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COMBINATORIAL THEORY OF PERMUTATION-INVARIANT RANDOM MATRICES II: CUMULANTS, FREENESS AND LEVY PROCESSES.

by

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Abstract. — In this article, we study random matrices in a framework based on the geometric study of partitions and some dualities as the Schur-Weyl's duality. This gives a unified and simple framework in order to understand families of random matrices which are invariant by conjugation in law by any group whose associated tensor category is spanned by partitions. This includes the unitary groups, the unitary reflection groups, the orthogonal groups, the bistochastic groups, the hyperoctahedral groups and the symmetric groups. For each choice of symmetry one can associate a subset of partitions A which allows us to define a notion of A-free cumulants. Besides, we introduce some observables on random matrices, namely the $\mathcal{P}$-moments, which generalize the normalized moments. One of the various by-products we get is that for any family of random matrices which is invariant by the unitary group, if it converges in non-commutative distribution then the $\mathcal{P}$-moments of this family converge in expectation. This implies a simple formula which allows us to compute the asymptotic of any product of the entries of a family of random matrices which is invariant in law by conjugation by the unitary group and which converges in non-commutative distribution. We prove similar results when the family is invariant in law by conjugation by the orthogonal group.

This setting leads to a unified way in order to define and study new notions of asymptotic freeness associated to each symmetry. As a by-product, we prove that independence and invariance in law by conjugation by the bistochastic group implies asymptotic Voiculescu's freeness. We show that there exist two formulations for each notion of asymptotic freeness: one uses some modified moments, and the other uses cumulants.

In this setting, a non-commutative central limit theorem is proved and the notion of asymptotic factorization is also studied. We also show how to inject the theory of classical probabilities in this new framework: as a consequence, the classical cumulants can be seen as a special case of the new free cumulants we defined.

At last, we give general theorems about convergence of matrix-valued additive and multiplicative Levy processes which are invariant in law by conjugation by the symmetric group. Using these results, we give a unified point of view on the study of matricial Levy processes on some Lie algebras and some Lie groups.

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1. Introduction

This article is the second one of a set of three, namely [**15**], [**16**] and [**17**]. Here we will not restate the results of [**15**], we will only refer to them. Thus we encourage the reader to read the first article before this one. Besides, the reader must not be surprised by the absence of introduction in this article since the introduction of [**15**] was a general introduction for the three articles, [**15**], [**16**] and [**17**], which are actually a unique article cut into three. In this article we apply the combinatorial framework developed in [**15**] to the study of random matrices. It has to be noticed that some results about the convergence of generalized observables for unitarily invariant random matrices were also proved independently in the paper [**13**]. They used the theory of traffics developped by C. Male [**26**], which is a generalization of the theory of free probability, and proved an equivalent result in this setting. We will explain the links between this article and the theory of traffics of C. Male in the forthcoming article [**14**].

1.1. Layout of the article. — The core of this article is the existence of dualities for some groups of matrices. This idea was first used in order to study random matrices in [**2**]. Let G be a group whose associated tensor category is spanned by partitions: to G is associated a subset of the partitions $\mathcal{P} = \bigcup_{k\in\mathbb{N}} \mathcal{P}_k$ that we will denote in the layout by A. An introduction to the dualities for subgroups of $O(N)$ can be found in [**3**]. For the unitary reflection group, the duality was proved by Tanabe in [**31**].

In Section 2, we recall some basic notions in random matrices theory such as the notion of empirical eigenvalues distribution. In order to prove convergence of this distribution, one can study observables. We generalize these observables to a family wich is called the A-moments. Then we begin our study of convergence of families of random matrices using these observables. A family of matrices is said to converge in expectation in A-moments if the expectation of any A-moments converges.

The link with the article [**15**] is explained in Section 3, where dualities like the Schur-Weyl's duality are explained. In the article [**3**], T. Banica and R. Speicher caracterized the easy orthogonal groups, which are roughly the groups G such that $\mathfrak{S}(N) \subset G \subset O(N)$ and which satisfy a duality with some sub-algebra of the partition algebra. These groups are interesting for our setting based on partitions. We focus in this article mainly on the unitary, orthogonal and symmetric groups.

We define, in Section 4, the notions of A-non-commutative (or free) cumulants for any family of random matrices which converges in expectation in A-moments. Using the results of Section 3, we also define non-commutative cumulants for any family of

random matrices with a given size: these cumulants are shown to converge to the first free cumulants we defined when the family of matrices converges in expectation in A-moments. This allows to prove in a simple way that for any family or random matrices which is invariant by conjugation by G, if it converges in expectation in A-moments then it converges in $\mathcal{P}$-moments (Theorem 4.5). For example this shows that a unitarily invariant family of random matrices converges in expectation in $\mathcal{P}$-moments if it converges in non-commutative distribution and satisfies an asymptotic factorization of its moments: for example Hadamard products between matrices of this family also converge in non-commutative distribution. We also study the development in $\frac{1}{N}$ of the observables, which is sometimes called the topological expansion.

We introduce the exclusive moments in Section 5: the study of the exclusive moments allows us to compute the asymptotic of any product of the entries of a family of random matrices which converges in non-commutative distribution and which is unitarily invariant (Theorem 5.4).

We generalize the notion of non-commutative law and $\mathcal{R}$-transform in Section 6. Then we define and study the notion of A-freeness in Section 7. We study the links between the different notions of A-freeness and the Voiculescu's notion of freeness: some of the links are summarized in Section 7.1.3. We prove for example, in Theorem 7.3, that asymptotic $\mathcal{P}$-freeness and a condition of asymptotic G-invariance imply A-freeness. Besides asymptotic $\mathcal{B}$-freeness implies asymptotic $\mathfrak{S}$-freeness which rhougly implies asymptotic Voiculescu's freeness. We prove Theorem 7.10 which is a generalization of a well-known result: independence and G-invariance implies asymptotic A-freeness. This is also generalized in Section 7.1.9, with Theorem 7.11, which allows us to suppose a weaker condition than the G-invariance. We recover in this setting a result from [**28**]: a unitarily invariant family of random matrices which converges in distribution is free from its transpose family. We also compute the $\mathcal{R}$-transform of the sum and the multiplication of two asymptotically A-free families of random matrices. This result extends the well-known formulas which use the Kreweras complement for non-crossing partitions. At the end, we define a notion of A-freeness of higher order in order to deal with the expansion in $\frac{1}{N}$ of the observables, and prove similar theorems for A-freeness of higher order.

Using the notion of asymptotically A-freeness, in Section 8 is proved a A-non-commutative central limit theorem, generalization of the non-commutative central limit theorem, which is shown in Section 9 to generalize also the classical central limit theorem.

In Section 9, we gather some simple and known facts about approximation of probability measures using the empirical eigenvalues distribution of random matrices. This gives two ways to inject the classical probabilities in the setting developed in the article: one of them allows us to express classical cumulants as non-commutative cumulants. This shows that probabilistic fluctuations of random matrices invariant by conjugation by the easy orthogonal groups could also be handled in this new setting.

In the last section, Section 10, we study Levy processes on $\mathcal{M}_N(\mathbb{C})$ and their asymptotic properties when the size N goes to infinity. We prove a new general result on the convergence in probability of the observables and the existence of expansion in series in $\frac{1}{N}$ in Section 10.2. In order to illustrate these theorems, we apply them to additive and multiplicative Brownian motions: for example we recover well-known results about

GUE and brownian motion on $U(N)$. A matricial Wick theorem is explained in order to illustrate the notion of N-finite dimension A-non-commutative cumulants we defined earlier. Then we apply the results of Section 10.2 to Levy processes with an application to free infinitely divisible measures: we extend known results ([**5**], [**10**]) to the real setting where one considers $\mathcal{M}_N(\mathbb{R})$-valued Levy processes.

1.2. Acknowledgements. — This work has been realized during my PhD at the LPMA which offered me the necessary liberty to complete this article. Many thanks to the researchers and administrative staff of the LPMA. I am grateful to my PhD advisor Thierry Levy for supporting this research, for his helpful comments and corrections which led to improvements in this manuscript and for the useful discussions about mathematics and other subjects. I would like to express my special gratitude to Terence Tao, particularly for his blog which, during a period of doubts, made me enjoy maths again. This project began when I wanted to understand the link between the work of T. Levy and the formulation given at the Pims summer school by David Brydges of Wick's theorem, I am really thankful to him for this. Also many thanks are due to Antoine Dahlqvist who explained me the duality between permutations and partitions and to Camille Male with whom I had great discussions about the partition and traffic point of views on random matrices. He brought to my knowledge the Proposition 7.1 which gives even more interest to the following article [**17**]. I would like to thank Tom Halverson and Arun Ram for answering my questions about partitions. I am also grateful to Guillaume Cébron for introducing me to Tanabe's result. I would like to thank T. Banica and J. Mingo for several useful discussions and R. Speicher for inviting me to Saarbrücken and Oberwolfach, invitations which allowed me to improve the papers. Thanks to Patrick Gabriel for his interest in my work and the discussions related to this work that we had together, Marie-Françoise Gabriel and Catherine Lam for trying to correct the English in this manuscript. At last, a thought to all the people which are supporting researchers and whose names never appear in the acknowledgements.

2. Random matrices and observables

2.1. Introduction to random matrices. — Random matrices are random variables which take values in a set of matrices $\mathcal{M}_{n,m}(\mathbb{C})$. In this article we will only consider square random matrices, we will say that M is of size N if M is a square matrix of size $N \times N$ and we will write $M \in \mathcal{M}_N(\mathbb{C})$.

Actually any random matrix M of size N that we will consider in this article is automatically supposed to be in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$. This means that for any i, j in $\{1, ..., N\}$ and for any positive integer k, $\mathbb{E}[\mid M_{i,j} \mid^k] < \infty$. Besides, all the random matrices we consider are defined on the same probability space. In fact, one can only suppose that for each integer N, all the random matrices of size N that we consider are defined on the same probability space.

In this article, we are interested in the convergence of some observables when the size of the random matrices goes to infinity. For example, let $(M_N)_{N\in\mathbb{N}}$ be a sequence of random matrices such that $M_N \in \mathcal{M}_N(\mathbb{C})$. In the usual method of moments, one is

interested in the convergence of:

$$\frac{1}{N}Tr\left((M_N)^k\right), \tag{1}$$

or the convergence of the mean moments:

$$\mathbb{E}\left[\frac{1}{N}Tr\left((M_N)^k\right)\right], \tag{2}$$

when $k \in \mathbb{N}$.

The method of moments is justified by the fact that a $N \times N$ random matrix M_N has N random eigenvalues: $\lambda_1(M_N), ..., \lambda_N(M_N)$ and for any integer $k \in \mathbb{N}$:

$$\frac{1}{N}Tr\left((M_N)^k\right) = \frac{1}{N}\sum_{i=1}^{N}\lambda_i(M_N)^k.$$

Thus, if we define the empirical eigenvalues distribution of M_N by:

$$\eta_{M_N} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(M_N)},$$

we get that:

$$\frac{1}{N}Tr\left((M_N)^k\right) = \int_{\mathbb{C}} z^k \eta_{M_N}(dz).$$

Let us suppose that M_N is symmetric or Hermitian for any integer N: η_{M_N} is a measure supported by the real line. Using the Carleman's continuity theorem, Theorem 2.2.9 in [**32**], one can use the convergence of (1) or (2) to prove that almost surely or in expectation the random measures $(\eta_{M_N})_{N\in\mathbb{N}}$ converge when N goes to infinity. Likewise, we can do the same for unitary or orthogonal matrices.

Theorem 2.1. — *Let $(M_N)_{N\in\mathbb{N}}$ be a sequence of random matrices such that for any integer N, M_N is of size N. We suppose that for any positive integer k,*

$$\mathbb{E}\left[\frac{1}{N}Tr\left((M_N)^k\right)\right]$$

converges when N goes to infinity.

1. *If for any integer N, M_N is a unitary or orthogonal matrix then there exists μ a probability measure on the circle such that the mean empirical eigenvalues distribution $\mathbb{E}[\eta_{M_N}]$ of M_N converges to μ as N tends to infinity.*
2. *If for any integer N, M_N is symmetric or Hermitian (resp. skew-symmetric or skew-hermitian), under a condition of uniform subgaussianity on $(\mathbb{E}[\eta_{M_N}])_{N\in\mathbb{N}}$, the measure $\mathbb{E}[\eta_{M_N}]$ converges to a probability measure, named μ, supported by the real line (resp. the imaginary line) as N goes to infinity.*

Besides, for any integer k:

$$\int_{\mathbb{C}} z^k \mu(dz) = \lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\left((M_N)^k\right)\right].$$

Proof. — We will only remind the reader the proof for the unitary case, the symmetric case being deduced from a tightness argument and Theorem 2.2.9 of [**32**]. Let $(M_N)_{N\in\mathbb{N}}$ be a sequence of random matrices such that for any integer N, M_N is of size N. For any integer N, the measure $\mathbb{E}[\eta_{M_N}]$ is supported by the unit circle since the eigenvalues of a unitary matrix are on the unit circle $\mathbb{U}$. Thus the sequence $(\mathbb{E}[\eta_{M_N}])_{N\in\mathbb{N}}$ is tight: we only have to show that any limit of a subsequence is determined by its positive moments. Let μ and μ' be two probability measures on the circle such that for any integer k:

$$\int_{\mathbb{U}} z^k \mu(dz) = \int_{\mathbb{U}} z^k \mu'(dz). \tag{3}$$

Let us show that $\mu = \mu'$.

The unital algebra generated by the functions $z \mapsto z$ and $z \mapsto \overline{z}$ is dense in the set of continuous complex functions on $\mathbb{U}$. Indeed, it is stable by complex conjugation, and it separates points, thus dense by the complex Stone-Weierstrass's theorem. Yet, on $\mathbb{U}$, it is also the algebra of functions which are linear combination of the functions $z \mapsto z^k$ and $z \mapsto \overline{z}^k$ where $k \in \mathbb{N}$.

Yet, because of (3), for any integer k, $\int_{\mathbb{U}} \overline{z^k}\mu(dz) = \overline{\int_{\mathbb{U}} z^k \mu(dz)} = \overline{\int_{\mathbb{U}} z^k \mu'(dz)} = \int_{\mathbb{U}} \overline{z^k}\mu'(dz)$. By linearity and density, we see that the two measures μ and μ' are equal. □

Actually, the author's motivation in order to study the convergence of (2) does not come from the convergence of the empirical eigenvalues distributions. Indeed, the motivation comes from Yang-Mills theory and random holonomy fields ([**18**]) where one of the main problem is to prove convergence of the expectation of generalizations of observables of the form (2). Thus, in this article, we will mainly focus on the observables.

2.2. Observables. —

2.2.1. Definitions: the zero order case. — Let N and k be two positive integers. Let $M_1, ..., M_k$ be k random matrices of size N defined on the same probability space. We would like to measure the system $(M_1, ..., M_k)$. Let Tr^k be the trace on $\left(\mathbb{C}^N\right)^{\otimes k}$ defined according to the canonical basis $\{e_{i_1} \otimes ... \otimes e_{i_k}, (i_1, ..., i_k) \in N^k\}$ such that $Tr^k(Id^{\otimes k}) = N^k$. Recall the definition of the set of partitions $\mathcal{P}_k$ and Definition 2.15 in [**15**] where we defined the representation $\rho_N^{\mathcal{P}_k}$ of $\mathbb{C}[\mathcal{P}_k(N)]$. Let p be in $\mathcal{P}_k$.

Definition 2.1. — *The mean p-normalized moment of $(M_1, ..., M_k)$ is:*

$$\mathbb{E}m_p(M_1, ..., M_k) = \frac{1}{Tr^k(\rho_N^{\mathcal{P}_k}(p))}\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\rho_N^{\mathcal{P}_k}({}^t p)\right)\right].$$

We will denote $Tr_N(p) = Tr^k(\rho_N^{\mathcal{P}_k}(p))$ and from now on, we will forget about the $\rho_N^{\mathcal{P}_k}$: any partition in $\mathcal{P}_k$ will be seen as an endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$, where the parameter N has to be deduced from the formulas.

For any k-tuple $(M_1, ..., M_k)$ of matrices of size N:

$$Tr^k\left[(M_1 \otimes ... \otimes M_k)\ {}^t p\right] = \sum_{i_1,...,i_k,i_{1'},...,i_{k'}\in\{1,...,N\}/p_{i_{1'},...,i_{k'}}^{i_1,...,i_k}=1} (M_1)_{i_{1'}}^{i_1}...(M_k)_{i_{k'}}^{i_k}$$

where for any $l \in \{1,...,k\}$, and any $(i,j) \in \{1,...,N\}^2$, $M_l(e_j) = (M_l)_i^j e_i$ and the notation $p_{i_{1'},...,i_{k'}}^{i_1,...,i_k}$ was defined in Definition 2.14 of [**15**]. This implies the following lemma which shows that the notion of mean p-normalized moments generalizes the observables (2), as explained in [**23**].

Lemma 2.1. — *Let $\sigma \in \mathfrak{S}_k$, let M be a random of size N. Then:*

$$\mathbb{E}m_\sigma(M,...,M) = \mathbb{E}\left[\prod_{c \text{ cycle of } \sigma} \frac{1}{N} Tr\left(M^{\#c}\right)\right],$$

where $\#c$ stands for the length of the cycle c.

When p is an element of the Brauer algebra, then $\mathbb{E}m_p(M,...,M)$ is the expectation of a product of normalized traces of products in M and ${}^tM$. In general, when $p \in \mathcal{P}_k$, the calculation of $\mathbb{E}m_p(M,...,M)$ has to do with the computation of the graph-test observables of [**26**], where one also considers non-connected graph-tests. This link will be developed in the forthcoming article [**14**].

Let I be a set of indexation. We will study the convergence of observables of some family $(M_i^N)_{i\in I}$ of random matrices of size N when N goes to infinity. In this setting, we consider each sequence of random matrices $\left(M^N\right)_{N\in\mathbb{N}}$, such that for any integer N the matrix M^N is of size N, as an element of the algebra

$$\prod_{N=1}^{\infty}\left(L^{\infty^-}(\Omega)\otimes\mathcal{M}_N(\mathbb{C})\right)$$

which we will denote by $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$.

For any family of elements $(a_i)_{i\in I}$ in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$, the algebra generated by $(a_i)_{i\in I}$ is simply:

$$\mathcal{A}\left(a_i\right)_{i\in I} = \left\{P(a_{i_1},...,a_{i_k}) \mid (i_1,...,i_k)\in I^k, P\in\mathbb{C}\{X_1,...,X_k\}, k\in\mathbb{N}\right\}, \tag{4}$$

where $\mathbb{C}\{X_1,...,X_k\}$ is the algebra of non-commutative polynomials. Recall the notation that we set in [**15**]: the letter A stands either for $\mathcal{P}$, $\mathcal{B}$ or $\mathfrak{S}$ and thus for any integer k, A_k stands either for $\mathcal{P}_k$, $\mathcal{B}_k$ or $\mathfrak{S}_k$. For any $i\in I$, let $(M_i^N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$.

Definition 2.2. — *Let k be a positive integer, let $(i_1,...,i_k)\in I^k$. The sequence $\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments if for any $p\in A_k$,*

$$\mathbb{E}m_p\left(M_{i_1}^N,...,M_{i_k}^N\right)$$

converges when N goes to infinity.

The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments if for any positive integer k, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_k^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, the sequence $\left(B_1^N\otimes...\otimes B_k^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments.

Notation 2.1. — *Let us suppose that the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in A-moments. For any positive integer k, for any* $(i_1,...,i_k)\in I^k$*, for any* $p\in A_k$*, we set:*

$$\mathbb{E}m_p(M_{i_1},...,M_{i_k})=\lim_{N\to\infty}\mathbb{E}m_p(M_{i_1}^N,...,M_{i_k}^N).$$

In fact, it is an easy exercise to prove the following lemma.

Lemma 2.2. — *The family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in A-moments if and only if for any integer k, for any* $(i_1,...,i_k)\in I^k$*, the sequence* $\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ *converges in expectation in* A_k*-moments.*

Remark 2.1. — *If the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in:*

- *expectation in* $\mathfrak{S}$*-moments then it converges in non-commutative distribution,*
- *expectation in* $\mathcal{B}$*-moments if and only if* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\cup\left({}^t(M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$*, the family and the transpose of the family, converges in* $\mathfrak{S}$*-moments,*
- *expectation in* $\mathcal{B}$*-moments then* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\cup\left({}^t(M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in non-commutative distribution,*
- *expectation in* $\mathcal{P}$*-moments then* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in distribution of traffics (see* [**26**] *and the correspondence explained in* [**14**]*).*

Remark 2.2. — *Let us remark also that the convergence in expectation in* $\mathcal{P}$*-moments implies the convergence in non-commutative distribution of the algebra of traffics generated by* $\left(M_i^N\right)_{i\in I}$*. This is the point of view developed in the paper in preparation* [**13**] *; in this paper, we do not take this point of view. Yet let us make a similar remark than C. Male in* [**26**]*, if* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in* $\mathcal{P}$*-moments, then for any i and j in I, the Hadamard product* $(M_i^N\circ M_j^N)_{N\in\mathbb{N}}$ *converges in* $\mathcal{S}$*-moments, and in fact also in* $\mathcal{P}$*-moments.*

Let $(M_N)_{N\in\mathbb{N}}$ be in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Using Lemma 2.1, we see that $(M_N)_{N\in\mathbb{N}}$ converges in expectation in $\mathfrak{S}$-moments if for any positive integer k, any k-tuple $(n_1,...,n_k)$ of integers,

$$\mathbb{E}\left[\prod_{i=1}^k\frac{1}{N}Tr(M_N^{n_i})\right],$$

converges when N goes to infinity. This is slightly more demanding than the usual convergence of moments, where one only considers the convergence of the mean moments (2). This is due to the fact that usually one works with sequences of random matrices such that some concentration of measure happens and implies that the following asymptotic factorization property holds.

2.2.2. The A-factorization property. — Recall the operation $\otimes$ on partitions defined in Definition 2.1 of [**15**]. Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$.

Definition 2.3. — *Let us suppose that the family* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *converges in expectation in A-moments. We say that* $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ *satisfies the asymptotic A-factorization*

property if for any integer k_1 and k_2, for any $p_1 \in A_{k_1}$ and $p_2 \in A_{k_2}$, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_{k_1+k_2}^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\big(\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}\big)$,

$$\mathbb{E}m_{p_1\otimes p_2}\left(B_1,...,B_{k_1+k_2}\right) = \mathbb{E}m_{p_1}(B_1,...,B_{k_1})\mathbb{E}m_{p_2}(B_{k_1+1},...,B_{k_2}).$$

Let us explain the importance of the factorization property.

Theorem 2.2. — *Let us suppose that $(M_i)_{i\in I}$ is in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{R})$, converges in expectation in A-moments and satisfies the asymptotic A-factorization property, then the A-moments of $(M_i)_{i\in I}$ converge in probability. This means that for any integer k, any $p\in A_k$, any $i_1,...,i_k$ in I:*

$$m_p(M_{i_1}^N\otimes...\otimes M_{i_k}^N) = \frac{1}{N^{\mathsf{nc}(p\vee id)}}Tr^k\left((M_{i_1}^N\otimes...\otimes M_{i_k}^N)^t p\right)$$

converges in probability as N goes to infinity to $\mathbb{E}m_p\left(M_{i_1},...,M_{i_k}\right)$.

If $(M_i)_{i\in I}$ is a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$, the same results holds if we suppose that either $(M_i,\overline{M}_i)_{i\in I}$ or $(M_i,(M_i)^*)_{i\in I}$ satisfy the asymptotic A-factorization.

Proof. — Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{R})$ which converge in expectation in A-moments and which satisfies the asymptotic A-factorization property. We use an argument from T. Levy [**24**]. Let k be an integer, let $p\in A_k$, and let $i_1,...,i_k$ be in I. The variance $\mathbb{V}\mathsf{ar}\big[m_p\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)\big]$ is equal to:

$$\mathbb{E}\left[m_{p\otimes p}\big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\otimes M_{i_1}^N\otimes...\otimes M_{i_k}^N\big)\right]-\mathbb{E}\left[m_p\big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\big)\right]^2,$$

which, by definition, is equal to:

$$\mathbb{E}m_{p\otimes p}\big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\otimes M_{i_1}^N\otimes...\otimes M_{i_k}^N\big)-\left[\mathbb{E}m_p\big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\big)\right]^2.$$

Thus the variance has a limit which is given by:

$$\mathbb{E}m_{p\otimes p}(M_{i_1}\otimes...\otimes M_{i_k}\otimes M_{i_1}\otimes...\otimes M_{i_k})-[\mathbb{E}m_p(M_{i_1}\otimes...\otimes M_{i_k})]^2,$$

which is equal to zero since $(M_i)_{i\in I}$ satisfies the asymptotic A-factorization property.

The same argument can be applied for a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$, by computing $\mathbb{V}\mathsf{ar}\big[m_p\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)\big]$, which is equal to:

$$\mathbb{E}\Big[m_{p\otimes p}\Big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\otimes\overline{M_{i_1}^N}\otimes...\otimes\overline{M_{i_k}^N}\Big)\Big]$$
$$-\mathbb{E}\big[m_p\big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\big)\big]\,\mathbb{E}\Big[m_p\Big(\overline{M_{i_1}^N}\otimes...\otimes\overline{M_{i_k}^N}\Big)\Big],$$

or:

$$\mathbb{E}\Big[m_{p\otimes{}^tp}\Big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\otimes\left(M_{i_1}^N\right)^*\otimes...\otimes\left(M_{i_k}^N\right)^*\Big)\Big]$$
$$-\mathbb{E}\big[m_p\big(M_{i_1}^N\otimes...\otimes M_{i_k}^N\big)\big]\,\mathbb{E}\Big[m_{{}^tp}\Big(\left(M_{i_1}^N\right)^*\otimes...\otimes\left(M_{i_k}^N\right)^*\Big)\Big],$$

and by using the same arguments than before. □

As a consequence, for example, when one considers orthogonal (resp. unitary) matrices, the factorization property of $(M_i)_{i\in I}$ (resp. $(M_i,\overline{M_i})_{i\in I}$ or $(M_i,(M_i)^*)_{i\in I}$) implies the convergence of probability of the empirical eigenvalues distributions. Actually, one

can go further than the convergence in probability using combinatorial arguments: indeed, in [**15**] and [**16**] we developed the tools in order to compute algebraic fluctuations: let us suppose that the algebraic fluctuations are in fact in powers of $\frac{1}{N^2}$, then the variance of the moments would be in $\frac{1}{N^2}$, and it would imply that the moments converge almost surely.

2.2.3. Definitions: the higher order case. — We can also define the notion of convergence in expectation in A-moments up to any order of fluctuations for $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$. Let m be a non-negative integer.

Definition 2.4. — *Let k be a positive integer and let $(i_1,...,i_k)\in I^k$. The sequence $\left(M_{i_1}^N\otimes...\otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments up to order m of fluctuations if there exists a family of complex numbers*

$$\left(\mathbb{E}m_p^i\left(M_1^N,...,M_k^N\right)\right)_{p\in A_k,i\in\{0,...,m\},N\in\mathbb{N}},$$

such that:

- $\forall p\in A_k,\forall N\in\mathbb{N}^*,\mathbb{E}m_p(M_1^N,...,M_k^N)=\sum_{i=0}^m\frac{\mathbb{E}m_p^i(M_1^N,...,M_k^N)}{N^i}$,
- $\forall p\in A_k,\forall i\in\{0,...,m-1\},\mathbb{E}m_p^i(M_1^N,...,M_k^N)$ *does not depend on* N,
- $\forall p\in A_k,\mathbb{E}m_p^m(M_1^N,...,M_k^N)$ *converges as* N *goes to infinity.*

The family $\left(\mathbb{E}m_p^i(M_1^N,...,M_k^N)\right)_{p\in A_k,i\in\{0,...,m\},N\in\mathbb{N}}$ is uniquely defined and for any $i\in\{0,...,m\}$, any $p\in A_k$ and any $N\in\mathbb{N}$, $\mathbb{E}m_p^i(M_1^N,...,M_k^N)$ is called the ith-order fluctuations of the mean p-normalized moment.

The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments up to order m of fluctuations if for any positive integer k, for any element $(B_1^N)_{N\in\mathbb{N}},...,$ $(B_k^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, the sequence $(B_1^N\otimes...\otimes B_k^N)_{N\in\mathbb{N}}$ converges in expectation in A_k-moments up to order m of fluctuations.

Notation 2.2. — *Let us suppose that the family $\left(\left(M_i^N\right)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments up to order m of fluctuations. For any positive integer k, for any k-tuple $(i_1,...,i_k)\in I^k$, for any $j\in\{0,...,m\}$ and any $p\in A_k$, we set:*

$$\mathbb{E}m_p^j(M_{i_1},...,M_{i_k})=\lim_{N\to\infty}\mathbb{E}m_p^j(M_{i_1}^N,...,M_{i_k}^N).$$

2.2.4. Convergence and transposition. — In fact the convergence in expectation in $\mathcal{B}$ (resp. $\mathcal{P}$)-moments implies the convergence, of the family and its transpose family, in expectation in $\mathcal{B}$ (resp. $\mathcal{P}$)-moments. In order to prove so, we need the following definition. Let k be a positive integer.

Definition 2.5. — *Let p be a partition in $\mathcal{P}_k$, let $k_1<k$. The partition $S_{k_1}(p)$ is the partition obtained by permuting i with i' for any $i\in\{k_1+1,...,k\}$ in the definition of p.*

Let us give an example. Let $k=4$, $k_1=2$ and let $p=\{\{1,3,2'\},\{2,4',1'\},\{4,3'\}\}$, then $S_{k_1}(p)=\{\{1,3',2'\},\{2,4,1'\},\{4',3\}\}$.

Lemma 2.3. — *Let k_1 and k_2 be non-negative integers such that $k_1 + k_2 = k$. Let $E \in \mathsf{End}\left((\mathbb{C}^N)^{\otimes k_1}\right)$ and $F \in \mathsf{End}\left((\mathbb{C}^N)^{\otimes k_2}\right)$, for any $p \in \mathcal{P}_k$:*

$$m_p(E \otimes {}^tF) = m_{S_{k_1}(p)}(E \otimes F).$$

For any non-negative integer k_1 such that $k_1 < k$, the two sets $\mathcal{B}_k$ and $\mathcal{P}_k$ are stable by the operation S_{k_1}. This remark and Lemma 2.3 imply the following proposition.

Proposition 2.1. — *Let us suppose that A stands either for $\mathcal{B}$ or $\mathcal{P}$. Let m be a non-negative integer. Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments up to order m of fluctuations then $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I} \cup \left(({}^tM_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments up to order m of fluctuations.*

For example, this implies that for a family of random orthogonal matrices $\left((O_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$, in order to prove that the family $\left((O_i^N)_{N\in\mathbb{N}}\right)_{i\in I} \cup \left(((O_i^N)^{-1})_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in $\mathfrak{S}$-moments, we only have to prove that $\left((O_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in $\mathcal{B}$-moments.

2.2.5. Some special cases. — We will see along this paper, that in some special cases, the theory becomes even easier. Let m be a non-negative integer. Using the fact that for any symmetric of skew-symmetric matrix M, the matrix ${}^tM$ is equal either to M or $-M$, one has the following result.

Proposition 2.2. — *Let $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{R})$. If for any positive integer N and any $i \in I$, M_N is symmetric or skew-symmetric, the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in $\mathcal{B}$-moments up to order m of fluctuations as N goes to infinity if and only if it converges in expectation in $\mathfrak{S}$-moments up to order m of fluctuations as N goes to infinity.*

For orthogonal matrices, one has a similar result when considering only one element $(M^N)_{N\in\mathbb{N}}$ in $L^{\infty^-} \otimes \mathcal{M}(\mathbb{R})$. From now on, we denote by $\mathfrak{S}(N)$ the set of permutation matrices of size N.

Proposition 2.3. — *Let $(M^N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{R})$. If for any positive integer N, M^N is orthogonal, the matrix M^N converges in expectation in $\mathcal{B}$-moments up to order m of fluctuations as N goes to infinity if and only if it converges in expectation in $\mathfrak{S}$-moments up to order m of fluctuations as N goes to infinity.*

Besides if for any positive integer N, M^N is almost surely in $\mathfrak{S}(N)$, the matrix M^N converges in expectation in $\mathcal{P}$-moments up to order m of fluctuations as N goes to infinity if and only if it converges in expectation in $\mathfrak{S}$-moments up to order m of fluctuations as N goes to infinity.

The first part of this proposition comes from the fact that for any orthogonal matrix M, $M\,{}^tM = Id$. The second part is a consequence of the following general proposition. Recall the notion of convergence in moments that we defined in Definition 4.4 of [**15**].

Proposition 2.4. — *Let k be a positive integer. For each integer N, let E_N be in the algebra generated by the elements $S^{\otimes k}$ with $S \in \mathfrak{S}(N)$ and denoted by $\mathbb{C}\left[\rho^k_{\mathfrak{S}(N)}\right]$. Then $(E_N)_{N\in\mathbb{N}}$ converges in $\mathcal{P}_k$-moments if and only if it converges in $\mathfrak{S}_k$-moments.*

Again, this is a simple consequence of the following lemma.

Lemma 2.4. — *Let k and N be two positive integers. Let p be in $\mathcal{P}_k$, there exist an integer $l \leq k$ and $\sigma \in \mathfrak{S}_l$ such that for any $S \in \mathfrak{S}(N)$:*

$$m_p\left(S^{\otimes k}\right) = m_\sigma\left(S^{\otimes l}\right),$$

with the convention that $m_\emptyset\left(S^{\otimes 0}\right) = 1$.

Proof. — The proof uses a recurence argument. Let k and N be two positive integers, let p be in $\mathcal{P}_k$ and let S be a permutation in $\mathfrak{S}(N)$. If $k = 1$, then $m^{\{\{1\},\{1'\}\}}(S)$ is equal to one and thus equal to $m_\emptyset(S^{\otimes 0})$. Let us suppose that $k \neq 1$, then, with a slight abuse of notation:

$$m_p(S^{\otimes k}) = \frac{1}{Tr(p)} Tr(S^{\otimes k}\ {}^t p) = \frac{1}{N^{\mathsf{nc}(p \vee id)}} \sum_{i_1,...,i_k,i_{1'},...,i_{k'} \in \{1,...,N\}, p^{i_1,...,i_k}_{i_{1'},...,i_{k'}}=1} S^{i_1}_{i_{1'}} ... S^{i_k}_{i_{k'}}.$$

Let us suppose that $p \notin \mathfrak{S}_k$, one of the three assertions is true:

- there exist r, s in $\{1, ..., k\}$ such that r and s are in the same block of p,
- there exist r', s' in $\{1', ..., k'\}$ such that r' and s' are in the same block of p,
- there exist r in $\{1, ..., k\} \cup \{1', ..., k'\}$ such that $\{r\}$ is a block of p.

Let us remark that for any $i, j, l \in \{1, ..., N\}$,

$$S^j_i S^l_i = S^j_i \delta_{l=j} \text{ and } S^i_j S^i_l = S^i_j \delta_{l=j}.$$

Besides, for any $i_1, ..., i_k$ and $i_{1'}, ..., i_{k'}$ in $\{1, ..., N\}$, we have:

$$\sum_{i=1}^N S^i_{i'_1} S^{i_2}_{i'_2} ... S^{i_k}_{i'_k} = S^{i_2}_{i'_2} ... S^{i_k}_{i'_k} \text{ and } \sum_{i=1}^N S^{i_1}_i S^{i_2}_{i'_2} ... S^{i_k}_{i'_k} = S^{i_2}_{i'_2} ... S^{i_k}_{i'_k}.$$

The first two equations assert that, when there exists an horizontal line in p, which means that there exist r, s (or r', s') in $\{1, ..., k\}$ (resp. $\{1', ..., k'\}$) such that r and s are in the same block of p, then we can glue the two blocks r' and s' (resp. r and s) belong to and then remove the column of r or s (resp. r' or s') without changing the value of the sum we need to compute. We get a new partition $p' \in \mathcal{P}_{k-1}$ with the same number of cycles than p: we get then the equality $m_p(S^{\otimes k}) = m_{p'}(S^{\otimes k-1})$.

The last two equations assert that, each time there exists a block of p which is composed of one unique element, let say r in $\{1, ..., k\}$, one can delete the column of this element and the sum over i_r without changing the value of the sum we need to compute. Let us denote by p' the new partition. If $\{r'\}$ is not a block of p then the new partition has as many cycles as p. If $\{r'\}$ is a block of p, we lose one block by doing so, but, one must not forget that we still have to sum over $i_{r'}$ even if it does not appear any more. Thus, at the end we still get the equality $m_p(S^{\otimes k}) = m_{p'}(S^{\otimes k-1})$. □

3. Schur-Weyl's duality

In the last section, we introduced observables proportional to:

$$\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\ {}^t p\right)\right].$$

It is not difficult to see that for any $S \in \mathfrak{S}(N)$ and any $p \in \mathcal{P}_k$, $S^{\otimes k} p (S^{-1})^{\otimes k} = p$. Thus, using the tracial property, $\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\ {}^t p\right)\right]$ is equal to:

$$Tr^k\left(\left(\frac{1}{N!}\sum_{S\in\mathfrak{S}(N)}(S^{-1})^{\otimes k}\mathbb{E}[M_1\otimes ... \otimes M_k]S^{\otimes k}\right){}^t p\right).$$

Let us remark that $\frac{1}{N!}\sum_{S\in\mathfrak{S}(N)}(S^{-1})^{\otimes k}\mathbb{E}[M_1\otimes ... \otimes M_k]S^{\otimes k}$ commutes with $S^{\otimes k}$ for any $S \in \mathfrak{S}(N)$. The same discussion could have been made by replacing $\mathcal{P}_k$ by $\mathcal{B}_k$ (resp. $\mathfrak{S}_k$), $\mathfrak{S}(N)$ by $O(N)$ (resp. $U(N)$) and the sum $\frac{1}{N!}\sum_{S\in\mathfrak{S}(N)}$ by an integration against the Haar measure of $O(N)$ (resp. of $U(N)$).

This motivates the study of endomorphisms of $\left(\mathbb{C}^N\right)^{\otimes k}$ which commute with $M^{\otimes k}$, where M is either in $\mathfrak{S}(N)$, $O(N)$ or $U(N)$. We introduce for this the Schur-Weyl's duality and some similar statements. We will focus on the duality between:

- the unitary group $U(N)$ and the symmetric group $\mathfrak{S}_k$,
- the orthonormal group $O(N)$ and the Brauer algebra $\mathbb{C}[\mathcal{B}_k(N)]$,
- the symmetric group $\mathfrak{S}(N)$ and the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$.

One can find more informations about these dualities in [**19**], and for the last duality one can look at [**22**], [**27**] and [**20**].

Remark 3.1. — *Actually, all the paper can be applied not only to the three groups we considered but also to the unitary reflection groups and to the easy orthogonal groups. We will not explain the duality for the imprimitive unitary reflection group $G(m,p,n)$, defined in Section* 2.1 *of* [**31**] *since the partition subalgebra that one get is not simple to define, and we will never use it in the article. For a definition of easy orthogonal groups, one can have a look to* [**3**] *and there exists a duality between:*

- *the group $\mathfrak{S}(N)\times\mathbb{Z}/2\mathbb{Z}$ (group of permutation matrices multiplied by ± 1) and the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$,*
- *the hyperoctahedral group $H(N)$, which consists of matrices which have exactly one nonzero enty in each row and each column which is equal to ± 1, and the algebra $\mathbb{C}\left[\mathcal{H}_k(N)\right]$, where $\mathcal{H}_k$ is the subset of $\mathcal{P}_k$ of partitions which have blocks of even size,*
- *the bistochastic group $B(N)$, which consists of orthogonal matrices having sum* 1 *in each row and each column and the algebra $\mathbb{C}[\mathcal{B}s_k(N)]$, where $\mathcal{B}s_k$ is the subset of $\mathcal{P}_k$ of partitions which have blocks of size less of equal to two.*
- *the group $B(N)\times\mathbb{Z}/2\mathbb{Z}$ (group of bistochastic matrices multiplied by ± 1) and the algebra $\mathbb{C}[\mathcal{B}s_k(N)]$.*

Thus, in the following Notation 3.1, one could also consider (G,A) being one of the five couples:

$$(\mathfrak{S},\mathcal{P}),(O,\mathcal{B}),(U,\mathfrak{S}),(H(N),\mathcal{H}),(B(N),\mathcal{B}s).$$

We ommited in the list the two groups $\mathfrak{S}(N)\times\mathbb{Z}/2\mathbb{Z}$ and $B(N)\times\mathbb{Z}/2\mathbb{Z}$ since the invariance in law by conjugation by $\mathfrak{S}(N)\times\mathbb{Z}/2\mathbb{Z}$ (resp. $B(N)\times\mathbb{Z}/2\mathbb{Z}$) is equivalent to the invariance in law by $\mathfrak{S}(N)$ (resp. $B(N)$). We decided to focus on the three first couples since $(U,\mathfrak{S})$ is the usual setting for asymptotic freeness, $(O,\mathcal{B})$ is interesting as soon as we consider matrices and their transpose and $(\mathfrak{S},\mathcal{P})$ is the most general setting. Yet, it

has to be noticed that in Section 7.1.4, we prove a new assertion using the hyperoctahedral group stating that in fact, in order to get asymptotic freeness, it is enought to have independence and invariance by conjugation by the hyperoctahedral group.

We have made a distinction between $\mathfrak{S}_k$ and $\mathfrak{S}(N)$. Indeed, $\mathfrak{S}_k$ will stand for the symmetric group, seen as a group of permutations, and $\mathfrak{S}(N)$ will be seen as the subgroup of $GL(N)$ of permutation matrices.

3.1. Tensor action of a subgroup of $GL(N)$. — Let k and N be two positive integers. Let G be any subgroup of $GL(N)$, for example $U(N)$, $O(N)$ or $\mathfrak{S}(N)$. We can define a natural action of G on the vector space $\left(\mathbb{C}^N\right)^{\otimes k}$.

Definition 3.1. — *The tensor action of G on $\left(\mathbb{C}^N\right)^{\otimes k}$ is the application:*

$$\rho_G^k : G \to \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$$
$$U \mapsto \rho_G^k(U),$$

defined such that for any $U \in G$, any $x_1, ..., x_k \in \mathbb{C}^N$,

$$\rho_G^k(U)(x_1 \otimes ... \otimes x_n) = Ux_1 \otimes ... \otimes Ux_n.$$

In representation theory, it is common to study the set of endomorphisms which commute with the action of the group. Let us define the commutant and some algebras associated with the representations we have just defined.

Definition 3.2. — *Let C be a subalgebra of $\mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right)$. We denote by C' the commutant of C defined by:*

$$C' = \left\{E \in \mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right), \forall F \in C, EF = FE\right\}.$$

Definition 3.3. — *For any subgroup $G(N)$ of $GL(N)$, we define:*

$$\mathbb{C}\left[\rho_{G(N)}^k\right] = \mathbb{C}\left[\left\{\rho_G^k(g), g \in G\right\}\right] \subset \mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right).$$

For A_k being either $\mathcal{P}_k$, $\mathcal{B}_k$ or $\mathfrak{S}_k$, we define:

$$\mathbb{C}\left[\rho_N^{A_k}\right] = \mathbb{C}\left[\left\{\rho_N^{A_k}(p), p \in A_k\right\}\right] \subset \mathsf{End}\left((\mathbb{C}^N)^{\otimes k}\right).$$

The Schur-Weyl's duality and some similar dualities (as the partition-symmetric group duality proved in [**22**] and [**27**]) assert that the algebras we have just defined are mutual commutants.

