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Olivier Bernardi, Mireille Bousquet-Mélou

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# COUNTING COLOURED PLANAR MAPS: DIFFERENTIAL EQUATIONS

OLIVIER BERNARDI AND MIREILLE BOUSQUET-MÉLOU

**ABSTRACT.** We address the enumeration of  $q$ -coloured planar maps counted by the number of edges and the number of monochromatic edges. We prove that the associated generating function is *differentially algebraic*, that is, satisfies a non-trivial polynomial differential equation with respect to the edge variable. We give explicitly a differential system that characterizes this series. We then prove a similar result for planar triangulations, thus generalizing a result of Tutte dealing with their proper  $q$ -colourings. In statistical physics terms, we solve the  $q$ -state Potts model on random planar lattices.

This work follows a first paper by the same authors, where the generating function was proved to be algebraic for certain values of  $q$ , including  $q = 1, 2$  and  $3$ . It is known to be transcendental in general. In contrast, our differential system holds for an indeterminate  $q$ .

For certain special cases of combinatorial interest (four colours; proper  $q$ -colourings; maps equipped with a spanning forest), we derive from this system, in the case of triangulations, an explicit differential equation of order 2 defining the generating function. For general planar maps, we also obtain a differential equation of order 3 for the four-colour case and for the self-dual Potts model.

## 1. INTRODUCTION

A planar map is a connected planar graph, given with one of its proper embeddings in the sphere, taken up to continuous deformation (Figure 1). The enumeration of planar maps is a combinatorial problem that has attracted a lot of interest since the sixties, in connection with graph theory [41, 26], algebra [28, 31], theoretical physics [3, 16, 21], and computational geometry [18, 37]. Several important approaches have been developed: recursive [40], bijective [36], using matrix integrals [16], or using connections with the characters of the symmetric group [30]. We only give very few references, as a complete bibliography would take dozens of pages.

From the combinatorial and physical point of view, it is natural to count planar maps equipped with an additional structure: for instance a spanning tree [34], a proper colouring [45, 50], an independent set of vertices [11, 13, 15], a configuration of the Ising or Potts model [32, 23], a self-avoiding walk [22]. (Again, we give very few of the relevant references.) The first result of this nature probably dates back to 1967 with Mullin's enumeration of planar maps equipped with a spanning tree [34]. The second attempt is due to Tutte, who, in the early seventies, started to study maps — more precisely, triangulations — equipped with a proper colouring [45]. In the decade that followed, he devoted at least eight other papers to this problem [43, 42, 44, 46, 47, 48, 49, 50]. His work culminated in 1982, when he proved that the series  $H(w)$  counting  $q$ -coloured rooted triangulations by vertices satisfies a polynomial differential equation [49, 50]:

$$2q^2(1-q)w + (qw + 10H - 6wH')H'' + q(4-q)(20H - 18wH' + 9w^2H'') = 0. \quad (1)$$

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We say that  $H(w)$  is *differentially algebraic*. Equivalently, the number  $h_n$  of rooted triangulations with  $n$  vertices satisfies the following simple recurrence relation:

$$q(n+1)(n+2)h_{n+2} = q(q-4)(3n-1)(3n-2)h_{n+1} + 2 \sum_{i=1}^n i(i+1)(3n-3i+1)h_{i+1}h_{n+2-i},$$

with the initial condition  $h_2 = q(q-1)$ . For instance,  $h_3 = q(q-1)(q-2)$  is the number of proper  $q$ -colourings of a triangle. To date, this recursion remains entirely mysterious, and Tutte's *tour de force* has remained isolated.

