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EXPONENTIAL FUNCTIONALS OF SPECTRALLY ONE-SIDED LÉVY PROCESSES CONDITIONED TO STAY POSITIVE

GRÉGOIRE VÉCHAMBRE

ABSTRACT. We study the properties of the exponential functional $\int_0^{+\infty} e^{-X^\uparrow(t)} dt$ where X^\uparrow is a spectrally one-sided Lévy process conditioned to stay positive. In particular, we study finiteness, self-decomposability, existence of finite exponential moments, asymptotic tail at 0 and smoothness of the density.

1. INTRODUCTION

We consider a spectrally negative Lévy process V which is not the opposite of a subordinator. We denote its Laplace exponent by Ψ_V :

$$\forall t, \lambda \geq 0, \mathbb{E} \left[e^{\lambda V(t)} \right] = e^{t\Psi_V(\lambda)}.$$

In the case where V drifts to $-\infty$, it is well known that its Laplace exponent admits a non trivial zero that we denote here by κ , $\kappa := \inf\{\lambda > 0, \Psi_V(\lambda) = 0\}$. If V does not drift to $-\infty$, then 0 is the only zero of Ψ_V so we put $\kappa := 0$ in this case. We denote by (Q, γ, ν) the generating triplet of V so Ψ_V can be expressed as

$$\Psi_V(\lambda) = \frac{Q}{2}\lambda^2 - \gamma\lambda + \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x \mathbf{1}_{|x|<1}) \nu(dx). \quad (1.1)$$

In the end of the paper, we also consider Z , a spectrally positive Lévy process drifting to $+\infty$.

We are interested in the basic exponential functionals of V and Z conditioned to stay positive,

$$I(V^\uparrow) := \int_0^{+\infty} e^{-V^\uparrow(t)} dt \quad \text{and} \quad I(Z^\uparrow) := \int_0^{+\infty} e^{-Z^\uparrow(t)} dt.$$

For both we study finiteness, exponential moments and the asymptotic tail at 0. For $I(V^\uparrow)$, we also get self-decomposability, more precise estimates on the asymptotic tail at 0 and a condition for smoothness of the density.

Our first motivation is to extend to spectrally one-sided Lévy processes conditioned to stay positive the general study of the exponential functionals of Lévy processes. Those functionals have been widely studied because of their importance in probability theory. For example they are fundamental to the study of diffusions in random environments and appear in many applications such as mathematical finance, see [4] for a survey on those functionals and their applications. For a general Lévy process, equivalent conditions for the finiteness of the exponential functional are given in [4], the asymptotic tail at $+\infty$ of the functional is studied in [10], the absolute continuity is proved in [9] and properties of the density (such as regularity) are studied in [11] under some hypothesis on the jumps of the Lévy process and in [12]. In this paper we also obtain, as a

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by-product of our approach, some results on the exponential functionals of spectrally one-sided Lévy processes.

Our second motivation is the possibility to apply our results to the study of diffusions in a spectrally negative Lévy environment. Such processes, introduced by Brox [6] when the environment is given by a brownian motion have been specifically studied for the spectrally negative Lévy case by Singh [15]. In [2], they prove that the supremum of local time \mathcal{L}_X^* of a diffusion in a drifted brownian environment converges in law and they express the limit law in term of a subordinator and an exponential functional of the environment conditioned to stay positive. In order to generalize their result to a diffusion in a spectrally negative Lévy environment, knowledge on the exponential functionals involved is needed. These are precisely exponential functionals of the environment (which is spectrally negative) and its dual (which is spectrally positive) conditioned to stay positive.

Finally, we have hints that the almost sure asymptotic behavior of \mathcal{L}_X^* , for a diffusion in the spectrally negative Lévy environment V , is crucially linked to the right and left tails of the distribution of $I(V^\uparrow)$. This is why we study these tails here and give for the left tail a precise asymptotic estimate when it is possible, in particular, when $\Psi_V(\lambda) \sim c\lambda^\alpha$, for some constant c and $\alpha \in]1, 2]$. For the right tail, we are only interested in the existence of some finite exponential moments. The application of the present work to diffusions in random environment is a work in preparation by the author [17], [16].

For A a process and S a borelian set, we denote

$$\tau(A, S) := \inf \{t \geq 0, A(t) \in S\}, \quad \mathcal{R}(A, S) := \sup \{t \geq 0, A(t) \in S\}.$$

We shall only write $\tau(A, x)$ (respectively $\mathcal{R}(A, x)$) instead of $\tau(A, \{x\})$ (respectively $\mathcal{R}(A, \{x\})$) and $\tau(A, x+)$ instead of $\tau(A, [x, +\infty[)$. For example, since V^\uparrow has no positive jumps, we see that it reaches each positive level continuously : $\forall x > 0, \tau(V^\uparrow, x+) = \tau(V^\uparrow, x)$ and since moreover V^\uparrow converges to $+\infty$ we have $\forall x > 0, \mathcal{R}(V^\uparrow, [0, x]) = \mathcal{R}(V^\uparrow, x)$.

Also let $\underline{A}(t) := \inf\{A(s), s \in [0, t]\}$ be the infimum process of A . If A is Markovian and $x \in \mathbb{R}$ we denote A_x for the process A starting from x . For A_0 we shall only write A . For any (possibly random) time $T > 0$, we write A^T for the process A shifted and centered at time T : $\forall s \geq 0, A^T(s) := A(T + s) - A(T)$.

We now recall some facts about V^\uparrow , that is, V conditioned to stay positive.

For a spectrally negative Lévy process V , the Markov family $(V_x^\uparrow, x \geq 0)$ may be defined as in [3], Section VII.3. For any $x \geq 0$, the process V_x^\uparrow must be seen as V conditioned to stay positive and starting from x . We denote V^\uparrow for the process V_0^\uparrow . It is known that V_x^\uparrow converges in the Skorokhod space to V^\uparrow when x goes to 0.

For X a positive random variable, we denote V_X^\uparrow for the process V conditioned to stay positive and starting from the random variable X . More rigorously, V_X^\uparrow is the Markov process that conditionally on $\{X = x\}$ has law V_x^\uparrow .

For any positive x , we have from the Markov property and the absence of positive jumps that the process V^\uparrow , shifted at $\tau(V^\uparrow, x)$, its first passage time at x , is equal in law to V_x^\uparrow . In the case where V drifts to $+\infty$, it is known from [3], Section VII.3, that V_x^\uparrow has the same law as V_x conditioned in the usual sense to remain positive. This property, interesting for our study, is unfortunately not true when V oscillates or drifts to $-\infty$ (in these cases we have to do the conditioning until an hitting time).

In the case where V drifts to $-\infty$, we define V^\sharp to be " V conditioned to drift to $+\infty$ ", as in [3], Section VII.1. The Laplace exponent Ψ_{V^\sharp} of V^\sharp satisfies $\Psi_{V^\sharp} = \Psi_V(\kappa + \cdot)$ where κ is the non-trivial zero of Ψ_V . As a consequence $\Psi'_{V^\sharp}(0) > 0$, so V^\sharp drifts to infinity (this is deduced thanks to Corollary VII.2 in [3]) and it is also proven that $V^\uparrow = (V^\sharp)^\uparrow$.

In order to do our proofs in a systematic way, we often work with V^\sharp which is defined to be " V conditioned to drift to $+\infty$ " in the case where V drifts to $-\infty$ and "only V " in the other cases (when V oscillates or drifts to $+\infty$). As a consequence, V^\sharp always denotes a spectrally negative Lévy process that does not drifts to $-\infty$ (it oscillates if V does and it drifts to $+\infty$ if V drifts to $+\infty$ or $-\infty$). In any case we have that for all $0 < x < y$, $(V_x^\uparrow(t), 0 \leq t \leq \tau(V_x^\uparrow, y))$ is equal in law to $(V_x^\sharp(t), 0 \leq t \leq \tau(V_x^\sharp, y))$ conditionally on $\{\tau(V_x^\sharp, y) < \tau(V_x^\sharp, \cdot) - \infty, 0\}$. Note that the same identity is true with V instead of V^\sharp , but the advantage of dealing with V^\sharp is that $\tau(V_x^\sharp, y)$ is always finite (while $\tau(V_x, y)$ can possibly be infinite when V drifts to $-\infty$) which simplifies the argumentation.

Let W be the scale function of V , defined as in Section VII.2 of [3]. It satisfies

$$\forall 0 < x < y, \mathbb{P}(\tau(V_x, y) < \tau(V_x, \cdot) - \infty, 0) = W(x)/W(x+y).$$

According to Theorem VII.8 in [3], this function is continuous, increasing, and for any $\lambda > \Phi_V(0)$,

$$\int_0^{+\infty} e^{-\lambda x} W(x) dx = \frac{1}{\Psi_V(\lambda)} < +\infty.$$

1.1. Results. In the special case of the exponential functional of a drifted brownian motion conditioned to stay positive, all the properties that are established here are already known and sometimes more explicitly. We discuss this case in the next subsection.

Our first result is the finiteness of $I(V^\uparrow)$ and the fact that it admits exponential moments.

Theorem 1.1. *The random variable $I(V^\uparrow)$ is almost surely finite, has finite expectation $\mathbb{E}[I(V^\uparrow)]$ and*

$$\forall \lambda < 1/\mathbb{E}[I(V^\uparrow)], \mathbb{E}\left[e^{\lambda I(V^\uparrow)}\right] < +\infty. \quad (1.2)$$

Then, a fundamental point of our study is Proposition 3.2 which says that for any positive y , $I(V^\uparrow)$ satisfies the random affine equation

$$I(V^\uparrow) \stackrel{\mathfrak{L}}{=} A^y + e^{-y} I(V^\uparrow), \quad (1.3)$$

where A^y is independent of the second term and will be specified later. We see that $I(V^\uparrow)$ is a positive self-decomposable random variable and is therefore absolutely continuous and unimodal. It is well known (see for example expression (1.10) in [13]) that the exponential functional $I(V)$ of a spectrally negative Lévy process V is also self-decomposable (as long as it is finite), it can be seen by splitting the trajectory at $\tau(V, y)$, the first passage time at y . Another consequence of (1.3) is that for any positive y , $I(V^\uparrow)$ can be written as the random series

$$I(V^\uparrow) \stackrel{\mathfrak{L}}{=} \sum_{k \geq 0} e^{-ky} A_k^y,$$

where the random variables A_k^y are *iid* and have the same law as A^y . This decomposition is a very useful tool for the study of the random variable $I(V^\uparrow)$ and is also the base of the proofs of the results we present below.

Our next results make a link between the asymptotic behavior of Ψ_V and the properties of $I(V^\uparrow)$.

Theorem 1.2. *Assume that there is $\alpha > 1$ and a positive constant C such that for all λ large enough we have $\Psi_V(\lambda) \leq C\lambda^\alpha$. Then for all $\delta \in]0, 1[$ and x small enough we have*

$$\mathbb{P}\left(I(V^\uparrow) \leq x\right) \leq \exp\left(-\delta(\alpha - 1)/(Cx)^{1/(\alpha-1)}\right). \quad (1.4)$$

Assume that there is $\alpha > 1$ and a positive constant c such that for all λ large enough we have $\Psi_V(\lambda) \geq c\lambda^\alpha$. Then for all $\delta > 1$ and x small enough we have

$$\mathbb{P}\left(I(V^\uparrow) \leq x\right) \geq \exp\left(-\delta\alpha^{\alpha/(\alpha-1)}/(cx)^{1/(\alpha-1)}\right). \quad (1.5)$$

Let us now recall how is usually quantified the asymptotic behavior of Ψ_V . We define, as in [3], page 94,

$$\begin{aligned} \sigma &:= \sup \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = \infty \right\}, \\ \beta &:= \inf \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = 0 \right\}. \end{aligned}$$

Recall that $\Psi_{V^\#}(\cdot) = \Psi_V(\kappa + \cdot)$, so σ and β are identical whether they are defined from $\Psi_{V^\#}$ or Ψ_V .

If Ψ_V has α -regular variation for $\alpha \in [1, 2]$ (for example if V is a drifted α -stable Lévy process with no positive jumps), we have $\sigma = \beta = \alpha$. Recall that Q is the brownian component of V . It is well known that $\Psi_V(\lambda)/\lambda^2$ converges to $Q/2$ when λ goes to infinity so, when $Q > 0$, Ψ_V has 2-regular variation, and when $Q = 0$,

$$1 \leq \sigma \leq \beta \leq 2,$$

where $1 \leq \sigma$ comes from the convexity of Ψ_V .

Remark 1.3. *When V has bounded variation, we know (see for example [3] Section I.1) that the brownian component of V is null, the Lévy measure ν of V satisfies $\int_{-1}^0 |x|\nu(dx) < +\infty$ and $-\gamma - \int_{-1}^0 x\nu(dx)$, the factor of λ in the expression of $\Psi_V(\lambda)$, is positive (otherwise V would be the opposite of a subordinator). It is thus easy to see that in this case $\Psi_V(\lambda)/\lambda$ converges to $-\gamma - \int_{-1}^0 x\nu(dx)$ when λ goes to infinity, so $\sigma = \beta = 1$. In the remaining, we sometimes assume that $\sigma > 1$, the reader should be aware that it excludes the case where V has bounded variation. However, this case is quite easy and shall be treated in the remarks.*

We are now ready to state our general results on the asymptotic tails at 0 of $I(V^\uparrow)$:

Theorem 1.4. *We have*

$$\forall \beta' > \beta, \lim_{x \rightarrow 0} x^{1/(\beta'-1)} \log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) = -\infty, \quad (1.6)$$

$$\text{if } \sigma > 1, \forall \sigma' \in]1, \sigma[, \lim_{x \rightarrow 0} x^{1/(\sigma'-1)} \log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) = 0. \quad (1.7)$$

Theorem 1.4 gives for $\mathbb{P}(I(V^\uparrow) \leq x)$ a lower bound involving σ and an upper bound involving β . In the case of α -regular variation we can expect, under some extra hypothesis, to get a stronger result. We indeed have :

Theorem 1.5. *We assume that there is a positive constant C and $\alpha \in]1, 2]$ such that $\Psi_V(\lambda) \sim_{\lambda \rightarrow +\infty} C\lambda^\alpha$, then*

$$\log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) \underset{x \rightarrow 0}{\sim} -\frac{\alpha - 1}{(Cx)^{\frac{1}{\alpha-1}}}.$$

The above theorem is true in particular when, for some $\alpha \in]1, 2]$, V is an α -stable spectrally negative Lévy process (with adjunction or not of a drift). In particular, it agrees exactly with the tail (1.11) given in the next subsection for the particular case of a drifted brownian motion.

Remark 1.6. *Since $\Psi_V(\lambda)/\lambda^2$ has always a finite limit at $+\infty$ we get, from Theorem 1.2, that there is always a positive constant K (depending on V) such that for x small enough*

$$\mathbb{P}\left(I(V^\uparrow) \leq x\right) \leq e^{-K/x}.$$

Remark 1.7. *Note that Theorem 1.4 holds when $\beta = 1$ and $1/(\beta' - 1)$ can then equal any number in $]0, +\infty[$. When V has bounded variation, we even have a stronger result : $\mathbb{P}(I(V^\uparrow) \leq x)$ is null for x small enough.*

Remark 1.8. *Recall that $I(V^\uparrow)$ is unimodal. If 0 was a mode, then we would have $\mathbb{P}(I(V^\uparrow) \leq x) \geq cx$ for some positive constant c and x small enough, which is incompatible with (1.6). As a consequence the density of $I(V^\uparrow)$ is non-decreasing on a neighborhood of 0. This allows to remark that Theorems 1.2, 1.4, 1.5 and Remarks 1.6, 1.7 are true for the density of $I(V^\uparrow)$ in place of the repartition function $\mathbb{P}(I(V^\uparrow) \leq \cdot)$.*

When V drifts to $+\infty$ we prove Proposition 4.3 which says that the left tail of $I(V^\uparrow)$ is the same as the left tail of $I(V)$. This implies that all the results we prove for the left tail of $I(V^\uparrow)$ are true for the left tail of $I(V)$:

Proposition 1.9. *If V drifts to $+\infty$, then Theorems 1.2, 1.4, 1.5 and Remarks 1.6, 1.7 are true for $I(V)$ in place of $I(V^\uparrow)$.*

Proposition 1.9 is an example of how the study of the exponential functional of the Lévy process conditioned to stay positive can be useful for the study of the exponential functional of the corresponding Lévy process.

We already mentioned that the law of $I(V^\uparrow)$ is absolutely continuous but we do not know how smooth the density is in general. The following theorem provides a condition for smoothness :

Theorem 1.10. *If $\sigma > 1$ and β are such that*

$$2\beta^2 - 3\sigma\beta + \sigma + \beta - 1 < 0, \tag{1.8}$$

then the density of $I(V^\uparrow)$ belongs to the Schwartz space. All its derivatives converge to 0 at $+\infty$ and 0.

