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Abstract. In the last three decades, Fourier analysis methods have known a growing importance in the study of linear and nonlinear PDE's. In particular, techniques based on Littlewood-Paley decomposition and paradifferential calculus have proved to be very efficient for investigating evolutionary fluid mechanics equations in the whole space or in the torus. We here give an overview of results that we can get by Fourier analysis and paradifferential calculus, for the compressible Navier-Stokes equations. We focus on the Initial Value Problem in the case where the fluid domain is $\mathbb{R}^d$ (or the torus $\mathbb{T}^d$) with $d \geq 2$, and also establish some asymptotic properties of global small solutions. The time decay estimates in the critical regularity framework that are stated at the end of the survey are new, to the best of our knowledge.

1. Introduction

In the Eulerian description, a general compressible fluid evolving in some open set Ω of $\mathbb{R}^d$ is characterized at every material point x in Ω and time $t \in \mathbb{R}$ by its *velocity field* $u = u(t,x) \in \mathbb{R}^d$, *density* $\varrho = \varrho(t,x) \in \mathbb{R}_+$, *pressure* $p = p(t,x) \in \mathbb{R}$, *internal energy by unit mass* $e = e(t,x) \in \mathbb{R}$, *entropy by unit mass* $s = s(t,x)$ and *absolute temperature* $T = T(t,x)$. In the absence of external forces, those quantities are governed by:

- The mass balance:
$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0.$$
- The momentum balance[1]:
$$\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) = \operatorname{div}\boldsymbol{\tau} - \nabla p,$$
where $\boldsymbol{\tau}$ stands for the *viscous stress tensor*.
- The energy balance:
$$\partial_t\Big(\varrho\big(e + \tfrac{|u|^2}{2}\big)\Big) + \operatorname{div}\Big(\varrho\big(e + \tfrac{|u|^2}{2}\big)u\Big) = \operatorname{div}(\boldsymbol{\tau}\cdot u + pu) - \operatorname{div} q,$$
where q is the *heat flux* vector.
- The entropy inequality:
$$\partial_t(\varrho s) + \operatorname{div}(\varrho s u) \geq -\operatorname{div}\big(\tfrac{q}{T}\big)\cdot \tag{1}$$

In what follows, we concentrate on so-called *Newtonian fluids*[2]. Hence (see e.g.[2]) $\boldsymbol{\tau}$ is given by:
$$\boldsymbol{\tau} \overset{\text{def}}{=} \lambda \operatorname{div} u \,\mathrm{Id} + 2\mu D(u),$$

[1]With the convention that $\big(\operatorname{div}(a\otimes b)\big)^j \overset{\text{def}}{=} \sum_i \partial_i(a^i b^j)$.

[2]That is to say: the viscous stress tensor $\boldsymbol{\tau}$ is a linear function of $D_x u$, invariant under rigid transforms, there is no internal mass couples (and thus the *angular momentum* is conserved), and the fluid is isotropic (*viz.* the physical quantities depend only on (t,x)).

where the real numbers λ and μ are the *viscosity coefficients* and $D(u) \overset{\text{def}}{=} \frac{1}{2}(Du + {}^TDu)$ is the *deformation tensor.*

If we assume in addition that the *Fourier law* $q = -k\nabla T$ is satisfied then we get the following system of equations:

$$\left\{\begin{array}{l} \partial_t\varrho + \operatorname{div}(\varrho u) = 0, \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u\otimes u) - 2\operatorname{div}(\mu D(u)) - \nabla(\lambda\operatorname{div} u) + \nabla p = 0, \\ \partial_t(\varrho e) + \operatorname{div}(\varrho e u) + p\operatorname{div} u - \operatorname{div}(k\nabla T) = 2\mu D(u) : D(u) + \lambda(\operatorname{div} u)^2. \end{array}\right. \tag{2}$$

We shall further postulate that the entropy s is interrelated with p, T and e through the so-called *Gibbs relation*

$$Tds = de + p\,d\Big(\frac{1}{\varrho}\Big),$$

and thus we get the following evolution equation for s:

$$T\big(\partial_t(\varrho s) + \operatorname{div}(\varrho s u)\big) = \boldsymbol{\tau}\cdot D(u) - \operatorname{div} q. \tag{3}$$

For the entropy inequality to be satisfied, a necessary and sufficient condition is thus

$$\boldsymbol{\tau} : D(u) + k\frac{|\nabla T|^2}{T} \geq 0,$$

which yields the following constraints on λ, μ and k:

$$k \geq 0, \quad \mu \geq 0 \quad \text{and} \quad 2\mu + d\lambda \geq 0.$$

In order to close System (2) which is composed of $d+2$ equations for $d+4$ unknowns (namely ϱ, e, p, T and $u^1, \cdots, u^d$), we need another two *state equations* interrelating p, ϱ, e, s and T. In this survey, for simplicity we shall focus on *barotropic gases* that is p depends only on the density and λ and μ are independent of T. Therefore the system constituted by the first two equations in (2), the so-called *barotropic compressible Navier-Stokes system* :

$$\left\{\begin{array}{l} \partial_t\varrho + \operatorname{div}(\varrho u) = 0, \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u\otimes u) - \operatorname{div}\big(2\mu D(u) + \lambda\operatorname{div} u\,\mathrm{Id}\,\big) + \nabla p = 0 \end{array}\right. \tag{4}$$

where $p \overset{\text{def}}{=} P(\varrho)$ for some given smooth function P, is closed.

Our main goal is to solve the Initial Value Problem (or Cauchy Problem) for (4) supplemented with initial data (ϱ_0, u_0) at time $t = 0$ *in the case where the fluid domain* Ω *is the whole space or the torus.* We will concentrate on the local well-posedness issue for large data with no vacuum, on the global well-posedness issue for small perturbations of a constant stable equilibrium, and will give exhibit some of the qualitative properties of the constructed solutions. As regards global results, the concept of *critical regularity* is fundamental. Indeed, experience shows that whenever the PDE system under consideration possesses some scaling invariance with respect to space and time dilations (which is in general the case when it comes from mathematical physics) then appropriate so-called critical norms or quantities essentially control the (possible) finite time blow-up and the asymptotic properties of the solutions.

If modifying the pressure law accordingly then the barotropic system (4) we are here considering has the following scaling invariance:

$$\varrho(t,x) \rightsquigarrow \varrho(\ell^2 t, \ell x), \qquad u(t,x) \rightsquigarrow \ell u(\ell^2 t, \ell x), \qquad \ell > 0. \tag{5}$$

More precisely, (ϱ, u) is a solution to (4) if and only if so does (ϱ_ℓ, u_ℓ), with pressure function $\ell^2 P$. This means that we expect optimal solution spaces (and norms) for (4) to have the scaling invariance pointed out above.

The rest of these notes unfolds as follows. In the next section, we present the basic tools and estimates that will be needed to study System (26). Then we concentrate on the local well-posedness issue for (26) in critical spaces. Section 4 is dedicated to solving (4) globally for small data. The last section is devoted to asymptotic results for the system. We concentrate on the low Mach number limit and on the decay rates of global solutions in the critical regularity framework.

2. The Fourier analysis toolbox

We here shortly introduce the Fourier analysis tools needed in this survey, then state estimates for the heat and transport equations that will play a fundamental role. For the sake of conciseness, some proofs are just sketched or omitted. Unless otherwise specified, the reader will find details in [1], Chap. 2 or 3.

2.1. The Littlewood-Paley decomposition.

The Littlewood-Paley decomposition is a dyadic localization procedure in the frequency space for tempered distributions over $\mathbb{R}^d$. One of the main motivations for using it when dealing with PDEs is that the derivatives act almost as dilations on distributions with Fourier transform supported in a ball or an annulus, as regards L^p norms. This is exactly what is stated in the following proposition:

Proposition 2.1 (Bernstein inequalities)**.** *Let $0 < r < R$.*

- *There exists a constant C so that, for any $k \in \mathbb{N}$, couple (p, q) in $[1, \infty]^2$ with $q \geq p \geq 1$ and function u of L^p with $\widehat{u}$ supported in the ball $B(0, \lambda R)$ of $\mathbb{R}^d$ for some $\lambda > 0$, we have $D^k u \in L^q$ and*
$$\|D^k u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}.$$
- *There exists a constant C so that for any $k \in \mathbb{N}$, $p \in [1, \infty]$ and function u of L^p with Supp $\widehat{u} \subset \{\xi \in \mathbb{R}^d \,/\, r\lambda \leq |\xi| \leq R\lambda\}$ for some $\lambda > 0$, we have*
$$\lambda^k \|u\|_{L^p} \leq C^{k+1} \|D^k u\|_{L^p}.$$

As general solutions to nonlinear PDE's need not be spectrally localized in annuli, we want a device for splitting any function or distribution into a sum of spectrally localized functions. To this end, fix some smooth radial non increasing function χ with $\mathrm{Supp}\,\chi \subset B(0, \frac{4}{3})$ and $\chi \equiv 1$ on $B(0, \frac{3}{4})$, then set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. We thus have
$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\cdot) = 1 \ \text{ in } \ \mathbb{R}^d \setminus \{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined by
$$\dot{\Delta}_j u \stackrel{\text{def}}{=} \varphi(2^{-j}D)u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u) = 2^{jd} h(2^j \cdot) \star u \quad \text{with} \quad h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi.$$
We also introduce the low frequency cut-off operator $\dot{S}_j$:
$$\dot{S}_j u \stackrel{\text{def}}{=} \chi(2^{-j}D)u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}\cdot)\mathcal{F}u) = 2^{jd} \check{h}(2^j \cdot) \star u \quad \text{with} \quad \check{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1}\chi.$$
Let us emphasize that operators $\dot{\Delta}_j$ and $\dot{S}_j$ are continuous on L^p, with norm *independent* of j, a property that would fail if taking a rough cut-off function χ (unless $p = 2$ of course).

The price to pay for smooth cut-off is that $\dot{\Delta}_j$ is not an L^2 orthogonal projector. However the following important quasi-orthogonality property is fulfilled:

$$\dot{\Delta}_j\dot{\Delta}_k = 0 \ \text{ if } \ |j-k|>1. \tag{6}$$

The homogeneous Littlewood-Paley decomposition for u reads

$$u = \sum_j \dot{\Delta}_j u. \tag{7}$$

This equality holds *modulo polynomials* only. In order to have equality in the distributional sense, one may consider the set $\mathcal{S}'_h$ of tempered distributions u such that

$$\lim_{j\to-\infty} \|\dot{S}_j u\|_{L^\infty} = 0.$$

As distributions of $\mathcal{S}'_h$ tend to 0 at infinity, one can easily conclude that (7) holds true in $\mathcal{S}'$ whenever u is in $\mathcal{S}'_h$.

2.2. **Besov spaces.** It is obvious that for all $s\in\mathbb{R}$, we have

$$C^{-1}\|u\|_{\dot{H}^s}^2 \leq \sum_{j\in\mathbb{Z}} 2^{2js}\|\dot{\Delta}_j u\|_{L^2}^2 \leq C\|u\|_{\dot{H}^s}^2, \tag{8}$$

and it is also not very difficult to prove that for $s\in(0,1)$,

$$C^{-1}\|u\|_{\dot{C}^{0,s}} \leq \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^\infty} \leq C\|u\|_{\dot{C}^{0,s}}, \qquad s\in(0,1). \tag{9}$$

In (8) and (9), we observe that three parameters come into play: the regularity parameter s, the Lebesgue exponent that is used for bounding $\dot{\Delta}_j u$ and the type of summation that is done over $\mathbb{Z}$. This motivates the following definition:

Definition 2.1. *For $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$, we set*

$$\|u\|_{\dot{B}^s_{p,r}} \overset{def}{=} \biggl(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|_{L^p}^r\biggr)^{\frac{1}{r}} \ \text{ if } \ r<\infty \ \text{ and } \ \|u\|_{\dot{B}^s_{p,\infty}} \overset{def}{=} \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j u\|_{L^p}.$$

We then define the homogeneous Besov space $\dot{B}^s_{p,r}$ to be the subset of distributions $u\in\mathcal{S}'_h$ such that $\|u\|_{\dot{B}^s_{p,r}}<\infty$.

We shall often use the following classical properties:

- *Scaling invariance*: For any $s\in\mathbb{R}$ and $(p,r)\in[1,+\infty]^2$ there exists a constant C such that for all $\lambda>0$ and $u\in\dot{B}^s_{p,r}$, we have

$$C^{-1}\lambda^{s-\frac{d}{p}}\|u\|_{\dot{B}^s_{p,r}} \leq \|u(\lambda\cdot)\|_{\dot{B}^s_{p,r}} \leq C\lambda^{s-\frac{d}{p}}\|u\|_{\dot{B}^s_{p,r}}. \tag{10}$$

- *Completeness*: $\dot{B}^s_{p,r}$ is a Banach space whenever $s<d/p$ or $s\leq d/p$ and $r=1$.
- *Fatou property*: if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions of $\dot{B}^s_{p,r}$ that converges in $\mathcal{S}'$ to some $u\in\mathcal{S}'_h$ then $u\in\dot{B}^s_{p,r}$ and $\|u\|_{\dot{B}^s_{p,r}} \leq C\liminf\|u_n\|_{\dot{B}^s_{p,r}}$.
- *Duality*: If u is in $\mathcal{S}'_h$ then we have

$$\|u\|_{\dot{B}^s_{p,r}} \leq C\sup_\phi \langle u,\phi\rangle$$

where the supremum is taken over those ϕ in $\mathcal{S}\cap\dot{B}^{-s}_{p',r'}$ such that $\|\phi\|_{\dot{B}^{-s}_{p',r'}}\leq 1$.

- *Interpolation*: The following inequalities are satisfied for[3] all $1 \leq p, r_1, r_2, r \leq \infty$, $s_1 \neq s_2$ and $\theta \in (0,1)$:

$$\|u\|_{\dot{B}^{\theta s_2+(1-\theta)s_1}_{p,r}} \lesssim \|u\|^{1-\theta}_{\dot{B}^{s_1}_{p,r_1}} \|u\|^{\theta}_{\dot{B}^{s_2}_{p,r_2}}.$$

- *Action of Fourier multipliers*: If F is a smooth homogeneous of degree m function on $\mathbb{R}^d \setminus \{0\}$ then

$$F(D) : \dot{B}^s_{p,r} \to \dot{B}^{s-m}_{p,r}. \tag{11}$$

In particular, the gradient operator maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-1}_{p,r}$.

Proposition 2.2 (Embedding for Besov spaces on $\mathbb{R}^d$). (1) *For any $p \in [1,\infty]$ we have the continuous embedding $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$.*

(2) *If $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

(3) *For $s' < s$ and any $1 \leq p, r_1, r_2 \leq \infty$, the embedding of $\dot{B}^s_{p,r_1}$ in $\dot{B}^{s'}_{p,r_2}$ is locally compact, i.e. for any $\varphi \in \mathcal{S}$, the map $u \mapsto \varphi u$ is compact from $\dot{B}^s_{p_1,r_1}$ to $\dot{B}^{s'-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

(4) *The space $\dot{B}^{\frac{d}{p}}_{p,1}$ is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if, additionally, $p < \infty$).*

2.3. **Paraproduct and nonlinear estimates.** Formally, the product of two tempered distributions u and v may be decomposed into

$$uv = T_u v + R(u,v) + T_v u \tag{12}$$

with

$$T_u v \stackrel{\text{def}}{=} \sum_j \dot{S}_{j-1}u\,\dot{\Delta}_j v \quad \text{and} \quad R(u,v) \stackrel{\text{def}}{=} \sum_j \sum_{|j'-j|\leq 1} \dot{\Delta}_j u\,\dot{\Delta}_{j'} v.$$

The above operator T is called "paraproduct" whereas R is called "remainder". The decomposition (12) has been first introduced by J.-M. Bony in [3]. We observe that in Fourier variables the sum in $T_u v$ is *locally finite*, hence $T_u v$ is always defined. We shall see however that it cannot be smoother than what is given by high frequencies, namely v. As for the remainder, it may be not defined, but if it is then the regularity exponents add up. All that is detailed below:

Proposition 2.3. *Let $(s,r) \in \mathbb{R} \times [1,\infty]$ and $1 \leq p, p_1, p_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$.*

- *We have:*

$$\|T_u v\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^{p_1}} \|v\|_{\dot{B}^s_{p_2,r}} \quad \text{and} \quad \|T_u v\|_{\dot{B}^{s+t}_{p,r}} \lesssim \|u\|_{\dot{B}^t_{p_1,\infty}} \|v\|_{\dot{B}^s_{p_2,r}}, \quad \text{if } t < 0.$$

- *If $s_1 + s_2 > 0$ and $1/r = 1/r_1 + 1/r_2 \leq 1$ then*

$$\|R(u,v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

- *If $s_1 + s_2 = 0$ and $1/r_1 + 1/r_2 \geq 1$ then*

$$\|R(u,v)\|_{\dot{B}^0_{p,\infty}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

Putting together decomposition (12) and the above results, one may get the following product estimate that depends only *linearly* on the highest norm of u and v:

[3]With the convention that $A \lesssim B$ means that $A \leq CB$ for some 'harmless' positive constant C.

Corollary 2.1. *Let u and v be in $L^\infty\cap\dot B^s_{p,r}$ for some $s>0$ and $(p,r)\in[1,\infty]^2$. Then there exists a constant C depending only on d, p and s and such that*

$$\|uv\|_{\dot B^s_{p,r}}\leq C\big(\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}+\|v\|_{L^\infty}\|u\|_{\dot B^s_{p,r}}\big).$$

Remark 2.1. *Because $\dot B^{\frac dp}_{p,1}$ is embedded in L^∞, we deduce that whenever $p<+\infty$, the product of two functions in $\dot B^{\frac dp}_{p,1}$ is also in $\dot B^{\frac dp}_{p,1}$ and that for some constant $C=C(p,d)$:*

$$\|uv\|_{\dot B^{\frac dp}_{p,1}}\leq C\|u\|_{\dot B^{\frac dp}_{p,1}}\|v\|_{\dot B^{\frac dp}_{p,1}}.$$

Let us finally state a composition result.

Proposition 2.4. *Let $F:\mathbb{R}\to\mathbb{R}$ be smooth with $F(0)=0$. For all $1\leq p,r\leq\infty$ and $s>0$ we have $F(u)\in\dot B^s_{p,r}\cap L^\infty$ for $u\in\dot B^s_{p,r}\cap L^\infty$, and*

$$\|F(u)\|_{\dot B^s_{p,r}}\leq C\|u\|_{\dot B^s_{p,r}} \tag{13}$$

with C depending only on $\|u\|_{L^\infty}$, F' (and higher derivatives), s, p and d.

2.4. **Endpoint maximal regularity for the linear heat equation.** This paragraph is dedicated to *maximal regularity* issues for the basic heat equation

$$\partial_t u-\Delta u=f,\qquad u_{|t=0}=u_0. \tag{14}$$

In the case $u_0\equiv0$, we say that the functional space X endowed with norm $\|\cdot\|_X$ has the maximal regularity property if

$$\|\partial_t u,D_x^2u\|_X\leq C\|f\|_X. \tag{15}$$

Fourier-Plancherel theorem implies that (15) holds true for $X=L^2(\mathbb{R}_+\times\mathbb{R}^d)$. From more complicated tools based on singular integrals and heat kernel estimates, one may gather that (15) is true for $X=L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))$ whenever $1<p,r<+\infty$. However, the endpoint cases where one of the exponents p or r is 1 or $+\infty$ are false.

