



**HAL**  
open science

## The magnetic Laplacian acting on discrete cusps

Sylvain Golénia, Françoise Truc

► **To cite this version:**

Sylvain Golénia, Françoise Truc. The magnetic Laplacian acting on discrete cusps. 2015. hal-01174771v1

**HAL Id: hal-01174771**

**<https://hal.science/hal-01174771v1>**

Preprint submitted on 9 Jul 2015 (v1), last revised 8 Jun 2017 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THE MAGNETIC LAPLACIAN ACTING ON DISCRETE CUSPS

SYLVAIN GOLÉNIA AND FRANÇOISE TRUC

ABSTRACT. We study several toy-models of cups-like weighted graphs. We prove that the form-domain of the magnetic Laplacian and that of the non-magnetic Laplacian can be different. We establish the emptiness of the essential spectrum and compute the asymptotic of eigenvalues for the magnetic Laplacian.

## 1. INTRODUCTION

The spectral theory of discrete Laplacians on graphs has drawn a lot of attention for decades. The spectral analysis of the Laplacian associated to a graph is strongly related to the geometry of the graph. Moreover, graphs are discretized versions of manifolds. In [AbTr, GoMo], it is shown that for a manifold with cusps, adding a magnetic field can drastically destroy the essential spectrum of the Laplacian. The aim of this article is to go along this line in a discrete setting.

We recall some standard definitions of graph theory. A *graph* is a triple  $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ , where  $\mathcal{V}$  is countable set (the *vertices*),  $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+$  (the *edges*) is symmetric, and  $m : \mathcal{V} \rightarrow (0, \infty)$  is a weight. We say that  $\mathcal{G}$  is *simple* if  $m = 1$  and  $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow \{0, 1\}$ .

Given  $x, y \in \mathcal{V}$ , we say that  $x$  and  $y$  are *neighbors* if  $\mathcal{E}(x, y) > 0$ . We denote this relationship by  $x \sim y$  and the set of neighbors of  $x$  by  $N_{\mathcal{G}}(x)$ . We say there is a *loop* at  $x \in \mathcal{V}$  if  $\mathcal{E}(x, x) > 0$ . A graph is *connected* if for all  $x, y \in \mathcal{V}$ , there exists a path  $\gamma$  joining  $x$  and  $y$ . Here,  $\gamma$  is a sequence  $x_0, x_1, \dots, x_n \in \mathcal{V}$  such that  $x = x_0$ ,  $y = x_n$ , and  $x_j \sim x_{j+1}$  for all  $0 \leq j \leq n-1$ . In this case, we set  $|\gamma| := n$ . A graph  $\mathcal{G}$  is *locally finite* if  $|N_{\mathcal{G}}(x)|$  is finite for all  $x \in \mathcal{V}$ . In the sequel, we assume that:

**All graphs are locally finite, connected with no loops.**

We endow a graph  $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$  with the metric  $\rho_{\mathcal{G}}$  defined by

$$\rho_{\mathcal{G}}(x, y) := \inf\{|\gamma|, \gamma \text{ is a path joining } x \text{ and } y\}.$$

The space of complex-valued functions acting on the set of vertices  $\mathcal{V}$  is denoted by  $C(\mathcal{V}) := \{f : \mathcal{V} \rightarrow \mathbb{C}\}$ . Moreover,  $C_c(\mathcal{V})$  is the subspace of  $C(\mathcal{V})$  of functions with finite support. We consider the Hilbert space

$$\ell^2(\mathcal{V}, m) := \left\{ f \in C(\mathcal{V}), \sum_{x \in \mathcal{V}} m(x) |f(x)|^2 < \infty \right\}$$

with the scalar product  $\langle f, g \rangle := \sum_{x \in \mathcal{V}} m(x) \overline{f(x)} g(x)$ .

We equip  $\mathcal{G}$  with a *magnetic potential*  $\theta : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$  such that  $\theta_{x,y} := \theta(x, y) = -\theta_{y,x}$  and  $\theta(x, y) := 0$  if  $\mathcal{E}(x, y) = 0$ . We define the Hermitian form

$$Q_{\mathcal{G}, \theta}(f) := \frac{1}{2} \sum_{x, y \in \mathcal{V}} \mathcal{E}(x, y) |f(x) - e^{i\theta_{x,y}} f(y)|^2,$$

---

2010 *Mathematics Subject Classification.* 34L20, 47A10, 05C63, 47B25, 47A63, 81Q10.

*Key words and phrases.* discrete magnetic Laplacian, locally finite graphs, eigenvalues, asymptotic, form-domain.

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . There is a unique non-negative self-adjoint operator  $\Delta_{\mathcal{G},\theta}$  such that  $\langle f, \Delta_{\mathcal{G},\theta} f \rangle_{\ell^2(\mathcal{V},m)} = Q_{\mathcal{G},\theta}(f)$ , for all  $f \in \mathcal{C}_c(\mathcal{V})$ . Moreover, the *magnetic Laplacian* acts as follows:

$$(\Delta_{\mathcal{G},\theta} f)(x) = \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x,y)(f(x) - e^{i\theta_{x,y}} f(y)),$$

for all  $f \in \mathcal{C}_c(\mathcal{V})$ . Notice that  $\Delta_{\mathcal{G},\theta}$  is the Friedrichs extension of  $\Delta_{\mathcal{G},\theta}|_{\mathcal{C}_c(\mathcal{V})}$ , e.g., [RS]. We set

$$\deg_{\mathcal{G}}(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x,y),$$

the *degree* of  $x \in \mathcal{V}$ . We see easily that  $\Delta_{\mathcal{G},\theta} \leq 2 \deg_{\mathcal{G}}(\cdot)$  in the *form sense*, i.e.,

$$(1) \quad 0 \leq \langle f, \Delta_{\mathcal{G},\theta} f \rangle \leq \langle f, 2 \deg_{\mathcal{G}}(\cdot) f \rangle, \text{ for all } f \in \mathcal{C}_c(\mathcal{V}).$$

Moreover, since  $\langle \delta_x, \Delta_{\mathcal{G},\theta} \delta_x \rangle = \deg_{\mathcal{G}}(x)$ ,  $\Delta_{\mathcal{G},\theta}$  is bounded if and only if  $\sup_{x \in \mathcal{V}} \deg_{\mathcal{G}}(x)$  is finite, e.g. [KL, Go].