This theorem admits the following corollary :

Corollary 1.11. *If V drifts to $+\infty$, is such that $\sigma > 1$ and (1.8) is satisfied, then the density of $I(V)$ is of class \mathcal{C}^∞ and all its derivatives converge to 0 at $+\infty$ and 0.*

Here again, the study of the exponential functional of the Lévy process conditioned to stay positive implies results about the exponential functional of the corresponding Lévy process.

Remark 1.12. *If Ψ_V has α -regular variation with $\alpha > 1$, then $\sigma = \beta = \alpha$ so the condition (1.8) becomes $-(\alpha - 1)^2 < 0$, but this is always true for $\alpha > 1$, so Theorem 1.10 and Corollary 1.11 apply. In other words, α -regular variation for the Laplace exponent of V implies smoothness of the density for $I(V^\uparrow)$ (and $I(V)$ if it is finite) when $\alpha > 1$.*

In the spectrally positive case, the finiteness of the exponential functional is quite easy to obtain, but our argument also yields the existence of some finite exponential moments. We can state the result as follows :

Theorem 1.13. *The random variable $I(Z^\uparrow)$ is almost surely finite and admits some finite exponential moments.*

We also obtain a lower bound for the asymptotic tail at 0 of both $I(Z)$ and $I(Z^\uparrow)$. This tail is heavier than the one given for $I(V^\uparrow)$ and this comes from the positive jumps.

Theorem 1.14. *If Z has unbounded variation and non-zero Lévy measure then, there is a positive constant c such that*

$$e^{-c(\log(x))^2} \leq \mathbb{P}(I(Z) \leq x) \leq \mathbb{P}\left(I(Z^\uparrow) \leq x\right).$$

The lower bound for $\mathbb{P}(I(Z) \leq x)$ does not require the hypothesis of unbounded variation.

Remark 1.15. *If the Lévy measure of Z is the zero measure then it is known, from the Lévy-Khintchine formula, that Z is a drifted brownian motion. The exact asymptotic tail at 0 of $I(Z^\uparrow)$ is then given by Theorem 1.5 and it is thinner than the one provided by Theorem 1.14. The existence of jumps thus plays an important role for the asymptotic tail at 0 of the exponential functional and the proof of Theorem 1.14 indeed crucially relies on this hypothesis.*

The study of the spectrally positive case does not go as far as the study of the spectrally negative case. The reason for this is twofold. First, we do not have, in the spectrally positive case, a decomposition of the law of $I(Z^\uparrow)$ as in (1.3), which deprives us of an important tool for the study. Secondly we do not need, in the applications, the results on the exponential functional to be as precise, in the spectrally positive case, as in the spectrally negative case. Indeed, in the study of a diffusion in a spectrally negative Lévy environment V drifting to $-\infty$, a random variable \mathcal{R} appears. Its law is the convolution of the laws of $I(V^\uparrow)$ and $I(\hat{V}^\uparrow)$, where $\hat{V} := -V$ is the dual process of V and is thus spectrally positive. The combination of the above theorems shows that for some things the behavior of $I(V^\uparrow)$ is dominant in the study of \mathcal{R} when V has jumps. In particular, the asymptotic tail at 0 of \mathcal{R} is the same as the one of $I(V^\uparrow)$.

The rest of the paper is organized as follows. In Section 2 we prove some preliminary results on V^\uparrow . In Section 3 we prove Theorem 1.1 and establish Proposition 3.2 about the self-decomposability of $I(V^\uparrow)$. In Section 4 we prove Theorems 1.2, 1.4 and 1.5 by studying the asymptotic behavior of the Laplace transform of $I(V^\uparrow)$, and in the case where V drifts to $+\infty$, we establish a connection between the tails at 0 of the exponential functionals $I(V^\uparrow)$ and $I(V)$. In Section 5 we prove Theorem 1.10 and Corollary 1.11 via a study of excursions. Section 6 is devoted to the spectrally positive case and the proofs of Theorems 1.13 and 1.14.

1.2. The example of drifted brownian motion conditioned to stay positive. The most simple case is the intersection of the spectrally positive and the spectrally negative case, that is, when V is a drifted brownian motion. All the results mentioned here are already known in this case. We define the κ -drifted brownian motion by $W_\kappa(t) := W(t) - \frac{\kappa}{2}t$. It is known that the two processes W_κ^\uparrow and $W_{-\kappa}^\uparrow$ are equal in law. This follows, for example, from the expression of the generator of W_κ^\uparrow , or from the fact that for positive κ , the Laplace exponent of W_κ^\uparrow is equal to the Laplace exponent of $W_{-\kappa}$, so the processes conditioned to stay positive have the same law. We thus only consider positive κ .

It is known (see (4.6) in [1], see also Lemma 6.6 in [2]) that $I(W_k^\uparrow)$ is almost surely finite and has Laplace transform

$$\mathbb{E}[e^{-\lambda I(W_k^\uparrow)}] = \frac{1}{2^\kappa \Gamma(1+\kappa)} \frac{(2\sqrt{2\lambda})^\kappa}{I_\kappa(2\sqrt{2\lambda})},$$

where I_κ is a modified Bessel function. This expression can also be written

$$\mathbb{E}[e^{-\lambda I(W_k^\uparrow)}] = \frac{1}{\Gamma(1 + \kappa)} \frac{1}{\sum_{j=0}^{+\infty} \frac{(2\lambda)^j}{j! \Gamma(1+j+\kappa)}}, \quad (1.9)$$

and it is easy to see that it can be analytically extended in a neighborhood of 0, so the random variable $I(W_k^\uparrow)$ admits some finite exponential moments.

An easy calculation on the asymptotic of this expression when λ goes to infinity yields

$$-\log \left(\mathbb{E}[e^{-\lambda I(W_k^\uparrow)}] \right) \underset{\lambda \rightarrow +\infty}{\sim} 2\sqrt{2\lambda}. \quad (1.10)$$

Combining (1.10) and De Bruijn's Theorem (see Theorem 4.12.9 in [5]) we get

$$-\log \left(\mathbb{P} \left(I(W_k^\uparrow) \leq x \right) \right) \underset{x \rightarrow 0}{\sim} \frac{2}{x}. \quad (1.11)$$

This estimate can be seen as a particular case of Theorem 1.5 (when it is applied in the case of a drifted brownian motion).

As the expression (1.9) extends to a neighborhood of 0, we get the expression of the characteristic function of $I(W_k^\uparrow)$ which can be proved, using estimates on modified Bessel functions, to belong to the Schwartz space. Therefore, the density of $I(W_k^\uparrow)$, which is the Fourier transform of its characteristic function, belongs to the Schwartz space, but this is already included in Theorem 1.10.

2. PRELIMINARY RESULTS ON V^\uparrow AND FINITENESS OF $I(V^\uparrow)$

2.1. Exponential functionals and excursions theory. We fix $y > 0$. In this subsection, we use excursions to prove that the integral of exponential $V^\uparrow(\tau(V^\uparrow, y) + \cdot)$ or V^\sharp stopped at there last passage time at y and 0 respectively are equal in law to some subordinators stopped at independent exponential random variables.

It is easy to see that regularity of $\{y\}$ for the markovian processes V_y^\sharp and V_y^\uparrow is equivalent to the regularity of $\{0\}$ for V (or V^\sharp) which in turn, according to Corollary VII.5 in [3], is equivalent to the fact that V has unbounded variation. The property of $\{y\}$ being instantaneous for V_y^\sharp and V_y^\uparrow is equivalent to the same property of $\{0\}$ for V , but this is a well known property of spectrally negative Lévy processes. $\{y\}$ is thus always instantaneous for V_y^\sharp and V_y^\uparrow and the only alternative is whether it is regular or not, which corresponds to the fact that V has or not unbounded variation.

We apply excursions theory away from y (see [3]). Let us denote by L_y^\uparrow (respectively L_y^\sharp) a local time at y of the process V_y^\uparrow (respectively V_y^\sharp) and η_y^\uparrow (respectively η_y^\sharp) the associated excursions measure. We denote η^\sharp for η_0^\sharp . The inverse of the local time $L_y^{\uparrow,-1}$ (respectively $L_y^{\sharp,-1}$) is a subordinator and V_y^\uparrow (respectively V_y^\sharp) can be represented as a Poisson point process on the set of excursions, with intensity measure η_y^\uparrow (respectively η_y^\sharp). Note that this is also true in the irregular case (when V has bounded variation) if the local time L_y^\uparrow (respectively L_y^\sharp) is defined artificially as in [3], Section IV.5. In this case, the excursion measure is proportional to the law of the first excursion and in particular the total mass of the excursion measure is finite.

In the case where V drifts to $+\infty$, we also consider the excursions of V away from 0. Then, L denotes a local time at 0 of V and η the associated excursion measure.

Given $\xi : [0, \zeta] \rightarrow \mathbb{R}$ an excursion away from y , we define $\zeta(\xi)$ to be its life-time, $H_y(\xi) := \sup_{[0, \zeta(\xi)]} \xi - y$ its height and $G(\xi) := \int_0^{\zeta(\xi)} e^{-\xi(t)} dt$.

For any $h > 0$, we consider IP_h , FP_h and N three subsets that make a partition of the excursions of V^\sharp away from y . These three subsets are respectively : the set of excursions higher than h that stay positive, the set of excursions of height smaller than h that stay positive, the set of excursions that reach $]-\infty, 0]$:

$$IP_h := \{\xi, \forall t \in [0, \zeta(\xi)], \xi(t) > 0, H_y(\xi) \geq h\}, \quad FP_h := \{\xi, \forall t \in [0, \zeta(\xi)], \xi(t) > 0, H_y(\xi) < h\}, \\ N := \{\xi, \tau(\xi,]-\infty, 0]) < \zeta(\xi)\}.$$

N does not depend on h . IP_∞ and FP_∞ are defined as the monotone limits of the sets IP_h and FP_h : IP_∞ is the set of infinite excursions that stay positive and FP_∞ is the set of finite excursions that stay positive. $\eta_y^\sharp(IP_\infty)$ and $\eta_y^\sharp(N)$ are always finite whereas $\eta_y^\sharp(FP_\infty)$ is infinite in the regular case (when V has unbounded variation). Also, note that $\eta_y^\sharp(IP_\infty) = 0$ if V oscillates.

Lemma 2.1. *Let y be positive and let S be a pure jump subordinator with Lévy measure $G\eta_y^\sharp(\cdot \cap FP_\infty)$, the image measure of $\eta_y^\sharp(\cdot \cap FP_\infty)$ by G . Let T be an exponential random variable with parameter $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$, independent of S . We have*

$$\int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \stackrel{\mathcal{L}}{=} S_T, \quad (2.12)$$

where $\tau(\cdot, \cdot)$ and $\mathcal{R}(\cdot, \cdot)$ are defined in the introduction.

Proof. $V^\uparrow(\tau(V^\uparrow, y) + \cdot)$ has the same law as V_y^\uparrow , from the Markov property applied to V^\uparrow at time $\tau(V^\uparrow, y)$ and the absence of positive jumps. As a consequence, $\int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt$ is equal in law to $\int_0^{\mathcal{R}(V_y^\uparrow, y)} e^{-V_y^\uparrow(t)} dt$ and we are left to prove the result for the latter.

Then, let us fix $h > 0$. As it is mentioned in the introduction, $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$ is equal in law to $(V_y^\sharp(t), 0 \leq t \leq \tau(V_y^\sharp, y + h))$ conditionally on $\{\tau(V_y^\sharp, y + h) < \tau(V_y^\sharp,]-\infty, 0])\}$.

$(V_y^\sharp(t), 0 \leq t \leq \tau(V_y^\sharp, y + h))$ can be built from the Poisson point process on the set of excursions with intensity measure η_y^\sharp . $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$ can be built from this same process, conditioned not to have jumps in N before its first jump in IP_h . In other words, we build $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$ from the process of jumps in FP_h stopped at the exponential time (that has parameter $\eta_y^\sharp(IP_h) + \eta_y^\sharp(N)$) at which occurs the first jump in $IP_h \cup N$ and conditionally to the fact that this jump belongs to IP_h . Then, the process of jumps in FP_h and in $IP_h \cup N$ are independent and by a property of Poisson point processes, the fact that the first jump in $IP_h \cup N$ belongs to IP_h is independent of the time when this jump occurs. As a consequence, $(V_y^\uparrow(t), 0 \leq t \leq \tau(V_y^\uparrow, y + h))$ is built from a Poisson point process with intensity measure $\eta_y^\sharp(\cdot \cap FP_h)$, until an independent exponential time, T_h , of parameter $\eta_y^\sharp(IP_h) + \eta_y^\sharp(N)$ where we pick, independently, a jump following the law $\eta_y^\sharp(\cdot \cap IP_h) / \eta_y^\sharp(IP_h)$ (and we only keep the part of this excursion that is before its hitting time of $y + h$).

Let $\mathcal{R}^h(V_y^\uparrow, y)$ be the last passage time of V_y^\uparrow at y before $\tau(V_y^\uparrow, y + h)$:

$$\mathcal{R}^h(V_y^\uparrow, y) := \sup\{t \in [0, \tau(V_y^\uparrow, y + h)], V_y^\uparrow(t) = y\}.$$

From above, if $(p_s)_{s \geq 0}$ is a Poisson point process in FP_∞ with measure $\eta_y^\sharp(\cdot \cap FP_\infty)$ and if T_h is an independent exponential random variable with parameter $\eta_y^\sharp(IP_h) + \eta_y^\sharp(N)$, then $(V_y^\uparrow(t), 0 \leq$

$t \leq \mathcal{R}^h(V_y^\uparrow, y)$) is built by putting aside the excursions of the process $(p_s \mathbf{1}_{p_s \in FP_h}, 0 \leq s \leq T_h)$. Since V_y^\uparrow converges almost surely to $+\infty$, $\mathcal{R}^h(V_y^\uparrow, y)$ converges almost surely to $\mathcal{R}(V_y^\uparrow, y)$, the last passage time at y , when h goes to infinity. On the other hand, $IP_h \cup N$ decreases to $IP_\infty \cup N$ when h goes to infinity. As a consequence T_h increases to an exponential random variable T with parameter $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N) > 0$. Also, FP_h increases to FP_∞ when h goes to infinity. Then, identifying the limits when h goes to infinity of both $(V_y^\uparrow(t), 0 \leq t \leq \mathcal{R}^h(V_y^\uparrow, y))$ and $(p_s \mathbf{1}_{p_s \in FP_h}, 0 \leq s \leq T_h)$, we get that $(V_y^\uparrow(t), 0 \leq t \leq \mathcal{R}(V_y^\uparrow, y))$ is built by putting aside the excursions of the process $(p_s, 0 \leq s \leq T)$, where T is an exponential random variable with parameter $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$ and is independent from $(p_s, s \geq 0)$.

Now, remark that $\int_0^{\mathcal{R}(V_y^\uparrow, y)} e^{-V_y^\uparrow(t)} dt$ is the sum of the images by G of the excursions of $(V_y^\uparrow(t), 0 \leq t \leq \mathcal{R}(V_y^\uparrow, y))$ away from y . We thus have

$$\int_0^{\mathcal{R}(V_y^\uparrow, y)} e^{-V_y^\uparrow(t)} dt \stackrel{\mathfrak{L}}{=} \sum_{0 < s < T} G(p_s). \quad (2.13)$$

By properties of Poisson point processes, the process in the right hand side, $\sum_{0 < s < T} G(p_s)$, is the sum of the jumps of a Poisson point process on \mathbb{R}_+ , with intensity measure $G\eta_y^\sharp(\cdot \cap FP_\infty)$. Thus, from the Lévy-Ito decomposition, it has the same law as the subordinator S , which yields the result. \square

Remark 2.2. *In the case where V has bounded variation, the total mass of η_y^\sharp is finite so S is only a compound Poisson process. In particular S_T can then be null with positive probability.*

$y > 0$ is still fixed (and arbitrary), let $\mathcal{R}^y(V^\sharp, 0)$ be the last passage time of V^\sharp at 0 before $\tau(V^\sharp, y)$:

$$\mathcal{R}^y(V^\sharp, 0) := \sup\{t \in [0, \tau(V^\sharp, y)], V^\sharp(t) = 0\}.$$

In order to study the trajectory of V^\sharp before $\mathcal{R}^y(V^\sharp, 0)$, we now consider excursions away from 0. Let I_y and F_y denote respectively the subset of excursions higher than y and lower than y :

$$I_y := \{\xi, H_0(\xi) \geq y\}, \quad F_y := \{\xi, H_0(\xi) < y\}.$$

A similar proof as for Lemma 2.1 gives the following lemma.