One of the keys to the approach presented in these notes is that (15) is true for $X=L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^d))$, a consequence of the following lemma:

Lemma 2.1. *There exist two positive constants c and C such that for any $j\in\mathbb{Z}$, $p\in[1,\infty]$ and $\lambda\in\mathbb{R}_+$, we have for all $u\in\mathcal{S}'$ with $\dot\Delta_j u$ in L^p :*

$$\|e^{\lambda\Delta}\dot\Delta_j u\|_{L^p}\leq Ce^{-c_0\lambda2^{2j}}\|\dot\Delta_j u\|_{L^p}.$$

Proof. A suitable change of variable reduces the proof to the case $j=0$. Then consider a function ϕ in $\mathcal{D}(\mathbb{R}^d\setminus\{0\})$ with value 1 on a neighborhood of $\operatorname{Supp}\varphi$. We have

$$\begin{aligned}e^{\lambda\Delta}\dot\Delta_0u&=\mathcal{F}^{-1}\Big(\phi e^{-\lambda|\cdot|^2}\widehat{\dot\Delta_0u}\Big)\\&=g_\lambda\star\dot\Delta_0u\quad\text{with}\quad g_\lambda(x)\overset{\text{def}}{=}(2\pi)^{-d}\int e^{i(x|\xi)}\phi(\xi)e^{-\lambda|\xi|^2}d\xi.\end{aligned}$$

Integrating by parts yields

$$g_\lambda(x)=(1+|x|^2)^{-d}\int_{\mathbb{R}^d}e^{i(x|\xi)}(\mathrm{Id}-\Delta_\xi)^d\Big(\phi(\xi)e^{-\lambda|\xi|^2}\Big)d\xi.$$

Combining Leibniz and Faá-di-Bruno's formulae to bound the integrant, we get

$$|g_\lambda(x))|\leq C(1+|x|^2)^{-d}e^{-c_0\lambda},$$

and thus

$$\|g_\lambda\|_{L^1} \leq Ce^{-c_0\lambda}. \tag{16}$$

Now the desired inequality (with $j=0$) just follows from $L^1 \star L^p \to L^p$. □

Theorem 2.1. *Let u satisfy* (14). *Then for any $p \in [1,\infty]$ and $s \in \mathbb{R}$ the following inequality holds true for all $t>0$:*

$$\|u(t)\|_{\dot B^s_{p,1}} + \int_0^t \|\nabla^2 u\|_{\dot B^s_{p,1}}\,d\tau \leq C\biggl(\|u_0\|_{\dot B^s_{p,1}} + \int_0^t \|f\|_{\dot B^s_{p,1}}\,d\tau\biggr)\cdot \tag{17}$$

Proof. If u satisfies (14) then we have for any $j \in \mathbb{Z}$,

$$\dot\Delta_j u(t) = e^{t\Delta}\dot\Delta_j u_0 + \int_0^t e^{(t-\tau)\Delta}\dot\Delta_j f(\tau)\,d\tau.$$

Taking advantage of Lemma 2.1, we thus have

$$\|\dot\Delta_j u(t)\|_{L^p} \lesssim e^{-c_0 2^{2j}t}\|\dot\Delta_j u_0\|_{L^p} + \int_0^t e^{-c_0 2^{2j}(t-\tau)}\|\dot\Delta_j f(\tau)\|_{L^p}\,d\tau. \tag{18}$$

Multiplying by 2^{js} and summing up over j yields

$$\sum_j 2^{js}\|\dot\Delta_j u(t)\|_{L^p} \lesssim \sum_j e^{-c_0 2^{2j}t}2^{js}\|\dot\Delta_j u_0\|_{L^p} + \int_0^t e^{-c_0 2^{2j}(t-\tau)}\sum_j \|\dot\Delta_j f(\tau)\|_{L^p}\,d\tau$$

whence

$$\|u\|_{L^\infty(0,t;\dot B^s_{p,1})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|f\|_{L^1(0,t;\dot B^s_{p,1})}.$$

Note that integrating (18) with respect to time also yields

$$2^{2j}\|\dot\Delta_j u\|_{L^1(0,t;L^p)} \lesssim \Bigl(1-e^{-c_0 2^{2j}t}\Bigr)\Bigl(\|\dot\Delta_j u_0\|_{L^p} + \|\dot\Delta_j f\|_{L^1(0,t;L^p)}\Bigr).$$

Therefore, multiplying by 2^{js}, using Bernstein inequality and summing up over j yields

$$\|\nabla^2 u\|_{L^1(0,t;\dot B^s_{p,1})} \lesssim \sum_j \Bigl(1-e^{-c_0 2^{2j}t}\Bigr)2^{js}\Bigl(\|\dot\Delta_j u_0\|_{L^p} + \|\dot\Delta_j f\|_{L^1(0,t;L^p)}\Bigr), \tag{19}$$

which is even slightly better than what we wanted to prove □

Remark 2.2. *Starting from* (18) *and using general convolution inequalities in $\mathbb{R}_+$ gives a whole family of estimates for the heat equation. However, as time integration has been performed* before *summation over j, the norms that naturally appear are*

$$\|\cdot\|_{\widetilde L^a_t(\dot B^\sigma_{b,c})} \overset{def}{=} \Bigl\|2^{j\sigma}\|\cdot\|_{L^a_t(L^b)}\Bigr\|_{\ell^c} \quad \textit{where} \quad \|\cdot\|_{L^a_t(L^b)} \overset{def}{=} \|\cdot\|_{L^a((0,t);L^b(\mathbb{R}^d))}.$$

With this notation, (18) *implies that*

$$\|u\|_{\widetilde L^{\rho_1}_t(\dot B^{s+\frac{2}{\rho_1}}_{p,r})} \lesssim \|u_0\|_{\dot B^s_{p,r}} + \|f\|_{\widetilde L^{\rho_2}_t(\dot B^{s-2+\frac{2}{\rho_2}}_{p,r})} \quad \textit{for} \quad 1 \leq \rho_2 \leq \rho_1 \leq \infty.$$

The relevancy of the above norms in the maximal regularity estimates has been first noticed (in a particular case) in the pioneering work by J.-Y. Chemin and N. Lerner [6], *then extended to general Besov spaces in* [5]. *They will play a fundamental role in the proof of decay estimates, at the end of the paper.*

Let us point out that results in the spirit of Propositions 2.3 and 2.4 may be easily proved for $\widetilde L^\rho_t(\dot B^\sigma_{p,r})$ spaces, the general rule being just that the time exponents behave according to Hölder inequality.

2.5. **The linear transport equation.** Here we give estimates in Besov spaces for the following *transport equation*:

$$\left\{\begin{array}{ll} \partial_t a + v\cdot\nabla_x a + \lambda a = f & \text{in} \quad \mathbb{R}\times\mathbb{R}^d \\ a_{|t=0} = a_0 & \text{in} \quad \mathbb{R}^d, \end{array}\right. \tag{20}$$

where the initial data $a_0 = a_0(x)$, the source term $f = f(t,x)$, the damping coefficient $\lambda \geq 0$ and the time dependent transport field $v = v(t,x)$ are given.

Assuming that $a_0 \in X$ and $f \in L^1_{loc}(\mathbb{R}_+; X)$, the relevant assumptions on v for (20) to be uniquely solvable depend on the nature of the Banach space X. Broadly speaking, in the classical theory based on Cauchy-Lipschitz theorem, v has to be at least integrable in time with values in the set of Lipschitz functions, so that it has a flow ψ. This allows to get the following explicit solution for (20):

$$a(t,x) = e^{-\lambda t}a_0(\psi_t^{-1}(x)) + \int_0^t e^{-\lambda(t-\tau)} f(\tau,\psi_\tau(\psi_t^{-1}(x)))\,d\tau. \tag{21}$$

Theorem 2.2. *Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$ satisfy*

$$-\min\Big(\frac{d}{p_1}, \frac{d}{p'}\Big) < s < 1 + \frac{d}{p_1}\cdot$$

Then any smooth enough solution to (20) *fulfills*

$$\|a\|_{\widetilde{L}^\infty_t(\dot B^s_{p,r})} + \lambda\|a\|_{\widetilde{L}^1_t(\dot B^s_{p,r})} \leq e^{CV(t)}\Big(\|a_0\|_{\dot B^s_{p,r}} + \|f\|_{\widetilde{L}^1_t(\dot B^s_{p,r})}\Big)$$

with

$$V(t) \overset{def}{=} \int_0^t \|\nabla v(\tau)\|_{\dot B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}\,d\tau.$$

In the case $s = 1 + \frac{d}{p_1}$ and $r = 1$, the above inequality is true with $V'(t) = \|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}$.

Proof. Applying $\dot\Delta_j$ to (20) gives

$$\partial_t\dot\Delta_j a + v\cdot\nabla\dot\Delta_j a + \lambda\dot\Delta_j a = \dot\Delta_j f + \dot R_j \quad \text{with} \quad \dot R_j \overset{\text{def}}{=} [v\cdot\nabla, \dot\Delta_j]a. \tag{22}$$

Therefore, from classical L^p estimates for the transport equation, we get

$$\begin{aligned}\|\dot\Delta_j a(t)\|_{L^p} + \lambda\|\dot\Delta_j a\|_{L^1_t(L^p)} &\leq \|\dot\Delta_j a_0\|_{L^p} \\ &\quad + \int_0^t \Big(\|\dot\Delta_j f\|_{L^p} + \|\dot R_j\|_{L^p} + \frac{\|\operatorname{div} v\|_{L^\infty}}{p}\|\dot\Delta_j a\|_{L^p}\Big)\,d\tau.\end{aligned} \tag{23}$$

We claim that the remainder term $\dot R_j$ satisfies

$$\|\dot R_j(t)\|_{L^p} \leq Cc_j(t)2^{-js}\|\nabla v(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}\|a(t)\|_{\dot B^s_{p,r}} \quad \text{with} \quad \|(c_j(t))\|_{\ell^r} = 1. \tag{24}$$

Indeed, from Bony's decomposition, we infer that (with the summation convention over repeated indices):

$$\dot R_j = [T_{v^k}, \dot\Delta_j]\partial_k a + T_{\partial_k\dot\Delta_j a}v^k - \dot\Delta_j T_{\partial_k a}v^k + R(v^k, \partial_k\dot\Delta_j a) - \dot\Delta_j R(v^k, \partial_k a). \tag{25}$$

The first term is the only one where having a commutator improves the estimate. Indeed, owing to the properties of spectral localization of the Littlewood-Paley decomposition, we have

$$[T_{v^k}, \dot\Delta_j]\partial_k a = \sum_{|j-j'|\leq 4}[\dot S_{j'-1}v^k, \dot\Delta_j]\partial_k\dot\Delta_{j'}a.$$

Now, remark that

$$[\dot S_{j'-1}v^k,\dot\Delta_j]\partial_k\dot\Delta_{j'}a(x)=2^{jd}\int_{\mathbb{R}^d}h(2^j(x-y))\big(\dot S_{j'-1}v^k(x)-\dot S_{j'-1}v^k(y)\big)\partial_k\dot\Delta_{j'}a(y)\,dy.$$

Hence using the mean value formula and Bernstein inequalities yields

$$\|[T_{v^k},\dot\Delta_j]\partial_k a\|_{L^p}\lesssim 2^{-j}\|\dot S_{j'-1}\nabla v\|_{L^\infty}\sum_{|j'-j|\leq 4}\|\partial_k\dot\Delta_{j'}a\|_{L^p}\lesssim\|\nabla v\|_{L^\infty}\sum_{|j'-j|\leq 4}\|\dot\Delta_{j'}a\|_{L^p}.$$

Bounding the third and last term in (25) follows from Proposition 2.3. Regarding the second term, we may write

$$T_{\partial_k\dot\Delta_j a}v^k=\sum_{j'\geq j-3}\dot S_{j'-1}\partial_k\dot\Delta_j a\,\dot\Delta_{j'}v^k,$$

hence using Bernstein inequality,

$$\|T_{\partial_k\dot\Delta_j a}v^k\|_{L^p}\lesssim\sum_{j'\geq j-3}2^{j-j'}\|\dot\Delta_j a\|_{L^p}\,\|\nabla\dot\Delta_{j'}v\|_{L^\infty},$$

and convolution inequality for series thus ensures (24) for that term.

Finally, we have

$$\dot\Delta_j R(v^k,\partial_k a)=\sum_{|j'-j|\leq 1}\partial_k\dot\Delta_{j'}a\,(\dot\Delta_{j'-1}+\dot\Delta_{j'}+\dot\Delta_{j'+1})v^k,$$

whence, by virtue of Bernstein inequality,

$$\|R(v^k,\partial_k a)\|_{L^p}\lesssim\sum_{|j'-j|\leq 1}\|\dot\Delta_{j'}a\|_{L^p}\|\nabla v\|_{L^\infty},$$

and that term is thus also bounded by the r.h.s. of (24).

Let us resume to (22). Using (23) and (24), multiplying by 2^{js} then summing up over j and using the notations of Remark 2.2 yields

$$\|a\|_{\widetilde L^\infty_t(\dot B^s_{p,r})}+\lambda\|a\|_{\widetilde L^1_t(\dot B^s_{p,r})}\leq\|a_0\|_{\dot B^s_{p,r}}+\|f\|_{\widetilde L^1_t(\dot B^s_{p,r})}+C\int_0^t V'\|a\|_{\dot B^s_{p,r}}\,d\tau.$$

Then applying Gronwall's lemma gives the desired inequality for a. □

3. The local existence in critical spaces

This section is dedicated to solving (4) locally in time, in critical spaces. For simplicity, we focus on the case where the density goes to 1 at ∞. Setting $a\overset{\text{def}}{=}\varrho-1$ and looking for reasonably smooth solutions with positive density, System (4) is equivalent to

$$\text{(26)}\qquad\begin{cases}\partial_t a+u\cdot\nabla a=-(1+a)\mathrm{div}\,u,\\ \partial_t u+u\cdot\nabla u-\dfrac{\mathcal{A}u}{1+a}+\nabla G(a)=\dfrac{1}{1+a}\mathrm{div}\,\big(2\widetilde\mu(a)D(u)+\widetilde\lambda(a)\mathrm{div}\,u\,\mathrm{Id}\big),\end{cases}$$

where[4] $G'(a)\overset{\text{def}}{=}\frac{P'(1+a)}{1+a}$, $\mathcal{A}\overset{\text{def}}{=}\mu\Delta+(\lambda+\mu)\nabla\mathrm{div}$ with $\lambda\overset{\text{def}}{=}\lambda(1)$ and $\mu\overset{\text{def}}{=}\mu(1)$,

$$\widetilde\mu(z)\overset{\text{def}}{=}\mu(1+z)-\mu(1)\quad\text{and}\quad\widetilde\lambda(z)\overset{\text{def}}{=}\lambda(1+z)-\lambda(1).$$

[4]In our approach, the exact value of functions G, $\widetilde\lambda$ and $\widetilde\mu$ will not matter. We shall only need enough smoothness, and vanishing at 0 for $\widetilde\lambda$ and $\widetilde\mu$.

The scaling invariance properties for (a,u) are those pointed out for (4). Critical norms for the initial data are thus invariant for all $\ell>0$ by

$$a_0(x)\rightsquigarrow a_0(\ell x)\quad\text{and}\quad u_0(x)\rightsquigarrow \ell u_0(\ell x).$$

In all that follows, we shall only consider *homogeneous Besov spaces* having the above scaling invariance and last index 1. There are good reasons for that choice, which will be explained throughout. Remembering (10), we thus take

$$a_0\in\dot B^{\frac dp}_{p,1}\quad\text{and}\quad u_0\in\dot B^{\frac dp-1}_{p,1}. \tag{27}$$

In order to guess what is the relevant solution space, we just use the fact that a is governed by a transport equation and that u may be seen as the solution to the following *Lamé equation*:

$$\partial_t u-\mathcal{A}u=f,\qquad u|_{t=0}=u_0. \tag{28}$$

In the whole space case, the solutions to (28) also satisfy the estimates of Theorem 2.1 and Remark 2.2 whenever the following ellipticity condition is fulfilled:

$$\mu>0\quad\text{and}\quad \nu\overset{\text{def}}{=}\lambda+2\mu>0. \tag{29}$$

Indeed, if we denote by $\mathcal{P}$ and $\mathcal{Q}$ the orthogonal projectors over divergence-free and potential vector fields, then we have

$$\partial_t\mathcal{P}u-\mu\Delta\mathcal{P}u=\mathcal{P}f\quad\text{and}\quad\partial_t\mathcal{Q}u-\nu\Delta\mathcal{Q}u=\mathcal{Q}f.$$

In particular, applying Theorem 2.1 yields for all $t\geq0$:

$$\|\mathcal{P}u(t)\|_{\dot B^s_{p,1}}+\mu\int_0^t\|\nabla^2\mathcal{P}u\|_{\dot B^s_{p,1}}\,d\tau\leq C\biggl(\|\mathcal{P}u_0\|_{\dot B^s_{p,1}}+\int_0^t\|\mathcal{P}f\|_{\dot B^s_{p,1}}\,d\tau\biggr),$$
$$\|\mathcal{Q}u(t)\|_{\dot B^s_{p,1}}+\nu\int_0^t\|\nabla^2\mathcal{Q}u\|_{\dot B^s_{p,1}}\,d\tau\leq C\biggl(\|\mathcal{Q}u_0\|_{\dot B^s_{p,1}}+\int_0^t\|\mathcal{Q}f\|_{\dot B^s_{p,1}}\,d\tau\biggr).$$

As $\mathcal{P}$ and $\mathcal{Q}$ are continuous on $\dot B^s_{p,1}$ (being 0 order multipliers) we conclude that

$$\|u(t)\|_{\dot B^s_{p,1}}+\min(\mu,\nu)\int_0^t\|\nabla^2u\|_{\dot B^s_{p,1}}\,d\tau\leq C\biggl(\|u_0\|_{\dot B^s_{p,1}}+\int_0^t\|f\|_{\dot B^s_{p,1}}\,d\tau\biggr). \tag{30}$$

So, in short, starting from $u_0\in\dot B^{\frac dp-1}_{p,1}$, we expect u in (26) to be in

$$E_p(T)\overset{\text{def}}{=}\bigl\{u\in\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1}),\ \partial_t u,\nabla^2u\in L^1(0,T;\dot B^{\frac dp-1}_{p,1})\bigr\}.$$

In particular ∇u has exactly the regularity needed in Theorem 2.2 to ensure the conservation of the initial regularity for a, and we thus expect $a\in\mathcal{C}([0,T];\dot B^{\frac dp}_{p,1})$.

The rest of this section is devoted to the proof of the following statement:

Theorem 3.1. *Let the viscosity coefficients λ and μ depend smoothly on ϱ, and satisfy (29). Assume that (a_0,u_0) fulfills (27) for some $1\leq p<2d$, and that $d\geq2$. If in addition $1+a_0$ is bounded away from 0 then (26) has a unique local-in-time solution[5] and (a,u) with a in $\mathcal{C}([0,T];\dot B^{\frac dp}_{p,1})$ and u in $E_p(T)$.*

[5]Owing to Remark 2.2 and to Theorem 2.2 the constructed solution will have the additional property that $u\in\widetilde L^\infty_T(\dot B^{\frac dp-1}_{p,1})$ and that $a\in\widetilde L^\infty_T(\dot B^{\frac dp}_{p,1})$.

We propose two different proofs for Theorem 3.1. The first one uses an iterative scheme for approximating (26) which consists in solving a linear transport equation and the Lamé equation with appropriate right-hand sides. Taking advantage of Theorem 2.2 and of (30), it is easy to bound the sequence in the expected solution space on some fixed time interval $[0,T]$ with $T>0$. However, because the whole system is not fully parabolic, the strong convergence of the sequence is shown for a weaker norm corresponding to 'a loss of one derivative'. For that reason, that approach works only[6] if $1\leq p<d$ and $d\geq 3$. The same restriction occurs as regards the uniqueness issue, although the limit case $p=d$, or $d=2$ and $p\leq 2$ is tractable by taking advantage of a logarithmic interpolation inequality combined with Osgood lemma (see the end of this section).

The second proof consists in rewriting (26) in *Lagrangian coordinates.* Then the density becomes essentially time independent, and one just has to concentrate on the velocity equation that is of parabolic type for small enough time, and can thus be solved by the contracting mapping argument.

3.1. **The classical proof in Eulerian coordinates.** We here present the direct approach for solving (26). Our proof covers only the case $d\geq 3$ and $1\leq p<d$ as regards existence, and $1\leq p\leq d$ with $d\geq 2$ for uniqueness (variations on the method would allow to get existence for the full range $1\leq p<2d$ with $d\geq 2$, though). To simplify the presentation, we assume that λ and μ are density independent so that (26) rewrites

$$\left\{\begin{array}{l}\partial_t a+u\cdot\nabla a=-(1+a)\mathrm{div}\,u,\\ \partial_t u-\mathcal{A}u=-u\cdot\nabla u-I(a)\mathcal{A}u-\nabla(G(a)),\end{array}\right.$$

with $I(a)\overset{\text{def}}{=}\frac{a}{1+a}$ and $G'(a)\overset{\text{def}}{=}\frac{P'(1+a)}{1+a}$.

Furthermore, we suppose that for a small enough constant $c=c(p,d,G)$,

$$\|a_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\leq c. \tag{31}$$

Step 1: An iterative scheme. We set $a_0^n\overset{\text{def}}{=}\dot{S}_n a_0$ and $u_0^n\overset{\text{def}}{=}\dot{S}_n u_0$, and define the first term (a^0,u^0) of the sequence of approximate solutions to be

$$a^0\overset{\text{def}}{=}a_0^0\quad\text{and}\quad u^0\overset{\text{def}}{=}e^{t\mathcal{A}}u_0^0,$$

where $(e^{t\mathcal{A}})_{t\geq 0}$ stands for the semi-group of operators associated to (28).

Next, once (a^n,u^n) has been constructed, we define a^{n+1} and u^{n+1} to be the solutions to the following *linear* transport and Lamé equations:

$$\left\{\begin{array}{l}\partial_t a^{n+1}+u^n\cdot\nabla a^{n+1}=-(1+a^n)\mathrm{div}\,u^n,\\ \partial_t u^{n+1}-\mathcal{A}u^{n+1}=-u^n\cdot\nabla u^n-I(a^n)\mathcal{A}u^n-\nabla(G(a^n)),\end{array}\right. \tag{32}$$

supplemented with initial data a_0^{n+1} and u_0^{n+1}.

[6]Let us emphasize however that one may modify the iterative scheme for constructing solutions, then resort to compactness arguments to get existence for the full range $1\leq p<2d$ and $d\geq 2$.

Step 2: Uniform estimates in the case $1\leq p<2d$ *and* $d\geq 2$. As the data are smooth, it is not difficult to check (by induction) that a^n and u^n are smooth and globally defined. We claim that there exists some $T>0$ such that $(a^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}([0,T];\dot{B}^{\frac{d}{p}}_{p,1})$ and $(u^n)_{n\in\mathbb{N}}$ is bounded in the space $E_p(T)$. Indeed, Theorem 2.2 and the fact that $\dot{B}^{\frac{d}{p}}_{p,1}$ is stable by product imply that for some $C\geq 1$,

$$\|a^{n+1}(T)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\leq\|a_0^{n+1}\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+C\int_0^T(1+\|a^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|\mathrm{div}\,u^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,dt$$
$$+C\int_0^T\|\nabla u^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|a^{n+1}\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,dt.$$

Let $U^n(T)\overset{\text{def}}{=}\|\nabla u^n\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}$. Applying Gronwall's lemma and using the definition of a_0^{n+1}, we thus get

$$\|a^{n+1}\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\leq Ce^{CU^n(T)}\|a_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+(1+\|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})})\Bigl(e^{CU^n(T)}-1\Bigr)\cdot\tag{33}$$

Therefore, assuming that a_0 fulfills (31) with some small enough c, that

$$\|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\leq 4Cc\tag{34}$$

and that

$$CU^n(T)\leq\log(1+c),\tag{35}$$

we conclude from (33) that a^{n+1} also satisfies (34) *for the same* T. At this point, let us observe that as $\dot{B}^{\frac{d}{p}}_{p,1}$ is continuously embedded in L^∞, one may take c so small as

$$\|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\leq 4Cc\quad\text{implies}\quad\|a^n\|_{L^\infty([0,T]\times\mathbb{R}^d)}\leq 1/2.\tag{36}$$

Let us now prove estimates for the velocity. From (30), we get for some constant C depending only on λ and μ,

$$\|u^{n+1}\|_{E_p(T)}\leq C\biggl(\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\int_0^T\|u^n\cdot\nabla u^n+I(a^n)\mathcal{A}u^n+\nabla(G(a^n))\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\,dt\biggr)\cdot$$

The terms in the r.h.s. may be bounded by means of Propositions 2.3 and 2.4 (remembering (36)) if $d\geq 2$ and $1\leq p<2d$. We get for some $C'=C'(p,d,G)$:

$$\|u^{n+1}\|_{E_p(T)}\leq C'\biggl(\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\bigl(\|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}+\|u^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\bigr)\|\nabla u^n\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}$$
$$+T\|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\biggr)\cdot$$

Using (34) and the definition of U^n, this implies that

$$\|u^{n+1}\|_{E_p(T)}\leq C'\bigl(\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+(4Cc+U^n(T))\|u^n\|_{E_p(T)}+4CcT\bigr).$$

Therefore, assuming that (35) is fulfilled and taking smaller c if needed, we get

$$\|u^{n+1}\|_{E_p(T)}\leq\frac{1}{2}\|u^n\|_{E_p(T)}+C'(\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+4CcT),$$

and thus, if

$$\|u^n\|_{E_p(T)} \leq 2C'\big(\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + 4cCT\big) \tag{37}$$

then u^{n+1} also satisfies (37).