Another consequence of (1) is

$$(2) \quad \mathcal{D}(\deg_{\mathcal{G}}^{1/2}(\cdot)) \subset \mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}).$$

However, the equality of the form-domains

$$(3) \quad \mathcal{D}(\deg_{\mathcal{G}}^{1/2}(\cdot)) = \mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2})$$

is wrong in general for a simple graph, see [Go, BGK]. In fact if  $\theta = 0$ , (2) is equivalent to a sparseness condition and holds true for planar simple graphs, see [BGK]. On a general weighted graph, if (3) holds true,

$$\sigma_{\text{ess}}(\Delta_{\mathcal{G},\theta}) = \emptyset \Leftrightarrow (\Delta_{\mathcal{G},\theta} + 1)^{-1} \text{ is compact} \Leftrightarrow \lim_{|x| \rightarrow \infty} \deg_{\mathcal{G}}(x) = \infty,$$

where  $|x| := \rho_{\mathcal{G}}(x_0, x)$  for a given  $x_0 \in \mathcal{V}$ . Note that the limit is independent of the choice of  $x_0$ . Besides if the latter is true and if the graph is sparse (simple and planar for instance), [BGK] ensures the following asymptotic of eigenvalues,

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n(\Delta_{\mathcal{G},\theta})}{\lambda_n(\deg_{\mathcal{G}}(\cdot))} = 1,$$

where  $\lambda_n(H)$  denotes the  $n$ -th eigenvalue, counted with multiplicity, of a self-adjoint operator  $H$ , which is bounded from below.

The technique used in [BGK] does not apply when the graph is cusp-like (thin at infinity). The aim of this article is to establish new behaviors for the asymptotic of eigenvalues, in the magnetic case and also to prove that the form of domains of the non-magnetic Laplacian is different from that of the magnetic Laplacian. We find the inspiration by mimicking the continuous case, which was studied in [AbTr, GoMo].

We present part of our results through the following theorem.

**Theorem 1.1.** *Let  $n \geq 3$  be an integer. There exist a graph  $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ , a magnetic potential  $\theta$ , a constant  $\nu > 0$  such that for all  $\kappa \in \mathbb{R}$*

$$\sigma_{\text{ess}}(\Delta_{\mathcal{G},\kappa\theta}) = \emptyset \Leftrightarrow \mathcal{D}(\Delta_{\mathcal{G},\kappa\theta}^{1/2}) = \mathcal{D}(\deg_{\mathcal{G}}^{1/2}(\cdot)) \Leftrightarrow \kappa \notin \mathbb{R}/\nu\mathbb{Z}.$$

Moreover:

1) *When  $\kappa \notin \mathbb{R}/\nu\mathbb{Z}$ , we have:*

$$\lim_{\lambda \rightarrow \infty} \frac{\mathcal{N}_{\lambda}(\Delta_{\mathcal{G},\kappa\theta})}{\mathcal{N}_{\lambda}(\deg_{\mathcal{G}}(\cdot))} = 1,$$

where  $\mathcal{N}_{\lambda}(H) := \dim \text{ran} 1_{]-\infty, \lambda]}(H)$  for a self-adjoint operator  $H$ .

2) When  $\kappa \in \mathbb{R}/\nu\mathbb{Z}$ , the absolutely continuous part of the  $\Delta_{\mathcal{G},\kappa\theta}$  is

$$\sigma_{\text{ac}}(\Delta_{\mathcal{G},\kappa\theta}) = \left[ e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2 \right],$$

with multiplicity 1 and

$$\lim_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(\Delta_{\mathcal{G},\kappa\theta} P_{\text{ac},\kappa}^\perp)}{\mathcal{N}_\lambda(\text{deg}_{\mathcal{G}}(\cdot))} = \frac{n-1}{n},$$

where  $P_{\text{ac},\kappa}$  denotes the projection onto the a.c. part of  $\Delta_{\mathcal{G},\kappa\theta}$ .

We now describe heuristically the phenomenon. Compared with the first point, the constant  $(n-1)/n$  that appears in the second point encodes the fact that a part of the wave packet diffuses. Moreover, switching on the magnetic field is not a gentle perturbation because the form domain of the operator is changed.