Theorem 3.1. — *We have the set of equalities:*

1. *$U(N) - \mathfrak{S}_k$-duality:*

$$\left(\mathbb{C}\left[\rho_{U(N)}^k\right]\right)' = \mathbb{C}\left[\rho_N^{\mathfrak{S}_k}\right] \text{ and } \left(\mathbb{C}\left[\rho_N^{\mathfrak{S}_k}\right]\right)' = \mathbb{C}\left[\rho_{U(N)}^k\right].$$

2. *$O(N) - \mathcal{B}_k$-duality:*

$$\left(\mathbb{C}\left[\rho_{O(N)}^k\right]\right)' = \mathbb{C}\left[\rho_N^{\mathcal{B}_k}\right] \text{ and } \left(\mathbb{C}\left[\rho_N^{\mathcal{B}_k}\right]\right)' = \mathbb{C}\left[\rho_{O(N)}^k\right].$$

3. $\mathfrak{S}(N)-\mathcal{P}_k$*-duality:*

$$\left(\mathbb{C}\left[\rho^k_{\mathfrak{S}(N)}\right]\right)'=\mathbb{C}\left[\rho^{\mathcal{P}_k}_N\right] \text{ and } \left(\mathbb{C}\left[\rho^{\mathcal{P}_k}_N\right]\right)'=\mathbb{C}\left[\rho^k_{\mathfrak{S}(N)}\right].$$

The last duality is the simplest to show, one can look at the proof in [**20**], and it uses the exclusive basis defined in Definition 2.16 of [**15**].

The main implication of this theorem for this article is that we can extend the definition of coordinate numbers for any endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$ which commutes with the action of $\mathfrak{S}(N)$. As we have seen in [**15**], the representation $\rho^{\mathcal{P}_k}_N$ is injective as soon as $N\geq 2k$, and the representations $\rho^{\mathcal{B}_k}_N$ and $\rho^{\mathfrak{S}_k}_N$ are injective as soon as $N\geq k$. Let us suppose that $N\geq 2k$, let E be an endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$ which commutes with the action of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$. There exists a unique family of reals $\left(\kappa^p(E)\right)_{p\in\mathcal{P}_k}$ such that:

$$E=\sum_{p\in\mathcal{P}_k}\frac{\kappa^p(E)}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}\rho^{\mathcal{P}_k}_N(p).$$

This family is the coordinate numbers of E.

The $U(N)-\mathfrak{S}_k$ (resp. $O(N)-\mathcal{B}_k$) duality asserts that if E commutes with the action of $U(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$, then we can define the coordinate numbers as soon as $N\geq k$ and in this case:

$$\forall p\notin\mathfrak{S}_k(\text{resp. } \forall p\notin\mathcal{B}_k),\kappa^p(E)=0.$$

Using this discussion, we can extend all the results of the article [**15**] to endomorphisms of $(\mathbb{C}^N)^{\otimes k}$ which commute with the action of either $\mathfrak{S}(N)$, $O(N)$ or $U(N)$ on $(\mathbb{C}^N)^{\otimes k}$. Let us define a notation which will be used all along the article.

Notation 3.1. — *The letter G will denote either $\mathfrak{S}$, O or U. Thus for any integer N, $G(N)$ stands either for $\mathfrak{S}(N)$, $O(N)$ or $U(N)$. Besides, by (G,A) we denote one of the three couples $(\mathfrak{S},\mathcal{P})$, $(O,\mathcal{B})$ or $(U,\mathfrak{S})$.*

Let us finish with some definitions about endomorphisms of $\left(\mathbb{C}^N\right)^{\otimes k}$ which will be useful in Section 10. We remind the reader that the permutation σ_I was defined in Definition 2.6 of [**15**]. Let l and k be two integers, such that $l\leq k$.

Definition 3.4. — *Let $A\in\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes l}\right)$, let $B\in\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k-l}\right)$ and let $I\subset\{0,...,k\}$ such that $\#I=l$. The endomorphism $\mathcal{I}_I(A,B)$ is the endomorphism in $\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$ defined by:*

$$\mathcal{I}_I(A,B)=\rho^{\mathfrak{S}_k}_N\left(\sigma_I^{-1}\right)(A\otimes B)\,\rho^{\mathfrak{S}_k}_N(\sigma_I).$$

Let us state a simple lemma that will be often used without referring to it.

Lemma 3.1. — *The application:*

$$\begin{array}{rcl}\mathcal{I}_I:\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes l}\right)\times\mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k-l}\right) & \to & \mathsf{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)\\ (A,B) & \mapsto & \mathcal{I}_I(A,B),\end{array}$$

is bi-linear and for any $(p,p')\in\mathcal{P}_l\times\mathcal{P}_{k-l}$,

$$\mathcal{I}_I(\rho_N^{\mathcal{P}_l}(p)\otimes\rho_N^{\mathcal{P}_{k-l}}(p'))=\rho_N^{\mathcal{P}_k}\left(\sigma_I^{-1}(p\otimes p')\sigma_I\right).$$

Proof. — Let $(p,p')\in\mathcal{P}_l\times\mathcal{P}_{k-l}$. Then:

$$\mathcal{I}_I(\rho_N^{\mathcal{P}_l}(p)\otimes\rho_N^{\mathcal{P}_{k-l}}(p'))=\rho_N^{\mathfrak{S}_k}\left(\sigma_I^{-1}\right)\left(\rho_N^{\mathcal{P}_l}(p)\otimes\rho_N^{\mathcal{P}_{k-l}}(p')\right)\rho_N^{\mathfrak{S}_k}\left(\sigma_I\right).$$

Using Lemma 2.2 of [**15**] which asserts that the representations are compatible with the operation $\otimes$:

$$\mathcal{I}_I(\rho_N^{\mathcal{P}_l}(p)\otimes\rho_N^{\mathcal{P}_{k-l}}(p'))=\rho_N^{\mathcal{P}_k}\left(\sigma_I^{-1}\right)\left(\rho_N^{\mathcal{P}_l}(p\otimes p')\right)\rho_N^{\mathcal{P}_k}\left(\sigma_I\right)=\rho_N^{\mathcal{P}_k}\left(\sigma_I^{-1}(p\otimes p')\sigma_I\right),$$

hence the lemma. □

4. Random Matrices and cumulants

4.1. Cumulants. — Let k be a positive integer, let $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ be a k-tuple of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in expectation in A-moments. We will define three notions of cumulants up to any order of fluctuations associated with $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$: the $\mathfrak{S}$, the $\mathcal{B}$ and the $\mathcal{P}$-cumulants. Recall the results about the distance on $\mathcal{P}_k$, the geodesic order denoted by $\leq$ and the set-geodesic $[id,p]_{A_k}$ which were the core of the article [**15**].

4.1.1. Zero order. —

Definition 4.1. — *The family of A-cumulants*

$$\left(\mathbb{E}\kappa_A^p\left[\left(M_1^N\right)_{N\in\mathbb{N}},...,\left(M_k^N\right)_{N\in\mathbb{N}}\right]\right)_{p\in A_k}$$

is the unique family of complex numbers such that for any $p\in A_k$*:*

$$\mathbb{E}m_p\left(M_1,...,M_k\right)=\sum_{p'\in A_k,p'\leq p}\mathbb{E}\kappa_A^{p'}\left[\left(M_1^N\right)_{N\in\mathbb{N}},...,\left(M_k^N\right)_{N\in\mathbb{N}}\right].$$

We will denote them by $\left(\mathbb{E}\kappa_A^p\left[M_1,...,M_k\right]\right)_{p\in A_k}$.

These numbers are well defined since $\leq$ is a partial order. Besides, since we computed the Mobius function for the geodesic order in Theorem 3.5 of [**15**], one can invert the moment-cumulant formula.

Let us also remark that if the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in $\mathcal{P}$-moments, the restriction of the $\mathcal{P}$-cumulants to $\mathfrak{S}_k$ is not in general equal to the $\mathfrak{S}$-cumulants. This is due to the fact that a geodesic in $\mathcal{P}_k$ between two permutations can go through $\mathcal{P}_k\setminus\mathfrak{S}_k$. Yet, if we suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in $\mathcal{B}$-moments the following assertion is true.

Lemma 4.1. — *For any* $\sigma\in\mathfrak{S}_k$, $\mathbb{E}\kappa_{\mathcal{B}}^{\sigma}\left[M_1,...,M_k\right]=\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma}\left[M_1,...,M_k\right]$.

This lemma is a simple consequence of Lemma 3.9 in [**15**] which asserts that for any $\sigma\in\mathfrak{S}_k$, $[id_k,\sigma]_{\mathcal{B}_k}=[id_k,\sigma]_{\mathfrak{S}_k}$.

Remark 4.1. — *Let us recall the definitions in Section* 4.4 *of* [**15**]. *The cumulants that we defined are linked by the fact that, if A stands either for $\mathfrak{S}$ or $\mathcal{B}$:*

$$\mathsf{E}^{A_k}\left[\left(\mathbb{E}\kappa_A^p[M_1,...,M_k]\right)_{p\in A_k}\right]=\mathcal{C}_\kappa^A\left[\left(\mathbb{E}\kappa_{\mathcal{P}}^p[M_1,...,M_k]\right)_{p\in\mathcal{P}_k}\right],$$

and

$$\mathsf{E}^{\mathfrak{S}_k}\left[\left(\mathbb{E}\kappa_A^p[M_1,...,M_k]\right)_{p\in\mathfrak{S}_k}\right]=\mathcal{C}_\kappa^{\mathfrak{S}}\circ\mathsf{E}^{\mathcal{B}_k}\left[\left(\mathbb{E}\kappa_{\mathcal{B}}^p[M_1,...,M_k]\right)_{p\in\mathcal{B}_k}\right].$$

We can also give a new characterization of the asymptotic A-factorization property whose proof is similar to the proof of the fact that the $\mathcal{R}_A$-transform defined in [**15**] is a bijection between multiplicative elements. Thus, we will not provide a proof for it.

Proposition 4.1. — *The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ satisfies the asymptotic A-factorization property if and only if for any integer k_1 and k_2, for any $p_1\in A_{k_1}$ and $p_2\in A_{k_2}$, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_{k_1+k_2}^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k\right)$,*

$$\mathbb{E}\kappa_A^{p_1\otimes p_2}(B_1,...,B_{k_1+k_2})=\mathbb{E}\kappa_A^{p_1}(B_1,...,B_{k_1})\mathbb{E}\kappa_A^{p_1}(B_{k_1+1},...,B_{k_2}).$$

We stated the result for a finite family, yet it implies a similar statement for infinite families. For any integer k, let $(1,...,k)$ be the permutation in $\mathfrak{S}_k$ which, for any $i\in\{1,...,k\}$, sends i on $i+1$ modulo k.

Proposition 4.2. — *Using the bijection between the poset of non-crossing partitions and elements of $([id_k,(1,...,k)]_{\mathfrak{S}_k},\leq)$, we see that this definition of cumulants is a generalization of the usual free cumulants. Indeed, if the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in $\mathfrak{S}$-moments and satisfies the asymptotic $\mathfrak{S}$-factorization, then for any $\sigma\in\mathfrak{S}_k$:*

$$\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma}[M_1,...,M_k]=\prod_{c\in\mathsf{C}(\sigma)}\kappa^{\#c}\left[(M_i)_{i\in c}\right],$$

where $\mathsf{C}(\sigma)$ is the set of cycles of σ and for any positive integer n, κ^n stands for the usual free cumulant.

For a definition of the usual free cumulants, one can read Chapter 2.5 of [**21**].

Proposition 4.3. — *Let us suppose that the $\mathfrak{S}$-factorization property asymptotically holds for $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ then $\mathbb{E}\kappa_{\mathfrak{S}}^{(1,...,k)}(M_1,...,M_k)$ is the usual non-commutative cumulant $\kappa^k(M_1,...,M_k)$.*

4.1.2. Higher order. — Let m be an integer. Let us suppose that $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in A-moments up to order m of fluctuations. Recall the definition of the defect $\mathsf{df}(p',p)$ defined in Definition 3.5 in [**15**].

Definition 4.2. — *The family of A-cumulants up to order m of fluctuations*

$$\left(\mathbb{E}\kappa_{i,A}^p\left[\left(M_1^N\right)_{N\in\mathbb{N}},...,\left(M_k^N\right)_{N\in\mathbb{N}}\right]\right)_{p\in A_k,i\in\{0,...,m\}}$$

is the unique family of complex numbers such that for any $i \in \{0, ..., m\}$ and any $p \in A_k$:

$$\mathbb{E}m_p^i(M_1, ..., M_k) = \sum_{p' \in A_k, \mathsf{df}(p',p) \leq i} \mathbb{E}\kappa^{p'}_{i-\mathsf{df}(p',p),A}\left[\left(M_1^N\right)_{N\in\mathbb{N}}, ..., \left(M_k^N\right)_{N\in\mathbb{N}}\right].$$

We will denote them by $\left(\mathbb{E}\kappa^p_{i,A}[M_1, ..., M_k]\right)_{p\in A_k, i\in\{0,...,m\}}$.

We recall that for any p and p' in A_k, $\mathsf{df}(p', p) = 0$ if and only if $p' \in [id_k, p]_{A_k}$. Thus we get the following lemma.

Lemma 4.2. — *For any $p \in A_k$, $\mathbb{E}\kappa^p_A[M_1, ..., M_k] = \mathbb{E}\kappa^p_{0,A}[M_1, ..., M_k]$.*

Again, the restriction to $\mathfrak{S}_k$ or $\mathcal{B}_k$ of the $\mathcal{P}$-cumulants of higher order is not equal to the $\mathfrak{S}_k$ or $\mathcal{B}_k$-cumulants of higher order. Besides, the Lemma 4.1 does not hold anymore for $m > 0$.

The Definitions 4.1 and 4.2 only allow us to define cumulants at the limit $N = \infty$: we can define cumulants only for k-tuple of elements in $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converges in expectation in A-moments. In the following, we define a notion of N-dimensional cumulants, defined for any element of $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$ and we show that these N-dimensional cumulants converge to the cumulants we have just defined.

4.2. N-dimensional cumulants. —

4.2.1. Zero order. — Let N and k be two positive integers. Let $M_1, ..., M_k$ be k random matrices in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$.

Let us choose a possibility for (G, A) as explained in Notation 3.1. Let dg be the Haar measure on $G(N)$. As explained in the beginning of Section 3, we are interested in the element:

$$\int_{G(N)} g^{\otimes k}\mathbb{E}[M_1 \otimes ... \otimes M_k](g^{-1})^{\otimes k}dg, \tag{5}$$

which is an element of $\left(\mathbb{C}\left[\rho^k_{G(N)}\right]\right)'$. Using Theorem 3.1, this endomorphism is an element of $\mathbb{C}\left[\rho_N^{A_k}\right]$. Let us remind that if A is equal to $\mathcal{B}$ or $\mathfrak{S}$, then $\rho_N^{A_k}$ is injective as soon as $N \geq k$, and if $A = \mathcal{P}$, then $\rho_N^{A_k}$ is injective as soon as $N \geq 2k$. This allows us to define the N-dimensional cumulants as the coordinate numbers of (5).

Definition 4.3. — *Let us suppose that $N \geq k$ if $A \in \{\mathcal{B}, \mathfrak{S}\}$ or that $N \geq 2k$ if $A = \mathcal{P}$. The family of N-dimensional A-cumulants of $(M_1, ..., M_k)$ is the family $\left(\mathbb{E}\kappa^p_A(M_1, ..., M_k)\right)_{p\in A_k}$ such that, for any $p \in A_k$,*

$$\mathbb{E}\kappa^p_A(M_1, ..., M_k) = \kappa^p\left(\int_{G(N)} g^{\otimes k}\mathbb{E}[M_1 \otimes ... \otimes M_k](g^{-1})^{\otimes k}dg\right).$$

We warn the reader that the three notions of cumulants are really distinct: for any $\sigma \in \mathfrak{S}_k$ in general $\mathbb{E}\kappa^\sigma_{\mathcal{P}}(M_1, ..., M_k)$, $\mathbb{E}\kappa^\sigma_{\mathcal{B}}(M_1, ..., M_k)$ and $\mathbb{E}\kappa^\sigma_{\mathfrak{S}}(M_1, ..., M_k)$ are distinct. Thus there is no similar result to Lemma 4.1 for the N-dimensional A-cumulants.

In the following we define the convergence in expectation in A-cumulants. For any $i \in I$, let $(M_i^N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$.

Definition 4.4. — *The family $\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}$ converges in expectation in A-cumulants if for any positive integer k, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_k^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\big(\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}\big)$, for any $p \in A_k$, the sequence of cumulants:*

$$\big(\mathbb{E}\kappa_A^p(B_1^N, ..., B_k^N)\big)_{N\geq 2k}$$

converges when N goes to infinity.

The first main and direct consequence of Theorem 4.1 of [**15**] is the following result which links the two notions of convergence.

Theorem 4.1. — *The family $\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}$ converges in expectation in A-moments if and only if it converges in expectation in A-cumulants. If so, we say that $\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}$ converges in A-expectation.*

Let us suppose that $\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}$ converges in A-expectation. For any positive integer k, for any $(i_1, ..., i_k) \in I^k$, for any $p \in A_k$:

$$\mathbb{E}m_p(M_{i_1}, ..., M_{i_k}) = \sum_{p'\in[id,p]_{A_k}} \lim_{N\to\infty} \mathbb{E}\kappa_A^{p'}\left(M_{i_1}^N, ..., M_{i_k}^N\right). \tag{6}$$

Remark 4.2. — *Let us remark that the convergence in $\mathcal{P}$-expectation implies the convergence in $\mathcal{B}$-expectation which implies the convergence in $\mathfrak{S}$-expectation. The easiest way to see this, is to consider the convergence in expectation in moments.*

Looking at Theorem 4.1 and Definition 4.1, we see that the N-dimensional A-cumulants of $(M_i^N)_{i\in I}$ converge to the A-cumulants of $\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}$

Theorem 4.2. — *Let us suppose that the family $\big((M_i^N)_{N\in\mathbb{N}}\big)_{i\in I}$ converges in A-expectation. For any integer k, for any $(i_1, ..., i_k) \in I^k$, for any $p \in A_k$,*

$$\lim_{N\to\infty} \mathbb{E}\kappa_A^p(M_{i_1}^N, ..., M_{i_k}^N) = \mathbb{E}\kappa_A^p\left[\left(M_{i_1}^N\right)_{N\in\mathbb{N}}, ..., \left(M_{i_k}^N\right)_{N\in\mathbb{N}}\right].$$

Let us remark that this theorem allows us to define in a more natural way the A-cumulants $\left(\mathbb{E}\kappa_A^p\left[\left(M_{i_1}^N\right)_{N\in\mathbb{N}}, ..., \left(M_{i_k}^N\right)_{N\in\mathbb{N}}\right]\right)_{p\in A_k}$.

4.2.2. Higher order. — One can define also the N-dimensional A-cumulants of higher order. Actually, one can not define them just for a family of random matrices of size N as we did for the zero order: the higher order N-dimensional A-cumulants are defined for families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. For each $i \in I$, let $(M_i^N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. The definition of higher order N-dimensional A-cumulants can not be disjoint from the definition of convergence in expectation in A-cumulants up to a given order of fluctuations. Let m be a non-negative integer.

Definition 4.5. — *Let k be a positive integer and let $(i_1, ..., i_k) \in I^k$. The sequence $\left(M_{i_1}^N \otimes ... \otimes M_{i_k}^N\right)_{N\in\mathbb{N}}$ converges in expectation in A_k-cumulants up to order m of fluctuations if there exists a family of complex numbers*

$$\left(\mathbb{E}\kappa_{i,A}^p(M_{i_1}^N, ..., M_{i_k}^N)\right)_{p\in A_k, i\in\{0,...,m\}, N\geq 2k},$$

such that:

- $\forall p \in A_k, \mathbb{E}\kappa_A^p(M_{i_1}^N, ..., M_{i_k}^N) = \sum_{i=0}^m \frac{\mathbb{E}\kappa_{i,A}^p(M_{i_1}^N,...,M_{i_k}^N)}{N^i}$,
- $\forall p \in A_k, \forall i \in \{0, ..., m-1\}, \mathbb{E}\kappa_{i,A}^p(M_{i_1}^N, ..., M_{i_k}^N)$ *does not depend on* N,
- $\forall p \in A_k, \mathbb{E}\kappa_{m,A}^p(M_{i_1}^N, ..., M_{i_k}^N)$ *converges as* N *goes to infinity.*

The family $\left(\mathbb{E}\kappa_{i,A}^p(M_{i_1}^N, ..., M_{i_k}^N)\right)_{p\in A_k, i\in\{0,...,m\}, N\geq 2k}$ *is uniquely defined.*

The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-cumulants up to order m of fluctuations if for any positive integer k, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_k^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, the sequence $(B_1^N \otimes ... \otimes B_k^N)_{N\in\mathbb{N}}$ converges in expectation in A_k-cumulants up to order m of fluctuations.

Actually, there is a slight abuse of notation as for any $p \in A_k$, for any $i \in \{0, ..., m\}$ and any $N \in \mathbb{N}$, $\mathbb{E}\kappa_{i,A}^p(M_1^N, ..., M_k^N)$ does not depend only on $(M_1^N, ..., M_k^N)$, it depends on all the sequences $(M_1^N)_{N\in\mathbb{N}}$,..., $(M_k^N)_{N\in\mathbb{N}}$. In particular, we warn the reader that $\mathbb{E}\kappa_{0,A}^p(M_1^N, ..., M_k^N)$ is not equal to $\mathbb{E}\kappa_A^p(M_1^N, ..., M_k^N)$.

Let us focus on a generalization of Theorem 4.1, which is a consequence of Theorem 9.1 of the article [**15**].

Theorem 4.3. — *The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in A-moments up to order m of fluctuations if and only if it converges in expectation in A-cumulants up to order m of fluctuations. If so, we say that $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in A-expectation up to order m of fluctuations.*

Let us suppose that $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in A-expectation up to order m of fluctuations. For any positive integer k, for any $(i_1, ..., i_k) \in I^k$, for any $i \in \{0, ..., m\}$ and any $p \in A_k$:

$$\lim_{N\to\infty} \mathbb{E}m_p^i\left(M_{i_1}^N, ..., M_{i_k}^N\right) = \sum_{p'\in A_k, \mathsf{df}(p',p)\leq i} \lim_{N\to\infty} \mathbb{E}\kappa_{i-\mathsf{df}(p',p),A}^{p'}\left(M_{i_1}^N, ..., M_{i_k}^N\right). \tag{7}$$

Remark 4.3. — *Let us remark that again the convergence in $\mathcal{P}$-expectation up to order m of fluctuations implies the convergence in $\mathcal{B}$-expectation up to order m of fluctuations which implies the convergence in $\mathfrak{S}$-expectation up to order m of fluctuations.*

Looking at Theorem 4.3 and Definition 4.2, we get the following theorem.

Theorem 4.4. — *Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in A-expectation up to order m of fluctuations. For any positive integer k, for any $(i_1, ..., i_k) \in I^k$, for any $i \in \{0, ..., m\}$ and any $p \in A_k$,*

$$\lim_{N\to\infty} \mathbb{E}\kappa_{i,A}^p\left[M_{i_1}^N, ..., M_{i_k}^N\right] = \mathbb{E}\kappa_{i,A}^p\left[(M_{i_1}^N)_{N\in\mathbb{N}}, ..., (M_{i_k}^N)_{N\in\mathbb{N}}\right].$$

4.3. Special case: G-invariant families. — Let N and k be two positive integers.

Definition 4.6. — *Let $M_1, ..., M_k$ be k random matrices in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$. The k-tuple $(M_1, ..., M_k)$ is $G(N)$-invariant if for any $g \in G(N)$, we have the equality in law:*

$$\left(gM_1g^{-1}, ..., gM_kg^{-1}\right) = (M_1, ..., M_k).$$

In general, if $(M_i)_{i\in I}$ is a family of random matrices of size N, then $(M_i)_{i\in I}$ is $G(N)$-invariant if for any integer k, for any $(i_1, ..., i_k) \in I^k$, $(M_{i_1}, ..., M_{i_k})$ is $G(N)$-invariant.

Let $((M_i^N)_{N\in\mathbb{N}})_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. We say that $((M_i^N)_{N\in\mathbb{N}})_{i\in I}$ is G-invariant if for any integer N, $(M_i^N)_{i\in I}$ is $G(N)$-invariant.

Let us remark that the invariance in law by conjugation by $U(N)$ implies the invariance in law by conjugation by $O(N)$ which implies the invariance in law by conjugation by $\mathfrak{S}(N)$. Recall the notation $p_{i_1,...,i_k}^{i_{1'},...,i_{k'}}$ defined in Section 2.3 of [**15**]. In order to define some micro-observables, we need to define the kernel of a partition.

Definition 4.7. — *Let $(i_1, ..., i_k, i_{1'}, ..., i_{k'})$ be a $2k$-tuple of $\{1, ..., N\}$. We denote by $\mathsf{Ker}\left((i_1, ..., i_k, i_{1'}, ..., i_{k'})\right)$ the unique partition $p \in \mathcal{P}_k$ such that $(p_{i_{1'},...,i_{k'}}^{i_1,...,i_k})^{ex} = 1$: for any r and s in $\{1, ..., k, 1', ..., k'\}$, r and s are in the same block if and only if $i_r = i_s$.*

This definition allows to define some observables $(\delta_p)_{p\in\mathcal{P}_k}$ for family of random matrices which is invariant in law by conjugation by the symmetric group.

Definition 4.8. — *Let $(M_1, ..., M_k)$ be a k-tuple of random matrices of size N. Let us suppose that $(M_1, ..., M_k)$ is invariant in law by conjugation by the symmetric group $\mathfrak{S}(N)$. Let p be a partition in $\mathcal{P}_k$, we define:*

$$\delta_p\left(M_1, ..., M_k\right) = \mathbb{E}\left[\,(M_1)_{i_{1'}}^{i_1} ...(M_k)_{i_{k'}}^{i_k}\right],$$

where $(i_1, ..., i_k, i_{1'}, ..., i_{k'})$ is any $2k$-tuple of $\{1, ..., N\}$ such that $\mathsf{Ker}\left((i_1, ..., i_k, i_{1'}, ..., i_{k'})\right) = p$.

We will study the asymptotics of these observables in Section 5 when $(M_i)_{i=1}^k$ is a family which converges in A-expectation.

4.3.1. Zero order. — Let us choose a possibility for (G, A) as explained in Notation 3.1. Let k and N be two positive integers such that $N \geq k$ if $A \in \{\mathcal{B}, \mathfrak{S}\}$ or such that $N \geq 2k$ if $A = \mathcal{P}$. Let $M_1, ..., M_k$ be k random matrices in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$. Let us suppose that $(M_1, ..., M_k)$ is $G(N)$-invariant. The notion of N-dimensional A-cumulants is simpler as one can remove the integration $\int_{G(N)} dg$ in (5). Indeed, by definition, we have the following lemma.

Lemma 4.3. — *Let us suppose that $(M_1, ..., M_k)$ is $G(N)$-invariant. Then:*

$$\mathbb{E}[M_1 \otimes ... \otimes M_k] = \sum_{p\in A_k} \mathbb{E}\kappa_A^p(M_1, ..., M_k)\frac{p}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}.$$

This lemma shows that the three notions of N-dimensional cumulants are linked for such k-tuple.

Proposition 4.4. — *If $G = U$, then:*

1. *for any* $p \notin \mathfrak{S}_k$, $\mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{B}}(M_1, ..., M_k) = 0$,
2. *for any* $p \in \mathfrak{S}_k$, $\mathbb{E}\kappa^p_{\mathfrak{S}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{B}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k)$.

If $G = O$, *then:*

1. *for any* $p \notin \mathcal{B}_k$, $\mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k) = 0$,
2. *for any* $p \in \mathcal{B}_k$, $\mathbb{E}\kappa^p_{\mathcal{B}}(M_1, ..., M_k) = \mathbb{E}\kappa^p_{\mathcal{P}}(M_1, ..., M_k)$.

Let $((M_i^N)_{N\in\mathbb{N}})_{i\in 1}^k$ be a G-invariant family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. Using Theorem 4.2, the Proposition 4.4 implies the following result on the $\mathfrak{S}$, $\mathcal{B}$ and $\mathcal{P}$-cumulants of $((M_i^N)_{N\in\mathbb{N}})_{i\in 1}^k$. Let us remark that we will use Lemma 4.1 for the third assertion when G is equal to O. Recall the notation that we explained at the end of Definition 4.1.

Theorem 4.5. — *Let us suppose that the family* $((M_i^N)_{N\in\mathbb{N}})_{i\in 1}^k$ *converges in* A*-expectation, then it converges in* $\mathcal{P}$*-expectation. Besides, for any* $p \in \mathcal{P}_k$*:*

$$\lim_{N\to\infty} \mathbb{E}m_p(M_1^N, ..., M_k^N) = \sum_{p'\in A_k, p'\leq p} \mathbb{E}\kappa^{p'}_A (M_1, ..., M_k) . \tag{8}$$

Moreover, if $G = U$, *then:*

1. *for any* $p \notin \mathfrak{S}_k$*:*
   - *if* $p \in \mathcal{P}_k$, $\mathbb{E}\kappa^p_{\mathcal{P}} [M_1, ..., M_k] = 0$,
   - *if* $p \in \mathcal{B}_k$, $\mathbb{E}\kappa^p_{\mathcal{B}} [M_1, ..., M_k] = 0$,
2. *for any* $p \in \mathfrak{S}_k$,

$$\begin{aligned}\mathbb{E}\kappa^p_{\mathfrak{S}} [M_1, ..., M_k] &= \mathbb{E}\kappa^p_{\mathcal{B}} [M_1, ..., M_k] \\ &= \mathbb{E}\kappa^p_{\mathcal{P}} [M_1, ..., M_k] .\end{aligned}$$

If $G = O$, *then:*

1. *for any* $p \in \mathcal{P}_k \setminus \mathcal{B}_k$, $\mathbb{E}\kappa^p_{\mathcal{P}} \left[(M_1^N)_{N\in\mathbb{N}}, ..., (M_k^N)_{N\in\mathbb{N}}\right] = 0$,
2. *for any* $p \in \mathcal{B}_k$,
$$\mathbb{E}\kappa^p_{\mathcal{B}} [M_1, ..., M_k] = \mathbb{E}\kappa^p_{\mathcal{P}} [M_1, ..., M_k] ,$$
3. *for any* $p \in \mathfrak{S}_k$,
$$\mathbb{E}\kappa^p_{\mathfrak{S}} \left[(M_1^N)_{N\in\mathbb{N}}, ..., (M_k^N)_{N\in\mathbb{N}}\right] = \mathbb{E}\kappa^p_{\mathcal{B}} \left[(M_1^N)_{N\in\mathbb{N}}, ..., (M_k^N)_{N\in\mathbb{N}}\right] .$$

Proof. — If $A = \mathcal{P}$, there is nothing to prove. Let us suppose that A is equal to $\mathcal{B}$ or $\mathfrak{S}$. Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in A-expectation. Using the Theorem 4.1, it converges in expectation in A-cumulants. Yet Proposition 4.4 implies that it converges in expectation in $\mathcal{P}$-cumulants. Indeed, for any positive integer l, for any element B_1,..., B_l of $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, for any positive integer N, and for any $p \in \mathcal{P}_l$:

- If $p \in A_k$, $\mathbb{E}\kappa^p_{\mathcal{P}}[B_1^N, ..., B_l^N] = \mathbb{E}\kappa^p_A[B_1^N, ..., B_l^N]$.
- If $p \notin A_k$, $\mathbb{E}\kappa^p_{\mathcal{P}}[B_1^N, ..., B_l^N] = 0$.

Thus using Theorem 4.1, the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in $\mathcal{P}$-expectation.

We have already noticed that the convergence in $\mathcal{P}$-expectation implies the convergence in $\mathcal{B}$-expectation: thus we can consider, for any $p \in \mathcal{B}_k$, the free cumulant

$\mathbb{E}\kappa^p_{\mathcal{B}}[M_1, ..., M_k]$. Taking the limit, when N goes to infinity, of the equalities in Proposition 4.4 implies the second part of the assertions. The Equality (8) is a consequence of our discussions and Theorem 4.1 of [**15**]. □

Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in A-moments. If it satisfies the asymptotic A-factorization property, then due to Proposition 4.1, for any integer k_1 and k_2, for any $p_1 \in A_{k_1}$ and $p_2 \in A_{k_2}$, for any element $(B_1^N)_{N\in\mathbb{N}}$,..., $(B_{k_1+k_2}^N)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k\right)$

$$\mathbb{E}\kappa_A^{p_1\otimes p_2}(B_1, ..., B_{k_1+k_2}) = \mathbb{E}\kappa_A^{p_1}(B_1, ..., B_{k_1})\mathbb{E}\kappa_A^{p_2}(B_{k_1+1}, ..., B_{k_2}).$$

Using Theorem 4.5, we see that this last equality holds for p_1 and p_2 in respectively $\mathcal{P}_{k_1}$ and $\mathcal{P}_{k_2}$. Using again Proposition 4.1, we deduce that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ satisfies the asymptotic $\mathcal{P}$-factorization property.

Proposition 4.5. — *The family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ satisfies the asymptotic A-factorization property if and only if it satisfies the asymptotic $\mathcal{P}$-factorization property.*

This proposition can be extended to any family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$, with I being any indexation set. Using the Theorem 4.5 and our discussion on the asymptotic A-factorization property, one has for example the following corollary.

Corollary 4.1. — *For any positive integer N, let $(M_i^N)_{i\in I}$ be a family of random matrices of size N which is invariant by conjugation by $U(N)$. If $(M_i^N)_{i\in I}$ converges in non-commutative distribution as N goes to infinity and satisfies the asymptotically $\mathfrak{S}$-factorization property, then $(M_i^N)_{i\in I} \cup \left({}^tM_i^N\right)_{i\in I}$ converges in non-commutative distribution. Besides, for any positive integer k, for any $p \in \mathcal{P}_k$, for any $(i_1, ..., i_k) \in I^k$, $\mathbb{E}m_p(M_{i_1}^N, ..., M_{i_k}^N)$ converges as N goes to infinity to:*

$$\sum_{\sigma\in\mathfrak{S}_k|\sigma\leq p}\ \prod_{c=(j_1,...,j_r)\ cycle\ of\ \sigma} \kappa\left(\left(M_{i_{j_1}}^N\right)_{N\in\mathbb{N}}, ..., \left(M_{i_{j_r}}^N\right)_{N\in\mathbb{N}}\right)$$

where the κ stand for the usual non-commutative cumulants in free probability.

A similar corollary can be written for families of random matrices which are invariant in law by conjugation by the orthogonal group. Recall the notion of cycles in Definition 2.9 of [**15**] and the notion of extraction in Definition 2.12 of the same article.

Corollary 4.2. — *Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is O-invariant. If $(M_i)_{i\in I} \cup ({}^tM_i)_{i\in I}$ converges in non-commutative distribution and if the $\mathfrak{S}$-factorization property holds for this family, then for any positive integer k, for any $p \in \mathcal{P}_k$, for any $(i_1, ..., i_k) \in I^k$, $\mathbb{E}m_p\left(M_{i_1}^N, ..., M_{i_k}^N\right)$ converges as N goes to infinity to:*

$$\sum_{b\in\mathcal{B}_k|b\leq p}\ \prod_{c=\{j_1,...,j_r\}\cup\{(j_1)',...,(j_r)'\}\ cycle\ of\ b, j_1<...<j_r} \mathbb{E}\kappa_{\mathcal{B}}^{b_c}\left((M_{i_{j_1}}^N)_{N\in\mathbb{N}}, ..., (M_{i_{j_r}}^N)_{N\in\mathbb{N}}\right).$$

4.3.2. Higher order. — Let m be a non negative integer, let us choose a possibility for (G, A) as explained in Notation 3.1. Let k be a positive integer, let $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in 1}^k$ be a G-invariant family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. The following proposition is a consequence of Proposition 4.4 and Theorem 4.3. The proof which is similar to the one of Theorem 4.5 will be omitted. Recall the notation that we explained in Definition 4.2.

Proposition 4.6. — *Let us suppose that the family $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ converges in expectation in A-moments up to order m of fluctuations, then it converges in expectation in $\mathcal{P}$-moments up to order m of fluctuations. Besides, for any $i \in \{0, ..., m\}$ and any $p \in \mathcal{P}_k$:*

$$\lim_{N\to\infty} \mathbb{E} m_p^i(M_1^N, ..., M_k^N) = \sum_{p'\in A_k, \mathsf{df}(p',p)\leq i} \mathbb{E}\kappa^{p'}_{i-\mathsf{df}(\mathsf{p}',\mathsf{p}),A}\left(M_1, ..., M_k\right).$$

Moreover, if $G = U$, then:

1. *For any $p \notin \mathfrak{S}_k$, for any $i \in \{0, ..., m\}$:*
$$\mathbb{E}\kappa^p_{i,\mathcal{P}}\left[M_1, ..., M_k\right] = \mathbb{E}\kappa^p_{i,\mathcal{B}}\left[M_1, ..., M_k\right] = 0.$$
2. *For any $p \in \mathfrak{S}_k$, for any $i \in \{0, ..., m\}$:*
$$\begin{aligned}\mathbb{E}\kappa^p_{i,\mathfrak{S}}\left[M_1, ..., M_k\right] &= \mathbb{E}\kappa^p_{i,\mathcal{B}}\left[M_1, ..., M_k\right]\\ &= \mathbb{E}\kappa^p_{i,\mathcal{P}}\left[M_1, ..., M_k\right].\end{aligned}$$

If $G = O$, then:

1. *For any $p \notin \mathcal{B}_k$, for any $i \in \{0, ..., m\}$:*
$$\mathbb{E}\kappa^p_{i,\mathcal{P}}\left[M_1, ..., M_k\right] = 0.$$
2. *For any $p \in \mathcal{B}_k$, for any $i \in \{0, ..., m\}$:*
$$\mathbb{E}\kappa^p_{i,\mathcal{B}}\left[M_1, ..., M_k\right] = \mathbb{E}^p_{i,\mathcal{P}}\left[M_1, ..., M_k\right].$$
3. *For any $p \in \mathfrak{S}_k$:*
$$\mathbb{E}\kappa^p_{0,\mathfrak{S}}\left[M_1, ..., M_k\right] = \mathbb{E}\kappa^p_{0,\mathcal{B}}\left[M_1, ..., M_k\right].$$

5. Exclusive moments and cumulants

5.1. Exclusive moments. — Let N and k be positive integers. Let $M_1, ..., M_k$ be k random matrices of size N defined on the same probability space, we are going to define the mean exclusive normalized moments. Let p in $\mathcal{P}_k$.

Definition 5.1. — *The mean exclusive p-normalized moment of the k-tuple $(M_1, ..., M_k)$ is:*

$$\mathbb{E} m_{p^c}(M_1, ..., M_k) = \frac{1}{Tr^k(\rho_N^{\mathcal{P}_k}(p))}\mathbb{E}\left[Tr^k\left((M_1 \otimes ... \otimes M_k)\rho_N^{\mathcal{P}_k}({}^t p^c)\right)\right].$$

We will define a second notion of mean exclusive normalized moments. Let us consider a choice of (G, A) as explained in Notation 3.1.

Definition 5.2. — *Let p in $\mathcal{P}_k$, the mean exclusive p-normalized A-moment of the k-tuple $(M_1, ..., M_k)$, denoted by $\mathbb{E}m^A_{p^c}(M_1, ..., M_k)$, is:*

$$\frac{1}{Tr^k(\rho_N^{\mathcal{P}_k}(p))}\mathbb{E}\left[Tr^k\left(\left(\int_{G(N)} g^{\otimes k}(M_1 \otimes ... \otimes M_k)(g^{-1})^{\otimes k}dg\right)\rho_N^{\mathcal{P}_k}({}^t p^c)\right)\right].$$

Let us remark that for any $p \in \mathcal{P}_k$, the mean exclusive p-normalized moment is equal to the mean exclusive p-normalized $\mathcal{P}$-moment.

Lemma 5.1. — *For any $p \in \mathcal{P}_k$:*

$$\mathbb{E}m_{p^c}(M_1, ..., M_k) = \mathbb{E}m^{\mathcal{P}}_{p^c}(M_1, ..., M_k).$$

One can define the convergence in expectation in exclusive A-moments, almost as we did in Definition 2.2, except that we ask for the convergence of the observables for any $p \in \mathcal{P}_k$. Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$.

Definition 5.3. — *Let k be a positive integer, let $(i_1, ..., i_k) \in I^k$. The sequence $\left(M^N_{i_1} \otimes ... \otimes M^N_{i_k}\right)_{N\in\mathbb{N}}$ converges in expectation in exclusive A-moments if for any $p \in \mathcal{P}_k$,*

$$\mathbb{E}m^A_{p^c}(M^N_{i_1}, ..., M^N_{i_k})$$

converges when N goes to infinity.

The family $\left((M^N_i)_{N\in\mathbb{N}}\right)_{i\in I}$ converges in expectation in exclusive A-moments if for any positive integer k, for any element $(B^N_1)_{N\in\mathbb{N}}$,..., $(B^N_k)_{N\in\mathbb{N}}$ in $\mathcal{A}\left(\left((M^N_i)_{N\in\mathbb{N}}\right)_{i\in I}\right)$, the sequence $(B^N_1 \otimes ... \otimes B^N_k)_{N\in\mathbb{N}}$ converges in expectation in exclusive A_k-moments.

Again, if $(M^N_1, ..., M^N_k)$ converges in expectation in exclusive A-moments, we denote by $\mathbb{E}m^A_{p^c}(M_1, ..., M_k)$ the limit of $\mathbb{E}m^A_{p^c}(M^N_1, ..., M^N_k)$.

Let us recall the notations and definitions in the Sections 3.2 and 3.3 of [**15**]. In these sections, we defined the set of admissible splittings of a partition p, denoted by $\mathsf{Sp}(p)$, the set of admissible gluings of p, denoted by $\mathsf{Gl}_c(p)$, and two partial orders $\sqsupset$ and $\dashv$. Then Proposition 4.2 of [**15**] implies the following proposition.

Proposition 5.1. — *The family $(M_i)_{i\in I}$ converges in expectation in A-moments if and only if it converges in expectation in A-exclusive moments. Besides if one condition holds then for any positive integer k, for any $p \in A_k$, for any $i_1, ..., i_k$ in I:*

$$\mathbb{E}m_p(M_{i_1}, ..., M_{i_k}) = \sum_{p'\in\mathcal{P}_k,\ p'\dashv p} \mathbb{E}m^A_{p'^c}(M_{i_1}, ..., M_{i_k}).$$

Proof. — This is due to the fact that, using Theorem 3.1, if p is in A_k, then:

$$\mathbb{E}m_p(M_{i_1}, ..., M_{i_k}) = m_p\left(\int_{G(N)} g^{\otimes}\mathbb{E}[M_{i_1} \otimes ... \otimes M_{i_k}](g^{-1})^{\otimes k}dg\right),$$

and with a slight abuse of notation:

$$\int_{G(N)} g^{\otimes}\mathbb{E}[M_{i_1} \otimes ... \otimes M_{i_k}](g^{-1})^{\otimes k}dg \in \mathbb{C}[A_k(N)].$$

The proposition now is a consequence of Proposition 4.2 of [**15**]. □

By looking at the proof, one can understand why the equality in Proposition 5.1 does not hold for any $p \in \mathcal{P}_k$. If $p \in \mathcal{P}_k \setminus A_k$, the same equality would hold if one changes the left hand side by $\mathbb{E}m_p(HM_{i_1}H^{-1}, ..., HM_{i_k}H^{-1})$ with H being a sequence of Haar random variable on the groups of symmetry, namely $(G(N))_{N\in\mathbb{N}}$, associated with A with the condition that for every positive integer N, H^N is independent of $(M_i^N)_{i\in I}$. This discussion implies that if the family $(M_i)_{i\in I}$ is G-invariant, then Proposition 5.1 does hold for any $p \in \mathcal{P}_k$.

Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ and let us suppose until the end of this section that $(M_i)_{i\in I}$ converges in expectation in A-moments. Using the Theorem 4.6 of [**15**], we get the following statement.

Theorem 5.1. — *For any positive integer k, any $i_1, ..., i_k$ in I and any $p \in \mathcal{P}_k$,*

$$\mathbb{E}m^A_{p^c}(M_{i_1}, ..., M_{i_k}) = \sum_{p'\in A_k, p'\sqsupset p} \mathbb{E}\kappa^{p'}_A(M_{i_1}, ..., M_{i_k}).$$

Besides, as a consequence of Corollary 4.1 of [**15**], we get that the limits of exclusive moments and finite-dimensional cumulants are equal for some special partitions.