To complete the proof, we still have to justify that (35) is fulfilled at rank $n+1$. From the definition of $\|\cdot\|_{E_p(T)}$, embedding and (37) (at rank $n+1$), we know that there exists some constant C'' so that

$$U^{n+1}(T) \leq C''\big(\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + cT\big).$$

Hence there exists a constant $c'>0$ such that if T and u_0 satisfy

$$\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + cT \leq c' \tag{38}$$

then both (35) and (37) are fulfilled at rank $n+1$.

If $\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \geq c'$ then we split u^n into $u^n_L + \widetilde{u}^n$ with $u^n_L(t) \overset{\text{def}}{=} e^{t\mathcal{A}}u^n_0$. Denoting $u_L \overset{\text{def}}{=} e^{t\mathcal{A}}u_0$ and observing that $u^n_L = \dot{S}_n u_L$, we discover that

$$U^n(T) \leq \|\nabla u^n_L\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} + \|\nabla\widetilde{u}^n\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} \leq C\|\nabla u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} + \|\nabla\widetilde{u}^n\|_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

The term with u_L goes to 0 for T tending to 0 with a speed of convergence that may be described according to (19). To handle the second term, we observe that $\widetilde{u}^{n+1}$ satisfies

$$\partial_t\widetilde{u}^{n+1} - \mathcal{A}\widetilde{u}^{n+1} = -\widetilde{u}^n\cdot\nabla u^n - u^n_L\cdot\nabla\widetilde{u}^n - u^n_L\cdot\nabla u^n_L - \nabla(G(a^n)) - I(a^n)\mathcal{A}u^n.$$

Because $\widetilde{u}^{n+1}(0)=0$, combining (30), product laws in Besov spaces and (31), we get

$$\begin{aligned}\|\widetilde{u}^{n+1}\|_{E_p(T)} \leq C\bigg(&\int_0^T \|\widetilde{u}^n\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\nabla u^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,dt + \int_0^T \|u_L\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\widetilde{u}^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,dt\\ &+ \|u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\|u_L\|_{L^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\|u^n\|_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + T\|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\bigg).\end{aligned} \tag{39}$$

Arguing by interpolation yields for any $\beta>0$,

$$\int_0^T \|u_L\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\widetilde{u}^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,dt \leq \beta\|u_L\|_{L^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\widetilde{u}^n\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} + C\beta^{-1}\|u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\|\widetilde{u}^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}.$$

Besides, as $\|u_L\|_{L^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}$, Inequality (39) implies that

$$\begin{aligned}\|\widetilde{u}^{n+1}\|_{E_p(T)} \leq C\bigg(&\big(U^n(T) + \beta\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \beta^{-1}\|u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}})}\big)\|\widetilde{u}^n\|_{E_p(T)}\\ &+\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} + (T + \|u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})})\|a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\bigg).\end{aligned}$$

Choosing $\beta = 1/(4C\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}})$, remembering (34) and (35) (taking c smaller if needed), and that

$$(1+\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}})\|u_L\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} \leq c, \tag{40}$$

we conclude that there exists C''' so that

$$\|\widetilde{u}^{n+1}\|_{E_p(T)} \leq \frac{1}{2}\|\widetilde{u}^{n}\|_{E_p(T)} + C'''c.$$

Hence $\|\widetilde{u}^{n}\|_{E_p(T)} \leq 2C'''c$ implies that $\|\widetilde{u}^{n+1}\|_{E_p(T)} \leq 2C'''c$, and thus (37) is fulfilled at rank $n+1$ if c has been chosen small enough.

Finally, let us notice that there exists some $T>0$ so that

$$\sum_j \Bigl(1-e^{-c_0 2^{2j}T}\Bigr)2^{j(\frac{d}{p}-1)}\|\dot{\Delta}_j u_0\|_{L^p} \leq \frac{c}{1+\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}}\cdot$$

Hence (19) ensures that (40) is satisfied for this choice of T.

Step 3: Convergence in the case $1\leq p<d$ and $d\geq 3$. Let $\delta a^n \overset{\text{def}}{=} a^{n+1}-a^n$ and $\delta u^n \overset{\text{def}}{=} u^{n+1}-u^n$. The couple $(\delta a^{n+1},\delta u^{n+1})$ satisfies

$$\left\{\begin{array}{l} \partial_t \delta a^{n+1} + u^{n+1}\cdot\nabla\delta a^{n+1} = \sum_{i=1}^3 \delta F_i^n,\\ \partial_t \delta u^{n+1} - \mathcal{A}\delta u^{n+1} = \sum_{i=1}^5 \delta G_i^n,\end{array}\right. \tag{41}$$

with $\quad \delta F_1^n \overset{\text{def}}{=} -\delta u^n\cdot\nabla a^{n+1}, \qquad \delta F_2^n \overset{\text{def}}{=} -\delta a^n \,\mathrm{div}\, u^{n+1}, \qquad \delta F_3^n \overset{\text{def}}{=} -(1+a^n)\mathrm{div}\,\delta u^n,$

$$\delta G_1^n \overset{\text{def}}{=} \bigl(I(a^n)-I(a^{n+1})\bigr)\mathcal{A}u^{n+1}, \quad \delta G_2^n \overset{\text{def}}{=} -I(a^n)\,\mathcal{A}\delta u^n, \quad \delta G_3^n \overset{\text{def}}{=} \nabla(G(a^n)-G(a^{n+1})),$$

$$\delta G_4^n \overset{\text{def}}{=} -u^{n+1}\cdot\nabla\delta u^n, \qquad \delta G_5^n \overset{\text{def}}{=} -\delta u^n\cdot\nabla u^n.$$

Owing to the first equation, one can perform estimates for $(\delta a^n,\delta u^n)$ only in a space with one less derivative, namely in $\mathcal{C}([0,T];\dot{B}^{\frac{d}{p}-1}_{p,1})\times F_p(T)$ with

$$F_p(T) \overset{\text{def}}{=} \mathcal{C}([0,T];\dot{B}^{\frac{d}{p}-2}_{p,1})\cap L^1(0,T;\dot{B}^{\frac{d}{p}}_{p,1}).$$

Now, using the same type of computations as in Step 3, we get for all $t\in[0,T]$,

$$\begin{aligned}\|\delta a^{n+1}(t)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \leq \|\delta a^{n+1}(0)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} &+ C\int_0^t \|\nabla u^{n+1}\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\bigl(\|\delta a^n\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|\delta a^{n+1}\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\bigr)\,d\tau\\ &+ \int_0^t \bigl(1+C\|a^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + C\|a^{n+1}\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\bigr)\|\delta u^n\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau.\end{aligned}$$

Using the bounds provided by the previous step, we thus get, taking c smaller if needed,

$$\|\delta a^{n+1}\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq \|\delta a^{n+1}(0)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \frac{1}{8}\|\delta a^n\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + 2\|\delta u^n\|_{F_p(t)}. \tag{42}$$

As in the previous step, bounding δu^{n+1} in $F_p(T)$ follows from (30) and product laws. However, as less regularity is available, one has to make the stronger assumption

$$d\geq 3 \quad\text{and}\quad 1\leq p<d. \tag{43}$$

Taking c smaller if needed, we eventually get thanks to (34), (35) and (37)

$$\|\delta u^{n+1}\|_{F_p(T)} \leq C\|\delta u^{n+1}(0)\|_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + \frac{1}{8}\bigl(\|\delta a^n\|_{L^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\delta u^n\|_{F_p(T)}\bigr).$$

Combining with (42) yields

$$\|\delta a^{n+1}\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+4\|\delta u^{n+1}\|_{F_p(T)}\leq C\big(\|\delta a^{n+1}(0)\|_{\dot B^{\frac dp-1}_{p,1}}+4\|\delta u^{n+1}(0)\|_{\dot B^{\frac dp-2}_{p,1}}\big)\\ +\frac58\big(\|\delta a^n\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}+4\|\delta u^n\|_{F_p(T)}\big).$$

Summing up over $n\in\mathbb{N}$, we conclude that $(a^n-a^0)_{n\in\mathbb{N}}$ and $(u^n-u^0)_{n\in\mathbb{N}}$ converge in $\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1})$ and in $F_p(T)$, respectively.

Step 4: Checking that the limit is a solution and upgrading regularity. From Step 3, we know that there exists a and u so that

$$a^n-a_0\to a-a_0\quad\text{in}\quad L^\infty(0,T;\dot B^{\frac dp-1}_{p,1})\quad\text{and}\quad u^n-u_0\to u-u_0\quad\text{in}\quad F_p(T).$$

The bounds of Step 2 combined with Banach-Alaoglu theorem imply that in addition

$$a^n\rightharpoonup a\ \text{ in }\ L^\infty(0,T;\dot B^{\frac dp}_{p,1})\ \text{ weak }*\quad\text{and}\quad u^n\rightharpoonup u\ \text{ in }\ L^\infty(0,T;\dot B^{\frac dp-1}_{p,1})\ \text{ weak }*.$$

Routine verifications thus allow to pass to the limit in (32).

The previous step tells us that ∇^2u is in $L^1(0,T;\dot B^{\frac dp-2}_{p,1})$. To upgrade the regularity exponent by 1, let us write that for all $J\in\mathbb{N}$:

$$\sum_{|j|\leq J}\int_0^T2^{j(\frac dp-1)}\|\dot\Delta_j\nabla^2u\|_{L^p}\,dt\leq\sum_{|j|\leq J}\int_0^T2^{j(\frac dp-1)}\|\dot\Delta_j\nabla^2u^n\|_{L^p}\,dt\\ +2^J\sum_{|j|\leq J}\int_0^T2^{j(\frac dp-2)}\|\dot\Delta_j\nabla^2u-\dot\Delta_j\nabla^2u^n\|_{L^p}\,dt.$$

The first term is bounded by the r.h.s. of (37) while, by virtue of Step 3, the second one tends to 0 when n goes 0. Hence, letting J tend to $+\infty$ ensures that $\|\nabla^2u\|_{L^1_T(\dot B^{\frac dp-1}_{p,1})}$ is finite.

Next, as (a,u) satisfies (26), Theorem 2.2 and Inequality (30) imply that $a\in\widetilde L^\infty_T(\dot B^{\frac dp}_{p,1})$ and that $u\in\widetilde L^\infty_T(\dot B^{\frac dp-1}_{p,1})$, which, combined with the fact that $a-a_0\in\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1})$ and $u-u_0\in\mathcal{C}([0,T];\dot B^{\frac dp-2}_{p,1})$ implies that $a\in\mathcal{C}([0,T];\dot B^{\frac dp}_{p,1})$ and $u\in\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1})$.

Step 5: Uniqueness. Consider two solutions (a^1,u^1) and (a^2,u^2) of (26) with the above regularity. The difference $(\delta a,\delta u)\overset{\text{def}}{=}(a^2-a^1,u^2-u^1)$ satisfies

$$\left\{\begin{array}{l}\partial_t\delta a+u^2\cdot\nabla\delta a=\sum_{i=1}^3\delta F_i,\\ \partial_t\delta u-\mathcal{A}\delta u=\sum_{i=1}^5\delta G_i,\end{array}\right.\tag{44}$$

with $\quad\delta F_1\overset{\text{def}}{=}-\delta u\cdot\nabla a^1,\qquad\delta F_2\overset{\text{def}}{=}-\delta a\,\mathrm{div}\,u^2,\qquad\delta F_3\overset{\text{def}}{=}-(1+a^1)\mathrm{div}\,\delta u,$

$$\delta G_1\overset{\text{def}}{=}\big(I(a^1)-I(a^2)\big)\mathcal{A}u^2,\quad\delta G_2\overset{\text{def}}{=}-I(a^1)\,\mathcal{A}\delta u,\quad\delta G_3\overset{\text{def}}{=}\nabla(G(a^1)-G(a^2)),$$

$$\delta G_4\overset{\text{def}}{=}-u^2\cdot\nabla\delta u,\qquad\delta G_5\overset{\text{def}}{=}-\delta u\cdot\nabla u^1.$$

Mimicking the computations of Step 3, it is easy to see that if (43) is fulfilled then $(\delta a,\delta u)\equiv0$ in $\mathcal{C}([0,T];\dot B^{\frac dp-1}_{p,1})\times F_p(T)$.

It turns out that the limit case $d=2$ or $p=d$ tractable even though

$$\Bigl(\delta a\in\dot B^0_{d,1}\ \text{ and }\ \mathcal{A}u^2\in\dot B^0_{d,1}\Bigr)\quad\text{implies}\quad\delta a\mathcal{A}u^2\in\dot B^{-1}_{d,\infty}\quad\text{only.}\tag{45}$$

Now, applying Theorem 2.2 and product laws (see Proposition 2.3) gives

$$\|\delta a\|_{L^\infty_T(\dot B^0_{d,\infty})}\leq\Bigl(\|\delta a(0)\|_{\dot B^0_{d,\infty}}+(1+\|a^1\|_{L^\infty_T(\dot B^1_{d,1})})\|\delta u\|_{L^1_T(\dot B^1_{d,1})}\Bigr)e^{\|u^2\|_{L^1_T(\dot B^2_{d,1})}}.$$

Regarding δu, owing to (45), one has to apply Remark 2.2 rather than Theorem 2.1, which enables us to control the following quantity:

$$\|\delta u\|_{\widetilde L^1_T(\dot B^1_{d,\infty})}\stackrel{\text{def}}{=}\sup_j 2^j\|\dot\Delta_j\delta u\|_{L^1_T(L^d)},$$

which is slightly weaker than $\|\delta u\|_{L^1_T(\dot B^1_{d,1})}$.

Inserting the following logarithmic interpolation inequality (see [15]):

$$\|\delta u\|_{L^1_T(\dot B^1_{d,1})}\lesssim\|\delta u\|_{\widetilde L^1_T(\dot B^1_{d,\infty})}\log\biggl(e+\frac{\|\delta u\|_{\widetilde L^1_T(\dot B^0_{d,\infty})}+\|\delta u\|_{\widetilde L^1_T(\dot B^2_{d,\infty})}}{\|\delta u\|_{\widetilde L^1_T(\dot B^1_{d,\infty})}}\biggr)\tag{46}$$

in the estimate for δa and using Osgood lemma (see e.g. [1], Chap. 3), we end up with

$$\|\delta a\|_{L^\infty_t(\dot B^0_{d,\infty})}+\|\delta u\|_{L^\infty_t(\dot B^{-1}_{d,\infty})\cap\widetilde L^1_t(\dot B^1_{d,\infty})}\lesssim\Bigl(\|\delta a(0)\|_{\dot B^0_{d,\infty}}+\|\delta u(0)\|_{\dot B^{-1}_{d,\infty}}\Bigr)^{\exp(-\int_0^t\alpha\,d\tau)}$$

where α is in $L^1(0,T)$ and depends only on the high norms of the two solutions. This yields uniqueness on $[0,T]$.

3.2. **The Lagrangian approach.** We now propose another proof of the local well-posedness of (26), which will provide us with the statement of Theorem 3.1 in its full generality. It is based on the Lagrangian formulation of the system under consideration.

To make it more precise, we need to introduce more notation. First, we agree that for a C^1 function $F:\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^m$ then $\operatorname{div}F:\mathbb{R}^d\to\mathbb{R}^m$ is defined by

$$(\operatorname{div}F)^j\stackrel{\text{def}}{=}\sum_{i=1}^d\partial_iF_{ij}\quad\text{for}\ \ 1\leq j\leq m,$$

and that for $A=(A_{ij})_{1\leq i,j\leq d}$ and $B=(B_{ij})_{1\leq i,j\leq d}$ two $d\times d$ matrices,

$$A:B=\operatorname{Tr}AB=\sum_{i,j}A_{ij}B_{ji}.$$

The notation $\operatorname{adj}(A)$ designates the adjugate matrix that is the transposed cofactor matrix of A. Of course if A is invertible then we have $\operatorname{adj}(A)=(\det A)\,A^{-1}$. Finally, given some matrix A, we define the "twisted" deformation tensor and divergence operator (acting on vector fields z) by the formulae

$$D_A(z)\stackrel{\text{def}}{=}\frac12\bigl(Dz\cdot A+{}^TA\cdot\nabla z\bigr)\quad\text{and}\quad\operatorname{div}_A z\stackrel{\text{def}}{=}{}^TA:\nabla z=Dz:A.$$

We recall the following classical result (see the proof in e.g. [16]).

Lemma 3.1. *Let K be a C^1 scalar function over $\mathbb{R}^d$ and H, a C^1 vector-field. Let X be a C^1 diffeomorphism such that $J \overset{def}{=} \det(D_y X) > 0$. Then we have*

$$\overline{\nabla_x K} = J^{-1}\,\mathrm{div}_y\,(\mathrm{adj}\,(D_y X)\bar{K}), \tag{47}$$

$$\overline{\mathrm{div}_x\, H} = J^{-1}\,\mathrm{div}_y\,(\mathrm{adj}\,(D_y X)\bar{H}). \tag{48}$$

Let X be the flow associated to the vector-field u, that is the solution to

$$X(t,y) = y + \int_0^t u(\tau, X(\tau,y))\,d\tau. \tag{49}$$

Let $\bar{\varrho}(t,y) \overset{\text{def}}{=} \varrho(t,X(t,y))$ and $\bar{u}(t,y) = u(t,X(t,y))$. Formally, we see from the chain rule and Lemma 3.1 above that (ϱ,u) satisfies (4) if and only if $(\bar{\varrho},\bar{u})$ fulfills

$$\begin{cases} \partial_t(J\bar{\varrho}) = 0 \\ \varrho_0\partial_t\bar{u} - \mathrm{div}\Big(\mathrm{adj}\,(DX)\big(2\mu(\bar{\varrho})D_A(\bar{u}) + \lambda(\bar{\varrho})\,\mathrm{div}_A\,\bar{u}\,\mathrm{Id}\, + P(\bar{\varrho})\mathrm{Id}\,\big)\Big) = 0 \end{cases} \tag{50}$$

with $J \overset{\text{def}}{=} \det DX$, $A \overset{\text{def}}{=} (D_y X)^{-1}$ and

$$X(t,y) = y + \int_0^t \bar{u}(\tau,y)\,d\tau. \tag{51}$$

The first equation means that $\varrho = J^{-1}\varrho_0$, and the velocity equation thus recasts in:

$$L_{\varrho_0}(\bar{u}) = \varrho_0^{-1}\mathrm{div}\,\big(I_1(\bar{u},\bar{u}) + I_2(\bar{u},\bar{u}) + I_3(\bar{u},\bar{u}) + I_4(\bar{u})\big)$$

with

$$L_{\varrho_0}(u) \overset{\text{def}}{=} \partial_t u - \varrho_0^{-1}\mathrm{div}\,\big(2\mu(\varrho_0)D(u) + \lambda(\varrho_0)\mathrm{div}\,u\,\mathrm{Id}\,\big) \tag{52}$$

and

$$\begin{aligned}
I_1(v,w) &\overset{\text{def}}{=} (\mathrm{adj}\,(DX_v) - \mathrm{Id}\,)\big(\mu(J_v^{-1}\varrho_0)(Dw\,A_v + {}^TA_v\,\nabla w) + \lambda(J_v^{-1}\varrho_0)({}^TA_v : \nabla w)\mathrm{Id}\,\big)\\
I_2(v,w) &\overset{\text{def}}{=} (\mu(J_v^{-1}\varrho_0) - \mu(\varrho_0))(Dw\,A_v + {}^TA_v\,\nabla w) + (\lambda(J_v^{-1}\varrho_0) - \lambda(\varrho_0))({}^TA_v : \nabla w)\mathrm{Id}\\
I_3(v,w) &\overset{\text{def}}{=} \mu(\varrho_0)\big(Dw(A_v - \mathrm{Id}\,) + {}^T(A_v - \mathrm{Id}\,)\nabla w\big) + \lambda(\varrho_0)({}^T(A_v - \mathrm{Id}\,) : \nabla w)\mathrm{Id}\\
I_4(v) &\overset{\text{def}}{=} -\mathrm{adj}\,(DX_v)P(\varrho_0 J_v^{-1}),
\end{aligned}$$

where X_v is given by (49) with v instead of u, $A_v \overset{\text{def}}{=} (DX_v)^{-1}$ and $J_v \overset{\text{def}}{=} \det DX_v$.

So finally, in order to solve (50) locally, it suffices to show that the map

$$\Phi : v \longmapsto u \tag{53}$$

with u the solution to

$$\begin{cases} L_{\varrho_0}(u) = \varrho_0^{-1}\mathrm{div}\,\big(I_1(v,v) + I_2(v,v) + I_3(v,v) + I_4(v)\big), \\ u|_{t=0} = u_0 \end{cases}$$

has a fixed point in $E_p(T)$ for small enough T.

In order to treat the case where ϱ is just bounded away from zero, we need to generalize (30) to the following Lamé system with *nonconstant* coefficients:

$$\partial_t u - 2a\mathrm{div}\,(\mu D(u)) - b\nabla(\lambda\mathrm{div}\,u) = f, \tag{54}$$

where a, b, λ and μ satisfy the following uniform ellipticity condition:

$$\alpha \overset{\text{def}}{=} \min\Big(\inf_{(t,x)\in[0,T]\times\mathbb{R}^d}(a\mu)(t,x), \inf_{(t,x)\in[0,T]\times\mathbb{R}^d}(2a\mu + b\lambda)(t,x)\Big) > 0. \tag{55}$$

In [16], the following statement has been proved.