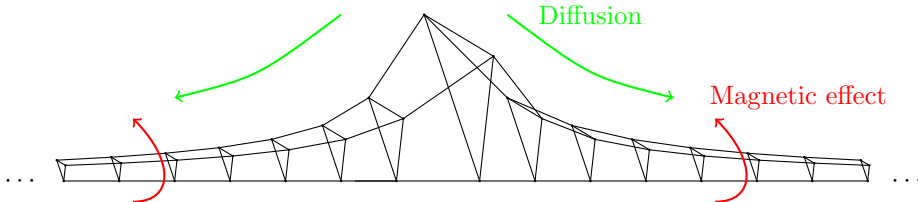
By Riemann-Lebesgue Theorem, the particle, which is localized in the a.c. part of the operator, escapes to every compact set. More precisely, for a finite subset  $X \subset \mathcal{V}$  and all  $f \in \mathcal{D}(\Delta_{\mathcal{G},0})$

$$\|1_X(\cdot) e^{it\Delta_{\mathcal{G},0}} P_{\text{ac},0} f\| \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Whereas, when the magnetic potential is active, the spectrum of  $\Delta_{\mathcal{G},\kappa\theta}$  is purely discrete. The particle cannot diffuse anymore. More precisely, for a finite subset  $X \subset \mathcal{V}$  and an eigenvalue  $f$  of  $\Delta_{\mathcal{G},\kappa\theta}$  such that  $f|_X \neq 0$ , there is  $c > 0$  such that:

$$\frac{1}{T} \int_0^T \|1_X(\cdot) e^{it\Delta_{\mathcal{G},\kappa\theta}} f\|^2 dt \rightarrow c, \text{ as } T \rightarrow \infty.$$

The particle is *trapped* by the magnetic field.



A cusp-like representation of  $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ :

The magnetic field traps the particle by spinning it,  
whereas its absence let the particle diffuse.

We now describe the structure of the paper. In Section 2.1, we recall some properties of the holonomy of a magnetic potential. Besides, in Section 2.2 we present our main hypotheses and several notions of (weighted) product fo graphs. Then in Section 2.3 we give a criteria of absence of essential spectrum. Next, in Section 2.4, we refine the analysis and compute the asymptotic of eigenvalues. We present the central theorem and finish the section by proving Theorem 1.1.

**Notation:**  $\mathbb{N}$  denotes the set of non negative integers and  $\mathbb{N}^*$  that of the positive integers. We denote by  $\mathcal{D}(H)$  the domain of an operator  $H$ . Its (essential) spectrum is denoted by  $\sigma(H)$  (by  $\sigma_{\text{ess}}(H)$ ). We set  $\delta_{x,y}$  equals 1 if and only if  $x = y$  and 0 otherwise and given a set  $X$ ,  $1_X(x)$  equals 1 if  $x \in X$  and 0 otherwise.

**Acknowledgments:** We would like to thank Michel Bonnefont, Yves Colin de Verdière, and Sergiu Moroianu for useful discussions. SG and FT were partially supported by the ANR project GeRaSic (ANR-13-BS01-0007-01) and by SQFT (ANR-12-JS01-0008-01).

## 2. MAIN RESULTS

**2.1. Holonomy of a magnetic potential.** We recall some facts about the gauge theory of magnetic fields, see [CTT3, HiSh] for more details. We recall that a *gauge transform*  $U$  is the unitary map on  $\ell^2(\mathcal{V}, m)$  defined by

$$(Uf)(x) = u_x f(x),$$

where  $(u_x)_{x \in \mathcal{V}}$  is a sequence of complex numbers with  $|u_x| \equiv 1$  (we write  $u_x = e^{i\sigma_x}$ ). The map  $U$  acts on the quadratic forms  $Q_{\mathcal{G}, \theta}$  by  $U^*(Q_{\mathcal{G}, \theta})(f) = Q_{\mathcal{G}, \theta}(Uf)$ , for all  $f \in \mathcal{C}_c(\mathcal{V})$ . The magnetic potential  $U^*(\theta)$  is defined by:

$$U^*(Q_{\mathcal{G}, \theta}) = Q_{\mathcal{G}, U^*(\theta)}.$$

More explicitly, we get:

$$U^*(\theta)_{xy} = \theta_{x,y} + \sigma_y - \sigma_x.$$

We turn to the definition of the flux of a magnetic potential, the *Holonomy*.

**Proposition 2.1.** *Let us denote by  $Z_1(\mathcal{G})$  the space of cycles of  $\mathcal{G}$ . It is a free  $\mathbb{Z}$ -module with a basis of geometric cycles  $\gamma = (x_0, x_1) + (x_1, x_2) + \dots + (x_{N-1}, x_N)$  with, for  $i = 0, \dots, N-1$ ,  $\mathcal{E}(x_i, x_{i+1}) \neq 0$ , and  $x_N = x_0$ . We define the holonomy map  $\text{Hol}_\theta : Z_1(\mathcal{G}) \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ , by*

$$\text{Hol}_\theta((x_0, x_1) + (x_1, x_2) + \dots + (x_N, x_0)) := \theta_{x_0, x_1} + \dots + \theta_{x_N, x_0}.$$

Then

- 1) The map  $\theta \mapsto \text{Hol}_\theta$  is surjective onto  $\text{Hom}_{\mathbb{Z}}(Z_1(\mathcal{G}), \mathbb{R}/2\pi\mathbb{Z})$ .
- 2)  $\text{Hol}_{\theta_1} = \text{Hol}_{\theta_2}$  if and only if there exists a gauge transform  $U$  so that  $U^*(\theta_2) = \theta_1$ .

The magnetic Schrödinger operator  $\Delta_{\mathcal{G}, \theta}$  is uniquely defined up to unitary conjugation by  $\mathcal{G}$  and  $\text{Hol}_\theta$ .

**Lemma 2.2.** *Let  $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$  be a connected graph of finite volume, i.e., such that  $\sum_{x \in \mathcal{V}} m(x) < \infty$  and let  $\theta$  be magnetic potential. We have  $\ker \Delta_{\mathcal{G}, \theta} \neq \{0\}$  if and only if  $\text{Hol}_\theta = 0$ .*

*Proof.* If  $\text{Hol}_\theta = 0$  then  $\Delta_{\mathcal{G}, \theta}$  is unitarily equivalent to  $\Delta_{\mathcal{G}, 0}$  by Proposition 2.1. Since the volume is finite, the constant function is in the kernel of the latter.