Let us be more precise about the content of this *tour de force*. Tutte started from a generating function  $G(w; x, y) \equiv G(x, y)$  which counts a larger family of  $q$ -coloured maps according to three parameters (as before,  $w$  counts vertices). The above series  $H(w)$  is  $G(w; 1, 0)$ . Using the deletion/contraction properties of the chromatic polynomial, he easily established the following functional equation:

$$G(x, y) = xq(q-1)w^2 + \frac{xy}{qw}G(1, y)G(x, y) - x^2yw \frac{G(x, y) - G(1, y)}{x-1} + x \frac{G(x, y) - G(x, 0)}{y}. \quad (2)$$

Observe that the divided differences (or discrete derivatives)

$$\frac{G(x, y) - G(1, y)}{x-1} \quad \text{and} \quad \frac{G(x, y) - G(x, 0)}{y}$$

prevent us from simply specializing  $x$  to 1 and  $y$  to 0 to obtain an equation for  $H = G(1, 0)$ . In fact, the variables  $x$  and  $y$ , called nowadays *catalytic variables* [54, 11], are essential to write this equation, even though they are not very important combinatorially. The whole challenge is to determine the nature of the series  $H = G(1, 0)$ : is it algebraic, as the generating function of many classes of planar maps? Is it D-finite, that is, a solution of a linear differential equation, as the generating function of maps equipped with a spanning tree [34]? Is it at least differentially algebraic? Tutte answered the latter question positively by deriving from the functional equation (2) the differential equation (1) satisfied by  $H$ .

It is known that the solutions of polynomial equations with *one* catalytic variable are systematically algebraic [11]. Such equations commonly occur in the enumeration of families of planar maps with no additional structure. Moreover, in the past decade, progress has been made on *linear* equations with *two* catalytic variables, which one typically encounters when counting plane lattice walks confined to a quadrant. In this context, it is precisely understood when the generating function is D-finite (see e.g. [7, 12, 33] and references therein). Hence the key difficulty with Tutte's equation (2) is the occurrence of two catalytic variables, combined with non-linearity.

A few years ago, we started to resurrect Tutte's technique in order to solve a more general problem: *the Potts model on planar maps*. In combinatorial terms, this means that we count all  $q$ -colourings, not necessarily proper, but with a weight  $\nu^m$ , where  $\nu$  is an indeterminate and  $m$  is the number of *monochromatic* edges (edges whose endpoints have the same colour). The case  $\nu = 0$  is thus the problem solved by Tutte. Moreover, we do not only study degree-constrained maps (triangulations) as Tutte did, but also general planar maps (counted by edges and vertices).

In a first paper on this topic [2], we wrote the counterpart of (2) for each of these two problems. Then, we performed a first step, by establishing an equation with only *one* catalytic variable,  $y$ . But this equation holds only for values of  $q$  of the form  $2 + 2\cos(j\pi/m)$ , for integers  $j$  and  $m$  (with  $q \neq 0, 4$ ). Moreover, the size of this equation grows with  $m$ . Nevertheless, equations with one catalytic variable are much better understood than those with two [11], and we were able to prove that the generating function of  $q$ -coloured maps is algebraic for all such values of  $q$ , including  $q = 2$  and  $q = 3$ . It is known to be transcendental in general [2].

In this paper, we perform the second step, and prove that for  $q$  an indeterminate, the generating function of  $q$ -coloured planar maps (or triangulations) is differentially algebraic. We give an explicit differential system that characterizes it. This system has the same form for general maps and triangulations. In some special cases (proper colourings; four colours; spanning forests;

self-dual Potts model) we derive from it an explicit differential equation of small order defining the generating function. In particular, we recover Tutte's result (1) for properly coloured triangulations. For the convenience of the reader, all calculations are performed in an accompanying Maple session, available on the web pages of the authors.

Here is an outline of the paper. In Section 2 we give definitions about maps, the Potts model and its combinatorial counterpart, the Tutte polynomial, as well as notation on power series. In Section 3 we state our main result and give an idea of Tutte's sophisticated approach (some would say obscure) on a simple example. In Sections 4 and 5 we establish differential systems for  $q$ -coloured planar maps and  $q$ -coloured triangulations, respectively. We simplify these systems in Section 6. The next sections deal with special cases: proper colourings (Section 7), four colours (Section 8), then the  $q = 0$  case, which corresponds to the enumeration of maps equipped with a spanning forest (Section 9), and finally the self-dual Potts model (Section 10).