Theorem 5.2. — *Let k be a positive integer, let p be a partition in $\mathcal{P}_k$ which does not have any pivotal block. For any $i_1, ..., i_k$ in I,*

$$\mathbb{E}m^A_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p\in A_k}\mathbb{E}\kappa^p_A(M_{i_1}, ..., M_{i_k}) \tag{9}$$

In particular, for any $p \in \mathcal{B}_k$, the Equality (9) is valid.

This last corollary is very crucial in order to study the asymptotic of the random walk on the symmetric group. The Corollary 4.2 of [**15**] allows us to assert the following proposition. Recall the definition of $\mathsf{Mb}(p)$ in Definition 3.9 of [**15**].

Theorem 5.3. — *Let us suppose that A is equal either to $\mathfrak{S}$ or $\mathcal{B}$. For any positive integer k, for any $i_1, ..., i_k$ in I, for any partition p in $\mathcal{P}_k$:*

$$\mathbb{E}m^A_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p\in\overline{A_k}}\mathbb{E}\kappa^{\mathsf{Mb}(p)}_A(M_{i_1}, ..., M_{i_k}).$$

5.1.1. The G-invariant case. — As always, when one imposes some invariance, the theory becomes simpler. Let us choose a possibility for (G, A) as explained in Notation 3.1. Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is G-invariant.

The Lemma 5.1 does not hold for general random matrices if one substitutes $\mathcal{P}$ in the r.h.s. by any other choice of A. Yet, in the setting where $(M_i)_{i\in I}$ is G-invariant we get the following lemma.

Lemma 5.2. — *For any positive integer k and N, for any k-tuple $(i_1, ..., i_k)$ of elements of I, for any $p \in \mathcal{P}_k$:*

$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \mathbb{E}m^A_{p^c}(M_{i_1}, ..., M_{i_k}).$$

Using this lemma, and using the discussion after Proposition 5.1, one can see that $(M_i)_{i\in I}$ converges in expectation in A-moments if and only if for any positive integer

k, for any $(i_1, ..., i_k) \in I^k$, for any $p \in \mathcal{P}_k$, $\mathbb{E}m_{p^c}(M_{i_1}^N, ..., M_{i_k}^N)$ converges as N goes to infinity. If so, then for any $p \in \mathcal{P}_k$:

$$\mathbb{E}m_p(M_{i_1}, ..., M_{i_k}) = \sum_{p' \in \mathcal{P}_k,\ p' \dashv p} \mathbb{E}m_{p'^c}(M_{i_1}, ..., M_{i_k}),$$
$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \sum_{p' \in A_k, p' \sqsupset p} \mathbb{E}\kappa_A^{p'}(M_{i_1}, ..., M_{i_k}).$$

If G is equal either to O of U, this last equality becomes much simpler, since it asserts that:

$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p \in \overline{A_k}} \mathbb{E}\kappa_A^{\mathsf{Mb}(p)}(M_{i_1}, ..., M_{i_k}),$$

moreover, if G equal to U, and if the asymptotic $\mathfrak{S}$-factorization holds for $(M_i)_{i \in I}$, this last equality asserts that:

$$\mathbb{E}m_{p^c}(M_{i_1}, ..., M_{i_k}) = \delta_{p \in \overline{\mathfrak{S}_k}} \prod_{c=(j_1,...,j_r) \text{ cycle of } \mathsf{Mb}(p)} \kappa(M_{i_{j_1}}, ..., M_{i_{j_r}}), \tag{10}$$

where κ is the usual non commutative cumulant in free probability theory.

We will see in the following that the symmetry by conjugation by G allows also to link the exclusive moments with the moments of the entries of the matrices. This will allow to see the Equation (10) as a kind of generalization of Theorem 2.6 of [**12**].

As noticed by C. Male in [**26**] we have the following fact.

Lemma 5.3. — *For any positive integer k and N, for any k-tuple $(i_1, ..., i_k)$ of elements of I, for any $p \in \mathcal{P}_k$, one has:*

$$\delta_p(M_{i_1}^N, ..., M_{i_k}^N) = \frac{(N - \mathsf{nc}(p))!}{N!} N^{\mathsf{nc}(p \vee id)} \mathbb{E}m_{p^c}(M_{i_1}^N, ..., M_{i_k}^N).$$

Proof. — Let N and k be two positive integers, let us consider $(i_1, ..., i_k)$ a k-tuple of elements of I, and let p be a partition in $\mathcal{P}_k$. Since $(M_{i_1}^N, ..., M_{i_k}^N)$ is invariant in law by conjugation by $G(N)$, it is invariant in law by conjugation by the symmetric group $\mathfrak{S}(N)$. We have already seen that $\mathbb{E}\big[\,(M_{i_1}^N)_{j_{1'}}^{j_1} ...(M_{i_k}^N)_{j_{k'}}^{j_k}\big]$ does not depend on the value of $(j_1, ..., j_k, j_{1'}, ...j_{k'})$ as long as $\mathsf{Ker}\,((j_1, ..., j_k, j_{1'}, ...j_{k'})) = p$. Since there exist exactly $\frac{N!}{(N-\mathsf{nc}(p))!}$ of $2k$-tuples $(j_1, ..., j_k, j_{1'}, ...j_{k'})$ such that $\mathsf{Ker}\,((j_1, ..., j_k, j_{1'}, ...j_{k'})) = p$, we get the following equalities:

$$\begin{aligned}
&\frac{(N - \mathsf{nc}(p))!}{N!} N^{\mathsf{nc}(p \vee id)} \mathbb{E}m_{p^c}(M_{i_1}^N, ..., M_{i_k}^N) \\
&= \frac{(N - \mathsf{nc}(p))!}{N!} \sum_{(j_1,...,j_k,j_{1'},...j_{k'}) \in \{1,...,N\}^{2k} | \mathsf{Ker}((j_1,...,j_k,j_{1'},...j_{k'}))=p} \mathbb{E}\big[\,(M_{i_1}^N)_{j_{1'}}^{j_1} ...(M_{i_k}^N)_{j_{k'}}^{j_k}\big] \\
&= \delta_p(M_{i_1}^N, ..., M_{i_k}^N).
\end{aligned}$$

□

We can state now a corollary of Theorem 5.3 and the discussion we had.

Theorem 5.4. — *Let k be a positive integer, let $(M_1, ...M_k)$ be a k-tuple of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is U-invariant, converges in $\mathfrak{S}$-expectation and satisfies the asymptotic $\mathfrak{S}$-factorization property. Then for any $2k$-tuple, $\mathbb{I} = (i_1, ..., i_k, i_{1'}, ..., i_{k'}) \in \{1, ..., N\}^{2k}$, one has that for any positive integer N:*

$$\mathbb{E}\left[\left(M_1^N\right)_{i_{1'}}^{i_1} ...\left(M_k^N\right)_{i_{k'}}^{i_k}\right] = \delta_{\mathsf{Ker}(\mathbb{I})\in\overline{\mathfrak{S}_k}} N^{\mathsf{nc}(\mathsf{Ker}(\mathbb{I})\vee id)-\mathsf{nc}(\mathsf{Ker}(\mathbb{I}))}\left(\kappa_{\mathsf{Mb}(\mathsf{Ker}(\mathbb{I}))}(M_1, ..., M_k)+o(1)\right).$$

where, as usual, for any $\sigma \in \mathfrak{S}_k$, $\kappa_\sigma\big(M_1, ..., M_k\big)$ is equal to

$$\prod_{c \text{ cycle of } \sigma, c=(i_1,...,i_r)} \kappa(M_{i_1}, ..., M_{i_r}),$$

and κ stands for the free cumulants.

One can state also a similar theorem when G is equal to O or $\mathfrak{S}$ but it would not use the usual free cumulants but the $\mathcal{B}$ and $\mathcal{P}$-cumulants that we defined. Yet, it would show that one can compute cumulants as limits of some normalized moments of the entries of the matrices or one can compute the asymptotics of any moments of the entries of the matrices by using the cumulants.

5.2. Exclusive cumulants. — Recall the notion of exclusive coordinate numbers defined in Definition 4.5 of [**15**]. We can define the N-dimensional mean A-exclusive cumulants.

Definition 5.4. — *Let k and N be two positive integers such that $N \geq 2k$. Let $M_1, ..., M_k$ be k random matrices in $L^{\infty^-}(\Omega) \otimes \mathcal{M}_N(\mathbb{C})$. The family of N-dimensional mean A-exclusive cumulants of $(M_1, ..., M_k)$ is the family $\left(\mathbb{E}\kappa_A^{p^c}(M_1, ..., M_k)\right)_{p\in\mathcal{P}_k}$ such that, for any $p \in \mathcal{P}_k$,*

$$\mathbb{E}\kappa_A^{p^c}(M_1, ..., M_k) = \kappa_c^p\left(\int_{G(N)} g^{\otimes k}\mathbb{E}[M_1 \otimes ... \otimes M_k](g^{-1})^{\otimes k}dg\right).$$

Let us remark that the A-exclusive cumulants are defined for any $p \in \mathcal{P}_k$. A consequence of Theorem 4.5 of [**15**] is the following theorem.

Theorem 5.5. — *Let $(M_i)_{i\in I}$ be a family of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converges in A-moments. Then, for any positive integer k, any $p \in \mathcal{P}_k$, any $i_1, ..., i_k$ in I:*

$$\mathbb{E}m_{p^c}^A(M_{i_1}, ..., M_{i_k}) = \mathbb{E}\kappa_A^{p^c}(M_{i_1}, ..., M_{i_k}).$$

6. The $\mathcal{R}$-transform and the non-commutative law

6.1. The $\mathcal{R}$-transform. —

6.1.1. Zero order. — Let $(M_N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation. We define the A-law and the $\mathcal{R}_A$-functional of $(M_N)_{N\in\mathbb{N}}$ as following. Recall the notation $\mathfrak{E}[A]$ defined in Definition 10.1 of [**15**].

Definition 6.1. — *The A-law of $(M_N)_{N\in\mathbb{N}}$, denoted by $\mathcal{M}_A\left((M_N)_{N\in\mathbb{N}}\right)$ or $\mathcal{M}_A(M)$, is the element of $\mathfrak{E}[A]$ such that for any $k\geq 1$, for any $p\in A_k$:*

$$\left(\mathcal{M}_A(M)\right)_p=\mathbb{E}m_p[M,...,M].$$

The $\mathcal{R}_A$-functional of $(M_N)_{N\in\mathbb{N}}$, denoted by $\mathcal{R}_A\left((M_N)_{N\in\mathbb{N}}\right)$ or $\mathcal{R}_A(M)$, is the element of $\mathfrak{E}[A]$ such that for any $k\geq 1$, for any $p\in A_k$:

$$\left(\mathcal{R}_A(M)\right)_p=\mathbb{E}\kappa^p[M,...,M].$$

Recall the definition of the $\mathcal{M}_A$ and $\mathcal{R}_A$-transforms, transformations defined on $\mathfrak{E}[A]$, given in Definitions 10.9 and 10.10 of [**15**]. Using all the definitions we have just made and Theorem 4.1, we get the following theorem. By definition, the following equalities hold:

$$\mathcal{R}_A(M)=\mathcal{R}_A[\mathcal{M}_A(M)],\ \mathcal{M}_A(M)=\mathcal{M}_A[\mathcal{R}_A(M)].$$

With these definitions, one can reformulate the definition of the asymptotic A-factorization property for $(M_N)_{N\in\mathbb{N}}$. Recall the definition of $\mathfrak{ME}[A]$ given in Definition 10.4 in [**15**].

Proposition 6.1. — *The asymptotic A-factorization property holds for $(M_N)_{N\in\mathbb{N}}$ if and only if $\mathcal{M}_A(M)$ or $\mathcal{R}_A(M)$ is in $\mathfrak{ME}[A]$.*

6.1.2. Higher order. — Let m be a non-negative integer, let us suppose that the sequence $(M_N)_{N\in\mathbb{N}}$ converges in A-expectation up to order m of fluctuations. We can define the A-law and the $\mathcal{R}_A^{(m)}$-functional of $(M_N)_{N\in\mathbb{N}}$ up to order m of fluctuations. Recall Definition 10.14 of [**15**] where we defined the algebra $\mathfrak{E}_{(m)}[A]$.

Definition 6.2. — *We define the A-law of $(M_N)_{N\in\mathbb{N}}$ up to order m of fluctuations, denoted by $\mathcal{M}_A^{(m)}\left((M_N)_{N\in\mathbb{N}}\right)$ or $\mathcal{M}_A^{(m)}(M)$, as the element of $\mathfrak{E}_{(m)}[A]$ such that for any positive integer k, any $p\in A_k$, any $i\in\{0,...,m\}$:*

$$\left(\mathcal{M}_A^{(m)}(M)\right)_{p,i}=\mathbb{E}m_p^i[M,...,M].$$

We define the $\mathcal{R}_A^{(m)}$-functional of $(M_N)_{N\in\mathbb{N}}$ up to order m of fluctuations, denoted by $\mathcal{R}_A^{(m)}\left((M_N)_{N\in\mathbb{N}}\right)$ or $\mathcal{R}_A^{(m)}(M)$ as the element of $\mathfrak{E}_{(m)}[A]$ such that for any integer k, any $p\in A_k$, any $i\in\{0,...,m\}$:

$$\left(\mathcal{R}_A^{(m)}(M)\right)_{p,i}=\mathbb{E}\kappa_{i,A}^p[M,...,M].$$

Using these new definitions, Definitions 10.16 and 10.17 of [**15**], the following equalities hold:

$$\mathcal{R}_A^{(m)}(M)=\mathcal{R}_A^{(m)}\left[\mathcal{M}^{(m)}(M)\right],\ \mathcal{M}_A^{(m)}(M)=\mathcal{M}_A^{(m)}\left[\mathcal{R}^{(m)}(M)\right].$$

6.2. The non-commutative law. —

6.2.1. Zero order. — We have defined in the last section the law of an element $(M_N)_{N\in\mathbb{N}}$ in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation. We generalize this for a family of elements in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation. Let $(M_i)_{i\in I}$ be a family of elements in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation.

Definition 6.3. — *The A-law of $(M_i)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\to\mathbb{C},$$
$$\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p\right)\mapsto\lim_{N\to\infty}\mathbb{E}m_p\left[\left(P_n\left[M_{i_1}^N,...,M_{i_{k_n}}^N\right]\right)_{n=1}^l\right].$$

One can also define the $\mathcal{R}_A$-functional of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$.

Definition 6.4. — *The $\mathcal{R}_A$-functional of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\to\mathbb{C},$$
$$\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p\right)\mapsto\mathbb{E}\kappa_A^p\left[\left(\left(P_n\left[M_{i_1}^N,...,M_{i_{k_n}}^N\right]\right)_{N\in\mathbb{N}}\right)_{n=1}^l\right].$$

6.2.2. Higher order. — One can also define the A-law and the $\mathcal{R}_A$-functional of the fluctuations up to order m. Let m be an integer, let us suppose that $(M_i)_{i\in I}$ converges in A-expectation up to order m of fluctuations.

Definition 6.5. — *The A-law up to order m of fluctuations of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\times(\{0,...,m\})\to\mathbb{C}$$

which sends $\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p,i_0\right)$ *on* $\mathbb{E}m_p^{i_0}\left[\left(P_n\left[M_{i_1},...,M_{i_{k_n}}\right]\right)_{n=1}^l\right]$.

Definition 6.6. — *The $\mathcal{R}_A$-functional up to order m of fluctuations of $\left((M_i^N)_{N\in\mathbb{N}}\right)_{i\in I}$ is the application:*

$$\bigcup_{l\in\mathbb{N}}\left(\left(\bigcup_{k=1}^{\infty}\left(\mathbb{C}\{X_1,...,X_k\}\times I^k\right)\right)^l\times A_l\right)\times(\{0,...,m\})\to\mathbb{C}$$

which sends $\left(\left(P_n,\{i_1^n,...,i_{k_n}^n\}\right)_{n=1}^l,p,i_0\right)$ *on* $\mathbb{E}\kappa_{i_0,A}^p\left[\left(\left(P_n\left[M_{i_1}^N,...,M_{i_{k_n}}^N\right]\right)_{N\in\mathbb{N}}\right)_{n=1}^l\right]$.

7. A-Freeness

Let $\left(M_1^N\right)_{N\in\mathbb{N}}$ and $\left(M_2^N\right)_{N\in\mathbb{N}}$ be two elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converge in A-expectation. Can we recover the A-law of the couple $\left(\left(M_1^N\right)_{N\in\mathbb{N}},\left(M_2^N\right)\right)_{N\in\mathbb{N}}$ knowing the A-law of $\left(M_1^N\right)_{N\in\mathbb{N}}$ and $\left(M_2^N\right)_{N\in\mathbb{N}}$? Actually, there is no general formula to do so, except in some particular cases. For example, this is possible if $\left(M_1^N\right)_{N\in\mathbb{N}}$ and $\left(M_2^N\right)_{N\in\mathbb{N}}$ are asymptotically A-free. In this section, we define the notion of asymptotic A-freeness.

7.1. Zero order. —

7.1.1. Definitions and general results. — Let I and J be two indexation sets. Recall Definition 2.9 in [**15**] where the notion of irreducible partitions is defined. Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation. We will need some notations in order to define the notion of asymptotic A-freeness.

Notation 7.1. — *Let k be a positive integer, let $(B_i)_{i=1}^k$ be a k-tuple of elements of*

$$\mathcal{A}\left((L_i)_{i\in I}\right)\cup\mathcal{A}\left((M_j)_{j\in J}\right).$$

We denote by $L_{(B_i)_{i=1}^k}$ the set of $i\in\{1,...,k\}$ such that $\left(B_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((L_i)_{i\in I}\right)$. We denote by $M_{(B_i)_{i=1}^k}$ the set of $i\in\{1,...,k\}$ such that $\left(B_i^N\right)_{N\in\mathbb{N}}\in\mathcal{A}\left((M_j)_{j\in J}\right)$. We say that $(B_i)_{i=1}^k$ is mixed *if $L_{(B_i)_{i=1}^k}$ and $M_{(B_i)_{i=1}^k}$ are not empty.*

Let p be in A_k. The decomposition of $\{1,...,k\}\cup\{1',...,k'\}$ by the cycles of p induces a partition π of $\{1,...,k\}$. The partition p is compatible with $(B_i)_{i=1}^k$ if any block of π is included in $L_{(B_i)_{i=1}^k}$ or in $M_{(B_i)_{i=1}^k}$.

Recall the definition of the extraction of a partition, defined in Definition 2.12 of [**15**]. Let us define the notion of A-freeness.

Definition 7.1. — *The families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if for any positive integer k, for any partition $p\in A_k$, for any mixed k-tuple $(B_i)_{i=1}^k$ of elements of $\mathcal{A}\left((L_i)_{i\in I}\right)\cup\mathcal{A}\left((M_j)_{j\in J}\right)$, the sequence $\left(\mathbb{E}\kappa_A^p(B_1^N,...,B_k^N)\right)_N$ converges as N goes to infinity and the two following conditions hold:*

1-Compatibility condition : *If p is not compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty}\mathbb{E}\kappa_A^p(B_1^N,...,B_k^N)=0.$$

2-Compatible factorization property : *If p is compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty}\mathbb{E}\kappa_A^p\left(B_1^N,...,B_k^N\right)=\mathbb{E}\kappa_A^{p_{L_{(B_i)_{i=1}^k}}}\left((B_i)_{i\in L_{(B_i)_{i=1}^k}}\right)\mathbb{E}\kappa_A^{p_{M_{(B_i)_{i=1}^k}}}\left((B_i)_{i\in M_{(B_i)_{i=1}^k}}\right).$$

In particular, if the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free, for any integers k and l, any $(i_1,...,i_k)\in I^k$, any $(j_1,...,j_l)\in I^l$ and any $p\in A_{k+l}$,

$$\mathbb{E}\kappa_A^p(M_{i_1}^N,...,M_{i_k}^N,L_{j_1}^N,...,L_{j_l}^N)$$

converges when N goes to infinity. By Theorem 4.1 of [**15**], for any positive integers k and l, any $(i_1,...,i_k)\in I^k$, any $(j_1,...,j_l)\in I^l$, $M_{i_1}^N\otimes...\otimes M_{i_k}^N\otimes L_{j_1}^N\otimes...\otimes L_{j_l}^N$ converges

in expectation in A_{k+l}-moments as N goes to infinity. Using Lemma 2.2, the family $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ converges in expectation in A-moments. We just proved the first part of the following theorem.

Theorem 7.1. — *Let us suppose that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. The family $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ converges in A-expectation. Besides, the A-law of $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ only depends on the A-law of $(L_i)_{i\in I}$ and the A-law of $(M_j)_{j\in J}$.*

The key idea in order to prove the second part of this theorem is the following lemma. In this article, by monomial, we understand a product of powers of the formal variables with nonnegative integer exponents: the coefficient is equal to one.

Lemma 7.1. — *Let k be a positive integer, let $p\in A_k$ and let $\big(P_1,(i_1^1,...,i_{n_1}^1)\big)$,..., $\big(P_k,(i_1^k,...,i_{n_k}^k)\big)$ be elements of $\bigcup_{m\in\mathbb{N}}(\mathbb{C}\{X_1,...,X_m\}\times I^m)$ such that for any $j\in\{1,...,k\}$, P_j is a monomial.*

There exist a positive integer l, a partition $p'\in A_l$ and a l-tuple $(j_1,...,j_l)\in I^l$ such that for any positive integer N, any family of random matrices $(M_i)_{i\in I}$ of size N:

$$\mathbb{E}m_p[P_1(M_{i_1},...,M_{i_{n_1}}),...,P_k(M_{i_1},...,M_{i_{n_k}})]=\mathbb{E}m_{p'}[M_{j_1},...,M_{j_l}].$$

In order to illustrate the last lemma, let us give a simple example: for any positive integer N, for any matrices B and C of size N:

$$\mathbb{E}m_{id_2}[BCBB,BB]=\mathbb{E}m_{(1,2,3,4)(5,6)}[B,C,B,B,B,B].$$

Proof of Theorem 7.1. — It remains to prove that the A-law of $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ only depends on the A-laws of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$. Let us denote by $(N_i)_{i\in I\sqcup J}$ the family such that for any $i\in I$, $N_i=L_i$ and for any $j\in J$, $N_j=M_j$. Let k be a positive integer, and let $\big(P_1,(i_1^1,...,i_{n_1}^1)\big)$,..., $\big(P_k,(i_1^k,...,i_{n_k}^k)\big)$ be elements of $\cup_{m\in\mathbb{N}}\mathbb{C}\{X_1,...,X_m\}\times(I\sqcup J)^m$, let $p\in A_k$. By multi-linearity, we can suppose that each polynomial is a monomial. Let us use Lemma 7.1, let l be an integer and let $(j_1,...,j_l)\in(I\sqcup J)^l$ be a l-tuple such that for any integer N, any family of random matrices $(C_i)_{i\in I\sqcup J}$ of size N:

$$\mathbb{E}m_p[P_1(C_{i_1},...,C_{i_{n_1}}),...,P_k(C_{i_1},...,C_{i_{n_k}})]=\mathbb{E}m_{p'}[C_{j_1},...,C_{j_l}].$$

Using Theorem 4.1:

$$\mathbb{E}m_p[P_1(N_{i_1},...,N_{i_{n_1}}),...,P_k(N_{i_1},...,N_{i_{n_k}})]=\sum_{p''\in[id_l,p']_{A_k}}\mathbb{E}\kappa_A^{p''}[N_{j_1},...,N_{j_l}].$$

By definition of A-freeness, for any $p''\in A_k$, $\mathbb{E}\kappa_A^{p''}[N_{j_1},...,N_{j_l}]$ only depends on the $\mathcal{R}_A$-functionals of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, and thus it only depends on the A-law of $(L_i)_{i\in I}$ and the A-law of $(M_j)_{j\in J}$. □

The dependance is actually polynomial.

7.1.2. The special case of G-invariant random matrices. — As seen already, the case where one considers G-invariant random matrices is quite special, and often easier. Let us choose a possibility for (G, A) as explained in Notation 3.1. Let us consider two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. Let us suppose that both families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation. Using Theorem 4.5, we already know that if $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are G-invariant then the two families converge in $\mathcal{P}$-expectation. The following theorem is a straightforward consequence of our discussions in Section 4.3.

Theorem 7.2. — *Let us suppose that the family $(L_i)_{i\in I} \cup (M_j)_{j\in J}$ is G-invariant. The two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if they are asymptotically $\mathcal{P}$-free.*

We can prove an other version of this theorem which use the notion of asymptotically G-invariance.

Definition 7.2. — *Let us suppose that $(M_j)_{j\in J}$ converges in $\mathcal{P}$-expectation. We say that $(M_j)_{j\in J}$ is asymptotically G-invariant if for any integer k, for any $p \in \mathcal{P}_k$, for any $(j_1, ..., j_k) \in J^k$,*

$$\mathbb{E}m_p\left(M_{j_1}, ..., M_{j_k}\right) = \lim_{N\to\infty} \mathbb{E}m_p\left(U^N M_{j_1}^N (U^N)^{-1}, ..., U^N M_{j_k}^N (U^N)^{-1}\right),$$

where, for any integer N, U^N is a Haar random variable on $G(N)$ which is independent from $(M_j^N)_{j\in J}$.

We could have also written this definition as a condition on the $\mathcal{P}$-law of $(M_i)_{i\in I}$, since we have the following lemma which is a consequence of Proposition 4.3, Lemma 4.3 and 4.4 of [**15**]. Recall the Definition 3.9 of [**15**] where we defined the notation Mb.

Lemma 7.2. — *Let us suppose that G is equal to O or U. The family $(M_j)_{j\in J}$ is asymptotically G-invariant if and only if for any positive integer k, for any $(j_1, ..., j_k) \in J^k$ one the following equivalent condition holds:*

- *for any $p \in \mathcal{P}_k \setminus A_k$, $\mathbb{E}\kappa_{\mathcal{P}}^p\left(M_{j_1}, ..., M_{j_k}\right) = 0$,*
- *$\left(\mathbb{E}m_p\left(M_{j_1}, ..., M_{j_k}\right)\right)_{p\in\mathcal{P}_k}$ is G-invariant (see Definition 4.10 of [**15**]),*
- *for any $p \in \mathcal{P}_k$, $\mathbb{E}m_{p^c}\left(M_{j_1}, ..., M_{j_k}\right) = \delta_{p\in\overline{A_k}}\mathbb{E}m_{(\mathsf{Mb}(p))^c}\left(M_{j_1}, ..., M_{j_k}\right)$.*

If $(M_j)_{j\in J}$ is asymptotically G-invariant then for any $p \in A_k$:

$$\mathbb{E}\kappa_{\mathcal{P}}^p(M_{j_1}, ..., M_{j_k}) = \mathbb{E}\kappa_A^p(M_{j_1}, ..., M_{j_k}).$$

As noticed by C. Male, the following theorem is a generalization of the "rigidity of freeness" theorem of [**26**]. A consequence of the Theorem 7.3 is that if two families are $\mathcal{P}$-free, satisfy the asymptotic $\mathfrak{S}$-factorization property and and that one of the two families is U, O or B-asymptotically invariant, then the two families are free in the meaning of Voiculescu.

Theorem 7.3. — *Let us suppose that the family $(M_j)_{j\in J}$ is asymptotically G-invariant, if the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically $\mathcal{P}$-free then they are asymptotically A-free.*

Proof. — Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ satisfy the conditions of the theorem. It is enough to prove that for any positive integers k_1 and k_2, for any $b \in A_{k_1+k_2}$, any $(i_1, ..., i_{k_2}) \in I^{k_1}$ and $(j_1, ..., j_{k_1}) \in J^{k_2}$, $\mathbb{E}\kappa_A^b(L_{i_1} \otimes ... \otimes L_{i_{k_1}} \otimes M_{j_1} \otimes ... \otimes M_{j_{k_2}})$ is equal to:

$$\delta_{\exists (b_1,b_2)\in A_{k_1}\times A_{k_2}, b=b_1\otimes b_2}\mathbb{E}\kappa_A^{b_1}(L_{i_1} \otimes ... \otimes L_{i_{k_1}})\mathbb{E}\kappa_A^{b_2}(M_{j_1} \otimes ... \otimes M_{j_{k_2}}).$$

Let us denote by k the integer k_1+k_2. In order to compute $\mathbb{E}\kappa_A^b(L_{i_1}\otimes...\otimes L_{i_{k_1}}\otimes M_{j_1}\otimes...\otimes M_{j_{k_2}})$, we need to compute $\int_{G(N)} g^{\otimes k}\mathbb{E}\left[L_{i_1}^N \otimes ... \otimes L_{i_{k_1}}^N \otimes M_{j_1}^N \otimes ... \otimes M_{j_{k_2}}^N\right](g^{-1})^{\otimes k}dg$ with N greater than $2k$. This is equal to:

$$\int_{G(N)} g^{\otimes k}\left(\frac{1}{N!}\sum_{S\in\mathfrak{S}(N)} S^{\otimes k}\mathbb{E}\Big[L_{i_1}^N \otimes ... \otimes L_{i_{k_1}}^N \otimes M_{j_1}^N \otimes ... \otimes M_{j_{k_2}}^N\Big](S^{-1})^{\otimes k}\right)(g^{-1})^{\otimes k}dg,$$

and thus this is equal to:

$$\sum_{p\in\mathcal{P}_k} \mathbb{E}\kappa_{\mathcal{P}}^p(L_{i_1}^N, ..., L_{i_{k_1}}^N, M_{j_1}^N, ..., M_{j_{k_2}}^N)\frac{1}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}\int_{G(N)} g^{\otimes k}p(g^{-1})^{\otimes k}dg. \tag{11}$$

Using Proposition 4.3 of [**15**], for any $p \in \mathcal{P}_k$, the sequence:

$$\left(E_N^p = \frac{1}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}\int_{G(N)} g^{\otimes k}p(g^{-1})^{\otimes k}dg\right)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$$

converges: we can neglect in the sum 11 any element of the form $o(1)\frac{p}{N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}}$. Using the asymptotic $\mathcal{P}$-freeness of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, this implies that we can study the following expression instead of the expression (11):

$$\sum_{(p_1,p_2)\in\mathcal{P}_{k_1}\times\mathcal{P}_{k_2}} \mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N, ..., L_{i_{k_1}}^N\right)\mathbb{E}\kappa_{\mathcal{P}}^{p_2}\left(M_{j_1}^N, ..., M_{j_{k_2}}^N\right)$$
$$\frac{1}{N^{\mathsf{nc}(p_1\otimes p_2)-\mathsf{nc}((p_1\otimes p_2)\vee id)}}\int_{G(N)} g^{\otimes k}(p_1\otimes p_2)(g^{-1})^{\otimes k}dg,$$

and using the fact that the family $(M_j)_{j\in J}$ is asymptotically G-invariant, which implies that for any $p \in \mathcal{P}\setminus A$, $\mathbb{E}\kappa_{\mathcal{P}}^p(M_{i_1}, ..., M_{i_k}) = 0$ and for any $p \in A$, $\mathbb{E}\kappa_A^p(M_{i_1}, ..., M_{i_k}) = \mathbb{E}\kappa_{\mathcal{P}}^p(M_{i_1}, ..., M_{i_k})$, we can study in fact:

$$\sum_{(p_1,p_2)\in\mathcal{P}_{k_1}\times A_{k_2}} \mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N, ..., L_{i_{k_1}}^N\right)\mathbb{E}\kappa_A^{p_2}\left(M_{j_1}^N, ..., M_{j_{k_2}}^N\right)$$
$$\frac{1}{N^{\mathsf{nc}(p_1\otimes p_2)-\mathsf{nc}((p_1\otimes p_2)\vee id)}}\int_{G(N)} g^{\otimes k}(p_1\otimes p_2)(g^{-1})^{\otimes k}dg.$$

Yet, we remind that for any $(p_1, p_2) \in \mathcal{P}_{k_1} \times A_{k_2}$,

$$\begin{aligned}\int_{G(N)} g^{\otimes k}(p_1\otimes p_2)(g^{-1})^{\otimes k}dg &= \int_{G(N)}\left(g^{\otimes k_1}p_1(g^{-1})^{\otimes k_1}\right)\otimes\left(g^{\otimes k_2}p_2(g^{-1})^{\otimes k_2}\right)dg\\ &= \left(\int_{G(N)} g^{\otimes k_1}p_1(g^{-1})^{\otimes k_1}dg\right)\otimes p_2.\end{aligned}$$

Thus we have to study:

$$\sum_{p_1\in\mathcal{P}_{k_1}} \mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\frac{\int_{G(N)} g^{\otimes k_1}p_1(g^{-1})^{\otimes k_1}dg}{N^{\mathsf{nc}(p_1)-\mathsf{nc}(p_1\vee id)}}$$
$$\otimes \sum_{p_2\in A_{k_2}} \mathbb{E}\kappa_A^{p_2}\left(M_{j_1}^N,...,M_{j_{k_2}}^N\right)\frac{p_2}{N^{\mathsf{nc}(p_2)-\mathsf{nc}(p_2\vee id)}}.$$

At last, let us remark that $\sum_{p_1\in\mathcal{P}_{k_1}} \mathbb{E}\kappa_{\mathcal{P}}^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\frac{\int_{G(N)} g^{\otimes k_1+k_2}p_1(g^{-1})^{\otimes k_1}dg}{N^{\mathsf{nc}(p_1)-\mathsf{nc}(p_1\vee id)}}$ is in fact equal to:

$$\sum_{p_1\in A_k} \mathbb{E}\kappa_A^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\frac{p_1}{N^{\mathsf{nc}(p_1)-\mathsf{nc}(p_1\vee id)}},$$

and thus we get that the limit, as N goes to infinity, of the coordinate numbers of $\int_{G(N)} g^{\otimes k}\mathbb{E}\left[L_{i_1}^N\otimes...\otimes L_{i_{k_1}}^N\otimes M_{j_1}^N\otimes...\otimes M_{j_{k_2}}^N\right](g^{-1})^{\otimes k}dg$ is equal to the limit, as N goes to infinity, of the coordinate numbers of:

$$\sum_{(p_1,p_2)\in A_{k_1}\times A_{k_2}} \mathbb{E}\kappa_A^{p_1}\left(L_{i_1}^N,...,L_{i_{k_1}}^N\right)\mathbb{E}\kappa_A^{p_2}\left(M_{j_1}^N,...,M_{j_{k_2}}^N\right)\frac{p_1\otimes p_2}{N^{\mathsf{nc}(p_1\otimes p_2)-\mathsf{nc}((p_1\otimes p_2)\vee id)}}.$$

This implies that for any $b\in A_{k_1+k_2}$,

$$\mathbb{E}\kappa_A^b(L_{i_1}\otimes...\otimes L_{i_{k_1}}\otimes M_{j_1}\otimes...\otimes M_{j_{k_2}})$$
$$=\delta_{\exists(b_1,b_2)\in A_{k_1}\times A_{k_2},b=b_1\otimes b_2}\mathbb{E}\kappa_A^{b_1}(L_{i_1}\otimes...\otimes L_{i_{k_1}})\mathbb{E}\kappa_A^{b_2}(M_{j_1}\otimes...\otimes M_{j_{k_2}}),$$

and thus $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. □

Using the Theorem 4.5 and the Definition 7.1 of asymptotic freeness, we can already see that a unitarily invariant family of random matrices which converges in expectation in $\mathfrak{S}$-moments is asymptotically $\mathfrak{S}$-free from its transpose family: this gives us the first example of families of random matrices which are $\mathfrak{S}$-free. This result was proved in [**28**] as a consequence of a more general result on second-order freeness ; this result is equivalent to the following theorem.

Theorem 7.4. — *Let us suppose that the family $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ is U-invariant, converges in expectation in $\mathfrak{S}$-moments and satisfies the asymptotic $\mathfrak{S}$-factorization property. Then the families $(L_i)_{i\in I}$ and $({}^tM_j)_{j\in J}$ are asymptotically $\mathfrak{S}$-free.*

Proof. — Since $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ is U-invariant, by Corollary 4.1, $(L_i)_{i\in I}\cup({}^tM_j)_{j\in J}$ converges in expectation in $\mathcal{P}$-moments. Let us prove that for any positive integer k and l, any $\sigma\in\mathfrak{S}_{k+l}$ and any $(i_1,...,i_k)\in I^k$ and any $(j_1,...,j_l)\in J^l$, $\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})$ is equal to:

$$\delta_{\exists(\sigma_1,\sigma_2)\in\mathfrak{S}_k\times\mathfrak{S}_l|\sigma=\sigma_1\otimes\sigma_2}\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma_1}(L_{i_1},...,L_{i_k})\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma_2}({}^tM_{j_1},...,{}^tM_{j_l}).$$

It has to be noticed that the family $(L_i)_{i\in I}\cup({}^tM_j)_{j\in J}$ is O-invariant: using the point 3. of Theorem 4.5 in the case where $G=O$, we know that:

$$\mathbb{E}\kappa_{\mathfrak{S}}^{\sigma}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})=\mathbb{E}\kappa_{\mathcal{B}}^{\sigma}(L_{i_1},...,L_{i_k},{}^tM_{j_1},...,{}^tM_{j_l})$$

Besides, using the definition of S_k given in Definition 2.5, the fact that the family $(L_{i_1}, ..., L_{i_k}, {}^tM_{j_1}, {}^tM_{j_l})$ is O-invariant, and using Lemma 4.3, we get:

$$\mathbb{E}\kappa^{\sigma}_{\mathcal{B}}(L_{i_1}, ..., L_{i_k}, {}^tM_{j_1}, ..., {}^tM_{j_l}) = \mathbb{E}\kappa^{S_k(\sigma)}_{\mathcal{B}}(L_{i_1}, ..., L_{i_k}, M_{j_1}, ..., M_{j_l}).$$

Since $(L_{i_1}, ..., L_{i_k}, M_{j_1}, ..., M_{j_l})$ is U-invariant, because of Theorem 4.5, we know that the right-hand-side is equal to zero if $S_k(\sigma)$ is not in $\mathfrak{S}_{k+l}$. Let us remark that $S_k(\sigma)$ is in $\mathfrak{S}_{k+l}$ if and only if there exists $(\sigma_1, \sigma_2) \in \mathfrak{S}_k \times \mathfrak{S}_l$ such that $\sigma = \sigma_1 \otimes \sigma_2$. This proves that if σ can not be decomposed as $\sigma_1 \otimes \sigma_2$ with $(\sigma_1, \sigma_2) \in \mathfrak{S}_k \times \mathfrak{S}_l$, $\mathbb{E}\kappa^{\sigma}_{\mathfrak{S}}(L_{i_1}, ..., L_{i_k}, {}^tM_{j_1}, ..., {}^tM_{j_l}) = 0$.

It remains to consider the case when σ can be decomposed as $\sigma_1 \otimes \sigma_2$ with $(\sigma_1, \sigma_2) \in \mathfrak{S}_k \times \mathfrak{S}_l$. In this case, using the asymptotic $\mathfrak{S}$-factorization property, we get:

$$\begin{aligned}\mathbb{E}\kappa^{\sigma_1\otimes\sigma_2}_{\mathfrak{S}}(L_{i_1}, ..., L_{i_k}, {}^tM_{j_1}, ..., {}^tM_{j_l}) &= \mathbb{E}\kappa^{\sigma_1\otimes{}^t\sigma_2}_{\mathfrak{S}}(L_{i_1}, ..., L_{i_k}, M_{j_1}, ..., M_{j_l}) \\ &= \mathbb{E}\kappa^{\sigma_1}_{\mathfrak{S}}(L_{i_1}, ..., L_{i_k})\mathbb{E}\kappa^{{}^t\sigma_2}_{\mathfrak{S}}(M_{j_1}, ..., M_{j_l}) \\ &= \mathbb{E}\kappa^{\sigma_1}_{\mathfrak{S}}(L_{i_1}, ..., L_{i_k})\mathbb{E}\kappa^{\sigma_2}_{\mathfrak{S}}({}^tM_{j_1}, ..., {}^tM_{j_l}).\end{aligned}$$

This ends the proof. □

Let us remark that the proof has nothing special to do with random matrices and would work in an abstract setting. Also, we could have use a weaker condition on $(M_i)_{i\in I}$, namely the fact that $(M_i)_{i\in I}$ is asymptotically U-invariant.

7.1.3. Links between the different notion of asymptotic freeness. — We defined three notions of asymptotic freeness, namely the $\mathfrak{S}$, $\mathcal{B}$ and $\mathcal{P}$-asymptotic freeness. Besides, since the work of Voiculescu, a notion of asymptotic freeness was defined. In this section, we will explain the links between these notions using results that one can find in this paper, and we will explain the following diagram. In order to shorten the notations, we will ommit to specify that the notions of freeness that we consider in this section are "asymptotic" notions.

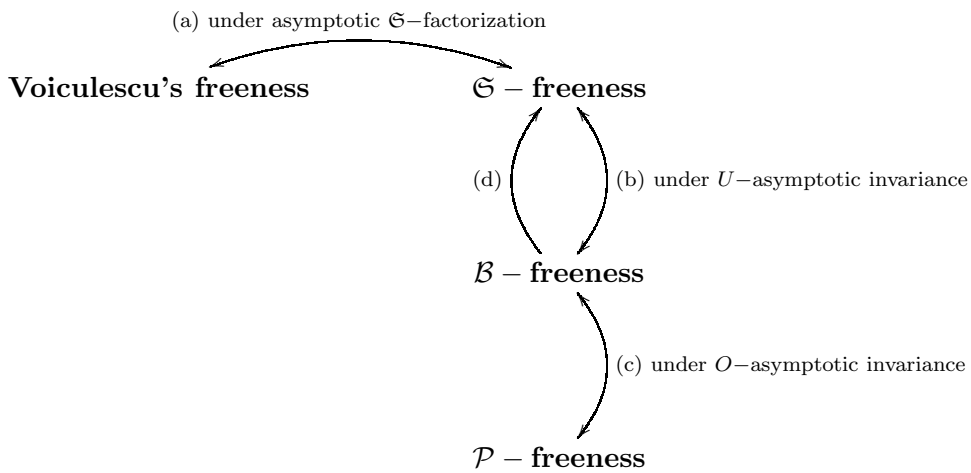

Diagram 1. The different notions of freeness.

Let us focus on the implications which are true. The equivalence denoted by (a) in Table 1 is a consequence of Proposition 4.3, and the fact that the freeness in free probability theory can be stated as a property of vanishing cumulants (Speicher's condition, Theorem 11.16 in [**29**]). The equivalences denoted by (b) and (c) in Diagram 1 are consequences of Theorems 7.3. The implication denoted by (d) is a direct consequence of Lemma 4.1.

Now, let us focus on the implications which are in general not true:

1. the $\mathcal{P}$-freeness does not imply the $\mathfrak{S}$-freeness. Let us considers two diagonal matrices which are independent and which have i.i.d. diagonal entries, then they are $\mathcal{P}$-free, but they are not $\mathfrak{S}$-free. Indeed, if they were, the limit of the eigenvalues distribution of their sum would be the free convolution of the limiting eigenvalues distributions of the two matrices, yet it is clear that it is the usual convolution of the limiting eigenvalues distributions of the two matrices. Since free convolution and usual convolution are not equivalent, one can conclude. More generaly, one can see the fact that $\mathcal{P}$-freeness does not imply the $\mathfrak{S}$-freeness as a consequence of Proposition 7.1.
2. The $\mathcal{P}$-freeness does not imply the $\mathcal{B}$-freeness: if it was the case, since $\mathcal{B}$-freeness implies $\mathfrak{S}$-freeness the first point would be true.
3. the $\mathfrak{S}$-freness does not implies the $\mathcal{B}$-freeness. By Theorem 7.4, if one considers $M \in L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is U invariant and which converges in $\mathfrak{S}$-moments, then M and M^t are asymptotically $\mathfrak{S}$-free. It M and M^t were $\mathcal{B}$-free, this would imply that M is asymptotically $\mathfrak{S}$-free with itself: this is usually not true.

