



**HAL**  
open science

# Controllability of a $2 \times 2$ parabolic system by one force with space-dependent coupling term of order one

M Duprez

► **To cite this version:**

M Duprez. Controllability of a  $2 \times 2$  parabolic system by one force with space-dependent coupling term of order one. 2015. hal-01174812v1

**HAL Id: hal-01174812**

**<https://hal.science/hal-01174812v1>**

Preprint submitted on 9 Jul 2015 (v1), last revised 9 Aug 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Controllability of a $2 \times 2$ parabolic system by one force with space-dependent coupling term of order one.

M. Duprez\*

July 9, 2015

## Abstract

This paper is devoted to the controllability of linear systems of two coupled parabolic equations when the coupling involves a space dependent first order term. This system is set on an bounded interval  $I \subset \subset \mathbb{R}$ , and the first equation is controlled by a force supported in a subinterval of  $I$  or on the boundary. In the case where the intersection of the coupling and control domains is nonempty, we prove null controllability at any time. Otherwise, we provide a minimal time for null controllability. Finally we give a necessary and sufficient condition for the approximate controllability. The main technical tool for obtaining these results is the moment method.

## 1 Introduction and main results

Let  $T > 0$  and  $\omega := (a, b) \subseteq (0, \pi)$ . We consider in the present paper the following distributed control system

$$\begin{cases} \partial_t y - \partial_{xx} y + A_0(p(x)\partial_x y + q(x)y) = B\mathbb{1}_\omega v & \text{in } Q_T := (0, \pi) \times (0, T), \\ y(0, \cdot) = y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi) \end{cases} \quad (1.1)$$

and boundary control system

$$\begin{cases} \partial_t z - \partial_{xx} z + A_0(p(x)\partial_x z + q(x)z) = 0 & \text{in } Q_T, \\ z(0, \cdot) = Bu, z(\pi, \cdot) = 0 & \text{on } (0, T), \\ z(\cdot, 0) = z^0 & \text{in } (0, \pi), \end{cases} \quad (1.2)$$

where  $y^0 \in L^2(0, \pi)^2$  and  $z^0 \in H^{-1}(0, \pi)^2$  are the initial conditions,  $v \in L^2(Q_T)$  and  $u \in L^2(0, T)$  are the controls,  $p \in W_\infty^1(0, \pi)$ ,  $q \in L^\infty(0, \pi)$  and the matrices  $A_0$  and  $B$  are given by

$$A_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is known (see [20, 21] (resp. [16, 25])) that for given initial data  $y^0 \in L^2(0, \pi)^2$  (resp.  $z^0 \in H^{-1}(0, \pi)^2$ ) and a control  $v \in L^2(Q_T)$  (resp.  $u \in L^2(0, T)$ ) System (1.1) (resp. (1.2)) has a unique solution  $y$  (resp.  $z$ ) in

$$\begin{aligned} & L^2(0, T; H_0^1(0, \pi)^2) \cap \mathcal{C}([0, T]; L^2(0, \pi)^2) \\ & (\text{resp. } L^2(Q_T)^2 \cap \mathcal{C}([0, T]; H^{-1}(0, \pi)^2)), \end{aligned}$$

---

\*Laboratoire de Mathématiques de Besançon UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex, France, E-mail: michel.duprez@univ-fcomte.fr, Tel.: 33 (0) 3.81.66.63.26

which depends continuously on the initial data and the control, that is

$$\begin{aligned} & \|y\|_{L^2(0,T;H_0^1(0,\pi)^2)} + \|y\|_{C([0,T];L^2(0,\pi)^2)} \leq C_T(\|y_0\|_{L^2(0,\pi)^2} + \|v\|_{L^2(Q_T)}) \\ & (\text{resp. } \|z\|_{L^2(Q_T)^2} + \|z\|_{C([0,T];H^{-1}(0,\pi)^2)} \leq C_T(\|z_0\|_{H^{-1}(0,\pi)^2} + \|u\|_{L^2(0,T)})). \end{aligned}$$

Let us introduce the notion of null and approximate controllability for this kind of systems.

- System (1.1) (resp. System (1.2)) is *null controllable* at time  $T$  if for every initial condition  $y^0 \in L^2(0,\pi)^2$  (resp.  $z^0 \in H^{-1}(0,\pi)^2$ ) there exists a control  $v \in L^2(Q_T)$  (resp.  $u \in L^2(0,T)$ ) such that the solution to System (1.1) (resp. System (1.2)) satisfies

$$y(T) \equiv 0 \quad (\text{resp. } z(T) \equiv 0) \quad \text{in } (0,\pi).$$

- System (1.1) (resp. System (1.2)) is *approximately controllable* at time  $T$  if for all  $\varepsilon > 0$  and all  $y^0, y^T \in L^2(0,\pi)^2$  (resp.  $z^0, z^T \in H^{-1}(0,\pi)^2$ ) there exists a control  $v \in L^2(Q_T)$  (resp.  $u \in L^2(0,T)$ ) such that the solution to System (1.1) (resp. System (1.2)) satisfies

$$\|y(T) - y^T\|_{L^2(0,\pi)^2} \leq \varepsilon \quad (\text{resp. } \|z(T) - z^T\|_{H^{-1}(0,\pi)^2} \leq \varepsilon).$$

The main goal of this article is to provide a complete answer to the null and approximate controllability issues for System (1.1) and (1.2). For a survey and some applications in physics, chemistry or biology concerning the controllability of this kind of systems, we refer to [6]. In the last decade, many papers studied this problem. However most of them are relating to some parabolic systems with zero order coupling terms. Without first order coupling terms, some Kalman coupling conditions are made explicit in [3], [4] and [16] for distributed null controllability of systems in higher space dimension of more than two equations with time-dependent and constant matrices.

Concerning the null and approximate controllability of Systems (1.1) and (1.2) in the case  $p \equiv 0$  and  $q \neq 0$  in  $(0,\pi)$ , a partial answer is given in [1, 2, 13, 24] under the sign condition

$$q \leq 0 \quad \text{or} \quad q \geq 0 \quad \text{in } (0,\pi).$$

These results are obtained as a consequence of controllability results of a hyperbolic system using the transmutation method (see [22]). One can find a necessary and sufficient condition in [7] when

$$\int_0^\pi q(x)dx \neq 0.$$

Finally in a recent work [8, 9], a complete study for any  $q \in L^\infty(0,\pi)$  is given.

Let us now remind known results of null controllability for systems of the following more general form. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \in \mathbb{N}^*$ ) of class  $\mathcal{C}^2$  and  $\omega_0$  an arbitrary nonempty subset of  $\Omega$ . We denote by  $\partial\Omega$  the boundary of  $\Omega$ . Consider the system of two coupled linear parabolic equations

$$\begin{cases} \partial_t y_1 = \Delta y_1 + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbf{1}_{\omega_0} v & \text{in } \Omega \times (0, T), \\ \partial_t y_2 = \Delta y_2 + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T), \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (1.3)$$

where  $y^0 \in L^2(\Omega)^2$ ,  $g_{ij} \in L^\infty(\Omega \times (0, T))^N$  and  $a_{ij} \in L^\infty(\Omega \times (0, T))$  for all  $i, j \in \{1, 2\}$ .

As a particular case of the result in [17] (see also [5]), System (1.3) is null controllable whenever

$$g_{21} \equiv 0 \quad \text{and} \quad (a_{21} > C \text{ or } a_{21} < -C) \quad \text{in } \omega_1 \subseteq \omega_0, \quad (1.4)$$

for a positive constant  $C$ .

In [18], the author supposes that  $a_{11}$ ,  $g_{11}$ ,  $a_{22}$ ,  $g_{22}$  are constant and the first order coupling operator  $g_{21} \cdot \nabla + a_{21}$  can be written

$$g_{21} \cdot \nabla + a_{21} = P_1 \circ \theta \quad \text{in } \Omega \times (0, T), \quad (1.5)$$

where  $\theta \in \mathcal{C}^2(\bar{\Omega})$  satisfies  $|\theta| > C$  in  $\omega_1 \subseteq \omega_0$  for a positive constant  $C$  and  $P_1$  is given by

$$P_1 := m_0 \cdot \nabla + m_1,$$

for some  $m_0, m_1 \in \mathbb{R}$ . Moreover the operator  $P_1$  satisfies

$$\|u\|_{H^1(\Omega)} \leq C \|P_1^* u\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega).$$

Under these assumptions, the author proves the null controllability of System (1.3) at any time.

In [10], the same property holds true for System (1.3) if we assume that  $a_{ij} \in \mathcal{C}^4(\Omega \times (0, T))$ ,  $g_{ij} \in \mathcal{C}^1(\Omega \times (0, T))^N$  for all  $i, j \in \{1, 2\}$ ,  $g_{21} \in \mathcal{C}^3(\bar{\Omega} \times (0, T))$  and the geometrical condition

$$\begin{cases} \partial\omega \cap \partial\Omega \text{ contains a nonempty open subset } \gamma \text{ s.t. } \dot{\gamma} \neq \emptyset, \\ \exists x_0 \in \gamma \text{ s.t. } g_{21}(t, x_0) \cdot \nu(x_0) \neq 0 \text{ for all } t \in [0, T], \end{cases} \quad (1.6)$$

where  $\nu$  represents the exterior normal unit vector to the boundary  $\partial\Omega$ .

Last, for constant coefficients, it is proved in [14] that System (1.3) is null/approximately controllable at any time  $T$  if and only if

$$g_{21} \neq 0 \quad \text{or} \quad a_{21} \neq 0.$$

Now let us go back to Systems (1.1) and (1.2) for which we will provide a complete description of the null and approximate controllability. Our first and main result is the following

**THEOREM 1.1.** *Let us suppose that  $p \in W_\infty^1(0, \pi) \cap W_\infty^2(\omega)$ ,  $q \in L^\infty(0, \pi) \cap W_\infty^1(\omega)$  and*

$$(\text{Supp}(p) \cup \text{Supp}(q)) \cap \omega \neq \emptyset. \quad (1.7)$$

*Then System (1.1) is null controllable at any time  $T$ .*

Let us compare this result with the previously described results to highlight our main contribution:

1. Even though System (1.1) is considered in one space dimension, we remark first that our coupling operator has a more general form than the one in (1.5) assumed by Guerrero [18]. Moreover unlike to [14], its coefficients are non constant with respect to the space variable.
2. We do not have the geometrical restriction (1.6) assumed in [10] by A. Benabdallah and al. More precisely we do not require the control support to be a neighbourhood of a part of the boundary.

For all  $x \in (0, \pi)$  and  $k \in \mathbb{N}^*$ , we denote by  $\varphi_k(x) := \sqrt{\frac{2}{\pi}} \sin(kx)$  the eigenvector of the Laplacian operator, with Dirichlet boundary condition, and consider the two following quantities

$$\begin{cases} I_{a,k}(p, q) := \int_0^a (q - \frac{1}{2} \partial_x p) \varphi_k^2, \\ I_k(p, q) := \int_0^\pi (q - \frac{1}{2} \partial_x p) \varphi_k^2, \end{cases} \quad (1.8)$$

for all  $k \in \mathbb{N}^*$ . Combined with the Hautus test ([15, Cor. 3.3] or Th. 5.1 in the present paper), Theorem 1.1 leads to the following characterization:

**THEOREM 1.2.** *Let us suppose that  $p \in W_\infty^1(0, \pi) \cap W_\infty^2(\omega)$  and  $q \in L^\infty(0, \pi) \cap W_\infty^1(\omega)$ . System (1.1) is approximately controllable at time  $T$  if and only if*

$$(\text{Supp}(p) \cup \text{Supp}(q)) \cap \omega \neq \emptyset \quad (1.9)$$

or

$$|I_k(p, q)| + |I_{a,k}(p, q)| \neq 0 \text{ for all } k \in \mathbb{N}^*. \quad (1.10)$$

This last result recovers the case  $p \equiv 0$  studied in [11] for  $\text{Supp}(q) \cap \omega = \emptyset$ . In [19], the authors prove the approximate controllability at any time  $T$  of System (1.1) under the condition  $p \equiv 0$  and  $q \equiv \mathbb{1}_{\omega_0}$  with  $\omega_0$  a nonempty open subset of  $(0, \pi)$ , which implies (1.10).

*Remark 1.* We will see in the prove of Theorems 1.1 and 1.2 that only the following regularity are needed for  $p$  and  $q$

$$\begin{cases} p \in W_\infty^1(0, \pi) \cap W_\infty^2(\tilde{\omega}), \\ q \in L^\infty(0, \pi) \cap W_\infty^1(\tilde{\omega}), \end{cases}$$

for an open subinterval  $\tilde{\omega}$  of  $\omega$ . These hypothesis are used in Definition (1.8) of  $I_k$  and  $I_{a,k}$  and the change of variable described in Section 3.2. For more general coupling terms, these control problems are open.