Conversely, let  $f \neq 0$  with  $\Delta_{\mathcal{G}, \theta} f = 0$  and hence  $Q_{\mathcal{G}, \theta}(f) = 0$ . This implies that all terms in the expression of  $Q_{\mathcal{G}, \theta}(f)$  vanish. In particular, if  $\mathcal{E}(x, y) \neq 0$  we have

$$(5) \quad f(x) = e^{i\theta_{x,y}} f(y).$$

Assume that there is a cycle  $\gamma = (x_0, x_1, \dots, x_N = x_0)$ , such that  $\text{Hol}_\theta(\gamma) \neq 0$ . Using (5), we obtain that

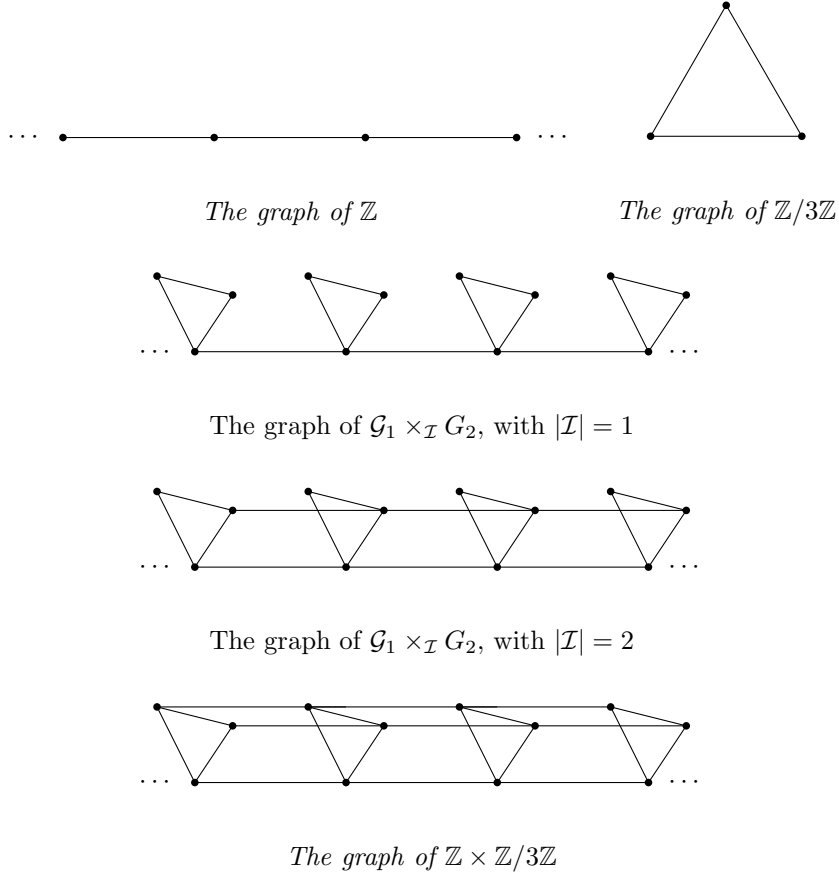
$$f(x_i) = e^{-i\text{Hol}_\theta(\gamma)} f(x_i) .$$

for all  $i = 0, \dots, N-1$ . Therefore  $f|_\gamma = 0$ . Then, since  $f \neq 0$ , there is  $x \in \mathcal{V}$  such that  $f(x) \neq 0$ . Using again (5) and by connectedness between  $x$  and  $\gamma$ , it yields that  $f(x) = 0$ . Contradiction. Therefore if there exists  $f \in \ker(\Delta_{\mathcal{G}, \theta}) \setminus \{0\}$  then  $\text{Hol}_\theta = 0$ .  $\square$

**2.2. The setting.** Given  $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$  and  $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$  and  $\mathcal{I} \subset \mathcal{V}_2$ , we define the *product of  $\mathcal{G}_1$  by  $\mathcal{G}_2$  through  $\mathcal{I}$*  by  $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ , where  $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$  and

$$\begin{cases} m(x, y) := m_1(x) \times m_2(y), \\ \mathcal{E}((x, y), (x', y')) := \mathcal{E}_1(x, x') \times \delta_{y, y'} (\sum_{z \in \mathcal{I}} \delta_{y, z}) + \delta_{x, x'} \times \mathcal{E}_2(y, y'), \\ \theta((x, y), (x', y')) := \theta_1(x, x') \times \delta_{y, y'} + \delta_{x, x'} \times \theta_2(y, y'), \end{cases}$$

for all  $x, x' \in \mathcal{V}_1$  and  $y, y' \in \mathcal{V}_2$ . We denote  $\mathcal{G}$  by  $\mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ . If  $\mathcal{I}$  is empty, the graph is disconnected and of no interest for our purpose. If  $|\mathcal{I}| = 1$ ,  $\mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$  is the graph  $\mathcal{G}_1$  decorated by  $\mathcal{G}_2$ , see [ScAi] for its spectral analysis in the unweighted case. If  $\mathcal{I} = \mathcal{V}_2$ ,  $\mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2 = \mathcal{G}_1 \times \mathcal{G}_2$ , the *Cartesian product* of  $\mathcal{G}_1$  by  $\mathcal{G}_2$ .



Under the representation  $\ell^2(\mathcal{V}) \simeq \ell^2(\mathcal{V}_1) \otimes \ell^2(\mathcal{V}_2)$ ,

$$(6) \quad \deg_{\mathcal{G}}(\cdot) = \deg_{\mathcal{G}_1}(\cdot) \otimes \frac{1_{\mathcal{I}}(\cdot)}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot)$$

and

$$(7) \quad \Delta_{\mathcal{G},\theta} = \Delta_{\mathcal{G}_1,\theta_1} \otimes \frac{1_{\mathcal{I}}(\cdot)}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2}.$$

We refer to [RS][Section VIII.10] for an introduction to tensor product of self-adjoint operators.