Proposition 3.1. *Let a, b, λ and μ be bounded functions satisfying (55). Assume that $a\nabla\mu$, $b\nabla\lambda$, $\mu\nabla a$ and $\lambda\nabla b$ are in $L^\infty(0,T;\dot B^{\frac dp-1}_{p,1})$ for some $1<p<\infty$. There exist $\eta>0$ and $\alpha>0$ such that if for some $m\in\mathbb{Z}$ we have*

$$\min\Big(\inf_{(t,x)\in[0,T]\times\mathbb{R}^d}\dot S_m(2a\mu+b\lambda)(t,x),\inf_{(t,x)\in[0,T]\times\mathbb{R}^d}\dot S_m(a\mu)(t,x)\Big)\geq\frac{\alpha}{2}, \tag{56}$$

$$\|(\mathrm{Id}-\dot S_m)(\mu\nabla a,a\nabla\mu,\lambda\nabla b,b\nabla\lambda)\|_{L^\infty_T(\dot B^{\frac dp-1}_{p,1})}\leq\eta\alpha \tag{57}$$

then the solutions to (54) satisfy for all $t\in[0,T]$,

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot B^s_{p,1})}+\alpha\|u\|_{L^1_t(\dot B^{s+2}_{p,1})}\\&\qquad\leq C\big(\|u_0\|_{\dot B^s_{p,1}}+\|f\|_{L^1_t(\dot B^s_{p,1})}\big)\exp\biggl(\frac{C}{\alpha}\int_0^t\|\dot S_m(\mu\nabla a,a\nabla\mu,\lambda\nabla b,b\nabla\lambda)\|^2_{\dot B^{\frac dp}_{p,1}}\,d\tau\biggr)\end{aligned}$$

whenever $-\min(d/p,d/p')<s\leq d/p-1$.

In order to show that Φ in (53) admits a fixed point in $E_p(T)$, we introduce, as in the previous subsection, the solution u_L in $E_p(T)$ to

$$L_1u_L=0,\qquad u|_{t=0}=u_0.$$

We want to apply Banach fixed point theorem to Φ in some suitable closed ball $\bar B_{E_p(T)}(u_L,R)$. Let v be in $\bar B_{E_p(T)}(u_L,R)$ and $u\overset{\text{def}}{=}\Phi(v)$. Denoting $\widetilde u\overset{\text{def}}{=}u-u_L$, we see that

$$\begin{cases}L_{\varrho_0}\widetilde u=\varrho_0^{-1}\mathrm{div}\,\big(I_1(v,v)+I_2(v,v)+I_3(v,v)+I_4(v)\big)+(L_1-L_{\varrho_0})u_L,\\ \widetilde u|_{t=0}=0.\end{cases} \tag{58}$$

The existence of some $m\in\mathbb{Z}$ so that

$$\min\biggl(\inf_{\mathbb{R}^d}\dot S_m\Big(2\frac{\mu(\varrho_0)}{\varrho_0}+\frac{\lambda(\varrho_0)}{\varrho_0}\Big),\inf_{\mathbb{R}^d}\dot S_m\Big(\frac{\mu(\varrho_0)}{\varrho_0}\Big)\biggr)>\frac{\alpha}{2}$$

$$\text{and}\quad\Big\|(\mathrm{Id}-\dot S_m)\Big(\frac{\mu(\varrho_0)}{\varrho_0^2}\nabla\varrho_0,\frac{\mu'(\varrho_0)}{\varrho_0}\nabla\varrho_0,\frac{\lambda(\varrho_0)}{\varrho_0^2}\nabla\varrho_0,\frac{\lambda'(\varrho_0)}{\varrho_0}\nabla\varrho_0\Big)\Big\|_{\dot B^{\frac dp-1}_{p,1}}\leq\eta\alpha$$

is ensured by the fact that all the coefficients (minus some constant) belong to the space $\dot B^{\frac dp}_{p,1}$ which is defined in terms of a convergent series and embeds continuously in the set of bounded continuous functions. Hence, if one can show that the right-hand side of (58) is in $L^1(0,T;\dot B^{\frac dp-1}_{p,1})$ (which will be carried out in the next step) then we will be allowed to apply Proposition 3.1 to bound $\widetilde u$ in $E_p(T)$.

First step: Stability of $\bar B_{E_p(T)}(u_L,R)$. Let $v\in\bar B_{E_p(T)}(u_L,R)$ and $\widetilde u$ be given by (58). Let $a_0\overset{\text{def}}{=}\varrho_0-1$. Proposition 3.1, product laws in Besov spaces and Proposition 2.4 imply that

$$\begin{aligned}\|\widetilde u\|_{E_p(T)}\leq Ce^{C_{\varrho_0}T}\Big(&\|(L_1-L_{\varrho_0})u_L\|_{L^1_T(\dot B^{\frac dp-1}_{p,1})}\\&+\big(1+\|a_0\|_{\dot B^{\frac dp}_{p,1}}\big)\big(\|I_4(v)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}+\sum_{i=1}^3\|I_i(v,v)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\big)\Big)\end{aligned} \tag{59}$$

for some constant C_{ϱ_0} depending only on ϱ_0.

In what follows, we assume that T and R have been chosen so that, for a small enough positive constant c,

$$\int_0^T \|\nabla v\|_{\dot B^{\frac{d}{p}}_{p,1}}\,dt \leq c. \tag{60}$$

Now, using the decomposition

$$\begin{aligned}(L_1-L_{\varrho_0})u_L = (\varrho_0^{-1}-1)\mathrm{div}\,\big(2\mu(\varrho_0)D(u_L)+\lambda(\varrho_0)\mathrm{div}\,u_L\,\mathrm{Id}\,\big)\\ +\mathrm{div}\,\big(2(\mu(\varrho_0)-\mu(1))D(u)+(\lambda(\varrho_0)-\lambda(1))\mathrm{div}\,u\,\mathrm{Id}\,\big),\end{aligned}$$

and Proposition 2.4, we see that $(L_1-L_{\varrho_0})u_L \in L^1(0,T;\dot B^{\frac{d}{p}-1}_{p,1})$ and

$$\|(L_1-L_{\varrho_0})u_L\|_{L^1_T(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}(1+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}})\|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}. \tag{61}$$

Likewise, flow and composition estimates (see the appendix) ensure that

$$\|I_i(v,w)\|_{L^1_T(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim (1+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}})\|Dv\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}\|Dw\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})} \quad \text{for } i=1,2,3, \tag{62}$$

$$\|I_4(v)\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})} \lesssim T(1+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}})(1+\|Dv\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}). \tag{63}$$

So plugging the above inequalities in (59) and keeping in mind that v satisfies (60), we get after decomposing v into $\widetilde v + u_L$:

$$\begin{aligned}\|\widetilde u\|_{E_p(T)} \leq Ce^{C_{\varrho_0,m}T}(1+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}})^2\Big((T+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})})\\ +\|Du_L\|^2_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}+\big(\|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}+\|D\widetilde v\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}\big)\|D\widetilde v\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}\Big).\end{aligned}$$

Now, because $\widetilde v \in \bar B_{E_p(T)}(0,R)$,

$$\begin{aligned}\|\widetilde u\|_{E_p(T)} \leq Ce^{C_{\varrho_0}T}(1+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}})^2\Big((T+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})})\\ +(R+\|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})})\|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})}+R^2\Big).\end{aligned}$$

Therefore, if we first choose R so that for a small enough constant η,

$$(1+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}})^2 R \leq \eta \tag{64}$$

and then take T so that

$$C_{\varrho_0}T \leq \log 2, \quad T\leq R^2, \quad \|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})} \leq R^2, \quad \|Du_L\|_{L^1_T(\dot B^{\frac{d}{p}}_{p,1})} \leq R, \tag{65}$$

then we may conclude that Φ maps $\bar B_{E_p(T)}(u_L,R)$ into itself.

Second step: contraction estimates. Let us now establish that, under Condition (65), the map Φ is contractive. We consider two vector-fields v^1 and v^2 in $\bar B_{E_p(T)}(u_L,R)$, and set $u^1 \overset{\text{def}}{=} \Phi(v^1)$ and $u^2 \overset{\text{def}}{=} \Phi(v^2)$. Let $\delta u \overset{\text{def}}{=} u^2-u^1$ and $\delta v \overset{\text{def}}{=} v^2-v^1$. We have

$$\begin{aligned}L_{\varrho_0}\delta u = \varrho_0^{-1}\mathrm{div}\Big((I_1(v^2,v^2)-I_1(v^1,v^1))\\ +(I_2(v^2,v^2)-I_2(v^1,v^1))+(I_3(v^2,v^2)-I_3(v^1,v^1))+(I_4(v^2)-I_4(v^1))\Big).\end{aligned}$$

So applying Proposition 3.1 (recall that $C_{\varrho_0}T\leq\log 2$), we get

$$(66)\ \|\delta u\|_{E_p(T)}\leq C(1+\|a_0\|_{\dot B^{\frac dp}_{p,1}})\Big(\sum_{i=1}^3\|I_i(v^2,v^2)-I_i(v^1,v^1)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}+\|I_4(v^2)-I_4(v^1)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\Big).$$

In order to deal with the first term of the right-hand side, we use the decomposition

$$\begin{aligned}I_1(v^2,v^2)-I_1(v^1,v^1)&=\lambda(J_{v^2}^{-1}\varrho_0)\big({}^TA_{v^2}:\nabla v^2\big)\big(\mathrm{adj}\,(DX_{v^2})-\mathrm{adj}\,(DX_{v^1})\big)\\&+\big(\mathrm{adj}\,(DX_{v^1})-\mathrm{Id}\,\big)\big(\lambda(J_{v^2}^{-1}\varrho_0)-\lambda(J_{v^1}^{-1}\varrho_0)\big)\big({}^TA_{v^2}:\nabla v^2\big)\\&+\big(\mathrm{adj}\,(DX_{v^1})-\mathrm{Id}\,\big)\lambda(J_{v^1}^{-1}\varrho_0)\big(({}^TA_{v^2}-{}^TA_{v^1}):\nabla v^1+{}^TA_{v^2}:\nabla\delta v\big)\\&+\text{terms pertaining to }\ \mu.\end{aligned}$$

Taking advantage of product laws in Besov spaces, of Proposition 2.4 and of the flow estimates in the appendix, we deduce that for some constant C_{ϱ_0} depending only on ϱ_0:

$$\|I_1(v^2,v^2)-I_1(v^1,v^1)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\leq C_{\varrho_0}\|(Dv^1,Dv^2)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\|D\delta v\|_{L^1_T(\dot B^{\frac dp}_{p,1})}.$$

Similar estimates may be proved for the next two terms of the right-hand side of (66). Concerning the last one, we use the decomposition

$$I_4(v^2)-I_4(v^1)=\big(\mathrm{adj}\,(DX_{v^1})-\mathrm{adj}\,(DX_{v^2})\big)P(J_{v^2}^{-1}\varrho_0)-\mathrm{adj}\,(DX_{v^1})\big(P(J_{v^2}^{-1}\varrho_0)-P(J_{v^1}^{-1}\varrho_0)\big).$$

Hence

$$\|I_4(v^2)-I_4(v^1)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\leq C(1+\|a_0\|_{\dot B^{\frac dp}_{p,1}})T\|D\delta v\|_{L^1_T(\dot B^{\frac dp}_{p,1})}.$$

We end up with

$$\|\delta u\|_{E_p(T)}\leq C(1+\|a_0\|_{\dot B^{\frac dp}_{p,1}})^2\big(T+\|(Dv^1,Dv^2)\|_{L^1_T(\dot B^{\frac dp}_{p,1})}\big)\|D\delta v\|_{L^1_T(\dot B^{\frac dp}_{p,1})}.$$

Given that v^1 and v^2 are in $\bar B_{E_p(T)}(u_L,R)$, our hypotheses over T and R (with smaller η in (64) if need be) thus ensure that,

$$\|\delta u\|_{E_p(T)}\leq\frac12\|\delta v\|_{E_p(T)}.$$

Hence Φ admits a unique fixed point in $\bar B_{E_p(T)}(u_L,R)$.

Third step: Regularity of the density. Set $\varrho\overset{\text{def}}{=}J_u^{-1}\varrho_0$. By construction (ϱ,u) satisfies (50) and $a\overset{\text{def}}{=}\varrho-1$ is given by

$$a=(J_u^{-1}-1)a_0+a_0.$$

From the appendix, as $Du\in L^1(0,T;\dot B^{\frac dp}_{p,1})$, we have $J_u^{-1}-1$ belongs to $\mathcal{C}([0,T];\dot B^{\frac dp}_{p,1})$. Hence a is in $\mathcal{C}([0,T];\dot B^{\frac dp}_{p,1})$, too. Because $\dot B^{\frac dp}_{p,1}$ is continuously embedded in L^∞, the density remains bounded away from 0 on $[0,T]$ (taking T smaller if needed).

Last step: Uniqueness and continuity of the flow map. Let the data (ϱ_0^1, u_0^1) and (ϱ_0^2, u_0^2) fulfill the assumptions of Theorem 3.1, and let (ϱ^1, u^1) and (ϱ^2, u^2) be the corresponding solutions. Setting $\delta u \overset{\text{def}}{=} u^2 - u^1$, we see that

$$L_{\varrho_0^1}(\delta u) = (L_{\varrho_0^1} - L_{\varrho_0^2})(u^2) + (\varrho_0^1)^{-1}\mathrm{div}\Big(\sum_{j=1}^{3}\big((I_j^2(u^2,u^2) - I_j^2(u^1,u^1)\big) + (I_4^2(u^2) - I_4^2(u^1))\Big)$$
$$+(\varrho_0^1)^{-1}\mathrm{div}\Big(\sum_{j=1}^{3}((I_j^2 - I_j^1)(u^1,u^1) + (I_4^2 - I_4^1)(u^1)\Big),$$

where I_1^i, I_2^i, I_3^i and I_4^i correspond to the quantities that have been defined just above (53), with density ϱ_0^i. Compared to the second step, the only definitely new terms are $(L_{\varrho_0^1} - L_{\varrho_0^2})(u^2)$ and the last line. As regards $(L_{\varrho_0^1} - L_{\varrho_0^2})(u^2)$, we have for $t \leq T$,

$$\|(L_{\varrho_0^1} - L_{\varrho_0^2})(u^2)\|_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})} \leq C_{\varrho_0^1,\varrho_0^2}\|\delta\rho_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\|Du^2\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}.$$

The other new terms satisfy analogous estimates. Hence, applying Proposition 3.1 yields if $\delta\rho_0$ is small enough:

$$\|\delta u\|_{E_p(t)} \leq C_{\varrho_0^1}\big((t + \|Du^1\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})} + \|\delta u\|_{E_p(t)})\|\delta u\|_{E_p(t)}$$
$$+\|\delta u_0\|_{\dot B^{\frac{d}{p}}_{p,1}} + \|\delta\rho_0\|_{\dot B^{\frac{d}{p}}_{p,1}}(t + \|Du^1\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})})\big).$$

An obvious bootstrap argument thus shows that if t, δu_0 and $\delta\rho_0$ are small enough then

$$\|\delta u\|_{E_p(t)} \leq 2C_{\varrho_0}\big(\|\delta u_0\|_{\dot B^{\frac{d}{p}}_{p,1}} + \|\delta\rho_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\big).$$

As regards the density, we have $\delta a = J_{u^1}^{-1}\delta a_0 + (J_{u^2}^{-1} - J_{u^1}^{-1})a_0^2$. Hence for all $t \in [0,T]$,

$$\|\delta a(t)\|_{\dot B^{\frac{d}{p}}_{p,1}} \leq C(1 + \|Du^1\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})})\|\delta a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\|D\delta u\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}.$$

So we get uniqueness and continuity of the flow map on a small time interval. Then iterating the proof yields uniqueness on the initial time interval $[0,T]$, as well as Lipschitz continuity of the flow map.

It is now easy to conclude to Theorem 3.1 in its full generality, as a mere corollary of the following proposition which states the equivalence of Systems (26) and (50) in our functional setting (see the proof in [16]).

Proposition 3.2. *Let $1 \leq p < 2d$. Assume that the couple (ϱ, u) with $(\varrho - 1) \in \mathcal{C}([0,T]; \dot B^{\frac{d}{p}}_{p,1})$ and $u \in E_p(T)$ is a solution to (26) such that*

$$\int_0^T \|\nabla u\|_{\dot B^{\frac{d}{p}}_{p,1}}\,dt \leq c. \tag{67}$$

Let X be the flow of u defined in (49). Then the couple $(\bar\varrho, \bar u) \overset{def}{=} (\varrho \circ X, u \circ X)$ belongs to the same functional space as (ϱ, u), and satisfies (50).

Conversely, if $(\bar\varrho-1,\bar u)$ *belongs to* $\mathcal{C}([0,T];\dot B^{\frac dp}_{p,1})\times E_p(T)$ *and* $(\bar\varrho,\bar u)$ *satisfies* (50) *and, for a small enough constant* c,

$$\int_0^T \|\nabla\bar u\|_{\dot B^{\frac dp}_{p,1}}\,dt\leq c \tag{68}$$

then the map $X_t\overset{def}{=}X(t,\cdot)$ *defined in* (51) *is a* C^1 *diffeomorphism on* $\mathbb{R}^d$ *and the couple* $(\varrho,u)(t)\overset{def}{=}(\bar\varrho(t)\circ X_t^{-1},\bar u(t)\circ X_t^{-1})$ *satisfies* (26) *and has the same regularity as* $(\bar\varrho,\bar u)$.

4. The global existence issue

This section is devoted to the proof of global existence of strong solutions for small perturbations of the constant state $(\varrho,u)=(1,0)$, under the stability assumption $P'(1)>0$. For simplicity, we assume that the viscosity functions λ and μ are constant.

Let us emphasize that the approach we used so far to solve (26) cannot provide us with global-in-time estimates (even if both a_0 and u_0 are small) because we completely ignored the coupling between the mass and momentum equation through the pressure term and looked at it as a low order source term, just writing

$$\partial_t u-\mathcal{A}u=-u\cdot\nabla u-I(a)\mathcal{A}u-\nabla(G(a)).$$

Then, applying Inequality (30) and product laws in Besov spaces led to

$$\|u\|_{E_p(t)}\leq C\biggl(\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\|u\|_{E_p(t)}\bigl(\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}+\|u\|_{E_p(t)}\bigr)+\int_0^t\|a\|_{\dot B^{\frac dp}_{p,1}}\,d\tau\biggr). \tag{69}$$

At the same time, as a is a solution to a transport equation, we can only get bounds on $\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}$ and the last term of (69) is thus out of control for $t\to+\infty$.

4.1. The linearized compressible Navier-Stokes system, and main result.

The key to proving global results is a refined analysis of the linearized system (26) about $(a,u)=(0,0)$ taking the coupling between the mass and momentum equation through the pressure term into account. The system in question reads:

$$\begin{cases}\partial_t a+\operatorname{div}u=f,\\ \partial_t u-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u+P'(1)\nabla a=g.\end{cases} \tag{70}$$

Applying the orthogonal projectors $\mathcal{P}$ and $\mathcal{Q}$ over divergence-free and potential vector-fields, respectively, to the second equation, and setting $\alpha\overset{\text{def}}{=}P'(1)$ and $\nu\overset{\text{def}}{=}\lambda+2\mu$, System (70) translates into

$$\begin{cases}\partial_t a+\operatorname{div}\mathcal{Q}u=f,\\ \partial_t\mathcal{Q}u-\nu\Delta\mathcal{Q}u+\alpha\nabla a=\mathcal{Q}g,\\ \partial_t\mathcal{P}u-\mu\Delta\mathcal{P}u=\mathcal{P}g.\end{cases} \tag{71}$$

We see that $\mathcal{P}u$ satisfies an ordinary heat equation, which is uncoupled from a and $\mathcal{Q}u$. For studying the coupling between a and $\mathcal{Q}u$, it is convenient to set $v\overset{\text{def}}{=}|D|^{-1}\operatorname{div}u$ (with $\mathcal{F}(|D|^s u)(\xi)\overset{\text{def}}{=}|\xi|^s\widehat{u}(\xi)$), keeping in mind that, according to (11), bounding v or $\mathcal{Q}u$ is equivalent, as one can go from v to $\mathcal{Q}u$ or from $\mathcal{Q}u$ to v by means of a 0 order homogeneous Fourier multiplier.

For notational simplicity, we assume from now on that[7] $\alpha=\nu=1$. Hence (a,v) satisfies the following 2×2 system:

$$\left\{\begin{array}{l}\partial_t a+|D|v=f,\\ \partial_t v-\Delta v-|D|a=h\overset{\text{def}}{=}|D|^{-1}\operatorname{div} g.\end{array}\right. \tag{73}$$

Taking the Fourier transform with respect to x, and denoting $\rho\overset{\text{def}}{=}|\xi|$ with $\xi\in\mathbb{R}^d$ the Fourier variable, System (73) translates into

$$\frac{d}{dt}\left(\begin{array}{c}\widehat{a}\\ \widehat{v}\end{array}\right)=M_\rho\left(\begin{array}{c}\widehat{a}\\ \widehat{v}\end{array}\right)+\left(\begin{array}{c}\widehat{f}\\ \widehat{h}\end{array}\right)\quad\text{with}\quad M_\rho\overset{\text{def}}{=}\left(\begin{array}{cc}0&-\rho\\ \rho&-\rho^2\end{array}\right). \tag{74}$$

- In the low frequency regime $\rho<2$, M_ρ has two *complex conjugated* eigenvalues:
$$\lambda_\pm(\rho)\overset{\text{def}}{=}-\frac{\rho^2}{2}(1\pm iS(\rho))\quad\text{with}\quad S(\rho)\overset{\text{def}}{=}\sqrt{\frac{4}{\rho^2}-1}$$
which have real part $-\rho^2/2$, exactly as for the heat equation with diffusion $1/2$.
- In the high frequency regime $\rho>2$, there are two distinct real eigenvalues:
$$\lambda_\pm(\rho)\overset{\text{def}}{=}-\frac{\rho^2}{2}(1\pm R(\rho))\quad\text{with}\quad R(\rho)\overset{\text{def}}{=}\sqrt{1-\frac{4}{\rho^2}}\cdot$$
As $1-R(\rho)\sim 2/\rho^2$ for $\rho\to+\infty$, we have $\lambda_+(\rho)\sim-\rho^2$ and $\lambda_-(\rho)\sim-1\cdot$ In other words, a parabolic and a damped mode coexist.