When the supports of the control and the coupling terms are disjoint in System (1.1), following the ideas in [9], we obtain a minimal time of null controllability:

**THEOREM 1.3.** *Let  $p \in W_\infty^1(0, \pi)$ ,  $q \in L^\infty(0, \pi)$ . Suppose that Condition (1.10) holds and*

$$(\text{Supp}(p) \cup \text{Supp}(q)) \cap \omega = \emptyset. \quad (1.11)$$

Let  $T_0(p, q)$  be given by

$$T_0(p, q) := \limsup_{k \rightarrow \infty} \frac{\min(-\log |I_k(p, q)|, -\log |I_{a,k}(p, q)|)}{k^2}. \quad (1.12)$$

One has

1. If  $T > T_0(p, q)$ , then System (1.1) is null controllable at time  $T$ .
2. If  $T < T_0(p, q)$ , then System (1.1) is not null controllable at time  $T$ .

*Remark 2.* For  $p \equiv 0$  in  $(0, \pi)$ , it is proved in [9] that for any  $\tau_0 \in [0, \infty]$  there exists a function  $q \in L^\infty(0, \pi)$  such that the minimal time of null controllability  $T_0(p, q)$  associated with System (1.1) is given by

$$T_0(p, q) = \tau_0.$$

The authors give explicit functions and one can easily adapt them to the case  $p \neq 0$  in  $(0, \pi)$ .

Concerning the boundary controllability, in [23, Th. 3.3], using the Hautus test, the author proves that System (1.2) is approximately controllable at time  $T$  if and only if

$$I_k(p, q) \neq 0 \text{ for all } k \in \mathbb{N}^*. \quad (1.13)$$

About null controllability of System (1.2) we obtain here a minimal time:

**THEOREM 1.4.** *Let  $p \in W_\infty^1(0, \pi)$ ,  $q \in L^\infty(0, \pi)$  and suppose that Condition (1.13) is satisfied. Let us define*

$$T_1(p, q) := \limsup_{k \rightarrow \infty} \frac{-\log |I_k|}{k^2}. \quad (1.14)$$

One has

1. If  $T > T_1(p, q)$ , then System (1.2) is null controllable at time  $T$ .
2. If  $T < T_1(p, q)$ , then System (1.2) is not null controllable at time  $T$ .

**Remark 3.** A simple computation leads to the convergence of sequences  $(I_k)_{k \in \mathbb{N}^*}$  and  $(I_{a,k})_{k \in \mathbb{N}^*}$ , more precisely

$$\lim_{k \rightarrow \infty} I_k(p, q) = I := \frac{1}{\pi} \int_0^\pi (q - \frac{1}{2} \partial_x p) \text{ and } \lim_{k \rightarrow \infty} I_{a,k}(p, q) = I_a := \frac{1}{\pi} \int_0^a (q - \frac{1}{2} \partial_x p).$$

We remark first that  $T_0(p, q) = T_1(p, q) = 0$  when

$$q > \partial_x p \text{ in } (0, \pi) \text{ or } q < \partial_x p \text{ in } (0, \pi).$$

Secondly, under the condition

$$\int_0^\pi q \neq \int_0^\pi \partial_x p,$$

System (1.1) (resp. (1.2)) is null controllable if and only if (1.10) (resp. (1.13)) holds.

In higher space dimension, even for this simplified system, distributed and boundary controllability are open problems. The cases  $T = T_0$  in Theorem 1.3 and  $T = T_1$  in Theorem 1.4 are also open.

This article is organized as follows. In the first section we present some preliminary results useful to reduce the null controllability issues to the moment problem. In the second and third sections we study the null controllability issue of System (1.1) in the two cases when the intersection of the coupling and control supports is nonempty or not. We finish with the proof of Theorems 1.2 and 1.4 in Section 4 and 5.

## 2 Preliminary results

Consider the differential operator

$$\begin{aligned} L : D(L) \subset L^2(0, \pi)^2 &\rightarrow L^2(0, \pi)^2 \\ f &\mapsto -\partial_{xx} f + A_0(p \partial_x f + q f), \end{aligned}$$

where the domain of  $L$  and its adjoint  $L^*$  is given by  $D(L) = D(L^*) = H^2(0, \pi)^2 \cap H_0^1(0, \pi)^2$ . In this section, we will first establish some properties of the operator  $L$  useful in the moment method. Secondly we will recall the unique continuation property and the observability inequality related to the approximate and null controllability, respectively.

### 2.1 Biorthogonal basis

Let us first analyze the spectrum of the operators  $L$  and  $L^*$ .

**PROPOSITION 2.1.** *For all  $k \in \mathbb{N}^*$  consider the two vectors*

$$\Phi_{1,k}^* := \begin{pmatrix} \psi_k^* \\ \varphi_k \end{pmatrix}, \Phi_{2,k}^* := \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix},$$

where  $\psi_k^*$  is defined for all  $x \in (0, \pi)$  by

$$\begin{cases} \psi_k^*(x) = \alpha_k^* \varphi_k(x) - \frac{1}{k} \int_0^x \sin(k(x - \xi)) [I_k(p, q) \varphi_k(\xi) + \partial_x(p(\xi) \varphi_k(\xi)) - q(\xi) \varphi_k(\xi)] d\xi, \\ \alpha_k^* = \frac{1}{k} \int_0^\pi \int_0^x \sin(k(x - \xi)) [I_k(p, q) \varphi_k(\xi) + \partial_x(p(\xi) \varphi_k(\xi)) - q(\xi) \varphi_k(\xi)] \varphi_k(x) d\xi dx, \end{cases}$$

One has

1. The spectrum of  $L^*$  is given by  $\sigma(L^*) = \{k^2 : k \in \mathbb{N}^*\}$ .
2. For  $k \geq 1$ , the eigenvalue  $k^2$  of  $L^*$  is simple if and only if  $I_k(p, q) \neq 0$ . In this case,  $\Phi_{2,k}^*$  and  $\Phi_{1,k}^*$  are an eigenfunction and a generalized eigenfunction of the operator  $L^*$  associated with the eigenvalue  $k^2$ , more precisely

$$\begin{cases} (L^* - k^2 Id)\Phi_{1,k}^* = I_k\Phi_{2,k}^*, \\ (L^* - k^2 Id)\Phi_{2,k}^* = 0. \end{cases} \quad (2.1)$$

3. For  $k \geq 1$ , the eigenvalue  $k^2$  of  $L^*$  is double if and only if  $I_k(p, q) = 0$ . In this case,  $\Phi_{1,k}^*$  and  $\Phi_{2,k}^*$  are two eigenfunctions of the operator  $L^*$  associated with the eigenvalue  $k^2$ , that is for  $i = 1, 2$

$$(L^* - k^2 Id)\Phi_{i,k}^* = 0.$$

*Proof.* The adjoint operator  $L^*$  of  $L$  is given by

$$\begin{aligned} L^* : D(L) \subset L^2(0, \pi)^2 &\rightarrow L^2(0, \pi)^2 \\ f &\mapsto -\partial_{xx}f + A_0(-\partial_x(pf) + qf). \end{aligned}$$

We can remark first that the inverse of  $L^*$  is compact. Thus the spectrum of  $L^*$  reduces to its point spectrum. The eigenvalue problem associated with the operator  $L^*$  is

$$\begin{cases} -\partial_{xx}\psi - \partial_x(p(x)\varphi) + q(x)\varphi = \lambda\psi & \text{in } (0, \pi), \\ -\partial_{xx}\varphi = \lambda\varphi & \text{in } (0, \pi), \\ \varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0, \end{cases} \quad (2.2)$$

where  $(\psi, \varphi) \in D(L^*)$  and  $\lambda \in \mathbb{C}$ . For  $\varphi \equiv 0$  in  $(0, \pi)$  and  $\psi = \varphi_k$  in  $(0, \pi)$ ,  $\lambda = k^2$  is an eigenvalue of  $L^*$  and the vector  $\Phi_{2,k}^* := (\varphi_k, 0)$  is an associated eigenfunction. If now  $\varphi \not\equiv 0$  in  $(0, \pi)$ , then  $\lambda = k^2$  is an eigenvalue and  $\varphi = \kappa\varphi_k$  with  $\kappa \in \mathbb{R}^*$ . We remark that System (2.2) has a solution if and only if  $I_k(p, q) = 0$ . If  $I_k(p, q) = 0$ ,  $\Phi_{1,k}^* := (\psi_k^*, \varphi_k)$  is a second eigenfunction of  $L^*$  linearly independent of  $\Phi_{2,k}^*$ , where  $\psi_k^*$  is the unique solution to the non-homogeneous Sturm-Liouville problem

$$\begin{cases} -\partial_{xx}\psi - k^2\psi = f & \text{in } (0, \pi), \\ \psi(0) = \psi(\pi) = 0, \end{cases} \quad (2.3)$$

with

$$f := \partial_x(p(x)\varphi_k) - q(x)\varphi_k$$

and is such that

$$\int_0^\pi \psi(x)\varphi_k(x) dx = 0. \quad (2.4)$$

A solution to System (2.3) can be written for all  $x \in (0, \pi)$

$$\psi(x) = \alpha\varphi_k(x) - \frac{1}{k} \int_0^x \sin(k(x-\xi))f(\xi)d\xi,$$

with  $\alpha \in \mathbb{R}$ . Under Condition (2.4), we obtain the expression of  $\psi_k^*$  given in Proposition 2.1. Thus, in the case  $I_k(p, q) = 0$ ,  $\lambda = k^2$  is a double eigenvalue of  $L^*$ . Items 1 and 3 are now proved.

Let us now suppose that  $I_k(p, q) \neq 0$ . The eigenvalue  $\lambda = k^2$  is simple,  $\Phi_{2,k}^* := (\varphi_k, 0)$  is an eigenfunction and a solution  $\Phi_{1,k}^* := (\psi, \varphi)$  to  $(L^* - k^2 Id)\Phi_{1,k}^* = I_k(p, q)\Phi_{2,k}^*$ , that is

$$\begin{cases} -\partial_{xx}\psi - \partial_x(p(x)\varphi) + q(x)\varphi = k^2\psi + I_k(p, q)\varphi_k & \text{in } (0, \pi), \\ -\partial_{xx}\varphi = k^2\varphi & \text{in } (0, \pi), \\ \varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0, \end{cases} \quad (2.5)$$

is a generalized eigenfunction of  $L^*$ . We deduce that  $\varphi = \kappa\varphi_k$  in  $(0, \pi)$  for a constant  $\kappa \in \mathbb{R}^*$ . Again System (2.5) has a solution if and only if  $\kappa = 1$ . Then  $\psi$  is solution to the Sturm-Liouville problem (2.3) with

$$f = I_k(p, q)\varphi_k + \partial_x(p(x)\varphi_k) - q(x)\varphi_k$$

and satisfying (2.4). Again, using (2.4), we obtain the expression of  $\psi_k^*$  given in Proposition 2.1.  $\square$

The function  $\psi_k^*$  given in Proposition 2.1 will play an important role in this paper and we will need the following immediate property

**Lemma 2.1.** *There exists a positive constant  $C$  such that*

$$|\alpha_k^*| \leq \frac{C}{k}, \quad \|\psi_k^*\|_{L^\infty(0, \pi)} \leq \frac{C}{k}, \quad \|\partial_x \psi_k^*\|_{L^\infty(0, \pi)} \leq C, \quad \forall k \in \mathbb{N}^*. \quad (2.6)$$

Since the eigenvalues of the operator  $L^*$  are real, we deduce that  $L$  and  $L^*$  have the same spectrum and the associated eigenspaces have the same dimension. The eigenfunctions and the generalized eigenfunctions of  $L$  can be found as previously.