In the sequel, we will deal with three types of hypotheses.

- (H1)  $m_1(x)$  tend to 0 as  $|x| \rightarrow \infty$ ,
- (H2)  $\mathcal{G}_2$  is finite,
- (H3)  $\Delta_{\mathcal{G}_1,\theta_1}$  is bounded (or equivalently  $\sup_{x \in \mathcal{V}_1} \deg_{\mathcal{G}_1}(x) < \infty$ ).

**2.3. Destroying the essential spectrum.** We have a first result of absence of essential spectrum. We refer to [CTT3] for related results based on the non-triviality of  $\text{Hol}_{\theta}$  in the context of non-complete graphs.

**Theorem 2.3.** *Let  $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$ , with  $|\mathcal{I}| > 0$ , so that  $\mathcal{G}_2$  is of finite volume,  $\inf \sigma_{\text{ess}}(\Delta_{\mathcal{G}_2,0}) > 0$ , and  $\text{Hol}_{\theta_2} \neq 0$ . If moreover (H1) holds, then  $\Delta_{\mathcal{G},\theta}$  has a compact resolvent, and*

$$\mathcal{N}_{\lambda}(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}) \geq \mathcal{N}_{\lambda}(\Delta_{\mathcal{G},\theta}), \text{ for all } \lambda \geq 0.$$

**Remark 2.4.** *With the convention that  $\inf \emptyset = \infty$ , we see that the hypothesis (H2) ensures that  $\inf \sigma_{\text{ess}}(\Delta_{\mathcal{G}_2,0}) > 0$ .*

*Proof.* Kato's inequality, e.g., [DM][Lemma 2.1] gives:

$$\langle |f|, \Delta_{\mathcal{G},0}(|f|) \rangle \leq \langle f, \Delta_{\mathcal{G},\theta} f \rangle, \text{ for all } f \in \mathcal{C}_c(\mathcal{V}).$$

Combined with Persson's Lemma, e.g., [KL][Proposition 18], we deduce:

$$\inf \sigma_{\text{ess}}(\Delta_{\mathcal{G}_2,0}) \leq \inf \sigma_{\text{ess}}(\Delta_{\mathcal{G}_2,\theta}), \text{ for all } \theta.$$

Therefore, recalling Lemma 2.2, we infer that  $\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2}) > 0$ . Then, note that

$$\Delta_{\mathcal{G},\theta} \geq \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2} \geq (\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2})) \frac{1}{m_1(\cdot)} \otimes \text{id}_{\mathcal{V}_2},$$

in the form sense on  $\mathcal{C}_c(\mathcal{V})$ . Then by (H1), the spectrum of the r.h.s. is purely discrete. Hence, by min-max theory, e.g., [Go], the two other operators have a compact resolvent. The inequality follows also by min-max theory.  $\square$

**2.4. The asymptotic of the eigenvalues.** From now on, we suppose (H1), (H2), and (H3). We aim at a more precise result. We set  $M := \sup_{x \in \mathcal{V}_1} \deg_{\mathcal{G}_1}(x) \times \max_{y \in \mathcal{V}_2} (1/m_2(y)) < \infty$ . Notice that:

$$(8) \quad \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) \leq \deg_{\mathcal{G}}(\cdot) \leq \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) + M$$

and

$$(9) \quad \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2} \leq \Delta_{\mathcal{G},\theta} \leq 2M + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2,\theta_2},$$

in the form sense on  $\mathcal{C}_c(\mathcal{V})$ , where we used (1), (6), and (7). Using again (1),

$$\frac{\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2})}{\max_{y \in \mathcal{V}_2} \deg_{\mathcal{G}_2}(y)} \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot) \leq \Delta_{\mathcal{G},\theta} \leq 2M + \frac{2}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot),$$

and therefore, combining this result with (8):

$$(10) \quad \frac{\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2})}{\max_{y \in \mathcal{V}_2} \deg_{\mathcal{G}_2}(y)} (\deg_{\mathcal{G}}(\cdot) - M) \leq \Delta_{\mathcal{G},\theta} \leq 2M + 2 \deg_{\mathcal{G}}(\cdot),$$

Hence, if  $\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2}) > 0$ , we get:  $\mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}(\deg_{\mathcal{G}}^{1/2}(\cdot))$ . Furthermore, since  $\lim_{|x| \rightarrow \infty} \deg_{\mathcal{G}}(x) = \infty$ , the min-max theory, e.g., [Go], yields that  $\Delta_{\mathcal{G},\theta}$  has a compact resolvent and

$$0 < \frac{\inf \sigma(\Delta_{\mathcal{G}_2,\theta_2})}{\max_{y \in \mathcal{V}_2} \deg_{\mathcal{G}_2}(y)} \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n(\Delta_{\mathcal{G},\theta})}{\lambda_n(\deg_{\mathcal{G}}(\cdot))} \leq \limsup_{n \rightarrow \infty} \frac{\lambda_n(\Delta_{\mathcal{G},\theta})}{\lambda_n(\deg_{\mathcal{G}}(\cdot))} \leq 2.$$

We now refine the asymptotic for  $\Delta_{\mathcal{G},\theta}$ .

