem 3.** *For every nice bipartite graph  $G$ , we have  $\chi_{\Sigma}^e(G) \leq 3$ .*

We now prove our result on bipartite digraphs.

**Theorem 4.** *For every nice bipartite digraph  $D$ , we have  $\chi_{\Sigma\pm}^e(D) \leq 3$ .*

*Proof.* Assume  $A \cup B$  is the bipartition of  $V(\text{und}(D))$ , where  $\text{und}(D)$  denotes the underlying undirected graph of  $D$ . Let  $D_{\uparrow\uparrow}$  and  $D_{\downarrow\downarrow}$  be the subdigraphs of  $D$  induced the arcs going from, say,  $A$  to  $B$ , and from  $B$  to  $A$ , respectively. We note that both  $D_{\uparrow\uparrow}$  and  $D_{\downarrow\downarrow}$  are nice, since otherwise  $D$  itself would not be nice. So both  $D_{\uparrow\uparrow}$  and  $D_{\downarrow\downarrow}$  admit a sum-colouring arc-weighting. Of course, since  $D$  is bipartite, so are  $D_{\uparrow\uparrow}$  and  $D_{\downarrow\downarrow}$ .

Let  $w$  be an arc-weighting of  $D$  obtained as follows. Consider  $D_{\uparrow\uparrow}$  first. Since  $\text{und}(D_{\uparrow\uparrow})$  is bipartite, it admits a sum-colouring 3-edge-weighting  $w'$  (according to Theorem 3) which we directly report to  $D_{\uparrow\uparrow}$  (*i.e.* if  $uv$  and  $\vec{uv}$  are corresponding edge and arc of  $\text{und}(D_{\uparrow\uparrow})$  and  $D_{\uparrow\uparrow}$ , we give them the same weight, that is  $w'(uv)$ ). Note that if  $\vec{uv}$  is an arc of  $D_{\uparrow\uparrow}$  then, because  $\sigma_{w'}(u) \neq \sigma_{w'}(v)$ , and  $u$  has in-degree 0 and  $v$  has out-degree 0 in  $D_{\uparrow\uparrow}$ , clearly we have  $\sigma_{w'}^+(u) \neq \sigma_{w'}^-(v)$  in  $D_{\uparrow\uparrow}$ . So  $w'$  is sum-colouring in  $D_{\uparrow\uparrow}$ . Analogously,  $D_{\downarrow\downarrow}$  admits such an arc-weighting  $w''$  as well, according to the same arguments.

To obtain  $w$ , it now suffices to directly report the weights by  $w'$  and  $w''$  of  $D_{\uparrow\uparrow}$  and  $D_{\downarrow\downarrow}$ , respectively, to  $D$  (that is, if  $\vec{uv}$  belongs to  $D_{\uparrow\uparrow}$ , just set  $w(\vec{uv}) = w'(\vec{uv})$ , or  $w(\vec{uv}) = w''(\vec{uv})$  otherwise). Then  $w$  retains its property of being sum-colouring, since, for every vertex  $v$  in  $A$  (resp.  $B$ ), the in-degree of  $v$  in  $D$  is exactly its in-degree in  $D_{\downarrow\downarrow}$  (resp.  $D_{\uparrow\uparrow}$ ), while the out-degree of  $v$  in  $D$  is exactly its out-degree in  $D_{\uparrow\uparrow}$  (resp.  $D_{\downarrow\downarrow}$ ). So if  $w$  were not sum-colouring, then one of  $w'$  and  $w''$  would not be sum-colouring.  $\square$

Following the proof of Theorem 4, we get that, for some nice bipartite digraph  $D$ , we have

$$\chi_{\Sigma\pm}^e(D) = \max \{ \chi_{\Sigma}^e(\text{und}(D_{\downarrow\downarrow})), \chi_{\Sigma}^e(\text{und}(D_{\uparrow\uparrow})) \},$$

where  $D_{\uparrow\uparrow}$  and  $D_{\downarrow\downarrow}$  have the same meaning as in the previous proof. This in particular means that if one of  $D_{\uparrow\uparrow}$  and  $D_{\downarrow\downarrow}$  has its value of  $\chi_{\Sigma}^e$  being equal to 3, then  $\chi_{\Sigma\pm}^e(D) = 3$ . Since many bipartite graphs have their value of  $\chi_{\Sigma}^e$  being exactly 3 (refer to [10] for a summary of all known such families), we get that a wide bunch of nice bipartite digraphs have their  $\chi_{\Sigma\pm}^e$  being exactly 3.

We note also that the proof of Theorem 4 gives something more general. Namely, if a nice digraph  $D$  has a partition  $V_1, \dots, V_k$  of its vertex set such that:

1. every  $V_i$  is a stable set,

2. all vertices in some  $V_i$  have all of their in-neighbours in a same  $V_j$ , and
3. all vertices in some  $V_i$  have all of their out-neighbours in a same  $V_j$ ,

then a sum-colouring 3-arc-weighting of  $D$  can be obtained by independently edge-weighting all of its underlying nice bipartite graphs of the form  $\text{und}(D[V_i \cup V_j])$  for every  $i \neq j$  (some of which may have no edge). In what follows, we call such a partition of  $V(D)$  a *circular vertex-colouring* of  $D$ , and, from the existence of such a vertex-colouring, we call  $D$  *circular*. Then we can formally state the following.

**Corollary 1.** *For every nice circular digraph  $D$ , we have  $\chi_{\Sigma^\pm}^e(D) \leq 3$ .*

When having a proper vertex-colouring  $V_1, \dots, V_k$  of  $\text{und}(D)$ , we note that, in between two different parts  $V_i$  and  $V_j$ , the arcs can be in any direction. Furthermore, every two parts  $V_i$  and  $V_j$  must be joined by arcs since contracting the  $V_i$ 's in  $\text{und}(D)$  should result in a complete graph. Mainly for these reasons, extending Theorem 4 to digraphs whose underlying graph is not 2-colourable does not seem to be a viable direction.

So that the range of considered digraphs is more important, one may, instead of starting from some proper vertex-colouring of the underlying graph of some digraph  $D$ , rather consider an *oriented colouring* of  $D$ , that is a proper vertex-colouring  $V_1, \dots, V_k$  in which all arcs in between two distinct parts  $V_i$  and  $V_j$  are towards the same direction. The least number of colours in an oriented vertex-colouring of  $D$  is denoted by  $\chi_o(D)$ , and is sometimes called the *oriented chromatic number*<sup>2</sup> of  $D$ . Note that digraphs with digons do not admit oriented vertex-colouring – so, from now on, we will focus on oriented graphs only.