**PROPOSITION 2.2.** *For all  $k \in \mathbb{N}^*$  consider the two vectors*

$$\Phi_{1,k} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \quad \Phi_{2,k} := \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix},$$

where  $\psi_k$  is defined for all  $x \in (0, \pi)$  by

$$\begin{cases} \psi_k(x) := \alpha_k \varphi_k(x) - \frac{1}{k} \int_0^x \sin(k(x-\xi)) [I_k(p, q)\varphi_k(\xi) - p(\xi)\partial_x(\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)] d\xi, \\ \alpha_k := \frac{1}{k} \int_0^\pi \int_0^x \sin(k(x-\xi)) [I_k(p, q)\varphi_k(\xi) - p(\xi)\partial_x(\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)] \varphi_k(x) d\xi dx, \end{cases}$$

One has

1. The spectrum of  $L$  is given by  $\sigma(L) = \sigma(L^*) = \{k^2 : k \in \mathbb{N}^*\}$ .
2. For  $k \geq 1$ , the eigenvalue  $k^2$  of  $L$  is simple if and only if  $I_k(p, q) \neq 0$ . In this case,  $\Phi_{1,k}$  and  $\Phi_{2,k}$  are an eigenfunction and a generalized eigenfunction of the operator  $L$  associated with the eigenvalue  $k^2$ , more precisely

$$\begin{cases} (L - k^2 Id)\Phi_{1,k} = 0, \\ (L - k^2 Id)\Phi_{2,k} = I_k \Phi_{1,k}. \end{cases} \quad (2.7)$$

3. For  $k \geq 1$ , the eigenvalue  $k^2$  of  $L$  is double if and only if  $I_k(p, q) = 0$ . In this case,  $\Phi_{1,k}$  and  $\Phi_{2,k}$  are two eigenfunctions of the operator  $L$  associated with the eigenvalue  $k^2$ , that is for  $i = 1, 2$

$$(L - k^2 Id)\Phi_{i,k} = 0.$$

Lemma 2.3 and Corollary 2.6 in [9] can be adapted easily to prove the following property.

**Property 2.1.** *Consider the families*

$$\mathcal{B} := \{\Phi_{1,k}, \Phi_{2,k} : k \in \mathbb{N}^*\} \quad \text{and} \quad \mathcal{B}^* := \{\Phi_{1,k}^*, \Phi_{2,k}^* : k \in \mathbb{N}^*\}.$$

Then

1. The sequences  $\mathcal{B}$  and  $\mathcal{B}^*$  are biorthogonal Riesz bases of  $L^2(0, \pi)^2$ .
2. The sequence  $\mathcal{B}^*$  is a basis of  $H_0^1(0, \pi)^2$  and  $\mathcal{B}$  is its biorthogonal basis in  $H^{-1}(0, \pi)$ .



## 2.2 Duality

As is well known, the controllability has a dual concept called *observability* (see for instance [6], [12, Th. 2.44, p. 56–57]). Consider the dual system associated with System (1.1)

$$\begin{cases} -\partial_t \theta - \partial_{xx} \theta + A_0^*(-\partial_x(p(x)\theta) + q(x)\theta) = 0 & \text{in } Q_T, \\ \theta(0, \cdot) = \theta(\pi, \cdot) = 0 & \text{on } (0, T), \\ \theta(\cdot, T) = \theta^0 & \text{in } (0, \pi), \end{cases} \quad (2.8)$$

where  $\theta^0 \in L^2(0, \pi)^2$ . The approximate controllability is equivalent to a *unique continuation property*:

**PROPOSITION 2.3.** 1. System (1.1) is approximately controllable at time  $T$  if and only if for all initial condition  $\theta^0 \in L^2(0, \pi)^2$  the solution to System (2.8) satisfies

$$\mathbf{1}_\omega B^* \theta \equiv 0 \text{ in } Q_T \Rightarrow \theta \equiv 0 \text{ in } Q_T. \quad (2.9)$$

2. System (1.2) is approximately controllable at time  $T$  if and only if for all initial condition  $\theta^0 \in H_0^1(0, \pi)^2$  the solution to System (2.8) satisfies

$$B^* \partial_x \theta(0, t) \equiv 0 \text{ in } (0, T) \Rightarrow \theta \equiv 0 \text{ in } Q_T. \quad (2.10)$$

The null controllability is characterized by an *observability inequality*:

**PROPOSITION 2.4.** 1. System (1.1) is null controllable at time  $T$  if and only if there exists a constant  $C_{obs}$  such that for all initial condition  $\theta^0 \in L^2(0, \pi)^2$  the solution to System (2.8) satisfies the observability inequality

$$\|\theta(0)\|_{L^2(0, \pi)^2}^2 \leq C_{obs} \iint_{Q_T} |\mathbf{1}_\omega B^* \theta|^2 dx dt. \quad (2.11)$$

2. System (1.1) is null controllable at time  $T$  if and only if there exists a constant  $C_{obs}$  such that for all initial condition  $\theta^0 \in H_0^1(0, \pi)^2$  the solution to System (2.8) satisfies the observability inequality

$$\|\theta(0)\|_{H_0^1(0, \pi)^2}^2 \leq C_{obs} \int_0^T |B^* \partial_x \theta(0, t)|^2 dt. \quad (2.12)$$

## 3 Proof of Theorem 1.1

In this section, we first establish the moment problem related to the null controllability for System (1.1) and then we will solve it in section 3.2. The strategy involves finding a equivalence system (see Definition 3.1) to System (1.1), which has a associated quantity  $I_k$  satisfying "some good properties".

### 3.1 The moment problem

Let  $y^0 := (y_1^0, y_2^0) \in L^2(0, \pi)^2$ . For  $i \in \{1, 2\}$  and  $k \in \mathbb{N}^*$ , if we consider  $\theta^0 := \Phi_{i,k}^*$  in the dual System (2.8), we get after an integration by part

$$\iint_{Q_T} v(x, t) \mathbf{1}_\omega B^* \theta(x, t) dx dt = \langle y(T), \Phi_{i,k}^* \rangle_{L^2(0, \pi)^2} - \langle y^0, \theta(0) \rangle_{L^2(0, \pi)^2}.$$

Since  $\mathcal{B}^*$  is a basis of  $L^2(0, \pi)^2$ , System (1.1) is null controllable if and only if for all  $y^0 \in L^2(0, \pi)^2$ , there exists a control  $v \in L^2(Q_T)$  such that for all  $k \in \mathbb{N}^*$  and  $i \in \{1, 2\}$  the solution  $y$  to System (1.1) satisfies the following equality

$$\iint_{Q_T} v(x, t) \mathbf{1}_\omega B^* \theta_{i,k}(x, t) dx dt = -\langle y^0, \theta_{i,k}(0) \rangle_{L^2(0, \pi)^2}, \quad (3.1)$$

where  $\theta_{i,k}$  is the solution to the dual system (2.8) with the initial condition  $\theta^0 := \Phi_{i,k}^*$ .

In the moment problem (3.1), we will look for a control  $v$  of the form

$$v(x, t) := f^{(1)}(x)v^{(1)}(T-t) + f^{(2)}(x)v^{(2)}(T-t) \text{ for all } (x, t) \in Q_T, \quad (3.2)$$

with  $v^{(1)}, v^{(2)} \in L^2(0, T)$  and  $f^{(1)}, f^{(2)} \in L^2(0, \pi)$  satisfying

$$\text{Supp}(f^{(1)}), \text{Supp}(f^{(2)}) \subseteq \omega.$$

The solutions  $\theta_{1,k}$  and  $\theta_{2,k}$  to the dual System (2.8) with the initial condition  $\Phi_{1,k}^*$  and  $\Phi_{2,k}^*$  are given for all  $(x, t) \in Q_T$  by

$$\begin{cases} \theta_{1,k}(x, t) = e^{-k^2(T-t)} \left( \Phi_{1,k}^*(x) - (T-t)I_k(p, q)\Phi_{2,k}^*(x) \right), \\ \theta_{2,k}(x, t) = e^{-k^2(T-t)}\Phi_{2,k}^*(x). \end{cases} \quad (3.3)$$

Plugging (3.2) and (3.3) in the moment problem (3.1), we get for all  $k \geq 1$

$$\begin{cases} \tilde{f}_k^{(1)} \int_0^T v^{(1)}(t)e^{-k^2t} dt + \tilde{f}_k^{(2)} \int_0^T v^{(2)}(t)e^{-k^2t} dt \\ \quad - I_k(p, q) f_k^{(1)} \int_0^T v^{(1)}(t)te^{-k^2t} dt - I_k(p, q) f_k^{(2)} \int_0^T v^{(2)}(t)te^{-k^2t} dt \\ \quad = -e^{-k^2T} \{y_{1,k}^0 - TI_k(p, q)y_{2,k}^0\}, \\ f_k^{(1)} \int_0^T v^{(1)}(t)e^{-k^2t} dt + f_k^{(2)} \int_0^T v^{(2)}(t)e^{-k^2t} dt = -e^{-k^2T}y_{2,k}^0, \end{cases}$$

where  $f_k^{(i)}, \tilde{f}_k^{(i)}$  and  $y_{i,k}^0$  are given for all  $i \in \{1, 2\}$  and  $k \in \mathbb{N}^*$  by

$$f_k^{(i)} := \int_0^\pi f^{(i)}(x)\varphi_k(x)dx, \quad \tilde{f}_k^{(i)} := \int_0^\pi f^{(i)}(x)\psi_k^*(x)dx, \quad (3.4)$$

and

$$y_{i,k}^0 := \langle y^0, \Phi_{i,k}^* \rangle_{L^2(0, \pi)} \quad (3.5)$$

In [16], the authors prove that the family  $\left\{ e_{1,k} := e^{-k^2t}, e_{2,k} := te^{-k^2t} \right\}_{k \geq 1}$  admits a biorthogonal family  $\{q_{1,k}, q_{2,k}\}_{k \geq 1}$  in the space  $L^2(0, T)$ , *i.e.* a family satisfying

$$\int_0^T e_{i,k}q_{j,l}(t) dt = \delta_{ij}\delta_{kl}, \quad \forall k, l \geq 1, \quad 1 \leq i, j \leq 2. \quad (3.6)$$

Moreover for all  $\varepsilon > 0$  there exists a constant  $C_{\varepsilon, T} > 0$  such that

$$\|q_{i,k}\|_{L^2(0, T)} \leq C_{\varepsilon, T} e^{\varepsilon k^2}, \quad \forall k \geq 1, \quad i = 1, 2. \quad (3.7)$$

We will look for  $v^{(1)}$  and  $v^{(2)}$  of the form

$$v^{(i)}(t) = \sum_{k \geq 1} \{v_{1,k}^{(i)}q_{1,k}(t) + v_{2,k}^{(i)}q_{2,k}(t)\}, \quad i = 1, 2. \quad (3.8)$$

Thus the moment problem (3.1) can be written as

$$A_{1,k}V_{1,k} + A_{2,k}V_{2,k} = F_k, \quad \text{for all } k \geq 1, \quad (3.9)$$

with for all  $k \in \mathbb{N}^*$

$$A_{1,k} = \begin{pmatrix} \tilde{f}_k^{(1)} & \tilde{f}_k^{(2)} \\ f_k^{(1)} & f_k^{(2)} \end{pmatrix}, \quad A_{2,k} = \begin{pmatrix} -I_k(p, q)f_k^{(1)} & -I_k(p, q)f_k^{(2)} \\ 0 & 0 \end{pmatrix}, \quad (3.10)$$

$$V_{1,k} := \begin{pmatrix} v_{1,k}^{(1)} \\ v_{1,k}^{(2)} \end{pmatrix}, \quad V_{2,k} := \begin{pmatrix} v_{2,k}^{(1)} \\ v_{2,k}^{(2)} \end{pmatrix} \quad (3.11)$$

and

$$F_k = \begin{pmatrix} -e^{-k^2 T} \begin{pmatrix} y_{1,k}^0 - T I_k(p, q) y_{2,k}^0 \\ -e^{-k^2 T} y_{2,k}^0 \end{pmatrix} \end{pmatrix}. \quad (3.12)$$

The next sections will be devoted to solving problem (3.9) and prove that the corresponding solution  $v^{(1)}$ ,  $v^{(2)}$  belong to  $L^2(0, T)$ .

### 3.2 Resolution of the moment problem

In this section, we will prove the null controllability of System (1.1) at any time  $T$  when the supports of  $p$  or  $q$  intersects the control domain  $\omega$ . In [17], the authors obtain the null controllability of System (1.1) at any time under Condition (1.4), so we will not consider this case and we will always suppose that  $|p| > C$  in  $\tilde{\omega}$  for a positive constant  $C$  and an open subinterval  $\tilde{\omega}$  of  $\omega$ .

Let us first introduce the following notion of equivalent systems.

**DEFINITION 3.1.** Let  $p_1, p_2 \in W_\infty^1(0, \pi)$  and  $q_1, q_2 \in L^\infty(0, \pi)$ . Consider the systems given for  $i \in \{1, 2\}$  by

$$\left\{ \begin{array}{ll} \text{For given } y^0 \in L^2(0, \pi)^2, v \in L^2(Q_T), \\ \text{Find } y := (y_1, y_2) \in W(0, T)^2 \text{ such that :} \\ \partial_t y_1 - \partial_{xx} y_1 = \mathbf{1}_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 + p_i(x) \partial_x y_1 + q_i(x) y_1 = 0 & \text{in } Q_T, \\ y(0, \cdot) = y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi). \end{array} \right. \quad (\mathcal{S}_i)$$

We say that System  $(\mathcal{S}_1)$  is *equivalent* to System  $(\mathcal{S}_2)$  if System  $(\mathcal{S}_1)$  is null controllable at time  $T$  if and only if System  $(\mathcal{S}_2)$  is null controllable at time  $T$ .