**Theorem 2.5.** *Setting  $\mathcal{G} := \mathcal{G}_1 \times_{\mathcal{I}} \mathcal{G}_2$  with  $|\mathcal{I}| > 0$  and assuming (H1), (H2), and (H3), we obtain*

$$(11) \quad \mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}\left(m_1^{-1/2}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}^{1/2}\right).$$

Moreover, we have:

- 1)  $\Delta_{\mathcal{G},\theta}$  has a compact resolvent if and only if  $\text{Hol}_{\theta_2} \neq 0$ .
- 2) Assuming that  $\text{Hol}_{\theta_2} \neq 0$ ,

$$\mathcal{D}(\Delta_{\mathcal{G},\theta}^{1/2}) = \mathcal{D}\left(\deg_{\mathcal{G}}^{1/2}(\cdot)\right)$$

and

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(\Delta_{\mathcal{G},\theta})}{\lambda_n(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2})} = 1.$$

Furthermore,

$$(12) \quad \mathcal{N}_{\lambda-2M}(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}) \leq \mathcal{N}_{\lambda}(\Delta_{\mathcal{G},\theta}) \leq \mathcal{N}_{\lambda}(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2,\theta_2}),$$

for all  $\lambda \geq 0$ .

3) Suppose that  $\deg_{\mathcal{G}_2}$  is constant on  $\mathcal{V}_2$ . Take  $\theta_2$  such that  $\text{Hol}_{\theta_2} \neq 0$ . Then, for all  $a \in (0, 1]$ , one can choose  $m_1$  such that

$$(13) \quad \lim_{\lambda \rightarrow \infty} \frac{\mathcal{N}_\lambda(\Delta_{\mathcal{G}, \theta})}{\mathcal{N}_\lambda(\deg_{\mathcal{G}}(\cdot))} = a.$$

*Proof.* First note that (11) follows directly from (9). Denoting by  $\{g_i\}_{i=1, \dots, |\mathcal{V}_2|}$  the eigenfunctions associated to the eigenvalues  $\{\lambda_i\}_{i=1, \dots, |\mathcal{V}_2|}$  of  $\Delta_{\mathcal{G}_2, \theta_2}$ , where  $\lambda_j \leq \lambda_{j+1}$ , we see that the eigenfunctions of  $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$  are given by  $\{\delta_x \otimes g_i\}$ , where  $x \in \mathcal{V}_1$  and  $i = 1, \dots, |\mathcal{V}_2|$ . Then, using (H1), we observe that

$$\sigma(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}) = \overline{m_1^{-1}(\mathcal{V}_1) \times \{\lambda_1, \dots, \lambda_{|\mathcal{V}_2|}\}} = m_1^{-1}(\mathcal{V}_1) \times \{\lambda_1, \dots, \lambda_{|\mathcal{V}_2|}\}.$$

Besides,  $0 \in \sigma(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2})$  if and only if 0 is an eigenvalue of  $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$  of infinite multiplicity if and only if  $\lambda_1 = 0$  if and only if  $\text{Hol}_{\theta_2} = 0$ , by Lemma 2.2. Moreover, recalling (H1), we see that all the eigenvalues of  $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$  which are not 0 are of finite multiplicity. Therefore,  $m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2}$  has a compact resolvent if and only if  $\text{Hol}_{\theta_2} \neq 0$ . Combining the latter and (9), the min-max theory yields the first point.

We turn to the second point and assume that  $\text{Hol}_{\theta_2} \neq 0$ . The equality of the form-domains is given by (10). Taking in account (9), the min-max theory ensures the asymptotic in  $\lambda_n$  and the inequalities (12).

We finish with the third point. Note that (8) yields

$$\mathcal{N}_{\lambda-M}(m_1^{-1}(\cdot) \otimes \deg_{\mathcal{G}_2}) \leq \mathcal{N}_\lambda(\deg_{\mathcal{G}}(\cdot)) \leq \mathcal{N}_\lambda(m_1^{-1}(\cdot) \otimes \deg_{\mathcal{G}_2}), \text{ for all } \lambda \geq 0.$$

Recalling (12), we compare the l.h.s. with  $\mathcal{N}_\lambda(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2, \theta_2})$ . Since  $\text{Hol}_{\theta_2} \neq 0$ ,  $\lambda_i \neq 0$  for all  $i = 1, \dots, |\mathcal{V}_2|$ . We have:

$$\mathcal{N}_\lambda\left(\frac{1}{m_1}(\cdot) \otimes \Delta_{\mathcal{G}_2, \theta_2}\right) = \left| \left\{ (x, i), \frac{\lambda_i}{m_1(x)} \leq \lambda \right\} \right| = \sum_{i=1}^{|\mathcal{V}_2|} \left| \left( \frac{1}{m_1} \right)^{-1} \left( \left[ 0, \frac{\lambda}{\lambda_i} \right] \right) \right|.$$

On the other hand,

$$\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right) = |\mathcal{V}_2| \times \left| \left( \frac{1}{m_1} \right)^{-1} \left( \left[ 0, \frac{\lambda}{\deg_{\mathcal{G}_2}} \right] \right) \right|.$$

Take  $(\mathcal{E}_1, \mathcal{V}_1)$  to be the simple graph of  $\mathbb{N}^*$  and choose  $m_1(x) := \exp(-x)$ . We get:

$$\frac{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2, \theta_2}\right)}{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right)} \sim \frac{\sum_{i=1}^{|\mathcal{V}_2|} \ln(\lambda) - \ln(\lambda_i)}{|\mathcal{V}_2|(\ln(\lambda) - \ln(\deg_{\mathcal{G}_2}))} \rightarrow 1, \text{ as } \lambda \rightarrow \infty$$

and for all  $c \in \mathbb{R}$ ,

$$(14) \quad \lim_{\lambda \rightarrow \infty} \frac{\mathcal{N}_{\lambda-c}\left(\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right)}{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right)} = 1.$$

This gives the third point with  $a = 1$ .