It was proved in [7] that the 1-2-3 Conjecture holds for 3-colourable graphs. In our context, a plausible equivalent statement could hence be to prove that the Directed 1-2-3 Conjecture holds for oriented graphs whose oriented chromatic number is at most 3. Theorem 4 and Corollary 1 are already almost sufficient to prove such a claim – we just additionally need the following result.

**Lemma 1.** *Let  $\vec{G}$  be a nice oriented graph admitting a proper vertex-colouring  $V_1, V_2, V_3$  such that all arcs incident to  $V_1$  are directed away from  $V_1$ , while all arcs incident to  $V_3$  are directed towards  $V_3$ . Then  $\chi_{\Sigma^\pm}^e(\vec{G}) \leq 3$ .*

*Proof.* Such digraphs are clearly acyclic, so the result follows from Theorem 2. □

We can now prove our result on oriented graphs with oriented chromatic number at most 3.

**Theorem 5.** *For every nice oriented graph  $\vec{G}$  verifying  $\chi_o(\vec{G}) \leq 3$ , we have  $\chi_{\Sigma^\pm}^e(\vec{G}) \leq 3$ .*

*Proof.* If  $\chi_o(\vec{G}) = 2$ , then the result follows from Theorem 4 since  $\text{und}(\vec{G})$  is hence bipartite. If  $\chi_o(\vec{G}) = 3$ , then  $\vec{G}$  admits an oriented 3-vertex-colouring  $V_1, V_2, V_3$ , which is either circular, or as described in the statement of Lemma 1. In these cases, the result follows from Corollary 1 and Lemma 1, respectively. □

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<sup>2</sup>Refer to [11] for a comprehensive survey on this parameter.

## 2.2. One first step towards the Directed 1-2-3 Conjecture

In this section, we prove a first constant upper bound on  $\chi_{\Sigma^\pm}^e$  holding for all nice digraphs. Namely, we prove that  $\chi_{\Sigma^\pm}^e(D) \leq 17$  for every nice digraph  $D$ . Our proof of this statement is inspired from a weighting algorithm originally designed by Kalkowski [4], from which are due the best known general upper bound on the irregularity strength of graphs [6], on  $\chi_{\Sigma}^e$  [5], and on  $\chi_{\Sigma}^t$  [4].

The main ideas behind Kalkowski's algorithm are the following, assuming an undirected graph  $G$  is weighted and the distinguishing aggregate is  $\sigma$ . We start from all edges of  $G$  being weighted in a particular way fulfilling some properties described below. The vertices of  $G$  are then processed linearly from first to last following some ordering  $v_1, \dots, v_n$ , with never reconsidering an already treated vertex. Whenever a new vertex  $v_i$  is considered during the algorithm's course, a set  $\Phi(v_i)$  of somewhat *safe possible sums* is assigned to  $v_i$ , meaning possible values as  $\sigma(v_i)$  for which there cannot be any conflict with the previously treated vertices (basically because  $\Phi(v_i) \cap \Phi(v_j) = \emptyset$  for every  $v_j$  neighbouring  $v_i$  with  $j < i$ ). So basically, in order to obtain a sum-colouring edge-weighting of  $G$ , the only things the algorithm needs to ensure is, when dealing with  $v_i$ :

1. at every moment of the algorithm, we have  $\sigma(v_j) \in \Phi(v_j)$  for every  $j < i$ , and
2.  $\Phi(v_i)$  can be defined without spoiling Condition (1) nor the next algorithm's steps.

Modifying the main lines of Kalkowski's ideas, we prove the following result.

**Theorem 6.** *For every nice digraph  $D$ , we have  $\chi_{\Sigma^\pm}^e(D) \leq 17$ .*

*Proof.* We may assume that  $D$  is connected. Let  $v_1, v_2, \dots, v_n$  denote the vertices of  $D$  following an arbitrary ordering. This ordering yields a characterization of the vertices of  $D$  into four types. First, we say that a vertex  $v_i$  is a *-source* (resp. *+source*) if  $v_i$  has a forward in- (resp. out-) neighbour, namely if there is a  $j > i$  such that  $\overrightarrow{v_j v_i}$  (resp.  $\overrightarrow{v_i v_j}$ ) is an arc. In case  $v_i$  is not a *-source* (resp. *+source*), we call  $v_i$  a *-sink* (resp. *+sink*), meaning  $v_i$  does not have any such forward neighbour. According to these definitions, every  $v_i$  is either a *-source* or a *-sink*, and either a *+source* or a *+sink*.

In order to deduce a sum-colouring 17-arc-weighting  $w$  of  $D$ , following Kalkowski's ideas, we will process the vertices linearly, *i.e.* from  $v_1$  to  $v_n$ . Whenever considering a *-source* (resp. *+source*) vertex  $v_i$ , we will associate one couple  $\Phi^-(v_i) = \{\phi^-(v_i), \phi^-(v_i) + 6\}$  (resp.  $\Phi^+(v_i) = \{\phi^+(v_i), \phi^+(v_i) + 6\}$ ) to  $v_i$  such that  $\Phi^-(v_i) \cap \Phi^+(v_j) = \emptyset$  (resp.  $\Phi^+(v_i) \cap \Phi^-(v_j) = \emptyset$ ) for every  $j < i$  such that  $\overrightarrow{v_j v_i}$  (resp.  $\overrightarrow{v_i v_j}$ ) is a backward arc. Furthermore, we will locally modify  $w$  to make sure that  $\sigma_w^-(v_i) \in \Phi^-(v_i)$  (resp.  $\sigma_w^+(v_i) \in \Phi^+(v_i)$ ) at any moment of the algorithm for every *-source* (resp. *+source*)  $v_i$ . Note that if these conditions are met once the algorithm is finished, then the only possible conflicts making  $w$  not sum-colouring involve *-sinks* and *+sinks*. At the end of the proof, we will explain how to handle these vertices without spoiling  $w$ .

Start from  $w$  assigning 9 to all arcs of  $D$ . We then have  $\sigma^-(v_1) = 9d^-(v_1)$  and  $\sigma^+(v_1) = 9d^+(v_1)$ . So that the weighting algorithm can run the next steps, unless  $d^-(v_1) = 0$  or  $d^+(v_1) = 0$ , respectively, we set  $\phi^-(v_1) = 9d^-(v_1)$  and  $\phi^+(v_1) = 9d^+(v_1)$ , as well as  $\Phi^-(v_1) = \{\phi^-(v_1), \phi^-(v_1) + 6\}$  and  $\Phi^+(v_1) = \{\phi^+(v_1), \phi^+(v_1) + 6\}$ . Note that if  $v_1$  is a *-sink* (resp. *+sink*) then it has in-degree (resp. out-degree) 0 and its value of  $\sigma_w^-$  (resp.  $\sigma_w^+$ ) cannot cause any trouble later.