Optimal a priori estimates may be easily derived by computing the explicit solution of (73) explicitly in the Fourier space. Below, we present an alternative method which is generalizable to much more complicated systems where explicit computations are no longer possible (see e.g. [17]).

Fix some $\rho\geq 0$ and consider the corresponding solution (A,V) of (74) in the case $\widehat{f}=\widehat{h}=0$. We easily get the following three identities:

$$\frac12\frac{d}{dt}|A|^2+\rho\operatorname{Re}(A\bar V)=0, \tag{75}$$

$$\frac12\frac{d}{dt}|V|^2+\rho^2|V|^2-\rho\operatorname{Re}(A\bar V)=0, \tag{76}$$

$$\frac{d}{dt}\operatorname{Re}(A\bar V)+\rho|V|^2-\rho|A|^2+\rho^2\operatorname{Re}(A\bar V)=0, \tag{77}$$

from which we deduce

$$\frac12\frac{d}{dt}\mathcal{L}_\rho^2+\rho^2|(A,V)|^2=0\quad\text{with}\quad\mathcal{L}_\rho^2\overset{\text{def}}{=}2|(A,V)|^2+|\rho A|^2-2\rho\operatorname{Re}(A\bar V). \tag{78}$$

Using Young inequality, we discover that there exists some constant $C_0>0$ independent of ρ so that

$$C_0^{-1}\mathcal{L}_\rho^2\leq|(A,\rho A,V)|^2\leq C_0\mathcal{L}_\rho^2. \tag{79}$$

Combining with (78), we conclude that there exists a universal constant $c_0>0$ so that

$$\mathcal{L}_\rho^2(t)\leq e^{-c_0\min(1,\rho^2)t}\mathcal{L}_\rho^2(0)\quad\text{for all}\quad t\geq 0. \tag{80}$$

[7] Which is not restrictive as the rescaling

$$a(t,x)=\widetilde{a}\big(\tfrac{\alpha}{\nu}t,\tfrac{\sqrt{\alpha}}{\nu}x\big)\quad\text{and}\quad u(t,x)=\sqrt{\alpha}\,\widetilde{u}\big(\tfrac{\alpha}{\nu}t,\tfrac{\sqrt{\alpha}}{\nu}x\big) \tag{72}$$

ensures that $(\widetilde{a},\widetilde{u})$ satisfies (71) with $\alpha=\nu=1$.

In the case of general source terms $\widehat{f}$ and $\widehat{h}$ in (74), using Duhamel formula and applying the above inequality to $A(t)=\widehat{a}(t,\rho)$ and $V(t)=\widehat{v}(t,\rho)$ thus leads to

$$|(\widehat{a},\rho\widehat{a},\widehat{v})(t)|+\min(1,\rho^2)\int_0^t|(\widehat{a},\rho\widehat{a},\widehat{v})|\,d\tau\leq C\biggl(|(\widehat{a}_0,\rho\widehat{a}_0,\widehat{v}_0)|+\int_0^t|(\widehat{f},\rho\widehat{f},\widehat{h})|\,d\tau\biggr). \tag{81}$$

Note that as

$$\partial_t\widehat{v}+\rho^2\widehat{v}=\widehat{h}+\rho\widehat{a},$$

we also have

$$\rho^2\int_0^t|\widehat{v}(\tau)|\,d\tau\leq|\widehat{v}_0|+\int_0^t|\widehat{h}(\tau)|\,d\tau+\int_0^t|\rho\widehat{a}(\tau)|\,d\tau.$$

And thus, bounding the last term according to (81), we get the following inequality which provides the full parabolic smoothing for v:

$$\begin{aligned}|(\widehat{a},\rho\widehat{a},\widehat{v})(t)|+\min(\rho,\rho^2)\int_0^t|\widehat{a}|\,d\tau+\rho^2\int_0^t|\widehat{v}|\,d\tau&\\ \leq C\biggl(|(\widehat{a}_0,\rho\widehat{a}_0,\widehat{v}_0)|+\int_0^t|(\widehat{f},\rho\widehat{f},\widehat{h})|\,d\tau\biggr).&\end{aligned} \tag{82}$$

From Inequality (80) and Fourier-Plancherel theorem, it is easy to obtain estimates of L^2 type for the solutions to (70). Optimal informations will be obtained if splitting the unknowns into frequency packets of comparable sizes. To this end, one may apply $\dot{\Delta}_k$ to (70) and get

$$\left\{\begin{array}{l}\partial_t\dot{\Delta}_k a+\operatorname{div}\dot{\Delta}_k\mathcal{Q}u=\dot{\Delta}_k f,\\ \partial_t\dot{\Delta}_k\mathcal{Q}u-\Delta\dot{\Delta}_k\mathcal{Q}u+\nabla\dot{\Delta}_k a=\dot{\Delta}_k\mathcal{Q}g,\\ \partial_t\dot{\Delta}_k\mathcal{P}u-\mu\Delta\dot{\Delta}_k\mathcal{P}u=\dot{\Delta}_k\mathcal{P}g.\end{array}\right. \tag{83}$$

In the case with no source term then using (80) combined with Fourier-Plancherel theorem readily yields for some universal constant C_0, and c_0 depending only on μ,

$$\|(\dot{\Delta}_k a,\dot{\Delta}_k\nabla a,\dot{\Delta}_k u)(t)\|_{L^2}\leq C_0e^{-c_0\min(1,2^{2k})t}\|(\dot{\Delta}_k a,\dot{\Delta}_k\nabla a,\dot{\Delta}_k u)(0)\|_{L^2}.$$

Then, for general source terms, using Duhamel's formula and repeating the computations leading to (82), we end up with

$$\begin{aligned}\|(\dot{\Delta}_k a,\dot{\Delta}_k\nabla a,\dot{\Delta}_k u)(t)\|_{L^2}+\min(1,2^k)\int_0^t\|\dot{\Delta}_k\nabla a\|_{L^2}\,d\tau+2^{2k}\int_0^t\|\dot{\Delta}_k u\|_{L^2}\,d\tau&\\ \leq C\biggl(\|(\dot{\Delta}_k a_0,\dot{\Delta}_k\nabla a_0,\dot{\Delta}_k u_0)\|_{L^2}+\int_0^t\|(\dot{\Delta}_k f,\dot{\Delta}_k\nabla f,\dot{\Delta}_k g)\|_{L^2}\,d\tau\biggr).&\end{aligned} \tag{84}$$

Multiplying both sides by 2^{ks}, taking the supremum on $[0,t]$ then summing up on $k\geq k_0$ or $k\leq k_0$, we conclude to the following:

Proposition 4.1. *Let $s\in\mathbb{R}$ and (a,u) satisfy (70) with $P'(1)=\nu=1$. Let $k_0\in\mathbb{Z}$. Then we have for some constant C depending only on k_0 and μ, and all $t\geq0$,*

$$\|(a,u)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^s_{2,1})}+\|(a,u)\|^\ell_{L^1_t(\dot{B}^{s+2}_{2,1})}\leq C\bigl(\|(a_0,u_0)\|^\ell_{\dot{B}^s_{2,1}}+\|(f,g)\|^\ell_{L^1_t(\dot{B}^s_{2,1})}\bigr),$$

$$\begin{aligned}\|a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{s+1}_{2,1})}+\|a\|^h_{L^1_t(\dot{B}^{s+1}_{2,1})}+\|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^s_{2,1})}+\|u\|^h_{L^1_t(\dot{B}^{s+2}_{2,1})}&\\ \leq\bigl(\|a_0\|^h_{\dot{B}^{s+1}_{2,1}}+\|u_0\|^h_{\dot{B}^s_{2,1}}+\|f\|^h_{L^1_t(\dot{B}^{s+1}_{2,1})}+\|g\|^h_{L^1_t(\dot{B}^s_{2,1})}\bigr),&\end{aligned}$$

where we used the notation

$$\|z\|^{\ell}_{\dot B^{\sigma}_{p,1}} = \sum_{k\leq k_0} 2^{k\sigma}\|\dot\Delta_k z\|_{L^p} \quad \textit{and} \quad \|z\|^{h}_{\dot B^{\sigma}_{p,1}} = \sum_{k\geq k_0} 2^{k\sigma}\|\dot\Delta_k z\|_{L^p}. \tag{85}$$

The high frequencies inequality means that in order to get optimal estimates, it is suitable to work with the same regularity for ∇a and u. In contrast, for low frequencies, one has to work in the same space for a and u, a fact which does not follow from our rough scaling considerations (5) but is fundamental to keep the pressure term under control in (26).

Granted with the above proposition, it is now natural to look at (26) as System (70) with right-hand side

$$f=-\mathrm{div}\,(au) \quad \text{and} \quad g=-u\cdot\nabla u - I(a)\mathcal{A}u - k(a)\nabla a \quad \text{where} \quad k(a)\overset{\text{def}}{=} G'(a)-G'(0).$$

The problem is that f will cause a loss of one derivative as there is no smoothing effect for a in high frequency. A second limitation of Proposition 4.1 is that it concerns Besov spaces related to L^2 whereas we know the system to be locally well-posed in more general Besov spaces (see Theorem 3.1). To overcome the first problem, let us include the convection terms in our linear analysis, thus considering:

$$\left\{\begin{array}{l} \partial_t a + v\cdot\nabla a + \mathrm{div}\,u = f,\\ \partial_t u + v\cdot\nabla u - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\,u + \alpha\nabla a = g,\end{array}\right. \tag{86}$$

where v stands for a given time-dependent vector field.

Proposition 4.2. *Let $-d/2<s\leq d/2$ and (a,u) satisfy (86) with $\alpha=\nu=1$. Let $k_0\in\mathbb{Z}$. Then we have for some constant C depending only on k_0 and μ, and all $t\geq 0$,*

$$\begin{aligned}\|(a,\nabla a,u)\|_{\widetilde L^\infty_t(\dot B^s_{2,1})} + \|a\|^{\ell}_{L^1_t(\dot B^{s+2}_{2,1})} + \|\nabla a\|^{h}_{L^1_t(\dot B^{s}_{2,1})} + \|u\|_{L^1_t(\dot B^{s+2}_{2,1})}\\ \leq C\biggl(\|(a_0,\nabla a_0,u_0)\|_{\dot B^s_{2,1}} + \|(f,\nabla f,g)\|_{L^1_t(\dot B^s_{2,1})} + \int_0^t \|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|(a,\nabla a,u)\|_{\dot B^s_{2,1}}\,d\tau\biggr).\end{aligned}$$

Proof. Applying $\dot\Delta_k$ to (86) yields:

$$\left\{\begin{array}{l} \partial_t a_k + \dot\Delta_k(v\cdot\nabla a) + \mathrm{div}\,u_k = f_k,\\ \partial_t u_k + \dot\Delta_k(v\cdot\nabla u) - \mu\Delta u_k - (\lambda+\mu)\nabla\mathrm{div}\,u_k + \nabla a_k = g_k,\end{array}\right.$$

with $a_k\overset{\text{def}}{=}\dot\Delta_k a$, $u_k\overset{\text{def}}{=}\dot\Delta_k u$, $f_k\overset{\text{def}}{=}\dot\Delta_k f$ and $g_k\overset{\text{def}}{=}\dot\Delta_k g$.

Keeping in mind the proof of Proposition 4.1, we introduce

$$\mathcal{L}^2_k \overset{\text{def}}{=} 2\|(a_k,u_k)\|^2_{L^2} + \|\nabla a_k\|^2_{L^2} + (u_k\mid\nabla a_k)_{L^2}.$$

Now, remembering that $\lambda+2\mu=1$, we get

$$\begin{aligned}\frac12\frac{d}{dt}\mathcal{L}^2_k + \mu\|\nabla\mathcal{P}u_k\|^2_{L^2} + \|(\nabla\mathcal{Q}u_k,\nabla a_k)\|^2_{L^2} &= \bigl(g_k\mid(2u_k+\nabla a_k)\bigr)_{L^2} + 2(f_k\mid a_k)_{L^2}\\ &+ \bigl(\nabla f_k\mid\nabla a_k\bigr)_{L^2} - 2\bigl(\dot\Delta_k(v\cdot\nabla a)\mid a_k\bigr)_{L^2} - 2\bigl(\dot\Delta_k(v\cdot\nabla u)|u_k\bigr)_{L^2}\\ &\bigl(\dot\Delta_k\nabla(v\cdot\nabla a)\mid\nabla a_k\bigr)_{L^2} - \bigl(\dot\Delta_k(v\cdot\nabla u)\mid\nabla a_k\bigr)_{L^2} - \bigl(\dot\Delta_k\nabla(v\cdot\nabla a)\mid u_k\bigr)_{L^2}.\end{aligned} \tag{87}$$

Let us explain how to bound the convection terms. To handle the second and third terms of the second line, we proceed as explained below, taking $b\in\{a,u^1,\cdots,u^d\}$.

Integrating by parts and setting

$$R_k(v,b) \overset{\text{def}}{=} \dot\Delta_k(v\cdot\nabla b) - v\cdot\nabla\dot\Delta_k b,$$

we discover that

$$\begin{aligned}\big(\dot\Delta_k(v\cdot\nabla b)\mid b_k\big)_{L^2} &= \int (v\cdot\nabla b_k)\cdot b_k\,dx + \int R_k(v,b)\,b_k\,dx\\ &\leq -\frac12\int |b_k|^2\mathrm{div}\,v\,dx + \|R_k(v,b)\|_{L^2}\|b_k\|_{L^2}.\end{aligned}$$

Bounding the last term according to (24), we thus get

$$\big|\big(\dot\Delta_k(v\cdot\nabla b)\mid b_k\big)_{L^2}\big| \leq Cc_k2^{-ks}\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|b\|_{\dot B^s_{2,1}}\|b_k\|_{L^2}$$

with $(c_k)_{k\in\mathbb{Z}}$ in the unit sphere of $\ell^1(\mathbb{Z})$.

Next, we use the fact that for $i\in\{1,\cdots,d\}$,

$$\partial_i\dot\Delta_k(v\cdot\nabla a) = v\cdot\nabla\partial_i a_k + \widetilde R^i_k(v,a) \quad\text{with}\quad \widetilde R^i_k(v,a)\overset{\text{def}}{=}[\partial_i\dot\Delta_k, v]\cdot\nabla a.$$

By adapting the proof of (24), it is easy to prove that

$$\|\widetilde R_k(v,a)\|_{L^2}\leq Cc_k2^{-ks}\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|\nabla a\|_{\dot B^s_{2,1}}.$$

Then using an integration by parts, exactly as above, we conclude that

$$\Big|\big(\dot\Delta_k\nabla(v\cdot\nabla a)|\nabla a_k\big)_{L^2}\Big| \leq Cc_k2^{-ks}\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|\nabla a\|_{\dot B^s_{2,1}}\|\nabla a_k\|_{L^2}.$$

Finally, to handle the last two convection terms, we use the fact that

$$\begin{aligned}&\big(\dot\Delta_k(v\cdot\nabla u)\mid\nabla a_k\big)_{L^2} + \big(\dot\Delta_k\nabla(v\cdot\nabla a)\mid u_k\big)_{L^2}\\ &\qquad= \big(v\cdot\nabla u_k|\nabla a_k\big)_{L^2} + \big((v\cdot\nabla)\nabla a_k)|u_k\big)_{L^2} + \big(R_k(v,u)\mid\nabla a_k\big)_{L^2} + \big(\widetilde R_k(v,a)\mid u_k\big)_{L^2}.\end{aligned}$$

Integrating by parts in the first two terms of the second line and using again (24) to bound the last two terms eventually leads to

$$\begin{aligned}&\Big|\big(\dot\Delta_k(v\cdot\nabla u)\mid\nabla a_k\big)_{L^2} + \big(\dot\Delta_k\nabla(v\cdot\nabla a)\mid u_k\big)_{L^2}\Big|\\ &\qquad\leq Cc_k2^{-ks}\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\big(\|\nabla a\|_{\dot B^s_{2,1}}\|u_k\|_{L^2} + \|u\|_{\dot B^s_{2,1}}\|\nabla a_k\|_{L^2}\big).\end{aligned}$$

Because $\mathcal{L}_k\approx\|(a_k,\nabla a_k,u_k)\|_{L^2}$, we thus conclude that

$$\begin{aligned}(88)\quad &\frac12\frac{d}{dt}\mathcal{L}_k^2 + \mu\|\nabla\mathcal{P}u_k\|_{L^2}^2 + \|(\nabla\mathcal{Q}u_k,\nabla a_k)\|_{L^2}^2\\ &\qquad\leq\big(\|(f_k,\nabla f_k,g_k)\|_{L^2} + Cc_k2^{-ks}\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|(a,\nabla a,u)\|_{\dot B^s_{2,1}}\big)\mathcal{L}_k,\end{aligned}$$

which after time integration and multiplication by 2^{ks} yields

$$\begin{aligned}2^{ks}\mathcal{L}_k(t) + c_02^{ks}\min(1,2^{2k})\int_0^t\|(a_k,\nabla a_k,u_k)\|_{L^2}\,d\tau &\leq 2^{ks}\mathcal{L}_k(0) + \int_0^t 2^{ks}\|g_k\|_{L^2}\,d\tau\\ &\quad+\int_0^t c_k\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|(a,\nabla a,u)\|_{\dot B^s_{2,1}}\,d\tau.\end{aligned}$$

Taking the supremum on $[0,t]$ then summing up over k, we thus get

$$\|(a,\nabla a,u)\|_{\widetilde{L}^\infty_t(\dot B^s_{2,1})}+\int_0^t\|(a,u)\|^\ell_{\dot B^{s+2}_{2,1}}\,d\tau+\int_0^t\|(\nabla a,u)\|^h_{\dot B^s_{2,1}}\,d\tau \tag{89}$$
$$\lesssim\|(a,\nabla a,u)(0)\|_{\dot B^s_{2,1}}+\int_0^t\|(f,\nabla f,g)\|_{\dot B^s_{2,1}}\,d\tau+\int_0^t\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|(a,\nabla a,u)\|_{\dot B^s_{2,1}}\,d\tau.$$

Finally, using the fact that

$$\partial_t u+v\cdot\nabla u-\mu\Delta u-(\lambda+\mu)\nabla\mathrm{div}\,u=g-\nabla a,$$

localizing according to $\dot\Delta_k$, and arguing as above, we find out that

$$\|u\|_{\widetilde{L}^\infty_t(\dot B^s_{2,1})}+\int_0^t\|u\|_{\dot B^{s+2}_{2,1}}\lesssim\|u(0)\|_{\dot B^s_{2,1}}+\int_0^t\|g-\nabla a\|_{\dot B^s_{2,1}}\,d\tau+\int_0^t\|\nabla v\|_{\dot B^{\frac d2}_{2,1}}\|u\|_{\dot B^s_{2,1}}\,d\tau.$$

Then bounding ∇a according to (89) completes the proof of the proposition. □ □

It turns out to be possible to extend the above proposition to more general Besov spaces related to the L^p spaces with $p\neq 2$. The proof relies on a paralinearized version of System (86) combined with a Lagrangian change of variables (see [4, 7]). Here, in order to solve (26) globally, we shall follow a more elementary approach based on the paper by B. Haspot [22] : we use Proposition 4.1 only for bounding low frequencies, and perform a suitable quasi-diagonalization of the system to handle high frequencies. This eventually leads to the following statement[8] that will be proved in the rest of this section:

Theorem 4.1. *Let $d\geq 2$. Let $p\in[2,\min(4,2d/(d-2)]$ with, additionally, $p\neq 4$ if $d=2$. Assume with no loss of generality that $P'(1)=1$ and $\nu=1$. There exists a universal integer $k_0\in\mathbb{N}$ and a small constant $c=c(p,d,\mu,G)$ such that if $a_0\in\dot B^{\frac dp}_{p,1}$ and $u_0\in\dot B^{\frac dp-1}_{p,1}$ with besides (a_0^ℓ,u_0^ℓ) in $\dot B^{\frac d2-1}_{2,1}$ (with the notation $z^\ell=\dot S_{k_0+1}z$ and $z^h=z-z^\ell$) satisfy*

$$X_{p,0}\overset{def}{=}\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\|u_0\|^h_{\dot B^{\frac dp-1}_{p,1}}\leq c \tag{90}$$

then (26) has a unique global-in-time solution (a,u) in the space X_p defined by

$$(a,u)^\ell\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{\frac d2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac d2+1}_{2,1}),\quad a^h\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{\frac dp}_{p,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac dp}_{p,1}),$$
$$u^h\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})\cap L^1(\mathbb{R}_+;\dot B^{\frac dp+1}_{p,1})$$

where we agree that $\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^s_{q,1})\overset{def}{=}\mathcal{C}(\mathbb{R}_+;\dot B^s_{q,1})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot B^s_{q,1})$, $s\in\mathbb{R}$, $1\leq q\leq\infty$.