Setting  $\alpha > 0$  and  $m_1(x) := x^{-1/\alpha}$ , we obtain:

$$\frac{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2, \theta_2}\right)}{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}\right)} \underset{\lambda \rightarrow \infty}{\sim} \frac{1}{|\mathcal{V}_2|} \sum_{i=1}^{|\mathcal{V}_2|} \frac{1}{(\lambda_i \deg_{\mathcal{G}_2})^\alpha} \rightarrow \begin{cases} 1, & \text{if } \alpha \rightarrow 0, \\ 0, & \text{if } \alpha \rightarrow \infty. \end{cases}$$

Moreover, (14) still holds true. By continuity with respect to  $\alpha$ , we obtain the third point for all  $a \in (0, 1)$ .  $\square$



**Remark 2.6.** *In the case when  $\text{Hol}_{\theta_2} = 0$ , for instance when  $\theta_2 = 0$ , we see that the form-domain is  $m_1^{-1/2} \otimes P_{\ker(\Delta_{\mathcal{G}_2, \theta_2})}^\perp$ . In particular, the form-domain is **not** that of  $\text{deg}_{\mathcal{G}}(\cdot)$ . Indeed if the two form-domains are the same, the closed graph theorem yields the existence of  $c_1 > 0$  and  $c_2 > 0$  so that*

$$c_1 \text{deg}_{\mathcal{G}}(\cdot) - c_2 \leq m_1^{-1/2} \otimes P_{\ker(\Delta_{\mathcal{G}_2, \theta_2})}^\perp,$$

*in the form sense on  $\mathcal{C}_c(\mathcal{V})$ . However, note that  $0 \in \sigma_{\text{ess}}\left(m_1^{-1/2} \otimes P_{\ker(\Delta_{\mathcal{G}_2, \theta_2})}^\perp\right)$ , whereas  $\text{deg}(\cdot)$  has a compact resolvent. This is a contradiction with the min-max theory. We obtain:*

$$\begin{aligned} \mathcal{D}\left(\Delta_{\mathcal{G}, \theta}^{1/2}\right) &= \mathcal{D}\left(\text{deg}^{1/2}(\cdot)\right) \Leftrightarrow \text{Hol}_{\theta_2} \neq 0 \\ &\Leftrightarrow \Delta_{\mathcal{G}, \theta} \text{ has a compact resolvent.} \end{aligned}$$

**Remark 2.7.** *In [Go, BGK], the asymptotic in  $\mathcal{N}_\lambda$  was not discussed since the estimates that they obtain seem too weak to conclude. Being able to compute  $\mathcal{N}_\lambda$  in an explicit way, as in (13), is a new phenomenon.*

We end this section by proving the results stated in the introduction.

*Proof of Theorem 1.1.* Take  $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ , where

$$\mathcal{V}_1 := \mathbb{N}, \quad m_1(n) := \exp(-n), \quad \text{and } \mathcal{E}_1(n, n+1) := \exp(-(2n+1)/2),$$

for all  $n \in \mathbb{N}$  and  $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, 1)$  to be the simple graph of  $\mathbb{Z}/n\mathbb{Z}$ . Set  $\mathcal{G} := \mathcal{G}_1 \times \mathcal{G}_2$ ,  $\theta_1 := 0$  and  $\theta_2$  such that  $\text{Hol}_{\theta_2} \neq 0$ .

We have that  $\Delta_{\mathcal{G}, \kappa\theta} := \Delta_{\mathcal{G}, \kappa\theta}^{\text{le}} \oplus \Delta_{\mathcal{G}, \kappa\theta}^{\text{he}}$ , where

$$\Delta_{\mathcal{G}, \kappa\theta}^{\text{le}} := \Delta_{\mathcal{G}_1, 0} \otimes P_\kappa^{\text{le}},$$

on  $(1 \otimes P_\kappa^{\text{le}})\ell^2(\mathcal{V}, m)$ , where  $P_\kappa^{\text{le}}$  is the projection on  $\ker(\Delta_{\mathcal{G}_2, \kappa\theta_2})$  and where

$$\Delta_{\mathcal{G}, \kappa\theta}^{\text{he}} := \Delta_{\mathcal{G}_1, 0} \otimes P_\kappa^{\text{he}} + \frac{1}{m_1(\cdot)} \otimes P_\kappa^{\text{he}} \Delta_{\mathcal{G}_2, \kappa\theta_2},$$

on  $(1 \otimes P_\kappa^{\text{he}})\ell^2(\mathcal{V}, m)$ , where  $P_\kappa^{\text{he}}$  is the projection on  $\ker(\Delta_{\mathcal{G}_2, \kappa\theta_2})^\perp$ . Here *le* stands for *low energy* and *he* for *high energy*.

By Lemma 2.2 and Remark 2.6, there exists  $\nu > 0$  such that

$$P_\kappa^{\text{le}} = 0 \quad \Leftrightarrow \quad \text{Hol}_{\kappa\theta_2} \neq 0 \quad \Leftrightarrow \quad \kappa \notin \mathbb{R}/\nu\mathbb{Z} \quad \Leftrightarrow \quad \mathcal{D}\left(\Delta_{\mathcal{G}, \kappa\theta}^{1/2}\right) = \mathcal{D}\left(\text{deg}_{\mathcal{G}}^{1/2}(\cdot)\right).$$

The proof of Theorem 2.5 gives the first point. Assume that  $\kappa \in \mathbb{R}/\nu\mathbb{Z}$ . Let  $U : \ell^2(\mathbb{N}, m_1) \rightarrow \ell^2(\mathbb{N}, 1)$  be the unitary map given by  $Uf(n) := \sqrt{m_1(n)}f(n)$ . We see that:

$$U\Delta_{\mathcal{G}, \kappa\theta}^{\text{le}}U^{-1} = \Delta_{\mathbb{N}, 0} + (e^{-1/2} - 1)\delta_0 + e^{1/2} + e^{-1/2} - 2 \text{ in } \ell^2(\mathbb{N}),$$

where  $\Delta_{\mathbb{N}, 0}$  is related to the simple graph of  $\mathbb{N}$ . The essential spectrum of  $\Delta_{\mathcal{G}, \kappa\theta}^{\text{le}}$  is purely absolutely continuous and equal to  $[e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2]$ . It has a unique eigenvalue.