Now that these initial values have been fixed, we can describe the general behaviour of the algorithm. Prior to that, let us formally state the properties we want the algorithm to preserve at any step, assuming we are considering vertex  $v_k$  with  $k \geq 2$ :

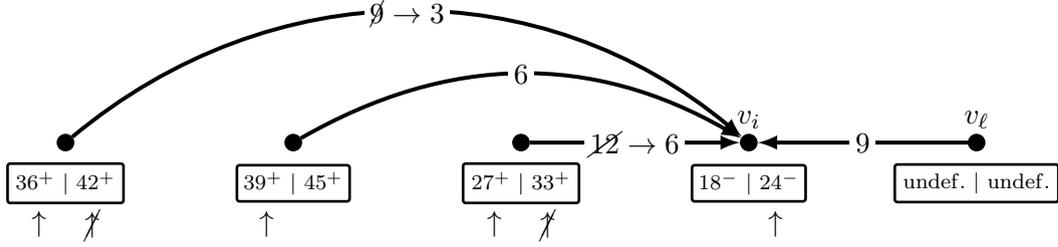


Figure 3: Illustration of the first part of the proof of Theorem 6. Valid moves are performed around  $v_i$  so that  $\Phi^-(v_i)$  does not intersect the incident already defined  $\Phi^+$ 's, and this with keeping all  $\sigma^+$ 's in their corresponding  $\Phi^+$ .

1. For all  $i < k$ , if  $v_i$  is a  $-$ source (resp.  $+$ source), then  $\sigma_w^-(v_i) \in \Phi^-(v_i)$  (resp.  $\sigma_w^+(v_i) \in \Phi^+(v_i)$ ).
2. For all  $i > k$ , the possible arcs joining  $v_k$  and  $v_i$  are weighted 9 (whatever their direction is).
3. For all  $i < k$ , the possible arcs joining  $v_i$  and  $v_k$  have weight among  $\{6, 9, 12\}$ . Besides, if any of these arcs, say,  $\overrightarrow{v_k v_i}$  (resp.  $\overrightarrow{v_i v_k}$ ) is weighted 6, then  $\sigma_w^-(v_i) = \phi^-(v_i)$  (resp.  $\sigma_w^+(v_i) = \phi^+(v_i)$ ). Conversely, if  $\overrightarrow{v_k v_i}$  (resp.  $\overrightarrow{v_i v_k}$ ) is weighted 12, then  $\sigma_w^-(v_i) = \phi^-(v_i) + 6$  (resp.  $\sigma_w^+(v_i) = \phi^+(v_i) + 6$ ).
4. Every arc is weighted with a weight among  $\{3, 6, 9, 12, 15\}$ .

Property (3) in particular implies that, when considering vertex  $v_k$  and one of its backward neighbours, say  $v_i$ , there are two possibilities concerning the weight on the arc between  $v_k$  and  $v_i$  that will not spoil Property (1), namely leave this weight unchanged, or either subtract or add 6 to it. Property (2) just means that when starting considering  $v_k$ , all weights on its incident forward arcs are neutral, meaning that we can freely either subtract or add something to it, that is with remaining in  $\{1, 2, \dots, 17\}$ . The only reason why we impose Property (4) is that we want all  $\phi(v_i)$ 's (and so all  $(\phi(v_i) + 6)$ 's) to be multiple of 3 once the algorithm is finished – this will then be useful to deal with  $-$ sinks and  $+$ sinks.

Assume we are dealing with vertex  $v_k$  with  $k \geq 2$  and that, so far, all of Properties (1) to (4) are met. We describe the algorithm's behaviour on the incident arcs in-coming to  $v_k$  only – but the incident out-going arcs have to be independently treated similarly. First, if  $v_k$  is a  $-$ sink, then we skip this vertex and consider  $v_{k+1}$ . Clearly, this does not alter the four properties above. Now suppose  $v_k$  is a  $-$ source. For every arc  $\overrightarrow{v_i v_k}$  with  $i < k$ , we note that  $v_i$  is a  $+$ source, meaning that either subtracting or adding 6 to  $w(\overrightarrow{v_i v_k})$  makes  $\sigma_w^+(v_i)$  stay in  $\Phi^+(v_i)$ , and is hence a “valid move”. Assuming  $v_k$  has  $d$  such backward in-neighbours, this means that, untouched or performing valid moves on the  $d$  incident backward in-coming arc weights, we can generate  $d + 1$  possible values as  $\sigma_w^-(v_k)$  (this can easily be proved by induction on  $d$ ). Furthermore, since  $v_k$  is a  $-$ source, by definition it has at least one forward in-neighbour. Suppose  $v_\ell$  with  $\ell > k$  is the such in-neighbour with the lowest index. Since  $\ell > k$ , vertex  $v_\ell$  has not been treated yet, so  $w(\overrightarrow{v_\ell v_k}) = 9$ . Note that by also allowing to subtract or add 3 to this weight, the range of possible values as  $\sigma_w^-(v_k)$  rises to  $2d + 3$  – again, this can be proved through an easy induction argument. In what follows, we consider this weight modification as a valid move as well.

Now the conclusion is that, performing only valid modifications on the incident in-coming weights at  $v_k$ , we can fix  $\Phi^-(v_k) = \{\phi^-(v_k), \phi^-(v_k) + 6\}$  so that:

1.  $\Phi^-(v_k) \cap \Phi^+(v_i) = \emptyset$  for every  $i < k$  such that  $\overrightarrow{v_i v_k}$  is an arc,
2.  $\sigma_w^-(v_k) \in \Phi^-(v_k)$ , and
3. if  $w(\overrightarrow{v_\ell v_k}) = 6$  (resp. 12), then  $\sigma_w^-(v_k) = \phi^-(v_k)$  (resp.  $\phi^-(v_k) + 6$ ).

Figure 3 depicts an example of how to perform valid moves in a particular situation. It should be clear that all of Properties (1) to (4) are fulfilled from the point of view of the vertices succeeding  $v_k$ . So the algorithm can go on, and all vertices are eventually successfully treated.