Lemma 2.3. *Let S be a pure jump subordinator with Lévy measure $G\eta^\sharp(\cdot \cap F_y)$, the image measure of $\eta^\sharp(\cdot \cap F_y)$ by G . Let T be an exponential random variable with parameter $\eta^\sharp(I_y)$ which is independent of S . We have*

$$\int_0^{\mathcal{R}^y(V^\sharp, 0)} e^{-V^\sharp(t)} dt \stackrel{\mathfrak{L}}{=} S_T. \quad (2.14)$$

In the case where V drifts to $+\infty$ we need to study the trajectory before $\mathcal{R}(V, 0)$, the last passage time of V at 0. We still consider excursions away from 0. Let I and F denote respectively the subsets of infinite and finite excursions :

$$I := \{\xi, \zeta(\xi) = +\infty\}, \quad F := \{\xi, \zeta(\xi) < +\infty\}.$$

A similar proof as for Lemma 2.1 gives the following lemma.

Lemma 2.4. *We assume that V drifts to $+\infty$. Let S be a pure jump subordinator with Lévy measure $G\eta(\cdot \cap F)$, the image measure of $\eta(\cdot \cap F)$ by G . Let T be an exponential random variable*

with parameter $\eta(I)$ which is independent of S . We have

$$\int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt \stackrel{\mathfrak{L}}{=} S_T. \quad (2.15)$$

2.2. V^\uparrow and V^\sharp shifted at a last passage time. To obtain decomposition (1.3) of the law of $I(V^\uparrow)$, we split V^\uparrow at its last passage time at a point y and obtain two independent trajectories that we can identify.

Lemma 2.5. (Corollary VII.19 of [3])

For any positive y , the two trajectories

$$\left(V^\uparrow(t), 0 \leq t \leq \mathcal{R}(V^\uparrow, y) \right) \text{ and } \left(V^\uparrow(t + \mathcal{R}(V^\uparrow, y)) - y, t \geq 0 \right)$$

are independent and the second is equal in law to V^\uparrow .

Lemma 2.6. • The two trajectories $(V^\sharp(t), 0 \leq t \leq \mathcal{R}^y(V^\sharp, 0))$

$$\text{and } \left(V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0) \right)$$

are independent and the second is equal in law to $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y))$. As a consequence we have $\tau(V^\uparrow, y) \stackrel{\mathfrak{L}}{=} \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0) \leq \tau(V^\sharp, y)$.

• We assume that V drifts to $+\infty$. The two trajectories

$$(V(t), 0 \leq t \leq \mathcal{R}(V, 0)) \text{ and } (V(t + \mathcal{R}(V, 0)), t \geq 0)$$

are independent and the second is equal in law to V^\uparrow .

Proof. We fix $y > 0$ and $a \in]0, y[$. Let us denote by $(e(s), s \geq 0)$ the excursions process of V^\sharp away from 0. Recall the notations I_y and F_y , $T_y := \inf\{s \geq 0, e(s) \in I_y\}$ is the time when occurs the first excursion higher than y and ξ_y is this excursion.

Decomposing V^\sharp as its excursions away from 0, we see that $\mathcal{R}^y(V^\sharp, 0)$ is the instant when begins the first excursion higher than y , so

$$\left(V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0) \right) = (\xi_y(t), 0 \leq t \leq \tau(\xi_y, y)). \quad (2.16)$$

$(V^\sharp(t), 0 \leq t \leq \mathcal{R}^y(V^\sharp, 0))$ is thus a function of $(e(s)\mathbb{1}_{e(s) \in F_y}, 0 \leq s \leq T_y)$ while $(V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0))$ is a function of ξ_y . By properties of Poisson point processes, T_y is an exponential random variable independent of ξ_y and the process of finite excursions $(e(s)\mathbb{1}_{e(s) \in F_y}, s \geq 0)$ is also independent of ξ_y . Therefore the objects $(e(s)\mathbb{1}_{e(s) \in F_y}, 0 \leq s \leq T_y)$ and ξ_y are independent. From this independence we deduce that

$$\left(V^\sharp(t), 0 \leq t \leq \mathcal{R}^y(V^\sharp, 0) \right) \perp\!\!\!\perp \left(V^\sharp(t + \mathcal{R}^y(V^\sharp, 0)), 0 \leq t \leq \tau(V^\sharp, y) - \mathcal{R}^y(V^\sharp, 0) \right),$$

which is the required independence. It only remains to prove that the right hand side in (2.16) has the same law as $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y))$.

Using the Markov property at time $\tau(\xi_a, a)$, for an excursion $\xi_a \in I_a$, we have that $\xi_a(\cdot + \tau(\xi_a, a))$ equals in law V_a^\sharp killed when it ever reaches 0.

Since $I_y \subset I_a$ we can apply this to an excursion $\xi_y \in I_y$ and get that $(\xi_y(t + \tau(\xi_y, a)), 0 \leq t \leq \tau(\xi_y, y) - \tau(\xi_y, a))$ is equal in law to $(V_a^\sharp(t), 0 \leq t \leq \tau(V_a^\sharp, y))$ conditioned to reach y before 0. Since V_a^\sharp has no positive jumps, reaching y before 0 is the same as reaching y before $]-\infty, 0]$.

As we mentioned in the introduction, $(V_a^\sharp(t), 0 \leq t \leq \tau(V_a^\sharp, y))$ conditioned to reach y before $] -\infty, 0]$ is equal in law to $(V_a^\uparrow(t), 0 \leq t \leq \tau(V_a^\uparrow, y))$. Putting all this together we get

$$(\xi_y(t + \tau(\xi_y, a)), 0 \leq t \leq \tau(\xi_y, y) - \tau(\xi_y, a)) \stackrel{\mathfrak{L}}{=} (V_a^\uparrow(t), 0 \leq t \leq \tau(V_a^\uparrow, y)).$$

Since $\tau(\xi_y, a)$ converges almost surely to 0 when a goes to 0 and V_a^\uparrow converges in law to V^\uparrow according to Proposition VII.14 in [3], we can let a go to 0 in both members and get

$$(\xi_y(t), 0 \leq t \leq \tau(\xi_y, y)) \stackrel{\mathfrak{L}}{=} (V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y)). \quad (2.17)$$

As a consequence the right hand side in (2.16) has the same law as $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y))$, which concludes the proof of the first point of the lemma.

We now assume that V drifts to $+\infty$ and prove the second point. For the independence, the arguments of the proof of the first point can be repeated, just replacing y by $+\infty$ (we consider ξ_∞ , the infinite excursion away from 0, instead of the first excursion higher than y). To prove that $(V(t + \mathcal{R}(V, 0)), t \geq 0)$ is equal in law to V^\uparrow , it suffices to prove that ξ_∞ is equal in law to V^\uparrow . Let y be finite, we know from the proof of the first point that (2.17) is true for any excursion in I_y . Since $\xi_\infty \in I_y$ we have

$$(\xi_\infty(t), 0 \leq t \leq \tau(\xi_\infty, y)) \stackrel{\mathfrak{L}}{=} (V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow, y)).$$

Since y is arbitrary and $\tau(V^\uparrow, y)$ converges almost surely to $+\infty$ when y goes to $+\infty$, we get

$$(\xi_\infty(t), t \geq 0) \stackrel{\mathfrak{L}}{=} (V^\uparrow(t), t \geq 0),$$

which gives the result. □

3. FINITENESS, EXPONENTIAL MOMENTS, AND SELF-DECOMPOSABILITY

3.1. Finiteness and exponential moments : Proof of Theorem 1.1. We are grateful to an anonymous referee for the following proof that is considerably simpler than the proof given by the author in the previous versions of this paper.

Proof. of Theorem 1.1

The idea of the proof is to provide finite upper bounds for the moments of $I(V^\uparrow)$. The first step is to prove that $\mathbb{E}[I(V^\uparrow)] < +\infty$. Using Fubini's Theorem and Corollary VII.16 of [3] we have

$$\begin{aligned} \mathbb{E}[I(V^\uparrow)] &= \int_0^{+\infty} \mathbb{E}[e^{-V^\uparrow(t)}] dt = \int_0^{+\infty} \int_0^{+\infty} e^{-y} \mathbb{P}(V^\uparrow(t) \in dy) dt \\ &= \int_0^{+\infty} \int_0^{+\infty} e^{-y} \frac{yW(y)}{t} \mathbb{P}(V(t) \in dy) dt. \end{aligned}$$

Now, using Corollary VII.3 of [3] we get

$$\mathbb{E}[I(V^\uparrow)] = \int_0^{+\infty} \int_0^{+\infty} e^{-y} W(y) \mathbb{P}(\tau(V, y) \in dt) dy = \int_0^{+\infty} e^{-y} W(y) \mathbb{P}(\tau(V, y) < +\infty) dy.$$

Since $\mathbb{P}(\tau(V, y) < +\infty) = \mathbb{P}(\sup_{[0, +\infty[} V \geq y) = e^{-\kappa y}$ we obtain

$$\mathbb{E}[I(V^\uparrow)] = \int_0^{+\infty} e^{-(1+\kappa)y} W(y) dy < +\infty,$$

where the finiteness comes from the fact that $\int_0^{+\infty} e^{-\lambda y} W(y) dy < +\infty$ for $\lambda > \kappa$. As a consequence, the exponential functional $I(V^\uparrow)$ is almost surely finite and has finite expectation.

We now turn to the proof of the finiteness of the Laplace transform. We proceed by bounding the moments of the exponential functional. For any $x \geq 0$ let us define $h(x) := \mathbb{E}[I(V_x^\uparrow)]$. For any $x > 0$ we have

$$I(V^\uparrow) = \int_0^{+\infty} e^{-V^\uparrow(t)} dt \geq \int_{\tau(V,x)}^{+\infty} e^{-V^\uparrow(t)} dt \stackrel{\mathcal{L}}{=} \int_0^{+\infty} e^{-V_x^\uparrow(t)} dt = I(V_x^\uparrow),$$

where, for the equality in law, we used the Markov property for V^\uparrow at time $\tau(V,x)$. As a consequence we have

$$\forall x > 0, h(x) = \mathbb{E}[I(V_x^\uparrow)] \leq \mathbb{E}[I(V^\uparrow)] < +\infty. \quad (3.18)$$

Now, note that for any $k \geq 1$,

$$\begin{aligned} \mathbb{E} \left[\left(I(V^\uparrow) \right)^k \right] &= \mathbb{E} \left[\int_0^{+\infty} \dots \int_0^{+\infty} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right] \\ &= k! \mathbb{E} \left[\int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right], \end{aligned}$$

so that

$$\forall k \geq 1, \mathbb{E} \left[\left(I(V^\uparrow) \right)^k \right] / k! = \mathbb{E} \left[\int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right]. \quad (3.19)$$

Let us prove by induction that for any $k \geq 1$,

$$\mathbb{E} \left[\left(I(V^\uparrow) \right)^k \right] \leq k! \left(\mathbb{E}[I(V^\uparrow)] \right)^k < +\infty \quad (3.20)$$

(3.20) is clearly true for $k = 1$. Let us assume that it is true for some arbitrary rank k . According to (3.19), $\mathbb{E}[(I(V^\uparrow))^{k+1}] / (k+1)!$ equals

$$\begin{aligned} &\mathbb{E} \left[\int_{0 \leq t_1 < \dots < t_{k+1}} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} \times e^{-V^\uparrow(t_{k+1})} dt_1 \dots dt_k dt_{k+1} \right] \\ &= \mathbb{E} \left[\int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} \left(\int_{t_k}^{+\infty} e^{-V^\uparrow(s)} ds \right) dt_1 \dots dt_k \right] \\ &= \mathbb{E} \left[\int_{0 \leq t_1 < \dots < t_k} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} \mathbb{E} \left[\int_{t_k}^{+\infty} e^{-V^\uparrow(s)} ds \mid \sigma(V^\uparrow(u), 0 \leq u \leq t_k) \right] dt_1 \dots dt_k \right]. \end{aligned}$$

From the Markov property at time t_k , the conditional expectation in the above expression equals $h(V^\uparrow(t_k))$ which, according to (3.18), is almost surely less than $\mathbb{E}[I(V^\uparrow)]$. We thus get

$$\begin{aligned} \mathbb{E}[(I(V^\uparrow))^{k+1}] / (k+1)! &\leq \mathbb{E}[I(V^\uparrow)] \times \mathbb{E} \left[\int_0^{+\infty} e^{-V^\uparrow(t_1)} \times \dots \times e^{-V^\uparrow(t_k)} dt_1 \dots dt_k \right] \\ &= \mathbb{E}[I(V^\uparrow)] \times \mathbb{E} \left[\left(I(V^\uparrow) \right)^k \right] / k! \leq \left(\mathbb{E}[I(V^\uparrow)] \right)^{k+1}, \end{aligned}$$

where we used (3.19) and the induction hypothesis. Thus the induction is proved. As a consequence for all $\lambda \in]0, 1/\mathbb{E}[I(V^\uparrow)][$ we have

$$\mathbb{E} \left[e^{\lambda I(V^\uparrow)} \right] = \sum_{k \geq 0} \lambda^k \mathbb{E} \left[\left(I(V^\uparrow) \right)^k \right] / k! \leq \sum_{k \geq 0} \left(\lambda \mathbb{E}[I(V^\uparrow)] \right)^k < +\infty.$$

The finiteness of the Laplace transform is obvious for $\lambda \leq 0$, so the result is proved. \square

Remark 3.1. If V does not oscillate, the finiteness of $I(V^\uparrow)$ can be derived as a consequence of Lemma 2.6. Indeed, from the second statement of Lemma 2.6 applied to V^\sharp , we have

$$I(V^\uparrow) \stackrel{\mathfrak{I}}{=} \int_0^{+\infty} e^{-V^\sharp(t+\mathcal{R}(V^\sharp,0))} dt = \int_{\mathcal{R}(V^\sharp,0)}^{+\infty} e^{-V^\sharp(t)} dt \leq \int_0^{+\infty} e^{-V^\sharp(t)} dt = I(V^\sharp),$$

and V^\sharp drifts to $+\infty$, so Theorem 1 in [4] ensures that $I(V^\sharp)$ is almost surely finite which yields the result.

3.2. Decomposition of the law of $I(V^\uparrow)$. In this subsection, we prove that the law of $I(V^\uparrow)$ is solution of the random affine equation (1.3) and we give a decomposition of its non-trivial coefficient A^y . This is a key point of our analysis of the law of $I(V^\uparrow)$.

Proposition 3.2. For any $y > 0$, the law of $I(V^\uparrow)$ satisfies the random affine equation

$$I(V^\uparrow) \stackrel{\mathfrak{I}}{=} \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)} dt + S_T + e^{-y} I(\tilde{V}^\uparrow), \quad (3.21)$$

where the three terms of the right hand side are independent, S_T is as in Lemma 2.1, and \tilde{V}^\uparrow is an independent copy of V^\uparrow .

We define

$$A^y := \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)} dt + S_T$$

to lighten notations.

As a consequence, $I(V^\uparrow)$ has the same law as the sum of a random series :

$$I(V^\uparrow) \stackrel{\mathfrak{I}}{=} \sum_{k \geq 0} e^{-ky} A_k^y, \quad (3.22)$$

where the random variables A_k^y are iid and have the same law as A^y .

Remark 3.3. The almost sure convergence of the random series in (3.22) is a consequence of the almost sure finiteness, given by Theorem 1.1, of the positive random variable $I(V^\uparrow)$. Also, it is a well known fact on random power series with iid coefficients that their radius of convergence is almost surely equal to a constant belonging to $\{0,1\}$. Since this constant, in the case of the power series in (3.22), has been proved to be greater than e^{-y} , we deduce that it equals 1.

Proof. of Proposition 3.2

We fix $y > 0$. As V^\uparrow has no positive jumps and goes to infinity we have $\tau(V^\uparrow, y) \leq \mathcal{R}(V^\uparrow, y) < +\infty$ and $V^\uparrow(\tau(V^\uparrow, y)) = V^\uparrow(\mathcal{R}(V^\uparrow, y)) = y$.