Our differential system for planar maps appeared (without proof) in a survey on map enumeration published in 2011 [9].

## 2. DEFINITIONS AND NOTATION

### 2.1. PLANAR MAPS

A *planar map* is a proper embedding of a connected planar graph in the oriented sphere, considered up to orientation preserving homeomorphism. Loops and multiple edges are allowed (Figure 1). The *faces* of a map are the connected components of its complement. The numbers of vertices, edges and faces of a planar map  $M$ , denoted by  $v(M)$ ,  $e(M)$  and  $f(M)$ , are related by Euler's relation  $v(M) + f(M) = e(M) + 2$ . The *degree* of a vertex or face is the number of edges incident to it, counted with multiplicity. A *corner* is a sector delimited by two consecutive edges around a vertex; hence a vertex or face of degree  $k$  defines  $k$  corners. Alternatively, a corner can be described as an incidence between a vertex and a face. The *dual* of a map  $M$ , denoted  $M^*$ , is the map obtained by placing a vertex of  $M^*$  in each face of  $M$  and an edge of  $M^*$  across each edge of  $M$ ; see Figure 1. A *triangulation* is a map in which every face has degree 3. Duality transforms triangulations into *cubic* maps, that is, maps in which every vertex has degree 3.

For counting purposes it is convenient to consider *rooted* maps. A map is rooted by choosing a corner, called the *root-corner*. The vertex and face that are incident at this corner are respectively the *root-vertex* and the *root-face*. The *root-edge* is the edge that follows the root-corner in counterclockwise order around the root-vertex. In figures, we indicate the rooting by an arrow pointing to the root-corner, and take the root-face as the infinite face (Figure 1). This explains why we often call the root-face the *outer face* and its degree the *outer degree* (denoted  $\text{drf}(M)$ ).

From now on, every map is *planar* and *rooted*. By convention, we include among rooted planar maps the *atomic map* having one vertex and no edge.



FIGURE 1. A rooted planar map and its dual (rooted at the dual corner).

## 2.2. THE POTTS MODEL

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . Let  $\nu$  be an indeterminate, and take  $q \in \mathbb{N}$ . A *colouring* of the vertices of  $G$  in  $q$  colours is a map  $c : V(G) \rightarrow \{1, \dots, q\}$ . An edge of  $G$  is *monochromatic* if its endpoints share the same colour. Every loop is thus monochromatic. The number of monochromatic edges is denoted by  $m(c)$ . The *partition function of the Potts model* on  $G$  counts colourings by the number of monochromatic edges:

$$P_G(q, \nu) = \sum_{c: V(G) \rightarrow \{1, \dots, q\}} \nu^{m(c)}.$$

The Potts model is a classical magnetism model in statistical physics, which includes (when  $q = 2$ ) the famous Ising model (with no magnetic field) [53, 52]. Of course,  $P_G(q, 0)$  is the chromatic polynomial of  $G$ , which counts proper colourings (no monochromatic edge).

It is not hard to see that  $P_G(q, \nu)$  is a polynomial in  $q$  and  $\nu$ . We call it the *Potts polynomial* of  $G$ . Observe that it is a multiple of  $q$ . We will often consider  $q$  as an indeterminate, or evaluate  $P_G(q, \nu)$  at real values  $q$ .