Let us present the main technique used all along this section. Suppose that System (1.1) is null controllable at time  $T$ . Let  $v$  a control such that the solution  $y$  to System (1.1) verifies  $y(T) = 0$  in  $(0, \pi)$  and  $\omega_0 := (\alpha, \beta)$  a subinterval of  $\omega = (a, b)$ . Consider a function  $\theta \in W_\infty^2(0, \pi)$  satisfying

$$\left\{ \begin{array}{ll} \theta \equiv \kappa_1 & \text{in } (0, \alpha), \\ \theta \equiv \kappa_2 & \text{in } (\beta, \pi), \\ |\theta| > \kappa_3 & \text{in } (0, \pi), \end{array} \right. \quad (3.13)$$

with  $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}_+^*$ . Thus if we consider the change of variable

$$\hat{y} := (\hat{y}_1, y_2) \quad \text{with} \quad \hat{y}_1 := \theta^{-1} y_1, \quad (3.14)$$

then  $\hat{y}$  is solution in  $L^2(0, T; H_0^1(0, \pi)^2) \cap \mathcal{C}([0, T]; L^2(0, \pi)^2)$  to the system

$$\left\{ \begin{array}{ll} \partial_t \hat{y}_1 - \partial_{xx} \hat{y}_1 = \mathbf{1}_\omega \hat{v} & \text{in } Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 + \hat{p} \partial_x \hat{y}_1 + \hat{q} y_1 = 0 & \text{in } Q_T, \\ \hat{y}(0, \cdot) = \hat{y}(\pi, \cdot) = 0 & \text{on } (0, T), \\ \hat{y}(\cdot, 0) = \hat{y}_0 & \text{in } (0, \pi), \end{array} \right. \quad (3.15)$$

where the initial condition is  $\widehat{y}_0 := (\theta^{-1}y_1^0, y_2^0) \in L^2(0, \pi)^2$ , the control is  $\widehat{v} := -\partial_{xx}(\theta^{-1})y_1 - 2\partial_x(\theta^{-1})\partial_x y_1 + \theta^{-1}v \in L^2(Q_T)$  and the coupling terms are given by

$$\begin{cases} \widehat{p} := p\theta, \\ \widehat{q} := p\partial_x\theta + q\theta. \end{cases} \quad (3.16)$$

Indeed,  $\theta$  is constant in  $(0, \pi) \setminus \omega_0$ , then

$$\text{Supp } \widehat{v} \subseteq \omega \times (0, T).$$

Since  $y$  is controlled, then  $\widehat{y}$  also. The converse is clearly true: starting from the controlled System (3.15) the same process leads to the construction of a controlled solution of System (1.1). Thus through the change of variable (3.14), following Definition 3.1, Systems (1.1) and (3.15) are equivalent.

The next main result of this section is Proposition 3.1 that will be introduced after some lemmas. The first of them is the following.

**Lemma 3.1.** *Let  $p \in W_\infty^1(0, \pi) \cap W_\infty^2(\omega)$  and  $q \in L^\infty(0, \pi) \cap W_\infty^1(\omega)$  with  $|p| > C$  in an open subinterval  $\widetilde{\omega}$  of  $\omega$  for a positive constant  $C$ . There exists a subinterval  $\omega_0 := (\alpha, \beta) \subset \widetilde{\omega}$  and a function  $\theta \in W_\infty^2(0, \pi)$  satisfying (3.13) such that System (1.1) is equivalent to System (3.15) with  $\widehat{q} \equiv 0$  in  $\omega_0$ . Moreover for all  $\epsilon > 0$  the interval  $\omega_0$  can be chosen in order to have for all  $k \in \mathbb{N}^*$*

$$|I(p, q) - I(\widehat{p}, \widehat{q})| \leq \epsilon \quad \text{and} \quad |I_k(p, q) - I_k(\widehat{p}, \widehat{q})| \leq \epsilon. \quad (3.17)$$

*Proof.* Let  $\omega_0 := (\alpha, \beta)$  be an interval strictly included in  $\widetilde{\omega} := (\widetilde{a}, \widetilde{b})$  and  $\theta \in W_\infty^2(0, \pi)$  satisfying

$$\begin{cases} p\partial_x\theta + q\theta = 0 & \text{in } \omega_0, \\ \theta \equiv 1 & \text{in } (0, \pi) \setminus \widetilde{\omega}, \\ |\theta| > C & \text{in } (0, \pi), \end{cases} \quad (3.18)$$

for a positive constant  $C$ . In the intervals  $(\widetilde{a}, \alpha]$  and  $[\beta, \widetilde{b})$ , we can take  $\theta$  of class  $\mathcal{C}^\infty$  in order to have  $\theta \in W_\infty^2(0, \pi)$ . Thus the function  $\theta$  verifies (3.13) and, following the change of variable described in (3.14), System (1.1) is equivalent to System (3.15) with  $\widehat{q} \equiv 0$  in  $\omega_0$  (see (3.16)). The estimates in (3.17) are obtained taking the interval  $\omega_0$  small enough.  $\square$

Let us first study System (1.1) in a particular case.

**Lemma 3.2.** *Consider  $p \in W_\infty^1(0, \pi) \cap W_\infty^2(\omega)$  and  $q \in L^\infty(0, \pi) \cap W_\infty^1(\omega)$ . Let us suppose that  $p \equiv C \in \mathbb{R}^*$  and  $q \equiv 0$  in an open subinterval  $\widetilde{\omega}$  of  $\omega$ . Then System (1.1) is equivalent to a system of the form (3.15) with coupling terms  $\widehat{p}$ ,  $\widehat{q}$  satisfying*

$$|I_k(\widehat{p}, \widehat{q})| > C/k^6, \quad \forall k \in \mathbb{N}^*.$$

To prove this result we will need this lemma:

**Lemma 3.3.** *Let  $(u_k)_{k \in \mathbb{N}^*}$  be a real sequence. Then there exists  $\kappa \in \mathbb{R}_+^*$  such that*

$$|u_k + \kappa| \geq 1/k^2.$$

*Proof of Lemma 3.3.* By contradiction let us suppose that for all  $\kappa \in \mathbb{R}_+^*$  there exists  $k \in \mathbb{N}^*$  such that for all  $k \in \mathbb{N}^*$

$$|u_k + \kappa| < 1/k^2.$$

Then

$$\mathbb{R}_+^* \subseteq \bigcup_{k \in \mathbb{N}^*} (u_k - 1/k^2, u_k + 1/k^2). \quad (3.19)$$

The convergence of the series  $\sum_{k \in \mathbb{N}^*} 1/k^2$  implies that the measure of the set in the right hand-side in (3.19) is finite and leads to the conclusion.  $\square$

*Proof of Lemma 3.2.* Let  $(\alpha, \beta)$  an open subinterval of  $\tilde{\omega}$  with  $\alpha$  and  $\beta$  to be determined later,  $\kappa \in \mathbb{R}_+^*$  and  $\theta \in W_\infty^2(0, \pi)$  satisfying

$$\begin{cases} \theta \equiv 1 & \text{in } (0, \pi) \setminus (\alpha, \beta), \\ \theta \equiv 1 + \kappa\xi & \text{in } (\alpha, \beta), \end{cases} \quad (3.20)$$

where

$$\xi = \sin^2 \left( \frac{\pi(x - \alpha)}{\beta - \alpha} \right) \text{ in } (\alpha, \beta).$$

In particular, we have  $\theta \geq 1$  in  $(0, \pi)$ . Let  $k \in \mathbb{N}^*$ ,  $\hat{y}_1 := \theta^{-1}y_1$  and  $\hat{y} := (\hat{y}_1, y_2)$  the solution to System (3.15). For System (3.15) the quantity  $I_k$  defined in the introduction is given by

$$\begin{aligned} I_k(\hat{p}, \hat{q}) &= \int_0^\pi \left\{ \hat{q} - \frac{1}{2} \partial_x \hat{p} \right\} \varphi_k^2 \\ &= I_k(p, q) + \kappa J_k, \end{aligned}$$

with  $\hat{p}$ ,  $\hat{q}$  given in (3.16) and  $J_k$  defined by

$$J_k := \frac{1}{2} \int_\alpha^\beta \partial_x(\xi) \varphi_k^2.$$

Then, after a simple calculation, we obtain

$$J_k = \frac{\frac{2\pi}{(\beta - \alpha)^2}}{\left(2k + \frac{2\pi}{\beta - \alpha}\right)\left(2k - \frac{2\pi}{\beta - \alpha}\right)} \sin(k(\beta + \alpha)) \sin(k(\beta - \alpha)). \quad (3.21)$$

Let  $n \in \mathbb{N}^*$  and  $\ell$  an algebraic number of order two satisfying

$$\frac{a}{n} < \ell < \frac{b}{n+1} \quad \text{and} \quad \ell \neq \frac{\pi}{j} \text{ for all } j \in \mathbb{N}^*.$$

Let us take  $\alpha := n\ell$  and  $\beta := (n+1)\ell$ . Thus  $\alpha, \beta \in (a, b)$  and

$$k(\beta + \alpha) = k(2n+1)\ell \quad \text{and} \quad k(\beta - \alpha) = k\ell. \quad (3.22)$$

Moreover

$$\left| 2k + \frac{2\pi}{\beta - \alpha} \right| \times \left| 2k - \frac{2\pi}{\beta - \alpha} \right| < Rk^2,$$

with  $R > 0$ . Since  $\ell$  is an algebraic number of order two, using diophantine approximations it can be proved that

$$\inf_{j \geq 1} (j |\sin(j\ell)|) \geq \gamma, \quad (3.23)$$

for a positive constant  $\gamma$  (see [9]). The expressions (3.21)-(3.23) give

$$|J_k| \geq \frac{2\pi}{(\beta - \alpha)^2} \frac{\gamma^2}{R(2n+1)k^4}. \quad (3.24)$$

Using Lemma 3.3, there exists  $\kappa \in \mathbb{R}_+^*$  satisfying

$$\left| \frac{I_k(p, q)}{J_k} + \kappa \right| \geq 1/k^2.$$

Combining the last inequality with Estimate (3.24),

$$|I_k(\hat{p}, \hat{q})| = |I_k(p, q) + \kappa J_k| \geq J_k/k^2 \geq C/k^6.$$

□

The next lemma is proved in [9] but, for the sake of completeness, we will include the proof in the annexe A.

**Lemma 3.4.** *There exist functions  $f^{(1)}, f^{(2)} \in L^2(0, \pi)$  satisfying*

$$\text{Supp} \left( f^{(1)} \right), \text{Supp} \left( f^{(2)} \right) \subseteq \omega$$

and such that for all  $k \in \mathbb{N}^*$

$$\begin{cases} \min\{|f_k^{(1)}|, |f_k^{(2)}|\} \geq \frac{C}{k^3}, \\ |B_k| := |\widehat{f}_k^{(1)} f_k^{(2)} - \widehat{f}_k^{(2)} f_k^{(1)}| \geq \frac{C}{k^5}, \end{cases} \quad (3.25)$$

where for  $i \in \{1, 2\}$  the terms  $f_k^{(i)}$  and  $\widehat{f}_k^{(i)}$  are given by

$$f_k^{(i)} := \int_0^\pi f^{(i)}(x) \varphi_k(x) dx \quad \text{and} \quad \widehat{f}_k^{(i)} := \int_0^\pi f^{(i)}(x) \cos(kx) dx. \quad (3.26)$$

**PROPOSITION 3.1.** *Consider  $p \in W_\infty^1(0, \pi) \cap W_\infty^2(\omega)$  and  $q \in L^\infty(0, \pi) \cap W_\infty^1(\omega)$ . Let us suppose that  $|p| > C$  in an open subinterval  $\widetilde{\omega}$  of  $\omega$  for a positive constant  $C$ . Then System (1.1) is equivalent to a system of the form (3.15) with coupling terms  $\widehat{p}, \widehat{q}$  satisfying for all  $k \in \mathbb{N}^*$*

$$\begin{cases} |I_k(\widehat{p}, \widehat{q})| > C/k^6, & \text{if } I_k(\widehat{p}, \widehat{q}) \neq 0, \\ |\det A_{1,k}| > C, & \text{if } I_k(\widehat{p}, \widehat{q}) = 0, \\ \#\{j \in \mathbb{N}^* : I_j(\widehat{p}, \widehat{q}) = 0\} \leq 1, \end{cases} \quad (3.27)$$

for a positive constant  $C$  (the notion of equivalent systems is defined at the beginning of Section 3.2).