We write :

$$\begin{aligned} I(V^\uparrow) &= \int_0^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)} dt + \int_{\mathcal{R}(V^\uparrow,y)}^{+\infty} e^{-V^\uparrow(t)} dt \\ &= \int_0^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)} dt + e^{-y} \int_0^{+\infty} e^{-(V^\uparrow(t+\mathcal{R}(V^\uparrow,y))-y)} dt \\ &\stackrel{\mathfrak{I}}{=} \int_0^{\mathcal{R}(V^\uparrow,y)} e^{-V^\uparrow(t)} dt + e^{-y} I(\tilde{V}^\uparrow), \end{aligned}$$

where we used Lemma 2.5 for the last equality in which \tilde{V}^\uparrow is an independent copy of V^\uparrow . We now decompose :

$$\int_0^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt = \int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt + \int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \quad (3.23)$$

Since $V^\uparrow(\tau(V^\uparrow, y)) = y$, combining with the Markov property at time $\tau(V^\uparrow, y)$, the two terms in the right hand side of (3.23) are independent :

$$\int_0^{\tau(V^\uparrow, y)} e^{-V^\uparrow(t)} dt \perp\!\!\!\perp \int_{\tau(V^\uparrow, y)}^{\mathcal{R}(V^\uparrow, y)} e^{-V^\uparrow(t)} dt.$$

Now, thanks to Lemma 2.1, the second term has the same law as S_T with S_T as in the lemma. This achieves the proof. \square

We have two remarks here :

Remark 3.4. *It is possible to prove Theorem 1.1 (the fact that $I(V^\uparrow)$ is finite and admits some finite exponential moments) by invoking (3.22) and proving that each of the two terms composing A^y admit some finite exponential moments.*

Remark 3.5. *Let S and T be as in Proposition 3.2 and $\epsilon \in]0, \mathbb{E}[S_1][$ (where we do not bother with the fact that $\mathbb{E}[S_1]$ is finite or not). Then we have*

$$\mathbb{P}(S_T \geq t) \geq \mathbb{P}(S_{t/\epsilon} \geq t) \times \mathbb{P}(T \geq t/\epsilon).$$

*The first factor in the right hand side converges to 1 thanks to the law of large numbers for Lévy processes (see for example Theorem 36.5 in [14]) and the second is equal to $e^{-pt/\epsilon}$, where p is the parameter of the exponential random variable T . Therefore, the Laplace transform of S_T is not finite everywhere, so neither is the Laplace transform of $I(V^\uparrow)$ (because of (3.22)). This is why we can not say better than " $I(V^\uparrow)$ admits **some** finite exponential moments".*

4. ASYMPTOTIC TAIL AT 0 : PROOF OF THEOREMS 1.2, 1.4 AND 1.5

First, let us prove that Theorem 1.2 easily implies Theorem 1.4 and prove Remark 1.7.

Proof. of Theorem 1.4

Assume that Theorem 1.2 is proved.

We first prove (1.6). Let us fix $\beta' > \beta$ and $\epsilon > 0$. From the definition of β we have that $\Psi_V(\lambda) \leq \epsilon \lambda^{\beta'}$ for all λ large enough. Using the first point of Theorem 1.2 we deduce that

$$\limsup_{x \rightarrow 0} x^{1/(\beta'-1)} \log \left(\mathbb{P} \left(I(V^\uparrow) \leq x \right) \right) \leq -(\beta' - 1)/\epsilon^{1/(\beta'-1)}.$$

Since ϵ can be chosen as small as we want we obtain (1.6).

We now assume that $\sigma > 1$ and prove (1.7). Let us fix $\sigma' \in]1, \sigma[$ and $M > 0$. From the definition of σ we have that $\Psi_V(\lambda) \geq M \lambda^{\sigma'}$ for all λ large enough. Using the second point of Theorem 1.2 we deduce that

$$0 \geq \liminf_{x \rightarrow 0} x^{1/(\sigma'-1)} \log \left(\mathbb{P} \left(I(V^\uparrow) \leq x \right) \right) \geq -\sigma'^{\sigma'/(\sigma'-1)} / M^{1/(\sigma'-1)}.$$

Since M can be chosen as large as we want we obtain (1.7). \square

Proof. of Remark 1.7

Let us assume that V has bounded variation. As it can be seen from Remark 1.3, it is the difference of a positive drift dt and a pure jump subordinator $S_t : \forall t > 0, V(t) = dt - S_t \leq dt$. Let us fix $y > 0$, we have almost surely

$$\int_0^{\tau(V,y)} e^{-V(t)} dt \geq \int_0^{\tau(V,y)} e^{-dt} dt = \frac{1}{d} \left(1 - e^{-\tau(V,y)}\right). \quad (4.24)$$

Since V has bounded variation, we have $\mathbb{P}(V(t) > 0, 0 \leq t \leq \tau(V,y)) > 0$ (see for example (47.1) in [14]) and we can see that $(V^\uparrow(t), 0 \leq t \leq \tau(V^\uparrow,y))$ is equal in law to $(V, 0 \leq t \leq \tau(V,y))$ conditioned in the usual sense to remain positive. Combining with (4.24), we see that almost surely

$$I(V^\uparrow) \geq \int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)} dt \geq \frac{1}{d} \left(1 - e^{-\tau(V^\uparrow,y)}\right).$$

Then, since $\tau(V^\uparrow,y)$ converges almost surely to $+\infty$ when y goes to $+\infty$, we deduce that $I(V^\uparrow)$ is more than the positive constant $1/d$ almost surely. As a consequence $\mathbb{P}(I(V^\uparrow) \leq x)$ is null for $x \leq 1/d$. \square

In the next Subsection we prepare the proofs of Theorems 1.2 and 1.5.

4.1. Laplace transform of $I(V^\uparrow)$. In order to prove asymptotic estimates on $\mathbb{P}(I(V^\uparrow) \leq x)$, we first study the Laplace transform of $I(V^\uparrow)$ via the decomposition given by Proposition 3.2. It is thus natural that we need first to study the Laplace transform of A^y .

First, let us define a notation. V^\sharp is a spectrally negative Lévy process, so, according to Theorem VII.1 in [3], the process $\tau(V^\sharp, \cdot)$ is a subordinator which Laplace exponent Φ_{V^\sharp} is defined for $\lambda \geq 0$ by

$$\Phi_{V^\sharp}(\lambda) := -\log \left(\mathbb{E} \left[e^{-\lambda \tau(V^\sharp, 1)} \right] \right),$$

and we have $\Phi_{V^\sharp} = \Psi_{V^\sharp}^{-1}$.

Proposition 4.1. *We fix $y > 0$. Let A^y be as in Proposition 3.2, then, for all $\epsilon > 0$ and λ large enough we have*

$$(1 - \epsilon)y\Phi_{V^\sharp}(e^{-y}\lambda) \leq -\log \left(\mathbb{E} \left[e^{-\lambda A^y} \right] \right) \leq (1 + \epsilon)y\Phi_{V^\sharp}(\lambda). \quad (4.25)$$

Proof. of Proposition 4.1

According to the definition of A^y in the Proposition 3.2, A^y can be decomposed as the sum of two independent random variables, one having the same law as $\int_0^{\tau(V^\uparrow,y)} e^{-V^\uparrow(t)} dt$ and another having the same law as S_T , defined as in Lemma 2.1. Let Φ_S be the Laplace exponent of the subordinator S :

$$\forall \lambda > 0, \Phi_S(\lambda) := -\log \left(\mathbb{E} \left[e^{-\lambda S_1} \right] \right). \quad (4.26)$$

We can see that the Laplace transform of the random variable S_T is given by

$$\forall \lambda > 0, \mathbb{E} \left[e^{-\lambda S_T} \right] = \frac{\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)}{\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N) + \Phi_S(\lambda)}.$$

We thus have

$$\begin{aligned} -\log\left(\mathbb{E}\left[e^{-\lambda A^y}\right]\right) &= -\log\left(\mathbb{E}\left[\exp\left(-\lambda\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt\right)\right]\right) - \log\left(\mathbb{E}\left[e^{-\lambda S_T}\right]\right) \\ &= -\log\left(\mathbb{E}\left[\exp\left(-\lambda\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt\right)\right]\right) - \log\left(\frac{p}{p+\Phi_S(\lambda)}\right), \end{aligned} \quad (4.27)$$

where we denoted p for the constant $\eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$. Using the fact that V^\uparrow is non-negative and the first point of Lemma 2.6 we have

$$\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt \leq \tau(V^\uparrow,y) \stackrel{sto}{\leq} \tau(V^\sharp,y), \quad (4.28)$$

where $\stackrel{sto}{\leq}$ denotes a stochastic inequality. As a consequence of (4.28) and of the definition of Φ_{V^\sharp} we have

$$-\log\left(\mathbb{E}\left[\exp\left(-\lambda\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt\right)\right]\right) \leq -\log\left(\mathbb{E}\left[e^{-\lambda\tau(V^\sharp,y)}\right]\right) = y\Phi_{V^\sharp}(\lambda).$$

Combining this inequality with (4.27) we obtain

$$-\log\left(\mathbb{E}\left[e^{-\lambda A^y}\right]\right) \leq y\Phi_{V^\sharp}(\lambda) - \log\left(\frac{p}{p+\Phi_S(\lambda)}\right). \quad (4.29)$$

Using the first point of Lemma 2.6 and Lemma 2.3 we have

$$-\log\left(\mathbb{E}\left[\exp\left(-\lambda\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt\right)\right]\right) = -\log\left(\mathbb{E}\left[e^{-\lambda\int_0^{\tau(V^\sharp,y)}e^{-V^\sharp(t)}dt\right]/\mathbb{E}\left[e^{-\lambda\tilde{S}_T}\right]\right),$$

where \tilde{S}_T is as S_T from Lemma 2.3. Here again, if $\Phi_{\tilde{S}}$ denotes the Laplace exponent of the subordinator \tilde{S} as in (4.26) we have

$$\forall \lambda > 0, \mathbb{E}\left[e^{-\lambda\tilde{S}_T}\right] = \frac{\eta^\sharp(I_y)}{\eta^\sharp(I_y) + \Phi_{\tilde{S}}(\lambda)}.$$

Moreover we have

$$\int_0^{\tau(V^\sharp,y)}e^{-V^\sharp(t)}dt \geq e^{-y}\tau(V^\sharp,y).$$

Putting together the above three expressions and the definition of Φ_{V^\sharp} we obtain

$$-\log\left(\mathbb{E}\left[\exp\left(-\lambda\int_0^{\tau(V^\uparrow,y)}e^{-V^\uparrow(t)}dt\right)\right]\right) \geq y\Phi_{V^\sharp}(e^{-y}\lambda) + \log\left(\frac{\eta^\sharp(I_y)}{\eta^\sharp(I_y) + \Phi_{\tilde{S}}(\lambda)}\right).$$

Combining the above inequality with (4.27) and the fact that the term $-\log(p/(p+\Phi_S(\lambda)))$ is non-negative we get

$$-\log\left(\mathbb{E}\left[e^{-\lambda A^y}\right]\right) \geq y\Phi_{V^\sharp}(e^{-y}\lambda) + \log\left(\frac{\eta^\sharp(I_y)}{\eta^\sharp(I_y) + \Phi_{\tilde{S}}(\lambda)}\right). \quad (4.30)$$

According to the Lévy-Khintchine formula for subordinators. The Laplace exponent Φ_S can be written

$$\Phi_S(\lambda) = \gamma_S\lambda + \int_0^{+\infty}(1 - e^{-\lambda x})\nu_S(dx),$$

so by dominated convergence, there exists a positive constant C_S such that for large λ , $\Phi_S(\lambda) \leq C_S\lambda$. Similarly, there is a positive constant $C_{\tilde{S}}$ such that for large λ , $\Phi_{\tilde{S}}(\lambda) \leq C_{\tilde{S}}\lambda$. On the other hand, since $\Psi_V(\lambda)/\lambda^2$ is bounded when λ goes to infinity, there is a positive constant c

such that $\Phi_{V^\#}(\lambda) \geq c\lambda^{1/2}$ for large λ . Combining all this with (4.29) and (4.30) we get (4.25) for any fixed $\epsilon > 0$ and λ large enough.

□

Proposition 4.2. *Assume that there is $\alpha \geq 1$ and a positive constant C such that for all λ large enough we have $\Psi_V(\lambda) \leq C\lambda^\alpha$, then we have*

$$\liminf_{\lambda \rightarrow +\infty} -\log \left(\mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \right) / \lambda^{1/\alpha} \geq \alpha / C^{1/\alpha}. \quad (4.31)$$

Assume that there is $\alpha \geq 1$ and a positive constant c such that for all λ large enough we have $\Psi_V(\lambda) \geq c\lambda^\alpha$, then we have

$$\limsup_{\lambda \rightarrow +\infty} -\log \left(\mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \right) / \lambda^{1/\alpha} \leq \alpha / c^{1/\alpha}. \quad (4.32)$$

Proof. Let us fix $y > 0$ for which we apply the decomposition (3.22). Let us denote

$$\mathcal{M}(\lambda) := -\log \left(\mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \right) = \sum_{k \geq 0} -\log \left(\mathbb{E} \left[e^{-\lambda e^{-ky} A^y} \right] \right),$$

where the second equality comes from (3.22) and from the fact that the sequence $(A_k^y)_{k \geq 0}$ is *iid*. To establish the left tail of $I(V^\uparrow)$ we study the asymptotic behavior of $\mathcal{M}(\lambda)$ and the latter, thanks to the above expression, is related to the asymptotic behavior of $-\log(\mathbb{E}[e^{-\lambda A^y}])$.

Unfortunately, Proposition 4.1 can not be applied simultaneously to all the terms of the sum defining $\mathcal{M}(\lambda)$. We separate this sum into three parts : a sum over a finite number of small indices for which we can apply (4.25) to each term, a sum over an infinite number of large indices that can be neglected, and a sum over the remaining indices (in finite number) that can be neglected.

We now prove the second point of the proposition. We assume that there is $\alpha \geq 1$ and a positive constant c such that for all λ large enough we have $\Psi_V(\lambda) \geq c\lambda^\alpha$, and prove (4.32). Let us fix $\delta > 1$. Since $\Psi_{V^\#}(\cdot) = \Psi_V(\kappa + \cdot)$ and $\Phi_{V^\#} = \Psi_{V^\#}^{-1}$, we have for all λ large enough that $\Phi_{V^\#}(\lambda) \leq \delta\lambda^{1/\alpha}/c^{1/\alpha}$.

According to Proposition 4.1 there exists $\lambda_\delta > 1$ such that (4.25) is satisfied with $\epsilon = \delta - 1$ for all $\lambda \geq \lambda_\delta$. By increasing λ_δ if necessary, we can also assume that $\Phi_{V^\#}(\lambda) \leq \delta\lambda^{1/\alpha}/c^{1/\alpha}$ for all $\lambda \geq \lambda_\delta$. Putting all this together we get

$$\lambda \geq \lambda_\delta \Rightarrow -\log \left(\mathbb{E} \left[e^{-\lambda A^y} \right] \right) \leq \delta^2 y \lambda^{1/\alpha} / c^{1/\alpha}. \quad (4.33)$$

Also, let us choose $M \in]0, 1[$ small enough so that

$$\forall \lambda \in [0, M], \quad 0 \leq -\log \left(\mathbb{E} \left[e^{-\lambda A^y} \right] \right) \leq 2\lambda \mathbb{E}[A^y].$$

For any $\lambda > \lambda_\delta$, we define $n_1(\lambda) := \lfloor \log(\lambda/\lambda_\delta)/y \rfloor$ and $n_2(\lambda) := \lfloor \log(\lambda/M)/y \rfloor$. From the definition of $\mathcal{M}(\lambda)$ we can write

$$\forall \lambda > \lambda_\delta, \quad \mathcal{M}(\lambda) = T_1(\lambda) + T_2(\lambda) + T_3(\lambda), \quad (4.34)$$

with

$$T_1(\lambda) := \sum_{k=0}^{n_1(\lambda)} -\log \left(\mathbb{E} \left[e^{-\lambda e^{-ky} A^y} \right] \right), \quad T_2(\lambda) := \sum_{k=n_1(\lambda)+1}^{n_2(\lambda)} -\log \left(\mathbb{E} \left[e^{-\lambda e^{-ky} A^y} \right] \right),$$

$$T_3(\lambda) := \sum_{k=n_2(\lambda)+1}^{+\infty} -\log \left(\mathbb{E} \left[e^{-\lambda e^{-ky} A^y} \right] \right).$$

From the definition of $n_1(\lambda)$, (4.33) can be applied to each term of the sum defining $T_1(\lambda)$, we thus have

$$T_1(\lambda) \leq \delta^2 y (\lambda/c)^{1/\alpha} \sum_{k=0}^{n_1(\lambda)} e^{-ky/\alpha} = \delta^2 y (\lambda/c)^{1/\alpha} \frac{1 - e^{-y(n_1(\lambda)+1)/\alpha}}{1 - e^{-y/\alpha}}.$$

We get that for λ large enough

$$T_1(\lambda) \leq \delta^2 y (\lambda/c)^{1/\alpha} \frac{1}{1 - e^{-y/\alpha}}. \quad (4.35)$$

Using the monotony of the Laplace transform and the definitions of $n_1(\lambda)$ and $n_2(\lambda)$ we get

$$\begin{aligned} 0 \leq T_2(\lambda) &\leq -(n_2(\lambda) - n_1(\lambda)) \log \left(\mathbb{E} \left[e^{-\lambda e^{-(n_1(\lambda)+1)y} A^y} \right] \right) \\ &\leq -\frac{y + \log(\lambda_\delta/M)}{y} \log \left(\mathbb{E} \left[e^{-\lambda_\delta A^y} \right] \right) < +\infty. \end{aligned} \quad (4.36)$$

From the definitions of $n_2(\lambda)$ and M we have

$$\begin{aligned} 0 \leq T_3(\lambda) &\leq 2\lambda \mathbb{E} [A^y] \sum_{k=n_2(\lambda)+1}^{+\infty} e^{-ky} = 2\lambda e^{-y(K_2(\lambda)+1)} \mathbb{E} [A^y] / (1 - e^{-y}) \\ &\leq 2M \mathbb{E} [A^y] / (1 - e^{-y}) < +\infty. \end{aligned} \quad (4.37)$$

Putting (4.35), (4.36), and (4.37) into (4.34) we obtain that for λ large enough

$$\limsup_{\lambda \rightarrow +\infty} \mathcal{M}(\lambda) / \lambda^{1/\alpha} \leq \frac{\delta^2 y}{c^{1/\alpha} (1 - e^{-y/\alpha})}.$$

Since, from the definition $\mathcal{M}(\lambda) = -\log(\mathbb{E}[e^{-\lambda I(V^\uparrow)}])$ which does not depend on δ nor on y , we can let δ go to 1 and then y go to 0 in the above expression. We obtain

$$\limsup_{\lambda \rightarrow +\infty} -\log \left(\mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \right) / \lambda^{1/\alpha} \leq \alpha / c^{1/\alpha}.$$

For the first point of the proposition, we proceed exactly as for the second point, using the lower bound of (4.25) instead of the upper bound and noticing that $n_1(\lambda)$ converges toward $+\infty$ as λ goes to infinity, so that the factor $(1 - e^{-y(n_1(\lambda)+1)/\alpha}) / (1 - e^{-y/\alpha})$ converges to $1 / (1 - e^{-y/\alpha})$. \square

4.2. Tail at 0 of $I(V^\uparrow)$: proof of Theorems 1.2 and 1.5. We now prove Theorems 1.2 and 1.5 by using Proposition 4.2 together with the deep link that exists between the left tail of a random variable and the asymptotic behavior of its Laplace transform.