For the moment, we do not have a good argument which would explain why the asymptotic $\mathcal{B}$-freeness does not imply the $\mathcal{P}$-freeness. Besides, as explained in Remark 3.1, one can define other notions of asymptotic freeness, for exemple the $\mathcal{B}s$ and $\mathcal{H}$ notions of asymptotic freeness. It would be interesting to know the links between all of them.

Let us finish with a proposition which is totally inspired by a result of C. Male, namely the first point of Corollary 3.5 of [**26**]. The proof that we give here differs from the one of C. Male as it is based on the cumulants we introduced in this article and the new Theorem 7.8.

Proposition 7.1. — *Let* $\mathsf{0}_2$ *be the partition* $\{\{1, 2, 1', 2'\}\}$. *Let* $\mathbf{M} = (M_1^N, M_2^N)_{N\in\mathbb{N}}$ *and* $\mathbf{L} = (L_1^N, L_2^N)_{N\in\mathbb{N}}$ *be two families of elements of* $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ *which converges in* $\mathcal{P}$*-expectation. Let us suppose that* $\mathbf{M}$ *and* $\mathbf{L}$ *are asymptotically* $\mathcal{P}$*-free. If*

$$\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}(M_1 \otimes M_2)\mathbb{E}\kappa_{\mathcal{P}}^{\mathsf{0}_2}(L_1 \otimes L_2) \neq 0,$$

then $\mathbf{M}$ *and* $\mathbf{L}$ *are not asymptotically* $\mathfrak{S}$*-free.*

Proof. — Let $\mathbf{M} = (M_1^N, M_2^N)_{N\in\mathbb{N}}$ and $\mathbf{L} = (L_1^N, L_2^N)_{N\in\mathbb{N}}$ be two families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converge in $\mathcal{P}$-expectation and which are asymptotically $\mathcal{P}$-free. We will apply a theorem that we will prove in a more general setting, namely Theorem

7.8. It shows that:

$$\mathbb{E}m_{(1,2,3,4)}\left[M_1,L_1,M_2,L_2\right]=\mathbb{E}m_{(1,2)}\left[M_1L_1,M_2L_2\right]$$
$$=\sum_{p\in\mathcal{P}_k,p\in[id,(1,2)]_{\mathcal{P}_2}}\mathbb{E}\kappa^p_{\mathcal{P}}[M_1,M_2]\mathbb{E}m_{\iota_{p\circ(1,2)}}[L_1,L_2].$$

Yet, $[id,(1,2)]_{\mathcal{P}_2}=\{id,\mathsf{0}_2,(1,2)\}$, thus $\mathbb{E}m_{(1,2,3,4)}\left[M_1,L_1,M_2,L_2\right]$ is equal to:

$$\mathbb{E}\kappa^{id}_{\mathcal{P}}[M_1,M_2]\mathbb{E}m_{(1,2)}[L_1,L_2]+\mathbb{E}\kappa^{\mathsf{0}_2}_{\mathcal{P}}[M_1,M_2]\mathbb{E}m_{\mathsf{0}_2}[L_1,L_2]+\mathbb{E}\kappa^{(1,2)}_{\mathcal{P}}[M_1,M_2]\mathbb{E}m_{id}[L_1,L_2].$$

Using the links between cumulants and moments, we can write again this expression as:

$$\mathbb{E}m_{id}[M_1,M_2]\mathbb{E}m_{(1,2)}[L_1,L_2]+\mathbb{E}m_{(1,2)}[M_1,M_2]\mathbb{E}m_{id}[L_1,L_2]-\mathbb{E}m_{id}[M_1,M_2]\mathbb{E}m_{id}[L_1,L_2]$$
$$+\mathbb{E}\kappa^{\mathsf{0}_2}_{\mathcal{P}}[M_1,M_2]\mathbb{E}\kappa^{\mathsf{0}_2}_{\mathcal{P}}[L_1,L_2].$$

If $\mathbf{M}$ and $\mathbf{L}$ were asymptotically $\mathfrak{S}$-free, using the same arguments but using the set-geodesic in $\mathfrak{S}_k$ would lead to the following expression for $\mathbb{E}m_{(1,2,3,4)}\left[M_1,L_1,M_2,L_2\right]$:

$$\mathbb{E}m_{id}[M_1,M_2]\mathbb{E}m_{(1,2)}[L_1,L_2]+\mathbb{E}m_{(1,2)}[M_1,M_2]\mathbb{E}m_{id}[L_1,L_2]-\mathbb{E}m_{id}[M_1,M_2]\mathbb{E}m_{id}[L_1,L_2].$$

Thus, we see that if $\mathbb{E}\kappa^{\mathsf{0}_2}_{\mathcal{P}}(M_1,M_2)\mathbb{E}\kappa^{\mathsf{0}_2}_{\mathcal{P}}(L_1,L_2)\neq 0$, then $\mathbf{M}$ and $\mathbf{L}$ are not asymptotically $\mathfrak{S}$-free. □

Remark 7.1. — *Recall that in Remark 3.1, we defined for any positive integer k, $\mathcal{H}_k$ as the subset of $\mathcal{P}_k$ of partitions which have blocks of even size. Let us remark that in Proposition 7.1 one can replace the condition of $\mathcal{P}$-free by $\mathcal{H}$-free since $\mathsf{0}_2\in[id,(1,2)]_{\mathcal{H}_2}$.*

Using this result, we will show in the article [**17**] that the $\mathcal{P}$-free Levy process which is the limit of random walks on the symmetric group is not a $\mathfrak{S}$-free Levy process: its increments are not $\mathfrak{S}$-free, and thus since the $\mathfrak{S}$-asymptotic factorization will hold, this is not a free Levy process.

7.1.4. The $\mathcal{B}s$-asymptotic freeness implies $\mathfrak{S}$-asymptotic. — In Section 3.1, in Remak 3.1, we explained that the paper could be generalized to any of the easy orthogonal groups and the category of partitions associated to them. We also explained by we decided to focus on the three groups $U(N)$, $O(N)$, $\mathfrak{S}(N)$. Yet, we would like to give an example of the use of the other groups in order to explain that a deeper study of the other notions of asymptotic freeness, namely the $\mathcal{B}s$ and the $\mathcal{H}$-asymptotic freeness, could be interesting.

Lemma 7.3. — *Let k be a positive integer. Let $\sigma\in\mathfrak{S}_k$, then we have the equality:*

$$[id,\sigma]_{\mathfrak{S}_k}=[id,\sigma]_{\mathcal{B}s_k}$$

Proof. — The proof is similar to the proof of Lemma 3.9 of [**15**]. One has just to check that for any permutation $\sigma\in\mathfrak{S}_k$ and any $p\in\mathcal{B}s_k$ such that $d(id,p)=1$, p is not in the set-geodesic $[id,\sigma]_{\mathcal{P}_k}$. □

A consequence of this is that for any family $(M_i)_{i\in I}$ of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in $\mathcal{B}s$-expectation, for any positive integer k, for any $\sigma\in\mathfrak{S}_k$ and any $(i_1,...,i_k)\in I^k$:

$$\mathbb{E}\kappa^{\sigma}_{\mathcal{B}s}\left[M_{i_1},...,M_{i_k}\right]=\mathbb{E}\kappa^{\sigma}_{\mathfrak{S}}\left[M_{i_1},...,M_{i_k}\right].$$

This fact implies the following theorem.

Theorem 7.5. — *Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converge in $\mathcal{B}s$-expectation. Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically $\mathcal{B}s$-free, then they are $\mathfrak{S}$-free. In particular, if each family satisfy the asymptotic $\mathfrak{S}$-factorization property, then $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are free in the meaning of Voiculescu.*

Using this theorem and Theorem 7.10, one gets the following theorem.

Theorem 7.6. — *Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converge in $\mathcal{B}s$-expectation and which satisfy the asymptotic $\mathfrak{S}$-factorization. Let us suppose that L_i is B-invariant and let us suppose that for every positive integer N, the two families $\left(L_i^N\right)_{i\in I}$ and $(M_j^N)_{j\in J}$ are independent. Then the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically free in the meaning of Voiculescu.*

7.1.5. Definition of freeness using exclusive moments. — Let us give an other definition of asymptotically A-freeness. Recall Definition 3.13 of [**15**], where we defined for any positive integer $l\leq k$ and $p\in\mathcal{P}_k$, the left and right parts p_l^g and p_l^d.

Theorem 7.7. — *The families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if for any positive integer k, for any partition $p\in\mathcal{P}_k$, for any integer $l\in]0,...,k[$, for any k-tuple $(B_i)_{i=1}^k$ such that $(B_i)_{i=1}^l$ is in $\mathcal{A}\left((L_i)_{i\in I}\right)$ and $(B_i)_{i=l+1}^k$ is in $\mathcal{A}\left((M_j)_{j\in J}\right)$, $\mathbb{E}m_{p^c}^A\left((B_i^N)_{i=1}^k\right)$ converges as N goes to infinity, and:*

$$\lim_{N\to\infty}\mathbb{E}m_{p^c}^A\left((B_i^N)_{i=1}^k\right)=\delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m_{(p_l^g)^c}^A\left[(B_i)_{i=1}^l\right]\mathbb{E}m_{(p_l^d)^c}^A\left[(B_i)_{i=l+1}^k\right].$$

For sake of clarity, we will say that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free in exclusive moments if the conditions on the exclusive moments hold. In the upcoming paper [**14**], the author and his coauthors explain the link between this definition of asymptotic A-freeness in exclusive moments and the definition of C. Male based on traffics and free-product of traffic defined in [**26**].

Proof of Theorem 7.7. — Let k be a positive integer, let l be an integer in $]0,...,k[$ and let $p\in\mathcal{P}_k$. Let $(B_i)_{i=1}^l$ be a l-tuple of elements of $\mathcal{A}\left((L_i)_{i\in I}\right)$ and let $(B_i)_{i=l+1}^k$ be a $(k-l)$-tuple of elements of $\mathcal{A}\left((M_j)_{j\in J}\right)$. Let us prove that:

$$\sum_{p_1'\in A_l,p_2'\in A_{k-l},p_1'\otimes p_2'\sqsupset p}\mathbb{E}\kappa_A^{p_1'}\left((B_i)_{i=1}^l\right)\mathbb{E}\kappa_A^{p_2'}\left((B_i)_{i=l+1}^k\right) \tag{12}$$

$$=\delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m_{(p_l^g)^c}^A\left[(B_i)_{i=1}^l\right]\mathbb{E}m_{(p_l^d)^c}^A\left[(B_i)_{i=l+1}^k\right]. \tag{13}$$

Let us consider the left hand side: the Proposition 3.4 of [**15**] shows that this is also equal to:

$$\delta_{p_l^g\otimes p_l^d\sqsupset p}\sum_{p_1\in A_l,p_1\sqsupset p_l^g}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^l\right)\sum_{p_2\in A_{k-l},p_2\sqsupset p_l^d}\mathbb{E}\kappa_A^{p_2}\left((B_i)_{i=l+1}^k\right),$$

and using Theorem 5.1, it is thus equal to:

$$\mathbb{E}m_{p^c}^A\left((B_i)_{i=1}^k\right)=\delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m_{(p_l^g)^c}^A\left[(B_i)_{i=1}^l\right]\mathbb{E}m_{(p_l^d)^c}^A\left[(B_i)_{i=l+1}^k\right].$$

Let us suppose that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. Then the left hand side (12) is equal to:

$$\sum_{p'\in A_k, p'\sqsupset p} \mathbb{E}\kappa_A^{p'}\left((B_i)_{i=1}^k\right).$$

which, by Theorem 5.1 is equal to $\mathbb{E}m_{p^c}^A\left((B_i)_{i=1}^k\right)$. This implies that

$$\mathbb{E}m_{p^c}^A\left((B_i)_{i=1}^k\right) = \delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m_{(p_l^g)^c}^A\left[(B_i)_{i=1}^l\right]\mathbb{E}m_{(p_l^d)^c}^A\left[(B_i)_{i=l+1}^k\right],$$

and thus $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free in exclusive moments.

Now, let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free in exclusive moments. Then the right hand side (13) is equal to $\mathbb{E}m_{p^c}^A\left((B_i)_{i=1}^k\right)$. Thus:

$$\sum_{p_1'\in A_l, p_2'\in A_{k-l}, p_1'\otimes p_2'\sqsupset p} \mathbb{E}\kappa_A^{p_1'}\left((B_i)_{i=1}^l\right)\mathbb{E}\kappa_A^{p_2'}\left((B_i)_{i=l+1}^k\right) = \mathbb{E}m_{p^c}^A\left((B_i)_{i=1}^k\right).$$

This equality is valid for any partition $p\in\mathcal{P}_k$, yet for any $p\in\mathcal{P}_k$, we know that:

$$\sum_{p'\in A_k, p'\sqsupset p} \mathbb{E}\kappa_A^{p'}\left((B_i)_{i=1}^k\right) = \mathbb{E}m_{p^c}^A\left((B_i)_{i=1}^k\right).$$

By inversion of the matrix order or $\sqsupset$, we get that for any $p\in A_k$:

$$\mathbb{E}\kappa_A^{p}\left((B_i)_{i=1}^k\right) = \delta_{\exists(p_1,p_2)\in A_l\times A_{k-l}, p=p_1\otimes p_2}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^l\right)\mathbb{E}\kappa_A^{p_2}\left((B_i)_{i=l+1}^k\right).$$

This shows that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. □

Remark 7.2. — *We proved Theorem 7.7 under the assumption that we were working with sequences of random matrices. Yet, it has to be noticed that we did use only a combinatorial argument: this implies that the proof would work exactly the same for general A-free non-commutative random variables.*

7.1.6. Factorization and freeness. — Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that both families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation and satisfy the asymptotic A-factorization property.

Proposition 7.2. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free, then the asymptotic A-factorization property holds for $(L_i)_{i\in I}\cup(M_j)_{j\in J}$.*

Proof. — This is a direct consequence of Proposition 4.1 and the definition of asymptotic A-freeness. □

7.1.7. Sum and product of families of A-free elements. — In this new setting, we can generalize some well-known theorems about the sum and the multiplication of asymptotic free elements. Recall the Definitions 6.1 and 6.2 of [**15**] about the notion $\prec$ and the generalized Kreweras complement. Recall the notation $\mathfrak{F}_2(p)$ defined in Definition 2.11 in [**15**]. We need an other notation.

Notation 7.2. — *Let k be a positive integer. Let $(C_i)_{i=1}^k$ be a k-tuple of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which converges in expectation in A-moments. Let K be a subset of $\mathcal{P}_k$, we set:*

$$\mathbb{E}\kappa_A^K\left[(C_i)_{i=1}^k\right] = \sum_{p\in K\cap A_k} \mathbb{E}\kappa_A^p\left[(C_i)_{i=1}^k\right].$$

Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$.

Theorem 7.8. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. Let k be a positive integer. Let us consider $\left((B_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ and $\left((C_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ such that for any $i \in \{1, ..., k\}$, $\left(B_i^N\right)_{N\in\mathbb{N}} \in \mathcal{A}\left((L_i)_{i\in I}\right)$ and $\left(C_i^N\right)_{N\in\mathbb{N}} \in \mathcal{A}\left((M_j)_{j\in J}\right)$.*

Then, for any $p \in A_k$:

$$\mathbb{E}\kappa_A^p\left[B_1 + C_1, ..., B_k + C_k\right] = \sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i\in I}\right)\mathbb{E}\kappa_A^{p_2}\left((C_j)_{j\in\{1,...,k\}\setminus I}\right),$$

$$\mathbb{E}\kappa_A^p\left[B_1C_1, ..., B_kC_k\right] = \sum_{p_1\in A_k, p_1\prec p} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_p(p_1)}\left((C_i)_{i=1}^k\right),$$

$$\mathbb{E}m_p\left[B_1C_1, ..., B_kC_k\right] = \sum_{p_1\in A_k, p_1\in[id,p]_{A_k}} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}m_{t_{p_1}\circ p}\left((C_i)_{i=1}^k\right).$$

Proof. — Let k be a positive integer, let us consider $\left((B_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ and $\left((C_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ such that for any $i \in \{1, ..., k\}$, $\left(B_i^N\right)_{N\in\mathbb{N}} \in \mathcal{A}\left((L_i)_{i\in I}\right)$, and $\left(C_i^N\right)_{N\in\mathbb{N}} \in \mathcal{A}\left((M_j)_{j\in J}\right)$. Let p be in A_k. Let us prove first that:

$$\mathbb{E}\kappa_A^p\left[B_1 + C_1, ..., B_k + C_k\right] = \sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i\in I}\right)\mathbb{E}\kappa_A^{p_2}\left((C_j)_{j\in\{1,...,k\}\setminus I}\right).$$

By multilinearity, we have:

$$\mathbb{E}\kappa_A^p\left[B_1 + C_1, ..., B_k + C_k\right] = \sum_{(U^{(1)},...,U^{(k)})\in\{B,C\}^k} \mathbb{E}\kappa_A^p\left[U_1^{(1)}, ..., U_k^{(k)}\right].$$

Using the asymptotic A-freeness of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, we see that every term is equal to zero except for the choices of $\left(U^{(1)}, ..., U^{(k)}\right) \in \{B, C\}^k$ such that for any cycle of p and for any i and j in this cycle, $U^{(i)} = U^{(j)}$. Such a choice gives a decomposition of p in two partitions. Recall the notation $\mathfrak{F}_2(p)$ defined in Definition 2.11 in [**15**]. We get:

$$\mathbb{E}\kappa_A^p\left[B_1 + C_1, ..., B_k + C_k\right] = \sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i\in I}\right)\mathbb{E}\kappa_A^{p_2}\left((C_j)_{j\in\{1,...,k\}\setminus I}\right).$$

Let us prove now that:

$$\mathbb{E}\kappa_A^p\left[B_1C_1, ..., B_kC_k\right] = \sum_{p_1\in A_k, p_1\prec p} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_p(p_1)}\left((C_i)_{i=1}^k\right).$$

For this, we only have to prove that for any $p \in A_k$,

$$\sum_{p'\in[id,p]_{A_k}} \sum_{p_1\in A_k, p_1\prec p'} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right) = \mathbb{E}m_p\left[B_1C_1, ..., B_kC_k\right].$$

Let p be in A_k, we have:

$$\sum_{p'\in[id,p]_{A_k}}\sum_{p_1\in A_k,p_1\prec p'}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right)$$
$$=\sum_{p'\in[id,p]_{A_k}}\sum_{p_1,p_2\in A_k,p_1\prec p_1\circ p_2=p'}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right)$$
$$=\sum_{p_1,p_2\in A_k,p_1\prec p_1\circ p_2,p_1\circ p_2\in[id,p]_{A_k}}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right).$$

We recall that we proved in Proposition 8.1 of [**15**], that for any p,p_1,p_2 in A_k:

$$\delta_{p_1\otimes p_2\in[id,(p\otimes id_k)\tau]_{A_{2k}}}=\delta_{p_1\circ p_2\in[id,p]_{A_k}}\delta_{p_1\prec p_1\circ p_2}.$$

where $\tau=(1,k+1)(2,k+2)...(k,2k)$. Thus,

$$\sum_{p'\in[id,p]_{A_k}}\sum_{p_1\in A_k,p_1\prec p'}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right)$$
$$=\sum_{p_1,p_2\in A_k,p_1\otimes p_2\in[id,(p\otimes id_k)\tau]_{A_{2k}}}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right).$$

Using the asymptotic A-freeness of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$, we know that for any p_1 and p_2 in A_k,

$$\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{p_2}\left((C_i)_{i=1}^k\right)=\mathbb{E}\kappa_A^{p_1\otimes p_2}\left(B_1,...,B_k,C_1,...,C_k\right).$$

Besides, if $p'\in A_{2k}$ is not compatible with $(B_1,...,B_k,C_1,...,C_k)$, which means that there exist no partitions p_1 and p_2 in A_k such that $p_1\otimes p_2=p'$ then:

$$\mathbb{E}\kappa_A^{p'}\left(B_1,...,B_k,C_1,...,C_k\right)=0.$$

This implies that:

$$\sum_{p'\in[id,p]_{A_k}}\sum_{p_1\in A_k,p_1\prec p'}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right)$$
$$=\sum_{p_1,p_2\in A_k,p_1\otimes p_2\in[id,(p\otimes id_k)\tau]_{A_{2k}}}\mathbb{E}\kappa_A^{p_1\otimes p_2}\left(B_1,...,B_k,C_1,...,C_k\right)$$
$$=\sum_{p'\in A_k,p'\in[id,(p\otimes id_k)\tau]_{A_{2k}}}\mathbb{E}\kappa_A^{p'}\left(B_1,...,B_k,C_1,...,C_k\right)$$
$$=\mathbb{E}m_{(p\otimes id_k)\tau}\left(B_1,...,B_k,C_1,...,C_k\right)$$
$$=\mathbb{E}m_p(B_1C_1,...,B_kC_k).$$

Let us prove now that:

$$\mathbb{E}m_p\left[B_1C_1,...,B_kC_k\right]=\sum_{p_1\in A_k,p_1\in[id,p]_{A_k}}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}m_{t_{p_1\circ p}}\left((C_i)_{i=1}^k\right).$$

For this, we use the equality we just proved, the Definition 4.1 and the Proposition 8.3 of [**15**]. Indeed,

$$\begin{aligned}
\mathbb{E}m_p\left[B_1C_1,...,B_kC_k\right] &= \sum_{p'\in A_k, p'\in[id,p]_{A_k}} \mathbb{E}\kappa_A^p\left[B_1C_1,...,B_kC_k\right] \\
&= \sum_{p'\in A_k, p'\in[id,p]_{A_k}} \sum_{p_1\in A_k, p_1\prec p'} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right) \\
&= \sum_{p_1\in A_k} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right) \sum_{p'\in A_k, p'\in[id,p]_{A_k}} \delta_{p_1\prec p'}\mathbb{E}\kappa_A^{K_{p'}(p_1)}\left((C_i)_{i=1}^k\right) \\
&= \sum_{p_1\in A_k} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\delta_{p_1\in[id,p]_{A_k}}\mathbb{E}m_{t_{p_1\circ p}}\left((C_i)_{i=1}^k\right) \\
&= \sum_{p_1\in A_k, p_1\in[id,p]_{A_k}} \mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}m_{t_{p_1\circ p}}\left((C_i)_{i=1}^k\right).
\end{aligned}$$

This finishes the proof. □

Let us remark that if p is the cycle $(1,...,k)$, using the fact that the set-geodesic $([id,(1,...,k)]_{\mathfrak{S}_k},\leq)$ is isomorphic to the poset of non-crossing partitions of $\{1,...,k\}$, we recover the classical formula, given in the Appendix of [**23**], which asserts that

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\big(B_1^NC_1^N...B_k^NC_k^N\big)\right] = \sum_{\pi\in NC_k}\tau_\pi(B_1,...,B_k)\kappa_{\pi^\vee}(C_1,...,C_k),$$

where we used the notation NC_k for the non-crossing partitions of k elements, the notation τ_π for the non-commutative moments and $\pi^\vee$ for the Kreweras complement of a non-crossing partition π.

Recall the definition of the operations $\boxplus$ and $\boxtimes$ defined in Section 10.1 of [**15**]. As a consequence of Theorem 7.8, we get the following theorem.

Theorem 7.9. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. Let us consider two sequences $(B_N)_{N\in\mathbb{N}}$ and $(C_N)_{N\in\mathbb{N}}$ such that $(B_N)_{N\in\mathbb{N}}\in\mathcal{A}\big((L_i)_{i\in I}\big)$, $(C_N)_{N\in\mathbb{N}}\in\mathcal{A}\big((M_j)_{j\in J}\big)$.*

The $\mathcal{R}_A$-transform of $(B_N+C_N)_{N\in\mathbb{N}}$ and $(B_NC_N)_{N\in\mathbb{N}}$ are given by:

$$\begin{aligned}
\mathcal{R}_A\left[(B_N+C_N)_{N\in\mathbb{N}}\right] &= \mathcal{R}_A\left[(B_N)_{N\in\mathbb{N}}\right]\boxplus\mathcal{R}_A\left[(C_N)_{N\in\mathbb{N}}\right], \\
\mathcal{R}_A\left[(B_NC_N)_{N\in\mathbb{N}}\right] &= \mathcal{R}_A\left[(B_N)_{N\in\mathbb{N}}\right]\boxtimes\mathcal{R}_A\left[(C_N)_{N\in\mathbb{N}}\right].
\end{aligned}$$

Actually, by using the up-coming Theorem 7.10, one could apply directly the results from the article [**15**], instead of the proof given in this section. We will explain more after Therorem 7.10. Yet the resulting proof could not be applied abstractly to two families which are A-free: both families should be families in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Instead, the proof that we gave for Theorem 7.8 is general enough to be applied in an abstract setting.

7.1.8. G-invariance and independence imply A-asymptotic freeness. — The following theorem generalizes the well-known fact that U and O invariance and independence implies freeness in the meaning of Voiculescu. Let us choose a possibility for (G, A) as explained in Notation 3.1. Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$. Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation.

Theorem 7.10. — *Let us suppose that $(L_i)_{i\in I}$ is G-invariant and let us suppose that for every positive integer N, the two families $(L_i^N)_{i\in I}$ and $(M_j^N)_{j\in I}$ are independent. Then the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free.*

Actually, the compatibility condition and the compatible factorization property hold for any integer N big enough so that the N-dimensional A-cumulants are defined: for any positive integer k, for any partition $p \in A_k$, for any positive integer $N \geq 2k$, for any mixed k-tuple $\left(B_i^N\right)_{i=1}^k$ of elements of $\mathcal{A}\left((L_i^N)_{i\in I}\right) \cup \mathcal{A}\left((M_j^N)_{j\in J}\right)$,

1-Compatibility condition : *If p is not compatible with $(B_i^N)_{i=1}^k$:*

$$\mathbb{E}\kappa_A^p\left(B_1^N, ..., B_k^N\right) = 0.$$

2-Compatible factorization property : *If p is compatible with $(B_i^N)_{i=1}^k$:*

$$\mathbb{E}\kappa_A^p\left(B_1^N, ..., B_k^N\right) = \mathbb{E}\kappa_A^{p_{L_{(B_i)_{i=1}^k}}}\left(\left(B_i^N\right)_{i\in L_{(B_i)_{i=1}^k}}\right)\mathbb{E}\kappa_A^{p_{M_{(B_i)_{i=1}^k}}}\left(\left(B_i^N\right)_{i\in M_{(B_i)_{i=1}^k}}\right).$$

Proof. — Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which satisfy the hypotheses stated in Theorem 7.10. Let k be a positive integer, let p be in A_k, let $(B_i)_{i=1}^k$ be any mixed k-tuple of elements of $\mathcal{A}\left((L_i)_{i\in I}\right) \cup \mathcal{A}\left((M_j)_{j\in J}\right)$.

We have to show that $\mathbb{E}\kappa^p\left[B_1^N, ..., B_k^N\right]$ converges as N goes to infinity, and we have to compute its limit. For this, we need to compute, for every positive integer N, $\int_{G(N)} g^{\otimes k}\mathbb{E}\left[B_1^N \otimes ... \otimes B_k^N\right](g^{-1})^{\otimes k}dg$. Using the independence for each positive integer N of the families $\left\{\left(B_i^N\right)/i \in L_{(B_i)_{i=1}^k}\right\}$ and $\left\{\left(B_i^N\right)/i \in M_{(B_i)_{i=1}^k}\right\}$, one can cut the expectation $\mathbb{E}\left[B_1^N \otimes ... \otimes B_k^N\right]$ in two. Thus, $\mathbb{E}\left[B_1^N \otimes ... \otimes B_k^N\right]$ is equal to:

$$\mathbb{E}\left[\mathcal{I}_{L_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in L_{(B_i)_{i=1}^k}} \otimes Id^{\otimes k-\#L_{(B_i)_{i=1}^k}}\right]\right]$$
$$\mathbb{E}\left[\mathcal{I}_{M_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in M_{(B_i)_{i=1}^k}} \otimes Id^{\otimes k-\#M_{(B_i)_{i=1}^k}}\right]\right].$$

For example, if X and Y are two independent random matrices:

$$\mathbb{E}[X \otimes Y \otimes X] = \mathbb{E}[X \otimes Id \otimes X]\mathbb{E}[Id \otimes Y \otimes Id].$$

Since for any positive integer N, $(L_i^N)_{i\in I}$ is invariant by conjugation by $G(N)$, the family $(B_i^N)_{i\in L_{(B_i)_{i=1}^k}}$ is invariant by conjugation by $G(N)$. Thus

$$\mathbb{E}\left[\mathcal{I}_{L_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in L_{(B_i)_{i=1}^k}} \otimes Id^{\otimes k-\#L_{(B_i)_{i=1}^k}}\right]\right]$$

commutes with the action of G on $\left(\mathbb{C}^N\right)^{\otimes k}$.

This implies that $\int_{G(N)} g^{\otimes k}\mathbb{E}\left[B_1^N\otimes ...\otimes B_k^N\right](g^{-1})^{\otimes k}dg$ is equal to:

$$\int_{G(N)}(g')^{\otimes k}\mathbb{E}\left[\mathcal{I}_{L_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in L_{(B_i)_{i=1}^k}}\otimes Id^{\otimes k-\#L_{(B_i)_{i=1}^k}}\right]\right](g'^{-1})^{\otimes k}dg'$$
$$\int_{G(N)}g^{\otimes k}\mathbb{E}\left[\mathcal{I}_{M_{(B_i)_{i=1}^k}}\left[(B_i^N)_{i\in M_{(B_i)_{i=1}^k}}\otimes Id^{\otimes k-\#M_{(B_i)_{i=1}^k}}\right]\right](g^{-1})^{\otimes k}dg.$$

It is now obvious that when one writes everything in terms of N-dimensional cumulants and partitions, the only partitions which will appear will be compatible with $(B_i)_{i=1}^k$, and if p is compatible, the compatible factorization property will hold for any $N\geq 2k$ if $A=\mathcal{P}$ and for any $N\geq k$ if $A=\mathfrak{S}$ or $A=\mathcal{B}$. □

Remark 7.3. — *Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families which satisfy the hypotheses of Theorem 7.10. Then for any integer N, the family $(L_i^N)_{i\in I}\cup((L_i^N)^*)_{i\in I}$ is also invariant by conjugation by $G(N)$, and the two families $(L_i^N)_{i\in I}\cup((L_i^N)^*)_{i\in I}$ and $(M_j^N)_{j\in J}$ are independent. Thus, if $(L_i)_{i\in I}\cup(L_i^*)_{i\in I}$ converges in A-expectation, then $(L_i^N)_{i\in I}\cup(L_i^*)_{i\in I}$ and $(M_j)_{j\in J}$ are also asymptotically A-free.*

Now we can recover the Theorem 7.8 as a simple consequence of the results in [**15**]. Indeed, let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families which are A-free. We have seen that the A-law of $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ only depends on the A-law of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$. Thus it remains unchanged if we suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are G-invariant. Besides because of Theorem 7.10, we can always suppose that, for any-positive integer N, $(L_i^N)_{i\in I}$ and $(M_j^N)_{j\in J}$ are independent. Let k be a positive integer. Let us consider $\left((B_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ and $\left((C_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ such that for any $i\in\{1,...,k\}$, $(B_i^N)_{N\in\mathbb{N}}\in\mathcal{A}\big((L_i)_{i\in I}\big)$, $(C_i^N)_{N\in\mathbb{N}}\in\mathcal{A}\big((M_j)_{j\in J}\big)$. Then, for any $p\in A_k$, for any positive integer N,

$$\begin{aligned}\mathbb{E}\left[B_1^NC_1^N\otimes ...\otimes B_k^NC_k^N\right]&=\mathbb{E}\left[(B_1^N\otimes ...\otimes B_k^N)(C_1^N\otimes ...\otimes C_k^N)\right]\\&=\mathbb{E}\left[B_1^N\otimes ...\otimes B_k^N\right]\mathbb{E}\left[C_1^N\otimes ...\otimes C_k^N\right].\end{aligned}$$

Recall Lemma 4.3 and Theorem 4.2: an application of the Equation (15) of Theorem 7.1 in [**15**] gives:

$$\mathbb{E}\kappa_A^p\left[B_1C_1,...,B_kC_k\right]=\sum_{p_1\in A_k,p_1\prec p}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}\kappa_A^{K_p(p_1)}\left((C_i)_{i=1}^k\right).$$

Besides, for any positive integer N,

$$\mathbb{E}m_p\left[B_1^NC_1^N,...,B_k^NC_k^N\right]=m_p\left[\mathbb{E}\left[B_1^N\otimes ...\otimes B_k^N\right]\mathbb{E}\left[C_1^N\otimes ...\otimes C_k^N\right]\right].$$

The Equation (16) of Theorem 7.1 in [**15**] gives:

$$\mathbb{E}m_p\left[B_1^NC_1^N,...,B_k^NC_k^N\right]\sum_{p_1\in A_k,p_1\leq p}\mathbb{E}\kappa_A^{p_1}\left((B_i)_{i=1}^k\right)\mathbb{E}m_{t_{p_1\circ p}}\left((C_i)_{i=1}^k\right).$$

7.1.9. Strong asymptotically G-invariance. — Let $(L_i)_{i\in I}\cup(M_j)_{j\in J}$ be a family of elements in $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in $\mathcal{P}$-expectation. Seeing that in Theorem 7.3 we only need the asymptotic G-invariance of one of the two families, one can wonder if it is possible to state a version of Theorem 7.10 where one could replace the condition of G-invariance imposed for $(L_i)_{i\in I}$ by the condition that $(L_i)_{i\in I}$ is asymptotically G-invariant.

One can not hope that such generalization is true, at least with the definition of asymptotically G-invariant that we gave. Indeed, let us suppose that $(A,G)=(\mathcal{P},\mathfrak{S})$: $(L_i)_{i\in I}$ is asymptotically $\mathfrak{S}$-invariant. Thus, if the generalization of Theorem 7.3 was true, this would imply that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically $\mathcal{P}$-free. Let us consider the special case where for any positive integer N, M_N is the diagonal matrix with $\lfloor\frac{N}{2}\rfloor$ zeros followed by $\lceil\frac{N}{2}\rceil$ ones and L_N is the diagonal matrix with $\lfloor\frac{N}{2}\rfloor$ ones follows by $\lceil\frac{N}{2}\rceil$ zeros. The sequences of random matrices $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$ converges in $\mathcal{P}$-expectation, have the same $\mathcal{P}$-law, and it is easy to see that:

$$m_{id_1}(M)=m_{id_1}(L)=\frac{1}{2}.$$

Let us suppose that $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$ are asymptotically $\mathcal{P}$-free. Let us denote by $1\!\!1_2$ the partition in $\mathcal{P}_2$ equal to $\{\{1,2,1',2'\}\}$: one can see that $id_1\otimes id_1\sqsupset 1\!\!1_2$. Thus, we should have the following equality:

$$m_{1\!\!1_2}(M,L)=m_{id_1}(M)m_{id_1}(L),$$

yet the left hand-side is equal to zero, and the right hand side is equal to $\frac{1}{4}$. Thus $(M_N)_{N\in\mathbb{N}}$ and $(L_N)_{N\in\mathbb{N}}$ are not $\mathcal{P}$-free.

Actually, one can state a generalization of Theorem 7.10 but one has to give a stronger definition of the asymptotic G-invariance.

Notation 7.3. — *Let N and k be two positive integers, let $\mathbb{I}=(i_1,...i_k)$ be a k-tuple of elements of I. Let $\mathbb{J}=(j_1,j_{1'},...,j_k,j_{k'})$ be a $2k$-tuple of $\{1,...,N\}^{2k}$. We denote by $L^N_{\mathbb{I},\mathbb{J}}$ the product:*

$$(L^N_{i_1})^{j_1}_{j_{1'}}...(L^N_{i_k})^{j_k}_{j_{k'}}.$$

Definition 7.3. — *The family $(L_i)_{i\in I}$ is asymptotically strongly G-invariant if it is asymptotically G-invariant and for any positive integer k, for any k-tuple $\mathbb{I}=(i_1,...i_k)$ of elements of I:*

$$\sup_{\mathbb{J},\mathbb{J}'\in\{1,...,N\}^{2k}|\mathsf{Ker}(\mathbb{J})=\mathsf{Ker}(\mathbb{J}')} N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id)}\mid\mathbb{E}\left[M^N_{\mathbb{I},\mathbb{J}}-M^N_{\mathbb{I},\mathbb{J}'}\right]\mid\underset{N\to\infty}{\longrightarrow}0.$$

Let us choose a possibility for (G,A) as explained in Notation 3.1. We recall that we supposed that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in $\mathcal{P}$-expectation. We can state a generalization of Theorem 7.10 which proof is similar to the proofs in [**26**].

Theorem 7.11. — *Let us suppose that $(L_i)_{i\in I}$ is asymptotically strongly G-invariant and that for every positive integer N, the two families $(L^N_i)_{i\in I}$ and $(M^N_j)_{j\in I}$ are independent. Then the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free.*

Proof. — Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ satisfy the hypotheses stated in the theorem. Let us suppose that we proved that the two families are asymptotically $\mathcal{P}$-free then, by definition, $(L_i)_{i\in I}$ is asymptotically G-invariant, and thus, by Theorem 7.3, we get that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free. It remains to prove that the two families are asymptotically $\mathcal{P}$-free. Let k be a positive integer, let l be an integer in $]0,...,k[$ and let $p\in\mathcal{P}_k$. Let $(B_i)_{i=1}^l$ be a l-tuple of elements of $(L_i)_{i\in I}$ and let $(B_i)_{i=l+1}^k$ be a $(k-l)$-tuple of elements of $(M_j)_{j\in J}$. It is enough to prove that:

$$\lim_{N\to\infty}\mathbb{E}m_{p^c}\left((B_i^N)_{i=1}^k\right)=\delta_{p_l^g\otimes p_l^d\sqsupset p}\mathbb{E}m_{(p_l^g)^c}\left[(B_i)_{i=1}^l\right]\mathbb{E}m_{(p_l^d)^c}\left[(B_i)_{i=l+1}^k\right].$$

Let N be a positive integer, we have:

$$\mathbb{E}m_{p^c}\left((B_i^N)_{i=1}^k\right)=\frac{1}{N^{\mathsf{nc}(p\vee id)}}\sum_{\mathbb{J}\in\{1,...,N\}^{2k}|\mathsf{Ker}(\mathbb{J})=p}\mathbb{E}\left[B^N_{(1,...,k),\mathbb{J}}\right].$$

Any $\mathbb{J}\in\{1,...,N\}^{2k}$ can be writen as the concatenation of $\mathbb{J}_1\in\{1,...,N\}^{2l}$ and $\mathbb{J}_2\in\{1,...,N\}^{2(k-l)}$, that we denote by $\mathbb{J}_1\mathbb{J}_2$.The right hand side can be written as:

$$\frac{1}{N^{\mathsf{nc}(p\vee id)}}\sum_{\mathbb{J}_1\in\{1,...,N\}^{2l},\mathbb{J}_2\in\{1,...,N\}^{2(k-l)}|\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p}\mathbb{E}\left[B^N_{(1,...,l),\mathbb{J}_1}\right]\mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right].$$

Let $\mathbb{J}_1$ be in $\{1,...,N\}^{2l}$. Since $(L_i)_{i\in I}$ is asymptotically G-invariant, using the same arguments as for Lemma 5.3, one gets:

$$\mathbb{E}\left[B^N_{(1,...,l),\mathbb{J}_1}\right]=N^{\mathsf{nc}(\mathsf{Ker}(\mathbb{J}_1)\vee id)-\mathsf{nc}(\mathsf{Ker}(\mathbb{J}_1))}\left[\mathbb{E}m_{\mathsf{Ker}(\mathbb{J}_1)^c}\left[B_1^N,...,B_l^N\right]+o(1)\right]$$

where the $o(1)$ is uniform in $\mathbb{J}_1$. Let us remark also that if $\mathbb{J}_1\in\{1,...,N\}^{2l}$ and $\mathbb{J}_2\in\{1,...,N\}^{2(k-l)}$ satisfy that $\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p$, then $\mathsf{Ker}(\mathbb{J}_1)=p_l^g$ and $\mathsf{Ker}(\mathbb{J}_2)=p_l^d$. This implies that $\mathbb{E}m_{p^c}\left((B_i^N)_{i=1}^k\right)$ is equal to:

$$\frac{1}{N^{\mathsf{nc}(p\vee id)}}\sum_{\mathbb{J}_1,\mathbb{J}_2|\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p}N^{\mathsf{nc}(p_l^g\vee id)-\mathsf{nc}(p_l^g)}\left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right]+o(1)\right]\mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right]$$

or to:

$$N^{-\mathsf{nc}(p\vee id)+\mathsf{nc}(p_l^g\vee id)-\mathsf{nc}(p_l^g)}$$
$$\sum_{\mathbb{J}_2|\mathsf{Ker}(\mathbb{J}_2)=p_l^d}\mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right]\sum_{\mathbb{J}_1|\mathsf{Ker}(\mathbb{J}_1)=p_l^g,\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p}\left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right]+o(1)\right].$$

When $\mathbb{J}_2\in\{1,...,N\}^{2(k-l)}$ is fixed then there exists exactly $N^{\mathsf{nc}(p)-\mathsf{nc}(p_l^d)}$ elements $\mathbb{J}_1$ in $\{1,...,N\}^{2l}$ such that $\mathsf{Ker}(\mathbb{J}_1)=p_l^g$ and $\mathsf{Ker}(\mathbb{J}_1\mathbb{J}_2)=p$, thus we can go on our calculations, and $\mathbb{E}m_{p^c}\left((B_i^N)_{i=1}^k\right)$ is equal to:

$$N^{-\mathsf{nc}(p\vee id)+\mathsf{nc}(p_l^g\vee id)-\mathsf{nc}(p_l^g)+\mathsf{nc}(p)-\mathsf{nc}(p_l^d)}$$
$$\left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right]+o(1)\right]\sum_{\mathbb{J}_2|\mathsf{Ker}(\mathbb{J}_2)=p_l^g}\mathbb{E}\left[B^N_{(l+1,...,k),\mathbb{J}_2}\right]$$

or:

$$N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id))-\left(\mathsf{nc}(p_l^g\otimes p_l^d)-\mathsf{nc}((p_l^g\otimes p_l^d)\vee id)\right)}$$
$$\left[\mathbb{E}m_{(p_l^g)^c}\left[B_1^N,...,B_l^N\right]+o(1)\right]\mathbb{E}m_{(p_l^d)^c}\left[B_{l+1}^N,...,B_k^N\right].$$

Since $p_l^g\otimes p_l^d$ is finer than p, using Lemma 3.4 of [**15**], one gets that:

$$N^{\mathsf{nc}(p)-\mathsf{nc}(p\vee id))-\left(\mathsf{nc}(p_l^g\otimes p_l^d)-\mathsf{nc}((p_l^g\otimes p_l^d)\vee id)\right)}\underset{N\to\infty}{\longrightarrow}\delta_{p_l^g\otimes p_l^d\sqsupset p},$$

this allows to conclude the proof of the theorem. □

Remark 7.4. — *As a corollary of the Theorem 7.11, we see that if one can prove that $(M_i)_{i\in I}$ and $(L_i)_{i\in I}$ converges in $\mathcal{P}$-moments, that the asymptotic $\mathfrak{S}$-factorization holds for both family and that one of the family is asymptotically strongly U or O-invariant, then $(M_i)_{i\in I}$ and $(L_i)_{i\in I}$ are asymptotically free in the meaning of Voiculescu.*

In particular, this is general enough to be used for the general Wigner real and compex matrices, where the matrices are symmetric or Hermitian, with the variance's entries equal to $1/N$ and with a condition of bounded moments on the entries of the matrix, but where one does not ask for an identically distribution property.

7.1.10. Matricial A-free Levy processes. — Let $(M_t)_{t\geq 0}$ be a family of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. We suppose that for any positive integer N, $M_0^N=Id$. Let us suppose that the family $(M_t)_{t\geq 0}$ converges in A-expectation.