We now have to handle  $-$ sinks (and  $+$ sinks) to make sure that their value of  $\sigma_w^-$  (resp.  $\sigma_w^+$ ) is not equal to the value of  $\sigma_w^+$  (resp.  $\sigma_w^-$ ) of some of their in-neighbours (resp. out-neighbours). We here describe how to deal with  $-$ sinks, but  $+$ sinks can be treated in a very same fashion. The tricky part here is that several  $-$ sinks can have common in-neighbours, so locally modifying  $w$  to fix the possible conflicts has to be done at one shot. The main argument we are going to use is that, so far, all non-sink vertices have their value of  $\sigma_w^-$  and  $\sigma_w^+$  being multiple of 3 according to Property (4). This essentially implies that, for any, say,  $+$ source  $v_i$ , we can safely modify  $w$  so that  $\sigma_w^+(v_i)$  increases by 1 or 2 since the resulting values have not been assigned to sources so far. If all  $\sigma_w^+$ 's increase by at most 2, and there is no conflicting involving sinks, we will eventually make  $w$  being sum-colouring.

In the sequel, we will make use of the following result on sum-colouring edge-weighting of bipartite graphs.

**Lemma 2.** *Every nice bipartite graph  $G = (A, B)$  admits a sum-colouring edge-weighting  $w$  such that*

1.  $w$  assigns weights among  $\{-2, -1, 0, 1, 2\}$ ,
2.  $\sigma_w(a) \equiv 2 \pmod{3}$  for every  $a \in A$ ,
3.  $\sigma_w(b) \equiv 0, 1 \pmod{3}$  for every  $b \in B$ ,
4.  $\sigma_w(a) \in \{0, 1, 2\}$  for every  $a \in A$ .

*Proof.* We may assume  $G$  is connected. Let us set  $A := \{a_1, a_2, \dots, a_x\}$ , where we assume  $d(a_x) \geq 2$ . We start from  $w$  assigning 0 to all edges of  $G$ . We first modify  $w$  so that Property (2) holds for every  $a \in \{a_1, a_2, \dots, a_{x-1}\}$ . Consider every such  $a$  consecutively. If already  $\sigma_w(a) \equiv 2 \pmod{3}$ , then we skip this vertex. Otherwise, consider an  $(a, a_x)$ -path  $P$  in  $G$ , which exists since  $G$  is connected. Let  $k \in \{0, 1\}$  be the value for which  $\sigma_w(a) \equiv k \pmod{3}$ . Note that, by adding  $2 - k$ , then  $1 + k$ , then  $2 - k$ , etc. to the weights on the successive edges of  $P$  starting from the one incident to  $a$ , only the remainder modulo 3 of  $\sigma_w(a)$  and  $\sigma_w(a_x)$  get modified. Furthermore, in doing so we obtain  $\sigma_w(a) \equiv 2 \pmod{3}$  as required, while we have no control on  $\sigma_w(a_x)$ . Hence repeating this argument for every  $a \in \{a_1, a_2, \dots, a_{x-1}\}$ , we eventually get all of these vertices meeting Property (2). Note also that the edge weights may not belong to  $\{-2, -1, 0, 1, 2\}$  after this step (so Property (1) may not hold), but we will handle this later. Last, we also have  $\sigma_w(b) \equiv 0 \pmod{3}$  for every  $b \in B$ .

Now, in case  $\sigma_w(a_x) \equiv 0 \pmod{3}$ , we choose two edges incident to  $a_x$ , say  $a_x b_1$  and  $a_x b_2$ , and add 1 to both  $w(a_x b_1)$  and  $w(a_x b_2)$ . In doing so, we get  $\sigma_w(a_x) \equiv 2 \pmod{3}$  as desired. Furthermore, we have  $\sigma_w(b_1), \sigma_w(b_2) \equiv 1 \pmod{3}$ . So far, clearly Properties (2) and (3) are met. A similar conclusion can be obtained if  $\sigma_w(a_x) \equiv 1 \pmod{3}$  (but with using, say,  $a_x b_1$  only).

We now have to make Properties (1) and (4) hold without spoiling the other two properties. So that Property (1) holds, we consider every edge  $ab$  of  $G$ . Assuming  $k$  is the remainder of  $w(ab)$  modulo 3, note that by just replacing  $w(ab)$  by  $k$ , we do not spoil Property (2) because of  $a$ , nor Property (3) because of  $b$ . Repeating this procedure for every edge, we eventually make  $w$  fulfil Property (1). To now make Property (4) hold, we consider any vertex  $a$  whose  $\sigma_w$  is out of  $\{0, 1, 2\}$ . Since all edges incident to  $a$  are weighted with weights among  $\{0, 1, 2\}$  and  $\sigma_w(a) > 2$ , there is some edge  $ab$  incident to  $a$  with weight 1 or 2. Note that by subtracting 3 to  $w(ab)$ , we cannot spoil any of Properties (1) to (3), while we make  $\sigma_w(a)$  decrease by 3. Repeating this argument as long as necessary, we eventually make  $a$  fulfil Property (4). Since this can be repeated for every  $a \in A$ , we eventually get all  $a$ 's satisfying Property (4) – and this without spoiling the other properties.  $\square$

Back to the proof of Theorem 6, we now consider “equivalent”  $-$ sinks (with respect to the explanations above) and handle them simultaneously. So the procedure below has to be performed successively for all equivalent  $-$ sinks.

We first make the notion of equivalent  $-$ sinks clear. Consider a  $-$ sink  $v_k$ . The  $-$ sinks of  $D$  we consider equivalent to  $v_k$  are those among

$$S := \{v_k\} \cup \{v_i \in V(D) : v_i \text{ is a } -\text{sink, and } v_i \text{ and } v_k \text{ have a common in-neighbour}\}.$$

From  $S$ , one can deduce the set  $N$  of vertices forbidding values as  $\sigma_w^-$ 's of the  $-$ sinks in  $S$ . Namely

$$N := \{v_i \in V(D) : v_i \text{ has an out-neighbour in } S\}.$$

We note in particular that it may be the case that  $S \cap N \neq \emptyset$ , since two  $-$ sinks  $v_i$  and  $v_j$  with, say,  $i < j$  may be adjacent (but then we know that the arc joining  $v_i$  and  $v_j$  is directed from  $v_i$  to  $v_j - v_i$  would not be a  $-$ sink otherwise).