Proof. of Theorem 1.2

We assume that there is $\alpha > 1$ and a positive constant C such that for all λ large enough we have $\Psi_V(\lambda) \leq C\lambda^\alpha$. We fix $\delta \in]0, 1[$. According to the first point of Proposition 4.2, there exists $\lambda_\delta > 0$ such that for all $\lambda > \lambda_\delta$ we have

$$\mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \leq \exp \left(-\delta^{1-1/\alpha} \alpha \lambda^{1/\alpha} / C^{1/\alpha} \right). \quad (4.38)$$

Let us fix $x \in]0, \delta^{(\alpha-1)/\alpha} C^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$. Using Markov inequality and (4.38) we get that for any $\lambda > \lambda_\delta$,

$$\mathbb{P} \left(I(V^\uparrow) \leq x \right) \leq e^{\lambda x} \mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \leq \exp \left(\lambda x - \delta^{1-1/\alpha} \alpha \lambda^{1/\alpha} / C^{1/\alpha} \right).$$

Since $x \in]0, \delta^{(\alpha-1)/\alpha} C^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$ we have $\delta C^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)} \geq \lambda_\delta$ so, in the above inequality, we can replace λ by $\delta C^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)}$. We obtain

$$\mathbb{P} \left(I(V^\uparrow) \leq x \right) \leq \exp \left(-\delta(\alpha-1)/(Cx)^{1/(\alpha-1)} \right),$$

which is (1.4).

We now prove the second point. Assume that there is $\alpha > 1$ and a positive constant c such that for all λ large enough we have $\Psi_V(\lambda) \geq c\lambda^\alpha$. We fix $\delta > 1$ and $r \in]0, 1[$. According to the second point of Proposition 4.2, there exists $\lambda_\delta > 0$ such that for all $\lambda > \lambda_\delta$ we have

$$\mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \geq \exp \left(-\delta^{r(\alpha-1)/\alpha} \alpha \lambda^{1/\alpha} / c^{1/\alpha} \right). \quad (4.39)$$

Let us fix $x \in]0, \delta^{(\alpha-1)/\alpha} \alpha c^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$. Using (4.39) we get that for any $\lambda > \lambda_\delta$,

$$\begin{aligned} \exp \left(-\delta^{r(\alpha-1)/\alpha} \alpha \lambda^{1/\alpha} / c^{1/\alpha} \right) &\leq \mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] = \mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \mathbf{1}_{\{I(V^\uparrow) \leq x\}} \right] + \mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \mathbf{1}_{\{I(V^\uparrow) > x\}} \right] \\ &\leq \mathbb{P} \left(I(V^\uparrow) \leq x \right) + \exp(-\lambda x), \end{aligned}$$

so we get

$$\mathbb{P} \left(I(V^\uparrow) \leq x \right) \geq \exp \left(-\delta^{r(\alpha-1)/\alpha} \alpha \lambda^{1/\alpha} / c^{1/\alpha} \right) - \exp(-\lambda x).$$

Since $x \in]0, \delta^{(\alpha-1)/\alpha} \alpha c^{-1/\alpha} \lambda_\delta^{(1-\alpha)/\alpha}[$ we have $\delta \alpha^{\alpha/(\alpha-1)} c^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)} \geq \lambda_\delta$ so, in the above inequality, we can replace λ by $\delta \alpha^{\alpha/(\alpha-1)} c^{-1/(\alpha-1)} x^{-\alpha/(\alpha-1)}$. We obtain

$$\mathbb{P} \left(I(V^\uparrow) \leq x \right) \geq \exp \left(-\delta^{(1+r(\alpha-1))/\alpha} \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)} \right) - \exp \left(-\delta \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)} \right).$$

Since $\delta > \delta^{(1+r(\alpha-1))/\alpha}$, the second term converges to 0 faster than the first one when 0 goes to 0 so we get, for x large enough,

$$\mathbb{P} \left(I(V^\uparrow) \leq x \right) \geq \frac{1}{2} \exp \left(-\delta^{(1+r(\alpha-1))/\alpha} \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)} \right) \geq \exp \left(-\delta \alpha^{\alpha/(\alpha-1)} / (cx)^{1/(\alpha-1)} \right),$$

where the last inequality is true for x small enough and comes from the fact that $r \in]0, 1[$. This is precisely (1.5). □

Proof. of Theorem 1.5

We assume that there is a positive constant C and $\alpha \in]1, 2]$ such that $\Psi_V(\lambda) \sim_{\lambda \rightarrow +\infty} C\lambda^\alpha$. From Proposition 4.2 we deduce that

$$-\log \left(\mathbb{E} \left[e^{-\lambda I(V^\uparrow)} \right] \right) \underset{\lambda \rightarrow +\infty}{\sim} \alpha \lambda^{1/\alpha} / C^{1/\alpha},$$

and the application of De Bruijn's Theorem (see Theorem 4.12.9 in [5]) yields the result. □

4.3. Connection between $I(V^\uparrow)$ and $I(V)$: proof of Proposition 1.9. In this subsection, we assume that V drifts to $+\infty$ (so that $I(V) < +\infty$) and we prove a simple connection between the asymptotic tails at 0 of $I(V^\uparrow)$ and $I(V)$.

Proposition 4.3. *If V drifts to $+\infty$, there is a positive constant c such that for all positive ϵ and x small enough,*

$$\mathbb{P}(I(V) \leq x) \leq \mathbb{P}(I(V^\uparrow) \leq x) \leq \mathbb{P}(I(V) \leq (1 + \epsilon)x) / c\epsilon x.$$

As in the proof of Proposition 3.2, we decompose $I(V)$ as the sum of two independent random variables, one having the same law as a subordinator stopped at an independent exponential time and the other having the same law as $I(V^\uparrow)$. We first need an easy lemma about the asymptotic tail at 0 of a subordinator stopped at an independent exponential time.

Lemma 4.4. *Let S be a subordinator and T an independent exponential random variable, there exists a positive constant c such that for all x small enough*

$$\mathbb{P}(S_T < x) \geq cx.$$

Proof. We prove in fact a stronger result : the function $x \mapsto \mathbb{P}(S_T < x)$ is sub-additive, that is

$$\forall x, y \geq 0, \mathbb{P}(S_T < x + y) \leq \mathbb{P}(S_T < x) + \mathbb{P}(S_T < y) \quad (4.40)$$

and the lemma follows easily. Recall from the introduction the notation $\tau(S, h+)$ for $\tau(S, [h, +\infty[)$. Let $x, y > 0$ (the case when $x = 0$ or $y = 0$ is obvious), we have

$$\begin{aligned} \mathbb{P}(S_T < x + y) &= \mathbb{P}(S_T < x) + \mathbb{P}(T \geq \tau(S, x+), T < \tau(S, (x + y)+)) \\ &\leq \mathbb{P}(S_T < x) + \mathbb{P}(T \geq \tau(S, x+), T < \tau(S, [S_{\tau(S, x+)} + y, +\infty[))) \end{aligned}$$

because $S_{\tau(S, x+)} \geq x$ almost surely,

$$= \mathbb{P}(S_T < x) + \mathbb{P}(T \geq \tau(S, x+)) \times \mathbb{P}(T < \tau(S, y+)),$$

from the characteristic property of the exponential distribution and the Markov property applied to S at time $\tau(S, x+)$. Since $\mathbb{P}(T \geq \tau(S, x+)) \leq 1$ and $\mathbb{P}(T < \tau(S, y+)) = \mathbb{P}(S_T < y)$ we obtain (4.40). □

We now prove the proposition

Proof. of Proposition 4.3.

We write :

$$\begin{aligned} I(V) &= \int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt + \int_{\mathcal{R}(V,0)}^{+\infty} e^{-V(t)} dt \\ &= \int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt + \int_0^{+\infty} e^{-(V(t+\mathcal{R}(V,0)))} dt \\ &\stackrel{\text{L}}{=} \int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt + I(V^\uparrow), \end{aligned} \quad (4.41)$$

where we used the second point of Lemma 2.6 for the last inequality in which the two terms $\int_0^{\mathcal{R}(V,0)} e^{-V(t)} dt$ and $I(V^\uparrow)$ are independent.

We thus have

$$\mathbb{P}(I(V) \leq x) \leq \mathbb{P}(I(V^\uparrow) \leq x). \quad (4.42)$$

According to Lemma 2.4, the term $\int_0^{\mathcal{R}(V;0)} e^{-V(t)} dt$ has the same law as S_T where S is a pure jump subordinator with Lévy measure $G\eta(\cdot \cap F)$, the image measure by G of $\eta(\cdot \cap F)$, and T an independent exponential random variable with parameter $\eta(I)$.

For $\epsilon > 0$ and $x \geq 0$, combining the equality in law of lemma 2.4 with (4.41) we obtain

$$\mathbb{P}(I(V) \leq (1 + \epsilon)x) \geq \mathbb{P}(I(V^\uparrow) \leq x) \times \mathbb{P}(S_T \leq \epsilon x) \geq c\epsilon x \mathbb{P}(I(V^\uparrow) \leq x),$$

for an appropriate constant $c > 0$, when x is small enough, according to lemma 4.4. Combining with (4.42) we get the result. \square

Now, if V drifts to $+\infty$, Proposition 4.3 easily implies that the results of Theorems 1.2, 1.4, 1.5 and of Remarks 1.6, 1.7 are also true for $I(V)$, as long as they are for $I(V^\uparrow)$. Proposition 1.9 is thus proved.

5. SMOOTHNESS OF THE DENSITY : PROOF OF THEOREM 1.10

According to Proposition 3.2, $I(V^\uparrow)$ contains, as a convolution factor, the sum of infinitely many independent multiples of random variables having the same law as S_T (S_T being as in Lemma 2.1). We can thus use a condition on the Lévy measure of S to have the existence of the smooth density for $I(V^\uparrow)$. Actually, the condition that we check for S is the one of Proposition 28.3 in [14], which is a condition on the Lévy measure of a Lévy process for it to have a \mathcal{C}^∞ density with bounded derivatives.

As a jump of S is the image by the mapping G of an excursion of V^\sharp , we start by lemmas on the excursions of V^\sharp .

Lemma 5.1. *Assume $\sigma > 1$ and choose σ' such that $1 < \sigma' < \sigma$. For all h small enough we have*

$$\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) > h) \geq h^{-(\sigma'-1)}.$$

Proof. We consider excursions away from 0. Let $M > 0$ be a fixed level and, for any $h \in]0, M[$, p_h denotes the probability that V^\sharp has no finite excursion of height in $]h, M[$ before its first excursion higher than M . Since the set of finite excursions of height in $]h, M[$ and the set of excursions higher than M are disjoint, we have, by a property of Poisson point processes

$$p_h = \frac{\eta^\sharp(\xi, H_0(\xi) > M)}{\eta^\sharp(\xi, H_0(\xi) > M) + \eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) \in]h, M[)}, \quad (5.43)$$

so we only need to give an upper bound for p_h . Now, note that p_h is only the probability that $\tau(V^\sharp, h)$ and $\tau(V^\sharp, M)$ belong to the same excursion of V^\sharp away from 0, so

$$p_h = \mathbb{P}\left(\forall s \in]\tau(V^\sharp, h), \tau(V^\sharp, M)[, V^\sharp(s) \neq 0\right) = \mathbb{P}\left(\forall s \in]0, \tau(V_h^\sharp, M)[, V_h^\sharp(s) \neq 0\right),$$

where we used the Markov property at time $\tau(V^\sharp, h)$.

$$= \mathbb{P}\left(\tau(V_h^\sharp, M) < \tau(V_h^\sharp,]-\infty, 0])\right) = W^\sharp(h)/W^\sharp(M+h) \underset{h \rightarrow 0}{\sim} W^\sharp(h)/W^\sharp(M),$$

where W^\sharp is the scale function of V^\sharp , and where the last equivalence comes from the continuity of the scale function (see the Introduction). Recall that W^\sharp has a Laplace transform given by the expression

$$\int_0^{+\infty} W^\sharp(x) e^{-\lambda x} dx = \frac{1}{\Psi_{V^\sharp}(\lambda)}, \quad \forall \lambda \geq 0. \quad (5.44)$$

Now, for any $h > 0$, by increases of W^\sharp we have

$$\begin{aligned} W^\sharp(h) &\leq h^{-1} \int_0^{2h} W^\sharp(x) dx \leq e^2 h^{-1} \int_0^{2h} W^\sharp(x) e^{-x/h} dx \\ &\leq e^2 h^{-1} \int_0^{+\infty} W^\sharp(x) e^{-x/h} dx \\ &= \frac{e^2}{h \Psi_{V^\sharp}(h^{-1})}, \end{aligned}$$

because of (5.44). From the definition of σ and the fact that $\sigma' < \sigma$, we know that $\Psi_{V^\sharp}(h^{-1})/e^2 \geq h^{-\sigma'}$ provided h is small enough. We deduce that whenever h is small enough,

$$W^\sharp(h) \leq h^{\sigma'-1}.$$

We thus get for h small enough,

$$p_h \leq 2h^{\sigma'-1}/W^\sharp(M), \quad (5.45)$$

and taking the inverse in (5.43),

$$\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) \in]h, M[) \geq \eta^\sharp(\xi, H_0(\xi) > M) \times \left(\frac{1}{p_h} - 1 \right). \quad (5.46)$$

Since $\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) > h) \geq \eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) \in]h, M[)$, the combination of (5.45) and (5.46) yields the result. We got rid of the constants since the same result with the same constant is true for σ' increased a little bit. \square

We now need the following lemma which states that for a spectrally negative Lévy process, the excursions of a given height can be split into two independents parts of which the law are known.

Lemma 5.2. *For any $h > 0$, assume that the process X follows the law $\eta^\sharp(\cdot | H(\cdot) > h)$. Then we have :*

- $(X(s), 0 \leq s \leq \tau(X, h)) \sim (V^\uparrow(s), 0 \leq s \leq \tau(V^\uparrow, h))$,
- $(X(s), \tau(X, h) \leq s \leq \tau(X(\cdot + \tau(X, h)), 0)) \sim (V_h^\sharp(s), 0 \leq s \leq \tau(V_h^\sharp, 0))$,
- $(X(s), 0 \leq s \leq \tau(X, h)) \perp\!\!\!\perp (X(s), \tau(X, h) \leq s \leq \tau(X(\cdot + \tau(X, h)), 0))$.

Note that the time $\tau(V_h^\sharp, 0)$ may possibly be infinite, but this is unlikely when h is small.

Proof. of Lemma 5.2

The first point is a consequence of the first point of Lemma 2.6. The second and third points come from the Markov property. \square

We can now prove the main lemma of this section, it will allow us to check the condition on the Lévy measure of S .