Definition 7.4. — *The family $(M_t)_{t\geq 0}$ is a matricial A-free additive Levy process if both following assertions hold:*

- **A-freeness of the additive increments :** *for any $0\leq t<s$, the two families $\left(M_s^N-M_t^N\right)_{N\in\mathbb{N}}$ and $\left((M_u^N)_{u\leq t}\right)_{N\in\mathbb{N}}$ are asymptotically A-free,*
- **Translation invariance :** *for any $0\leq t<s$, $\left(M_s^N-M_t^N\right)_{N\in\mathbb{N}}$ has the same A-law as $\left(M_{s-t}^N\right)_{N\in\mathbb{N}}$.*

We also give a multiplicative version.

Definition 7.5. — *The family $(M_t)_{t\geq 0}$ is a matricial A-free multiplicative Levy process if both following assertions hold:*

- **A-freeness of the multiplicative increments :** *for any $0\leq t<s$, the two families $\left(M_s^N(M_t^N)^{-1}\right)_{N\in\mathbb{N}}$ and $\left((M_u^N)_{u\leq t}\right)_{N\in\mathbb{N}}$ are asymptotically A-free,*
- **Translation invariance :** *for any $0\leq t<s$, $\left(M_s^N(M_t^N)^{-1}\right)_{N\in\mathbb{N}}$ has the same A-law as $\left(M_{t-s}^N\right)_{N\in\mathbb{N}}$*

7.2. Higher order. — We can define a notion of freeness up to higher order of fluctuations for some families of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let m be a non-negative integer. Let $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ be two families of elements of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that the two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ converge in A-expectation up to order m of fluctuations.

Definition 7.6. — *The families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations if and only if for any positive integer k, for any partition $p\in\mathcal{P}_k$, for any mixed k-tuple $(B_i)_{i=1}^k$ of elements of $\mathcal{A}\left((L_i)_{i\in I}\right)\cup\mathcal{A}\left((M_j)_{j\in J}\right)$, the sequence*

$\left(B_1^N, ..., B_k^N\right)_N$ *converges in expectation in A_k-cumulants up to order m of fluctuations as N goes to infinity and for any $i_0 \in \{0, ..., m\}$, the two conditions hold:*

Compatibility condition : *If p is not compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty} \mathbb{E}\kappa^p_{i_0,A}\left(B_1^N, ..., B_k^N\right) = 0.$$

Compatible factorization property : *If p is compatible with $(B_i)_{i=1}^k$:*

$$\lim_{N\to\infty} \mathbb{E}\kappa^p_{i_0,A}\Big(B_1^N, ..., B_k^N\Big) = \sum_{i_1=0}^{i_0} \mathbb{E}\kappa^{p_{|L_{(B_j)_{j=1}^k}}}_{i_1,A}\left((B_i)_{i\in L_{(B_j)_{j=1}^k}}\right) \mathbb{E}\kappa^{p_{|M_{(B_j)_{j=1}^k}}}_{i_0-i_1,A}\left((B_i)_{i\in M_{(B_j)_{j=1}^k}}\right).$$

Two families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free if and only if they are asymptotically A-free up to order 0 of fluctuations. All the theorems we proved in the zero order case can be easily generalized for the notion of A-freeness up to order m of fluctuations.

Theorem 7.12. — *Let us suppose that the families $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations. The family $(L_i)_{i\in I} \cup (M_j)_{j\in J}$ converges in A-expectation up to order m of fluctuations. Besides, the A-law of $(L_i)_{i\in I} \cup (M_j)_{j\in J}$ up to order m of fluctuations only depends on the A-law of $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ up to order m of fluctuations.*

The dependence is again polynomial. As for the zero order of fluctuations, we can state a theorem for higher order of fluctuations which is similar to Theorem 7.8.

Theorem 7.13. — *Let us suppose that $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ are asymptotically A-free up to order m of fluctuations. Let k be a positive integer. Let us consider $\left((B_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ and $\left((C_i^N)_{N\in\mathbb{N}}\right)_{i=1}^k$ such that for any $i \in \{1, ..., k\}$, $\left(B_i^N\right)_{N\in\mathbb{N}} \in \mathcal{A}\big((L_i)_{i\in I}\big)$, $\left(C_i^N\right)_{N\in\mathbb{N}} \in$*

$\mathcal{A}\big((M_j)_{j\in J}\big)$. *Then, for any* $p \in A_k$ *and any* $i_0 \in \{0, ..., m\}$, *one has:*

$$\mathbb{E}\kappa^p_{i_0,A}[B_1 + C_1, ..., B_k + C_k] = \sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)} \sum_{i=0}^{i_0} \mathbb{E}\kappa^{p_1}_{i,A}(B_1, ..., B_k)\mathbb{E}\kappa^{p_2}_{i_0-i,A}(C_1, ..., C_k),$$

$$\mathbb{E}\kappa^p_{i_0,A}[B_1C_1, ..., B_kC_k] = \sum_{p',p''\in A_k,\eta(p',p'')\leq i_0,p'\circ p''=p} \sum_{i=0}^{i_0-\eta(p',p'')} \mathbb{E}\kappa^{p'}_{i,A}(B_1, ..., B_k)\mathbb{E}\kappa^{p''}_{i_0-\eta(p',p'')-i,A}(C_1, ..., C_k),$$

$$\mathbb{E}m^{i_0}_p[B_1C_1, ..., B_kC_k] = \sum_{p'\in A_k} \sum_{i,j,i+j+\mathsf{df}(p',p)=i_0} \mathbb{E}\kappa^{p'}_{i,A}(B_1, ..., B_k)\mathbb{E}m^j_{\iota_{p'\circ p}}(C_1, ..., C_k).$$

Proof. — One could do a combinatorial proof as we did for Theorem 7.8, using results such that Proposition 8.2 in [**15**]. In order to give a shorter proof, one can use the same arguments as explained after Remark 7.3. Using Theorem 7.12 and the up-coming Theorem 7.15, one can suppose that the families we consider satisfy the hypotheses of Theorem 7.15. In this case the theorem is only a consequence of a simple calculation for the first equality and a consequence of Theorem 9.2 of [**15**] for the others equalities. □

Recall the definition of the operations $\boxplus$ and $\boxtimes$ defined in Section 10.1 of [**15**]. As a consequence of this theorem, we have the following theorem.

Theorem 7.14. — *Let us suppose that* $(L_i)_{i\in I}$ *and* $(M_j)_{j\in J}$ *are asymptotically* A*-free up to order* m *of fluctuations. Let us consider two sequences* $(B_N)_{N\in\mathbb{N}}$ *and* $(C_N)_{N\in\mathbb{N}}$ *such that* $(B_N)_{N\in\mathbb{N}} \in \mathcal{A}\big((L_i)_{i\in I}\big)$, $(C_N)_{N\in\mathbb{N}} \in \mathcal{A}\big((M_j)_{j\in J}\big)$.
The $\mathcal{R}^{(m)}_A$*-transform of* $(B_N + C_N)_{N\in\mathbb{N}}$ *and* $(B_NC_N)_{N\in\mathbb{N}}$ *are given by:*

$$\mathcal{R}^{(m)}_A\left[(B_N + C_N)_{N\in\mathbb{N}}\right] = \mathcal{R}^{(m)}_A\left[(B_N)_{N\in\mathbb{N}}\right] \boxplus \mathcal{R}^{(m)}_A\left[(C_N)_{N\in\mathbb{N}}\right],$$
$$\mathcal{R}^{(m)}_A\left[(B_NC_N)_{N\in\mathbb{N}}\right] = \mathcal{R}^{(m)}_A\left[(B_N)_{N\in\mathbb{N}}\right] \boxtimes \mathcal{R}^{(m)}_A\left[(C_N)_{N\in\mathbb{N}}\right].$$

Besides, the following theorem allows us to construct examples of families of sequences of matrices which are asymptotically A-free up to order m of fluctuations.

Theorem 7.15. — *Let us choose a possibility for* (G, A) *as explained in Notation 3.1. Let us suppose that the two families* $(L_i)_{i\in I}$ *and* $(M_j)_{j\in J}$ *converge in* A*-expectation up to order* m *of fluctuations.*

Let us suppose that $(L_i)_{i\in I}$ *is* G*-invariant. Besides let us suppose that for every integer* N, *the two families* $(L^N_i)_{i\in I}$ *and* $(M^N_j)_{j\in I}$ *are independent. Then the two families* $(L_i)_{i\in I}$ *and* $(M_j)_{j\in J}$ *are asymptotically* A*-free up to order* m *of fluctuations.*

Actually, the compatibility condition and the compatible factorization property hold again for any integer N big enough so that the N-dimensional A-cumulants are defined. The Theorem 7.15 is general enough so that one can apply it to families of random matrices of the form $(L_i,(L_i)^*)_{i\in I}$ and $(M_j,(M_j)^*)_{j\in J}$ as long as one can prove that $(L_i,(L_i)^*)_{i\in I}$ and $(M_j,(M_j)^*)_{j\in J}$ converge in expectation in A-expectation.

Moreover, we can also define the notion of matricial A-free Levy process up to order m of fluctuations by changing, in Definitions 7.4 and 7.5, the notion of asymptotic A-freeness, by the notion of asymptotic A-freeness up to order m of fluctuations, and using the A-law up to order m of fluctuations in order to define the translation invariance property.

8. The A-non-commutative central limit theorem

In this section, we prove a A-non-commutative central limit theorem. Recall the definition of $e^E_{\boxplus}$ defined in Definition 10.7 of [**15**]. Recall also the notation E_k, for $E\in\mathfrak{E}[A]$ and $k\in\mathbb{N}$, set in Section 10 of the same article.

Theorem 8.1. — *Let $(L_N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$ which converges in A-expectation and such that $(\mathcal{R}_A[L])_1=0$. For any integer $i\in\mathbb{N}$, let $(L^i_N)_{N\in\mathbb{N}}\in L^{\infty^-}\otimes\mathcal{M}(\mathbb{C})$. Let us suppose that:*

- *for any $i\in\mathbb{N}$, $(L^i_N)_{N\in\mathbb{N}}$ has the same A-law as $(L_N)_{N\in\mathbb{N}}$,*
- *for any $i\in\mathbb{N}$, $(L^i_N)_{N\in\mathbb{N}}$ is asymptotically A-free from $\left((L^j_N)_{N\in\mathbb{N}}\right)_{j\neq i}$.*

Then, as n goes to infinity, the $\mathcal{R}_A$-transform of $\frac{1}{\sqrt{n}}\sum_{i=1}^n L^i$ converges to:

$$e_{\boxplus}^{(\mathcal{R}_A[L])_2}. \tag{14}$$

Proof. — Let us consider $L^i=(L^i_N)_{N\in\mathbb{N}}$ as in the theorem. By multi-linearity, for any integer $n,k\in\mathbb{N}$, and any $p\in A_k$:

$$\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n L^i\right]\right)_p=\frac{1}{n^{\frac{2}{k}}}\left(\mathcal{R}_A\left[\sum_{i=1}^n L^i\right]\right)_p.$$

Recall the notion of cycle defined in Definition 2.9 of [**15**] and the notation $\mathsf{C}(p)$ for the set of cycles of p. Using the asymptotic A-freeness property of the sequences $(L^i_N)_{N\in\mathbb{N}}$ and using the fact that for any positive integer i, $\left(L^i_N\right)$ has the same A-law as $(L_N)_{N\in\mathbb{N}}$, one has:

$$\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n L^i\right]\right)_p=\frac{1}{n^{\frac{k}{2}}}\sum_{f:\mathsf{C}(p)\to\{1,\dots,n\}}\prod_{i=1}^n(\mathcal{R}_A[L])_{p_{\left(\bigcup_{c\in f^{-1}(i)}c\right)}}$$

where we recall that for any subset J, p_J is the extraction of p to J. We recall the convention $(\mathcal{R}_A[L])_\emptyset=1$. Let us consider a function $f:\mathsf{C}(p)\to\{1,\dots,n\}$. This defines a partition of the cycles of p by considering $\{f^{-1}(i),i\in\{1,\dots,n\}\}$. We say that $f\sim f'$

if the underlying partitions of $\mathsf{C}(p)$ are equal. It is obvious that, if $f \sim f'$, then:

$$\prod_{i=1}^{n} (\mathcal{R}_A[L])_{p_{\left(\bigcup_{c\in f^{-1}(i)} c\right)}} = \prod_{i=1}^{n} (\mathcal{R}_A[L])_{p_{\left(\bigcup_{c\in f'^{-1}(i)} c\right)}}.$$

Let π be a partition of $\mathsf{C}(p)$, we denote by $\mathsf{nc}(\pi)$ the number of blocks of π. By a counting argument, there exist approximatively $n^{\mathsf{nc}(\pi)}$ functions $f : C(p) \to \{1, ..., n\}$ such that the underlying partition of f is π. Thus we can write:

$$\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i\right]\right)_p \simeq \frac{1}{n^{\frac{k}{2}}} \sum_{\pi \text{ partition of } \mathsf{C}(p)} n^{\mathsf{nc}(\pi)} \prod_{t \text{ block of } \pi} (\mathcal{R}_A[L])_{p_{\bigcup_{c\in t} c}}. \tag{15}$$

The condition $(\mathcal{R}[A])_1 = 0$ allows us to show that for any partition π of $\mathsf{C}(p)$, if $\mathsf{nc}(\pi) > \frac{k}{2}$ then $\prod_{t \text{ block of } \pi} (\mathcal{R}_A[L])_{p_{\bigcup_{c\in t} c}} = 0$. Indeed let us suppose that $\mathsf{nc}(\pi) > \frac{k}{2}$, then one block of π must be a cycle of size 1 of p. Thus in the considered product, one would have an element of the form $\mathcal{R}_A[L]_{p'}$ with $p' \in \mathcal{P}_1$: the product is then equal to zero. Thus, in the r.h.s., in the sum, one can impose the condition $\mathsf{nc}(\pi) \leq \frac{k}{2}$. This shows that $\left(\mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i\right]\right)_p$ converges when n goes to infinity.

Let us understand what the limit is: if $\mathsf{nc}(\pi) < \frac{k}{2}$, then the term associated with π disappears when n goes to infinity. Only remain the partitions π of $\mathsf{C}(p)$ such that $\mathsf{nc}(\pi) = \frac{k}{2}$ and such that none of the block of π is a cycle of length 1 of p. Thus, in order that the limit is not equal to zero, p must consist in some cycles of size 2 and an even number of cycles of length 1 ; the blocks of π are either two cycles of size 1 of p or a cycle of size 2 of p. This implies, by sending n to infinity in (15), that:

$$\lim_{n\to\infty} \mathcal{R}_A\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i\right] = e_{\boxplus}^{(\mathcal{R}_A[L])_2},$$

which is the equality we wanted to prove. □

Using the intuition we will develop in Section 10.3, one can show easily the following theorem which proof will be given in a later version of the article.

Theorem 8.2. — *Let $E \in (\mathbb{C}[A_2])^{\mathfrak{S}}$ be an element of $\mathbb{C}[A_2]$ which is invariant by conjugation by $\mathfrak{S}_2$. This means that for any permutation $\sigma \in \mathfrak{S}_2$, for any positive integer N, $\sigma E \sigma^{-1} = E$ in $\mathbb{C}[A_k(N)]$.*

There exists a sequence of random matrices $(M_N)_{N\in\mathbb{N}}$, such that for any positive integer N, $M_N \in \mathcal{M}_N(\mathbb{C})$ and such that:

$$\mathcal{R}_A[M] = e_{\boxplus}^{E}.$$

Using Theorem 8.2, the Theorem 8.1 asserts now that the A-law of $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L^i$ converges to the A-law of a sequence of random matrices which $\mathcal{R}_A$-transform is given by (14). The Theorem 8.1 is a generalization of the usual non-commutative central limit theorem that one recovers when $A = \mathfrak{S}$ and when the $\mathfrak{S}$-factorization is assumed: this is due to the link between the usual $\mathcal{R}_u$-transform and the $\mathcal{R}_{\mathfrak{S}}$-transform explained in Theorem 10.3 of [**15**]. We will see that it also gives the usual central limit theorem given

in Theorem 9.1. Yet, we warn the reader that Theorem 8.1 is not just an interpolation between the usual and the non-commutative central limit theorem since the space of possible limits is, in some sense, of dimension $\mathsf{dim}\left(\mathbb{C}^{\mathfrak{S}}[A_2]\right)$ equal to eleven if $A = \mathcal{P}$.

One could easily extend Theorem 8.1 to families of elements of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$.

9. Classical probabilities

The goal of this section is to show that the classical probabilities are in some sense included in the theory we developed: for example, one can recover in a new way the usual cumulants as a particular case of $\mathcal{P}$-cumulants. This is partly based on easy and common knowledge, shared by researchers, that we found interesting to gather here.

9.1. Approximation of a measure by random matrices. — We consider the set $P^\infty(\mathbb{C})$ of probability measures such that for any integer $k \in \mathbb{N}$,

$$\int_{\mathbb{C}} |z|^k \, d\mu(z) < \infty.$$

Let μ be in $P^\infty(\mathbb{C})$. Let us see how to construct a sequence of random matrices $(M_N)_{N\in\mathbb{N}}$ in $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ such that for any positive integer n,

$$\mathbb{E}m_{(1,\dots,n)}\left[M_N^{\otimes n}\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{C}} z^n d\mu(z).$$

If $(M_N)_{N\in\mathbb{N}}$ is such a sequence of random matrices, we say that $(M_N)_{N\in\mathbb{N}}$ is a weak approximation of μ. If, besides, the factorization property holds for $(M_N)_{N\in\mathbb{N}}$, we say that $(M_N)_{N\in\mathbb{N}}$ is a strong approximation of μ. In this last case, for any non-negative integer n and any $\sigma \in \mathfrak{S}_n$,

$$\mathbb{E}m_\sigma\left[M_N^{\otimes n}\right] \underset{N\to\infty}{\longrightarrow} \prod_{c\in\mathsf{C}(\sigma)} \int_{\mathbb{C}} z^{\#c} d\mu(z),$$

where, here, $\mathsf{C}(\sigma)$ is the set of cycles of σ as defined in the usual way. Let us give two simple approximations of μ.

Lemma 9.1. — *Let X be a random variable which law is μ. The sequence $(XId_N)_{N\in\mathbb{N}}$ is a weak approximation of μ. It will be called the natural weak approximation of μ.*

Proof. — Indeed, for any non negative integer n, $\mathbb{E}m_{(1,\dots,n)}[(XId_N)^{\otimes n}] = \mathbb{E}[X^n] = \int_{\mathbb{C}} z^n d\mu(z)$. Thus, the equality holds for any integer N and not only at the limit $N = \infty$. □

The last example does not provide a strong approximation of μ: for any non-negative integer n, for any $\sigma \in \mathfrak{S}_n$:

$$\mathbb{E}m_\sigma\left[(XId_N)^{\otimes n}\right] = \mathbb{E}[X^n] = \int_{\mathbb{C}} z^k d\mu(z) \neq \prod_{c\in\mathsf{C}(\sigma)} \int_{\mathbb{C}} z^{\#c} d\mu(z).$$

Thus $\mathcal{R}_{\mathfrak{S}}(XId)$ is not multiplicative. Yet, this is the simplest way to inject the classical probabilities in the theory we developed. For example, if μ and ν are two probability

measures in $P^\infty(\mathbb{C})$, let us consider $(XId_N)_{N\in\mathbb{N}}$ and $(YId_N)_{N\in\mathbb{N}}$ two independent natural weak approximations associated respectively with μ and ν. Then $(XId_N)_{N\in\mathbb{N}}$ and $(YId_N)_{N\in\mathbb{N}}$ are asymptotically $\mathfrak{S}$-free, besides the sum $((X+Y)Id_N)_{N\in\mathbb{N}}$ is a natural weak approximation of the classical convolution $\mu*\nu$: this implies the usual central limit theorem for random variables with moments of any order.

Theorem 9.1. — *Let $(X_i)_{i=1}^\infty$ be a sequence of i.i.d. real random variables such that, for any integer k, $\mathbb{E}\left[|X_1|^k\right]<\infty$, and such that $\mathbb{E}[X_1]=0$. The moments of $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$ converge to the moments of a Gaussian of law $\mathcal{N}(0,\mathbb{E}\left[X_1^2\right])$.*

Proof. — Let $(X_i)_{i=1}^\infty$ be a sequence of i.i.d. random variables whose law admits moments up to any order and such that $\mathbb{E}[X_1]=0$. Let us consider for any integer i, the sequences of matrices $(X_iId_N)_{N=0}^\infty$. Using Theorem 7.10, these sequences of matrices are asymptotically $\mathfrak{S}$-free, as they are independent and each of them is invariant by conjugation by the unitary group. Applying the central limit theorem proved in Theorem 8.1, one has that the $\mathcal{R}$-transform of:

$$\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)Id_N\right)_{N\in\mathbb{N}}$$

converges when n goes to infinity. Yet, using the N-dimensional cumulants, we see that its $\mathcal{R}$-transform is equal to:

$$\left(\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)^k\right]id_k\right)_{k\in\mathbb{N}}.$$

Thus for any positive integer k, when n goes to infinity, $\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)^k\right]$ converges to:

$$\left(e_{\boxplus}^{\mathbb{E}\left[X_1^2\right]id_2}\right)_{id_k}=\sum_{\text{pairings of }\{1,\dots,k\}}\mathbb{E}[X_1^2]=\mathbb{E}[X_1^2].\#\{\text{pairings of }\{1,\dots,k\}\}.$$

Hence, for any positive integer k, $\mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)^k\right]$ converges to the moments of a random variable of law $\mathcal{N}(0,\mathbb{E}\left[X_1^2\right])$. □

It would be interesting to have also a strong approximation of μ. For this, we need the following notation.

Notation 9.1. — *Let N be a positive integer, let $(\alpha_i)_{i=1}^N$ be N complex numbers. We denote by $\mathsf{Diag}\left[(\alpha_i)_{i=1}^N\right]$ the matrix $\sum_i\alpha_iE_i^i$, where for any $i\in\{1,\dots,N\}$, E_i^i is the elementary matrix whose elements are equal to zero except in position (i,i) where it is equal to one.*

Proposition 9.1. — *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables which are independent with same law μ. The sequence $\left(\mathsf{Diag}\left[(X_i)_{i=1}^N\right]\right)_{N\in\mathbb{N}}$ is a strong approximation of μ. It will be called the natural strong approximation of μ.*

Proof. — Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables which are independent with same law μ. Let us denote by $(M_N)_{N\in\mathbb{N}}$ the sequence $\left(\mathsf{Diag}\left[(X_i)_{i=1}^N\right]\right)_{N\in\mathbb{N}}$. For any positive integer k, for any permutation $\sigma\in\mathfrak{S}_k$:

$$m^\sigma\left[M_N\otimes ...\otimes M_N\right]=\prod_{c\in\mathsf{C}(\sigma)}\frac{1}{N}Tr\left(M_N^{\#c}\right)=\prod_{c\in\mathsf{C}(\sigma)}\left[\frac{1}{N}\sum_{i=1}^N X_i^{\#c}\right].$$

By the law of large numbers $\mathbb{E}m_\sigma\left[M_N^{\otimes n}\right]\underset{N\to\infty}{\longrightarrow}\prod_{c\in\mathsf{C}(\sigma)}\int_{\mathbb{C}}z^{\#c}d\mu(z)$. This concludes the proof. □

9.2. A Schur-Weyl interpretation of classical cumulants. —

9.2.1. A simple duality. — Let N and k be positive integers. Let $\mathcal{D}_N(\mathbb{C})$ be the set of matrices $M\in\mathcal{M}_N(\mathbb{C})$ which are diagonal. We will study elements of $(\mathcal{D}_N(\mathbb{C}))^{\otimes k}$ which commute with the action of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$.

Definition 9.1. — *Let J be a finite subset, we denote by P_J the set of partitions of J. For any positive integer k, we simply denote by P_k the set of partitions of $\{1,...,k\}$.*

Definition 9.2. — *Let $\pi=\{\pi^1,...,\pi^r\}$ be in P_k. Let $\pi'^i=\{j'\in\{1',...,k'\},j\in\pi_i\}$. We define f_π in $\mathcal{P}_k$ equal to $\{\{\pi_i\cup\pi'_i\},i\in\{1,...,n\}\}$. We set:*

$$\mathcal{D}_k=\{f_\pi,\pi\in\mathsf{P}_k\}.$$

One example is given in Figure 1.

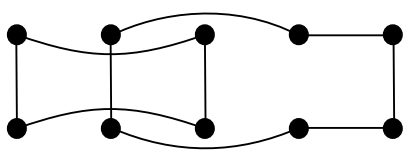

FIGURE 1. Partition f_π with $\pi=\{\{1,3\},\{2,4,5\}\}$.

One of the main feature of this new set is that for any f in $\mathcal{D}_k$, any f' which is coarser than f is in $\mathcal{D}_k$; besides f_π is coarser than $f_{\pi'}$ if and only if π is coarser than π'. Let us recall the notion of exclusive basis defined in Definition 2.16 of [**15**]. From the discussion we just had, we obtain the following lemma.

Lemma 9.2. — *For any partition π of $\{1,...,k\}$,*

$$f_\pi=\sum_{\pi'\ \text{coarser than}\ \pi}(f_{\pi'})^c.$$

A consequence is that for any $\pi\in\mathsf{P}_k$, $(f_{\pi'})^c\in\mathbb{C}[\mathcal{D}_k]$.

In the following, we denote by $f_{\pi'}^c$ the element $(f_{\pi'})^c$. Using this lemma, it is easy to prove the following result.

Lemma 9.3. — *Let E be an element of $\mathcal{D}_N(\mathbb{C})^{\otimes k}$ seen as an endomorphism of $\left(\mathbb{C}^N\right)^{\otimes k}$. Let us suppose that E commutes with the action $\rho^k_{\mathfrak{S}(N)}$ of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$. Then:*

$$E \in \rho_N^{\mathcal{P}_k}\left[\mathbb{C}\left[\mathcal{D}_k\right]\right].$$

Besides, if $N \geq k$, the restriction of $\rho_N^{\mathcal{P}_k}$ to $\mathbb{C}\left[\mathcal{D}_k\right]$ is a bijection.

Proof. — Using the Lemma 9.2, it is enough to show that E can be written as a linear combination of elements of the form $\rho_N^{\mathcal{P}_k}(f_\pi^c)$ with π being any partition of $\{1,...,k\}$. Any element of $\mathcal{D}_N(\mathbb{C})^{\otimes k}$ is a linear combination of elements of the form $E_{i_1}^{i_1} \otimes ... \otimes E_{i_k}^{i_k}$. Let $E \in \mathcal{D}_N(\mathbb{C})^{\otimes k}$:

$$E = \sum_{(i_1,...,i_k)\in\{1,...,N\}^k} c_{i_1,...,i_k} E_{i_1}^{i_1} \otimes ... \otimes E_{i_k}^{i_k}.$$

Any choice $(i_1,...,i_k) \in \{1,...,N\}^k$ defines a partition of $\{1,...,k\}$ denoted also by $\mathsf{Ker}((i_1,...,i_k))$ which is the unique partition of $\{1,...,k\}$ such that two elements u and v of $\{1,...,k\}$ are in the same block if and only if $i_u = i_v$.

If E commutes with the action $\rho^k_{\mathfrak{S}(N)}$ of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$, then $c_{i_1,...,i_k}$ only depends on the partition of k which is induced by $(i_1,...,i_k)$. Thus:

$$\begin{aligned} E &= \sum_{\pi\in\mathsf{P}_k} c_\pi \sum_{(i_1,...,i_k)|\mathsf{Ker}((i_1,...,i_k))=\pi} E_{i_1}^{i_1} \otimes ... \otimes E_{i_k}^{i_k} \\ &= \sum_{\pi\in\mathsf{P}_k} c_\pi \rho_N^{\mathcal{P}_k}\left[f_\pi^c\right]. \end{aligned}$$

Let us suppose that $N \geq k$, let $(c_\pi)_{\pi\in\mathsf{P}_k} \in \mathbb{C}^{\mathsf{P}_k}$ such that $\sum_{\pi\in\mathsf{P}_k} c_\pi \rho_N^{\mathcal{P}_k}\left[f_\pi^c\right] = 0$. Let $\pi_0 \in \mathsf{P}_k$, and let us consider $(i_1,...,i_k) \in \{1,...,N\}^k$ such that $\mathsf{Ker}((i_1,...,i_k)) = \pi_0$. Such k-tuple exists because $N \geq k$. Then:

$$0 = \left(\sum_{\pi\in\mathsf{P}_k} c_\pi \rho_N^{\mathcal{P}_k}\left[f_\pi^c\right]\right)(e_{i_1} \otimes ... \otimes e_{i_k}) = c_{\pi_0}(e_{i_1} \otimes ... \otimes e_{i_k}),$$

and thus $c_\pi = 0$ for any $\pi \in \mathsf{P}_k$. □

On the set $\mathcal{D}_k$, the geodesic order becomes much simpler than on $\mathcal{P}_k$.

Proposition 9.2. — *Let π and π' be in P_k. The following assertions are equivalent:*

- *$f_{\pi'} \in [id, f_\pi]_{\mathcal{P}_k}$,*
- *f_π is coarser than $f_{\pi'}$,*
- *π is coarser than π'.*

As a consequence, the ordered set $(\mathcal{D}_k, \leq)$ is isomorphic to the set P_k endowed with the order that we will also denote by $\leq$ which is defined by the fact that for any π and π' in P_k, $\pi' \leq \pi$ if and only if π' is finer than π

Proof. — Let π and π' be in P_k. We have already noticed that f_π is coarser than $f_{\pi'}$ if and only if π is coarser than π'. Besides it is easy to see that $\mathsf{Gl}_c(f_\pi) = f_\pi$ and $f_{\pi'}$

is always in $\mathsf{Sp}(f_\pi \vee f_{\pi'})$. Using Theorem 3.2 of [**15**], we see that $f_{\pi'} \in [id, f_\pi]_{\mathcal{P}_k}$ if and only if $f_\pi \vee f_{\pi'} = f_\pi$, thus if and only if f_π is coarser than $f_{\pi'}$. □

We will give a kind of Schur-Weyl duality interpretation of classical cumulants. For any $\pi \in \mathsf{P}_k$, let us denote by $\mathsf{C}(\pi)$ the blocks of π. Let us recall the classical notion of cumulants.

Definition 9.3. — *Let $m = (m_I)_{I\subset\{1,...,k\}}$ be a family of complex numbers. The sequence of cumulants of $(\kappa_I)_{I\subset\{1,...,k\}}$ associated with m is the only sequence of complex numbers such that for any $J \subset \{1, ..., k\}$:*

$$m_J = \sum_{\pi\in\mathsf{P}_J} \prod_{c\in\mathsf{C}(\pi)} \kappa_c.$$

In classical probabilities, for any $I \subset \{1, ..., k\}$, we consider m_I equal to $\mathbb{E}\left[\prod_{i\in I} X_i\right]$ where $(X_i)_1^k$ is a family of random variables which has mixed moments of any order. In this case, we will denote by $\kappa_J\left((X_i)_{i=1}^k\right)$ the cumulant κ_J associated with $\left(\mathbb{E}\left[\prod_{i\in I} X_i\right]\right)_{I\subset\{1,...,k\}}$. If $J = \{1, ..., k\}$, we also denote the cumulant $\kappa_J\left((X_i)_{i=1}^k\right)$ by $\kappa_k\left((X_i)_{i=1}^k\right)$. In our framework, we prefer to work with partitions and not with sets: this leads us to the following lemma.

Lemma 9.4. — *Let $m = (m_I)_{I\subset\{1,...,k\}}$ be a family of complex numbers. For any $\pi \in \mathsf{P}_k$ we define $m_\pi = \prod_{c\in\mathsf{C}(\pi)} m_c$. Let $(\eta_\pi)_{\pi\in\mathsf{P}_k}$ be a family of complex numbers such that $m_\pi = \sum_{\pi'\leq\pi} \eta_{\pi'}$. Then for any $\pi \in \mathsf{P}_k$,*

$$\eta_\pi = \prod_{c\in\mathsf{C}(\pi)} \kappa_c,$$

where $(\kappa_I)_{I\subset\{1,...,k\}}$ is the family of cumulants associated with m.

Proof. — The proof relies on the fact that there exists a unique family $(\eta_\pi)_{\pi\in\mathsf{P}_k}$ such that for any $\pi \in \mathsf{P}_k$, $m_\pi = \sum_{\pi'\leq\pi} \eta_{\pi'}$. If $(\kappa_I)_{I\subset\{1,...,k\}}$ is the family of cumulants associated with m, then $\left(\prod_{c\in\mathsf{C}(\pi)} \kappa_c\right)_{\pi\in\mathsf{P}_k}$ satisfies these equalities. □

The Schur-Weyl duality interpretation of classical cumulants is given by the following theorem.

Theorem 9.2. — *Let $(X_1, ..., X_k)$ be a k-tuple of random variables. Let $l \geq k$, and let us consider $(X_1^n, ..., X_k^n)_{n\in\{1,...,l\}}$ a family of l i.i.d. k-tuple of random variables which have the same law as $(X_1, ..., X_k)$. Let us consider, for any $i \in \{1, ..., k\}$:*

$$M_i = \mathsf{Diag}\left[(X_i^n)_{n=1}^l\right].$$

We have the following formula:

$$\kappa_k(X_1, ..., X_k) = \mathbb{E}\kappa^{f_{\{\{1,...,k\}\}}}\left[M_1, ..., M_k\right].$$

Proof. — The endomorphism $\mathbb{E}\left[\bigotimes_{i=1}^{k} M_i\right]$ is in $(\mathcal{D}_N)^{\otimes k}$ and commutes with the action $\rho^k_{\mathfrak{S}(N)}$ of $\mathfrak{S}(N)$ on $\left(\mathbb{C}^N\right)^{\otimes k}$: by Lemma 9.3, it belongs to $\rho_N^{\mathcal{P}_k}\left[\mathbb{C}\left[\mathcal{D}_k\right]\right]$ and, since $l \geq k$, the numbers $\mathbb{E}\kappa^{f_{\{\{1,\dots,k\}\}}}\left[M_1,\dots,M_k\right]$ are well defined.

Following the calculation in the proof of Lemma 9.3, and using the independence of the $(X_1^n,\dots,X_k^n)_{n\in\{1,\dots,k\}}$, one gets:

$$\mathbb{E}\left[\bigotimes_{i=1}^{k} M_i\right] = \sum_{\pi\in\mathsf{P}_k}\left(\prod_{c\in\mathsf{C}(\pi)}\mathbb{E}\left[\prod_{i\in c} X_i\right]\right) f_\pi^c.$$

Using the definition of finite-dimensional cumulants, Lemma 9.3, Lemma 9.2 and Proposition 9.2, one gets:

$$\begin{aligned}\mathbb{E}\left[\bigotimes_{i=1}^{k} M_i\right] &= \sum_{\pi\in\mathsf{P}_k}\mathbb{E}\kappa^{f_\pi}\left[M_1,\dots,M_k\right] f_\pi \\ &= \sum_{\pi\in\mathsf{P}_k}\mathbb{E}\kappa^{f_\pi}\left[M_1,\dots,M_k\right]\left(\sum_{\pi\leq\pi'} f_{\pi'}^c\right) \\ &= \sum_{\pi\in\mathsf{P}_k}\left(\sum_{\pi'\leq\pi}\mathbb{E}\kappa^{f_{\pi'}}\left[M_1,\dots,M_k\right]\right) f_\pi^c.\end{aligned}$$

Using the second part of Lemma 9.3, we obtain the fact that for any $\pi\in\mathsf{P}_k$,

$$\sum_{\pi'\leq\pi}\mathbb{E}\kappa^{f_{\pi'}}\left[M_1,\dots,M_k\right] = \prod_{c\in\mathsf{C}(\pi)}\mathbb{E}\left[\prod_{i\in\pi} X_i\right].$$

An application of Lemma 9.4 and considering the partition $\pi = \{\{1,\dots,k\}\}$ allow us to conclude. □

In fact from the last proof, using the same notations, one has the more general formula:

$$\mathbb{E}\kappa^{f_\pi}[M_1,\dots,M_k] = \prod_{c\in\mathsf{c}(\pi)}\kappa_{\#c}\left((X_i)_{i\in c}\right).$$

The theorem 9.2 shows that one could be able to study the probabilities fluctuations of random matrices in the framework we developed. Let us remark that one can recover the classical central limit theorem using Theorem 9.2.

An other proof of Theorem 9.1. — Let $(X_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables whose law admits moments up to any order and such that $\mathbb{E}[X_1] = 0$. Let k be a positive integer. We have to understand $\kappa_k\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i,\dots,\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)$ when n goes to infinity. Let us consider $\left(Y_i^l\right)_{i,l\in\mathbb{N}}$ a family of independent identically distributed random variables which have the same law as X_1. Let us consider for any positive integers i and N:

$$M_i^N = \mathsf{Diag}\left[\left(Y_i^l\right)_{l=1}^N\right].$$

Using the $\mathfrak{S}$-invariance, the independence property and Theorem 7.10, for any $i \geq 1$, $\left(M_i^N\right)_{N\in\mathbb{N}}$ is asymptotically $\mathcal{P}$-free from $\left((M_j^N)_{j\neq i}\right)_{N\in\mathbb{N}}$ and for any positive integer n, $\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)_{N\in\mathbb{N}}$ is a strong approximation of the law of $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$. Let us apply the Theorem 8.1: using the fact that $\mathbb{E}[X_1] = 0$, the $\mathcal{R}$-transform of $\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)_{N\in\mathbb{N}}$ converges, when n goes to infinity, to $e_{\boxplus}^{\mathbb{E}[X_1^2]f_{\{\{1,2\}\}}}$. Yet, using Theorem 9.2, $\kappa_k\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i, ..., \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)$ is equal to:

$$\lim_{N\to\infty} \mathbb{E}\kappa^{f_{\{\{1,\dots,k\}\}}}\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)^{\otimes k}\right] = \left[\mathcal{R}_{\mathcal{P}}\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n M_i^N\right)_{N\in\mathbb{N}}\right)\right]_{f_{\{\{1,\dots,k\}\}}}.$$

Thus $\kappa_k\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i, ..., \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)$ converges, as n goes to infinity, to:

$$\left(e_{\boxplus}^{\mathbb{E}[X_1^2]f_{\{\{1,2\}\}}}\right)_{f_{\{\{1,\dots,k\}\}}}$$

which is equal to 0 if $k = 1$, $\mathbb{E}[X_1^2]$ if $k = 2$ and 0 if $k \geq 3$. □

10. Levy processes

10.1. Generalities about Levy processes. — Let us recall the notion of Levy processes.

Definition 10.1. — *Let E be a topological group. Let $(X_t)_{t\geq 0}$ be a cadlàg process in E which is stochastically continuous such that X_0 is the neutral element of E. The process $(X_t)_{t\geq 0}$ is a Levy process if for any $0 < t < s$, $X_sX_t^{-1}$ is independent of $(X_u)_{u\leq t}$ and $X_sX_t^{-1}$ has the same law as X_{s-t}.*

Let us remark that we consider actually right Levy processes as defined by M. Liao in [**25**]. From now on, we will only consider groups of matrices. If E is a subgroup of $(\mathcal{M}_N(\mathbb{C}), +)$, the Levy processes are called additive Levy processes. If E is a subgroup of $GL(N)$, the Levy processes are called multiplicative Levy processes.

Recall that the notation G stands either for O, U, or $\mathfrak{S}$. Following Definition 4.6, we define the notion of G-invariant Levy processes. Let N be a positive integer.

Definition 10.2. — *Let $(X_t)_{t\geq 0}$ be an additive or multiplicative Levy process in $\mathcal{M}_N(\mathbb{C})$, $(X_t)_{t\geq 0}$ is a G-invariant Levy process if for any $g \in G(N)$, we have the following equality in law: $\left(gX_tg^{-1}\right)_{t\geq 0} = (X_t)_{t\geq 0}$.*

In the following, we only consider Levy processes such that for any $t \geq 0$, X_t is in $L^{\infty^-} \otimes \mathcal{M}_N(\mathbb{C})$. In this case, for any $k \in \mathbb{N}$,

$$G_k = \frac{d}{dt}_{|t=0}\mathbb{E}\left[X_t^{\otimes k}\right]$$

is defined. In the next lemma, we show that this is the only data one needs in order to compute $\mathbb{E}[X_t^{\otimes k}]$ for any $t \in \mathbb{R}^+$. Recall the Definition 3.4.

Lemma 10.1. — *Let $(X_t)_{t\geq 0}$ be a matrix-valued Levy process such that for any positive real t, $X_t \in L^{\infty^-} \otimes \mathcal{M}_N(\mathbb{C})$. Let us define for any positive integer k:*

$$G_k = \frac{d}{dt}_{|t=0} \mathbb{E}\left[X_t^{\otimes k}\right].$$

For any positive integer k, for any positive real t_0,

– *if $(X_t)_{t\geq 0}$ is an additive Levy process:*

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[X_t^{\otimes k}\right] = \sum_{l=0}^{k-1} \sum_{I\subset\{1,\dots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[X_{t_0}^{\otimes l}\right], G_{k-l}\right],$$

– *if $(X_t)_{t\geq 0}$ is a multiplicative Levy process:*

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[X_t^{\otimes k}\right] = G_k \mathbb{E}\left[X_{t_0}^{\otimes k}\right].$$

Proof. — Let $(H_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ be two matrix-valued Levy processes which are respectively additive and multiplicative. Let us define for any integer k:

$$G_k^H = \frac{d}{dt}_{|t=0} \mathbb{E}\left[H_t^{\otimes k}\right],\ G_k^U = \frac{d}{dt}_{|t=0} \mathbb{E}\left[U_t^{\otimes k}\right].$$

Let t_0 be a non negative real. Using the independence and the stationarity properties of the processes $(H_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ we have the two following sets of equalities.
For the additive Levy process $(H_t)_{t\geq 0}$:

$$\begin{aligned}
\frac{d}{dt}_{|t=t_0} \mathbb{E}\left[H_t^{\otimes k}\right] &= \lim_{s\to 0} \frac{\mathbb{E}\left[H_{t_0+s}^{\otimes k}\right] - \mathbb{E}\left[H_{t_0}^{\otimes k}\right]}{s} \\
&= \lim_{s\to 0} \frac{\mathbb{E}\left[(H_{t_0} + (H_{t_0+s} - H_{t_0}))^{\otimes k}\right] - \mathbb{E}\left[H_{t_0}^{\otimes k}\right]}{s} \\
&= \lim_{s\to 0} \frac{\sum_{l=0}^{k-1} \sum_{I\subset\{1,\dots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes l}\right], \mathbb{E}\left[H_s^{\otimes k-l}\right]\right]}{s} \\
&= \sum_{l=0}^{k-1} \sum_{I\subset\{1,\dots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes l}\right], \lim_{s\to 0} \frac{\mathbb{E}\left[H_s^{\otimes k-l}\right]}{s}\right] \\
&= \sum_{l=0}^{k-1} \sum_{I\subset\{1,\dots,k\},\#I=l} \mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes l}\right], G_{k-l}^H\right].
\end{aligned}$$

For the multiplicative Levy process $(U_t)_{t\geq 0}$, let us consider $(U'_t)_{t\geq 0}$ a process which has the same law as $(U_t)_{t\geq 0}$ and which is independent of $(U_t)_{t\geq 0}$. Then we have:

$$\begin{aligned}\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[U_t^{\otimes k}\right] = \lim_{s\to 0}\frac{\mathbb{E}\left[U_{t_0+s}^{\otimes k}\right]-\mathbb{E}\left[U_{t_0}^{\otimes k}\right]}{s} &= \lim_{s\to 0}\frac{\mathbb{E}\left[(U'_sU_{t_0})^{\otimes k}\right]-\mathbb{E}\left[\left(U_{t_0}^{\otimes k}\right)\right]}{s}\\ &= \lim_{s\to 0}\frac{\mathbb{E}\left[(U'_s)^{\otimes k}U_{t_0}^{\otimes k}\right]-\mathbb{E}\left[\left(U_{t_0}^{\otimes k}\right)\right]}{s}\\ &= \lim_{s\to 0}\frac{\mathbb{E}\left[(U'_s)^{\otimes k}\right]-Id^{\otimes k}}{s}\mathbb{E}\left[U_{t_0}^{\otimes k}\right]\\ &= G_k^U\mathbb{E}\left[U_{t_0}^{\otimes k}\right].\end{aligned}$$

This allows us to conclude the proof. □

We will also use the more general fact that for any multiplicative matrix-valued Levy process $(U_t)_{t\geq 0}$, for any integers k and l, the family $\left(\mathbb{E}\left[U^{\otimes k}\otimes\overline{U}^{\otimes l}\right]\right)_{t\geq 0}$ is a semi-group. Indeed, using the same arguments, one can see that for any $t_0\geq 0$,

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right] = \left(\frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right]\right)\mathbb{E}\left[U_{t_0}^{\otimes k}\otimes\overline{U_{t_0}}^{\otimes l}\right]. \tag{16}$$

10.2. Convergence of Levy processes. —

10.2.1. Zero order. — In this section, we state and prove the main theorems for the convergence of Levy processes. Actually, we have decided to cut it in four parts for the sake of clarity. Yet, these theorems have to be read one after the other as if it was only one theorem.