*Proof.* The proof is divided in three steps:

- (i) In a first step we will see that System (1.1) is equivalent to a system with coupling terms  $p, q$  satisfying

$$I(p, q) := \int_0^\pi \left\{ q - \frac{1}{2} \partial_x p \right\} \neq 0.$$

- (ii) We will show in a second step that we can suppose that

$$\begin{cases} \text{For all } k \in \mathbb{N}^* : |I_k| > C \text{ if } I_k(p, q) \neq 0, \\ \#\{j \in \mathbb{N}^* : I_j(p, q) = 0\} \leq 1. \end{cases}$$

- (iii) Finally in a third step we will prove that System (1.1) is equivalent to a system which fulfils the three conditions described in (3.27).

**Step 1:** Using Lemma 3.1, without loss of generality, we can suppose that  $q \equiv 0$  and  $|p| > C$  in a subinterval  $\widehat{\omega}$  of  $\widetilde{\omega}$  for a positive constant  $C$ . If  $\partial_x p \equiv 0$  in  $\widehat{\omega}$ , Lemma 3.2 leads to

$$|I_k(p, q)| \geq C/k^6, \quad \forall k \in \mathbb{N}^*.$$

Otherwise, let  $(\alpha, \beta) \subseteq \widehat{\omega}$  such that  $\partial_x p > C$  in  $(\alpha, \beta)$  or  $\partial_x p < -C$  in  $(\alpha, \beta)$  for a positive constant  $C$ . Assume that  $I(p, q) = 0$  and consider  $\theta \in W_\infty^2(0, \pi)$  defined in (3.20), with  $\kappa := 1$ . We remark that  $|\theta| \geq 1$ . If we consider the change of variable described in (3.14), then for all  $k \in \mathbb{N}^*$ , using the definition of  $I_k$ , we obtain

$$\begin{aligned} I_k(\widehat{p}, \widehat{q}) &= I_k(p, q) + \int_\alpha^\beta \left\{ \frac{1}{2} \partial_x(\xi) p - \frac{1}{2} \xi \partial_x(p) \right\} \varphi_k^2 dx \\ &= I_k(p, q) + J_k(p, q), \end{aligned}$$

where

$$\begin{aligned} J_k(p, q) &= \frac{1}{2\pi} \int_{\alpha}^{\beta} \{\partial_x(\xi)p - \xi\partial_x(p)\} \{1 - \cos(2kx)\} dx \\ &\xrightarrow{k \rightarrow \infty} \frac{1}{2\pi} \int_{\alpha}^{\beta} \{\partial_x(\xi)p - \xi\partial_x(p)\} dx \\ &= -\frac{1}{\pi} \int_{\alpha}^{\beta} \xi\partial_x(p) dx =: J(p, q). \end{aligned}$$

Using the definition of  $\xi$ , we get

$$\begin{aligned} |J(p, q)| &\geq \frac{1}{\pi} \inf_{(\alpha, \beta)} |\partial_x p| \int_{\alpha}^{\beta} \sin^2 \left( \frac{\pi(x - \alpha)}{\beta - \alpha} \right) dx \\ &= \frac{1}{2\pi} \inf_{(\alpha, \beta)} |\partial_x p| \int_{\alpha}^{\beta} \left\{ 1 - \cos \left( \frac{2\pi(x - \alpha)}{\beta - \alpha} \right) \right\} dx \\ &= \frac{(\beta - \alpha)}{2\pi} \inf_{(\alpha, \beta)} |\partial_x p| \neq 0. \end{aligned}$$

We recall that  $I_k(p, q) \rightarrow I(p, q) = 0$ . Thus we obtain  $I_k(\widehat{p}, \widehat{q}) \rightarrow I(\widehat{p}, \widehat{q}) \neq 0$ .

**Step 2:** Let us now assume that  $I(p, q) \neq 0$ . Using Lemma 3.1, up to the change of variable (3.14) we can also suppose that  $q \equiv 0$  in an open subinterval  $\widehat{\omega}$  of  $\widetilde{\omega}$ . Indeed, by (3.17), the function  $\theta$  and  $\widehat{\omega}$  can be chosen in order to keep the quantity  $I$  different of zero. Let  $(\alpha, \beta) \subseteq \widehat{\omega}$  such that  $|p| > C > 0$  in  $(\alpha, \beta)$ . Since  $I(p, q) \neq 0$  and  $I_k(p, q) \rightarrow I(p, q)$ , there exists  $k_0 \in \mathbb{N}^*$  such that  $|I_k(p, q)| > C$  for a constant  $C > 0$  and all  $k \geq k_0$ . Let us define the set

$$S_0 := \{k \in \mathbb{N}^* : I_k(p, q) = 0 \text{ and } p\varphi_k \text{ non constant in } (\alpha, \beta)\}$$

and  $M := \#S_0 < \infty$ . Let  $\theta \in W_{\infty}^2(0, \pi)$  satisfying

$$\begin{cases} \theta = 1 + \sum_{m=1}^M \xi_m, \\ \xi_m \in W_{\infty}^2(0, \pi), & \text{for all } m \in \{1, \dots, M\}, \\ \text{Supp}(\xi_m) \subseteq (\alpha, \beta), & \text{for all } m \in \{1, \dots, M\}, \\ |\theta| > C > 0, \end{cases}$$

where  $\xi_1, \dots, \xi_M$  are to be determined. Again, if we consider the change of variable (3.14), then for all  $k \in \mathbb{N}^*$ , using the definition of  $I_k$ , we obtain

$$\begin{aligned} I_k(\widehat{p}, \widehat{q}) &= I_k(p, q) + \sum_{m=1}^M \int_{\alpha}^{\beta} \left\{ \frac{1}{2} \partial_x(\xi_m)p - \frac{1}{2} \xi_m \partial_x(p) \right\} \varphi_k^2 dx \\ &=: I_k(p, q) + \sum_{m=1}^M J_{m,k}(p, q). \end{aligned}$$

The goal is to choose the functions  $\xi_1, \dots, \xi_M$  such that  $|I_k(\widehat{p}, \widehat{q})| > C$  if  $I_k(p, q) = 0$  and  $\#\{k : I_k(\widehat{p}, \widehat{q}) = 1\} \leq 1$  for a positive constant  $C$ . We will construct  $\xi_1, \dots, \xi_M$  by induction. Let  $m \in \{1, \dots, M-1\}$  and let us assume that  $\xi_1, \dots, \xi_{m-1}$  are already constructed. Consider the set

$$S_{m-1} := \{k \in \mathbb{N}^* : I_k(p, q) + \sum_{j=1}^{m-1} J_{j,k}(p, q) = 0 \text{ and } p\varphi_k \text{ non constant in } (\alpha, \beta)\}.$$

If  $S_{m-1} = \emptyset$ , then we take  $\xi_m = 0$  in  $(0, \pi)$ . Otherwise, let  $k \in S_{m-1}$  and consider  $(f, \xi_m) \in W_{\infty}^1(\alpha, \beta) \cap W_{\infty}^2(\alpha, \beta)$  a solution to

$$\begin{cases} \frac{1}{2} \partial_x(\xi_m)p - \frac{1}{2} \xi_m \partial_x(p) = f & \text{in } (\alpha, \beta), \\ \xi_m(\alpha) = \xi_m(\beta) = \partial_x \xi_m(\alpha) = \partial_x \xi_m(\beta) = 0. \end{cases}$$

This system is equivalent to

$$\begin{cases} \xi_m(x) = p(x) \int_{\alpha}^x \frac{2f(s)}{p^2(s)} ds, & \text{for all } x \in (\alpha, \beta), \\ \int_{\alpha}^{\beta} \frac{2f(s)}{p^2(s)} ds = 0, \\ f(\alpha) = f(\beta) = 0. \end{cases}$$

We remark that we need that  $p \in W_{\infty}^2(\alpha, \beta)$ . Finding a function  $f$  satisfying

$$f(\alpha) = f(\beta) = 0, \int_{\alpha}^{\beta} \frac{2f(s)}{p^2(s)} ds = 0 \text{ and } J_{m,k}(p, q) = \int_{\alpha}^{\beta} f(s) \varphi_k^2(s) ds \neq 0, \quad (3.28)$$

is equivalent to finding a function  $g := 2f/p^2$  satisfying

$$g(\alpha) = g(\beta) = 0, \int_{\alpha}^{\beta} g(s) ds = 0 \text{ and } \int_{\alpha}^{\beta} g(s) p^2(s) \varphi_k^2(s) ds \neq 0.$$

Let  $\kappa_m \in \mathbb{R}$  and define for all  $j \in \mathbb{N}^*$  and all  $x \in (\alpha, \beta)$

$$g_j(x) := \kappa_m \sin\left(\frac{2\pi j(x - \alpha)}{\beta - \alpha}\right).$$

Using the fact that  $p\varphi_k$  is non constant in  $(\alpha, \beta)$ , without loss of generality we can suppose that

$$\varphi_k\left(\alpha + \frac{\beta - \alpha}{4}\right) p\left(\alpha + \frac{\beta - \alpha}{4}\right) \neq \varphi_k\left(\alpha + \frac{3(\beta - \alpha)}{4}\right) p\left(\alpha + \frac{3(\beta - \alpha)}{4}\right),$$

otherwise we adapt the interval  $(\alpha, \beta)$  at the beginning of Step 2. Since  $(g_j)_{j \in \mathbb{N}^*}$  is a basis of  $L^2(\alpha, \alpha + (\beta - \alpha)/2)$  and  $g_j(s) = g_j(s + \beta - \alpha)$  pour tout  $s \in (\alpha, \alpha + (\beta - \alpha)/2)$ , there exists  $j_0 \in \mathbb{N}^*$  such that

$$\int_{\alpha}^{(\beta - \alpha)/2} g_{j_0}(s) p^2(s) \varphi_k^2(s) ds \neq \int_{\alpha}^{(\beta - \alpha)/2} (s + \beta - \alpha) g_{j_0}(s) p^2(s + \beta - \alpha) \varphi_k^2(s) ds.$$

Thus

$$\int_{\alpha}^{\beta} g_{j_0}(s) p^2(s) \varphi_k^2(s) ds \neq 0.$$

Plugging  $g := g_{j_0}$  and  $f := \frac{g_{j_0} p^2}{2}$  in (3.28), we obtain

$$J_{m,k}(p, q) = \frac{\kappa_m}{2} \int_{\alpha}^{\beta} \sin\left(\frac{2\pi j_0(s - \alpha)}{\beta - \alpha}\right) p(s)^2 \varphi_k(s)^2 ds \neq 0$$

and we fix  $\kappa_m$  in order to have

$$\sup_{i \in \mathbb{N}^*} |J_{m,i}(p, q)| \leq \frac{1}{2} \inf_{i \in \mathbb{N}^*} \left| I_i(p, q) + \sum_{j=1}^{m-1} J_{j,i}(p, q) \right|.$$

Thus,  $R := \#\{j \in \mathbb{N}^* : I_j(\hat{p}, \hat{q}) = 0\} \leq 1$  and for a positive constant  $C$  and for all  $k \notin R$ ,  $|I_k(\hat{p}, \hat{q})| > C$ .

**Step 3:** Assume that

$$\begin{cases} R := \#\{j \in \mathbb{N}^* : I_j(p, q) = 0\} = 1, \\ \text{For all } k \notin R : |I_k(p, q)| > C > 0. \end{cases} \quad (3.29)$$



Again, using Lemma 3.1, up to the change of variable (3.14) described at the beginning of the section we can also suppose that  $q \equiv 0$  in a subinterval  $(\alpha, \beta)$  of  $\tilde{\omega}$ . Indeed, using (3.17), this change of variable can be chosen in order to keep property (3.29). Let  $m \in \mathbb{N}^*$  such that  $I_m(p, q) = 0$ . We also suppose that  $p\varphi_m$  is constant in a  $(\alpha, \beta)$ , otherwise we argue as in Step 2. Let  $\theta \in W_\infty^2(0, \pi)$  satisfying

$$\begin{cases} \theta = 1 + \xi & \text{in } (0, \pi), \\ \xi \in W_\infty^2(0, \pi), \\ \xi \equiv \xi_\alpha \in \mathbb{R}_+^* & \text{in } (0, \alpha), \\ \xi \equiv 0 & \text{in } (\beta, \pi), \\ |\theta| > C > 0. \end{cases}$$

Again, if we consider the change of variable described in (3.14), then for all  $k \in \mathbb{N}^*$

$$\begin{aligned} I_k(\widehat{p}, \widehat{q}) &= I_k(p, q) + \int_0^\beta \left\{ \frac{1}{2} \partial_x(\xi) p + \xi q - \frac{1}{2} \xi \partial_x(p) \right\} \varphi_k^2 dx \\ &=: I_k(p, q) + J_k(p, q). \end{aligned}$$

We will distinguish the cases  $I_{\alpha, m}(p, q) = 0$  and  $I_{\alpha, m}(p, q) \neq 0$  (see (1.8) for the definition of this quantity) for the new control domain  $\omega := (\alpha, \beta)$ .