Lemma 5.3. *Assume $\sigma > 1$ and choose σ' and β' such that $1 < \sigma' < \sigma \leq \beta < \beta'$. We also choose $\epsilon > \frac{\beta'-1}{\sigma'-1} - 1$ and fix $C > 0$ an arbitrary constant. Then, for r small enough we have*

$$\eta^\sharp(\xi, \zeta(\xi) \in]x, r]) \geq x^{-(\sigma'-1)/\beta'}, \quad \forall x \in]0, Cr^{1+\epsilon}].$$

Proof. Let us fix $r > 0$ and $x \in]0, Cr^{1+\epsilon}]$, then

$$\begin{aligned} \eta^\sharp(\xi, \zeta(\xi) \in]x, r]) &\geq \eta^\sharp\left(\xi, \zeta(\xi) \in [x, r], H_0(\xi) > x^{1/\beta'}\right) \\ &= \eta^\sharp\left(\xi, \zeta(\xi) \in]x, r] \mid H_0(\xi) > x^{1/\beta'}\right) \times \eta^\sharp\left(\xi, H_0(\xi) > x^{1/\beta'}\right), \end{aligned}$$

where $\eta^\sharp(\cdot \mid H(\cdot) > x^{1/\beta'})$ is the measure of excursions conditioned to be higher than $x^{1/\beta'}$. The last quantity thus equals

$$\left[\eta^\sharp\left(\xi, \zeta(\xi) > x \mid H_0(\xi) > x^{1/\beta'}\right) - \eta^\sharp\left(\xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'}\right)\right] \times \eta^\sharp\left(\xi, H_0(\xi) > x^{1/\beta'}\right) \quad (5.47)$$

We now study the three quantities appearing in (5.47) and show that this expression is of the same order as $\eta^\sharp(\xi, H_0(\xi) > x^{1/\beta'})$ for which Lemma 5.1 provides a lower bound. We start by proving that $\eta^\sharp(\xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'})$ converges to 0 uniformly in $x \in]0, Cr^{1+\epsilon}]$ when r goes to 0. First, Lemma 5.2 gives for all x in $]0, Cr^{1+\epsilon}]$ that

$$\eta^\sharp\left(\xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'}\right) \leq \mathbb{P}\left(\tau(V_{x^{1/\beta'}}^\sharp, 0) > r/2\right) + \mathbb{P}\left(\tau(V^\uparrow, x^{1/\beta'}) > r/2\right). \quad (5.48)$$

The first thing is thus to prove that

$$\sup_{x \in]0, Cr^{1+\epsilon}]} \mathbb{P}\left(\tau(V_{x^{1/\beta'}}^\sharp, 0) > r/2\right) \xrightarrow{r \rightarrow 0} 0. \quad (5.49)$$

Before reaching 0 (if it does) the process $V_{x^{1/\beta'}}^\sharp$ reaches $] -\infty, 0]$ (not necessarily at 0). We define $\tau_{x^{1/\beta'}}^N := \tau(V_{x^{1/\beta'}}^\sharp,] -\infty, 0])$. For all x in $]0, Cr^{1+\epsilon}]$ we have

$$\mathbb{P}\left(\tau(V_{x^{1/\beta'}}^\sharp, 0) > r/2\right) \leq \mathbb{P}\left(\tau_{x^{1/\beta'}}^N > r/4\right) + \mathbb{P}\left(\tau_{x^{1/\beta'}}^N \leq r/4, \tau(V_{x^{1/\beta'}}^\sharp(\cdot + \tau_{x^{1/\beta'}}^N), 0) > r/4\right) \quad (5.50)$$

Since $(x \mapsto \tau_{x^{1/\beta'}}^N)$ is stochastically increasing in the variable x we have

$$\sup_{x \in]0, Cr^{1+\epsilon}]} \mathbb{P}\left(\tau_{x^{1/\beta'}}^N > r/4\right) = \mathbb{P}\left(\tau_{C^{1/\beta'} r^{(1+\epsilon)/\beta'}}^N > r/4\right) = \mathbb{P}\left(\underline{V}^\sharp(r/4) > -C^{1/\beta'} r^{(1+\epsilon)/\beta'}\right).$$

We introduce T , an exponential random variable with parameter 1 independent of the process V , and a decreasing function $q :]0, +\infty[\rightarrow]0, +\infty[$ that converges to $+\infty$ as r goes to 0 and will be specified latter. We have

$$\begin{aligned} \mathbb{P}\left(\underline{V}^\sharp(r/4) > -C^{1/\beta'} r^{(1+\epsilon)/\beta'}\right) &\leq \mathbb{P}\left(\underline{V}^\sharp(T/q(r)) > -C^{1/\beta'} r^{(1+\epsilon)/\beta'}\right) + \mathbb{P}(T/q(r) > r/4) \\ &= \mathbb{P}\left(\exp\left(\frac{\underline{V}^\sharp(T/q(r))}{r^{(1+\epsilon)/\beta'}}\right) > \exp(-C^{1/\beta'})\right) + e^{-rq(r)/4} \\ &\leq e^{C^{1/\beta'}} \mathbb{E}\left[\exp\left(\frac{\underline{V}^\sharp(T/q(r))}{r^{(1+\epsilon)/\beta'}}\right)\right] + e^{-rq(r)/4}, \end{aligned}$$

from Markov inequality,

$$= \frac{1/\Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} - 1}{\Psi_{V^\sharp}(r^{-(1+\epsilon)/\beta'})/q(r) - 1} \times e^{C^{1/\beta'}} + e^{-rq(r)/4},$$

from the expression of the Laplace transform of the random variable $\underline{V}^\sharp(T/q(r))$ that can be found page 192 of [3], and where $\Phi_{V^\sharp} = \Psi_{V^\sharp}^{-1}$ as in Section 4.1. The last expression goes to 0 if these three conditions are satisfied :

$$\begin{aligned} q(r)r &\xrightarrow[r \rightarrow 0]{} +\infty, \\ \Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0, \\ q(r)/\Psi_{V^\sharp}(r^{-(1+\epsilon)/\beta'})\Phi_{V^\sharp}(q(r))r^{(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

From the definition of σ and β and the fact that $\sigma' < \sigma \leq \beta < \beta'$, we have $\Phi_{V^\sharp}(u) \leq u^{1/\sigma'}$, $\Psi_{V^\sharp}(u) \geq u^{\sigma'}$ and $\Phi_{V^\sharp}(u) \geq u^{1/\beta'}$, provided u is large enough. The three conditions can thus be simplified and we only need to have :

$$\begin{aligned} q(r)r &\xrightarrow[r \rightarrow 0]{} +\infty, \\ (q(r))^{1/\sigma'}r^{(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0, \\ (q(r))^{1-1/\beta'}r^{(\sigma'-1)(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

Elevating $(q(r))$ to the right power so it makes its exponent disappear, these three conditions become

$$\begin{aligned} q(r)r &\xrightarrow[r \rightarrow 0]{} +\infty, \\ q(r)r^{\sigma'(1+\epsilon)/\beta'} &\xrightarrow[r \rightarrow 0]{} 0, \\ q(r)r^{(\sigma'-1)(1+\epsilon)/(\beta'-1)} &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

Since $\sigma' < \beta'$ we can check that $\sigma'/\beta' > (\sigma'-1)/(\beta'-1)$, so the third condition implies the second. We then only need to verify the first and the third condition, but from the choice of ϵ , we have that $(\sigma'-1)(1+\epsilon)/(\beta'-1) > 1$, so $q(r)$ can be chosen such that the first and the third conditions are satisfied. This yields

$$\sup_{x \in]0, Cr^{1+\epsilon}] } \mathbb{P} \left(\tau_{x^{1/\beta'}}^N > r/4 \right) \xrightarrow[r \rightarrow 0]{} 0. \quad (5.51)$$

We now turn to the second term of (5.50). It is known that the jumps of V^\sharp is a Poisson process with intensity measure ν^\sharp , the Lévy measure of V^\sharp . The probability that V^\sharp has a jump smaller than $-r$ before time $r/4$ is thus $1 - e^{r\nu^\sharp([-\infty, -r])}/4 \sim r\nu^\sharp([-\infty, -r])/4 := \gamma(r)$ and this goes to 0 for any Lévy measure. As a consequence, on $\{\tau_{x^{1/\beta'}}^N \leq r/4\}$ we have $V_{x^{1/\beta'}}^\sharp(\tau_{x^{1/\beta'}}^N) \in [-r, 0]$, exepted on some event having probability less than $\gamma(r)$. We thus get

$$\sup_{x \in]0, Cr^{1+\epsilon}] } \mathbb{P} \left(\tau_{x^{1/\beta'}}^N \leq r/4, \tau(V_{x^{1/\beta'}}^\sharp(\cdot + \tau_{x^{1/\beta'}}^N), 0) > r/4 \right) \leq \mathbb{P} \left(\tau(V^\sharp, r) > r/4 \right) + \gamma(r),$$

and we now only need to show that $\tau(V^\sharp, r)/r$ converges to 0 in probability. Now recall that we know from Theorem VII.1 in [3] that the Laplace transform of this random variable is given by

$$\mathbb{E} \left[\exp \left(-\lambda \frac{\tau(V^\sharp, r)}{r} \right) \right] = e^{-r\Phi_{V^\sharp}(\lambda/r)}.$$

Since $\Phi_{V^\sharp}(u) \leq u^{1/\sigma'}$ for u large enough, the last quantity converges to 1 when r goes to 0 so we indeed have the convergence to 0 in probability of $\tau(V^\sharp, r)/r$ and as a consequence

$$\sup_{x \in]0, Cr^{1+\epsilon}] } \mathbb{P} \left(\tau_{x^{1/\beta'}}^N \leq r/4, \tau(V_{x^{1/\beta'}}^\sharp(\cdot + \tau_{x^{1/\beta'}}^N), 0) > r/4 \right) \xrightarrow{r \rightarrow 0} 0.$$

Putting this, together with (5.51), in (5.50) we get (5.49). We now deal with the second term of (5.48), more precisely we prove that

$$\sup_{x \in]0, Cr^{1+\epsilon}] } \mathbb{P} \left(\tau(V^\uparrow, x^{1/\beta'}) > r/2 \right) \xrightarrow{r \rightarrow 0} 0. \quad (5.52)$$

Because of the increases of the quantity $\mathbb{P}(\tau(V^\uparrow, x^{1/\beta'}) > r/2)$ we can write

$$\begin{aligned} \sup_{x \in]0, Cr^{1+\epsilon}] } \mathbb{P} \left(\tau(V^\uparrow, x^{1/\beta'}) > r/2 \right) &= \mathbb{P} \left(\tau(V^\uparrow, C^{1/\beta'} r^{(1+\epsilon)/\beta'}) > r/2 \right) \\ &\leq \mathbb{P} \left(\tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'}) > r/2 \right), \end{aligned}$$

according to the first point of Lemma 2.6 applied with $y = C^{1/\beta'} r^{(1+\epsilon)/\beta'}$.

Therefore, (5.52) will follow if we prove that the random variable $\tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'})/r$ converge to 0 in probability as r goes to 0. Again, we see from Theorem VII.1 in [3] that the Laplace transform of this random variable is given by

$$\mathbb{E} \left[\exp \left(-\lambda \frac{\tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'})}{r} \right) \right] = e^{-C^{1/\beta'} r^{(1+\epsilon)/\beta'} \Phi_{V^\sharp}(\lambda r^{-1})},$$

but, since $\Phi_{V^\sharp}(u) \leq u^{1/\sigma'}$ provided λ is large enough, we have that, for small r ,

$$0 \leq r^{(1+\epsilon)/\beta'} \Phi_{V^\sharp}(\lambda r^{-1}) \leq r^{(1+\epsilon)/\beta' - 1/\sigma'} \lambda^{1/\sigma'}.$$

By the choice of ϵ , we have $1 + \epsilon > (\beta' - 1)(\sigma' - 1) > \beta'/\sigma'$, so the last quantity converges to 0 as r goes to 0. This shows that the Laplace transform of $\tau(V^\sharp, C^{1/\beta'} r^{(1+\epsilon)/\beta'})/r$ converges to 1 as r goes to 0 so we get the asserted convergence to 0 in probability as r goes to 0 and (5.52) follows.

Putting (5.49) and (5.52) in (5.48) we get

$$\sup_{x \in]0, Cr^{1+\epsilon}] } \eta^\sharp \left(\xi, \zeta(\xi) > r \mid H_0(\xi) > x^{1/\beta'} \right) \xrightarrow{r \rightarrow 0} 0, \quad (5.53)$$

which means that among the excursions of heigh greater than $x^{1/\beta'}$, those of length greater than r are in negligible proportion, and it thus remains to show that those of length greater than x are in non-negligible proportion. We want to show that

$$\liminf_{r \rightarrow 0} \inf_{x \in]0, Cr^{1+\epsilon}] } \eta^\sharp \left(\xi, \zeta(\xi) > x \mid H_0(\xi) > x^{1/\beta'} \right) > 0. \quad (5.54)$$

From the second point of Lemma 5.2 we have for all x in $]0, Cr^{1+\epsilon}]$,

$$\eta^\sharp \left(\xi, \zeta(\xi) > x \mid H_0(\xi) > x^{1/\beta'} \right) \geq \mathbb{P} \left(\tau(V_{x^{1/\beta'}}^\sharp, 0) > x \right) \geq \mathbb{P} \left(\tau_{x^{1/\beta'}}^N > x \right). \quad (5.55)$$

As we did before, we introduce T , an exponential random variable with parameter 1 independent of the process V^\sharp , and a decreasing function $q :]0, +\infty[\rightarrow]0, +\infty[$ that converges to $+\infty$ at 0 and will be specified latter.

For any x in $]0, Cr^{1+\epsilon}]$ we have

$$\begin{aligned} \mathbb{P}\left(\tau_{x^{1/\beta'}}^N > x\right) &= \mathbb{P}\left(\underline{V}^\sharp(x) > -x^{1/\beta'}\right) \\ &\geq \mathbb{P}\left(\underline{V}^\sharp(T/q(x)) > -x^{1/\beta'}\right) - \mathbb{P}(T/q(x) < x) \\ &= \mathbb{P}\left(\frac{\underline{V}^\sharp(T/q(x))}{x^{1/\beta'}} > -1\right) - 1 + e^{-xq(x)} \end{aligned}$$

We choose $q(x) = 1/x$ and, because of (5.55), (5.54) will follow if we prove that the random variable $\underline{V}^\sharp(T/q(x))/x^{1/\beta'}$ converges in probability to 0 as x goes to 0. According to [3] p 192, the Laplace transform of this random variable is given by

$$\mathbb{E}\left[\exp\left(\lambda \frac{\underline{V}^\sharp(T/q(x))}{x^{1/\beta'}}\right)\right] = \frac{q(x) \left(\Phi_{V^\sharp}(q(x)) - \lambda/x^{1/\beta'}\right)}{\Phi_{V^\sharp}(q(x)) (q(x) - \Psi_{V^\sharp}(\lambda/x^{1/\beta'}))}.$$

This last quantity converges to 1 when x goes to 0 if

- $\lambda/x^{1/\beta'}$ is negligible compared to $\Phi_{V^\sharp}(q(x))$ when x goes to 0,
- $\Psi_{V^\sharp}(\lambda/x^{1/\beta'})$ is negligible compared to $q(x)$ when x goes to 0.

These two conditions can be written

$$\begin{aligned} x^{1/\beta'} \Phi_{V^\sharp}(q(x)) &\xrightarrow[r \rightarrow 0]{} +\infty, \\ \Psi_{V^\sharp}(\lambda/x^{1/\beta'})/q(x) &\xrightarrow[r \rightarrow 0]{} 0. \end{aligned}$$

Now, because of the definition of β , because $\beta' > \beta$, and because $q(x) = 1/x$, it is easy to see that these two conditions are satisfied. This shows that the Laplace transform of the random variable $\underline{V}^\sharp(T/q(x))/x^{1/\beta'}$ converges to 1 as x goes to 0, so this random variable converges to 0 in probability and (5.54) follows.

For the factor $\eta^\sharp(\xi, H_0(\xi) > x^{1/\beta'})$ of (5.47) we note that it is trivially more than $\eta^\sharp(\xi, \zeta(\xi) < +\infty, H_0(\xi) > x^{1/\beta'})$ and we can use Lemma 5.1 which yields

$$\eta^\sharp\left(\xi, H_0(\xi) > x^{1/\beta'}\right) \geq x^{-(\sigma'-1)/\beta'}, \quad (5.56)$$

for x small enough. Putting (5.53), (5.54) and (5.56) in (5.47), we get the asserted result. Here again, we actually obtain the result up to a multiplicative constant, but since the result is still true if, for example, we increase σ' a little bit, we can get rid of the constant. \square

We can now prove Theorem 1.10.