Let us choose a possibility for (G,A) as explained in Notation 3.1. Let $\left((X_t^N)_{t\geq 0}\right)_{N\geq 0}$ be a sequence of G-invariant Levy processes which are either all additive or multiplicative. Let us suppose that for any positive integer N, and any $t\geq 0$, $X_t^N\in L^{\infty^-}\otimes\mathcal{M}_N(\mathbb{C})$. We define, for any positive integer k and N :

$$G_k^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right].$$

Let k and N be two positive integers. The endomorphism G_k^N commutes with the tensor action of G on $\left(\mathbb{C}^N\right)^{\otimes k}$. We can apply Section 3 to G_k^N: with a slight abuse of notation G_k^N belongs to $\mathbb{C}[A_k(N)]$. Thus, we can define the coordinate numbers $\left(\kappa^p\left(G_k^N\right)\right)_{p\in A_k}$ of G_k^N. Recall the notion of convergence for elements of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$ defined in [**15**]. Every time we will consider the sequence $\left(G_k^N\right)_{N\geq 0}$, we will consider it as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$.

Theorem 10.1. — *If for any positive integer k, the sequence $\left(G_k^N\right)_{N\geq 0}$ converges, then the process $(X_t^N)_{t\geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. Besides, $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial A- and $\mathcal{P}$-non-commutative Levy process.*

Proof. — Let us prove that for any real $t \geq 0$, $\left(X_t^N\right)_{N\in\mathbb{N}}$ converges in A-expectation. Using Lemma 2.2, it is enough to show that for anypositive integer k, $\left(X_t^N\right)^{\otimes k}$ converges in expectation in A_k-moments as N tends to infinity. Let k be a positive integer. Recall that, since $\left(X_t^N\right)_{t\geq 0}$ is G-invariant, using the dualities in Theorem 3.1, $\mathbb{E}\left[(X_t^N)^{\otimes k}\right]$ and G_k^N are in $\mathbb{C}\left[\rho_N^{A_k}\right]$.

In the additive case, by Lemma 10.1, for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[(X_t^N)^{\otimes k}\right] = \sum_{l=0}^{k-1}\sum_{I\subset\{0,...,k\},\#I=l}\mathcal{I}_I\left[\mathbb{E}\left[(X_{t_0}^N)^{\otimes l}\right], G_{k-l}^N\right]. \tag{17}$$

Thus, for any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0}\kappa^p\left[\mathbb{E}\left[(X_t^N)^{\otimes k}\right]\right] = \sum_{(p_1,p_2,I)\in\mathfrak{F}_2(p)/p_1\neq p}\kappa^{p_1}\left[\mathbb{E}\left[(X_t^N)^{\otimes \mathsf{l}(p_1)}\right]\right]\kappa^{p_2}\left[G_{\mathsf{l}(p_2)}^N\right],$$

where for any partition p, $\mathsf{l}(p)$ is the lenght of p defined as the unique integer such that $p \in \mathcal{P}_{\mathsf{l}(p)}$. Since for any positive integer k, G_k^N converges when N goes to infinity, for any partition $p_2 \in \cup_{k\in\mathbb{N}}A_k$, the sequence $\left(\kappa^{p_2}\left[G_{\mathsf{l}(p_2)}^N\right]\right)_{N\in\mathbb{N}}$ converges when N goes to infinity. By recurrence on the length of p, we see that for any positive integer k, any $p \in \mathcal{P}_k$ and any $t \geq 0$, $\kappa^p\left[\mathbb{E}\left[X_t^{\otimes k}\right]\right]$ converges as N goes to infinity.

In the multiplicative case, by Lemma 10.1, for any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right] = G_k^N\mathbb{E}\left[\left(X_{t_0}^N\right)^{\otimes k}\right].$$

Thus, $\mathbb{E}\left[(X_t^N)^{\otimes k}\right]$ is a semi-group in $\mathbb{C}\left[\rho_N^{A_k}\right]$. The Theorem 7.2 of [**15**] allows us to conclude.

We must now prove that the family $\left(X_t^N\right)_{t\geq 0}$ converges in A-expectation. We will only explain the multiplicative case as the additive case is similar. We have to prove that for any $0 \leq t_1 \leq t_2 ... \leq t_n$, $\left(X_{t_i}^N\right)_{i=1}^n$ converges in A-expectation as N tends to infinity. Let us define $t_0 = 0$. Using the definition of Levy processes, we know that for any positive integer N, $(X_{t_i}^N(X_{t_{i-1}}^N)^{-1})_{i=1}^n$ is a family of independent random matrices, each of them is invariant in law by conjugation by $G(N)$. Besides for any $i \in \{1, ..., n\}$, $X_{t_i}^N(X_{t_{i-1}}^N)^{-1}$ has the same law as $X_{t_i-t_{i-1}}^N$. Using the first part of the proof, we know that for any $i \in \{1, ..., n\}$, $X_{t_i}^N(X_{t_{i-1}}^N)^{-1}$ converges in A-expectation as N tends to infinity. Using Theorem 7.10, one concludes that $(X_{t_i}^N)_{i=1}^n$ converges in A-expectation as N tends to infinity. Because of the discussion we just had, we also get that $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a A-non-commutative matricial Levy process.

An application of Theorem 4.5 allows us to prove that $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation. By definition, we have proved that it is a matricial A-free Levy process, and since $\left(X_t^N\right)_{t\geq 0}$ is G-invariant for any positive integer N, as a consequence of Theorem 7.2, it is also a $\mathcal{P}$-non-commutative Levy process. □

We define the $\mathcal{R}_A$-functional of the family $\left(\left(G_k^N\right)_{N\in\mathbb{N}}\right)_{k\in\mathbb{N}}$. Recall Definition 10.1 of [**15**].

Definition 10.3. — *Let us suppose that for any positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$ converges as N goes to infinity. We define $\mathcal{R}_A(G)$ as the only element in $\mathfrak{e}[A]$ such that for any positive integer k and any $p \in A_k$,*

$$\left(\mathcal{R}_A(G)\right)_p = \lim_{N\to\infty} \kappa^p\left(G_k^N\right).$$

From now on, we suppose that for any positive integer k, $(G_k^N)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[A_k(N)]$, converges. Recall the notions of $\mathfrak{me}_{\boxplus}[A]$ and $\mathfrak{me}_{\boxtimes}[A]$ defined in Definition 10.5 and 10.6 in [**15**].

Definition 10.4. — *We say that $\left(G_k^N\right)_{k,N}$ condensates (resp. weakly condensates) if $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxplus}[A]$ (resp. $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxtimes}[A]$).*

Let us compute the $\mathcal{R}_A$-functional of $\left(X_t^N\right)_{t\geq 0}$. Recall the notation $\mathfrak{E}^{(i)}[A]$ defined in Definition 10.3 of [**15**]. If E is an element of $\mathfrak{E}[A]$, we denote by $E_{|\mathfrak{E}^{(i)}[A]}$ the restriction of E to $\bigcup_{k\in\mathbb{N}^*} A_k^{(i)}$. Recall also Definitions 10.7 and 10.8 in [**15**].

Theorem 10.2. — *Let us suppose that for any postive integer N, $\left(X_t^N\right)_{t\geq 0}$ is an additive Levy process. For any real $t_0 \geq 0$,*

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}_A[X_t] = \mathcal{R}_A[G] \boxplus \mathcal{R}_A[X_{t_0}].$$

This equation can also be written in the following way, for any $t_0 \geq 0$:

$$\mathcal{R}_A[X_{t_0}] = e_{\boxplus}^{t_0 \mathcal{R}_A[G]}.$$

A consequence is that, for any real $t_0 \geq 0$:

$$\left(\mathcal{R}_A\left[X_{t_0}\right]\right)_{|\mathfrak{E}^{(i)}[A]} = t_0 \left(\mathcal{R}_A\left[G\right]\right)_{|\mathfrak{E}^{(i)}[A]}.$$

Let us suppose that for any positive integer N, $\left(X_t^N\right)_{t\geq 0}$ is a multiplicative Levy process. For any real $t_0 \geq 0$,

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}_A[X_t] = \mathcal{R}_A[G] \boxtimes \mathcal{R}_A[X_{t_0}].$$

This equation can also be written in the following way, for any $t_0 \geq 0$:

$$\mathcal{R}_A[X_{t_0}] = e_{\boxtimes}^{t_0 \mathcal{R}_A[G]}.$$

Besides, for any positive integer k, any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}_{|t=t_0} \mathbb{E}m_p(X_t, ..., X_t) = \sum_{p_1 \in [id,p]_{A_k}} \kappa_A^{p_1}(G_k) \mathbb{E}m_{t_{p_1 \circ p}}(X_{t_0}, ..., X_{t_0}).$$

Proof. — The first equation appears when one takes the limit of Equation (17) when N goes to infinity. Let k be a positive integer, let p be an irreducible partition in A_k. Using the first equality and using the fact that $(\mathcal{R}_A[G])_\emptyset = 0$, we get:

$$\frac{d}{dt}_{|t=t_0} (\mathcal{R}_A[X_t])_p = (\mathcal{R}_A[G])_p \,,$$

hence $(\mathcal{R}_A[X_t])_p = t\,(\mathcal{R}_A[G])_p$ for any $t \geq 0$.

The equations in the multiplicative case are consequences of Equations (18) and (19) in Theorem 7.2 of [**15**]. □

Theorem 10.3. — *In the additive case, if* $\left(G_k^N\right)_{k,N}$ *condensates, the asymptotic* $\mathcal{P}$*-factorization property holds for* $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$.

In the multiplicative case, if $\left(G_k^N\right)_{k,N}$ *weakly condensates, the asymptotic* $\mathcal{P}$*-factorization property holds for* $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$.

Proof. — Let us prove first that for any positive real t, the A-factorization property holds for $(X_t^N)_{N\in\mathbb{N}}$. Let t be a positive real. Using Theorem 10.2, in the additive case $\mathcal{R}_A[X_t] = e_{\boxplus}^{t\mathcal{R}[G]}$ and in the multiplicative case $\mathcal{R}_A[X_t] = e_{\boxtimes}^{t\mathcal{R}[G]}$. By Definition 10.4, $(G_k^N)_{k,N}$ condensates if $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxplus}[A]$, and $(G_k^N)_{k,N}$ weakly condensates if $\mathcal{R}_A(G) \in \mathfrak{me}_{\boxtimes}[A]$. By Theorem 10.2 of [**15**], we see that under the condensation property, in the additive case as for the multiplicative case, $\mathcal{R}_A[X_t] \in \mathfrak{ME}[A]$, which is equivalent, by Proposition 6.1, to say that the asymptotic A-factorization property holds for $(X_t^N)_{N\in\mathbb{N}}$.

As a consequence of Proposition 7.2, the asymptotic A-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ since the increments are asymptotically A-free. Then using the fact that the Levy processes we considered are G-invariant, an application of Proposition 4.5 allows us to assert that the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. □

In order to show that $(G_k^N)_{N\in\mathbb{N}}$ converges for any positive integer k, it is often easier to compute the normalized moments of G_k^N. Thus it would be interesting to have a criterion about normalized moments in order to know if $(G_k^N)_{k,N\in\mathbb{N}}$ condensates or weakly condensates.

Lemma 10.2. — *The family* $\left(G_k^N\right)_{k,N}$ *condensates if and only if for any positive integer* k *and* l*, for any* $p_1 \in A_k$ *and any* $p_2 \in A_l$*,*

$$m_{p_1\otimes p_2}\left(\left(G_{k+l}^N\right)_{N\in\mathbb{N}}\right) = 0.$$

The family $\left((G_k^N)_{N\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ *weakly condensates if and only if for any positive integer* k *and* l*, for any* $p_1 \in A_k$ *and any* $p_2 \in A_l$*,*

$$m_{p_1\otimes p_2}\left(\left(G_{k+l}^N\right)_{N\in\mathbb{N}}\right) = m_{p_1}\left(\left(G_k^N\right)_{N\in\mathbb{N}}\right) + m_{p_2}\left(\left(G_l^N\right)_{N\in\mathbb{N}}\right).$$

We will only give the proof for the weak condensation, the first assertion can be proved with a similar argument.

Proof. — Let us recall that we defined an application $\mathfrak{J} : \mathfrak{e}^{(i)}[A] \to \mathfrak{e}[A]$ in Definition 10.6 of [**15**]. Besides, we defined an application $\mathcal{M}_A$ in Definition 10.9 of [**15**] which can be restrained to $\mathfrak{e}[A]$. In this proof, we will consider thus $\mathcal{M}_A : \mathfrak{e}[A] \to \mathfrak{e}[A]$. Let

us define two others applications. The first application allows us to extend a function defined on the irreducible partitions to all the partitions:

$$\mathfrak{A}_A : \mathfrak{e}^{(i)}[A] \to \mathfrak{e}[A]$$

such that for any $E \in \mathfrak{e}^{(i)}[A]$, for any positive integer k, for any $p \in A_k$,

$$(\mathfrak{A}_A(E))_p = \sum_{C \in \mathsf{C}(p)} E_{p_c}.$$

The second one $\mathcal{M}_A^{(i)}$ is defined as the application:

$$\mathcal{M}_A^{(i)} : \mathfrak{e}^{(i)}[A] \to \mathfrak{e}^{(i)}[A]$$

such that for any $E \in \mathfrak{e}^{(i)}[A]$, for any positive integer k, for any $p \in A_k$ which is irreducible:

$$\left(\mathcal{M}_A^{(i)}(E)\right)_p = ((\mathcal{M}_A \circ \mathfrak{J})(E))_p$$

Let us remark that $\mathcal{M}_A^{(i)}$ is a bijection. Let us prove that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{e}^{(i)}[A] \arrow[r, "\mathfrak{J}"] \arrow[d, "\mathcal{M}_A^{(i)}"'] & \mathfrak{e}[A] \arrow[d, "\mathcal{M}_A"] \\
\mathfrak{e}^{(i)}[A] \arrow[r, "\mathfrak{A}_A"] & \mathfrak{e}[A]
\end{tikzcd}$$

Let us consider $E \in \mathfrak{e}^{(i)}[A]$, let k be a positive integer, let $p \in A_k$, let p_1, ..., p_r be the extraction of p on its cycles. We can suppose that $p = p_1 \otimes ... \otimes p_r$ with p_1, ..., p_r some irreducible partitions. Using the factorization of the geodesics and using the fact that for any $p' \in A_k$, $(\mathfrak{J}(E))_{p'}$ is equal to zero if p' is not weakly irreducible, we get:

$$\begin{aligned}
(\mathcal{M}_A(\mathfrak{J}(E)))_p = \sum_{p' \in [id, p_1 \otimes ... \otimes p_r]_{A_k}} (\mathfrak{J}(E))_{p'} &= \sum_{i=1}^{r} \sum_{p'_i \in [id, p_i]_{A_k}} (\mathfrak{J}(E))_{p'_i} \\
&= \sum_{i=1}^{r} \left(\mathcal{M}_A^{(i)}(E)\right)_{p_i} = \left(\left(\mathfrak{A}_A \circ \mathcal{M}_A^{(i)}\right)(E)\right)_p .
\end{aligned}$$

Using the commutative diagram, one gets the following equalities:

$$\mathfrak{A}_A\left[\mathfrak{e}^i[A]\right] = \mathfrak{A}_A \circ \mathcal{M}_A^{(i)}\left[\left(\mathcal{M}_A^{(i)}\right)^{-1}\left[\mathfrak{e}^i[A]\right]\right] = \mathcal{M}_A \circ \mathfrak{J}\left[\mathfrak{e}^i[A]\right] = \mathcal{M}_A\left[\mathfrak{me}_{\boxtimes}[A]\right].$$

Thus any element $E \in \mathfrak{e}[A]$ belongs to $\mathfrak{me}_{\boxtimes}[A]$ if and only if $\mathcal{M}_A(E) \in \mathfrak{A}_A\left[\mathfrak{e}^i[A]\right]$. By applying this to $\mathcal{R}_A(G)$ one gets the expected result. $\square$

10.2.2. The $*$-convergence. — Let us consider the same sequence $\left(\left(X_t^N\right)_{t\geq 0}\right)_{N\geq 0}$ of G-invariant Levy processes used in the last subsection. If the matrices X_t^N are complex-valued, one can not deduce any convergence in probability of normalized A-moments using the Theorem 10.3: one has to show that $\left(\left(X_t^N, \left(X_t^N\right)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ or $\left(\left(X_t^N, \overline{X_t^N}\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ converges as N goes to infinity and satisfies the A-factorization property in order to be able to apply the second part of Theorem 2.2. This is what we do in this section: we generalize Theorems 10.1, 10.2 and 10.3.

We will suppose that for any integer N, $(X_t^N)_{t\geq 0}$ is a multiplicative Levy process, since often the additive Levy processes considered are in the Hermitian matrices space for which the convergence of $\left(\left(X_t^N, \left(X_t^N\right)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ is a direct consequence of the convergence of $\left(\left(X_t^N\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$.

Let us recall that according to the Equation (16), $\left(\mathbb{E}\left[(X_t^N)^{\otimes k}\otimes\left(\overline{X_t^N}\right)^{\otimes l}\right]\right)_{t\geq 0}$ is a semi-group of endomorphisms. We define, for any integers k, l and N:

$$G_{k,l}^N = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left(\overline{X_t^N}\right)^{\otimes l}\right].$$

Recall the operation S_k defined in Definition 2.5. For any positive integer k and l, for any real $t\geq 0$, the endomorphism $\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left((X_t^N)^*\right)^{\otimes l}\right]$ commutes with the tensor action of G on $\left(\mathbb{C}^N\right)^{\otimes(k+l)}$: according to Theorem 3.1, it is an element of $\mathbb{C}\left[\rho_N^{A_{k+l}}\right]$, or with a slight abuse of notation it is an element of $\mathbb{C}[A_{k+l}(N)]$. This implies that $\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left(\overline{X_t^N}\right)^{\otimes l}\right]$ which is equal to $S_k\left[\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\otimes\left((X_t^N)^*\right)^{\otimes l}\right]\right]$ is in $\mathbb{C}[\mathcal{P}_{k+l}(N)]$ if $G=\mathfrak{S}$, in $\mathbb{C}[\mathcal{B}_{k+l}(N)]$ if $G=O$ and in $S_k\left[\mathbb{C}[\mathfrak{S}_k]\right]\subset\mathcal{B}_{k+l}$ if $G=U$. The same conclusion holds for $G_{k,l}^N$. The proof of the following theorem is similar to 10.1, thus we will omit it.

Theorem 10.4. — *If for any positive integer k and l, the sequence $\left(G_{k,l}^N\right)_{N\geq 0}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}\left[\mathcal{P}_{k+l}(N)\right]$ converges, then the family $\left(\left(X_t^N, (X_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ converges in $\mathcal{P}$-expectation as N goes to infinity.*

Let us suppose, from now on that for any positive integer k and l, the sequence $\left(G_{k,l}^N\right)_{N\geq 0}$ converges. We can define for any positive integer k and l,

$$(\mathcal{R}(G))_{k,l} = \sum_{p\in\mathcal{P}_{k+l}}\kappa^p(G_{k,l})p\in\mathbb{C}\left[\mathcal{P}_{k+l}(\infty,\infty)\right].$$

Since $\left(\left(X_t^N,(X_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ converges in $\mathcal{P}$-expectation as N goes to infinity, we can define for any positive integer k and l and any real $t\geq 0$:

$$(\mathcal{R}(X_t))_{k,l}=\sum_{p\in\mathcal{P}_{k+l}}\kappa^p\left(\left(X_t^N\right)_{N\in\mathbb{N}},...,\left(X_t^N\right)_{N\in\mathbb{N}},\left(\overline{X_t^N}\right)_{N\in\mathbb{N}},...,\left(\overline{X_t^N}\right)_{N\in\mathbb{N}}\right)p,$$

where we wrote k times $(X_t^N)_{N\in\mathbb{N}}$ and l times $\left(\overline{X_t^N}\right)_{N\in\mathbb{N}}$.

Definition 10.5. — *We say that $\left(G_{k,l}^N\right)_{k,l,N}$ weakly condensates if for any positive integer k and l, for any $p\in\mathcal{P}_{k+l}$, if p is not weakly irreducible,*

$$\kappa^p(G_{k,l})=0,$$

and

$$\kappa^{id_{k+l}}(G_{k,l})=k\kappa^{id_1}(G_{1,0})+l\kappa^{id_1}(G_{0,1}).$$

This definition allows us to state the following theorem which is a generalization of Theorems 10.2 and 10.3.

Theorem 10.5. — *For any real $t_0\geq 0$, the following equality holds in $\mathbb{C}[\mathcal{P}_{k+l}(\infty,\infty)]$:*

$$\frac{d}{dt}_{|t=t_0}\left(\mathcal{R}(X_t)\right)_{k,l}=(\mathcal{R}(G))_{k,l}\left(\mathcal{R}(X_{t_0})\right)_{k,l}.$$

Moreover, for any integer k and l, any $p\in\mathcal{P}_k$ and any $t_0\geq 0$:

$$\frac{d}{dt}_{|t=t_0}m_p\left[\mathbb{E}\left(X_t^{\otimes k}\otimes\overline{X_t}^{\otimes l}\right)\right]=\sum_{p_1\in[id,p]_{A_k}}\kappa^{p_1}(G_{k,l})m_{{}^t p_1\circ p}\left[\mathbb{E}\left(X_{t_0}^{\otimes k}\otimes\overline{X_{t_0}}^{\otimes l}\right)\right],$$

where $m_p\left[\mathbb{E}\left(X_{t_0}^{\otimes k}\otimes\overline{X_{t_0}}^{\otimes l}\right)\right]$ stands for the limit as N goes to infinity of the mean moment $m_p\left[\mathbb{E}\left(X_{t_0}^{N\otimes k}\otimes\left(\overline{X_{t_0}^N}\right)^{\otimes l}\right)\right]$.

Besides, if $\left(G_{k,l}^N\right)_{k,l,N}$ weakly condensates, the asymptotic $\mathcal{P}$-factorization property holds for $\left(\left(X_t^N,(X_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ and thus the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t\geq 0}$ converge in probability.

10.2.3. Higher order. — More generally, one can state similar results for the fluctuations than those we get in Section 10.2.1. The proofs are similar to the one for the zero order, and all the tools that one needs in order to prove them are included either in [**15**], namely Theorem 9.3, or in this article, thus we will omit them. Let m be a non negative integer.

Theorem 10.6. — *Let us suppose that for anypositive integer k, $(G_k^N)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$, converges up to order m of fluctuations. Then $(X_t^N)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to order m of fluctuations as N goes to infinity. Besides the family $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial A- and $\mathcal{P}$-non-commutative Levy process up to any order of fluctuations.*

Let us define the $\mathcal{R}_A^{(m)}$-functional of the family $\left(\left(G_k^N\right)_{N\in\mathbb{N}}\right)_{k\in\mathbb{N}}$.

Definition 10.6. — *Let us suppose that* $\left(G_k^N\right)_{N\in\mathbb{N}}$*, seen in* $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$*, converges up to order* m *of fluctuations. We define* $\mathcal{R}_A^{(m)}(G)$ *as the only element in* $\mathfrak{E}^{(m)}[A]$ *such that for any positive integer* k*, any* $p\in A_k$ *and any* $i\in\{0,...,m\}$*:*

$$\left(\mathcal{R}_A(G)\right)_{p,i}=\lim_{N\to\infty}\kappa_i^p\left(G_k^N\right),$$

and for any $i\in\{0,...,m\}$*,* $\left(\mathcal{R}_A(G)\right)_{\emptyset,i}=0$*.*

From now on, we suppose that $\left(G_k^N\right)_{N\in\mathbb{N}}$ converges up to order m of fluctuations. Let us recall the two operations $\boxplus$ and $\boxtimes$ defined in Section 10.1 of [**15**].

Theorem 10.7. — *Let us suppose that for any positive integer* N*,* $(X_t^N)_{t\geq 0}$ *is an additive Levy process. For any real* $t_0\geq 0$*,*

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}_A^{(m)}[X_t]=\mathcal{R}_A^{(m)}[G]\boxplus\mathcal{R}_A^{(m)}[X_{t_0}].$$

Let us suppose that for any integer N*,* $(X_t^N)_{t\geq 0}$ *is a multiplicative Levy process. For any real* $t_0\geq 0$*,*

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}_A^{(m)}[X_t]=\mathcal{R}_A^{(m)}[G]\boxtimes\mathcal{R}_A^{(m)}[X_{t_0}].$$

Besides, for any integer k*, any* $p\in\mathcal{P}_k$*, any* $t_0\geq 0$ *and any* $i_0\in\{0,...,m\}$*:*

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}m_p^{i_0}(X_t,...,X_t)=\sum_{p_1\in A_k}\sum_{i+j+\mathsf{df}(p_1,p)=i_0}\kappa_i^{p_1}(G_k)\mathbb{E}m_{t_{p_1\circ p}}^j(X_{t_0},...,X_{t_0}).$$

10.2.4. Special cases. — Using Lemma 3.9 in [**15**] which states that any geodesic in the set of Brauer elements between permutations is included in the set of permutations, we can state a slightly stronger theorem for sequence of O-invariant Levy processes. Let $\left((X_t^N)_{t\geq 0}\right)_{N\geq 0}$ be a sequence of O-invariant Levy processes which are either all multiplicative or additive. Let us suppose that for any positive integer N, for any positive real t, $X_t^N\in L^{\infty^-}\otimes\mathcal{M}_N(\mathbb{C})$. As usual, let us define for any positive integer k and N:

$$G_k^N=\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right].$$

Let us recall, with a slight abuse of notation, that using the dualities in Theorem 3.1, for any positive integers k and N, $G_k^N\in\mathbb{C}[\mathcal{B}_k]$. As a consequence of Proposition 2.3 in this article and Theorem 4.3, Theorem 7.3, Equations (18) and (20) in the article [**15**] one can prove the following theorem.

Theorem 10.8. — *Let us suppose that for any positive integer* k*, the* $\mathfrak{S}_k$*-normalized moments of* $\left(G_k^N\right)$ *converge as* N *goes to infinity and that the* $\mathcal{B}_k$*-normalized moments of* $\left(G_k^N\right)$ *are bounded, then* $(X_t^N)_{t\geq 0}$ *converges in* $\mathfrak{S}$*-expectation as* N *goes to infinity.*

As a special case, let us suppose that for any positive integer N*,* $\left(X_t^N\right)_{t\geq 0}$ *is a* O*-invariant Levy process which is defined on the set of orthogonal or symmetric matrices, then the condition of boundeness of the* $\mathcal{B}_k$*-normalized moments of* $\left(G_k^N\right)$ *is not necessary:*

the $\mathcal{B}$-normalized moments of $\left(G_k^N\right)_{N\in\mathbb{N}}$ will converge and because of Theorem 10.1, $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation.

Besides let $\left((Y_t^N)_{t\geq 0}\right)_{N\geq 0}$ be a sequence of U-invariant Levy processes which is of the same type than $\left((X_t^N)_{t\geq 0}\right)_{N\geq 0}$ (i.e. multiplicative of additive). Let us define for any positive integer k and N:

$$G_k'^N = \frac{d}{dt}_{|t=0} \mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right].$$

Let us suppose that $\left(G_k'^N\right)_{N\in\mathbb{N}}$ seen as an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathfrak{S}(N)]$ converges. Let us suppose that for any positive integer k and any $\sigma \in \mathfrak{S}_k$:

$$\lim_{N\to\infty} m_\sigma\left(G_k'^N\right) = \lim_{N\to\infty} m_\sigma\left(G_k^N\right) \text{ or } \lim_{N\to\infty} \kappa^\sigma\left(G_k'^N\right) = \lim_{N\to\infty} \kappa^\sigma\left(G_k^N\right),$$

then $\left(\left(X_t^N\right)_{t\geq 0}\right)_{N\geq 0}$ has the same $\mathfrak{S}$-law as $\left(\left(Y_t^N\right)_{t\geq 0}\right)_{N\geq 0}$.

The reader must be aware that Theorem 10.8 holds only for O-invariant Levy processes. It does not hold for a general $\mathfrak{S}$-invariant Levy process, due to the fact that the geodesics from a permutation to the identity might go through an element of $\mathcal{P}_k \setminus \mathfrak{S}_k$. Yet, a similar result holds for $\mathfrak{S}(N)$-invariant Levy processes which are defined on the set of permutation matrices: this is a consequence of Lemma 2.4.

Theorem 10.9. — *Let us suppose that for any integer N, $\left(X_t^N\right)_{t\geq 0}$ is a $\mathfrak{S}$-invariant Levy process which is defined on the set of permutation matrices.*

Let us suppose that for any integer k, the $\mathfrak{S}$-normalized moments of $\left(G_k^N\right)_{N\in\mathbb{N}}$ converge, then the $\mathcal{P}$-normalized moments of $\left(G_k^N\right)_{N\in\mathbb{N}}$ converge. As a consequence, because of Theorem 10.1, $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation.

10.3. Brownian motions, convergence and matricial Wick formula. — In this section, we study the convergence of Brownian motions on some Lie algebras and Lie groups. We decided to write a whole section about Brownian motions since we will explain different ideas than the ones which will be presented in the general setting of Levy processes in Section 10. Besides, the intuitions that we will develop in this section will allow us to prove Theorem 8.2.

10.3.1. General definitions. — Let us define some useful space of matrices. We will follow the presentation of Levy in [**23**]. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, let N be a positive integer.

Definition 10.7. — *The spaces of skew-symmetric and symmetric real matrices of size N are respectively:*

$$\mathfrak{a}_N = \{M \in \mathcal{M}_N(\mathbb{R}),\ {}^tM = -M\},$$
$$\mathfrak{s}_N = \{M \in \mathcal{M}_N(\mathbb{R}),\ {}^tM = M\}.$$

The space of skew-Hermitian matrices of size N is:

$$\mathfrak{u}_N = \{M \in \mathcal{M}_N(\mathbb{C}), M^* = -M\}.$$

It is easy to see that the space of Hermitian matrices of size N is $i\mathfrak{u}_N$. We will use the conventions:

$$\mathfrak{u}(N,\mathbb{K}) = \begin{cases} \mathfrak{a}_N & \text{if } \mathbb{K} = \mathbb{R}, \\ \mathfrak{u}_N & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

and:

$$\mathfrak{h}(N,\mathbb{K}) = \begin{cases} \mathfrak{s}_N & \text{if } \mathbb{K} = \mathbb{R}, \\ i\mathfrak{u}_N & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Even more generally, we will consider:

$$\mathfrak{g}^\epsilon(N,\mathbb{K}) = \begin{cases} \mathfrak{u}(N,\mathbb{K}) & \text{if } \epsilon = -1, \\ \mathfrak{h}(N,\mathbb{K}) & \text{if } \epsilon = 1. \end{cases}$$

Besides, we will also use the following notation:

$$\beta_{\mathbb{K}} = \begin{cases} 1 & \text{if } \mathbb{K} = \mathbb{R}, \\ 2 & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Let us consider $\epsilon \in \{-1, 1\}$.

Definition 10.8. — *We will always consider $\mathfrak{g}^\epsilon(N,\mathbb{K})$ as a Lie algebra endowed with the scalar product:*

$$\forall X, Y \in \mathfrak{g}^\epsilon(N,\mathbb{K}), \quad <X,Y>= \frac{\beta_{\mathbb{K}} N}{2} Tr(X^*Y).$$

Let us remark that the scalar product on $\mathbb{R} = \mathfrak{g}^1(1,\mathbb{R})$ is $<x,y>= \frac{1}{2}xy$ for any reals x and y.

Definition 10.9. — *Let d be the dimension of $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Let $(X_i)_{i=1}^d$ be an orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$. We define the Casimir of $\mathfrak{g}^\epsilon(N,\mathbb{K})$ as:*

$$C_{\mathfrak{g}^\epsilon(N,G)} = \sum_{i=1}^d X_i \otimes X_i.$$

This is an easy and well-known result that the definition of the Casimir does not depend on the orthonormal basis we have chosen. We remind the reader that we defined the transpositions and the Weyl contractions in Definition 2.8 of [**15**]. Using the results of Levy in [**23**], we know the value of the Casimir operator.

Lemma 10.3. — *The Casimir of $\mathfrak{g}^\epsilon(N,\mathbb{K})$ is equal to:*

$$\frac{1}{N}\rho_N^{\mathcal{B}_2}\big(\epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2]\big).$$

This lemma implies the following result.

Lemma 10.4. — *Let d be the dimension of $\mathfrak{g}^\epsilon(N,\mathbb{K})$, and let $(X_i)_{i=1}^d$ be an orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Then:*

$$\sum_{i=1}^d X_i X_i = c_{\mathfrak{g}^\epsilon(N,\mathbb{K})} Id,$$

with $c_{\mathfrak{g}^\epsilon(N,\mathbb{K})} = \epsilon + \frac{2-\beta_{\mathbb{K}}}{N}$.

In the following, we will use the following notation:

$$U(N,\mathbb{K}) = \begin{cases} O(N) & \text{if } \mathbb{K}=\mathbb{R}, \\ U(N) & \text{if } \mathbb{K}=\mathbb{C}. \end{cases}$$

The group N, $U(N,\mathbb{K})$ is a Lie group, whose Lie algebra is $\mathfrak{u}(N,\mathbb{K})$.

10.3.2. Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. — Let N be an integer and let ϵ be in $\{-1,1\}$. As Levy did in [**23**], we now define the notion of Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Let d be the dimension of $\mathfrak{g}^\epsilon(N,\mathbb{K})$, and let $(X_i)_{i=1}^d$ be an orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$.

Definition 10.10. — *Let $\left((B_t^i)_{t\geq 0}\right)_{i=1}^d$ be a d-tuple of independent real Brownian motions. The process $(H_t)_{t\geq 0}$ where for any $t\geq 0$:*

$$H_t = \sum_{i=1}^d B_t^i X_i.$$

is a Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. Any process which has the same law as $(H_t)_{t\geq 0}$ is called a Brownian motion on $g^\epsilon(N,\mathbb{K})$.

Remark 10.1. — *The law of the process $(H_t)_{t\geq 0}$ does not depend on the choice of the orthonormal basis of $\mathfrak{g}^\epsilon(N,\mathbb{K})$: this allows us to show that $(H_t)_{t\geq 0}$ is a $U(N,\mathbb{K})$-invariant process.*

A consequence of Definition 10.10 is that any Brownian motion in $\mathfrak{g}^\epsilon(N,\mathbb{K})$ is an additive Levy process. Besides, we can compute the bracket of H_t with itself:

$$dH_t \otimes dH_t = \sum_{i,j=1}^d dB_t^i dB_t^j X_i \otimes X_j = \left(\sum_{i=1}^d X_i \otimes X_i\right) dt = C_{\mathfrak{g}^\epsilon(N,\mathbb{K})} dt, \tag{18}$$

due to the fact that for any $i,j\in\{0,...,d\}$, $dB_t^i dB_t^j = \delta_{i,j} dt$. Let us remark that one can easily compute the bracket of H_t with ${}^tH_t$ and ${}^tH_t$ with itself since:

$$dH_t \otimes {}^t dH_t = S_1\left[dH_t \otimes dH_t\right]$$
$${}^t dH_t \otimes {}^t dH_t = S_0\left[dH_t \otimes dH_t\right],$$

where S_k is the operation defined in Definition 2.5. We will be interested in the case where $\mathbb{K}=\mathbb{C}$ and $\epsilon=-1$: let us suppose so just for this discussion. Then, we get the following equalities:

$$dH_t \otimes {}^t dH_t = -\frac{1}{N}\rho_N^{\mathcal{B}_k}\big([1,2]\big) dt, \tag{19}$$

$${}^t dH_t \otimes {}^t dH_t = -\frac{1}{N}\rho_N^{\mathfrak{S}_k}(1,2) dt. \tag{20}$$

We recall that we defined the insertion operator $\mathcal{I}_I$ in Definition 3.4. Using the Itô's formula, one has the following lemma.

Lemma 10.5. — *Let us consider $(H_t)_{t\geq 0}$ a Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$. For any integer $k\in\mathbb{N}\setminus\{2\}$, $G_k^N=\frac{d}{dt}_{|t=0}\mathbb{E}\left[H_t^{\otimes k}\right]=0$ and:*

$$G_2^N=\frac{d}{dt}_{|t=0}\mathbb{E}\left[H_t^{\otimes 2}\right]=C_{\mathfrak{g}^\epsilon(N,\mathbb{K})}.$$

10.3.3. Brownian motion on $U(N,\mathbb{K})$. — We can now define the multiplicative notion of Brownian Motion.

Definition 10.11. — *Let $(H_t)_{t\geq 0}$ be a Brownian motion on $\mathfrak{u}(N,\mathbb{K})$. The solution of the Stratonovich stochastic equation:*

$$\begin{cases} dU_t &= dH_t\circ U_t,\\ U_0 &= I_N,\end{cases}$$

is a Brownian motion on $U(N,\mathbb{K})$. We can reformulate this in terms of Itô integral. The solution of the Itô stochastic equation:

$$\begin{cases} dU_t &= dH_tU_t+\frac{c_{\mathfrak{u}(N,\mathbb{K})}}{2}U_tdt,\\ U_0 &= I_N,\end{cases}$$

is a Brownian motion on $U(N,\mathbb{K})$. Any process which has the same law as $(U_t)_{t\geq 0}$ is a Brownian motion on $U(N,\mathbb{K})$.

It is an important result that any Brownian motion on $U(N,\mathbb{K})$ is a multiplicative Levy process which is invariant by conjugation by $U(N,\mathbb{K})$: this is a consequence of the fact that the linear Brownian motion is also invariant by conjugation by $U(N,\mathbb{K})$.

We can also compute the bracket of U_t with itself. Let $(H_t)_{t\geq 0}$ be an additive Brownian motion on $\mathfrak{u}(N,\mathbb{K})$ and let $(U_t)_{t\geq 0}$ be the Brownian motion on $U(N,\mathbb{K})$ associated with $(H_t)_{t\geq 0}$. Then:

$$dU_t\otimes dU_t=dH_tU_t\otimes dH_tU_t=(dH_t\otimes dH_t)(U_t\otimes U_t)=C_{\mathfrak{u}(N,\mathbb{K})}(U_t\otimes U_t)dt,$$

where the last equality is a consequence of the additive case.

This remark allows us to compute the action of the infinitesimal generator on the tensor product. Using the Itô's formula and Lemma 10.4, we get the following formula given in [**23**] :

Lemma 10.6. — *Let us consider $(U_t)_{t\geq 0}$ a Brownian motion on $U(N,\mathbb{K})$. For any positive integer k:*

$$G_k^N=\frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k}\right]=\frac{kc_{\mathfrak{u}_{N,\mathbb{K}}}}{2}Id^{\otimes k}+\sum_{1\leq i<j\leq k}\mathcal{I}_{\{i,j\}}\left[C_{\mathfrak{u}(N,\mathbb{K})},Id^{\otimes k-2}\right].$$

We can also generalize this lemma in order to understand $\frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right]$ when $\mathbb{K}=\mathbb{C}$. Since H_t is skew-Hermitian, $\overline{dH_t}=-{}^tdH_t$. Besides, by Definition 10.11, we

know that $dU_t = dH_tU_t - \frac{1}{2}U_tdt$, thus:

$$d\overline{U_t} = -\,{}^t dH_t\overline{U_t} - \frac{1}{2}\overline{U_t}dt.$$

This implies that:

$$dU_t \otimes d\overline{U_t} = -dH_tU_t \otimes {}^t dH_t\overline{U_t} = \frac{1}{N}\rho_N^{\mathcal{B}_k}\left([1,2]\right)\left(U_t \otimes \overline{U_t}\right)dt,$$
$$d\overline{U_t} \otimes d\overline{U_t} = {}^t dH_t\overline{U_t} \otimes {}^t dH_t\overline{U_t} = -\frac{1}{N}\rho_N^{\mathfrak{S}_k}\left((1,2)\right)\left(\overline{U_t} \otimes \overline{U_t}\right)dt.$$

This remark, the Equations (19) and (20) allow us to use the Itô's formula in order to show the following result which was already proved by T. Levy in [**23**], and by A. Dahlqvist in [**1**].

Lemma 10.7. — *Let us consider $(U_t)_{t\geq 0}$ a Brownian motion on $U(N,\mathbb{C})$. For any positive integers k and k', we have, with a slight abuse of notations:*

$$\begin{aligned} G_{k,k'}^N &= \frac{d}{dt}_{|t=0}\mathbb{E}\left[U_t^{\otimes k} \otimes \overline{U_t}^{\otimes k'}\right] \\ &= -\frac{k+k'}{2}Id^{\otimes k} - \frac{1}{N}\left[\sum_{1\leq i<j\leq k+k'}(i,j) + \sum_{1\leq i\leq k<j\leq k+k'}([i,j]-(i,j))\right]. \end{aligned}$$

10.3.4. Convergence of additive Brownian motion. — Let us state the results about convergence of additive Brownian motions.Let $\epsilon \in \{-1,1\}$. Let A be equal to $\mathfrak{S}$ if $\mathbb{K} = \mathbb{C}$ and equal to $\mathcal{B}$ if $\mathbb{K} = \mathbb{R}$. For any integer N, let $\left(H_t^N\right)_{t\geq 0}$ be a Brownian motion on $\mathfrak{g}^\epsilon(N,\mathbb{K})$.

Theorem 10.10. — *The process $\left(H_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. The family $\left((H_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial A- and $\mathcal{P}$-non-commutative additive Levy process up to any order of fluctuations.*

Besides, the asymptotic $\mathcal{P}$-factorization property holds for $\left(\left(H_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that the $\mathcal{P}$-moments of $\left(H_t^N\right)_{t\geq 0}$ converges in probability to a non-random constant as N goes to infinity. Moreover, for any positive real t:

$$\mathcal{R}_A\left((H_t^N)_{N\in\mathbb{N}}\right) = e_{\boxplus}^{t\mathcal{R}_A(G)},$$

with:

$$\mathcal{R}_A(G) = \epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2].$$

Proof of Theorem 10.10. — We have seen that $\left(H_t^N\right)_{t\geq 0}$ is invariant by conjugation by $U(N,\mathbb{K})$. As an application of Theorems 10.2, 10.3 and 10.6, using the notations of Lemma 10.5, it remains to show that for any integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}\left[A_k(N)\right]$, converges up to any order of fluctuations, and that:

$$\mathcal{R}_A(G) = \epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2].$$

Indeed if so, by Theorem 10.6 the process $\left(H_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. Since $(1,2)$ and $[1,2]$ are

irreducible, $\mathcal{R}_A[G] \in \mathfrak{me}_{\boxplus}[A]$: by Theorem 10.3, the asymptotic $\mathcal{P}$-factorization holds for $\left(\left(H_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$. Using the fact that we are considering matrices which are in $\mathfrak{g}^{\epsilon}(N,\mathbb{K})$, this implies that the asymptotic $\mathcal{P}$-factorization also holds for $\left(\left(H_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0} \cup \left(\left(\overline{H_t^N}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$: by Theorem 2.2, the assertion on the convergence in probability of the $\mathcal{P}$-moments is proved. The remaining assertion is a consequence of Theorem 10.2.