Case 1: Assume that  $I_{\alpha, m}(p, q) = 0$ . Let  $(\xi, h) \in W_\infty^2(\alpha, \beta) \times W_\infty^1(\alpha, \beta)$  be a solution to the system

$$\begin{cases} \frac{1}{2} \partial_x(\xi) p - \frac{1}{2} \xi \partial_x(p) = h & \text{in } (\alpha, \beta), \\ \xi(\beta) = \partial_x \xi(\alpha) = \partial_x \xi(\beta) = 0, \\ \xi(\alpha) = \xi_\alpha \in \mathbb{R}^*. \end{cases}$$

This system is equivalent to

$$\begin{cases} \xi(x) = -p(x) \int_x^\beta \frac{2h(s)}{p^2(s)} ds, & \text{for all } x \in (\alpha, \beta), \\ \int_\alpha^\beta \frac{2h(s)}{p^2(s)} ds = \frac{-\xi_\alpha}{p(\alpha)}, \\ h(\alpha) = \frac{-\xi_\alpha \partial_x p(\alpha)}{2}, \\ h(\beta) = 0. \end{cases}$$

Taking in account that  $I_{\alpha, m}(p, q) = 0$ ,  $q \equiv 0$  in  $(\alpha, \beta)$  and  $p\varphi_m \equiv \gamma$  in  $(\alpha, \beta)$  for a  $\gamma \in \mathbb{R}^*$ , one gets

$$J_m(p, q) = \xi_\alpha \int_0^\alpha \left( q - \frac{1}{2} \partial_x(p) \right) \varphi_m^2 dx + \frac{\gamma^2}{2} \int_\alpha^\beta \partial_x \left( \frac{\xi}{p} \right) dx = -\frac{\gamma^2 \xi_\alpha}{2p(\alpha)}.$$

Let  $\xi_\alpha$  and  $h$  be such that

$$\sup_{k \in \mathbb{N}^*} |J_k| \leq \frac{1}{2} \inf_{k \in \mathbb{N}^*} |I_k|.$$

Then  $|I_k(\widehat{p}, \widehat{q})| > C$  for all  $k \in \mathbb{N}^*$  and a positive constant  $C$ .

Case 2: Let us now assume that  $I_{\alpha, m}(p, q) \neq 0$ . In this case we recall that in the moment problem described in the last section we have

$$\det A_{1, m} := \widetilde{f}_m^{(1)} f_m^{(2)} - \widetilde{f}_m^{(2)} f_m^{(1)},$$

where  $f_m^{(1)}$ ,  $f_m^{(2)}$ ,  $\widetilde{f}_m^{(1)}$  and  $\widetilde{f}_m^{(2)}$  are given in (3.4). Since  $p\varphi_m$  is constant in  $(\alpha, \beta)$ , the function  $\psi_m^*$  of Proposition 2.1 reads for all  $x \in (\alpha, \beta)$

$$\begin{aligned} \psi_m^*(x) &= \alpha_m^* \varphi_m - \frac{1}{m} \int_0^\alpha \sin(m(x - \xi)) [\partial_x(p(\xi) \varphi_m(\xi)) - q(\xi) \varphi_m(\xi)] d\xi \\ &= \tau_m \varphi_m(x) - \sqrt{\frac{\pi}{2}} \frac{1}{m} I_{\alpha, m}(p, q) \cos(mx), \end{aligned}$$

with

$$\tau_m := \alpha_m^* - \sqrt{\frac{\pi}{2}} \frac{1}{m} \int_0^\alpha \cos(m\xi) [\partial_x(p(\xi)\varphi_m(\xi)) - q(\xi)\varphi_m(\xi)] d\xi.$$

We deduce that

$$\det A_{1,m} = -\sqrt{\frac{\pi}{2}} \frac{1}{m} I_{\alpha,m}(p,q) (\widehat{f}_m^{(1)} f_m^{(2)} - \widehat{f}_m^{(2)} f_m^{(1)}),$$

where  $\widehat{f}_m^{(1)}$  and  $\widehat{f}_m^{(2)}$  are given in Lemma 3.4. We conclude with the help of Lemma 3.4.  $\square$

We recall that  $T_0(p,q)$  is given by

$$T_0(p,q) := \limsup_{k \rightarrow \infty} \frac{\min(-\log |I_k(p,q)|, -\log |I_{a,k}(p,q)|)}{k^2}. \quad (3.30)$$

In order to use the following proposition in the next section, we prove the null controllability of System (1.1) under Condition (3.31) which is more general than (3.27).

**PROPOSITION 3.2.** *Assume that  $T > T_0(p,q)$  and, for positive constants  $C_1$  and  $C_2$ ,*

$$|\det A_{1,k}| \geq \frac{C_1}{k^6} |I_{a,k}(p,q)| - \frac{C_2}{k} |I_k(p,q)|, \quad (3.31)$$

*Then System (1.1) is null controllable at time  $T$ .*

*Proof.* Let  $\varepsilon > 0$ . Using the definition of the minimal time  $T_0(p,q)$  in (3.30), there exists a positive integer  $k_\varepsilon$  for which

$$\min \left\{ \log |I_{a,k}(p,q)|^{-1}, \log |I_k(p,q)|^{-1} \right\} < k^2 (T_0(p,q) + \varepsilon), \quad \forall k > k_\varepsilon. \quad (3.32)$$

The goal is to solve the moment problem described in Section 3.1. We recall that we look for a control  $v$  of the form (3.2) and (3.8) with  $f^{(1)}$  and  $f^{(2)}$  defined in Lemma 3.4. We will solve the moment problem (3.9) depending on whether  $k$  belongs to  $\Lambda_1$  or  $\Lambda_2$ , where

$$\begin{cases} \Lambda_1 & := \{k \in \mathbb{N}^* : I_{a,k}(p,q) \neq 0\}, \\ \Lambda_2 & := \{k \in \mathbb{N}^* : I_{a,k}(p,q) = 0\}. \end{cases}$$

**Case 1 :** Consider the case  $k \in \Lambda_1$  with  $k \leq k_\varepsilon$ .

Let us take

$$v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0.$$

The moment problem (3.9) becomes

$$\begin{cases} \widehat{f}_k^{(1)} v_{1,k}^{(1)} - I_k(p,q) f_k^{(1)} v_{2,k}^{(1)} = -e^{-k^2 T} \left( y_{1,k}^0 - T I_k(p,q) y_{2,k}^0 \right), \\ f_k^{(1)} v_{1,k}^{(1)} = -e^{-k^2 T} y_{2,k}^0. \end{cases}$$

Since  $I_k(p,q) \neq 0$  and using the estimate of  $f_k^{(1)}$  and  $f_k^{(2)}$  in Lemma 3.4, the last system has a unique solution

$$\begin{cases} v_{1,k}^{(1)} = -e^{-k^2 T} \frac{y_{2,k}^0}{f_k^{(1)}}, \\ v_{2,k}^{(1)} = \frac{e^{-k^2 T}}{I_k(p,q) f_k^{(1)}} \left( y_{1,k}^0 - T I_k(p,q) y_{2,k}^0 - \widehat{f}_k^{(1)} \frac{y_{2,k}^0}{f_k^{(1)}} \right). \end{cases} \quad (3.33)$$

Moreover, since the set of the  $k$  considered in this case is finite, we get the inequality

$$\left| v_{j,k}^{(i)} \right| \leq C_\varepsilon e^{-k^2 T} \|y^0\|_{L^2(0,\pi)^2}, \quad i, j = 1, 2. \quad (3.34)$$

**Case 2:** Let  $k \in \Lambda_1$  such that  $k > k_\varepsilon$  and  $|I_k(p, q)|^{-1} \leq e^{k^2(T_0(p,q)+2\varepsilon)}$ . As in the previous case, we take  $v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0$  and the moment problem (3.9) has a unique solution, given by (3.33). Thanks to the property of  $\psi_k^*$  (see (2.6)) and Lemma 3.4, we get for  $i = 1, 2$  the following estimates

$$|f_k^{(1)}| \geq C/k^3, \quad |\tilde{f}_k^{(i)}| \leq \frac{C}{k}, \quad |y_{i,k}^0| \leq C \|y_0\|_{L^2(0,\pi)^2}, \quad \forall k \in \mathbb{N}^*. \quad (3.35)$$

Thus, using the assumptions on  $k$ , we obtain

$$\begin{cases} |v_{1,k}^{(1)}| \leq C k^3 e^{-k^2 T} \|y^0\|_{L^2(0,\pi)^2} \leq C_\varepsilon e^{-(T-\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \\ |v_{1,k}^{(2)}| \leq \frac{C_\varepsilon e^{-(T-\varepsilon)k^2}}{I_k(p, q)} \|y^0\|_{L^2(0,\pi)^2} \leq C_\varepsilon e^{-(T-T_0-3\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \end{cases}$$

where  $C_\varepsilon$  is a constant which is independent on  $k$  and  $y^0$ .

**Case 3:** Consider now  $k \in \Lambda_1$  such that  $k > k_\varepsilon$  and  $|I_k(p, q)|^{-1} > e^{k^2(T_0(p,q)+2\varepsilon)}$ . This implies with (3.32) that

$$|I_{a,k}(p, q)|^{-1} < e^{k^2(T_0(p,q)+\varepsilon)}. \quad (3.36)$$

The two last inequality lead to

$$|I_k(p, q)| < e^{-\varepsilon k^2} |I_{a,k}(p, q)|.$$

Combined with inequality (3.31), taking  $k_\varepsilon$  large enough, we get

$$|\det A_{1,k}| > C_\varepsilon e^{-\varepsilon k^2} |I_{a,k}(p, q)|, \quad (3.37)$$

with  $C_\varepsilon$  independent on  $k$ . To solve the moment problem (3.9), we take here

$$v_{2,k}^{(1)} = v_{2,k}^{(2)} = 0.$$

Then the moment problem (3.9) reads  $A_{1,k} V_{1,k} = F_k$ . Since  $\det A_{1,k} \neq 0$ , the inverse of  $A_{1,k}$  is given by

$$(A_{1,k})^{-1} = (\det A_{1,k})^{-1} \begin{pmatrix} f_k^{(2)} & -\tilde{f}_k^{(2)} \\ -f_k^{(1)} & \tilde{f}_k^{(1)} \end{pmatrix}.$$

We deduce that the solution to the moment problem (3.9) is

$$\begin{cases} v_{1,k}^{(1)} = \frac{e^{-k^2 T}}{\det A_{1,k}} \{-f_k^{(2)} y_{1,k}^0 + (T I_k(p, q) f_k^{(2)} + \tilde{f}_k^{(2)}) y_{2,k}^0\}, \\ v_{1,k}^{(2)} = \frac{e^{-k^2 T}}{\det A_{1,k}} \{f_k^{(1)} y_{1,k}^0 - (T I_k(p, q) f_k^{(1)} + \tilde{f}_k^{(1)}) y_{2,k}^0\}. \end{cases}$$

The last expression together with (3.36) and (3.37) gives

$$|v_{1,k}^{(i)}| \leq C_\varepsilon e^{-(T-T_0-2\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \quad i = 1, 2.$$

**Case 4:** Let us consider  $k \in \Lambda_2$ .

If  $k \leq k_\varepsilon$ , we can argue as in Case 1. Let us suppose that  $k > k_\varepsilon$ . In this case,  $I_{a,k}(p, q) = 0$ ,  $I_k(p, q) \neq 0$  and inequality (3.32) reads  $|I_k(p, q)|^{-1} < e^{k^2(T_0(p,q)+\varepsilon)}$ . We take here

$$v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0$$

and the solution of moment problem (3.9) is given by (3.33). We get

$$\left| v_{j,k}^{(i)} \right| \leq C_\varepsilon e^{-k^2(T-T_0(p,q)-2\varepsilon)} \|y^0\|_{L^2(0,\pi)^2}, \quad i, j = 1, 2.$$

**Conclusion:**

We have constructed a control  $v$  of the form (3.2) and (3.8), which satisfies

$$\left| v_{j,k}^{(i)} \right| \leq C_\varepsilon \sum_{k \geq 1} e^{-k^2(T-T_0(p,q)-3\varepsilon)} \|y^0\|_{L^2(0,\pi)^2}, \quad i, j = 1, 2, k \in \mathbb{N}^*.$$

The last inequality, the estimate (3.7) of  $q_{i,k}$  and the expression (3.8) of  $v^{(i)}$  ( $i = 1, 2$ ) lead to

$$\left\| v^{(i)} \right\|_{L^2(0,T)} \leq C_{\varepsilon,T} e^{-k^2(T-T_0(p,q)-4\varepsilon)}, \quad i = 1, 2.$$

Thus, taking

$$\varepsilon \in \left( 0, \frac{T - T_0(p, q)}{4} \right),$$

we have the absolute convergence of the series defining  $v^{(1)}$  and  $v^{(2)}$  in  $L^2(0, T)$ . This ends the proof.  $\square$

*Proof of Theorem 1.1.* Using Proposition 3.1, System (1.1) is equivalent to a system with coupling terms  $\hat{p}$  and  $\hat{q}$  satisfying Condition (3.27), that is

$$\begin{cases} |I_k(\hat{p}, \hat{q})| > C/k^6, & \text{if } I_k(\hat{p}, \hat{q}) \neq 0, \\ |\det A_{1,k}| > C, & \text{if } I_k(\hat{p}, \hat{q}) = 0, \\ \#\{j \in \mathbb{N}^* : I_j(\hat{p}, \hat{q}) = 0\} \leq 1, \end{cases}$$

This condition implies Condition (3.31):

$$|\det A_{1,k}| \geq \frac{C_1}{k^6} |I_{a,k}(p, q)| - \frac{C_2}{k} |I_k(p, q)|.$$

Proposition 3.2 leads to the null controllability of System (1.1) when  $T > T_0(\hat{p}, \hat{q})$ . We ends the proof of Theorems 1.1 remarking that  $T_0(\hat{p}, \hat{q}) = 0$ .  $\square$

## 4 Proof of Theorem 1.2

### 4.1 Positive null controllability result

Before studying the case where the intersection of the coupling and control domains is empty, we will first rewrite the function  $\psi_k^*$  given in Proposition 2.1.