We define the *Potts generating function* of planar maps by:

$$M(y) \equiv M(q, \nu, w, t; y) = \frac{1}{q} \sum_M P_M(q, \nu) w^{v(M)} t^{e(M)} y^{\text{drf}(M)}, \quad (3)$$

where the sum runs over all planar maps  $M$ . Since there are finitely many maps with a given number of edges, and  $P_M(q, \nu)$  is a multiple of  $q$ , the generating function  $M(y)$  is a power series in  $t$  with coefficients in  $\mathbb{Q}[q, \nu, w, y]$ , the ring of polynomials in  $q, \nu, w$  and  $y$  with rational coefficients. The expansion of  $M$  at order 2 reads

$$M(y) = w + (w^2 y^2 (q - 1 + \nu) + w y \nu) t + (2w^3 y^4 (q - 1 + \nu)^2 + w^2 y^2 (q - 1 + \nu^2) + w^2 \nu (y + 3y^3) (q - 1 + \nu) + w \nu^2 (y + y^2)) t^2 + O(t^3),$$

as illustrated in Figure 2. In combinatorial terms,  $M$  counts  $q$ -coloured planar maps by vertices, edges, monochromatic edges and outer degree, with the convention that the root-vertex is coloured in a prescribed colour (this accounts for the division by  $q$ ).


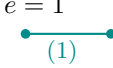

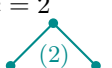



	$e = 0$	$e = 1$		$e = 2$			
							
$\frac{P_M(q)}{q}$	1	$(q-1+\nu)$	$\nu$	$(q-1+\nu)^2$	$(q-1+\nu^2)$	$\nu(q-1+\nu)$	$\nu^2$

FIGURE 2. The planar maps with  $e = 0, 1$ , and 2 edges, and their Potts polynomials (divided by  $q$ ). The maps are shown *unrooted*, and the numbers in parentheses indicate the number of corresponding *rooted* maps.

## 2.3. SPECIALIZATIONS

We consider in this paper several specializations of the Potts polynomial. For some of them, the combinatorial meaning is obvious: for instance,  $P_G(q, 0)$  counts proper  $q$ -colourings, and  $P_G(4, \nu)$  counts 4-colourings by monochromatic edges. To understand the significance of some other specializations, it is useful to relate  $P_G$  to another invariant of graphs: the *Tutte polynomial*  $T_G(\mu, \nu)$ . It is defined as follows (see e.g. [4]):

$$T_G(\mu, \nu) := \sum_{S \subseteq E(G)} (\mu - 1)^{c(S) - c(G)} (\nu - 1)^{e(S) + c(S) - v(G)}, \quad (4)$$

where the sum is over all spanning subgraphs of  $G$  (equivalently, over all subsets of edges) and  $v(\cdot)$ ,  $e(\cdot)$  and  $c(\cdot)$  denote respectively the number of vertices, edges and connected components. The equivalence with the Potts polynomial was established by Fortuin and Kasteleyn [25]:

$$P_G(q, \nu) = \sum_{S \subseteq E(G)} q^{c(S)} (\nu - 1)^{e(S)} = (\mu - 1)^{c(G)} (\nu - 1)^{v(G)} T_G(\mu, \nu), \quad (5)$$

for  $q = (\mu - 1)(\nu - 1)$ . The Tutte polynomial satisfies an interesting duality property: if  $G$  and  $G^*$  are dual connected planar graphs (that is, if  $G$  and  $G^*$  can be embedded as dual planar maps) then

$$T_{G^*}(\mu, \nu) = T_G(\nu, \mu). \quad (6)$$

This gives a duality relation on the Potts generating function defined by (3):

$$M(q, \nu, w, t; 1) = w^2 q M(q, \mu, (wq)^{-1}, tw(\nu - 1); 1). \quad (7)$$

The connection (5) between  $P_G$  and  $T_G$  allows us to understand combinatorially the limit  $q = 0$  of  $P_G/q$ : for a planar map  $M$ ,

$$\begin{aligned} \lim_{q \rightarrow 0} \frac{1}{q} P_M(q, \nu) &= (\nu - 1)^{v(M)-1} T_M(1, \nu) \\ &= \sum_{C \text{ connected on } M} (\nu - 1)^{e(C)} && \text{by (4),} \\ &= (\nu - 1)^{f(M^*)-1} T_{M^*}(\nu, 1) && \text{by (6),} \\ &= (\nu - 1)^{f(M^*)-1} \sum_{F \text{ forest on } M^*} (\nu - 1)^{c(F)-1} && \text{by (4) again,} \end{aligned}$$