By Lemma 10.5, $G_k^N = 0$ if $k \neq 2$ and $G_2^N = C_{\mathfrak{u}(N,\mathbb{K})}$. Recall the notation M_k^N defined in Definition 5.1 of [**15**]. Using Lemma 10.3 and Lemma 5.4 of [**15**], it is enough to show that:

$$(M_2^N)^{-1}\left(\frac{1}{N}\left(\epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2]\right)\right)$$

converges in $\mathbb{C}[A_2]$ and has an asymptotic development as an infinite power series in $\frac{1}{N}$. For any integer N :

$$(M_2^N)^{-1}\left(\frac{1}{N}\left(\epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2]\right)\right) = \epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2].$$

Thus for any positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}} \mathbb{C}\left[A_k(N)\right]$, converges up to any order of fluctuations, and using the discussion we just had:

$$\mathcal{R}_A[G] = \epsilon(1,2) + (2-\beta_{\mathbb{K}})[1,2].$$

This concludes the proof. □

When $\epsilon = 1$, it is well-known that the limit of the mean empirical eigenvalues distribution of H_1^N is the Wigner semicircular distribution.

Theorem 10.11. — *For any positive integer N, let $\left(H_t^N\right)_{t\geq 0}$ be a Brownian motion on $\mathfrak{g}^1(N,\mathbb{K})$. The mean empirical eigenvalues distribution of H_1^N converges in probability to the Wigner semicircle distribution:*

$$\mu_{sc} = \frac{1}{2\pi}\sqrt{4-|x|^2}\mathbb{1}_{[-2,2]}dx.$$

Proof. — Using the usual arguments we explained in the beginning of Section 2.1, namely Theorem 2.1, we only have to prove that for any positive integer k, $\frac{1}{N}\mathbb{E}\left[Tr\left(\left(H_1^{\mathfrak{g}^1(N,\mathbb{K})}\right)^k\right)\right]$ converges when N goes to infinity to:

$$\int_{\mathbb{R}} x^k \mu_{sc}(dx) = \#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\}.$$

Actually, one needs to prove some uniformity sub-Gaussian property for the mean empirical eigenvalues distribution, this can be proved using concentration of measure for the operator norm of a random matrix. We refer to the discussion in [**32**] at the beginning of Section 2.4.2.

Let us remark that, using the bijection between non-crossing pair-partitions of $\{1,...,k\}$ and involutions without fixed points in $[id,(1,...,k)]_{\mathfrak{S}_k}$, we see that:

$$\#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\} = \sum_{\sigma \text{ involution in } \mathfrak{S}_k, \#\mathsf{C}(\sigma)=\frac{k}{2}, \sigma\in[id,(1,...,k)]_{\mathfrak{S}_k}} 1.$$

Thus:

$$\#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\} = \Big(\mathcal{M}_{\mathfrak{S}}\left[e_{\boxplus}^{(1,2)}\right]\Big)_{(1,...,k)},$$

where the application $\mathcal{M}_{\mathfrak{S}}$ was defined in Definition 10.9 of [**15**]. Let A be equal to $\mathfrak{S}$ if $\mathbb{K}=\mathbb{C}$ or $\mathcal{B}$ if $\mathbb{K}=\mathbb{R}$. By Theorem 10.10, $\mathcal{R}_A\left[(H_1^N)_{N\in\mathbb{N}}\right] = e_{\boxplus}^{(1,2)+(2-\beta_{\mathbb{K}})[1,2]}$, thus:

$$\begin{aligned}\mathcal{M}_A\left[\left(H_1^N\right)_{N\in\mathbb{N}}\right] &= \mathcal{M}_A\left[\mathcal{R}_A\left[\left(H_1^N\right)_{N\in\mathbb{N}}\right]\right]\\ &= \mathcal{M}_A\left[e_{\boxplus}^{(1,2)+(2-\beta_{\mathbb{K}})[1,2]}\right].\end{aligned}$$

As an application of Proposition 10.4 of [**15**] and using the fact that $\left(e_{\boxplus}^{E}\right)_{|\mathfrak{E}[\mathfrak{S}]} = e_{\boxplus}^{E_{|\mathfrak{S}}}$, one gets, for any positive integer k:

$$\begin{aligned}\Big(\mathcal{M}_A\left[\left(H_1^N\right)_{N\in\mathbb{N}}\right]\Big)_{(1,...,k)} &= \Big(\mathcal{M}_{\mathfrak{S}}\left[\left(e_{\boxplus}^{(1,2)+(2-\beta_{\mathbb{K}})[1,2]}\right)_{|\mathfrak{E}[\mathfrak{S}]}\right]\Big)_{(1,...,k)}\\ &= \Big(\mathcal{M}_{\mathfrak{S}}\left[e_{\boxplus}^{(1,2)}\right]\Big)_{(1,...,k)}\\ &= \#\Big\{\text{Non-crossing pair-partitions of } \{1,...,k\}\Big\}\\ &= \int_{\mathbb{R}} x^k \mu_{sc}(dx),\end{aligned}$$

which is the equality we wanted to prove. □

10.3.5. Matricial Wick Formula. — In this section, we only write in a more usual way the equation stated in Theorem 10.10: one can recover a matricial Wick formula. In order to do so, we need the following definition. Let n be a positive integer.

Definition 10.12. — *A one-specie pairing of n is a partition of $\{1,...,n\}$ into pairs. A two-species pairing of n is a partition of $\{1,...,n\}$ into pairs, with a partition in two sets of these pairs: (T_m, W_m).*

We denote by $\mathcal{F}_n(1)$ the set of one-specie pairings of n and $\mathcal{F}_n(2)$ the set of two-species parings of n. By convention $\#W_m = 0$ for any one-specie pairing m.

Any one-specie pairing of n can be written as $\big\{\{i,j_i\}, i\in I\big\}$ where I is a subset of $\{1,...,n\}$ of size $n/2$. For example, the partition $\{\{1,4\},\{2,3\},\{5,6\}\}$ is a one-specie pairing of 6 and $\{\{1,4\},\{2,3\},\{5,6\}\}$ with the partition $T_m = \{\{1,4\},\{5,6\}\}$, $W_m = \{\{2,3\}\}$ is a two-species pairing of 6.

There is an obvious bijection between one-specie pairings of n and involutions in $\mathfrak{S}_n$, thus we will often consider a one-specie pairing as a permutation of $\{1,...,n\}$. We can also inject the set of two-species pairings of n in $\mathcal{B}_n$. Recall the definitions of the transposition (i,j) and the Weyl contraction $[i,j]$ stated in Definition 2.8 of [**15**].

Definition 10.13. — *Let* $m = \big(\{\{i, j_i\}, i \in I\}, T_m, W_m\big)$ *be a two-species pairing of* n*. We consider* $b_m \in \mathcal{B}_n$ *equal to:*

$$b_m = \prod_{i\in I, \{i,j_i\}\in T_m} (i, j_i) \prod_{i\in I, \{i,j_i\}\in W_m} [i, j_i].$$

The elements in the product commute, thus the order is not important.

Often we will consider a two-species pairing m of n either as a colored partition of $\{1, ..., n\}$ or as an element of $\mathcal{B}_n$.

Definition 10.14. — *Let* M *be a random matrix in* $\mathcal{M}_N(\mathbb{K})$*. It is a standard Gaussian* $\mathfrak{g}^\epsilon(N, \mathbb{K})$ *matrix if the equality in law holds:*

$$M = \sqrt{N} H_1,$$

where $(H_t)_{t\geq 0}$ *is a Brownian motion on* $\mathfrak{g}^\epsilon(N, K)$*.*

When $\epsilon = 1$ this is an equivalent definition for the G.O.E and G.U.E ensembles.

Theorem 10.12. — *For any integer* N*, let* M_N *be a standard Gaussian* $\mathfrak{g}^1(N, \mathbb{K})$ *random matrix. The mean empirical eigenvalues distribution of* $\frac{1}{\sqrt{N}} M_N$ *converges in probability to the Wigner semicircular distribution* μ_{sc} *as* N *goes to infinity.*

Besides, one has a matricial Wick theorem.

Theorem 10.13. — *Let* $(M_1, ..., M_k)$ *be a random vector of matrices such that there exist a* $k \times k'$ *complex matrix* A *and a* k'*-tuple of independent standard Gaussian* $\mathfrak{g}^\epsilon(N, \mathbb{K})$ *matrices* $(\tilde{M}_1, ..., \tilde{M}_{k'})$ *such that the following equality in law holds:*

$$(M_i)_{i=1}^k = \left(\sum_{j=1}^{k'} A_{i,j} \tilde{M}_j \right)_{i=1}^k,$$

or in a shorter way:

$${}^t(M_1, ..., M_k) = A.{}^t(\tilde{M}_1, ..., \tilde{M}_{k'}).$$

Let us define for any i, j *in* $\{1,...,k\}$*,* $C(M_i, M_j) = (A\,{}^tA)_{i,j}$*. Then:*

$$\mathbb{E}[M_1 \otimes ... \otimes M_k] = \sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})} \left(\epsilon^{\#T_m} \prod_{\{i,j\}\in m} C(M_i, M_j) \right) b_m,$$

where we have made a slight abuse of notation by writing b_m *instead of* $\rho_N^{\mathcal{B}_k}(b_m)$*.*

Proof. — First of all, the result holds if one considers only one matrix M. Indeed, if M is a standard Gaussian $\mathfrak{g}^\epsilon(N, \mathbb{K})$ matrix, then:

$$\mathbb{E}\left[M^{\otimes k}\right] = \sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})} \epsilon^{\#T_m} b_m.$$

Indeed, M has the same law than $\sqrt{N}H_1$ where $(H_t)_{t\geq 0}$ is a Brownian motion on $\mathfrak{g}^\epsilon(N,K)$. Yet, we have seen in the proof of Theorem 10.10 that for any positive integer k:

$$G_k = \frac{d}{dt}_{|t=0}\mathbb{E}[H_t^{\otimes k}] = \delta_{k=2}\left[M_2^N\left(\epsilon(1,2)+(2-\beta_{\mathbb{K}})[1,2]\right)\right].$$

Using Lemma 10.1, for any positive real t_0:

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}\left[H_t^{\otimes k}\right] = \sum_{I\subset\{1,\dots,k\},\#I=k-2}\mathcal{I}_I\left[\mathbb{E}\left[H_{t_0}^{\otimes(k-2)}\right],G_2\right].$$

Let $\left(E_t^k\right)_{k\in\mathbb{N},t\geq 0}$ be such that for any $k\in\mathbb{N}$, $(E_t^k)_{t\geq 0}\in\mathbb{C}[\mathcal{P}_k]^{\mathbb{R}^+}$ and such that $\left(E_t^k\right)_{k\in\mathbb{N},t\geq 0}$ is the solution of: $\forall k\in\mathbb{N}$:

$$\frac{d}{dt}_{|t=t_0}E_t^k = \sum_{I\subset\{0,\dots,k\},\#I=k-2}\mathcal{I}_I\left[E_t^{k-2},\epsilon(1,2)+(2-\beta_{\mathbb{K}})[1,2]\right], \tag{21}$$

with the initial condition: $\forall k\in\mathbb{N}, E_0^k = id_k$. Then for any positive integer k, for any positive real t,

$$\mathbb{E}\left[H_t^{\otimes k}\right] = \rho_N^{\mathcal{B}_k}\left[M_2^N\left(E_t^k\right)\right].$$

Thus, we only have to find a solution of Equation (21): this is given for any positive real t and any $k\in\mathbb{N}$ by:

$$E_t^k = \sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})}\epsilon^{\#T_m}t^{\frac{k}{2}}b_m.$$

Thus for any integer k:

$$\begin{aligned}\mathbb{E}\left[M^{\otimes k}\right] = N^{\frac{k}{2}}\mathbb{E}[H_1^{\otimes k}] &= N^{\frac{k}{2}}\rho_N^{\mathcal{B}_k}\left[M_2^N\left(\sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})}\epsilon^{\#T_m}b_m\right)\right]\\ &= \sum_{m\in\mathcal{F}_k(3-\beta_{\mathbb{K}})}\epsilon^{\#T_m}b_m.\end{aligned}$$

Now let $(M_1,\dots,M_k)$ be a random vector of matrices such that there exist a $k\times k'$ complex matrix A and a k'-tuple of independent standard Gaussian $\mathfrak{g}^\epsilon(N,\mathbb{K})$ matrices $(\tilde{M}_1,\dots,\tilde{M}_k)$ such that the following equality in law holds:

$${}^t(M_1,\dots,M_k) = A.{}^t(\tilde{M}_1,\dots,\tilde{M}_{k'}).$$

Let us define for any i,j in {1,...,k}, $C(M_i,M_j) = (A\,{}^tA)_{i,j}$. Then:

$$\begin{aligned}\mathbb{E}\left[M_1\otimes\dots\otimes M_k\right] &= \sum_{i_1,\dots,i_k=1}^{k'}A_1^{i_1}\dots A_k^{i_k}\mathbb{E}[\tilde{M}_{i_1}\otimes\dots\otimes\tilde{M}_{i_k}]\\ &= \sum_{\pi\in\mathsf{P}_k}\sum_{\mathsf{Ker}((i_1,\dots,i_k))=\pi}A_1^{i_1}\dots A_k^{i_k}\mathbb{E}[\tilde{M}_{i_1}\otimes\dots\otimes\tilde{M}_{i_k}].\end{aligned}$$

Using the independence of the $\left(\tilde{M}_i\right)_{i=1}^{k'}$, and using the result already proved for one matrix, one gets:

$$\begin{aligned}&\mathbb{E}\left[M_1 \otimes ... \otimes M_k\right]\\&= \sum_{\pi \in \mathsf{P}_k} \left(\sum_{\mathsf{Ker}((i_1,...,i_k))=\pi} A_1^{i_1} ... A_k^{i_k} \right) \sum_{m \in \mathcal{F}_k(3-\beta_{\mathbb{K}}), m \leq \pi} \epsilon^{\#T_m} b_m.\end{aligned}$$

We can go on the calculation:

$$\begin{aligned}&\mathbb{E}\left[M_1 \otimes ... \otimes M_k\right]\\&= \sum_{m \in \mathcal{F}_k(3-\beta_{\mathbb{K}})} \left(\sum_{\pi \in \mathsf{P}_k, m \leq \pi} \sum_{\mathsf{Ker}((i_1,...,i_k))=\pi} A_1^{i_1} ... A_k^{i_k} \right) \epsilon^{\#T_m} b_m,\end{aligned}$$

and the result follow from the easy equality which holds for any $m \in \mathcal{F}_k(3-\beta_{\mathbb{K}})$:

$$\begin{aligned}\sum_{\pi \in \mathsf{P}_k, m \leq \pi} \sum_{\mathsf{Ker}((i_1,...,i_k))=\pi} A_1^{i_1} ... A_k^{i_k} &= \prod_{\{i,j\} \in m} \sum_{l=1}^{k'} A_i^l A_j^l\\&= \prod_{\{i,j\} \in m} C(M_i, M_j).\end{aligned}$$

Hence the matricial Wick formula. □

We stated the result for the calculation of $\mathbb{E}[M_1 \otimes ... \otimes M_k]$, yet using Theorem 10.13, one can see that for any $l \in \{1, ..., k-1\}$, one can calculate the value of $\mathbb{E}\left[M_1 \otimes ... \otimes M_l \otimes M_l \otimes \overline{M_{l+1}} \otimes ... \otimes \overline{M_k}\right]$. One can use two ways to do it: one can either decompose any random matrices using its real part and imaginary part, and apply Theorem 10.13 for the real matrices, or one can use the fact that M^* is either equal to M or $-M$. Let us illustrate the second possibility: for example, let us consider $\mathbb{E}\left[M^{\otimes k} \otimes \overline{M}^{\otimes k}\right]$, M being a standard Gaussian $\mathfrak{g}^1(N, \mathbb{C})$. Using Theorem 10.13, we know that:

$$\mathbb{E}\left[M^{\otimes k} \otimes (M^*)^{\otimes k}\right] = \mathbb{E}\left[M^{\otimes k} \otimes M^{\otimes k}\right] = \sum_{m \in \mathcal{F}_k(1)} \epsilon^{\#T_m} b_m.$$

And thus:

$$\mathbb{E}\left[M^{\otimes k} \otimes \overline{M}^{\otimes k}\right] = S_k\left(\mathbb{E}[M^{\otimes k} \otimes (M^*)^{\otimes k}]\right) = \sum_{m \in \mathcal{F}_k(1)} \epsilon^{\#T_m} S_k\left(b_m\right),$$

where the operation S_k was defined in Definition 2.5.

10.3.6. Convergence of multiplicative Brownian motion. — Let us now state the results about convergence of multiplicative Brownian motions. Let A be equal to $\mathfrak{S}$ if $\mathbb{K} = \mathbb{C}$ or equal to $\mathcal{B}$ if $\mathbb{K} = \mathbb{R}$. For any integer N, let $\left(U_t^N\right)_{t \geq 0}$ be a Brownian motion on $U(N, \mathbb{K})$. The next theorem is inspired by the results of T. Lévy in **??**: we extend slightly his results to our setting.

Theorem 10.14. — *The process $\left(U_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. The family $\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial A and $\mathcal{P}$-non-commutative multiplicative Levy process up to any order of fluctuations.*

The asymptotic $\mathcal{P}$-factorization property holds for $\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$. For any $t\in\mathbb{R}^+$:

$$\mathcal{R}_A\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)=e_{\boxtimes}^{t\mathcal{R}_A(G)},$$

where, for any positive integer k,

$$\left(\mathcal{R}_A(G)\right)_k=-\frac{k}{2}id_k+\sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})\left[i,j\right]\right). \tag{22}$$

And for any $t_0\geq 0$, any positive integer k, and for any $p\in\mathcal{P}_k$:

$$\frac{d}{dt}_{|t=t_0}\mathbb{E}m_p(U_t,...,U_t)=\sum_{p_1\in A_k,p_1\leq p}\left(\mathcal{R}_A(G)\right)_{p_1}\mathbb{E}m_{t_{p_1\circ p}}(U_{t_0},...,U_{t_0}). \tag{23}$$

Let us suppose that $\mathbb{K}=\mathbb{C}$. Then the family $\left(U_t^N,\left(U_t^N\right)^*\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. Besides, the asymptotic $\mathcal{P}$-factorization property holds for $\left(\left(U_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}\cup\left(\left(\left(U_t^N\right)^*\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$. In particular, the normalized $\mathcal{P}$-moments of the family $\left(U_t^N,\left(U_t^N\right)^*\right)_{t\geq 0}$ converges in probability to the limit of their expectation as N goes to infinity.

Let us use the same notations as for Theorem 10.5. Then for any real $t_0\geq 0$, for any positive integer k and l, the following equality holds in $\mathbb{C}[\mathcal{P}_{k+l}(\infty,\infty)]$:

$$\frac{d}{dt}_{|t=t_0}\left(\mathcal{R}(U_t)\right)_{k,l}=\left(\mathcal{R}(G)\right)_{k,l}\left(\mathcal{R}(U_{t_0})\right)_{k,l},$$

where

$$\left(\mathcal{R}(G)\right)_{k,l}=-\frac{k+l}{2}id_{k+l}-\sum_{1\leq i<j\leq k+k'}(i,j)-\sum_{1\leq i\leq k<j\leq k+k'}\left([i,j]-(i,j)\right).$$

Moreover, for any positive integer k and l, any $p\in\mathcal{P}_{k+l}$ and any $t_0\geq 0$:

$$\frac{d}{dt}_{|t=t_0}m_p\left[\mathbb{E}\left(U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right)\right]=\sum_{p_1\in A_k,p_1\leq p}\left(\left(\mathcal{R}(G)\right)_{k,l}\right)_{p_1}m_{t_{p_1\circ p}}\left[\mathbb{E}\left(U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right)\right],$$

where $m_p\left[\mathbb{E}\left(U_t^{\otimes k}\otimes\overline{U_t}^{\otimes l}\right)\right]$ stands for the limit as N goes to infinity of the mean normalized moment $m_p\left[\mathbb{E}\left(U_t^{N\otimes k}\otimes\left(\overline{U_t^N}\right)^{\otimes l}\right)\right]$.

Proof of Theorem 10.14. — For any integer N, let $(U_t^N)_{t\geq 0}$ be a Brownian motion on $U(N,\mathbb{K})$. Let us prove first all the statements about $\left((U_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. We have seen that $\left(U_t^N\right)_{t\geq 0}$ is invariant by conjugation by $U(N,\mathbb{K})$. As an application of Theorems 10.6, 10.2 and 10.3, using the notations of Lemma 10.6, it remains to show that for any

positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}\left[A_k(N)\right]$, converges up to any order of fluctuations and that for any positive integer k,

$$(\mathcal{R}_A[G])_k = -\frac{k}{2}id_k + \sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right).$$

Indeed if so, by Theorem 10.6 the process $\left(U_t^N\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation up to any order of fluctuations as N tends to infinity. Since (i,j) and $[i,j]$ are weakly irreducible, $\mathcal{R}_A[G]\in\mathfrak{mc}_{\boxtimes}[A]$: by Theorem 10.3, the asymptotic $\mathcal{P}$-factorization holds for $\left((U_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. The remaining assertion is a consequence of Theorem 10.2.

Using Lemma 10.6, Lemma 10.3 of this article and Lemma 5.4 of [**15**], it is enough to show that for any positive integer k:

$$\left(M_k^N\right)^{-1}\left(\frac{kc_{\mathfrak{u}(N,\mathbb{K})}}{2}id_k + \frac{1}{N}\sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right)\right)$$

converges in $\mathbb{C}[A_k]$ and has an asymptotic development as an infinite power series in $\frac{1}{N}$. Using Lemma 10.4, for any positive integer N, this is equal to:

$$\frac{k}{2}\left(-1+\frac{2-\beta_{\mathbb{K}}}{N}\right)id_k + \sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right).$$

Thus for any positive integer k, $\left(G_k^N\right)_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}}\mathbb{C}[A_k(N)]$, converges up to any order of fluctuations, and using the discussion we just had:

$$\mathcal{R}_A(G) = \mathfrak{J}\Big(-\frac{1}{2}id_1 - (1,2) + (2-\beta_{\mathbb{K}})[1,2]\Big)$$

where $\mathfrak{J}$ was defined in Definition 10.6 of [**15**], or equivalently, for any integer k,

$$(\mathcal{R}_A(G))_k = -\frac{k}{2}id_k + \sum_{1\leq i<j\leq k}\left(-(i,j)+(2-\beta_{\mathbb{K}})[i,j]\right).$$

This finishes the first part of the theorem.

Let us suppose that $\mathbb{K}=\mathbb{C}$. We can prove the assertions about $\left(\left(U_t^N,(U_t^N)^*\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ and $\left(\left(U_t^N,\overline{U_t^N}\right)_{t\geq 0}\right)_{N\in\mathbb{N}}$ by using similar arguments to the ones we just used and using Lemma 10.7, Theorem 10.4, Theorem 10.5 and Theorem 2.2. □

One can also calculate for any positive real t, the limit of the moments of the mean empirical eigenvalues distribution of U_t^N. The first proof of the next theorem was given by Biane in [**8**], and then by Levy in [**24**]. The proof we give here is just a reformulation in our framework of the latter proof. For this, let us define, for any real $t\geq 0$ and any positive integer k,

$$m_k(t) = e^{-\frac{kt}{2}}\sum_{l=0}^{k-1}\frac{(-t)^l}{l!}k^{l-1}\binom{k}{l+1}.$$

Theorem 10.15. — *For any integer N, let $\left(U_t^N\right)_{t\geq 0}$ be a Brownian motion on $U(N,\mathbb{K})$. For any $t\geq 0$, the empirical eigenvalues distribution of U_t^N converges in probability to a measure ν_t as N goes to infinity and for any positive integer k:*

$$\int_{\mathbb{U}} z^k \nu_t(dz) = \int_{\mathbb{U}} z^{-k}\nu_t(dz) = m_k(t).$$

Thus for any integer $r\geq 1$, any integers $l_1,...,l_r$ in $\mathbb{Z}$, for any real $t\geq 0$,

$$\mathbb{E}\left[\frac{1}{N}Tr\left((U_t^N)^{l_1}\right)...\frac{1}{N}Tr\left((U_t^N)^{l_r}\right)\right] \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^{r} m_{l_i}(t).$$

Proof. — For any positive integer N, let $\left(U_t^{N,\mathbb{K}}\right)_{t\geq 0}$ be a Brownian motion on $U(N,\mathbb{K})$. Let $\mathcal{R}_A\left[G^{\mathbb{K}}\right]$ be the $\mathcal{R}_A$-functional of the family $\left(\left(G_k^{N,\mathbb{K}}\right)\right)_{k\in\mathbb{N}}$, where as usual,

$$G_k^{N,\mathbb{K}} = \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(U_t^{N,\mathbb{K}}\right)^{\otimes k}\right].$$

Using Theorem 10.14, $\left(\left(U_t^{N,\mathbb{K}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation and each normalized $\mathfrak{S}$-moment of $\left(\left(U_t^{N,\mathbb{K}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ converges in probability to the limit of their expectation. Using a similar argument to the one used in Theorem 2.1, for any positive real t, the empirical eigenvalues distribution of $U_t^{N,\mathbb{K}}$ converges in probability to a probability measure called $\nu_{\mathbb{K},t}$. Again using Theorem 10.14, the restriction on $\mathfrak{E}[\mathfrak{S}]$ of $\mathcal{R}_{\mathcal{B}}\left[G^{\mathbb{R}}\right]$ is equal to $\mathcal{R}_{\mathfrak{S}}\left[G^{\mathbb{C}}\right]$. Thus, by Theorem 10.8, $\left(\left(U_t^{N,\mathbb{R}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ and $\left(\left(U_t^{N,\mathbb{C}}\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ have the same $\mathfrak{S}$-law: in particular, for any positive real t,

$$\nu_{\mathbb{C},t} = \nu_{\mathbb{R},t}.$$

We denote for any positive real t, $\nu_t = \nu_{\mathbb{C},t}$. We have to show now that for any $k\geq 1$:

$$\int_{\mathbb{U}} z^k \nu_t(dz) = m_k(t).$$

For any integer $k\geq 1$, let us denote by $m_{\nu,k}(t)$ the moment $\int_{\mathbb{U}} z^k\nu_t(dz)$. Using the Equality (22) in Theorem 10.14, the family $(m_{\nu,k}(t))_{t\geq 0,k\in\mathbb{N}}$ satisfies the following system of differential equations:

$$\forall k\geq 1, \forall t_0\geq 0, \frac{d}{dt}_{|t=t_0} m_{\nu,k}(t) = -\frac{k}{2}m_{\nu,k}(t_0) - \frac{k}{2}\sum_{l=1}^{k-1} m_{\nu,l}(t_0)m_{\nu,k-l}(t_0).$$

In [**23**], Levy proved in Lemma 2.4 that $(m_k(t))_{t\geq 0,k\in\mathbb{N}}$ also satisfies the same system of linear differential equations with the same initial conditions. By unicity we get that:

$$\int_{\mathbb{U}} z^k \nu_t(dz) = m_k(t).$$

This allows us to conclude. □

10.4. Levy processes. —

10.4.1. The generator. — The main step in order to show the convergence of the Brownian motions in Section 10.3, was to compute the action of the generator at time $t = 0$ on the application $M \to M^{\otimes k}$. The Itô's formula allowed us to do so in Lemma 10.5 and Lemma 10.6. Fortunately, there exists an automatic way to compute the generator given by Theorem 31.5 in [**30**] and Hunt's Theorem, Theorem 1.1 in [**25**].

Theorem 10.16. — *Let E be a finite dimensional vector space of dimension d, let $(Y_i)_{i=1}^d$ be a basis of E. Let $(X_t)_{t\geq 0}$ be an additive Levy process in E. There exist:*

1. *$Y_0 \in E$,*
2. *a symmetric positive semidefinite matrix $(y_{i,j})_{i,j=1}^d$,*
3. *a Levy measure Π on E, that is a measure on E such that $\Pi(\{0\}) = 0$ and such that, if B is the ball of center 0 and radius 1 in E:*

$$\int_B \| x \|_E^2 \, \Pi(dx) \leq \infty \text{ and } \Pi(B^c) < \infty,$$

such that the generator G of $(X_t)_{t\geq 0}$ is given for any $f \in C_0^2(E)$ and any $y \in E$ by $Gf(y) = \frac{d}{dt}_{|t=0}\mathbb{E}[f(y + X_t)]$ which is equal to:

$$\partial_{X_0} f(y) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\partial_{Y_i}\partial_{Y_j} f(y) + \int_{\mathcal{H}_N} [f(y + x) - f(y) - 1\!\!1_B(x)\partial_x f(y)]\, \Pi(dx).$$

Conversely, every operator of this form is the generator of a unique Levy process $(X_t)_{t\geq 0}$.

Besides, let us suppose that E is equal to the Lie algebra $\mathfrak{g}$ of a compact Lie group G. Let H be a Lie subgroup of G. Let us suppose that Y_0, the operator $\sum_{i,j=1}^{N^2} y_{i,j}\partial_{Y_i}\partial_{Y_j}$ and Π are invariant by conjugation by any element of H, then the Levy process associated is invariant by conjugation by H.

Remark 10.2. — *One can change $1\!\!1_B$ in the form of the generator given by the last theorem by anything of the form $1\!\!1_V$ where V is a neighborhood of 0. This operation only changes the drift X_0. This remark will be important latter as we will work with a Levy measure which is compactly supported: it is then easier to suppose that $\mathsf{Supp}(\Pi) \subset B$.*

A similar result exists for compact Lie groups. Let G be a compact Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Let $A : G \to \mathfrak{g}$ be a smooth mapping such that:

$$A(Id) = 0 \text{ and } d_{Id} = id_{\mathfrak{g}}.$$

As explained by G. Cébron in [**10**] for the case of $U(N)$-valued Levy processes, one can use on $U(N, \mathbb{K})$ the mapping:

$$A : M \mapsto \frac{M - M^*}{2}.$$

From now on, we will denote this application $i\Im$, even when we are working on $O(N)$. Besides, it is also invariant by conjugation by $U(N, \mathbb{K})$: for any $U \in U(N, \mathbb{K})$, for any $M \in \mathcal{M}(N, \mathbb{K})$:

$$UA(M)U^{-1} = A(UMU^{-1}).$$

For any matrix M, we denote by $\Re(M)$ the matrix $\frac{M+M^*}{2}$. We recall also that any element X in the Lie algebra $\mathfrak{g}$ induces a right invariant vector field which is defined

for any g in $\mathfrak{g}$, by $X^r(g) = DR_g(X)$, with DR_g being the diffential map of the right multiplication operation $R_g : h \mapsto gh$. Hunt's theorem, see [**25**], allows to compute the generator of a Levy process on a compact Lie group.

Theorem 10.17. — *Let $(X_t)_{t\geq 0}$ be a Levy process on a compact Lie group G. Let d be the dimension of the Lie algebra $\mathfrak{g}$ of G. Let $(Y_i)_{i=1}^d$ be a basis of $\mathfrak{g}$. There exist:*

1. *$Y_0 \in \mathfrak{g}_N$,*
2. *a symmetric positive semidefinite matrix $(y_{i,j})_{i,j=1}^d$,*
3. *a Levy measure Π on G, that is a measure on G such that $\Pi(\{Id\}) = 0$ and for any neighborhood V of Id in G, we have:*

$$\int_V ||\, A(x)\,||_{\mathfrak{g}}^2\, \Pi(dx) \leq \infty \text{ and } \Pi(V^c) < \infty,$$

such that the generator G of $(X_t)_{t\geq 0}$ is given for any $f \in C^2(G)$ and any $h \in U_N$ by $Gf(h) = \frac{d}{dt}_{|t=0}\mathbb{E}[f(X_t h)]$ which is equal to:

$$Y_0^r f(h) + \frac{1}{2}\sum_{i,j=1}^d y_{i,j} Y_i^r Y_j^r f(h) + \int_G [f(gh) - f(h) - A(g)^r f(h)]\, \Pi(dg).$$

Conversely, every operator of this form is the generator of a unique Levy process $(X_t)_{t\geq 0}$.

Let H be a Lie subgroup of G. Let us suppose that Y_0, the measure Π and the operator $\sum_{i,j=1}^d y_{i,j} Y_i^l Y_j^l$ are invariant by conjugation by any element of H, then the Levy process associated is invariant by conjugation by H.

Remark 10.3. — *We will apply Theorem 10.16 and Theorem 10.17 to the case where E is one of the Lie algebra defined in Section 10.3.1 and where G is one of the Lie group defined in the same section. In Definition 10.8, we defined a scalar product on these Lie algebras: from now on, we will always assume that the basis $(Y_i)_{i=1}^d$ used in both theorems is an orthonormal basis for this scalar product.*

10.4.2. Free additive and multiplicative divisible measures. — One of our goal is to study the convergence of some natural Levy processes using the new setting we developed and to characterize the limit of the mean eigenvalues distribution. In order to do so, we need to introduce the well-known notions of free additive and multiplicative infinitely divisible measures. First, let us recall the notions of free additive and multiplicative convolutions.

We denote for any topological set K, $\mathcal{P}(K)$ the set of probability measure on K and $\mathcal{P}^c(K)$ the elements in $\mathcal{P}(K)$ which are compactly supported. Let $\mathbb{U}$ be the unit circle. Recall all the definitions in Section 10 of [**15**].

Definition 10.15. — *Let μ be in $\mathcal{P}^c(\mathbb{C})$. For any positive integer n, let $[(1,...,n)]_{\mathfrak{S}_n}$ be the conjugacy class of $(1,...,n)$ in $\mathfrak{S}_n$. Let $\mathcal{M}^{(i)}(\mu)$ be the element of $\mathfrak{E}^{(i)}[\mathfrak{S}]$ such that for any integer n, for any $\sigma \in [(1,...,n)]_{\mathfrak{S}_n}$, $\left(\mathcal{M}^{(i)}(\mu)\right)_\sigma = \int_{\mathbb{C}} z^n d\mu(z)$. The moment invariant of μ is:*

$$\mathcal{M}(\mu) = \mathfrak{M}\left(\mathcal{M}^{(i)}(\mu)\right).$$

The cumulant invariant of μ is:

$$\mathcal{R}(\mu) = \mathcal{R}_{\mathfrak{S}}\left[\mathcal{M}(\mu)\right].$$

The moment invariant and the cumulant invariant characterize the measures which are compactly supported either by the real line or by the unit circle, this is a well-known consequence of the Stone-Weierstrass's theorem.

Lemma 10.8. — *Let μ_1, μ_2 be two measures either both in $\mathcal{P}^c(\mathbb{R})$ or in $\mathcal{P}(\mathbb{U})$. If either $\mathcal{M}(\mu_1) = \mathcal{M}(\mu_2)$ or $\mathcal{R}(\mu_1) = \mathcal{R}(\mu_2)$ then $\mu_1 = \mu_2$.*

Recall the notion of strong approximation of a measure μ defined in Section 9.1. The following lemma is straightforward.

Lemma 10.9. — *Let $\mu \in \mathcal{P}^c(\mathbb{C})$. Let $(M_N)_{N\in\mathbb{N}}$ be an element of $L^{\infty^-} \otimes \mathcal{M}(\mathbb{C})$ which is a strong approximation of μ, then $\mathcal{M}_{\mathfrak{S}}[(M_N)_{N\in\mathbb{N}}] = \mathcal{M}[\mu]$.*

In order to define the free additive convolution, we need the following lemma.

Lemma 10.10. — *For any Hermitian matrix M of size N, we denote by $\lambda_1(M) \geq ... \geq \lambda_N(M)$ the eigenvalues of M. Let A and B be two Hermitian matrices, for any $i \in \{1, ..., N\}$:*

$$\lambda_N(A) + \lambda_N(B) \leq \lambda_i(A + B) \leq \lambda_1(A) + \lambda_1(B).$$

Now we can define the free additive convolution of two probability measures supported by the real line.

Proposition 10.1. — *Let μ and ν be in $\mathcal{P}^c(\mathbb{R})$. There exists a unique probability measure in $\mathcal{P}^c(\mathbb{R})$, denoted $\mu \boxplus \nu$, such that the equation:*

$$\mathcal{R}(\mu \boxplus \nu) = \mathcal{R}(\mu) \boxplus \mathcal{R}(\nu)$$

holds in $\mathfrak{E}[\mathfrak{S}]$. Moreover, the application $\boxplus : \mathcal{P}^c_{\mathbb{R}} \times \mathcal{P}^c_{\mathbb{R}} \to \mathcal{P}^c_{\mathbb{R}}$ can be extended by continuity from $\mathcal{P}^c_{\mathbb{R}}$ to $\mathcal{P}_{\mathbb{R}}$.

Proof. — We will only give a proof of the first assertion. The second result is a consequence of Proposition 4.13. of [**7**].

Let μ and ν be in $\mathcal{P}^c(\mathbb{R})$. Their moments are finite: let us consider two independent natural strong approximations of μ and ν, as defined in Proposition 9.1, respectively $\left(M^N_\mu\right)_{N\in\mathbb{N}}$ and $\left(M^N_\nu\right)_{N\in\mathbb{N}}$. For each positive integer N, let us consider U_N a matrix such that $\left(M^N_\mu, M^N_\nu, U_N\right)$ are independent, and U_N is Haar distributed on $U(N)$. Then for any integer N, M^N_μ is independent of $U_N M^N_\nu U^*_N$ which is invariant by conjugation by $U(N)$. By Theorem 7.10, $\left(M^N_\mu\right)_{N\in\mathbb{N}}$ and $\left(U_N M^N_\nu U^*_N\right)_{N\in\mathbb{N}}$ are asymptotically $\mathfrak{S}$-free. By Theorem 7.1,

$$M^N = M^N_\mu + U_N M^N_\nu U^*_N$$

converges in $\mathfrak{S}$-expectation when N goes to infinity. Besides, by Lemma 10.10, the mean eigenvalues distributions $\mathbb{E}\left[\eta_{M^N}\right]$ of M^N are uniformly compactly supported. By Theorem 2.1, there exists a measure that we denote $\mu \boxplus \nu$ such that:

$$\mathbb{E}\left[\eta_{M^N}\right] \underset{N\to\infty}{\longrightarrow} \mu \boxplus \nu.$$

By Theorem 7.9, we also get that:

$$\mathcal{R}_{\mathfrak{S}}\left(\left(M^N\right)_{N\in\mathbb{N}}\right) = \mathcal{R}_{\mathfrak{S}}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right) \boxplus \mathcal{R}_{\mathfrak{S}}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right).$$

Let us remark that $\left(M^N\right)_{N\in\mathbb{N}}$ is a strong approximation of $\mu \boxplus \nu$. Indeed, as $(M_\mu^N)_{N\in\mathbb{N}}$ and $(M_\nu^N)_{N\in\mathbb{N}}$ are strong approximations, $\mathcal{R}_{\mathfrak{S}}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right)$ and $\mathcal{R}_{\mathfrak{S}}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right)$ are multiplicative. By Theorem 10.1 of [**15**],

$$\mathcal{R}_{\mathfrak{S}}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right) \boxplus \mathcal{R}_{\mathfrak{S}}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right)$$

is also multiplicative. This implies that $\mathcal{R}_{\mathfrak{S}}\left(\left(M^N\right)_{N\in\mathbb{N}}\right)$ is multiplicative, and thus $\mathcal{M}_{\mathfrak{S}}\left(\left(M^N\right)_{N\in\mathbb{N}}\right)$ is multiplicative. Using Lemma 10.9, the measure $\mu \boxplus \nu$ satisfies:

$$\mathcal{R}(\mu \boxplus \nu) = \mathcal{R}(\mu) \boxplus \mathcal{R}(\nu),$$

which is the equality we wanted to prove. □

Proposition 10.2. — *Let μ and ν be in $\mathcal{P}(\mathbb{U})$. There exists a unique probability measure in $\mathcal{P}(\mathbb{U})$, denoted $\mu \boxtimes \nu$, such that the equation:*

$$\mathcal{R}(\mu \boxtimes \nu) = \mathcal{R}(\mu) \boxtimes \mathcal{R}(\nu)$$

holds in $\mathfrak{E}[\mathfrak{S}]$.

Proof. — The proof is similar to the proof of Proposition 10.1. Let μ and ν be in $\mathcal{P}(\mathbb{U})$. We can consider two independent natural strong matricial approximations of μ and ν, respectively $\left(M_\mu^N\right)_{N\in\mathbb{N}}$ and $\left(M_\nu^N\right)_{N\in\mathbb{N}}$. For each positive integer N, let us consider U_N a matrix such that $\left(M_\mu^N, M_\nu^N, U_N\right)$ are independent, and U_N is Haar distributed on $U(N)$. By the Theorems 7.1 and 7.10, the matrix:

$$M^N = M_\mu^N U_N M_\nu^N U_N^*$$

converges in $\mathfrak{S}$-expectation when N goes to infinity. For any integer N, M^N is a unitary matrix: the mean eigenvalues distribution $\mathbb{E}\left[\eta_{M^N}\right]$ of M^N is supported by $\mathbb{U}$. By Theorem 2.1, there exists a measure that we denote $\mu \boxtimes \nu$ such that:

$$\mathbb{E}\left[\eta_{M^N}\right] \underset{N\to\infty}{\longrightarrow} \mu \boxtimes \nu.$$

Besides, by Theorem 7.9, we also get that:

$$\mathcal{R}\left(\left(M^N\right)_{N\in\mathbb{N}}\right) = \mathcal{R}\left(\left(M_\mu^N\right)_{N\in\mathbb{N}}\right) \boxtimes \mathcal{R}\left(\left(M_\nu^N\right)_{N\in\mathbb{N}}\right),$$

in $\mathfrak{E}[\mathfrak{S}]$. Using similar arguments to those used in Proposition 10.1, $\left(M^N\right)_{N\in\mathbb{N}}$ is a strong approximation of $\mu \boxtimes \nu$. Using Lemma 10.9, the measure $\mu \boxtimes \nu$ satisfies:

$$\mathcal{R}(\mu \boxtimes \nu) = \mathcal{R}(\mu) \boxtimes \mathcal{R}(\nu),$$

which is the equality we wanted to prove. □

As one can do for the usual convolution, we can define the notion of free additive and multiplicative infinitely divisible measures.

Definition 10.16. — *Let μ be in $\mathcal{P}(\mathbb{R})$ and ν be in $\mathcal{P}(\mathbb{U})$. The measure μ is a free additive infinitely divisible measure if for any integer $n \geq 1$ there exists $\mu_{\frac{1}{n}} \in \mathcal{P}(\mathbb{R})$ such that $\mu = \mu_{\frac{1}{n}}^{\boxplus n}$. The measure ν is a free multiplicative infinitely divisible measure if for any integer $n \geq 1$ there exists $\nu_{\frac{1}{n}} \in \mathcal{P}(\mathbb{U})$ such that $\nu = \nu_{\frac{1}{n}}^{\boxtimes n}$.*

Let $\lambda_{\mathbb{U}}$ be the uniform probability measure on $\mathbb{U}$. Let μ be any measure in $\mathcal{P}(\mathbb{R})$ or $\mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$ which is a free additive (resp. multiplicative) infinitely divisible measure. There exists a continuous one parameter semi-group of measures $(\mu_t)_{t\geq 0}$ for the free additive (resp. multiplicative) convolution such that $\mu = \mu_1$ and $\mu_0 = \delta_0$ (resp. $\mu_0 = \delta_1$). This semi-group is unique in the additive case and in the multiplicative case there is a canonical way to associate a free multiplicative convolution semi-group. We will not explain in details how to construct these continuous one parameter semi-group of measures. Just let us explain briefly what is the canonical choice for the free multiplicative convolution.