**Lemma 4.1.** *Let  $k \in \mathbb{N}^*$ . Consider the function  $\psi_k^*$  defined in Proposition 2.1. If we suppose that Condition (1.11) holds, then for all  $x \in \omega$*

$$\psi_k^*(x) = \tau_k \varphi_k(x) + g_k(x) \text{ for all } x \in \omega,$$

where

$$\tau_k := \alpha_k^* - \sqrt{\frac{\pi}{2}} \frac{1}{k} \int_0^a \cos(k\xi) [\partial_x(p(\xi)\varphi_k(\xi) - q(\xi)\varphi_k(\xi))] d\xi$$

and

$$g_k(x) := -\frac{I_k(p, q)}{k} \int_0^x \sin(k(x - \xi)) \varphi_k(\xi) d\xi - \sqrt{\frac{\pi}{2}} \frac{1}{k} I_{a,k}(p, q) \cos(kx).$$

*Proof.* Since  $p = q \equiv 0$  in  $\omega$ , we get for all  $x \in \omega$ ,

$$\begin{aligned} \psi_k^*(x) &= \alpha_k^* \varphi_k(x) - \frac{I_k(p, q)}{k} \int_0^x \sin(k(x - \xi)) \varphi_k(\xi) d\xi \\ &\quad - \frac{1}{k} \int_0^a \sin(k(x - \xi)) [\partial_x(p(\xi) \varphi_k(\xi)) - q(\xi) \varphi_k(\xi)] d\xi. \end{aligned}$$

□

*Proof of Theorem 1.2.* Assume that Condition (1.11) holds. Consider the functions  $f^{(1)}$  and  $f^{(2)}$  defined in Lemma 3.4 and the matrix  $A_{1,k}$  given in (3.10). Let  $k \in \mathbb{N}^*$ . We recall that, in this case, we have

$$\det A_{1,k} := \tilde{f}_k^{(1)} f_k^{(2)} - \tilde{f}_k^{(2)} f_k^{(1)},$$

where, for  $i = 1, 2$ ,  $f_k^{(i)}$  and  $\tilde{f}_k^{(i)}$  are defined in (3.4). Since  $\text{Supp}(f^{(i)}) \subseteq \omega$ , using the expression of  $\psi_k^*$  given in Lemma 4.1, we obtain

$$\tilde{f}_k^{(i)} = \tau_k f_k^{(i)} + \int_0^\pi f^{(i)}(x) g_k(x) dx,$$

where for all  $x \in \omega$

$$g_k(x) = -\frac{I_k(p, q)}{k} \int_0^x \sin(k(x - \xi)) \varphi_k(\xi) d\xi - \sqrt{\frac{\pi}{2}} \frac{1}{k} I_{a,k}(p, q) \cos(kx).$$

We deduce that

$$\begin{aligned} \det A_{1,k} &= f_k^{(2)} \int_0^\pi f^{(1)}(x) g_k(x) dx - f_k^{(1)} \int_0^\pi f^{(2)}(x) g_k(x) dx \\ &= -\frac{I_k(p, q)}{k} \left( f_k^{(2)} \int_0^\pi \int_0^x f^{(1)}(x) \sin(k(x - \xi)) \varphi_k(\xi) d\xi dx \right. \\ &\quad \left. - f_k^{(1)} \int_0^\pi \int_0^x f^{(2)}(x) \sin(k(x - \xi)) \varphi_k(\xi) d\xi dx \right) \\ &\quad - \sqrt{\frac{\pi}{2}} \frac{1}{k} I_{a,k}(p, q) \left( \tilde{f}_k^{(1)} f_k^{(2)} - \tilde{f}_k^{(2)} f_k^{(1)} \right), \end{aligned}$$

where  $\tilde{f}_k^{(i)}$  are defined in (3.26). Since the integrals

$$\int_0^\pi \int_0^x f^{(i)}(x) \sin(k(x - \xi)) \varphi_k(\xi) d\xi dx$$

are uniformly bounded with respect to  $k$  and  $i$ , we conclude with the help of Lemma 3.4.

We deduce that Condition (3.31) holds. Thus, using Proposition 3.2, System (1.1) is null controllable at time  $T$ .

□

## 4.2 Negative null controllability result

Let us suppose that  $T < T_0(p, q)$  and System (1.1) is null controllable at time  $T$ . Using Proposition 2.4, there exists a constant  $C_{obs} > 0$  such that for all  $\theta^0 \in L^2(0, \pi)^2$ , the solution to System (2.8) satisfies the observability inequality

$$\|\theta(0)\|_{L^2(0, \pi)^2}^2 \leq C_{obs} \iint_{Q_T} |\mathbb{1}_\omega B^* \theta|^2 dx dt. \quad (4.1)$$

Using the Definition of  $T_0(p, q)$  (see (3.30)) there exists a sequence  $(k_n)_{n \in \mathbb{N}^*} \subseteq \mathbb{N}$  satisfying:

$$T_0(p, q) = \lim_{n \rightarrow \infty} \frac{\min(\log |I_{a, k_n}(p, q)^{-1}|, \log |I_{k_n}(p, q)^{-1}|)}{k_n^2}. \quad (4.2)$$

Let us fix  $n \geq 1$  and  $\theta^{0n} := a_n \Phi_{1, k_n}^* + b_n \Phi_{2, k_n}^*$  with  $(a_n, b_n) \in \mathbb{R}^2$  to be determined later and  $\Phi_{2, k_n}^*$ ,  $\Phi_{1, k_n}^*$  the eigenfunction and generalized eigenfunction associated with  $k_n^2$  given in Proposition 2.1. If we denote by  $\theta^n$  the solution to the dual System (2.8) for initial data  $\theta^{0n}$ , then

$$\theta^n(x, t) = e^{-k_n^2(T-t)} \{a_n \Phi_{1, k_n}^* + (b_n - (T-t)I_{k_n}(p, q)a_n) \Phi_{2, k_n}^*\},$$

thus we have

$$A_{1, n} := \|\theta^n(0)\|_{L^2(0, \pi)^2}^2 = e^{-2k_n^2 T} \left\{ |a_n|^2 |\psi_{k_n}|^2 + |a_n|^2 + (b_n - T I_{k_n}(p, q) a_n)^2 \right\}$$

and

$$A_{2, n} := \iint_{Q_T} |\mathbb{1}_\omega B^* \theta^n|^2 dx dt = \int_0^T \int_\omega e^{-2k_n^2 t} |a_n \psi_{k_n}(x) + (b_n - t I_{k_n}(p, q) a_n) \varphi_{k_n}(x)|^2 dx dt.$$

The observability inequality (4.1) reads

$$A_{1, n} \leq C_{obs} A_{2, n}. \quad (4.3)$$

By choosing  $a_n := 1$  and  $b_n := -\tau_{k_n}$ , we get

$$A_{1, n} \geq e^{-2k_n^2 T} \quad (4.4)$$

and the expression of  $\psi_{k_n}^*(x)$  given in Lemma 4.1 leads to

$$\begin{aligned} A_{2, n} &= \int_0^T \int_\omega e^{-2k_n^2 t} \left| -\sqrt{\frac{\pi}{2}} \frac{1}{k_n} I_{a, k_n}(p, q) \cos(k_n x) \right. \\ &\quad \left. - I_{k_n}(p, q) \frac{1}{k_n} \int_0^x \sin(k_n(x-\xi)) \varphi_{k_n}(\xi) d\xi - t I_{k_n}(p, q) \varphi_{k_n}(x) \right|^2 dx dt \\ &\leq C(I_{a, k_n}(p, q)^2 + I_{k_n}(p, q)^2). \end{aligned}$$

Let  $\varepsilon > 0$ . Equality (4.2) implies that there is  $k_\varepsilon \in \mathbb{N}^*$  such that for all  $k_n \geq k_\varepsilon$

$$\max(|I_{a, k_n}(p, q)|^2, |I_{k_n}(p, q)|^2) \leq e^{-2k_n^2(T_0(p, q) - \varepsilon)}.$$

We deduce that for  $\varepsilon := (T_0(p, q) - T)/2$ , we get

$$A_{2, n} \leq C e^{-2k_n^2(T+\varepsilon)}. \quad (4.5)$$

Thus estimates (4.4) and (4.5) are in contradiction with inequality (4.3) for  $n$  large enough.

## 5 Approximate controllability of system (1.1)

Theorem 1.2 can be proved with the same argument as in [9], but we propose to use here the Hautus test given below.

**THEOREM 5.1** (see [15], Cor. 3.3). *System (1.1) is approximatively controllable at time  $T$  if and only if for any  $s \in \mathbb{C}$  and for any  $u \in \mathcal{D}(L^*)$  we have*

$$\left. \begin{array}{l} L^*u = su \text{ in } (0, \pi) \\ B^*u = 0 \text{ in } \omega \end{array} \right\} \Rightarrow u = 0.$$

*Proof of Theorem 1.2.*

**Necessary condition:** Let us suppose that Conditions (1.9)-(1.10) do not hold *i.e.* there exists  $k_0 \in \mathbb{N}^*$  such that

$$I_{k_0}(p, q) = I_{a, k_0}(p, q) = 0$$

and

$$(\text{Supp}(p) \cup \text{Supp}(q)) \cap \omega = \emptyset.$$

Then the function  $\psi_{k_0}^*$  of Lemma 4.1 is given by  $\psi_{k_0}^* = \tau_{k_0} \varphi_{k_0}$ . Moreover

$$\Phi_{1, k_0}^* - \tau_{k_0} \Phi_{2, k_0}^* = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}$$

is an eigenfunction associated with the eigenvalue  $k_0^2$  of the operator  $L^*$  and satisfies

$$B^*(\Phi_{1, k_0}^* - \tau_{k_0} \Phi_{2, k_0}^*) \equiv 0 \text{ in } \omega.$$

Thus, using Theorem 5.1 System (1.1) is not approximately controllable at time  $T$ .