Furthermore, we have for some constant $C=C(p,d,\mu,G)$,

$$\|(a,u)\|_{X_p}\leq CX_{p,0}. \tag{91}$$

Remark 4.1. *Condition (90) is satisfied for small a_0 and* large *highly oscillating velocities: take $u_0^\varepsilon:x\mapsto\phi(x)\sin(\varepsilon^{-1}x\cdot\omega)\,n$ with ω and n in $\mathbb{S}^{d-1}$ and $\phi\in\mathcal{S}(\mathbb{R}^d)$. Then*

$$\|u_0^\varepsilon\|_{\dot B^{\frac dp-1}_{p,1}}\leq C\varepsilon^{1-\frac dp}\quad\text{if }\ p>d,$$

[8]The reader may refer to [18] for a slightly more general result.

and $\|u_0^\varepsilon\|^\ell_{\dot B^{\frac d2-1}_{2,1}}$ *has fast decay with respect to* ε*. Hence such data with small enough* ε *generate global unique solutions in dimension* $d=2,3$*.*

Remark 4.2. *One may extend the above global result to* $2d/(d+2)\leq p<2$ *provided the following smallness condition is fulfilled:*

$$\|a_0\|_{\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}}+\|u_0\|_{\dot B^{\frac d2-1}_{2,1}}\leq\eta.$$

Indeed, Theorem 5.1 provides a global small solution in X_2*. Therefore it is only a matter of checking that the constructed solution has additional regularity* X_p*. This may be achieved by following Steps 3 and 4 of the proof below, knowing already that the solution is in* X_2*. The condition that* $2d/(d+2)\leq p$ *comes from the part* $u^\ell\cdot\nabla a$ *of the convection term in the mass equation, as* ∇u^ℓ *is only in* $L^1(\mathbb{R}_+;\dot B^{\frac d2}_{2,1})$*, and the regularity to be transported is* $\dot B^{\frac dp}_{p,1}$*. Hence we need to have* $d/p\leq d/2+1$ *(see Theorem 2.2). The same condition appears when handling* $k(a)\nabla a$*.*

Remark 4.3. *Using space* $\widetilde C_b(\mathbb{R}_+;\dot B^s_{2,1})$ *rather than just* $C_b(\mathbb{R}_+;\dot B^s_{2,1})$ *is not essential in the proof of Theorem 4.1. We chose to present that slightly more accurate result, as it will be needed when investigating time decay estimates, at the end of the survey.*

4.2. **Global a priori estimates.** Consider a smooth solution (a,u) to (26) satisfying, say,

$$\|a\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)}\leq 1/2. \tag{92}$$

We want to find conditions under which the following quantity:

$$\begin{aligned}X_p(t)\overset{\text{def}}{=}\|(a,u)\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2-1}_{2,1})}+\|(a,u)\|^\ell_{L^1_t(\dot B^{\frac d2+1}_{2,1})}\\ +\|a\|^h_{\widetilde L^\infty_t(\dot B^{\frac dp}_{p,1})}+\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}+\|u\|^h_{\widetilde L^\infty_t(\dot B^{\frac dp-1}_{p,1})}+\|u\|^h_{L^1_t(\dot B^{\frac dp+1}_{p,1})}\end{aligned}$$

satisfies (91) for all $t\in\mathbb{R}_+$.

Rewriting System (26) as follows:

$$\begin{cases}\partial_t a+\operatorname{div}u=f\overset{\text{def}}{=}-\operatorname{div}(au),\\ \partial_t u-\mathcal{A}u+\nabla a=g\overset{\text{def}}{=}-u\cdot\nabla u-I(a)\mathcal{A}u-k(a)\nabla a,\end{cases}$$

we shall take advantage of Proposition 4.1 with $s'=d/2-1$ to bound the low frequency part of (a,u). To handle high frequencies, following [22], we shall use the facts that, up to low order terms:

- $\mathcal{P}u$ satisfies a heat equation (hence parabolic smoothing in any Besov space);
- The *effective velocity*

$$w\overset{\text{def}}{=}\nabla(-\Delta)^{-1}(a-\operatorname{div}u) \tag{93}$$

  satisfies a heat equation;
- The high frequencies of a have exponential decay.

First step: Low frequencies. From Proposition 4.1, we readily infer that

$$\|(a,u)\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2-1}_{2,1})}+\|(a,u)\|^\ell_{L^1_t(\dot B^{\frac d2+1}_{2,1})}\lesssim\|(a_0,u_0)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|(f,g)\|^\ell_{L^1_t(\dot B^{\frac d2-1}_{2,1})}. \tag{94}$$

Second step: high frequencies, the incompressible part of the velocity. To handle $\mathcal{P}u$, we just use the fact that

$$\partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = \mathcal{P}g.$$

Hence, according to Remark 2.2 (restricted to high frequencies)

$$\|\mathcal{P}u\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} + \mu\|\mathcal{P}u\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})} \leq C\big(\|\mathcal{P}u_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} + \|\mathcal{P}g\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\big). \tag{95}$$

Third step: high frequencies, the effective velocity and the density. On the one hand, the effective velocity w defined in (93) fulfills

$$\partial_t w - \Delta w = \nabla(-\Delta)^{-1}(f - \operatorname{div} g) + w - (-\Delta)^{-1}\nabla a.$$

Therefore, Theorem 2.1 and the fact that $\nabla(-\Delta)^{-1}$ is an homogeneous Fourier multiplier of degree -1 imply that

$$\begin{aligned}\|w\|^h_{\widetilde{E}_p(t)} \overset{\text{def}}{=} \|w\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} + \|w\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})} &\leq C\big(\|w_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} \\ &\quad + \|f - \operatorname{div} g\|^h_{L^1_t(\dot B^{\frac{d}{p}-2}_{p,1})} + \|w - (-\Delta)^{-1}\nabla a\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\big).\end{aligned} \tag{96}$$

On the other hand, we have

$$\partial_t a + \operatorname{div}(au) + a = -\operatorname{div} w. \tag{97}$$

We claim that

$$\|a\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{p}}_{p,1})} + \|a\|^h_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})} \leq C\bigg(\|a_0\|^h_{\dot B^{\frac{d}{p}}_{p,1}} + \|\operatorname{div} w\|^h_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})} + \int_0^t \|\nabla u\|_{\dot B^{\frac{d}{p}}_{p,1}} \|a\|_{\dot B^{\frac{d}{p}}_{p,1}}\, d\tau\bigg). \tag{98}$$

Indeed, as in the proof of Theorem 2.2, let us apply $\dot\Delta_k$ to (97). We get

$$\partial_t\dot\Delta_k a + u\cdot\nabla\dot\Delta_k a + \dot\Delta_k a = -\dot\Delta_k(a\operatorname{div} u) - \dot\Delta_k\operatorname{div} w + \dot R_k,$$

where, according to (24), the remainder term $\dot R_k$ satisfies:

$$\forall k\in\mathbb{Z},\ \|\dot R_k\|_{L^p} \leq Cc_k 2^{-k\frac{d}{p}}\|\nabla u\|_{\dot B^{\frac{d}{p}}_{p,1}} \|a\|_{\dot B^{\frac{d}{p}}_{p,1}} \quad\text{with}\quad \sum_{k\in\mathbb{Z}} c_k = 1$$

and

$$\|a\operatorname{div} u\|_{\dot B^{\frac{d}{p}}_{p,1}} \leq C\|\operatorname{div} u\|_{\dot B^{\frac{d}{p}}_{p,1}} \|a\|_{\dot B^{\frac{d}{p}}_{p,1}}.$$

Therefore evaluating the L^p norm of $\dot\Delta_k a$ seen as the solution to a transport equation, multiplying by $2^{k\frac{d}{p}}$ and summing up over $k\geq k_0$ yields (98).

Next, let us observe that, owing to the high frequency cut-off, we have for some universal constant C,

$$\|w\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} \leq C2^{-2k_0}\|w\|^h_{\dot B^{\frac{d}{p}+1}_{p,1}} \quad\text{and}\quad \|(-\Delta)^{-1}\nabla a\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} \leq C2^{-2k_0}\|a\|^h_{\dot B^{\frac{d}{p}}_{p,1}}. \tag{99}$$

In consequence, combining (96) and (98), and choosing k_0 large enough yields

$$\|w\|^h_{\widetilde{E}_p(t)}+\|a(t)\|^h_{\dot B^{\frac dp}_{p,1}}+\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}\leq\biggl(\|w_0\|^h_{\dot B^{\frac dp-1}_{p,1}}+\|a_0\|^h_{\dot B^{\frac dp}_{p,1}}+\|f-\operatorname{div}g\|^h_{L^1_t(\dot B^{\frac dp-2}_{p,1})}+\int_0^t\|\nabla u\|_{\dot B^{\frac dp}_{p,1}}\|a\|_{\dot B^{\frac dp}_{p,1}}\,d\tau\biggr)\cdot \tag{100}$$

Fourth step: end of the proof of the linear estimate. Putting Inequality (100) together with (94) and (95) and observing that

$$\|u\|^h_{\widetilde{E}_p(t)}\leq\|\mathcal{P}u\|^h_{E_p(t)}+\|w\|^h_{\widetilde{E}_p(t)}+C\bigl(\|a\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac dp-2}_{p,1})}+\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}\bigr),$$

we come to the conclusion (if k_0 has been taken large enough) that

$$X_p(t)\leq C_{k_0}\biggl(X_p(0)+\int_0^t\bigl(\|(f,g)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|f\|^h_{\dot B^{\frac dp-2}_{p,1}}+\|g\|^h_{\dot B^{\frac dp-1}_{p,1}}+\|\nabla u\|_{\dot B^{\frac dp}_{p,1}}\|a\|_{\dot B^{\frac dp}_{p,1}}\bigr)\,d\tau\biggr)\cdot$$

Fifth step : nonlinear estimates. It is only a matter of proving that under hypothesis (92), we have

$$\int_0^t\bigl(\|(f,g)\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|f\|^h_{\dot B^{\frac dp-2}_{p,1}}+\|g\|^h_{\dot B^{\frac dp-1}_{p,1}}+\|\nabla u\|_{\dot B^{\frac dp}_{p,1}}\|a\|_{\dot B^{\frac dp}_{p,1}}\bigr)\,d\tau\leq CX_p^2(t). \tag{101}$$

As $p\geq2$, it is clear that the last term of the r.h.s. of (101) is bounded by $CX_p^2(t)$. Next, arguing exactly as in the proof of the local existence, we easily get for $1\leq p<2d$,

$$\|f\|_{L^1_t(\dot B^{\frac dp-1}_{p,1})}\leq C\|a\|_{L^2_t(\dot B^{\frac dp}_{p,1})}\|u\|_{L^2_t(\dot B^{\frac dp}_{p,1})},$$

$$\|g\|_{L^1_t(\dot B^{\frac dp-1}_{p,1})}\leq C\bigl(\|u\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}\|\nabla u\|_{L^1_t(\dot B^{\frac dp}_{p,1})}+\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\|\nabla^2u\|_{L^1_t(\dot B^{\frac dp-1}_{p,1})}+\|a\|_{L^2_t(\dot B^{\frac dp}_{p,1})}\|\nabla a\|_{L^2_t(\dot B^{\frac dp-1}_{p,1})}\bigr)\cdot$$

Therefore, using the definition of $X_p(t)$ and embedding (recall that $p\geq2$), we get

$$\|(f,g)\|^h_{L^1_t(\dot B^{\frac dp-1}_{p,1})}\leq CX_p^2(t).$$

So we are left with the proof of

$$\|(f,g)\|^\ell_{L^1_t(\dot B^{\frac d2-1}_{2,1})}\leq CX_p^2(t). \tag{102}$$

Let us admit the following two inequalities (the first one being proved in [18] and the second one being a particular case of Proposition 2.3 followed by suitable embedding, owing to $1\leq p/2\leq2$) :

$$\|T_ab\|_{\dot B^{s-1+\frac d2-\frac dp}_{2,1}}\leq C\|a\|_{\dot B^{\frac dp-1}_{p,1}}\|b\|_{\dot B^s_{p,1}}\quad\text{if }\ d\geq2\ \text{ and }\ \tfrac{d}{d-1}\leq p\leq\min\bigl(4,\tfrac{2d}{d-2}\bigr), \tag{103}$$

$$\|R(a,b)\|_{\dot B^{s-1+\frac d2-\frac dp}_{2,1}}\leq C\|a\|_{\dot B^{\frac dp-1}_{p,1}}\|b\|_{\dot B^s_{p,1}}\quad\text{if }\ s>1-\min\bigl(\tfrac dp,\tfrac{d}{p'}\bigr)\ \text{ and }\ 1\leq p\leq4. \tag{104}$$

In order to prove (102) for f, it suffices to bound $(au)^\ell$ in $L^1(0,t;\dot B^{\frac d2}_{2,1})$. Now, using Bony's decomposition and the fact that $a=a^\ell+a^h$, we see that

$$(au)^\ell=\big(T_au\big)^\ell+\big(R(a,u)\big)^\ell+\big(T_ua^\ell\big)^\ell+\big(T_ua^h\big)^\ell. \tag{105}$$

The first three terms may be bounded thanks to Prop. 2.3 and Inequalities (103), (104) with $s=\frac dp-1$. Observing that $\|z\|^\ell_{\dot B^\sigma_{r,1}}\leq C\|z\|_{\dot B^\sigma_{r,1}}$ for any Besov norm, we get

$$\begin{aligned}
&\|(T_au)^\ell\|_{L^1_t(\dot B^{\frac d2}_{2,1})}\leq C\|a\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}\|u\|_{L^1_t(\dot B^{\frac dp+1}_{p,1})},\\
&\|(R(a,u))^\ell\|_{L^1_t(\dot B^{\frac d2}_{2,1})}\leq C\|a\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}\|u\|_{L^1_t(\dot B^{\frac dp+1}_{p,1})},\\
&\|(T_ua^\ell)^\ell\|_{L^1_t(\dot B^{\frac d2}_{2,1})}\leq C\|u\|_{L^\infty_t(\dot B^{-1}_{\infty,\infty})}\|a^\ell\|_{L^1_t(\dot B^{\frac d2+1}_{2,1})}.
\end{aligned}$$

Because $\dot B^{\frac dp-1}_{p,1}$ is embedded in $\dot B^{-1}_{\infty,\infty}$, the above right-hand sides may be bounded by $CX_p^2(t)$. To handle the last term of (105), we just have to observe that owing to the spectral cut-off, there exists a universal integer N_0 so that

$$\big(T_ua^h\big)^\ell=\dot S_{k_0+1}\Big(\sum_{|k-k_0|\leq N_0}\dot S_{k-1}u\,\dot\Delta_ka^h\Big)\cdot$$

Hence $\|T_ua^h\|_{\dot B^{\frac d2}_{2,1}}\approx 2^{k_0\frac d2}\sum_{|k-k_0|\leq N_0}\|\dot S_{k-1}u\,\dot\Delta_ka^h\|_{L^2}$. Now, if $2\leq p\leq\min(d,2d/(d-2))$ then we may use for $|k-k_0|\leq N_0$

$$2^{k_0\frac d2}\|\dot S_{k-1}u\,\dot\Delta_ka^h\|_{L^2}\leq C2^{k_0}\,\|\dot S_{k-1}u\|_{L^d}\,\big(2^{k\frac dp}\|\dot\Delta_ka^h\|_{L^p}\big),$$

and if $d\leq p\leq 4$ then

$$2^{k_0\frac d2}\|\dot S_{k-1}u\,\dot\Delta_ka^h\|_{L^2}\leq C2^{k_0}\big(2^{k(\frac dp-1)}\|\dot S_{k-1}u\|_{L^p}\big)\big(2^{k\frac dp}\|\dot\Delta_ka^h\|_{L^p}\big).$$

Hence one may conclude that f satisfies (102). Bounding g is similar (see [18]).

Last step: Global estimate. Putting all the previous estimates together, we get

$$X_p(t)\leq C\big(X_p(0)+X_p^2(t)\big). \tag{106}$$

Now it is clear that as long as

$$2CX_p(t)\leq 1, \tag{107}$$

Inequality (106) ensures that

$$X_p(t)\leq 2CX_p(0). \tag{108}$$

Using a bootstrap argument, one may conclude that if $X_p(0)$ is small enough then (92) and (107) are satisfied as long as the solution exists. Hence (108) holds globally in time.

4.2.1. *The proof of Theorem 4.1.* We just give the important steps. We fix some initial data so that X_0 is small enough. First, Theorem 3.1 implies that there exists a unique maximal solution (a,u) to (26) on some time interval $[0,T^*[$, with $a\in\mathcal{C}([0,T^*);\dot B^{\frac dp-1}_{p,1})$, $\|a\|_{L^\infty(0,T^*\times\mathbb{R}^d)}\leq 1/2$ and $u\in\mathcal{C}([0,T^*);\dot B^{\frac dp-1}_{p,1})\cap L^1_{loc}(0,T^*;\dot B^{\frac dp+1}_{p,1})$. From (26) and Proposition 4.1, one may check that the additional low frequency information is preserved on $[0,T^*)$: we have

$$a^\ell\in\mathcal{C}([0,T^*);\dot B^{\frac d2-1}_{2,1})\cap L^1(0,T^*;\dot B^{\frac d2+1}_{2,1})\quad\text{and}\quad u^\ell\in\mathcal{C}([0,T^*);\dot B^{\frac d2-1}_{2,1})\cap L^1_{loc}(0,T^*;\dot B^{\frac d2+1}_{2,1}).$$

Let us assume (by contradiction) that $T^*<\infty$. Then applying (108) for all $t<T^*$ yields

$$\|a\|_{\widetilde L^\infty_{T^*}(\dot B^{\frac dp}_{p,1})}+\|u\|_{\widetilde L^\infty_{T^*}(\dot B^{\frac dp-1}_{p,1})}\leq CX_0.$$

If X_0 is so small as (108) to imply that both (31) and (67) are fulfilled on $[0,T^*)$ then, for all $t_0\in[0,T^*)$, one can solve (26) starting with data $(a(t_0),u(t_0))$ at time $t=t_0$ and get a solution according to Theorem 3.1 on the interval $[t_0,T+t_0]$ with T *independent of* t_0. Choosing $t_0>T^*-T$ thus shows that the solution can be continued beyond T^*, a contradiction. □

5. Asymptotic results

In this section, we focus on two types of asymptotic issues for small global solutions to (4) that received a lot of attention since the eighties: the low Mach number asymptotic, and the long time behavior. We shall see that essentially optimal results may be obtained by very simple arguments from the global result we established in the previous section.

5.1. **The low Mach number limit.** This subsection is devoted to the rigorous justification of the convergence of (4) to the incompressible Navier-Stokes equations

$$\left\{\begin{array}{l}\partial_t u+u\cdot\nabla u-\mu\Delta u+\nabla\Pi=0,\\ \operatorname{div}u=0,\end{array}\right.\tag{109}$$

in the so-called *ill-prepared* data case, where we only assume that $\varepsilon^{-1}(\rho_0^\varepsilon-1)$ and u_0^ε are suitably bounded. In particular, if we set $a^\varepsilon\overset{\text{def}}{=}\varepsilon^{-1}(\rho^\varepsilon-1)$, this means that $(\partial_t a^\varepsilon,\partial_t u^\varepsilon)|_{t=0}$ is of order $1/\varepsilon$, and that one cannot exclude highly oscillating acoustic waves. More concretely, we have to pass to the limit $\varepsilon\to0$ in:

$$\left\{\begin{array}{l}\partial_t a^\varepsilon+\dfrac{\operatorname{div}u^\varepsilon}{\varepsilon}=-\operatorname{div}(a^\varepsilon u^\varepsilon),\\ \partial_t u^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon-\dfrac{\mathcal{A}u^\varepsilon}{1+\varepsilon a^\varepsilon}+\dfrac{\nabla a^\varepsilon}{\varepsilon}=\dfrac{k(\varepsilon a^\varepsilon)}{\varepsilon}\nabla a^\varepsilon\\ \qquad\qquad+\dfrac{1}{1+\varepsilon a^\varepsilon}\operatorname{div}\big(2\widetilde\mu(\varepsilon a^\varepsilon)D(u^\varepsilon)+\widetilde\lambda(\varepsilon a^\varepsilon)\operatorname{div}u^\varepsilon\mathrm{Id}\big).\end{array}\right.\tag{110}$$

Before stating our main results, let us introduce some notation. In this section, we agree that for $z\in\mathcal{S}'(\mathbb{R}^d)$,

$$z^{\ell,\beta}\overset{\text{def}}{=}\sum_{2^j\beta\leq2^{j_0}}\dot\Delta_j z\quad\text{and}\quad z^{h,\beta}\overset{\text{def}}{=}\sum_{2^j\beta>2^{j_0}}\dot\Delta_j z,\tag{111}$$

for some large enough nonnegative integer j_0 depending only on p, d, and on the functions k, λ/ν, μ/ν with $\nu \overset{\text{def}}{=} \lambda + 2\mu$. The corresponding "truncated" semi-norms are defined by

$$\|z\|^{\ell,\beta}_{\dot B^\sigma_{p,r}} \overset{\text{def}}{=} \|z^{\ell,\beta}\|_{\dot B^\sigma_{p,r}} \quad \text{and} \quad \|z\|^{h,\beta}_{\dot B^\sigma_{p,r}} \overset{\text{def}}{=} \|z^{h,\beta}\|_{\dot B^\sigma_{p,r}}. \tag{112}$$

Keeping in mind the linear analysis we performed for (70) in the case $\nu = 1$ and $\varepsilon = 1$, and combining with the change of variable

$$(a,u)(t,x) \overset{\text{def}}{=} \varepsilon(a^\varepsilon, u^\varepsilon)(\varepsilon^2\nu t, \varepsilon\nu x), \tag{113}$$

we expect the threshold between low and high frequencies to be at $1/\widetilde\varepsilon$ with $\widetilde\varepsilon \overset{\text{def}}{=} \varepsilon\nu$, and it is thus natural to consider families of data $(a_0^\varepsilon, u_0^\varepsilon)$ such that

$$\|(a_0^\varepsilon, u_0^\varepsilon)\|^{\ell,\widetilde\varepsilon}_{\dot B^{\frac d2-1}_{2,1}} + \|a_0^\varepsilon\|^{h,\widetilde\varepsilon}_{\dot B^{\frac dp}_{p,1}} + \|u_0^\varepsilon\|^{h,\widetilde\varepsilon}_{\dot B^{\frac dp-1}_{p,1}}$$

is bounded independently of ε. We expect the corresponding solutions of (110) to be uniformly in the space $X^p_{\varepsilon,\nu}$ defined by

- $(a^{\ell,\widetilde\varepsilon}, u^{\ell,\widetilde\varepsilon}) \in \widetilde{\mathcal C}_b(\mathbb{R}_+; \dot B^{\frac d2-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot B^{\frac d2+1}_{2,1})$,
- $a^{h,\widetilde\varepsilon} \in \widetilde{\mathcal C}_b(\mathbb{R}_+; \dot B^{\frac dp}_{p,1}) \cap L^1(\mathbb{R}_+; \dot B^{\frac dp}_{p,1})$,
- $u^{h,\widetilde\varepsilon} \in \widetilde{\mathcal C}_b(\mathbb{R}_+; \dot B^{\frac dp-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot B^{\frac dp+1}_{p,1})$,

and endowed with the norm:

$$\begin{aligned}\|(a,u)\|_{X^p_{\varepsilon,\nu}} \overset{\text{def}}{=} \|(a,u)\|^{\ell,\widetilde\varepsilon}_{\widetilde L^\infty(\dot B^{\frac d2-1}_{2,1})} + \|u\|^{h,\widetilde\varepsilon}_{\widetilde L^\infty(\dot B^{\frac dp-1}_{p,1})} + \widetilde\varepsilon\|a\|^{h,\widetilde\varepsilon}_{\widetilde L^\infty(\dot B^{\frac dp}_{p,1})} \\ + \nu\|(a,u)\|^{\ell,\widetilde\varepsilon}_{L^1(\dot B^{\frac d2+1}_{2,1})} + \nu\|u\|^{h,\widetilde\varepsilon}_{L^1(\dot B^{\frac dp+1}_{p,1})} + \varepsilon^{-1}\|a\|^{h,\widetilde\varepsilon}_{L^1(\dot B^{\frac dp}_{p,1})}.\end{aligned}$$

One can now state our main result of convergence in the small data case, the reader being referred to [13, 14] for the large data case and stronger results of convergence.