We turn to the high energy part. Denote by  $\{\lambda_i\}_{i=1, \dots, n}$ , with  $\lambda_i \leq \lambda_{i+1}$ , the eigenvalues of  $\Delta_{\mathcal{G}_2, \kappa\theta_2}$ . Recall that  $\lambda_1 = 0$  due to the fact that  $\text{Hol}_{\kappa\theta_2} = 0$ . By (9),

$$\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2, \kappa\theta_2} P_\kappa^{\text{he}} \leq \Delta_{\mathcal{G}, \kappa\theta} (1 \otimes P_\kappa^{\text{he}}) \leq 2M + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2, \kappa\theta_2} P_\kappa^{\text{he}}.$$

Hence,  $\Delta_{\mathcal{G}, \kappa\theta}(1 \otimes P_\kappa^{\text{he}})$  has a compact resolvent and

$$\begin{aligned} \mathcal{N}_{\lambda-2M}\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2, \kappa\theta_2} P_\kappa^{\text{he}}\right) &\leq \mathcal{N}_\lambda\left(\Delta_{\mathcal{G}, \kappa\theta}(1 \otimes P_\kappa^{\text{he}})\right) \\ &\leq \mathcal{N}_\lambda\left(m_1^{-1}(\cdot) \otimes \Delta_{\mathcal{G}_2, \kappa\theta_2} P_\kappa^{\text{he}}\right), \end{aligned}$$

for all  $\lambda \geq 0$ . Finally:

$$\frac{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2, \kappa \theta_2} P_\kappa^{\text{he}}\right)}{\mathcal{N}_\lambda\left(\frac{1}{m_1(\cdot)} \otimes \text{deg}_{\mathcal{G}_2}\right)} \sim \frac{\sum_{i=2}^n \ln(\lambda) - \ln(\lambda_i)}{n(\ln(\lambda) - \ln(\text{deg}_{\mathcal{G}_2}))} \rightarrow \frac{n-1}{n}, \text{ as } \lambda \rightarrow \infty.$$

We conclude with (14) for  $a = 1$ .  $\square$

#### REFERENCES

- [AbTr] A.Morame and F.Truc: *Magnetic bottles on geometrically finite hyperbolic surfaces*, J. Geom. Phys. **59** (2009), no. 7, 1079–1085.
- [BGK] M.Bonnefont, M.Keller, and S.Golénia: *Eigenvalue asymptotics for Schrödinger operators on sparse graphs*, to appear in Ann. Inst. Fourier.
- [CTT2] Y.Colin de Verdière, N.Torki-Hamza, and F.Truc: *Essential self-adjointness for combinatorial Schrödinger operators II: metrically non complete graphs*, Math. Phys. Anal. Geom. **14** (2011), no. 1, 21–38.
- [CTT3] Y.Colin de Verdière, N.Torki-Hamza, and F.Truc: *Essential self-adjointness for combinatorial Schrödinger operators III: Magnetic fields*, Ann. Fac. Sci. Toulouse Math. (6) **20** (2011), no. 3, 599–611.
- [DM] J. Dodziuk and V. Matthai: *Kato’s inequality and asymptotic spectral properties for discrete magnetic Laplacians. The ubiquitous heat kernel*, Cont. Math. **398**, Am. Math. Soc. (2006), 69–81.
- [Go] S.Golénia: *Hardy inequality and asymptotic eigenvalue distribution for discrete Laplacians*, J. Funct. Anal. **266** (2014), no. 5, 2662–2688.
- [GoMo] S.Golénia and S.Moroianu: *Spectral analysis of magnetic Laplacians on conformally cusp manifolds*, Ann. Henri Poincaré **9** (2008), no. 1, 131–179.
- [HiSh] Y.Higuchi and T.Shirai: *Weak Bloch property for discrete magnetic Schrödinger operators*, Nagoya Math. J. **161** (2001), 127–154.
- [KL] M.Keller and D.Lenz: *Unbounded Laplacians on graphs: basic spectral properties and the heat equation*, Math. Model. Nat. Phenom. **5**, no. 4, (2010) 198–224.
- [RS] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, Tome I–IV: Analysis of operators* Academic Press.
- [ScAi] J.Schenker and M.Aizenman: *The creation of spectral gaps by graph decoration*, Lett. Math. Phys. **53** (2000), no. 3, 253–262.

INSTITUT DE MATHÉMATIQUES DE BORDEAUX, 351, COURS DE LA LIBÉRATION F-33405 TALENCE CEDEX

*E-mail address:* `sylvain.golenia@math.u-bordeaux.fr`

GRENOBLE UNIVERSITY, INSTITUT FOURIER, UNITÉ MIXTE DE RECHERCHE CNRS-UJF 5582, BP 74, 38402-SAINTE MARTIN D’HÈRES CEDEX, FRANCE

*E-mail address:* `francoise.truc@ujf-grenoble.fr`