First assume  $|N| = |S| = 1$ , and let  $N = \{v_i\}$  and  $S = \{v_j\}$ . Since  $D$  is nice, necessarily there exists  $v_k \neq v_j$  such that  $\overrightarrow{v_i v_k}$ . It then follows that

$$\sigma_w^+(v_i) \geq w(\overrightarrow{v_i v_j}) + w(\overrightarrow{v_i v_k}) > w(\overrightarrow{v_i v_j}) = \sigma_w^-(v_j),$$

so there cannot be any conflict involving the ends of the arc  $\overrightarrow{v_i v_j}$ . Now assume  $|S| = 1$  and  $|N| > 1$ , let  $S = \{v_j\}$ , and let  $v_{i_1}$  and  $v_{i_2}$  be two distinct vertices in  $N$ . As a modification of  $w$ , just add 1 to both  $w(\overrightarrow{v_{i_1} v_j})$  and  $w(\overrightarrow{v_{i_2} v_j})$ . In doing so, we now have  $\sigma_w^+(v_{i_1}), \sigma_w^+(v_{i_2}) \equiv 1 \pmod{3}$ , while  $\sigma_w^-(v_j) \equiv 2 \pmod{3}$  – so all in-neighbours of  $v_j$  have their  $\sigma_w^+$  having remainder 0 or 1 modulo 3, hence different from  $\sigma_w^-(v_j)$ . Furthermore, all of the  $+$ -sources in-neighbouring  $v_j$  have had their  $\sigma_w^+$  risen by at most 2, as required.

We now use Lemma 2 to deal with the general case. Consider the nice undirected bipartite graph  $G_{v_i} = (A, B)$  obtained as follows. Every vertex in  $S \setminus N$  is added to  $B$  directly, while every vertex of  $N \setminus S$  is added to  $A$  directly. Now, for every vertex  $v_i \in N \cap S$ , we want to model both of the facts that  $v_i$  neighbours a  $-$ sink and is a  $-$ sink itself. To that end, we add one vertex  $v_i^A$  to  $A$ , and one vertex  $v_i^B$  to  $B$ . The edges of  $G_{v_i}$  are then added in the obvious way, namely, for every arc  $\overrightarrow{v_i v_j}$  with  $v_i \in N$  and  $v_j \in S$ , we add  $v_i v_j$  to  $G_{v_i}$  (where maybe  $v_i$  and  $v_j$  are actually  $v_i^A$  and  $v_i^B$ , respectively, depending on the context). According to Lemma 2 (applied with  $G = G_{v_i}$ ), there exists a sum-colouring  $\{-2, -1, 0, 1, 2\}$ -edge-weighting  $w'$  of  $G_{v_i}$  with the properties described in the statement of Lemma 2. Now just reverberate the weights by  $w'$  to the weights by  $w$  on the corresponding arcs (namely, for every arc  $\overrightarrow{v_i v_j}$  of  $D$  corresponding to an arc  $uv$  present in  $G$ , add  $w'(uv)$  to  $w(\overrightarrow{v_i v_j})$ ).

After having applied the above procedure for all equivalent  $-$ sinks and  $+$ sinks, the conclusions are the following:

- For every  $-$ source  $v_i$ , we have

$$\sigma_w^-(v_i) \in \{\phi^-(v_i), \phi^-(v_i) + 1, \phi^-(v_i) + 2, \phi^-(v_i) + 6, \phi^-(v_i) + 6 + 1, \phi^-(v_i) + 6 + 2\},$$

while we have

$$\sigma_w^+(v_i) \in \{\phi^+(v_i), \phi^+(v_i) + 1, \phi^+(v_i) + 2, \phi^+(v_i) + 6, \phi^+(v_i) + 6 + 1, \phi^+(v_i) + 6 + 2\}$$

for every  $+$ source  $v_i$ . Since  $\Phi^+(v_i) \cap \Phi^-(v_j) = \emptyset$  whenever  $\overrightarrow{v_i v_j}$  is an arc with  $i < j$ , and  $v_i$  being a  $+$ source and  $v_j$  being a  $-$ source, at the end we still have  $\sigma_w^+(v_i) \neq \sigma_w^-(v_j)$ .

- It cannot be the case that some  $+$ source  $v_j$  appears in two considered  $G_{v_i}$ 's. Indeed, if  $v_i$  and  $v'_i$  are two  $-$ sinks both having  $v_j$  as an in-neighbour, then actually  $G_{v_i}$  and  $G_{v'_i}$  could be merged. This means that a  $+$ source vertex can have its value of  $\sigma_w^+$  being augmented by 0, 1 or 2 at most once. This is important for the previous item.
- If  $\overrightarrow{v_i v_j}$  with  $i < j$  is an arc such that  $v_i$  is a  $+$ source and  $v_j$  is a  $-$ sink, then  $\sigma_w^+(v_i) \neq \sigma_w^-(v_j)$  since these two values have distinct remainders modulo 3 (due to how  $w'$  was obtained). The same argument applies when  $v_i$  is also a  $-$ sink.

According to these arguments (or analogous ones), it should be clear that we end up with  $w$  being sum-colouring. We note that, since a weight of  $w$  can be augmented or reduced by at most 2 in the last step of the procedure, the minimum weight used by  $w$  is 1 while its maximum weight is 17. This completes the proof.  $\square$

### 3. A directed 1-2 Conjecture

We now investigate how helpful it would be to be granted the possibility to locally modify  $\sigma^-$  and  $\sigma^+$  at every vertex of a given digraph. Let us define this problem formally. Let  $w$  be a total-weighting of some digraph  $D$ . For every vertex  $v$  of  $D$ , we define:

$$\sigma_w^-(v) := w(v) + \sum_{u \in N^-(v)} w(\overrightarrow{uv}) \quad \text{and} \quad \sigma_w^+(v) := w(v) + \sum_{u \in N^+(v)} w(\overrightarrow{vu})$$

that is, the local weight by  $w$  is counted in both  $\sigma_w^-(v)$  and  $\sigma_w^+(v)$  (as if we were weighting a loop at  $v$ ). Once more, we call  $w$  *sum-colouring* if  $\sigma_w^+(v_i) \neq \sigma_w^-(v_j)$  for every arc  $\overrightarrow{v_i v_j}$  of  $D$ . This time, it should be clear that all digraphs admit a sum-colouring total-weighting (if  $D$  is not nice, just use vertex weights to “destroy” its bad configurations). For every digraph  $D$ , the chromatic parameter  $\chi_{\Sigma^\pm}^t(D)$ , denoting the least number of weights in a sum-colouring  $k$ -total-weighting of  $D$ , is hence well defined.

As a consequence of some remark above, we have the following.

**Observation 1.** *For every nice digraph  $D$ , we have  $\chi_{\Sigma^\pm}^t(D) \leq \chi_{\Sigma^\pm}^e(D)$ .*

This in particular implies that some of our results from previous Section 2 directly give upper bounds on  $\chi_{\Sigma^\pm}^t(D)$  when  $D$  belongs to some families of nice digraphs. Overall, we do believe that the possibility of locally modifying both  $\sigma^-$  and  $\sigma^+$  is very handy. For this reason, we conjecture that the following, which is a direct analogue to the 1-2 Conjecture, should be true.