Theorem 5.1. *Assume that the fluid domain is either $\mathbb{R}^d$ or $\mathbb{T}^d$, that the initial data $(a_0^\varepsilon, u_0^\varepsilon)$ are as above and that p is as in Theorem 4.1. There exists a constant η independent of ε and of ν such that if*

$$C_0^{\varepsilon,\nu} \overset{def}{=} \|(a_0^\varepsilon, u_0^\varepsilon)\|^{\ell,\widetilde\varepsilon}_{\dot B^{\frac d2-1}_{2,1}} + \|u_0^\varepsilon\|^{h,\widetilde\varepsilon}_{\dot B^{\frac dp-1}_{p,1}} + \widetilde\varepsilon\|a_0^\varepsilon\|^{h,\widetilde\varepsilon}_{\dot B^{\frac dp}_{p,1}} \leq \eta\nu, \tag{114}$$

then System (110) *with initial data $(a_0^\varepsilon, u_0^\varepsilon)$ has a unique global solution $(a^\varepsilon, u^\varepsilon)$ in the space $X^p_{\varepsilon,\nu}$ with, for some constant C independent of ε and ν,*

$$\|(a^\varepsilon, u^\varepsilon)\|_{X^p_{\varepsilon,\nu}} \leq CC_0^{\varepsilon,\nu}. \tag{115}$$

In addition, $\mathcal Q u^\varepsilon$ converges weakly to 0 when ε goes to 0, and, if $\mathcal P u_0^\varepsilon \rightharpoonup v_0$ then $\mathcal P u^\varepsilon$ converges in the sense of distributions to the unique solution of (109) *supplemented with initial data v_0.*

Proof. Performing the change of unknowns given in (113) and the change of data

$$(a_0, u_0)(x) \overset{\text{def}}{=} \varepsilon(a_0^\varepsilon, u_0^\varepsilon)(\varepsilon\nu x) \tag{116}$$

reduces the proof of the global existence to the case $\nu=1$ and $\varepsilon=1$, which was done in Theorem 4.1. Back to the original variables will yield the desired uniform estimate (115) under Condition (114). Indeed, we notice that we have up to some harmless constant:

$$\|(a_0^\varepsilon,u_0^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\dot B^{\frac d2-1}_{2,1}}+\|u_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{\frac dp-1}_{p,1}}+\widetilde{\varepsilon}\|a_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{\frac dp}_{p,1}}=\nu\big(\|(a_0,u_0)\|^{\ell,1}_{\dot B^{\frac d2-1}_{2,1}}+\|u_0\|^{h,1}_{\dot B^{\frac dp-1}_{p,1}}+\|a_0\|^{h,1}_{\dot B^{\frac dp}_{p,1}}\big)$$

and

$$\|(a^\varepsilon,u^\varepsilon)\|_{X^p_{\varepsilon,\nu}}=\nu\|(a,u)\|_{X^p_{1,1}}.$$

Granted with the uniform estimates established in the previous section, it is now easy to pass to the limit in the system in the sense of distributions, by adapting the compactness arguments of P.-L. Lions and N. Masmoudi in [24].

More precisely, consider a family $(a_0^\varepsilon,u_0^\varepsilon)$ of data satisfying (114) and $\mathcal{P}u_0^\varepsilon\rightharpoonup v_0$ when ε goes to 0. Let $(a^\varepsilon,u^\varepsilon)$ be the corresponding solution of (110) given by the first part of Theorem 5.1. Because

$$\|a_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{\frac dp-1}_{p,1}}\lesssim\widetilde{\varepsilon}\|a_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{\frac dp}_{p,1}}, \tag{117}$$

the data $(a_0^\varepsilon,u_0^\varepsilon)$ are uniformly bounded in $\dot B^{\frac dp-1}_{p,1}\times\dot B^{\frac dp}_{p,1}$. Likewise, (115) ensures that $(a^\varepsilon,u^\varepsilon)$ is bounded in the space $\mathcal{C}_b(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})$. Therefore there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ decaying to 0 so that $(a_0^{\varepsilon_n},u_0^{\varepsilon_n})\rightharpoonup(a_0,u_0)$ in $\dot B^{\frac dp-1}_{p,1}$ (with $\mathcal{P}u_0=v_0$) and

$$(a^{\varepsilon_n},u^{\varepsilon_n})\rightharpoonup(a,u)\ \text{ in }\ L^\infty(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})\quad\text{weak }*. \tag{118}$$

The strong convergence of the density to 1 is obvious: we have $\varrho^{\varepsilon_n}=1+\varepsilon_n a^{\varepsilon_n}$, and $(a^{\varepsilon_n})_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}_+;\dot B^{\frac dp}_{p,1})$. In order to justify that $\operatorname{div}u=0$, we rewrite the mass equation as follows:

$$\operatorname{div}u^{\varepsilon_n}=-\varepsilon_n\operatorname{div}(a^{\varepsilon_n}u^{\varepsilon_n})-\varepsilon_n\partial_t a^{\varepsilon_n}.$$

Given that a^{ε_n} and u^{ε_n} are bounded in $L^2(\mathbb{R}_+;\dot B^{\frac dp}_{p,1})$ (use the definition of $X^p_{\varepsilon,\nu}$ and interpolation), the first term in the right-hand side is $\mathcal{O}(\varepsilon_n)$ in $L^1(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})$. As for the last term, it tends to 0 in the sense of distributions, by virtue of (118). We thus have $\operatorname{div}u^{\varepsilon_n}\rightharpoonup 0$, whence $\operatorname{div}u=0$.

To establish that u is a solution to (109), let us project the velocity equation onto divergence-free vector fields:

$$\partial_t\mathcal{P}u^{\varepsilon_n}-\mu\Delta\mathcal{P}u^{\varepsilon_n}=-\mathcal{P}(u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n})-\mathcal{P}\biggl(\frac{1}{1+\varepsilon_n a^{\varepsilon_n}}\mathcal{A}u^{\varepsilon_n}\biggr). \tag{119}$$

Because $\mathcal{Q}u=0$, the left-hand side weakly converges to $\partial_t u-\mu\Delta u$. To prove that the last term tends to 0, we use the fact that having $\widetilde{\varepsilon}(a^\varepsilon)^{h,\widetilde{\varepsilon}}$ and $(a^\varepsilon)^{\ell,\widetilde{\varepsilon}}$ bounded in $L^\infty(\dot B^{\frac dp}_{p,1})$ and $L^\infty(\dot B^{\frac dp-1}_{p,1})$, respectively, implies that, for all $\alpha\in[0,1]$,

$$\widetilde{\varepsilon}^\alpha a^\varepsilon\quad\text{is bounded in }\ L^\infty(\dot B^{\frac dp-1+\alpha}_{p,1}). \tag{120}$$

Now, $\mathcal{A}u^\varepsilon$ is bounded in $L^1(\dot{B}^{\frac{d}{p}-1}_{p,1})$ and $p<2d$. Hence, according to product laws in Besov spaces, composition inequality and (120), we get $(1+\varepsilon a^\varepsilon)^{-1}\mathcal{A}u^\varepsilon=\mathcal{O}(\widetilde{\varepsilon}^{1-\alpha})$ in $L^1(\dot{B}^{\frac{d}{p}-2+\alpha}_{p,1})$, whenever $2\max(0,1-\frac{d}{p})<\alpha\leq 1$. Hence the last term of (119) goes strongly to 0 for some appropriate norm.

In order to prove that $\mathcal{P}(u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n})\rightharpoonup\mathcal{P}(u\cdot\nabla u)$, we note that

$$u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n}=\frac{1}{2}\nabla|\mathcal{Q}u^{\varepsilon_n}|^2+\mathcal{P}u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n}+\mathcal{Q}u^{\varepsilon_n}\cdot\nabla\mathcal{P}u^{\varepsilon_n}.$$

Projecting the first term onto divergence free vector fields gives 0, and we also know that $\mathcal{P}u=u$. Hence we just have to prove that

$$\mathcal{P}(\mathcal{P}u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n})\rightharpoonup\mathcal{P}(\mathcal{P}u\cdot\nabla u)\quad\text{and}\quad\mathcal{P}(\mathcal{Q}u^{\varepsilon_n}\cdot\nabla\mathcal{P}u^{\varepsilon_n})\rightharpoonup 0. \tag{121}$$

This requires our proving results of strong convergence for $\mathcal{P}u^{\varepsilon_n}$. To this end, we shall exhibit uniform bounds for $\partial_t\mathcal{P}u^{\varepsilon_n}$ in a suitable space. First, arguing by interpolation, we see that $(\nabla^2u^{\varepsilon_n})$ is bounded in $L^m(\dot{B}^{\frac{d}{p}+\frac{2}{m}-3}_{p,1})$ for any $m\geq 1$. Choosing $m>1$ so that $\frac{2}{m}-3>-d\min(\frac{2}{p},1)$ (this is possible as $p<2d$) and remembering that $(\varepsilon^n a^{\varepsilon_n})$ is bounded in $L^\infty(\dot{B}^{\frac{d}{p}}_{p,1})$, we thus get $((1+\varepsilon_n a^{\varepsilon_n})^{-1}\mathcal{A}u^{\varepsilon_n})$ bounded in $L^m(\dot{B}^{\frac{d}{p}+\frac{2}{m}-3}_{p,1})$. Similarly, combining the facts that (u^{ε_n}) and $(\nabla u^{\varepsilon_n})$ are bounded in $L^\infty(\dot{B}^{\frac{d}{p}-1}_{p,1})$ and $L^m(\dot{B}^{\frac{d}{p}+\frac{2}{m}-2}_{p,1})$, respectively, we see that $(u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n})$ is bounded in $L^m(\dot{B}^{\frac{d}{p}+\frac{2}{m}-3}_{p,1})$, too. Computing $\partial_t\mathcal{P}u^{\varepsilon_n}$ from (119), it is now clear that $(\partial_t\mathcal{P}u^{\varepsilon_n})$ is bounded in $L^m(\dot{B}^{\frac{d}{p}+\frac{2}{m}-3}_{p,1})$. Hence $(\mathcal{P}u^{\varepsilon_n}-\mathcal{P}u_0^{\varepsilon_n})$ is bounded in $\mathcal{C}^{1-\frac{1}{m}}(\mathbb{R}_+;\dot{B}^{\frac{d}{p}+\frac{2}{m}-3}_{p,1})$. As $\mathcal{P}u^{\varepsilon_n}$ is also bounded in $\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{p}-1}_{p,1})$, and as the embedding of $\dot{B}^{\frac{d}{p}-1}_{p,1}$ in $\dot{B}^{\frac{d}{p}+\frac{2}{m}-3}_{p,1}$ is locally compact (see e.g. [1], page 108), we conclude by means of Ascoli theorem that, up to a new extraction, for all $\phi\in\mathcal{S}(\mathbb{R}^d)$ and $T>0$,

$$\phi\mathcal{P}u^{\varepsilon_n}\longrightarrow\phi\mathcal{P}u\quad\text{in}\quad\mathcal{C}([0,T];\dot{B}^{\frac{d}{p}+\frac{2}{m}-3}_{p,1}). \tag{122}$$

Interpolating with the bounds in $\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{p}-1}_{p,1})$, we can upgrade the strong convergence in (122) to the space $\mathcal{C}([0,T];\dot{B}^{\frac{d}{p}-1-\alpha}_{p,1})$ for all small enough $\alpha>0$, and all $T>0$. Combining with the properties of weak convergence for ∇u^{ε_n} to ∇u, and $\mathcal{Q}u^{\varepsilon_n}$ to 0 that may be deduced from the bounds of u^{ε_n}, it is now easy to conclude to (121). One can use for instance the fact that for all $m>1$, we have

$$\nabla u^{\varepsilon_n}\rightharpoonup\nabla u\ \text{ in }\ L^m(\dot{B}^{\frac{d}{p}+\frac{2}{m}-2}_{p,1})\ \text{ weak }\ *\quad\text{and}\quad\mathcal{Q}u^{\varepsilon_n}\rightharpoonup 0\ \text{ in }\ L^m(\dot{B}^{\frac{d}{p}+\frac{2}{m}-1}_{p,1})\ \text{ weak }\ *\,.$$

□

5.2. **Time decay estimates.** In the present subsection, we show that under a mild additional decay assumption that is satisfied if the data are in $L^1(\mathbb{R}^d)$ for instance, the L^2 norm (the $\dot{B}^0_{2,1}$ norm in fact) of the global solutions constructed in Theorem 4.1 decays like $t^{-\frac{d}{4}}$ for $t\to+\infty$, exactly as for the linearized equations. This fact has been first observed by A. Matsumura and T. Nishida in [25] in the case of solutions with high Sobolev regularity. The adaptation to the L^2 type critical regularity framework has been carried out recently by M. Okita in [27], in dimension $d\geq 3$. Below, we give a more accurate description of the

time decay, emphasizing a better decay for high frequencies. This is the key to handling any dimension $d \geq 2$. For simplicity, we concentrate on the L^2 type framework, even though we expect similar results to be true in the more general L^p framework of Theorem 4.1.

Theorem 5.2. *Let the data (a_0, u_0) satisfy the assumptions of Theorem 4.1 with $p = 2$ and assume with no loss of generality that $P'(1) = 1$ and that $\nu = 1$. Denote $\langle\tau\rangle \overset{def}{=} \sqrt{1+\tau^2}$ and $\alpha \overset{def}{=} \min(\frac{d}{4}+2, \frac{d}{2}+\frac{1}{2}-\varepsilon)$ with $\varepsilon > 0$ arbitrarily small. There exists a positive constant c so that if in addition*

$$D_0 \overset{def}{=} \sup_{k\leq k_0} \big(\|\mathcal{F}(\dot\Delta_k a_0)\|_{L^\infty} + \|\mathcal{F}(\dot\Delta_k u_0)\|_{L^\infty}\big) \leq c \tag{123}$$

then the global solution (a, u) given by Theorem 4.1 satisfies for all $t \geq 0$,

$$D(t) \leq C\big(D_0 + \|(a_0, \nabla a_0, u_0)\|_{\dot B^{\frac{d}{2}-1}_{2,1}}\big) \tag{124}$$

$$\text{with}\quad D(t) \overset{def}{=} \sup_{s\in(-\frac{d}{2},2]} \|\langle\tau\rangle^{\frac{d}{4}+\frac{s}{2}}(a,u)\|^\ell_{L^\infty_t(\dot B^s_{2,1})} + \|\langle\tau\rangle^\alpha(\nabla a, u)\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{2}-1}_{2,1})} + \|\tau\nabla u\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{2}}_{2,1})}.$$

Proof. Throughout the proof, we shall use repeatedly that for $0 < \sigma_1 \leq \sigma_2$, we have:

$$\int_0^t \langle t-\tau\rangle^{-\sigma_1}\langle\tau\rangle^{-\sigma_2}\,d\tau \lesssim \langle t\rangle^{-\sigma_1} \quad \text{if in addition} \quad \sigma_2 > 1. \tag{125}$$

Step 1: Bounds for the low frequencies. Denoting by $E(D)$ the semi-group associated to (70), we have for all $k \in \mathbb{Z}$,

$$\begin{pmatrix} \dot\Delta_k a(t) \\ \dot\Delta_k u(t)\end{pmatrix} = e^{tE(D)}\begin{pmatrix} \dot\Delta_k a_0 \\ \dot\Delta_k u_0\end{pmatrix} - \int_0^t e^{(t-\tau)E(D)}\begin{pmatrix} \dot\Delta_k f_1(\tau) \\ \dot\Delta_k(f_2+f_3+f_4)(\tau)\end{pmatrix}d\tau \tag{126}$$

with $f_1 \overset{\text{def}}{=} \operatorname{div}(au)$, $f_2 \overset{\text{def}}{=} u\cdot\nabla u$, $f_3 \overset{\text{def}}{=} k(a)\nabla a$ and $f_4 \overset{\text{def}}{=} I(a)\mathcal{A}u$.

From an explicit computation of the action of $e^{tE(D)}$ in Fourier variables (see e.g. [4]), we discover that there exist positive constants c and C depending only on k_0 and such that

$$|\mathcal{F}(e^{tA(D)}U)(\xi)| \leq Ce^{-c_0t|\xi|^2}|\mathcal{F}U(\xi)| \quad \text{for all} \quad |\xi| \leq 2^{k_0}.$$

Therefore, for all $k \leq k_0$,

$$\begin{aligned}\|e^{tE(D)}\dot\Delta_k U\|^2_{L^2} &\lesssim \int e^{-2c_0|\xi|^2t}|\mathcal{F}\dot\Delta_k U(\xi)|^2\,d\xi \\ &\lesssim \|\mathcal{F}\dot\Delta_k U\|^2_{L^\infty}2^{kd}\,e^{-c_0 2^{2k}t}.\end{aligned}$$

We thus get up to a change of c_0,

$$t^{\frac{d}{4}+\frac{s}{2}}\sum_{k\leq k_0}2^{ks}\|e^{tE(D)}\dot\Delta_k U\|_{L^2} \lesssim \Big(\sup_{k\leq k_0}\|\mathcal{F}\dot\Delta_k U\|_{L^\infty}\Big)\sum_{k\leq k_0}(\sqrt{t}\,2^k)^{\frac{d}{2}+s}\,e^{-c_0 2^{2k}t}. \tag{127}$$

As for any $\sigma > 0$ there exists a constant C_σ so that

$$\sup_{t\geq 0}\sum_{k\in\mathbb{Z}} t^{\frac{\sigma}{2}}2^{k\sigma}e^{-c_0 2^{2k}t} \leq C_\sigma, \tag{128}$$

we get from (127) that for $s > -d/2$,

$$\sup_{t\geq 0} t^{\frac{d}{4}+\frac{s}{2}}\|e^{tE(D)}U\|^\ell_{\dot B^s_{2,1}} \leq C_s \sup_{k\leq k_0}\|\mathcal{F}\dot\Delta_k U\|_{L^\infty}.$$

It is also obvious that for $s > -d/2$,

$$\|e^{tE(D)}U\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \|U\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \sup_{k\leq k_0} \|\mathcal{F}\dot{\Delta}_k U\|_{L^\infty}.$$

Hence we conclude that

$$\sup_{t\geq 0}\langle t\rangle^{\frac{d}{4}+\frac{s}{2}}\|e^{tE(D)}U\|^{\ell}_{\dot{B}^{s}_{2,1}} \lesssim \sup_{k\leq k_0} \|\mathcal{F}\dot{\Delta}_k U\|_{L^\infty}. \tag{129}$$

Next, we claim that for all $s \in (-d/2, 2]$ and $i \in \{1, \cdots, 4\}$, we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}} \sup_{k\leq k_0} \|\mathcal{F}\dot{\Delta}_k f_i(\tau)\|_{L^\infty}\, d\tau \lesssim \langle t\rangle^{-\frac{d}{4}-\frac{s}{2}}\big(D^2(t) + X^2(t)\big) \tag{130}$$

with $X(t) \overset{\text{def}}{=} \|(a, \nabla a, u)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \displaystyle\int_0^t \big(\|a\|^{\ell}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|a\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\big)\, d\tau.$

Of course, as the Fourier transform maps L^1 to L^∞, it suffices to prove (130) with $\|f_i\|_{L^1}$ instead of $\sup_{k\leq k_0}\|\mathcal{F}\dot{\Delta}_k f_i\|_{L^\infty}$.

To bound the term with f_1, we use the following decomposition:

$$f_1 = u\cdot\nabla a + a\,\mathrm{div}\,u^{\ell} + a\,\mathrm{div}\,u^{h}.$$

Now, from Cauchy-Schwarz inequality and the definition of $D(t)$, one may write

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|(u\cdot\nabla a)(\tau)\|_{L^1}\, d\tau &\leq \Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac{d}{4}}\|u(\tau)\|_{L^2}\Big)\Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac{d}{4}+\frac{1}{2}}\|\nabla a(\tau)\|_{L^2}\Big)\\ &\qquad\times\int_0^t\langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\langle\tau\rangle^{-\frac{d}{2}-\frac{1}{2}}\,d\tau\\ &\lesssim \langle t\rangle^{-\frac{d}{4}-\frac{s}{2}}D^2(t),\end{aligned}$$

where we used (125) and the fact that $0 < \frac{d}{4}+\frac{s}{2} \leq \frac{d}{2}+\frac{1}{2}$.