Proof. of Theorem 1.10

We make the assumption that (1.8) is satisfied so, by continuity of the left hand side of (1.8) in σ and β , we can choose σ' and β' to be as in Lemma 5.3, but close enough to respectively σ and β so that they also satisfy (1.8). We also choose ϵ as in Lemma 5.3.

We fix $y > 0$. Let S is a pure jump subordinator with Lévy measure $\mu(\cdot) := G\eta_y^\sharp(\cdot \cap FP_\infty)$, the image measure of $\eta_y^\sharp(\cdot \cap FP_\infty)$ by the mapping G . From the Lévy-Khintchine formula, the characteristic function of $S(t)$ is

$$\mathbb{E}\left[e^{i\xi S(t)}\right] = e^{t\Phi_S(\xi)},$$

where

$$\forall \xi \in \mathbb{R}, \quad \Phi_S(\xi) := \int_0^{+\infty} (e^{i\xi x} - 1) \mu(dx),$$

so, taking the real part,

$$\begin{aligned} \forall \xi \in \mathbb{R}, \quad |\mathcal{R}(\Phi_S(\xi))| &= \int_0^{+\infty} (1 - \cos(\xi x)) \mu(dx) \\ &\geq \int_0^{\pi/|\xi|} (1 - \cos(\xi x)) \mu(dx) \\ &\geq \frac{2\xi^2}{\pi^2} \int_0^{\pi/|\xi|} x^2 \mu(dx). \end{aligned} \quad (5.57)$$

We now prove that the measure μ satisfies the hypothesis of Proposition 28.3 of [14]. We have for any $r \in]0, 1[$:

$$\int_0^r x^2 \mu(dx) = 2 \int_0^r x \mu([x, r]) dx \geq 2 \int_0^{r^{1+\epsilon}} x \mu([x, r]) dx. \quad (5.58)$$

We thus need to minorate $\mu([x, r])$ for $x \in]0, r^{1+\epsilon}[$:

$$\begin{aligned} \mu([x, r]) &= \eta_y^\#(\{\xi, G(\xi) \in [x, r] \} \cap FP_\infty) \\ &= \eta_y^\# \left(\left\{ \xi, \int_0^{\zeta(\xi)} e^{-\xi(t)} dt \in [x, r] \right\} \cap FP_\infty \right), \end{aligned}$$

from the definition of G ,

$$\begin{aligned} &\geq \eta_y^\#(\{\xi, \zeta(\xi) \in [e^{2y}x, r], \sup \xi \leq 2y\} \cap FP_\infty) \\ &\geq \eta_y^\#(\{\xi, \zeta(\xi) \in [e^{2y}x, r]\}) - \eta_y^\#(\{\xi, \sup \xi > 2y\}) - \eta_y^\#(FP_\infty^c) \\ &= \eta_y^\#(\{\xi, \zeta(\xi) \in [e^{2y}x, r]\}) - \eta_y^\#(\{\xi, \sup \xi > 2y\}) - \eta_y^\#(IP_\infty) - \eta_y^\#(N) \\ &= \eta_y^\#(\{\xi, \zeta(\xi) \in [e^{2y}x, r]\}) - c, \end{aligned}$$

where we have put $c := \eta_y^\#(\{\sup \xi > 2y\}) + \eta_y^\#(IP_\infty) + \eta_y^\#(N)$. Note that c is well defined because the quantities $\eta_y^\#(\{\sup \xi > 2y\})$, $\eta_y^\#(IP_\infty)$ and $\eta_y^\#(N)$ are finite.

We now apply Lemma 5.3 (taking $C = e^{2y}$) and we deduce that for r small enough,

$$\forall x \in]0, r^{1+\epsilon}], \quad \mu([x, r]) \geq e^{-2y(\sigma'-1)/\beta'} x^{-(\sigma'-1)/\beta'} - c.$$

Combining this with (5.58), we get that whenever r is small enough :

$$\begin{aligned} \int_0^r x^2 \mu(dx) &\geq 2e^{-2y(\sigma'-1)/\beta'} \int_0^{r^{1+\epsilon}} x^{1-(\sigma'-1)/\beta'} dx - 2c \int_0^{r^{1+\epsilon}} x dx \\ &\geq \frac{2e^{-2y(\sigma'-1)/\beta'}}{2 - (\sigma' - 1)/\beta'} r^{(1+\epsilon)(2-(\sigma'-1)/\beta')} - cr^{2(1+\epsilon)}. \end{aligned}$$

If

$$(1 + \epsilon)(2 - (\sigma' - 1)/(3\beta' - 1)) < 2, \quad (5.59)$$

then, choosing $\delta \in]0, 2 - (1 + \epsilon)(2 - (\sigma' - 1)/\beta')[$ and combining the above estimate with (5.57), we get that

$$|\Phi_S(\xi)| \geq |\mathcal{R}(\Phi_S(\xi))| \geq |\xi|^\delta, \quad (5.60)$$

whenever $|\xi|$ is large enough. As the only assumption on ϵ is that it is greater than $(\beta' - 1)/(\sigma' - 1) - 1$, we can choose ϵ such that (5.59) is satisfied if and only if

$$\frac{\beta' - 1}{\sigma' - 1} (2 - (\sigma' - 1)/(3\beta' - 1)) < 2,$$

which is equivalent to the fact that σ' and β' satisfy (1.8). Therefore, we have proved that there exists $\delta > 0$ such that (5.60) is true.

Let T be an exponential random variable with parameter $p := \eta_y^\sharp(IP_\infty) + \eta_y^\sharp(N)$ which is independent of S , the Fourier transform of S_T is

$$\mathbb{E} \left[e^{i\xi S_T} \right] = \int_0^{+\infty} p e^{t\Phi_S(\xi)} e^{-pt} dt = \frac{p}{p - \Phi_S(\xi)} = \underset{|\xi| \rightarrow +\infty}{\mathcal{O}} \left(|\xi|^{-\delta} \right), \quad (5.61)$$

because of (5.60).

Proposition 3.2 gives the decomposition

$$I(V^\uparrow) = \sum_{k \geq 0} e^{-ky} B_k^y + \sum_{k \geq 0} e^{-ky} C_k^y$$

where all the random variables in the two series are mutually independent, and where each term B_k^y has the same law as S_T . Therefore, the characteristic function of $I(V^\uparrow)$ is the product of a characteristic function bounded by 1 and of $\prod_{k \geq 0} \mathbb{E}[e^{ie^{-ky}\xi S_T}]$ which, thanks to (5.61), goes to 0 faster than any negative power of $|\xi|$. This proves that the density of $I(V^\uparrow)$ is of class \mathcal{C}^∞ and that all its derivatives converge to 0 when x goes to $+\infty$. The derivatives of the density of $I(V^\uparrow)$ also converge to 0 when x goes to 0 since this density is of class \mathcal{C}^∞ and null on $] -\infty, 0[$.

To prove that $\phi_{I(V^\uparrow)}$ actually belongs to the Schwartz space, we have to study a little more deeply the infinite product. Let us denote by ψ the characteristic function of $\sum_{k \geq 0} e^{-ky} C_k^y$. Then

$$\phi_{I(V^\uparrow)}(\xi) = \psi(\xi) \prod_{k \geq 0} \mathbb{E} \left[e^{ie^{-ky}\xi S_T} \right]. \quad (5.62)$$

The random variable S_T admits moments of any positive order because it is a convolution factor of $I(V^\uparrow)$ which admits moments of any positive order thanks to Theorem 1.1. As a consequence the derivatives at any order of functions of the kind of $(\xi \mapsto \mathbb{E}[e^{ie^{-ky}\xi S_T}])$, for integers k , are defined and bounded. For any $n \in \mathbb{N}$ and $m > n$, we can see by induction that the n^{th} derivative

$$P_{m,n} := \left(\xi \mapsto \prod_{k=0}^m \mathbb{E} \left[e^{ie^{-ky}\xi S_T} \right] \right)^{(n)}$$

is a finite sum of products. In each of these products, there are at least $m - n$ factors of the form $\mathbb{E}[e^{ie^{-ky}\xi S_T}]$ for some integers k and the other factors are derivatives at some orders of the functions $(\xi \mapsto \mathbb{E}[e^{ie^{-ky}\xi S_T}])$ for some integers k . Therefore, from (5.61), we deduce that

$$P_{m,n}(\xi) = \underset{|\xi| \rightarrow +\infty}{\mathcal{O}} \left(|\xi|^{-(m-n)\delta} \right). \quad (5.63)$$

We decompose (5.62) into

$$\phi_{I(V^\uparrow)}(\xi) = \left(\prod_{k=0}^m \mathbb{E} \left[e^{ie^{-ky}\xi S_T} \right] \right) \times R_m(\xi),$$

where

$$R_m(\xi) := \psi(\xi) \prod_{k \geq m+1} \mathbb{E} \left[e^{ie^{-ky} \xi S_T} \right].$$

From the Leibniz formula applied to the product, we have

$$\phi_{I(V^\uparrow)}^{(n)}(\xi) = \sum_{k=0}^n C_n^k P_{m,k}(\xi) R_m^{(n-k)}(\xi).$$

R_m is the Fourier transform of a random variable that admits moments of any positive order (because it is a convolution factors of $I(V^\uparrow)$), so its derivatives at any order are defined and bounded. From (5.63) we thus get that

$$\phi_{I(V^\uparrow)}^{(n)}(\xi) = \mathcal{O} \left(|\xi|^{-(m-n)\delta} \right).$$

As m is arbitrary, $\phi_{I(V^\uparrow)}^{(n)}$ goes to 0 faster than any negative power of $|\xi|$. Therefore, $\phi_{I(V^\uparrow)}$ belongs to the Schwartz space and so does the density of $I(V^\uparrow)$, since the Schwartz space is stable by Fourier transform. □

Remark 5.4. *The case where V has bounded variation is not contained in Theorem 1.10. Moreover, Remark 2.2 shows that the law of S_T (S_T being as in Lemma 2.1) has an atom at 0 if V has bounded variation, so there is no hope to generalize our proof of Theorem 1.10 to this case.*

We also prove Corollary 1.11.

Proof. of Corollary 1.11

Since V drifts to $+\infty$ we have $V^\sharp = V$ so the expression (4.41) in the proof of Proposition 4.3 tells us that $I(V^\uparrow)$ is a convolution factor of $I(V)$. Now, under the assumptions of the corollary, Theorem 1.10 applies and we get the regularity of the density of $I(V)$ thanks to the boundedness of the derivatives of the density of $I(V^\uparrow)$ and the differentiation under the integral sign theorem. We get the convergence to 0 at $+\infty$ of the derivatives of the density of $I(V)$ thanks to the boundedness of the derivatives of the density of $I(V^\uparrow)$ and the dominated convergence theorem. The convergence to 0 at 0 of the derivatives of the density of $I(V)$ comes from the fact that this density is of class C^∞ and null on $] -\infty, 0[$. □

6. THE SPECTRALLY POSITIVE CASE

We now make a brief study of the exponential functional of Z^\uparrow where Z is a spectrally positive Lévy process drifting to $+\infty$. If Z is a subordinator, then it stays positive and $I(Z^\uparrow)$ is only $I(Z)$ which is already known to be finite and have some finite exponential moments (see for example Theorem 2 in [4]), so Theorem 1.13 is already known in this case.

We thus assume that Z is not a subordinator. Since, in this case, $-Z$ is spectrally negative and not the opposite of a subordinator (then, we denote by κ the non-trivial zero of Ψ_{-Z}), it is regular for $]0, +\infty[$ according to Theorem VII.1 in [3], so Z is regular for $] -\infty, 0[$. Moreover, Z drifts to $+\infty$. We can thus define the Markov family $(Z_x^\uparrow, x \geq 0)$ as in [7], Chapter 8. It can be seen from there that the processes such defined are Markov, have infinite life-time (this is where we need the hypothesis that Z drifts to $+\infty$) and that Z_0^\uparrow , that we denote by Z^\uparrow , is indeed well defined.

Here again, for any $x \geq 0$, the process Z_x^\uparrow must be seen as Z conditioned to stay positive and starting from x . Note that, since Z converges almost surely to infinity, for $x > 0$, Z_x^\uparrow is only Z_x conditioned in the usual sense to remain positive.

6.1. Finiteness, exponential moments : Proof of Theorem 1.13. The idea is that adding a small term of negative drift to Z^\uparrow does not change its convergence to $+\infty$. It makes Z^\uparrow ultimately greater than a deterministic linear function for which the exponential functional is defined and deterministically bounded. The key point is thus to control the time taken by Z^\uparrow to become greater than the linear function once and for good. We start with the following lemma.

Lemma 6.1. *For any $y > 0$, there exists $\epsilon > 0$ and positive constants c_1 and c_2 such that*

$$\forall s > 0, \mathbb{P}\left(\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot),] - \infty, 0]) > s\right) \leq c_1 e^{-c_2 s}.$$

Proof. We fix $y > 0$. From Corollary VII.2 in [3], a spectrally negative Lévy process X drifts to $-\infty$ if and only if $\mathbb{E}[X(1)] < 0$. Z is a spectrally positive Lévy process drifting to $+\infty$ so taking the dual in the theorem we get $\mathbb{E}[Z(1)] > 0$. Now $\mathbb{E}[(Z - \epsilon)(1)] = \mathbb{E}[Z(1)] - \epsilon$ which is positive for ϵ chosen small enough. Still taking the dual in Corollary VII.2 in [3], this implies that $Z - \epsilon$ is also a spectrally positive Lévy process that drifts to $+\infty$ and which is not a subordinator. We have

$$\begin{aligned} \mathbb{P}\left(\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot),] - \infty, 0]) > s\right) &= \mathbb{P}\left(\inf_{t \in [s, +\infty[} Z_y^\uparrow(t) - (y + \epsilon t) \leq 0\right) \\ &= \mathcal{C} \mathbb{P}\left(\inf_{t \in [s, +\infty[} Z_y(t) - (y + \epsilon t) \leq 0, \inf_{[0, +\infty[} Z_y > 0\right), \end{aligned}$$

where $\mathcal{C} := 1/\mathbb{P}(\inf_{[0, +\infty[} Z_y > 0) = 1/(1 - e^{-\kappa y})$. This comes from the fact that Z_y^\uparrow is only Z_y conditioned to stay positive in the usual sense. Now, noting that $Z_y = y + Z$, we bound the above quantity by

$$\begin{aligned} &\mathcal{C} \mathbb{P}\left(\inf_{t \in [s, +\infty[} Z(t) - \epsilon t \leq 0\right) = \mathcal{C} \mathbb{P}\left(\sup_{t \in [s, +\infty[} -Z(t) + \epsilon t \geq 0\right) \\ &\leq \mathcal{C} \mathbb{P}\left(\sup_{t \in [s, +\infty[} -Z(t) + \epsilon t \geq 0, -Z(s) + \epsilon s \leq 0\right) + \mathcal{C} \mathbb{P}(-Z(s) + \epsilon s \geq 0) \\ &= \mathcal{C} \mathbb{P}\left(\sup_{t \in [0, +\infty[} -Z^s(t) + \epsilon t \geq -(-Z(s) + \epsilon s), -Z(s) + \epsilon s \leq 0\right) + \mathcal{C} \mathbb{P}(-Z(s) + \epsilon s > 0). \end{aligned} \tag{6.64}$$

From the independence of the increments, the process $-Z^s + \epsilon \cdot$ is equal in law to $-Z + \epsilon \cdot$ and independent from $-Z(s) + \epsilon s$. From Corollary VII.2 in [3], the supremum over $[0, +\infty[$ of the process $-Z^s + \epsilon \cdot$ follows an exponential distribution with parameter α , where α is the non-trivial zero of $\Psi_{-Z+\epsilon}$. From this, combined with the independence from $-Z(s) + \epsilon s$, (6.64) becomes

$$\begin{aligned} &\mathcal{C} \mathbb{E}\left(e^{\alpha(-Z(s)+\epsilon s)} \mathbf{1}_{\{-Z(s)+\epsilon s \leq 0\}}\right) + \mathcal{C} \mathbb{P}(-Z(s) + \epsilon s > 0) \\ &\leq \mathcal{C} \mathbb{E}\left(e^{\alpha(-Z(s)+\epsilon s)/2} \mathbf{1}_{\{-Z(s)+\epsilon s \leq 0\}}\right) + \mathcal{C} \mathbb{P}\left(e^{\alpha(-Z(s)+\epsilon s)/2} > 1\right), \end{aligned}$$

from the decreases of negative exponential and composing by function $x \mapsto \exp(\frac{\alpha}{2}x)$ in the probability of the second term,

$$\leq \mathcal{C} \mathbb{E}\left(e^{\alpha/2(-Z(s)+\epsilon s)}\right) + \mathcal{C} \mathbb{P}\left(e^{\alpha(-Z(s)+\epsilon s)/2} > 1\right) \leq 2\mathcal{C} \mathbb{E}\left(e^{\alpha/2(-Z(s)+\epsilon s)}\right),$$

where we used Markov inequality in the second term,

$$= 2\mathcal{C} e^{s\Psi_{-Z+\epsilon}(\alpha/2)}.$$

As $\Psi_{-Z+\epsilon}$ is negative on $]0, \alpha[$, we get the result with $c_1 = 2\mathcal{C}$ and $c_2 = -\Psi_{-Z+\epsilon}(\alpha/2)$. \square

We can now prove Theorem 1.13.