**Directed 1-2 Conjecture.** For every digraph  $D$ , we have  $\chi_{\Sigma^\pm}^t(D) \leq 2$ .

As a support for this Directed 1-2 Conjecture, let us briefly show that it holds for some of the classes of digraphs considered in Section 2 for which  $\chi_{\Sigma^\pm}^e$  can even reach 3.

**Proposition 1.** For every acyclic digraph  $D$ , we have  $\chi_{\Sigma^\pm}^t(D) \leq 2$ .

*Proof.* Start by mimicking the beginning of the proof of Theorem 2 to get a 2-arc-weighting  $w$  of  $D$  such that all  $\sigma_w^-(v_i)$ 's are even and some  $\sigma_w^+(v_i)$ 's (but maybe not all) are odd. Put  $w(v_i) = 2$  for every vertex  $v_i$  of  $D$ . From now on we will use notation  $w$  to deal with the 2-total-weighting. In case there is no arc  $\overrightarrow{v_i v_j}$  such that  $\sigma_w^+(v_i) = \sigma_w^-(v_j)$ , we are done. Otherwise, there are some such arcs  $\overrightarrow{v_i v_j}$  such that  $\sigma_w^+(v_i) = \sigma_w^-(v_j)$  with  $\sigma_w^+(v_i)$  being even while it should be odd. From the point of view of  $\overrightarrow{v_i v_j}$ , we call  $v_i$  *bad*, while we call  $v_j$  *good*. Note that all sinks are good from the beginning.

To fix the conflicts, we prove that, starting from the rightmost conflicting arc (*i.e.* the one  $\overrightarrow{v_i v_j}$  with  $v_i$  having the largest index), we can make  $v_i$  good by considering another arc  $\overrightarrow{v_k v_i}$  and possibly making  $v_k$  bad. This means that we can basically “push” the conflicts towards the left, *i.e.* towards the sources, which we can handle easily (since we do not care about their value of  $\sigma_w^-$ ).

Consider a conflicting arc  $\overrightarrow{v_i v_j}$ . By definition, we have:

- $\sigma_w^-(v_j)$  even and  $\sigma_w^+(v_j)$  odd or null (good), and
- $\sigma_w^-(v_i)$  even and  $\sigma_w^+(v_i)$  even (bad).

If  $v_i$  is a source, then we can solve the problem by just switching  $w(v_i)$ , *i.e.* set this weight to  $3 - w(v_i)$ . Now, if  $v_i$  is not a source, then there is an arc  $\overrightarrow{v_k v_i}$  with  $k < i$ . Then by just switching the weight  $w(v_i)$ , as well as the weight  $w(\overrightarrow{v_k v_i})$ , note that  $v_i$  becomes good.

By just repeating this argument until only sources of  $D$  are bad, we eventually can solve all conflicts and make  $w$  sum-colouring.  $\square$

**Proposition 2.** For every bipartite digraph  $D$ , we have  $\chi_{\Sigma^\pm}^t(D) \leq 2$ .

*Proof.* Let  $A, B$  denote the bipartition of  $D$ . Start with assigning weight 2 to all arcs of  $D$  so that  $\sigma^-$  and  $\sigma^+$  are even for all vertices. Now assign weight 1 on all vertices of  $A$ , and assign weight 2 on all vertices in  $B$ . The result is that,

- for every  $a \in A$ , both  $\sigma^-(a)$  and  $\sigma^+(a)$  are odd, and
- for every  $b \in B$ , both  $\sigma^-(b)$  and  $\sigma^+(b)$  are even.

It then follows that, for every arc  $\overrightarrow{u v}$  of  $D$ , the parities of  $\sigma^+(u)$  and  $\sigma^-(v)$  are different – so the two values are different.  $\square$

We now prove our main result that, in the context of sum-colouring total-weighting, we do not need to use up to 17 weights as indicated by Theorem 6. Here, we prove that 10 weights suffice.

**Theorem 7.** For every digraph  $D$ , we have  $\chi_{\Sigma^\pm}^t(D) \leq 10$ .

*Proof.* We reuse some of the terminology from the proof of Theorem 6 throughout. Assuming some ordering  $v_1, v_2, \dots, v_n$  of the vertices of  $D$  is fixed, we additionally say that some vertex  $v_i$  is a *weak sink* (or *weak source*) if it is either both a  $-$ sink and a  $+$ source, or both a  $+$ sink and a  $-$ source. We conversely call  $v_i$  a *strong sink* if it is both a  $-$ sink and

a +sink. Similarly, we say that  $v_i$  is a *strong source* if it is both a  $-$ source and  $+$ source. Note in particular that two strong sinks cannot be adjacent.

Our upcoming arguments rely on the existence of a particular ordering  $v_1, v_2, \dots, v_n$  for every digraph  $D$ , that is proved to exist in the following lemma.

**Lemma 3.** *Every digraph  $D$  admits an ordering  $v_1, v_2, \dots, v_n$  of its vertices into three blocks  $A = \{v_1, v_2, \dots, v_a\}$ ,  $B = \{v_{a+1}, v_{a+2}, \dots, v_b\}$ ,  $C = \{v_{b+1}, v_{b+2}, \dots, v_n\}$  of consecutive vertices such that all vertices of  $A$  are strong sources, all vertices of  $B$  are weak sinks in which two sinks of a same type (i.e.  $-$ sinks or  $+$ sinks) are not adjacent, and all vertices of  $C$  are strong sinks.*

*Proof.* Note that moving a vertex  $v_i$  to the right cannot create new sinks different from  $v_i$   $-$  actually the only bad thing that can happen is that  $v_i$  was, say, a  $-$ sink but moving it to the right makes it become a  $+$ sink. This apart, the set of cannot grow when performing rightward moves. Furthermore, it should be clear that, at any moment, moving vertices in  $B$  at the end of  $A$  or at the beginning of  $C$  cannot spoil the main property of the strong sources in  $A$  and the strong sinks in  $C$ .

Start from  $A = C = \emptyset$  and  $B = \{v_1, v_2, \dots, v_n\}$ . As long as  $B$  contains vertices being strong sources, consider the one  $v_i$  with the lowest index (regarding its location in  $B$ ) and just move  $v_i$  at the end of  $A$ . Recall that, when performing such a move, strong sources in  $A$  are preserved. According to this argument, at the end of this first step all vertices in  $A$  are strong sources, while all vertices in  $B$  are weak or strong sinks (but their status may change later).