where the first sum runs over all connected subgraphs  $C$  of  $M$ , and the second over all spanning forests  $F$  of  $M^*$  (subsets of edges of  $M^*$  with no cycle). Hence, if  $S(q, \nu, w, t)$  is the Potts generating function of some family of planar maps  $\mathcal{M}$  (meaning that each map  $M$  is given a weight  $\frac{1}{q} P_M(q, \nu) w^{v(M)} t^{e(M)}$ ) and  $\mathcal{M}^*$  denotes the set of duals of maps of  $\mathcal{M}$ ,

$$\begin{aligned} S(0, 1 + \beta, w, t) &= \sum_{M \in \mathcal{M}, C} \beta^{e(C)} w^{v(M)} t^{e(M)} \\ &= \frac{1}{\beta} \sum_{M^* \in \mathcal{M}^*, F} \beta^{c(F)-1} (\beta w)^{f(M^*)} t^{e(M^*)}, \end{aligned} \quad (8)$$

where the first sum runs over all maps  $M$  of  $\mathcal{M}$  equipped with a connected subgraph  $C$ , and the second over all maps  $M$  of  $\mathcal{M}^*$  equipped with a spanning forest  $F$ .

Our final specialization is the self-dual Potts model, in which a map and its dual get the same weight. In view of the duality relation (7), this means that  $\mu = \nu$ , so that  $q = (\nu - 1)^2$ , and that  $w = 1/\sqrt{q} = 1/(\nu - 1)$ . Denoting  $\beta = \nu - 1$ , we will thus consider

$$M(\beta^2, \beta + 1, \beta^{-1}, t; 1) = \frac{1}{\beta} \sum_M T_M(1 + \beta, 1 + \beta) t^{e(M)} \quad (9)$$

where the sum runs over all planar maps. One motivation for studying this specialization is that it should coincide with a critical random-cluster model [38, 19], in which one chooses a map  $M$  and a subset  $S$  of its edges with probability proportional to  $q^{c(S)} \beta^{e(S)} t^{e(M)} \beta^{-v(M)}$ . The partition function of this model is

$$\hat{M}(q, \beta, t) := \sum_M t^{e(M)} \beta^{-v(M)} \sum_{S \subseteq E(M)} q^{c(S)} \beta^{e(S)} = q M(q, \beta + 1, \beta^{-1}, t; 1).$$

The duality relation (7) implies

$$\hat{M}(q, \beta, t) = q \beta^{-2} \hat{M}(q, q/\beta, t),$$

and one can thus expect the critical point  $\beta_c(q)$  to satisfy  $\beta_c(q)^2 = q$ . The critical partition function would then be given by (9). Note that the connection between criticality and self-duality has recently been proved for the regular square lattice [1].

#### 2.4. POWER SERIES

Let  $A$  be a commutative ring and  $x$  an indeterminate. We denote by  $A[x]$  (resp.  $A[[x]]$ ) the ring of polynomials (resp. formal power series) in  $x$  with coefficients in  $A$ . The coefficient of  $x^n$  in a series  $F(x)$  is denoted by  $[x^n]F(x)$ . If  $A$  is a field, then  $A(x)$  denotes the field of rational functions in  $x$ . This notation is generalized to polynomials, fractions and series in several indeterminates. For instance, the Potts generating function of planar maps  $M(q, \nu, w, t; y)$  defined by (3) belongs to  $\mathbb{Q}[q, \nu, w, y][[t]]$ . To lighten notation, we often omit the dependence of our series in  $q, \nu$  and  $w$ , writing for instance  $M(t; y)$ , or even  $M(y)$ .