Proof. of Theorem 1.13

We fix $y > 0$. Let m_y be the point where the process Z_y^\uparrow reaches its infimum, $m_y := \sup\{s \geq 0, Z_y^\uparrow(s-) \wedge Z_y^\uparrow(s) = \inf_{[0, +\infty[} Z_y^\uparrow\}$. Note that from the absence of negative jumps the infimum is always reached at least at m_y- so $Z_y^\uparrow(m_y-) = \inf_{[0, +\infty[} Z_y^\uparrow$. In order to get Z^\uparrow from Z_y^\uparrow , we use the decomposition given by Theorem 24 in [7], that is :

- The two processes $\left(Z_y^\uparrow(m_y + s) - Z_y^\uparrow(m_y-), s \geq 0\right)$ and $\left(Z_y^\uparrow(s), 0 \leq s < m_y\right)$ are independent,
- $\left(Z_y^\uparrow(m_y + s) - Z_y^\uparrow(m_y-), s \geq 0\right)$ is equal in law to Z^\uparrow .

Now,

$$\begin{aligned} I(Z_y^\uparrow) &= \int_0^{m_y} e^{-Z_y^\uparrow(u)} du + \int_{m_y}^{+\infty} e^{-Z_y^\uparrow(u)} du \\ &= \int_0^{m_y} e^{-Z_y^\uparrow(u)} du + e^{-Z_y^\uparrow(m_y-)} \int_0^{+\infty} e^{-(Z_y^\uparrow(m_y+u) - Z_y^\uparrow(m_y-))} du \\ &\geq e^{-y} \int_0^{+\infty} e^{-(Z_y^\uparrow(m_y+u) - Z_y^\uparrow(m_y-))} du, \end{aligned}$$

because almost surely $Z_y^\uparrow(m_y-) \leq y$,

$$\stackrel{\mathfrak{I}}{=} e^{-y} I(Z^\uparrow),$$

because of the above decomposition. We thus get

$$I(Z^\uparrow) \stackrel{sto}{\leq} e^y I(Z_y^\uparrow), \quad (6.65)$$

where $\stackrel{sto}{\leq}$ denotes a stochastic inequality. As a consequence we only need to prove the result for $I(Z_y^\uparrow)$. We now choose $\epsilon > 0$ as in Lemma 6.1. We have

$$\begin{aligned} 0 \leq I(Z_y^\uparrow) &= \int_0^{\mathcal{R}(Z_y^\uparrow(\cdot) - (y+\epsilon \cdot),]-\infty, 0])} e^{-Z_y^\uparrow(t)} dt + \int_{\mathcal{R}(Z_y^\uparrow(\cdot) - (y+\epsilon \cdot),]-\infty, 0])}^{+\infty} e^{-Z_y^\uparrow(t)} dt \\ &\leq \mathcal{R}(Z_y^\uparrow(\cdot) - (y+\epsilon \cdot),]-\infty, 0]) + \int_{\mathcal{R}(Z_y^\uparrow(\cdot) - (y+\epsilon \cdot),]-\infty, 0])}^{+\infty} e^{-Z_y^\uparrow(t)} dt, \end{aligned}$$

but for $t \geq \mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot),] - \infty, 0])$ we have $Z_y^\uparrow(t) \geq y + \epsilon t$, so

$$\begin{aligned} 0 \leq I(Z_y^\uparrow) &\leq \mathcal{R}\left(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot),] - \infty, 0]\right) + \int_{\mathcal{R}(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot),] - \infty, 0])}^{+\infty} e^{-y - \epsilon t} dt \\ &\leq \mathcal{R}\left(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot),] - \infty, 0]\right) + \int_0^{+\infty} e^{-y - \epsilon t} dt \\ &= \mathcal{R}\left(Z_y^\uparrow(\cdot) - (y + \epsilon \cdot),] - \infty, 0]\right) + e^{-y}/\epsilon. \end{aligned}$$

From Lemma 6.1, this is almost surely finite and admits some finite exponential moments. Thanks to (6.65), we have the same for $I(Z^\uparrow)$, which is the expected result. \square

6.2. Tails at 0 of $I(Z^\uparrow)$: Proof of Theorem 1.14. We need an analogous of Lemma 2.6 in order to compare, as we did in subsection 4.3, the exponential functionals $I(Z^\uparrow)$ and $I(Z)$. We define m , the point where the process Z reaches its infimum : $m := \sup\{s \geq 0, Z(s-) \wedge Z(s) = \inf_{]0, +\infty[} Z\}$. Here again, from the absence of negative jumps, the infimum is always reached at least at $m-$ so $Z(m-) = \inf_{]0, +\infty[} Z$.

Lemma 6.2. *If Z has unbounded variation, then $Z(m + \cdot) - Z(m-)$ has the same law as Z^\uparrow .*

Proof. $Z(m + \cdot) - Z(m-)$ is only the infinite excursion of the post-infimum process $Z - \underline{Z}$, so we only need to prove that this infinite excursion has the same law as Z^\uparrow , and for this we want to apply Proposition 4.7 of [8]. We already know that, because it is spectrally positive, Z is regular for $] - \infty, 0[$. Taking the dual of the process in Corollary VII.5 in [3], we get the regularity of $\{0\}$ and $]0, +\infty[$ for Z , thanks to the hypothesis of unbounded variation. The hypothesis of the proposition in [8] are thus fulfilled.

Let \mathcal{N} denote the excursion measure of the Markov process $Z - \underline{Z}$ and (L^{-1}, U) the ladder process of Z : L^{-1} is the inverse of the local time at 0 of $Z - \underline{Z}$ and for any positive t , $U(t) = \underline{Z}(L^{-1}(t))$.

Recall from the introduction the notation $\tau(A, h+)$ for $\tau(A, [h, +\infty[)$. We denote by \mathcal{U} the potential measure of U and, since $-Z$ is spectrally negative, the formula page 191 in [3] applies and yields that $\mathcal{U}(] - x, 0]) = (1 - e^{-\kappa x})/\kappa$ for any $x \geq 0$. Proposition 4.7 of [8] tells us that for any positive measurable function G defined on the space of càd-làg functions from $[0, +\infty[$ to \mathbb{R} with finite lifetime, we have

$$\begin{aligned} \mathbb{E} \left[G \left((Z^\uparrow(s))_{0 \leq s \leq \tau(Z^\uparrow, h+)} \right) \right] &= \mathcal{N} \left(G \left((\xi(s))_{0 \leq s \leq \tau(\xi, h+)} \right) \mathcal{U}(] - \xi(\tau(\xi, h+)), 0]) \mid \tau(\xi, h+) < \infty \right) \\ &\quad \times \mathcal{N}(\xi, \tau(\xi, h+) < \infty) \\ &= c_h \mathcal{N} \left((1 - e^{-\kappa \xi(\tau(\xi, h+))}) G \left((\xi(s))_{0 \leq s \leq \tau(\xi, h+)} \right) \mid \tau(\xi, h+) < \infty \right), \end{aligned} \tag{6.66}$$

replacing \mathcal{U} by its expression and where we set $c_h := \mathcal{N}(\xi, \tau(\xi, h+) < \infty) / \kappa$. Let ξ_∞ denote the infinite excursion of $Z - \underline{Z}$, then, for any positive measurable function F , we get that

$$\begin{aligned} \mathbb{E} [F((\xi_\infty(s))_{0 \leq s \leq \tau(\xi_\infty, h+)})] &= \mathcal{N} \left(\mathbb{P} \left(\inf_{[0, +\infty[} Z_{\xi(\tau(\xi, h+))} > 0 \right) F((\xi(s))_{0 \leq s \leq \tau(\xi, h+)}) \mid \tau(\xi, h+) < \infty \right) \\ &= \mathbb{E} \left[\frac{\mathbb{P} \left(\inf_{[0, +\infty[} Z_{Z^\uparrow(\tau(Z^\uparrow, h+))} > 0 \right)}{c_h(1 - e^{-\kappa Z^\uparrow(\tau(Z^\uparrow, h+))})} F((Z^\uparrow(s))_{0 \leq s \leq \tau(Z^\uparrow, h+)}) \right], \end{aligned} \quad (6.67)$$

where we used (6.66) with

$$G((\xi(s))_{0 \leq s \leq \tau(\xi, h+)}) := \frac{\mathbb{P}(\inf_{[0, +\infty[} Z_{\xi(\tau(\xi, h+))} > 0)}{c_h(1 - e^{-\kappa \xi(\tau(\xi, h+))})} \times F((\xi_\infty(s))_{0 \leq s \leq \tau(\xi_\infty, h+)}) .$$

It follows from (6.67) and from $Z^\uparrow(\tau(Z^\uparrow, h+)) \geq h$ that, $\mathbb{P}(\inf_{[0, +\infty[} Z_{\xi(\tau(\xi, h+))} > 0) / c_h(1 - e^{-\kappa \xi(\tau(\xi, h+))})$ is a bounded martingale with respect to the filtration $\mathcal{F}_h := \sigma(Z^\uparrow(s), 0 \leq s \leq \tau(Z^\uparrow, h+))$ and that it converges almost surely to some constant. As a consequence, this quantity is almost surely equal to 1 for any positive h , hence,

$$\forall h > 0, \mathbb{E} [F((\xi_\infty(s))_{0 \leq s \leq \tau(\xi_\infty, h+)})] = \mathbb{E} \left[F((Z^\uparrow(s))_{0 \leq s \leq \tau(Z^\uparrow, h+)}) \right],$$

so the infinite excursion of $Z - \underline{Z}$ indeed has the same law as Z^\uparrow . □

We can now prove Theorem 1.14.

Proof. of Theorem 1.14

We have

$$\begin{aligned} I(Z) &= \int_0^m e^{-Z(t)} dt + \int_m^{+\infty} e^{-Z(t)} dt \\ &= \int_0^m e^{-Z(t)} dt + e^{-Z(m-)} \int_m^{+\infty} e^{-(Z(m+t) - Z(m-))} dt \\ &\geq \int_m^{+\infty} e^{-(Z(m+t) - Z(m-))} dt, \end{aligned}$$

because almost surely $Z(m-) \leq 0$. Since Z has unbounded variation, we can use Lemma 6.2 which tells us that the last term is equal in law to $I(Z^\uparrow)$. We thus get

$$\mathbb{P}(I(Z) \leq x) \leq \mathbb{P}(I(Z^\uparrow) \leq x), \quad (6.68)$$

so we only need to prove the result for $I(Z)$. Obtaining (6.68) is the only thing for which we need the hypothesis of unbounded variation in this proof. The result that we now prove for $I(Z)$ is thus true without this hypothesis.

Let (Q, γ, ν) be the generating triplet of Z in the Lévy-Khintchine representation. Since ν is non zero, there exist $0 < \gamma_1 < \gamma_2 < +\infty$ such that $\nu([\gamma_1, \gamma_2]) > 0$. Then, for $\eta \in]0, 1[$ we define

$$\nu^{\eta, 1} := \eta \nu(\cdot \cap [\gamma_1, \gamma_2]) \quad \text{and} \quad \nu^{\eta, 2} := \nu - \eta \nu(\cdot \cap [\gamma_1, \gamma_2]),$$

and set $Z^{\eta, 1}$ and $Z^{\eta, 2}$ to be two independent Lévy processes which generating triplets are respectively $(0, 0, \nu^{\eta, 1})$ and $(Q, \gamma, \nu^{\eta, 2})$. We have $\nu = \nu^{\eta, 1} + \nu^{\eta, 2}$ so according to the Lévy-Khintchine

formula,

$$Z \stackrel{\mathcal{L}}{=} Z^{\eta,1} + Z^{\eta,2}.$$

According to Corollary VII.2 in [3], a spectrally negative Lévy process X drifts to $-\infty$ if and only if $\mathbb{E}[X(1)] < 0$. Z is a spectrally positive Lévy process drifting to $+\infty$ so taking the dual in the theorem we get $\mathbb{E}[Z(1)] > 0$. Now since $\mathbb{E}[Z(1)] = \mathbb{E}[Z^{\eta,1}(1)] + \mathbb{E}[Z^{\eta,2}(1)]$ and $\mathbb{E}[Z^{\eta,1}(1)] < \gamma_2 \eta \nu([\gamma_1, \gamma_2])$, we have that $\mathbb{E}[Z^{\eta,2}(1)]$ is positive for η small enough. Still taking the dual in Corollary VII.2 in [3], this implies that $Z^{\eta,2}$ drifts to $+\infty$ for η small enough. We thus choose such an $\eta_0 \in]0, 1[$ and denote by m_2 the point at which $Z^{\eta_0,2}$ reaches its minimum.

$Z^{\eta_0,1}$ is a compound Poisson process, we define N the counting process of its jumps : $N(t) := \#\{s \in [0, t], Z^{\eta_0,1}(s) - Z^{\eta_0,1}(s-) > 0\}$. N is thus a standard Poisson process with parameter $c_0 := \eta_0 \nu([\gamma_1, \gamma_2])$ and it is independent of $Z^{\eta_0,2}$. We have :

$$\begin{aligned} I(Z) &= \int_0^{+\infty} e^{-(Z^{\eta_0,1}(t) + Z^{\eta_0,2}(t))} dt \\ &\leq e^{-Z^{\eta_0,2}(m_2)} \int_0^{+\infty} e^{-Z^{\eta_0,1}(t)} dt \\ &\leq e^{-Z^{\eta_0,2}(m_2)} \int_0^{+\infty} e^{-\gamma_1 N(t)} dt \\ &= e^{-Z^{\eta_0,2}(m_2)} I(\gamma_1 N), \end{aligned}$$

so, from the independence between the two factors :

$$\mathbb{P}(I(\gamma_1 N) \leq x/2) \times \mathbb{P}(e^{-Z^{\eta_0,2}(m_2)} \leq 2) \leq \mathbb{P}(I(Z) \leq x). \quad (6.69)$$

We put $c_1 := \mathbb{P}(e^{-Z^{\eta_0,2}(m_2)} \leq 2) > 0$. Now by a property of standard Poisson processes, it is easy to see that $I(\gamma_1 N)$ has the same law as

$$\frac{1}{c_0} \sum_{k=0}^{+\infty} e^{-\gamma_1 k} e_k,$$

where $(e_k)_{k \in \mathbb{N}}$ is a sequence of *iid* exponential random variable with parameter 1. This allows us to compute the Laplace transform of $I(\gamma_1 N)$:

$$\forall \lambda \geq 0, \mathbb{E} \left[e^{-\lambda I(\gamma_1 N)} \right] = \prod_{k=0}^{+\infty} \frac{1}{1 + \frac{\lambda}{c_0} e^{-\gamma_1 k}}.$$

We put $K(\lambda) := \min\{k \in \mathbb{N}, \lambda e^{-\gamma_1 k} \leq 1\}$ and taking the logarithm we get

$$\begin{aligned} \log \left(\mathbb{E} \left[e^{-\lambda I(\gamma_1 N)} \right] \right) &= - \sum_{k=0}^{+\infty} \log \left(1 + \frac{\lambda}{c_0} e^{-\gamma_1 k} \right) \\ &\geq -K(\lambda) \log \left(1 + \frac{\lambda}{c_0} \right) - \sum_{k \geq K(\lambda)} \log \left(1 + \frac{\lambda}{c_0} e^{-\gamma_1 k} \right) \\ &\geq -K(\lambda) \log \left(1 + \frac{\lambda}{c_0} \right) - \sum_{k \geq 0} \log \left(1 + \frac{1}{c_0} e^{-\gamma_1 k} \right). \end{aligned}$$

Now, since $K(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} \log(\lambda)/\gamma_1$, we get that

$$\log \left(\mathbb{E} \left[e^{-\lambda I(\gamma_1 N)} \right] \right) \geq -2 (\log(\lambda))^2 / \gamma_1,$$

for λ large enough. Now, reasoning as in the proof of (1.5) (where, from a lower bound on the Laplace transform of $I(V^\uparrow)$, we deduced a lower bound for its asymptotic tail at 0) we get

$$e^{-c_2(\log(x))^2} \leq \mathbb{P}(I(\gamma_1 N) \leq x),$$

for some positive constant c_2 . Combining with (6.68) and (6.69), we get the sought result. \square

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