**Sufficient condition:** Let us suppose that Conditions (1.9)-(1.10) hold. If  $(\text{Supp}(p) \cup \text{Supp}(q)) \cap \omega \neq \emptyset$  we conclude using Theorem 1.1. Let us now suppose that

$$|I_k(p, q)| + |I_{a, k}(p, q)| \neq 0 \text{ for all } k \in \mathbb{N}^*$$

and

$$(\text{Supp}(p) \cup \text{Supp}(q)) \cap \omega = \emptyset.$$

If  $I_k(p, q) \neq 0$ , the set of the eigenvectors associated with the eigenvalue  $k^2$  of  $L^*$  is generated by  $\Phi_{2, k}^*$  (see Proposition 2.1). In this case, we remark that for all  $k \in \mathbb{N}^*$

$$B^* \Phi_{2, k}^* = \varphi_k \neq 0 \text{ in } \omega. \quad (5.1)$$

If  $I_k(p, q) = 0$ , the eigenvectors associated with the eigenvalue  $k^2$  of  $L^*$  are linear combinations of  $\Phi_{1, k}^*$  and  $\Phi_{2, k}^*$ . We remark first that (5.1) holds. Let us suppose that

$$B^* \Phi_{1, k}^* = \psi_k \equiv 0 \text{ in } \omega. \quad (5.2)$$

Using Lemma 4.1, it is equivalent to

$$\tau_k \varphi_k(x) - \sqrt{\frac{\pi}{2}} \frac{1}{k} I_{a, k}(p, q) \cos(kx) = 0 \text{ for all } x \in \omega.$$

Since  $I_{a, k}(p, q) \neq 0$ , we obtain a contradiction with (5.2), which ends the proof. □

## 6 Null controllability of System (1.2)

As in Section 3.1, System (1.2) is null controllable at time  $T$  if and only if for all  $y^0 \in H^{-1}(0, \pi)^2$ ,  $k \in \mathbb{N}^*$  and  $i \in \{1, 2\}$  the solution  $\theta_{i, k}$  to the dual System (2.8) for the initial data  $\Phi_{i, k}^*$  satisfies

$$\int_0^T u(t) B^* \partial_x \theta_{i, k}(0, t) dt = -\langle y^0, \theta_{i, k}(\cdot, 0) \rangle_{H^{-1}, H_0^1}. \quad (6.1)$$

We recall that, for all  $k \in \mathbb{N}^*$ ,  $\theta_{1,k}$  and  $\theta_{2,k}$  are given for all  $(x, t) \in Q_T$  by

$$\begin{cases} \theta_{1,k}(x, t) = e^{-k^2(T-t)} \left( \Phi_{1,k}^*(x) - (T-t)I_k(p, q)\Phi_{2,k}^*(x) \right), \\ \theta_{2,k}(x, t) = e^{-k^2(T-t)}\Phi_{2,k}^*(x). \end{cases}$$

*Proof of Theorem 1.4.* Assume that  $T > T_1$  and  $I_k(p, q) \neq 0$  for all  $k \in \mathbb{N}^*$ . We will look for the control  $u$  under the form

$$u(t) := \sum_{k \in \mathbb{N}^*} \{u_{1,k}q_{1,k}(T-t) + u_{2,k}q_{2,k}(T-t)\}, \quad (6.2)$$

for all  $t \in (0, T)$ , where  $q_{1,k}$  and  $q_{2,k}$  are defined in Section 3.1. Plugging the expressions of  $u$ ,  $\theta_{1,k}$  and  $\theta_{2,k}$  in Equality (6.1), we obtain the *moment problem*

$$\begin{cases} u_{1,k} = -e^{-k^2T} \frac{\langle y_1^0, \varphi_k \rangle_{H^{-1}, H_0^1}}{\partial_x \varphi_k(0)}, \\ u_{2,k} = \frac{e^{-k^2T}}{I_k \partial_x \varphi_k(0)} \{ \langle y_1^0, \psi_k \rangle_{H^{-1}, H_0^1} + \langle y_2^0, \varphi_k \rangle_{H^{-1}, H_0^1} - \left( I_k T + \frac{\partial_x \psi_k(0)}{\partial_x \varphi_k(0)} \right) \langle y_1^0, \varphi_k \rangle_{H^{-1}, H_0^1} \}. \end{cases}$$

Let  $\varepsilon > 0$ . Using the definition of  $T_1$  (see (1.14)), we have  $I_k(p, q) > C_\varepsilon e^{-k^2(T_1+\varepsilon)}$  for all  $k \in \mathbb{N}^*$ . Then, using the estimates (2.6) and (3.35), we get

$$|u_{1,k}| + |u_{2,k}| \leq C e^{-k^2(T-T_1-2\varepsilon)} \|y^0\|_{H^{-1}(0, \pi)^2}.$$

Thus for  $\varepsilon \in (T - T_1)/2$ , the control  $u$  defined in (6.2) is an element of  $L^2(0, T)$ .

Assume now that  $T < T_1$  and  $I_k(p, q) \neq 0$  for all  $k \in \mathbb{N}^*$ . By contradiction let us suppose that there exists a constant  $C_{obs}$  such that for all  $\theta^0 \in H_0^1(0, \pi)^2$  the solution to the dual System (2.8) satisfies

$$\|\theta(0)\|_{H_0^1(0, \pi)^2}^2 \leq C_{obs} \int_0^T |B^* \partial_x \theta(0, t)|^2 dt. \quad (6.3)$$

Let  $\varepsilon = (T_1 - T)/2$ . Using the definition of  $T_1$ , there exists a sequence  $(k_n)_{n \in \mathbb{N}^*}$  such that

$$I_{k_n}(p, q) < e^{-k_n^2(T+\varepsilon)}. \quad (6.4)$$

Let  $\theta_n^0 := a_n \Phi_{1, k_n}^* + b_n \Phi_{2, k_n}^*$  with  $(a_n, b_n) \in \mathbb{R}^2$ . We recall that

$$\theta^n(x, t) = e^{-k_n^2(T-t)} \{ a_n \Phi_{1, k_n}^* + (b_n - (T-t)I_k(p, q)a_n) \Phi_{2, k_n}^* \}.$$

Then, after calculation, we get

$$\|\theta(0)\|_{H_0^1(0, \pi)^2}^2 = e^{-2k_n^2T} (a_n^2 \|\psi_{k_n}\|_{H_0^1}^2 + a_n^2 k_n^2 + (b_n - T I_k(p, q)a_n)^2 k_n^2)$$

and

$$\int_0^T |B^* \partial_x \theta(0, t)|^2 dt = \int_0^T e^{-2k_n^2(T-t)} |a_n \partial_x \psi_{k_n}(0) + (b_n - (T-t)I_{k_n}(p, q)a_n)k_n|^2 dt.$$

For  $a_n := 1$  and  $b_n := -\partial_x \psi_{k_n}(0)/k_n$ , taking in account inequality (6.4) and using the estimate (2.6), we obtain

$$\|\theta(0)\|_{H_0^1(0, \pi)^2}^2 \geq k_n^2 e^{-2k_n^2T} \quad \text{and} \quad \int_0^T |B^* \partial_x \theta(0, t)|^2 dt \leq C k_n^2 e^{-2k_n^2(T+\varepsilon)}.$$

Thus for  $n$  large enough we get a contradiction with observability inequality (6.3). □



## A Proof of Lemma 3.4

*Proof of Lemma 3.4.* Let  $k \in \mathbb{N}^*$ . Consider  $a_1, b_1, a_2, b_2 \in \omega$  to be determined later with  $a_i < b_i$  for  $i \in \{1, 2\}$  and define the functions

$$\begin{cases} f^{(1)} := \mathbb{1}_{(a_1, b_1)}, \\ f^{(2)} := \mathbb{1}_{(a_2, b_2)}. \end{cases}$$

Thus, for  $i = 1, 2$ , we obtain

$$f_k^{(i)} = \int_0^\pi f^{(i)}(x) \varphi_k(x) dx = \frac{2}{k} \sqrt{\frac{2}{\pi}} \sin\left(k \frac{a_i + b_i}{2}\right) \sin\left(k \frac{b_i - a_i}{2}\right)$$

and

$$\widehat{f}_k^{(i)} = \int_0^\pi f^{(i)}(x) \cos(kx) dx = \frac{2}{k} \cos\left(k \frac{a_i + b_i}{2}\right) \sin\left(k \frac{b_i - a_i}{2}\right).$$

A simple computation leads to

$$|B_k| = \frac{4}{k^2} \sqrt{\frac{2}{\pi}} \left| \sin\left(k \frac{b_1 - a_1}{2}\right) \sin\left(k \frac{b_2 - a_2}{2}\right) \sin\left(k \frac{a_2 + b_2 - a_1 - b_1}{2}\right) \right|. \quad (\text{A.1})$$

Let  $n \in \mathbb{N}^*$  and  $\ell$  an algebraic number of order two satisfying

$$a/2n < \ell < b/(2n + 3).$$

Let us take  $a_1 := 2n\ell$ ,  $b_1 := a_1 + 2\ell$ ,  $a_2 := a_1 + \ell$  and  $b_2 := a_2 + 2\ell$ . Thus  $a_1, b_1, a_2, b_2 \in \omega$  and  $a_i < b_i$  for  $i = 1, 2$ . Furthermore

$$\begin{cases} b_1 - a_1 = b_2 - a_2 = a_2 + b_2 - a_1 - b_1 = 2\ell, \\ a_1 + b_1 = (4n + 2)\ell, \\ a_2 + b_2 = (4n + 4)\ell. \end{cases} \quad (\text{A.2})$$

Since  $\ell$  is an algebraic number of order 2, as said in the proof of Proposition 3.2, using diophantine approximations, there exists a constant  $\gamma > 0$  such that

$$\inf_{j \geq 1} (j |\sin(j\ell)|) \geq \gamma. \quad (\text{A.3})$$

Combining (A.1)-(A.3), we deduce that

$$|B_k| \geq \sqrt{\frac{2}{\pi}} \frac{4\gamma^3}{k^5}, \quad |f_k^{(1)}| \geq \sqrt{\frac{2}{\pi}} \frac{2\gamma^2}{(2n+1)k^3} \quad \text{and} \quad |f_k^{(2)}| \geq \sqrt{\frac{2}{\pi}} \frac{\gamma^2}{(n+1)k^3},$$

for all  $k \in \mathbb{N}^*$ . □

*Acknowledgements.* The author thanks professors Assia Benabdallah and Manuel González-Burgos for their interesting comments and suggestions.

## References

- [1] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled systems under geometric conditions. *Comptes Rendus Mathématique*, 349(7):395–400, 2011.
- [2] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled wave-type systems and applications. *J. Math. Pures Appl.*, 99:544–576, 2013.

- [3] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems. *Differ. Equ. Appl.*, 1(3):427–457, 2009.
- [4] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems. *J. Evol. Equ.*, 9(2):267–291, 2009.
- [5] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and I. Kostin. Null-controllability of some systems of parabolic type by one control force. *ESAIM Control Optim. Calc. Var.*, 11(3):426–448 (electronic), 2005.
- [6] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. Recent results on the controllability of linear coupled parabolic problems: a survey. *Math. Control Relat. Fields*, 1(3):267–306, 2011.
- [7] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. Controllability of some system of parabolic equations. In *Proceedings of the II Encuentro RSME-SMM*, 2012.
- [8] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. Minimal time of controllability of two parabolic equations with disjoint control and coupling domains. *C. R. Math. Acad. Sci. Paris*, 352(5):391–396, 2014.
- [9] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence. *Submitted*, 2015.
- [10] A. Benabdallah, M. Cristofol, P. Gaitan, and L. De Teresa. Controllability to trajectories for some parabolic systems of three and two equations by one control force. *Math. Control Relat. Fields*, 4(1):17–44, 2014.
- [11] F. Boyer and G. Olive. Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients. *Math. Control Relat. Fields*, 4(3):263–287, 2014.
- [12] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
- [13] B. Dehman and M. Léautaud. Controllability of two coupled wave equations on a compact manifold. *Arch. Rational Mech. Anal.*, 211:113–187, 2014.
- [14] M. Duprez and P. Lissy. Indirect controllability of some linear parabolic systems of  $m$  equations with  $m - 1$  controls involving coupling terms of zero or first order. *Submitted*, 2015.
- [15] H. O. Fattorini. Some remarks on complete controllability. *SIAM J. Control*, 4:686–694, 1966.
- [16] E. Fernández-Cara, M. González-Burgos, and L. de Teresa. Boundary controllability of parabolic coupled equations. *J. Funct. Anal.*, 259(7):1720–1758, 2010.
- [17] M. González-Burgos and L. de Teresa. Controllability results for cascade systems of  $m$  coupled parabolic PDEs by one control force. *Port. Math.*, 67(1):91–113, 2010.
- [18] S. Guerrero. Null controllability of some systems of two parabolic equations with one control force. *SIAM J. Control Optim.*, 46(2):379–394, 2007.
- [19] O. Kavian and L. de Teresa. Unique continuation principle for systems of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 16(2):247–274, 2010.
- [20] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Uralceva. Linear and quasilinear equations of parabolic type. pages xi+648, 1968.
- [21] J.-L. Lions. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Avant propos de P. Lelong. Dunod, Paris, 1968.
- [22] L. Miller. The control transmutation method and the cost of fast controls. *SIAM J. Control Optim.*, 45(2):762–772 (electronic), 2006.
- [23] G. Olive. Boundary approximate controllability of some linear parabolic systems. *Evol. Equ. Control Theory*, 3(1):167–189, 2014.
- [24] L. Rosier and L. de Teresa. Exact controllability of a cascade system of conservative equations. *C. R. Math. Acad. Sci. Paris*, 349(5-6):291–296, 2011.
- [25] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.