If, at any moment of the upcoming arguments, moving some vertex of  $B$  to  $A$  or to  $C$  creates some new strong sources or strong sinks in  $B$ , then again we can safely move them to  $A$  or  $C$  (provided we start with the ones with the lowest or largest, respectively, index with respect to  $B$ ). Next step is moving all strong sinks of  $B$  at the beginning of  $C$ . Once this cannot be performed further, the conditions of the statement are reached, unless  $B$  has, say, two  $-$ sinks  $v_i$  and  $v_j$  being adjacent with  $i < j$  (following the ordering with regards to  $B$ ). Necessarily the arc joining  $v_i$  and  $v_j$  is from  $v_i$  to  $v_j$  (otherwise  $v_i$  would not be a  $-$ sink). Then move  $v_i$  just on the right of  $v_j$  (with staying in  $B$ ). This makes  $v_j$  becomes a  $-$ source, so the local conflict is solved. Since we moved  $v_i$  to the right, the overall situation in  $B$  can only get better, unless  $v_j$  was the only forward neighbour of  $v_i$ , in which case  $v_i$  becomes a  $+$ sink and  $v_i$  was already adjacent to a backward  $+$ sink. But in such a situation, we note that  $v_i$  actually becomes a strong sink, and can hence freely moved at the beginning of  $C$ , solving the conflict. Repeating these moving arguments, eventually  $A$ ,  $B$  and  $C$  meet the claimed properties.  $\square$

We now describe how to prove Theorem 7. Let first  $v_1, v_2, \dots, v_n$  be an ordering of the vertices of  $D$  as described in the statement of Lemma 3 (we will in particular reuse the notation  $A$ ,  $B$  and  $C$ ). Using the very first algorithm described in the first part of the proof of Theorem 6 (i.e. everything done before treating the  $-$ sinks and  $+$ sinks) but with initially setting all vertex weights to 4 and using arc weights among  $\{2, 4, 6, 8, 10\}$  (instead of  $\{3, 6, 9, 12, 15\}$ ), we can obtain a 10-total-weighting  $w$  of  $D$  with the following properties:

1. All  $\sigma_w^-$ 's and  $\sigma_w^+$ 's are even.
2. To every  $-$ source (resp.  $+$ source)  $v_i$  is associated some couple  $\Phi^-(v_i) = \{\phi^-(v_i), \phi^-(v_i) + 4\}$  (resp.  $\Phi^+(v_i) = \{\phi^+(v_i), \phi^+(v_i) + 4\}$ ) such that  $\sigma_w^-(v_i) \in \Phi^-(v_i)$  (resp.  $\sigma_w^+(v_i) \in \Phi^+(v_i)$ ).

3. For every arc  $\overrightarrow{v_i v_j}$  such that  $v_i$  is a +source and  $v_j$  is a -source, we have  $\Phi^+(v_j) \cap \Phi^-(v_i) = \emptyset$  (so  $\sigma_w^+(v_i) \neq \sigma_w^-(v_j)$ ).
4. All backward arcs incident to -sinks or +sinks are assigned weights among  $\{4, 6, 8\}$ .

The main problem now is that we have to make sure that sinks' values of  $\sigma_w^-$  and  $\sigma_w^+$  do not create any problem, which seems hard to ensure by just manipulating arc weights. So we will now adjust the vertex weights to make sure that no conflict remains.

For the sake of clarity, we will first explain what final properties we would like  $w$  to meet, and then why these would make  $w$  sum-colouring. Then we will describe how to modify  $w$  so that all these properties are met.

The properties we desire are the following.

- A. For every vertex  $a \in A$ , we have  $\sigma_w^-(a) \in \Phi^-(a)$  and  $\sigma_w^+(a) \in \Phi^+(a)$ , both these values being even. Since vertices in  $A$  are strong sources, recall that they all have their own  $\Phi^-$  and  $\Phi^+$  defined, according to Property (2).
- B. For every vertex  $b \in B$ , if  $b$  is a -sink and a +source, we have  $\sigma_w^-(b)$  odd, and  $\sigma_w^+(b) \equiv 1, 3 \pmod{6}$  with  $\sigma_w^+(b) \in \{\phi^+(b) + 1, \phi^+(b) + 4 + 1\}$ . Conversely, if  $b$  is a +sink and a -source, we have  $\sigma_w^+(b)$  odd, and  $\sigma_w^-(b) \equiv 1, 3 \pmod{6}$  with  $\sigma_w^-(b) \in \{\phi^-(b) + 1, \phi^-(b) + 4 + 1\}$ .
- C. For every vertex  $c \in C$ , we have both  $\sigma_w^-(c)$  and  $\sigma_w^+(c)$  having remainder 5 modulo 6.

Assuming  $w$  meets all of Properties (A), (B) and (C), it has to be sum-colouring. Indeed, assume  $\overrightarrow{v_i v_j}$  is an arc of  $D$ . First, if both  $v_i$  and  $v_j$  belong to  $A$ , then  $\sigma_w^+(v_i)$  and  $\sigma_w^-(v_j)$  are different according to Properties (3) and (A). Now, since all vertices of  $B$  and  $C$  have their values of  $\sigma_w^-$  and  $\sigma_w^+$  being odd (while vertices in  $A$  have these values being even), clearly no conflict can involve a vertex in  $A$  and a vertex in  $B$  or  $C$ .

We now focus on the cases where  $v_i$  belongs to  $B$ . First consider the cases where  $v_j$  also belongs to  $B$ . If  $i < j$ , then  $v_i$  is a +source and hence a -sink. According to the ordering  $v_1, v_2, \dots, v_n$  (recall the statement of Lemma 3) it cannot be the case that  $v_i$  is adjacent to another -sink in  $B$ , so  $v_j$  is a -source. Then  $\sigma_w^+(v_i)$  and  $\sigma_w^-(v_j)$  are distinguished according to Properties (3) and (B). Now, if  $j < i$ , it means that  $v_j$  is a -source and a +sink, and the previous arguments apply in an obvious way. Now, if one of  $v_i$  and  $v_j$  belongs to  $B$  while the second one belongs to  $C$ , then the two parameters of interest are different since they have different residues modulo 6, recall Properties (B) and (C). Finally, since any two vertices of  $C$  cannot be adjacent, there are no parameters to be distinguished there. So, according to all of these arguments,  $w$  is sum-colouring as soon as it satisfies Properties (A), (B) and (C).

We finally describe how to modify  $w$  so that all of these properties hold. We recall that, so far, we have  $w(v_i) = 4$  for every vertex  $v_i$ . Note that Property (A) is already met according to Property (1), so nothing more needs to be done for vertices in  $A$ .