Let ν be a free multiplicative infinitely divisible measure in $\mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$. From the explanations given in Section 2.6 of [**10**], there exist a unique $\omega \in \mathbb{U}$ and a unique measure ρ on $\mathbb{U}$ such that $\rho(\mathbb{U}) < \infty$, such that the S-transform of ν is given by:

$$S(z) = \omega \exp\left(\int_{\mathbb{U}} \frac{1+z+\zeta z}{1+z-\zeta z} d\rho(\zeta)\right).$$

Conversely, any function of the form given above is the S-transform of a unique free multiplicative infinitely divisible measure in $\mathcal{P}(\mathbb{U})$. We did not define in the article what was the S-transform but one can read [**10**] for an explanation. The only thing to be known for this article is that it characterizes the measures defined on $\mathbb{U}$ and it transforms the $\boxtimes$ operation in a multiplication in the space of formal power series. Let $\mathsf{arg}(\omega)$ be the argument of ω defined in $]-\pi, \pi]$. Let us define for any $t \geq 0$,

$$S_t(z) = e^{it\mathsf{arg}(\omega)} \exp\left(\int_{\mathbb{U}} \frac{1+z+\zeta z}{1+z-\zeta z} d(t\rho)(\zeta)\right),$$

then for any positive real t there exists ν_t such that the S-transform of ν_t is S_t. Yet, for any positive real t_1 and t_2, $S_{t_1} S_{t_2} = S_{t_1+t_2}$ thus $\nu_{t_1} \boxtimes \nu_{t_2} = \nu_{t_1+t_2}$. The family $(\nu_t)_{t\geq 0}$ is a one-parameter semi-group of measures for the free multiplicative convolution which is in fact continuous and which satisfies $\nu_1 = \nu$. We see here that one could have chosen any $\mathsf{arg}(\omega) + 2k\pi$, with $k \in \mathbb{Z}$, instead of $\mathsf{arg}(\omega)$ in the definition of S_t: this would have given a different one parameter continuous semi-group hence the non-unicity property stated above. Each time we will consider the canonical semi-group associated with ν, the reader will have to understand that we consider the one constructed just above with the use of $\mathsf{arg}(\omega)$.

Let μ be in $\mathcal{P}(\mathbb{R})$ and let ν be in $\mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$. Let us suppose from now on that μ is a free additive infinitely divisible measure and that ν is a free multiplicative infinitely divisible measure. Let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be the canonical continuous semi-group associated respectively with μ and ν. By definition, $(\mathcal{R}(\mu_t))_{t\geq 0}$ and $(\mathcal{R}(\nu_t))_{t\geq 0}$ are continuous one parameter semi-groups in respectively $(\mathfrak{E}[\mathfrak{S}], \boxplus)$ and $(\mathfrak{E}[\mathfrak{S}], \boxtimes)$ which begin at the

neutral element. Thus there exist $\mathcal{G}(\mu)$ and $\mathcal{LR}(\nu)$ in $\mathfrak{E}[\mathfrak{S}]$ such that:

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}(\mu_t) = \mathcal{G}(\mu) \boxplus \mathcal{R}(\mu_{t_0}), \tag{24}$$

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}(\nu_t) = \mathcal{LR}(\nu) \boxtimes \mathcal{R}(\nu_{t_0}), \tag{25}$$

hold in $\mathfrak{E}[\mathfrak{S}]$. The element $\mathcal{G}(\mu)$ is easy to understand.

Lemma 10.11. — *The element $\mathcal{G}(\mu)$ is the unique element in $\mathfrak{me}_{\boxplus}[\mathfrak{S}]$ such that, for any positive integer k, for any $p \in \mathfrak{S}_k$ which is irreducible:*

$$(\mathcal{G}(\mu))_p = (\mathcal{R}(\mu))_p .$$

Definition 10.17. — *The log-cumulant invariant of ν is $\mathcal{LR}(\nu)$.*

This definition is equivalent to the one given by G. Cébron in [**11**] and [**10**]. Let us remark that in some sense, the S-transform is the exponentiation in the algebra of formal series of the log-cumulant. One can introduce the S-transform as a way to send the $\boxtimes$ operation on the simpler multiplication of formal series.

Using Lemma 10.8, we get the following lemma.

Lemma 10.12. — *Let ν_1 and ν_2 be two free multiplicative infinitely divisible measures in $\mathcal{P}(\mathbb{U})$ distinct from $\lambda_{\mathbb{U}}$. If $\mathcal{LR}(\nu_1)$ is equal to $\mathcal{LR}(\nu_2)$, then $\nu_1 = \nu_2$.*

It remains to characterize the cumulant invariant of free additive infinitely divisible measures and the log-cumulant invariant of free multiplicative infinitely divisible measures. This is given by the two following theorems. We warn the reader that these theorems are only reformulations of well-known results about the R- and S-transforms of free infinitely divisible measures that one can find in [**4**] and [**6**].

Let us recall that a measure ρ on $\mathbb{R}$ is a Levy measure if $\rho(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(x^2, 1)\rho(dx) < \infty.$$

Let us also recall that any multiplicative element E of $\mathfrak{E}[\mathfrak{S}]$ is determined by its restriction on the irreducible partitions: this gives an element of $\mathfrak{E}^{(i)}[\mathfrak{S}]$. Let us suppose that E is invariant by conjugation by any permutation: we recall that this means that for any positive integer k and any $\sigma \in \mathfrak{S}_k$, the following equality holds in $\mathbb{C}[A_k(N)]$ for any positive integer N:

$$\sigma E_k \sigma^{-1} = E_k.$$

The restriction of E on the irreducible partitions is an element of $\left(\mathfrak{E}^{(i)}[\mathfrak{S}]\right)^{\mathfrak{S}}$. Yet, $\mathfrak{E}^{(i)}[\mathfrak{S}]$ is isomorphic to the affine space of formal series of the form $\mathbb{C}_1[[z]]$ with constant coefficient equal to 1. Let us suppose that for a complex number z, the evaluation of this formal series converges, then we denote this evaluation by $E(z)$.

Theorem 10.18. — *Let μ be a free additive infinitely divisible measure. For any complex number z such that $Im(z) < 0$, $\mathcal{R}(\mu)(z)$ is defined, and there exist $\eta \in \mathbb{R}$, $a \in \mathbb{R}^+$*

and ρ a Levy measure on $\mathbb{R}$ which are unique and such that:

$$\mathcal{R}(\mu)(z) = 1 + \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1-tz} - 1 - tz\mathbb{1}_{[-1,1]}(t) \right) \rho(dt). \tag{26}$$

Conversely, for any $\eta \in \mathbb{R}$, $a \in \mathbb{R}^+$ and any Levy measure ρ on $\mathbb{R}$, there exists a free additive infinitely divisible measure $\mu \in \mathcal{P}(\mathbb{R})$ such that for any complex number z such that $Im(z) < 0$, the equality (26) is satisfied. In particular, if the Levy measure of μ is compactly supported, then there exist $\eta \in \mathbb{R}$, $a \in \mathbb{R}^+$ and ρ a Levy measure on $\mathbb{R}$ which are unique and such that:

$$\mathcal{R}(\mu) = \mathcal{M}\left(1 + \sum_{n \in \mathbb{N}^*} \kappa_n z^n\right),$$

where we used the bijection explained in Proposition 10.1 *of* [**15**]*, and where:*

- $\kappa_1 = \eta$,
- $\kappa_2 = a + \int_{\mathbb{R}} x^2 \rho(dx)$,
- $\forall n \geq 3$, $\kappa_n = \int_{\mathbb{R}} x^n \rho(dx)$.

The last triplet (η, a, ρ) will be called the $\boxplus$-characteristic triplet of μ.

A similar theorem exists for the free multiplicative infinitely divisible measures. Recall that ν, a measure on $\mathbb{U}$, is a Levy measure if $\nu(\{1\}) = 0$ and

$$\int_{\mathbb{U}} (1 + \Re(\zeta)) d\nu(\zeta) < \infty,$$

where $\Re(\zeta)$ is the real part of ζ. Recall the definition of $\mathfrak{J}$ in Definition 10.6 of [**15**].

Theorem 10.19. — *Let η be a free multiplicative infinitely divisible measure. There exist $\omega \in \mathbb{U}$, $b \in \mathbb{R}^+$ and ν a Levy measure on $\mathbb{U}$ which are unique and such that:*

$$\mathcal{LR}(\nu) = \mathfrak{J}\left[(L\kappa_n)_{n \in \mathbb{N}^*}\right] \tag{27}$$

where:

- $L\kappa_1 = i\mathsf{arg}(\omega) - \frac{b}{2} + \int_{\mathbb{U}} (\Re(\zeta) - 1) d\nu(\zeta)$,
- $L\kappa_2 = -b + \int_{\mathbb{U}} (\zeta - 1)^2 d\nu(\zeta)$,
- $\forall n \geq 3$, $L\kappa_n = \int_U (\zeta - 1)^n d\nu(\zeta)$.

Conversely, for any element $\mathcal{LR}$ of the form given by (27), there exists a free multiplicative infinitely divisible measure $\mu \in \mathcal{P}(\mathbb{U})$ such that $\mathcal{LR}(\mu) = \mathcal{LR}$. The triplet (ω, b, ν) is called the $\boxtimes$-characteristic triplet of ν.

Proof. — One could give a proof by using the characterization of the S-transform of a multiplicative infinitely divisible measure. Yet, G. Cébron showed already in [**10**] the forthcoming Theorem 10.26. In the proof of Theorem 10.26, we compute the log-cumulant invariant of μ given by Equation (29). This allows to conclude. □

In the following, we will give a new proof of results found in [**5**], [**9**] and [**10**], in the setting we have developed.

Theorem 10.20. — *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a free additive infinitely divisible measure. Let us suppose that the Levy measure ρ of μ is compactly supported. Let $(\mu_t)_{t\geq 0}$ be the continuous semi-group associated with μ. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative additive Levy process such that:*

- *for any positive integer N, for any positive real t, X_t^N is hermitian,*
- *for any positive integer N, $(X_t^N)_{t\geq 0}$ is U-invariant,*
- *for any $t \geq 0$, $\mathcal{M}(\mu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

In [**5**] it is proved that one can remove the condition on the measure μ. The Theorem 10.20 is a consequence of Theorem 10.24. In [**10**], a similar result for free multiplicative infinitely divisible probability measures is proved.

Theorem 10.21. — *Let $\nu \in \mathcal{P}(\mathbb{U}) \setminus \{\lambda_{\mathbb{U}}\}$ be a free multiplicative infinitely divisible measure. Let $(\nu_t)_{t\geq 0}$ be the canonical continuous semi-group associated with ν. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative multiplicative Levy process such that:*

- *for any positive integer N, for any positive real t, X_t^N is unitary,*
- *for any positive integer N, $(X_t^N)_{t\geq 0}$ is U-invariant,*
- *for any $t \geq 0$: $\mathcal{M}(\nu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

This theorem is a consequence of Theorem 10.26. We will also extend these theorems to the real setting.

Theorem 10.22. — *Let us use the same notations than those in Theorem 10.20. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative additive Levy process such that:*

- *for any integer N, for any positive real t, X_t^N is symmetric,*
- *for any integer N, $(X_t^N)_{t\geq 0}$ is O-invariant,*
- *for any $t \geq 0$: $\mathcal{M}(\mu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

This theorem is a consequence of Theorem 10.25. In order to state a similar theorem for multiplicative semi-groups of measures, we need to define the notion of being even according to the conjugation.

Definition 10.18. — *Let ν be a measure on $\mathbb{U}$. It is even according to the conjugation if for any continuous function $f : \mathbb{U} \to \mathbb{R}$, one has:*

$$\int_{\mathbb{U}} f(\overline{z})d\nu(z) = \int_{\mathbb{U}} f(z)d\nu(z).$$

Theorem 10.23. — *Let us use the same notations than those in Theorem 10.21. Let (ω, b, η) be the characteristic triplet of ν. Let us suppose that η is even according to the conjugation and ω is equal to one. There exists $\left(\left(X_t^N\right)_{N\in\mathbb{N}}\right)_{t\geq 0}$ a matricial $\mathcal{P}$-non-commutative multiplicative Levy process such that:*

- *for any integer N, for any positive real t, X_t^N is orthogonal,*
- *for any integer N, $(X_t^N)_{t\geq 0}$ is O-invariant,*
- *for any $t \geq 0$: $\mathcal{M}(\nu_t) = \mathcal{M}_{\mathfrak{S}}[X_t]$.*

This theorem is a consequence of Theorem 10.28.

10.4.3. Strong approximation in $\mathfrak{h}(N,\mathbb{C})$. —

Theorem 10.24. — *Let (η, a, ρ) be the characteristic triplet of μ, a free additive multiplicative infinitely divisible probability measure on $\mathbb{R}$. Let $(\mu_t)_{t\geq 0}$ be the one-parameter semi-group for the free additive convolution associated with μ. Let us suppose that the measure ρ has a compact support. Let us define for any integer N:*

$$a_N = aId_{N^2},$$

$$\rho_N(f) = N\int_{\mathbb{R}}\int_{U(N)} f\left(g\begin{pmatrix} x & 0 & \cdots & 0\\ 0 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 0\end{pmatrix}g^*\right)dgd\rho(x).$$

For any positive integer N, let $(X_t^N)_{t\geq 0}$ be a Levy process on $\mathfrak{h}(N,\mathbb{C})$ with characteristic triplet $\left(\eta I_N, a_N, \rho_N\right)$. The process $(X_t^N)_{t\geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. The family $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Levy process and the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t\geq 0}$ converges in probability when N goes to infinity to the limit of their expectation. Besides, for any $t\geq 0$ and any $k\in\mathbb{N}$,

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(X_t^N\right)^k\right)\right]\underset{N\to\infty}{\longrightarrow}\int_{\mathbb{R}} z^k d\mu_t.$$

Proof. — For any positive integer N, let $(X_t^N)_{t\geq 0}$ be a Levy process on $\mathfrak{h}(N,\mathbb{C})$ with characteristic triplet $\left(\eta I_N, a_N, \rho_N\right)$ defined in the theorem. Using the last assertion of Theorem 10.16, for any positive integer N, $(X_t^N)_{t\geq 0}$ is a Levy process invariant by conjugation by $U(N)$.

Let us prove that for any positive integer k and N:

$$\left(G_k^N = \frac{d}{dt}_{t=0}\mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right]\right)_{N\in\mathbb{N}}$$

converges in $\mathfrak{S}$-moments, and let us compute $\mathcal{R}_{\mathfrak{S}}(G)$. Let us suppose, without loss of generality, that for any $x\in\mathsf{Supp}(\rho)$, the matrix:

$$\begin{pmatrix} x & 0 & \cdots & 0\\ 0 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 0\end{pmatrix}$$

is in the ball of center 0 and radius 1. We can compute the action of the generator using Theorem 10.16. Depending on the value of k, we have:

– if $k=1$:

$$\frac{d}{dt}_{|t=0}\mathbb{E}\left[X_t^N\right] = \eta I_N,$$

– if $k = 2$:

$$\frac{d}{dt}_{|t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes 2}\right] = \frac{a}{2} C_{\mathcal{H}_N} + \int_{\mathcal{H}_N} g^{\otimes 2} \rho_N(dg),$$

– if $k \geq 3$:

$$\frac{d}{dt}_{|t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right] = \int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg).$$

The convergence of G_1^N is obvious, and we have already understood the convergence of $\frac{a}{2} C_{\mathcal{H}_N}$ in Section 10.3. Let k be an integer greater than two, it remains to study the sequence $\left(\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\right)_{N \in \mathbb{N}}$. Let σ be a permutation in $\mathfrak{S}_k$, we have the following equalities:

$$m_\sigma\left(\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\right) = \frac{1}{Tr_N(\sigma)} Tr^k\left[\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\ {}^t\sigma\right] = \frac{N}{Tr_N(\sigma)} \int_{\mathbb{R}} x^k d\rho(x).$$

We recall that $Tr_N(\sigma) = N^{\mathsf{nc}(\sigma \vee id)}$ and $\mathsf{nc}(\sigma \vee id)$ is the number of cycles of σ, thus:

$$m_\sigma\left(\int_{\mathcal{H}_N} g^{\otimes k} \rho_N(dg)\right) \underset{N \to \infty}{\longrightarrow} \delta_{\sigma \in [(1,\ldots,k)]_{\mathfrak{S}_k}} \int_{\mathbb{R}} x^k d\rho(x), \tag{28}$$

where we recall that $[(1, ..., k)]_{\mathfrak{S}_k}$ is the conjugacy class of $(1, ..., k)$ in $\mathfrak{S}_k$. Thus for any $k \geq 1$, $\left(G_k^N\right)_{N \in \mathbb{N}}$ converges in $\mathfrak{S}$-moments.

We would like to compute $\mathcal{R}_{\mathfrak{S}}[G]$: we see in (28) that the moments of G_k^N concentrate on the k-cycles as N goes to infinity. It is easy to compute the limit of the coordinate numbers of G_k^N: using Theorem 4.1 in the article [**15**], we get:

$$\begin{aligned}
(\mathcal{R}[G])_{(1)} &= \lim_{N \to \infty} \kappa^{(1)}(G_1^N) = \eta, \\
(\mathcal{R}[G])_{(1,2)} &= \lim_{N \to \infty} \kappa^{(1,2)}(G_2^N) = \frac{a}{2} + \int_{\mathbb{R}} x^2 d\rho(dx), \\
\forall k \geq 3, (\mathcal{R}[G])_{(1,\ldots,k)} &= \lim_{N \to \infty} \kappa^{(1,\ldots,k)}(G_k^N) = \int_{\mathbb{R}} x^k d\rho(dx),
\end{aligned}$$

and for any partition p which is not irreducible, $(\mathcal{R}[G])_p = 0$.

By Theorem 10.1, the process $(X_t^N)_{t \geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. Besides the family $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Levy process. Using the calculation of $\mathcal{R}_{\mathfrak{S}}[G]$, and using Theorem 10.3, the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$. Finally, using Theorem 10.2, we know that for any positive real t_0:

$$\frac{d}{dt}_{|t=t_0} \mathcal{R}_{\mathfrak{S}}\left[X_t\right] = \mathcal{R}_{\mathfrak{S}}[G] \boxplus \mathcal{R}_{\mathfrak{S}}\left[X_{t_0}\right].$$

Yet, using Theorem 10.18, Lemma 10.11 and Equation (24), we see that $(\mathcal{R}(\mu_t))_{t \geq 0}$ satisfies the same differential equation with same initial conditions. Thus, for any $t \geq 0$, $\mathcal{R}_{\mathfrak{S}}[X_t] = \mathcal{R}(\mu_t)$. Which is equivalent to the fact that $\mathcal{M}_{\mathfrak{S}}[X_t] = \mathcal{M}(\mu_t)$. If one considers

the evaluation of this equality to the cycles $(1,...,k)$ with k being any positive integer, one gets that for any $t \geq 0$ and any $k \in \mathbb{N}$:

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} Tr\left(\left(X_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} z^k d\mu_t.$$

It remains to prove that the normalized $\mathcal{P}$-moments of $(X_t^N)_{t\geq 0}$ converges in probability when N goes to infinity to the limit of their expectation. Since for any non-negative real t and any positive integer N the random matrix X_t^N is Hermitian, the $\mathcal{P}$-convergence and the asymptotic $\mathcal{P}$-factorization of $(X_t^N)_{t\geq 0}$ implies the $\mathcal{P}$-convergence and the asymptotic $\mathcal{P}$-factorization of $(X_t^N)_{t\geq 0} \cup \left(\left(X_t^N\right)^*\right)_{t\geq 0}$. Using Theorem 2.2, we get that the normalized $\mathcal{P}$-moments of $(X_t^N)_{t\geq 0}$ converges in probability when N goes to infinity to the limit of their expectation. □

10.4.4. Convergence in $\mathfrak{s}_N$. —

Theorem 10.25. — *Let (η, a, ρ) be the characteristic triplet of μ, a free additive multiplicative infinitely divisible measure on $\mathbb{R}$. Let $(\mu_t)_{t\geq 0}$ be the one-parameter semi-group for the free additive convolution associated with μ. Let us suppose that the measure ρ has a compact support. Let us define for any integer N:*

$$a_N = aId_{\frac{N(N+1)}{2}},$$

$$\rho_N(f) = N \int_{\mathbb{R}} \int_{O(N)} f\left(g \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} {}^t g\right) dg d\rho(x).$$

For any positive integer N, let $(X_t^N)_{t\geq 0}$ be a Levy process on $\mathfrak{h}(N,\mathbb{R})$ with characteristic triplet $\left(\eta I_N, a_N, \rho_N\right)$. The process $(X_t^N)_{t\geq 0}$ converges in expectation in $\mathcal{P}$-moments as N goes to infinity. The family $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Levy process and the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that as N goes to infinity, the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t\geq 0}$ converge in probability to the limit of their expectation. Besides, for any $t \geq 0$ and any $k \in \mathbb{N}$,

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} Tr\left(\left(X_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} z^k d\mu_t.$$

Proof. — The proof is similar to the proof of Theorem 10.24. Let us show that for any positive integer k,

$$\left(G_k^N = \frac{d}{dt}_{|t=0} \mathbb{E}\left[(X_t^N)^{\otimes k}\right]\right)_{N\in\mathbb{N}}$$

converges in $\mathcal{B}$-moments. Yet, using Theorem 10.8 or Proposition 2.2, it is enough to show that for any positive integer k G_k^N converges in $\mathfrak{S}$-moments as N goes to infinity. As for the proof of Theorem 10.24:

– if $k = 1$:

$$\frac{d}{dt}_{t=0} \mathbb{E}\left[X_t^N\right] = \eta I_N,$$

– if $k = 2$:

$$\frac{d}{dt}_{t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes 2}\right] = \frac{a}{2} C_{\mathfrak{s}_N} + \int_{\mathfrak{s}_N} g^{\otimes 2} \rho_N(dg),$$

– if $k \geq 3$:

$$\frac{d}{dt}_{t=0} \mathbb{E}\left[\left(X_t^N\right)^{\otimes k}\right] = \int_{\mathfrak{s}_N} g^{\otimes k} \rho_N(dg).$$

We have already handled the convergence of ηI_N and $\frac{a}{2} C_{\mathfrak{s}_N}$ in Section 10.3. It remains to show that for any $k \geq 2$, $\left(\int_{\mathfrak{s}_N} g^{\otimes k} \rho_N(dg)\right)_{N \in \mathbb{N}}$ converges in $\mathfrak{S}$-moments. But this calculation has already been made in the proof of Theorem 10.24.

By Theorem 10.8 and 10.1, the process $(X_t^N)_{t \geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. The family $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ is a matricial $\mathcal{P}$-non-commutative additive Levy process.

Besides, for any positive integer k, for any $\sigma \in \mathfrak{S}_k$, the limit of $m_\sigma \left(\frac{d}{dt}_{|t=0} \mathbb{E}\left[(X_t^N)^{\otimes k}\right]\right)$ are equal in the Hermitian and symmetric case. Using Theorem 10.8, $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$ has the same $\mathfrak{S}$-law as the matricial $\mathcal{P}$-non-commutative additive Levy process defined in the proof of Theorem 10.24. This proves that for any $t \geq 0$ and any $k \in \mathbb{N}$,

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} Tr\left(\left(X_t^N\right)^k\right)\right] \underset{N \to \infty}{\longrightarrow} \int_{\mathbb{R}} z^k d\mu_t.$$

Moreover, since the asymptotic $\mathfrak{S}$-factorization holds in the Hermitian case, the asymptotic $\mathfrak{S}$-factorization property holds for $(X_t^N)_{N \in \mathbb{N}}$ for any $t \geq 0$. But for any positive real t, for any positive integer N, X_t^N is symmetric: using a similar argument as for Proposition 2.2, the asymptotic $\mathcal{B}$-factorization property holds for $(X_t^N)_{N \in \mathbb{N}}$. By Proposition 4.5, it satisfies also the asymptotic $\mathcal{P}$-factorization property, and using the fact that the increments are independent and G-invariant, we get that the asymptotic $\mathcal{P}$-factorization property holds for $\left((X_t^N)_{N \in \mathbb{N}}\right)_{t \geq 0}$. An application of Theorem 2.2 allows to state that the normalized $\mathcal{P}$-moments of $\left(X_t^N\right)_{t \geq 0}$ converge in probability to the limit of their expectation as N goes to infinity. □

10.4.5. Convergence in $U(N)$. — Let us introduce the model of G. Cébron explained in [**10**].

Theorem 10.26. — *Let (ω, b, ν) be the characteristic triplet associated with μ, a free multiplicative infinitely divisible measure on the circle $\mathbb{U}$. Let $(\mu_t)_{t \geq 0}$ be the one-parameter semi-group for the free multiplicative convolution associated with μ. Let us*

define:

$$b_N = bId_{N^2},$$
$$\nu_N(f) = N\int_{\mathbb{U}}\int_{U(N)} f\left(g\begin{pmatrix}\zeta & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}g^*\right)dgd\nu(\zeta).$$

For any positive integer N, let $(Y_t^N)_{t\geq 0}$ be a Levy process with characteristic triplet $(i\mathsf{arg}(\omega)I_N, b_N, \nu_N)$. The process $(Y_t^N)_{t\geq 0}$ converges in expectation in $\mathcal{P}$-moments as N goes to infinity. The family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative multiplicative Levy process. Besides the asymptotic $\mathcal{P}$-factorization property holds for the family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$. For any $t\geq 0$ and any $k\in\mathbb{N}$,

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(Y_t^N\right)^k\right)\right]\underset{N\to\infty}{\longrightarrow}\int_{\mathbb{U}} z^k d\mu_t.$$

Proof. — The proof follows exactly the one of Theorem 10.24. For any positive integer N, let $(Y_t^N)_{t\geq 0}$ be a $U(N)$-valued Levy process with characteristic triplet $(i\mathsf{arg}(\omega)I_N, b_N, \nu_N)$. Using the last assertion of Theorem 10.17, for any positive integer N, $(Y_t^N)_{t\geq 0}$ is a Levy process invariant by conjugation by $U(N)$. We compute the action of the generator at time $t=0$ on the application $U\mapsto U^{\otimes k}$. This computation follows the work of Cébron in Propositions 5.2 and 5.3 of [**10**]. Applying Theorem 10.17, for any positive integer k:

$$\begin{aligned}G_k^N &= \frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right]\\ &= ki\mathsf{arg}(w)Id^{\otimes k} + \frac{b}{2}kc_{\mathfrak{g}}Id^{\otimes k} + \frac{1}{2}\sum_{i,j=1}^{k}\mathcal{I}_{\{i,j\}}(C_{\mathfrak{g}^1(N,\mathbb{K})}, Id^{\otimes k-2})\\ &+ \int_{U(N)}\left(M^{\otimes k} - Id^{\otimes k} - \sum_{i=1}^{k}\mathcal{I}_{\{i\}}(i\Im(M), Id^{\otimes k-1})\right)\nu_N(dM).\end{aligned}$$

Using in the last term the fact that:

$$M^{\otimes k} = (M - Id + Id)^{\otimes k} = Id^{\otimes k} + \sum_{m=1}^{k}\sum_{1\leq i_1<i_2<\ldots<i_m\leq k}\mathcal{I}_{\{i_1,\ldots,i_m\}}\left[(M-Id)^{\otimes m}, Id^{\otimes k-m}\right]$$

and the fact that $M - Id - i\Im(M) = \Re(M) - Id$, one gets:

$$\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right] = ki\mathsf{arg}(w)Id^{\otimes k} + \sum_{i=1}^{k}\mathcal{I}_{\{i\}}\left[\int_{U(N)}(\Re(M)-Id)\nu_N(dM), Id^{\otimes k-1}\right]$$
$$+\frac{b}{2}kc_{\mathfrak{g}}Id^{\otimes k} + \frac{b}{2}\sum_{i,j=1}^{k}\mathcal{I}_{\{i,j\}}(C_{\mathfrak{g}^1(N,\mathbb{K})}, Id^{\otimes k-2})$$
$$+\sum_{2<m\leq l}\sum_{1\leq i_1<...<i_m\leq k}\mathcal{I}_{\{i_1,...,i_m\}}\left[\int_{U(N)}(M-Id)^{\otimes m}\nu_N(dM), Id^{\otimes k-m}\right],$$

result obtained in [**10**], Proposition 5.3. The proof now differs from the one of C. Cébron. We want to understand the limit of $\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right]$ when N tends to infinity. Using our work on the Brownian motion and on the Casimir element, it remains to understand the limit of the $\mathfrak{S}$-cumulants or of the $\mathfrak{S}$-moments of:

$$\left(A_N = \sum_{i=1}^{k}\mathcal{I}_{\{i\}}\left[\int_{U(N)}(\Re(M)-Id)\nu_N(dM), Id^{\otimes k-1}\right]\right)_{N\in\mathbb{N}}$$

and of

$$\left(B_N = \mathcal{I}_{\{i_1,...,i_m\}}\left[\int_{U(N)}(M-Id)^{\otimes m}\nu_N(dM), Id^{\otimes k-m}\right]\right)_{N\in\mathbb{N}}$$

for any $m \in \{2, ..., k\}$ and for any $1 \leq i_1 < ... < i_m \leq k$.

It is easier to work with the $\mathfrak{S}$-moments. Let σ be in $\mathfrak{S}_k$, for any positive integer N:

$$m_\sigma(A_N) = k\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta).$$

Using the cumulant-moment equation:

$$\lim_{N\to\infty}\kappa^\sigma(A_N) = \left(k\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta)\right)\mathbb{1}_{\sigma=id_k},$$

or, written in a different way:

$$\lim_{N\to\infty}\left(M_N^k\right)^{-1}(A_N) = \left(k\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta)\right)id_k,$$

where we recall that M_N^k is the application defined in Definition 5.1 of [**15**].

Let m be a positive integer. Let us understand the sequence $(B_N)_{N\in\mathbb{N}}$. For this, it is enough to understand the sequence:

$$\left(\tilde{B}_N = \int_{U(N)}(M-Id)^{\otimes m}\nu_N(dM)\right)_{N\in\mathbb{N}}$$

For any $\sigma \in \mathfrak{S}_m$:

$$m_\sigma\left(\tilde{B}_N\right) = \frac{N}{N^{\mathsf{nc}(\sigma\vee id)}}\int_{\mathbb{U}}(\zeta-1)^m d\nu(\zeta).$$

Thus, as N goes to infinity, for any $\sigma \in \mathfrak{S}_k$:

$$m_\sigma\left(\tilde{B}_N\right) \underset{N\to\infty}{\longrightarrow} \delta_{\sigma\in[(1,\dots,m)]_{\mathfrak{S}_m}} \int_{\mathbb{U}} (\zeta-1)^m d\nu(\zeta),$$

where we recall that $[(1,\dots,m)]_{\mathfrak{S}_m}$ is the conjugacy class in $\mathfrak{S}_m$ of the cycle $(1,\dots,m)$. We can compute the limit of the coordinate numbers of $\tilde{B}_N$ using Theorem 4.1 in the article [**15**]:

$$\kappa^\sigma\left(\tilde{B}_N\right) \underset{N\to\infty}{\longrightarrow} \delta_{\sigma\in[(1,\dots,m)]_{\mathfrak{S}_m}} \int_{\mathbb{U}} (\zeta-1)^m d\nu(\zeta).$$

Hence:

$$\lim_{N\to\infty}\left(M_N^k\right)^{-1}(\tilde{B}_N) = \sum_{\sigma\in[(1,\dots,m)]_{\mathfrak{S}_m}} \left(\int_{\mathbb{U}} (\zeta-1)^m d\nu(\zeta)\right)\sigma.$$

Using all the results we just had, we can conclude that

$$\left(M_N^k\right)^{-1}\left[\frac{d}{dt}_{|t=0}\mathbb{E}\left[\left(Y_t^N\right)^{\otimes k}\right]\right]$$

converges as N goes to infinity and its limit is equal to:

$$k\left(i\mathsf{arg}(\omega)-\frac{b}{2}+\int_{\mathbb{U}}(\Re(\zeta)-1)d\nu(\zeta)\right)Id_k - b\sum_{1\le i<j\le k}(i,j) + \sum_{2<m\le k}\sum_{c\in\mathcal{C}_m}\left(\int_{\mathbb{U}}(\zeta-1)^m d\nu(\zeta)\right)c, \tag{29}$$

where $\mathcal{C}_m$ is the set of permutations in $\mathfrak{S}_k$ which have only one non-trivial cycle which is of length m.

By Theorem 10.1, the process $\left(Y_t^N\right)_{t\ge0}$ converges in $\mathcal{P}$-expectation as N goes to infinity. Besides the family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\ge0}$ is a matricial $\mathcal{P}$-non-commutative additive Levy process. Using the calculations we just did and using Theorem 10.3, since any $c\in\mathcal{C}_m$ is weakly irreducible, the asymptotic $\mathcal{P}$-factorization property holds for $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\ge0}$. Finally, using Theorem 10.2, we know that for any positive real t_0:

$$\frac{d}{dt}_{|t=t_0}\mathcal{R}_{\mathfrak{S}}\left[Y_t\right] = \mathcal{R}_{\mathfrak{S}}[G]\boxtimes\mathcal{R}_{\mathfrak{S}}\left[X_{t_0}\right].$$

Yet, using Theorem 10.19 and Equation (25), we see that $(\mathcal{R}(\mu_t))_{t\ge0}$ satisfies the same differential equation with same initial conditions. Thus, for any $t\ge0$, $\mathcal{R}_{\mathfrak{S}}[X_t]=\mathcal{R}(\mu_t)$. Which is equivalent to the fact that $\mathcal{M}_{\mathfrak{S}}[X_t]=\mathcal{M}(\mu_t)$. If one considers the evaluation of this equality on the cycles $(1,\dots,k)$ with k being any positive integer, one gets that for any $t\ge0$ and any $k\in\mathbb{N}$:

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(Y_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{U}} z^k d\mu_t.$$

This concludes the proof. □

In the theorem we just proved, we only proved the $\mathcal{P}$-asymptotic factorization for $(Y_t^N)_{t\geq 0}$, yet it would be interesting to have the convergence in $\mathcal{P}$-expectation and the asymptotic $\mathcal{P}$-asymptotic factorization of $\left(Y_t^N, \left(Y_t^N\right)^*\right)_{t\geq 0}$ as N goes to infinity. Indeed, this would allow to improve the convergence in expectation of the normalized moments into a convergence in probability. In [**10**], G. Cébron used some concentration of measure arguments in order to prove the almost sure convergence of the empirical measure. The goal of the following discussion is to show that actually, in our setting, no more work as to be done in order to get the weaker result of convergence in probability.

Let k and k' be two positive integers. In order to prove that the normalized $\mathcal{P}$-moments of $\left(Y_t^N\right)_{t\geq 0}$ converge in probability to the limit of their expectation, we need to understand $\left(Y_t^N, \left(Y_t^N\right)^*\right)_{t\geq 0}$. Yet it is easier to understand $\left(Y_t^N, \overline{Y_t^N}\right)_{t\geq 0}$ since $\mathbb{E}\left[(Y_t^N)^{\otimes k} \otimes (\overline{Y_t^N})^{\otimes k'}\right]$ is a semi-group. Let us define:

$$G_{k,k'} = \frac{d}{dt}_{|t=0} \mathbb{E}\left[(Y_t^N)^{\otimes k} \otimes (\overline{Y_t^N})^{\otimes k'}\right].$$

Using Theorem 10.17, we can compute $G_{k,k'}$ and we can see that it is composed of three parts: a drift and Brownian parts which were already studied in Section 10.3, and a third part which remains to be understood:

$$\int_{U(N,\mathbb{C})} \Bigg(M^{\otimes k} \otimes \overline{M}^{\otimes k'} - Id^{\otimes k+k'} - \sum_{i=1}^{k} \mathcal{I}_{\{i\}}(i\Im(M), Id^{\otimes k+k'-1}) \\ + \sum_{i=k+1}^{k+k'} \mathcal{I}_{(i)}(i^t\Im(M), Id^{\otimes k+k'-1}) \Bigg) \nu_N(dM),$$

where we used the fact that $\overline{i\Im(M)} = -^t(i\Im(M))$. Using the same argument as the one in the proof of Theorem 10.26, we know that $M^{\otimes k} \otimes \overline{M}^{\otimes k'} - Id^{\otimes k+k'}$ is equal to:

$$\sum_{m=1,l=1}^{k,k'} \sum_{1\leq i_1<\ldots<i_m\leq k<j_1\leq\ldots\leq j_l\leq k+k'} \mathcal{I}_{\{i_1,\ldots,i_m,j_1,\ldots,j_l\}} \\ \left[(M-Id)^{\otimes m} \otimes (\overline{M}-Id)^{\otimes l}, Id^{\otimes k+k'-m-l}\right].$$

Let us remark also that $\overline{M} + {}^t(i\Im(M)) = {}^t\Re(M)$. As for the proof of Theorem 10.26, it remains to understand the limit of the $\mathcal{B}$-moments or of the $\mathcal{B}$-cumulants of:

$$\int_{U(N,\mathbb{C})} (\Re(M) - Id)\nu_N(dM), \int_{U(N,\mathbb{C})} ({}^t\Re(M) - Id)\nu_N(dM), \\ \int_{U(N,\mathbb{C})} (M-Id)^{\otimes m} \otimes \left(\overline{M}-Id\right)^{\otimes l} \nu_N(dM),$$

for any $(m,l) \in \{1,...,k\}\times\{1,...,k'\}$. The two first sequences are easy to understand, only the last sequence is interesting, let (m,l) in $\{1,...,k\}\times\{1,...,k'\}$ and let us understand:

$$\left(C_N = \int_{U(N,\mathbb{C})} (M-Id)^{\otimes m} \otimes \left(\overline{M}-Id\right)^{\otimes l} \nu_N(dM)\right)_{N\in\mathbb{N}}.$$

For any positive integer N, C_N commutes with the tensor action of $O(N)$: this is an element of $\mathbb{C}[\mathcal{B}_{m+l}(N)]$. Let us recall the operation S_m defined in Definition 2.5. The element $S_m(C_N)$ is equal to:

$$\int_{U(N)} (M - Id)^{\otimes m} \otimes (M^* - Id)^{\otimes l} \nu_N(dM)$$

which commutes with the tensor action of $U(N)$: it is an element of $\mathbb{C}[\mathfrak{S}_{m+l}(N)]$, and the convergence in $\mathfrak{S}$-moments of $S_m(C_N)$ implies the convergence in $\mathbb{C}[\mathcal{B}_{m+l}(N)]$-moments of $(C_N)_{N\in\mathbb{N}}$. Yet, for any $\sigma \in \mathfrak{S}_{m+l}$:

$$\lim_{N\to\infty} m_\sigma(S_m(C_N)) = \delta_{[(1,...,m+l)]_{\mathfrak{S}_{m+l}}}(\sigma) \int_{\mathbb{U}} (\zeta - 1)^m (\overline{\zeta} - 1)^l d\nu(\zeta).$$

where we remind that $[(1, ..., m + l)]_{\mathfrak{S}_{m+l}}$ is the conjugacy class in $\mathfrak{S}_{m+l}$ of the $(m + l)$-cycle $(1, ..., m + l)$. Using the cumulant-moment relation, we get that for any $\sigma \in \mathfrak{S}_{m+l}$:

$$\lim_{N\to\infty} \kappa^\sigma(S_m(C_N)) = \delta_{[(1,...,m+l)]_{\mathfrak{S}_{m+l}}}(\sigma) \int_{\mathbb{U}} (\zeta - 1)^m (\overline{\zeta} - 1)^l d\nu(\zeta),$$

Hence:

$$\lim_{N\to\infty} \left(M_N^{m+l}\right)^{-1}(C_N) = \sum_{\sigma\in[(1,...,m+l)]_{\mathfrak{S}_{m+l}}} \left(\int_{\mathbb{U}} (\zeta - 1)^m (\overline{\zeta} - 1)^l d\nu(\zeta)\right) S_m(\sigma).$$

At the end, we get that $(G^N_{k,k'})_{N\in\mathbb{N}}$, seen as an element of $\prod_{N\in\mathbb{N}} \mathbb{C}[\mathcal{B}_{k+k'}(N)]$ converges as N goes to infinity, and $(G^N_{k,k'})_{k,k',N\in\mathbb{N}}$ weakly condensates.

Using Theorem 10.4 and 10.5, this implies that the family $\left(Y_t^N, (Y_t^N)^*\right)_{t\geq 0}$ converges in $\mathcal{P}$-expectation as N goes to infinity and the $\mathcal{P}$-asymptotic factorization property holds for this family. In particular for any positive integer m, any m-tuple $t_1, ..., t_m$ of non negative reals, for any m-tuple $(k_1, ..., k_m) \in \mathbb{Z}^m$:

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^m \frac{1}{N} Tr\left((Y_{t_i}^N)^{k_i}\right)\right] = \prod_{i=1}^m \int_{\mathbb{U}} z^{k_i} d\mu_t.$$

Besides, this shows that for any positive integer k, any k-tuple $t_1, ..., t_k$ of non negative reals, for any k-tuple $(l_1, ..., l_k) \in \mathbb{Z}^k$ and any $p \in \mathcal{P}_k$, $m_p\left(\bigotimes_{i=1}^k \left(Y_{t_i}^N\right)^{l_i}\right)$ converges in probability as N goes to infinity. This discussion allows to state the following theorem.

Theorem 10.27. — *For any positive integer N, let $\left(Y_t^N\right)_{t\geq 0}$ be the Levy process defined in Theorem 10.26. As N goes to infinity, the family $\left(\left(Y_t^N\right)_{t\geq 0}, \left((Y_t^N)^*\right)_{t\geq 0}\right)$ converges in expectation in $\mathcal{P}$-moments and satisfies the $\mathcal{P}$-factorization property.*

In particular the family $\left(\left(Y_t^N\right)_{t\geq 0}, \left((Y_t^N)^*\right)_{t\geq 0}\right)$ converges in probability in $\mathcal{P}$-moments. This means that for any $p \in \mathcal{P}$, for any positive integer k, for any k-tuple $(M_1, ..., M_k)$ of elements of $\mathcal{A}\left(\left(\left(Y_t^N\right)_{t\geq 0}, \left((Y_t^N)^*\right)_{t\geq 0}\right)\right)$,

$$m_p\left(M_1^N \otimes ... \otimes M_k^N\right)$$

converges in probability as N goes to infinity.

10.4.6. Convergence in $O(N)$. —

Theorem 10.28. — *Let μ be a free multiplicative infinitely divisible measure on the circle $\mathbb{U}$. Let (ω, b, ν) be the characteristic triplet of μ. Let $(\mu_t)_{t\geq 0}$ be the one-parameter semi-group for the free multiplicative convolution associated with μ. Let us suppose that ν is even according to the conjugation and that ω is equal to 1. Let us define:*

$$b_N = bId_{\frac{N(N-1)}{2}},$$

$$\nu_N(f) = N\int_{[-\pi,\pi]}\int_{O(N)} f\left(g\begin{pmatrix} cos\theta & -sin\theta & 0 & \cdots & 0 \\ sin\theta & cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}{}^t g\right) dgd\nu(\theta),$$

where in the last equality, we considered ν as a measure on $[-\pi, \pi]$ instead of a measure on $\mathbb{U}$.

For any positive integer N, let $(Y_t^N)_{t\geq 0}$ be a Levy process in $O(N)$ with characteristic triplet $(0, b_N, \nu_N)$. The process $(Y_t^N)_{t\geq 0}$ converges in expectation in $\mathcal{P}$-moments as N goes to infinity and the family $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$ is a matricial $\mathcal{P}$-non-commutative multiplicative Levy process.

Besides, the asymptotic $\mathcal{P}$-factorization property holds for $\left((Y_t^N)_{N\in\mathbb{N}}\right)_{t\geq 0}$: this implies that the normalized $\mathcal{P}$-moments of $\left(Y_t^N\right)_{t\geq 0}$ converge in probability to the limit of their expectation as N goes to infinity. Moreover for any $t \geq 0$ and any $k \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\frac{1}{N}Tr\left(\left(Y_t^N\right)^k\right)\right] \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{U}} z^k d\mu_t.$$

Proof. — The proof is the same as the proof of Theorem 10.26 and uses the same arguments as for Theorem 10.25. □

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