Bounding the term with $a\,\mathrm{div}\,u^{\ell}$ is totally similar. Regarding the term with $a\,\mathrm{div}\,u^{h}$, we use that if $t \geq 2$,

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|(a\mathrm{div}\,u^{h})(\tau)\|_{L^1}\,d\tau &\lesssim \int_0^1 \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|a(\tau)\|_{L^2}\|\mathrm{div}\,u^{h}(\tau)\|_{L^2}\,d\tau\\ &\quad+\int_1^t\langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\langle\tau\rangle^{-1-\frac{d}{4}}\big(\langle\tau\rangle^{\frac{d}{4}}\|a(\tau)\|_{L^2}\big)\big(\tau\|\mathrm{div}\,u^{h}(\tau)\|_{L^2}\big)\,d\tau.\end{aligned}$$

Therefore, as $-d/2 < s \leq 2$, we get

$$\begin{aligned}\langle t\rangle^{\frac{s}{2}+\frac{d}{4}}\int_0^t\langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|(a\mathrm{div}\,u^{h})(\tau)\|_{L^1}\,d\tau &\lesssim \Big(\sup_{\tau\in[0,t]}\|a(\tau)\|_{L^2}\Big)\int_0^t\|\mathrm{div}\,u^{h}(\tau)\|_{L^2}\,d\tau\\ &\quad+\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac{d}{4}}\|a(\tau)\|_{L^2}\Big)\Big(\sup_{\tau\in[0,t]}\tau\|\mathrm{div}\,u^{h}(\tau)\|_{L^2}\Big),\end{aligned}$$

and (130) is thus satisfied by the term with $a\,\mathrm{div}\,u^{h}$ if $t \geq 2$, the case $t \leq 2$ being obvious as $\langle t\rangle \approx 1$ and $\langle t-\tau\rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$ and one may write

$$\int_0^t \|a\,\mathrm{div}\,u^{h}\|_{L^1}\,d\tau \leq \|a\|_{L^2_t(L^2)}\|\mathrm{div}\,u^{h}\|_{L^2_t(L^2)} \lesssim X^2(t).$$

Handling the terms with f_2 and f_3 is totally similar: $k(a)\nabla a$ and $u\cdot\nabla u^\ell$ may be treated as $u\cdot\nabla a$, and $u\cdot\nabla u^h$, as $a\,\mathrm{div}\,u^h$. For f_4, we write that

$$f_4 = I(a)\mathcal{A}u^\ell + I(a)\mathcal{A}u^h.$$

Now, we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|I(a)\mathcal{A}u^\ell\|_{L^1}\,d\tau \\ \lesssim \Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac{d}{4}}\|a(\tau)\|_{L^2}\Big)\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac{d}{4}+1}\|\nabla^2u^\ell(\tau)\|_{L^2}\Big)\int_0^t\langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\langle\tau\rangle^{-1-\frac{d}{2}}\,d\tau.$$

Hence, thanks to (125), the term with $I(a)\mathcal{A}u^\ell$ fulfills (130). Finally, for $t\geq 2$,

$$\int_0^t \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|I(a)\mathcal{A}u^h\|_{L^1}\,d\tau \lesssim \langle t\rangle^{-\frac{d}{4}-\frac{s}{2}}\int_0^1\|a\|_{L^2}\|\nabla^2u^h\|_{L^2}\,d\tau \\ +\int_1^t\langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\langle\tau\rangle^{-1-\frac{d}{4}}\big(\langle\tau\rangle^{\frac{d}{4}}\|a(\tau)\|_{L^2}\big)\big(\tau\|\nabla^2u^h(\tau)\|_{L^2}\big)d\tau,$$

hence, because $-d/2<s\leq 2$ and $\|\tau\nabla^2u^h\|_{L^\infty_t(L^2)}\lesssim\|\tau\nabla u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}$,

$$\int_0^t \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|I(a)\mathcal{A}u^h\|_{L^1}\,d\tau \lesssim \langle t\rangle^{-\frac{d}{4}-\frac{s}{2}}\big(D^2(t)+X^2(t)\big)\quad\text{for}\quad t\geq 2.$$

Obviously, as $\langle t\rangle\approx 1$ and $\langle t-\tau\rangle\approx 1$ for $0\leq\tau\leq t\leq 2$, we have the following inequality:

$$\int_0^t \langle t-\tau\rangle^{-\frac{d}{4}-\frac{s}{2}}\|I(a)\mathcal{A}u^h\|_{L^1}\,d\tau \lesssim \langle t\rangle^{-\frac{d}{4}-\frac{s}{2}}X^2(t)\quad\text{for}\quad t\leq 2,$$

which completes the proof of (130). Combining with (129) and using Duhamel's formula, we conclude that for all $t\geq 0$ and $s\in(-d/2,2]$,

$$\langle t\rangle^{\frac{d}{4}+\frac{s}{2}}\|(a,u)\|^\ell_{\dot{B}^s_{2,1}}\lesssim D_0+D^2(t)+X^2(t). \tag{131}$$

Step 2: Decay estimates for the high frequencies of $(\nabla a,u)$. We here want to bound the second term of $D(t)$. Recall that Theorem 4.1 ensures that

$$\|(\nabla a,u)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\leq CX(0)\quad\text{for all}\quad T\geq 0.$$

Therefore it suffices to bound $\|t^\alpha(\nabla a,u)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}$ for, say, $T\geq 2$.

Now, the starting point is Inequality (87) which implies that for $k\geq k_0$ and for some $c_0=c(k_0)>0$, we have

$$\frac12\frac{d}{dt}\mathcal{L}_k^2+c_0\mathcal{L}_k^2\leq\Big(\|(\nabla f_k,g_k)\|_{L^2} \\ +\|R_k(u,a)\|_{L^2}+\|R_k(u,u)\|_{L^2}+\|\widetilde{R}_k(u,a)\|_{L^2}+\|\nabla u\|_{L^\infty}\mathcal{L}_k\Big)\mathcal{L}_k$$

with $f\overset{\text{def}}{=}-a\mathrm{div}\,u$, $g=-k(a)\nabla a-I(a)\mathcal{A}u$, $R_k(u,b)\overset{\text{def}}{=}\dot\Delta_k(u\cdot\nabla b)-u\cdot\nabla\dot\Delta_k b$ for $b\in\{a,u\}$, and $\widetilde{R}^i_k(u,a)\overset{\text{def}}{=}\partial_i\dot\Delta_k(u\cdot\nabla a)-u\cdot\nabla\partial_i\dot\Delta_k a$.

After time integration, we discover that

$$e^{c_0t}\mathcal{L}_k(t) \leq \mathcal{L}_k(0) + \int_0^t e^{c_0\tau}\Big(\|(\nabla f_k, g_k)\|_{L^2} + \|R_k(u,a)\|_{L^2} \\ +\|R_k(u,u)\|_{L^2} + \|\widetilde{R}_k(u,a)\|_{L^2} + \|\nabla u\|_{L^\infty}\mathcal{L}_k\Big)d\tau,$$

whence, remembering that $\mathcal{L}_k \approx \|(\dot\Delta_k\nabla a, \dot\Delta_k u)\|_{L^2}$ for $k \geq k_0$,

$$t^\alpha\|(\dot\Delta_k\nabla a, \dot\Delta_k u)(t)\|_{L^2} \lesssim t^\alpha e^{-c_0t}\|(\dot\Delta_k\nabla a, \dot\Delta_k u)(0)\|_{L^2} + t^\alpha\int_0^t e^{c_0(\tau-t)}\Big(\|(\nabla f_k, g_k)\|_{L^2} \\ +\|R_k(u,a)\|_{L^2} + \|R_k(u,u)\|_{L^2} + \|\widetilde{R}_k(u,a)\|_{L^2} + \|\nabla u\|_{L^\infty}\|(\dot\Delta_k\nabla a, \dot\Delta_k u)\|_{L^2}\Big)d\tau$$

and thus, multiplying both sides by $2^{k(\frac{d}{2}-1)}$, taking the supremum on $[0,T]$, and summing up over $k \geq k_0$,

$$\|t^\alpha(\nabla a, u)\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})} \lesssim \|(\nabla a_0, u_0)\|^h_{\dot B^{\frac{d}{2}-1}_{2,1}} + \sum_{k\geq k_0}\sup_{0\leq t\leq T}\Big(t^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k\,d\tau\Big) \tag{132}$$

with $S_k \overset{\text{def}}{=} \sum_{i=1}^5 S_k^i$ and

$$S_k^1 \overset{\text{def}}{=} \|(\nabla f_k, g_k)\|_{L^2}, \quad S_k^2 \overset{\text{def}}{=} \|R_k(u,a)\|_{L^2}, \quad S_k^3 \overset{\text{def}}{=} \|R_k(u,u)\|_{L^2}, \\ S_k^4 \overset{\text{def}}{=} \|\widetilde{R}_k(u,a)\|_{L^2}, \quad S_k^5 \overset{\text{def}}{=} \|\nabla u\|_{L^\infty}\|(\dot\Delta_k\nabla a, \dot\Delta_k u)\|_{L^2}.$$

In order to bound the sum, we first notice that

$$\sum_{k\geq k_0}\sup_{0\leq t\leq 2}\Big(t^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k(\tau)\,d\tau\Big) \lesssim \int_0^2\sum_{k\geq k_0}2^{k(\frac{d}{2}-1)}S_k(\tau)\,d\tau.$$

Hence taking advantage of (24) and of a similar inequality for $\widetilde{R}_k(u,a)$, we end up with

$$\sum_{k\geq k_0}\sup_{0\leq t\leq 2}t^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k\,d\tau \lesssim \int_0^2\Big(\|(\nabla f, g)\|_{\dot B^{\frac{d}{2}-1}_{2,1}} + \|\nabla u\|_{\dot B^{\frac{d}{2}}_{2,1}}\|(a, \nabla a, u)\|_{\dot B^{\frac{d}{2}-1}_{2,1}}\Big)d\tau.$$

Bounding ∇f and g as in the proof of Theorem 4.1 leads to

$$\sum_{k\geq k_0}\sup_{0\leq t\leq 2}t^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k\,d\tau \lesssim X^2(2). \tag{133}$$

To bound the supremum on $[2,T]$, we split the integral on $[0,t]$ into integrals on $[0,1]$ and $[1,t]$, respectively. The $[0,1]$ part of the integral is easy to handle: we have

$$\sum_{k\geq k_0}\sup_{2\leq t\leq T}t^\alpha\int_0^1 e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k(\tau)\,d\tau \leq \sum_{k\geq k_0}\sup_{2\leq t\leq T}t^\alpha e^{-\frac{c_0}{2}t}\int_0^1 2^{k(\frac{d}{2}-1)}S_k\,d\tau \\ \lesssim \int_0^1\sum_{k\geq k_0}2^{k(\frac{d}{2}-1)}S_k\,d\tau.$$

Hence

$$\sum_{k\geq k_0}\sup_{2\leq t\leq T}\Big(t^\alpha\int_0^1 e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k(\tau)\,d\tau\Big) \lesssim X^2(1). \tag{134}$$

Let us finally consider the $[1,t]$ part of the integral for $2\leq t\leq T$. We shall use repeatedly the following inequality

$$\|\tau\nabla u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim D(t), \tag{135}$$

which is straightforward as regards the high frequencies of u and stems from

$$\|\tau\nabla u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\|\langle\tau\rangle^{\frac{d}{4}+\frac12}\nabla u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\|\langle\tau\rangle^{\frac{d}{4}+\frac12}u\|^\ell_{L^\infty_t(\dot{B}^1_{2,1})}\leq D(t)$$

for the low frequencies of u.

Regarding the contribution of S^1_k, we first notice that, by virtue of (125),

$$\sum_{k\geq k_0}\sup_{2\leq t\leq T}t^\alpha\int_1^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S^1_k(\tau)\,d\tau\lesssim\|\tau^\alpha(\nabla f,g)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}. \tag{136}$$

Now, product laws in tilde spaces ensures that

$$\|\tau^\alpha\nabla f\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|\tau^{\alpha-1}a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\|\tau\mathrm{div}\,u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

The high frequencies of the first term of the r.h.s. is obviously bounded by $D(T)$. As for the low frequencies, we notice that if $d\leq4$ then for all small enough $\varepsilon>0$,

$$\|\tau^{\frac{d}{2}-\varepsilon}a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\|\tau^{\frac{d}{2}-\varepsilon}a\|^\ell_{L^\infty_T(\dot{B}^{\frac{d}{2}-2\varepsilon}_{2,1})}\leq D(T) \tag{137}$$

and if $d\geq5$, taking $s=2$ in the first term of $D(T)$,

$$\|\tau^{\frac{d}{4}+1}a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\|\tau^{\frac{d}{4}+1}a\|^\ell_{L^\infty_T(\dot{B}^2_{2,1})}\leq D(T). \tag{138}$$

Therefore, using (135) and remembering the definition of α, we get

$$\|\tau^\alpha\nabla f\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim D^2(T).$$

Next, we have

$$\|\tau^\alpha(k(a)\nabla a^h)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\|\tau^\alpha a\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\leq X(T)D(T)$$

and, according to (137) and (138),

$$\|\tau^\alpha(k(a)\nabla a^\ell)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|\tau^{1-\varepsilon}a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\|\tau^{\alpha-1+\varepsilon}a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim D^2(T).$$

We also see that

$$\|\tau^\alpha I(a)\mathcal{A}u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|\tau\nabla^2u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\big(\|\tau^{\alpha-1}a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}+\|\tau^{\alpha-1}a\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}\big).$$

The first term of the r.h.s. may be bounded by virtue of (135), and it is also clear that the last term is bounded by $D(T)$. As for the second one, we use again (137) and (138). Resuming to (136), we end up with

$$\sum_{k\geq k_0}\sup_{2\leq t\leq T}t^\alpha\int_1^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S^1_k(\tau)\,d\tau\lesssim D^2(T).$$

To bound the term with S_k^2, we use the fact that

$$\int_1^t e^{c_0(\tau-t)}\|R_k(u,a)\|_{L^2}\,d\tau \le \|R_k(\tau u,\tau^{\alpha-1}a)\|_{L^\infty_t(L^2)}\int_1^t e^{c_0(\tau-t)}\tau^{-\alpha}\,d\tau.$$

Hence, thanks to (125) and to (24) (adapted to tilde spaces),

$$\begin{aligned}\sum_{k\ge k_0}\sup_{2\le t\le T}\biggl(t^\alpha\int_1^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k^2(\tau)\,d\tau\biggr) &\lesssim \sum_{k\ge k_0}2^{k(\frac{d}{2}-1)}\|R_k(\tau u,\tau^{\alpha-1}a)\|_{L^\infty_T(L^2)}\\ &\lesssim \|\tau\nabla u\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\|\tau^{\alpha-1}a\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})}.\end{aligned}$$

The first term of the r.h.s. may be bounded thanks to (135), and the high frequencies of the last one are obviously bounded by $D(T)$. To bound $\|\tau^{\alpha-1}a\|^\ell_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})}$, we use the following two inequalities

$$\begin{aligned}\|\tau^{\alpha-1}a\|^\ell_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})} &\lesssim \|\tau^{\alpha-1}a\|^\ell_{L^\infty_T(\dot B^{\frac{d}{2}-1-2\varepsilon}_{2,1})} &&\text{if } d\le 6,\\ \|\tau^{\alpha-1}a\|^\ell_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})} &\lesssim \|\tau^{\alpha-1}a\|^\ell_{L^\infty_T(\dot B^{2}_{2,1})} &&\text{if } d\ge 7.\end{aligned}$$

Because $\alpha-1=\frac{d}{2}-\frac12-\varepsilon$ if $d\le 6$, and $\alpha-1=\frac{d}{4}+1$ if $d\ge 7$, the r.h.s. above are bounded by $D(T)$. We eventually get

$$\sum_{k\ge k_0}\sup_{2\le t\le T}t^\alpha\int_1^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k^2(\tau)\,d\tau\lesssim D^2(T).$$

The terms S_k^3 and S_k^4 may be treated along the same lines.

Finally, using product laws and (125), we get

$$\begin{aligned}&\sum_{k\ge k_0}\sup_{2\le t\le T}t^\alpha\int_1^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k^5(\tau)\,d\tau\\ &\qquad\lesssim \|\tau\nabla u\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\|\tau^{\alpha-1}(\nabla a,u)\|^h_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})}\sup_{2\le t\le T}t^\alpha\int_1^t e^{c_0(\tau-t)}\tau^{-\alpha}\,d\tau\lesssim D^2(T).\end{aligned}$$

Putting all the above inequalities together, we conclude that

$$\sum_{k\ge k_0}\sup_{2\le t\le T}\biggl(t^\alpha\int_1^t e^{c_0(\tau-t)}2^{k(\frac{d}{2}-1)}S_k(\tau)\,d\tau\biggr)\lesssim D(T)X(T)+D^2(T).$$

Then plugging this latter inequality, (133) and (134) in (132) yields

$$\|\langle\tau\rangle^\alpha(\nabla a,u)\|_{\widetilde{L}^\infty_T(\dot B^{\frac{d}{2}-1}_{2,1})}\lesssim\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac{d}{2}-1}_{2,1}}+X^2(T)+D^2(T). \tag{139}$$

Step 3: Decay estimates with gain of regularity for the high frequencies of ∇u. In order to bound the last term of $D(t)$, we shall use the fact that the velocity u satisfies

$$\partial_t u-\mathcal{A}u=F\overset{\text{def}}{=}-(1+k(a))\nabla a-u\cdot\nabla u-I(a)\mathcal{A}u,$$

whence

$$\partial_t(t\mathcal{A}u)-\mathcal{A}(t\mathcal{A}u)=\mathcal{A}u+t\mathcal{A}F.$$

Because the maximal regularity estimates for the Lamé semi-group are the same as for the heat semi-group (see the beginning of Section 3), we deduce from Remark 2.2 that

$$\|t\mathcal{A}u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|\mathcal{A}u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|t\mathcal{A}F\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-3}_{2,1})},$$

whence, using the bounds given by Theorem 5.1,

$$\|t\nabla u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim X(0)+\|\tau F\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}. \tag{140}$$

In order to bound the last term, we notice that, because $\alpha\geq 1$, we have

$$\|\tau\nabla a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|\langle\tau\rangle^\alpha a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

Next, product and composition estimates adapted to tilde spaces give

$$\|\tau\,k(a)\nabla a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|\tau^{\frac12}a\|^2_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\leq D^2(t),$$

as well as

$$\|\tau\,u\cdot\nabla u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\|\tau\nabla u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}$$

and

$$\|\tau I(a)\mathcal{A}u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\tau\nabla^2u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}.$$

Therefore, resuming to (140) and remembering (135), we get

$$\|t\nabla u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim X(0)+D(t)X(t)+D^2(t)+\|\langle\tau\rangle^\alpha a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

Finally, bounding the last term according to (139), and adding up the obtained inequality to (131) and (139) yields

$$D(t)\lesssim D_0+\|(\nabla a_0,u_0)\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+X^2(t)+D^2(t).$$

As Theorem 5.1 ensures that $X(t)$ is small, on can now conclude that (124) is fulfilled for all time if D_0 and $\|(\nabla a_0,u_0)\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}}$ are small enough. □

6. Appendix

Here we recall various estimates for the flow that have been used repeatedly in the proof of Theorem 3.1. More details may be found in [16] or [19].

Recall that if $v:[0,T)\times\mathbb{R}^d\to\mathbb{R}^d$ is measurable, such that $t\mapsto v(t,x)$ is in $L^1(0,T)$ for all $x\in\mathbb{R}^d$ and in addition $\nabla v\in L^1(0,T;L^\infty)$ then it has, by virtue of the Cauchy-Lipschitz theorem, a unique C^1 flow X_v satisfying

$$X_v(t,y)=y+\int_0^t v(\tau,X_v(\tau,y))\,d\tau\quad\text{for all}\quad t\in[0,T).$$

In addition, for all $t\in[0,T)$, the map $X_v(t,\cdot)$ is a C^1-diffeomorphism over $\mathbb{R}^d$.

Lemma 6.1. *Let $p \in [1,\infty)$. Let $\bar v(t,y) \overset{def}{=} v(t,X(t,y))$. Under Assumption* (60)*, we have for all $t \in [0,T]$,*

$$\|\mathrm{Id} - \mathrm{adj}\,(DX_v(t))\|_{\dot B^{\frac dp}_{p,1}} \lesssim \|D\bar v\|_{L^1_t(\dot B^{\frac dp}_{p,1})}, \tag{141}$$

$$\|\mathrm{Id} - A_v(t)\|_{\dot B^{\frac dp}_{p,1}} \lesssim \|D\bar v\|_{L^1_t(\dot B^{\frac dp}_{p,1})}, \tag{142}$$

$$\|\mathrm{adj}\,(DX_v(t))^T A_v(t) - \mathrm{Id}\,\|_{\dot B^{\frac dp}_{p,1}} \lesssim \|D\bar v\|_{L^1_t(\dot B^{\frac dp}_{p,1})}, \tag{143}$$

$$\|J_v^{\pm1}(t) - 1\|_{\dot B^{\frac dp}_{p,1}} \lesssim \|D\bar v\|_{L^1_t(\dot B^{\frac dp}_{p,1})}. \tag{144}$$

Proof. As an example, let us prove the last item. We have thanks to the chain rule,

$$J_v(t,y) = 1 + \int_0^t \mathrm{div}\, v(\tau, X_v(\tau,y))\, J_v(\tau,y)\, d\tau = 1 + \int_0^t (D\bar v : \mathrm{adj}\,(DX_v))(\tau,y)\, d\tau. \tag{145}$$

Hence, if Condition (60) holds then we have (144) for J_v, a consequence of the fact that $\dot B^{\frac dp}_{p,1}$ is an algebra, and of (141). In order to get the inequality for J_v^{-1}, it suffices to notice that

$$J_v^{-1}(t,y) - 1 = (1 + (J_v(t,y) - 1))^{-1} - 1 = \sum_{k\geq 1} (-1)^k \int_0^t D\bar v : \mathrm{adj}\,(DX_v)\, d\tau.$$

□

Lemma 6.2. *Let $\bar v_1$ and $\bar v_2$ be two vector-fields satisfying* (60)*, and $\delta v \overset{def}{=} \bar v_2 - \bar v_1$. Then we have for all $p \in [1,\infty)$ and $t \in [0,T]$:*

$$\|A_{v_2} - A_{v_1}\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})} \lesssim \|D\delta v\|_{L^1_t(\dot B^{\frac dp}_{p,1})}, \tag{146}$$

$$\|\mathrm{adj}\,(DX_{v_2}) - \mathrm{adj}\,(DX_{v_1})\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})} \lesssim \|D\delta v\|_{L^1_t(\dot B^{\frac dp}_{p,1})}, \tag{147}$$

$$\|J_{v_2} - J_{v_1}\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})} \lesssim \|D\delta v\|_{L^1_t(\dot B^{\frac dp}_{p,1})}. \tag{148}$$

Proof. In order to prove the first inequality, we use the fact that, for $i = 1, 2$, we have

$$A_{v_i} = (\mathrm{Id} + C_i)^{-1} = \sum_{k\geq 0} (-1)^k C_i^k \quad \text{with} \quad C_i(t) = \int_0^t D\bar v_i\, d\tau.$$

Hence

$$A_{v_2} - A_{v_1} = \sum_{k\geq 1} \Big(C_2^k - C_1^k\Big) = \Big(\int_0^t D\delta v\, d\tau\Big) \sum_{k\geq 1} \sum_{j=0}^{k-1} C_1^j C_2^{k-1-j}.$$

So using the fact that $\dot B^{\frac dp}_{p,1}$ is a Banach algebra, it is easy to conclude to (146). Proving the second inequality is similar. □

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