We now focus on vertices in  $C$ . First of all, for every vertex  $v_i \in C$ , we would like to locally modify the weights on the arcs incident to  $v_i$  so that  $\sigma_w^-(v_i)$  and  $\sigma_w^+(v_i)$  (which are even so far) have the same even residue modulo 6. In case  $v_i$  has, say, no incident in-coming arcs, then  $v_i$  has in-degree 0 so we do not care about the value of  $\sigma_w^-(v_i)$  and we can immediately jump to the conclusion. So  $v_i$  has both an incident in-coming arc  $\overrightarrow{v_j v_i}$  and an incident out-going arc  $\overrightarrow{v_i v_k}$ , where  $j, k < i$  and  $v_j$  and  $v_k$  are respectively +source and -source. As described in the proof of Theorem 6, recall that, according to

Property (4), either subtracting 4 or adding 4 to either of the weights on these arcs is a valid move (in particular, this cannot break Property (A), which already holds). Assume then  $\sigma_w^+(v_i) \equiv r \pmod{6}$ , where  $r \in \{0, 2, 4\}$ . If  $r' = \sigma_w^-(v_i) \pmod{6}$  is equal to  $r$ , we are done. Otherwise, one can perform the valid move on the arc  $\overrightarrow{v_j v_i}$ , hence subtracting or adding 4 to  $\sigma_w^-(v_i)$ , reaching another residue  $r''$  different from  $r'$ . If  $r'' = r$ , we are done. Otherwise, by performing the valid move on  $\overrightarrow{v_i v_k}$ , we will necessarily reach either of  $r'$  and  $r''$ , meaning that there is a way to perform valid moves on  $\overrightarrow{v_j v_i}$  and/or  $\overrightarrow{v_i v_k}$  so that  $\sigma_w^-(v_i)$  and  $\sigma_w^+(v_i)$  have the same even residue modulo 6. To now make Property (C) hold, we just have to modify the weight on  $v_i$  so that  $\sigma_w^-(v_i)$  and  $\sigma_w^+(v_i)$  both have residue 5 modulo 6. To that end, just set  $w(v_i)$  to 3, 1 or 5 when the original remainder of  $\sigma_w^-(v_i)$  and  $\sigma_w^+(v_i)$  is 0, 2 or 4, respectively.

We finally make Property (B) hold to complete the proof. Note first that, for every vertex  $v_i$  of  $B$ , changing  $w(v_i)$  to an odd weight will make the condition on  $\sigma_w^-(v_i)$  (resp.  $\sigma_w^+(v_i)$ ) hold whenever  $v_i$  is a  $-$ sink (resp.  $+$ sink). So, provided we eventually attribute an odd weight to  $v_i$ , we just have to focus on the second part of the condition. Assume  $v_i$  is a  $-$ source without loss of generality. So far its value of  $\sigma_w^-(v_i)$  hence belongs to  $\{\phi^-(v_i), \phi^-(v_i) + 4\}$ . We just go through a straight case distinction:

- Assume first  $\phi^-(v_i) \equiv 0 \pmod{6}$  and  $\phi^-(v_i) + 4 \equiv 4 \pmod{6}$ . If  $\sigma_w^-(v_i) = \phi^-(v_i)$ , we can just add 1 (*i.e.* assign 5) to  $w(v_i)$  so that  $\sigma_w^-(v_i) \equiv 1 \pmod{6}$ . On the second hand, if  $\sigma_w^-(v_i) = \phi^-(v_i) + 4$ , we can just subtract 3 (*i.e.* assign 1) to  $w(v_i)$  so that  $\sigma_w^-(v_i) \equiv 1 \pmod{6}$ .
- Suppose now  $\phi^-(v_i) \equiv 2 \pmod{6}$  and  $\phi^-(v_i) + 4 \equiv 0 \pmod{6}$ . First, if  $\sigma_w^-(v_i) = \phi^-(v_i)$ , we can add 1 (*i.e.* assign 5) to  $w(v_i)$  in order to get  $\sigma_w^-(v_i) \equiv 3 \pmod{6}$ . Second, if  $\sigma_w^-(v_i) = \phi^-(v_i) + 4$ , we can just add 1 (*i.e.* assign 5) to  $w(v_i)$  so that  $\sigma_w^-(v_i) \equiv 1 \pmod{6}$ .
- Finally suppose  $\phi^-(v_i) \equiv 4 \pmod{6}$  and  $\phi^-(v_i) + 4 \equiv 2 \pmod{6}$ . On the one hand, if  $\sigma_w^-(v_i) = \phi^-(v_i)$ , add 5 (*i.e.* assign 9) to  $w(v_i)$  so that  $\sigma_w^-(v_i) \equiv 3 \pmod{6}$ . On the other hand, if  $\sigma_w^-(v_i) = \phi^-(v_i) + 4$ , just add 1 (*i.e.* assign 5) to  $w(v_i)$  so that  $\sigma_w^-(v_i) \equiv 3 \pmod{6}$ .

□

#### 4. Conclusion

In this paper, we have introduced a new directed analogue of the 1-2-3 Conjecture, which seemingly catches the inherent difficulty behind the original conjecture. We have supported our Directed 1-2-3 Conjecture by showing it to hold in many contexts (namely for nice acyclic digraphs, and nice digraphs with particular colouring properties), before eventually proving a weakening of the conjecture where 3 is replaced with 17. We eventually studied the consequences on the problem to be allowed to locally modify the incident incoming and out-going sums (in the spirit of the 1-2 Conjecture), and proved that less weights, namely 10, suffice in this setting.

The main direction for future work would naturally be to step towards the Directed 1-2-3 and 1-2 Conjectures by proving better constant upper bounds on the number of necessary weights. This apart, it would be interesting to check whether the Directed 1-2-3 (or 1-2) Conjecture holds in more classes of digraphs. As appealing cases, we could raise the following questions, which have commonly been considered when dealing with related notions.

**Question 1.** Does the Directed 1-2-3 Conjecture hold for nice digraphs whose underlying graph has bounded maximum degree  $d$ ? Is it true for  $d \leq 3$ ?

**Question 2.** Does the Directed 1-2-3 Conjecture hold for nice digraphs with large minimum in- and out-degree?

Since there exist digraphs whose value of  $\chi_{\Sigma^\pm}^e$  is equal to 3, one another interesting question could be to wonder whether only a few digraphs need the three edge weights. This could read as follows.

**Question 3.** Is there an easy characterization of digraphs whose value of  $\chi_{\Sigma^\pm}^e$  is at most 2?

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