If  $\mathbb{K}$  is a field, a power series  $F(t) \in \mathbb{K}[[t]]$  is *algebraic* (over  $\mathbb{K}(t)$ ) if it satisfies a non-trivial polynomial equation  $P(t, F(t)) = 0$  with coefficients in  $\mathbb{K}$ . It is *differentially algebraic* if it satisfies a non-trivial polynomial differential equation  $P(t, F(t), F'(t), \dots, F^{(k)}(t)) = 0$  with coefficients in  $\mathbb{K}$ .

For a series  $F$  in several variables  $x, y, \dots$ , we denote by  $F'_x$  the derivative of  $F$  with respect to  $x$ .

### 3. MAIN RESULT, AND OUTLINE OF THE PROOF

Our study of coloured maps started in [2] with the following equation with two catalytic variables  $x$  and  $y$ :

$$\begin{aligned} \bar{M}(x, y) &= 1 + xywt((\nu - 1)(y - 1) + qy)\bar{M}(x, y)\bar{M}(1, y) \\ &\quad + xyt(x\nu - 1)\bar{M}(x, y)\bar{M}(x, 1) \\ &\quad + xywt(\nu - 1)\frac{x\bar{M}(x, y) - \bar{M}(1, y)}{x - 1} + xyt\frac{y\bar{M}(x, y) - \bar{M}(x, 1)}{y - 1}. \end{aligned}$$

It defines uniquely a power series in  $t$ , denoted  $\bar{M}(x, y) \equiv \bar{M}(q, \nu, w, t; x, y)$ , which has polynomial coefficients in  $q, \nu, w, x$  and  $y$ . The Potts generating function  $M(y)$  defined by (3) is essentially the specialization  $x = 1$  of  $\bar{M}(x, y)$ . More precisely,

$$M(y) \equiv M(q, \nu, w, t; y) = w\bar{M}(q, \nu, w, t; 1, y) = w\bar{M}(1, y).$$

We then proved that when  $q \neq 0, 4$  is of the form  $2 + 2\cos(j\pi/m)$ , for integers  $j$  and  $m$ , the series  $M(y)$  satisfies an equation of the form:

$$P(\nu, w, t, y, M(y), M_1, \dots, M_r) = 0, \quad (10)$$

for some polynomial  $P$  depending on  $j$  and  $m$ , and  $r$  (unknown) series in  $t$ , denoted by  $M_1 = M(1), \dots, M_r$ , which are independent of  $y$  [2, Cor. 10]. We call (10) a *polynomial equation with one catalytic variable*,  $y$ . In particular, when  $q = 1$ , every edge is monochromatic and (10) coincides with the standard functional equation obtained by deleting recursively the root-edge in planar maps [40]:

$$M(y) = w + y^2 t \nu M(y)^2 + \nu t y \frac{yM(y) - M_1}{y - 1}. \quad (11)$$

The classical way of solving (11) is Brown's *quadratic method*, presented below in Section 3.1.1. It derives from (11) a polynomial equation satisfied by  $M_1$ . The quadratic method was generalized in [11] to arbitrary (well-posed) polynomial equations with one catalytic variable: under minimal assumptions, their solutions are *always algebraic*. Using this approach, we proved in [2] that for  $q = 2 + 2\cos(j\pi/m)$ , the series  $M(q, \nu, w, t; y)$  is algebraic over  $\mathbb{R}(\nu, w, t, y)$  (and even over  $\mathbb{Q}(q, \nu, w, t, y)$ ). However, we could not obtain an explicit polynomial equation satisfied by  $M(y)$ , nor even by  $M(1) = M_1$ . We only constructed one for the three *integer* values of  $q$  of the above form, namely  $q = 1, 2$ , and  $3$ . And for  $q = 3$ , we were only able to do it when  $\nu = 0$ , that is,

when one counts *proper* 3-colourings. We expect the degree of  $M_1$  to grow with  $m$ . For  $q$  an indeterminate, the series  $M_1$  is transcendental.

The main result of this paper is that when  $q$  is an indeterminate, and consequently for any real  $q$ , the series  $M_1$  is differentially algebraic over  $\mathbb{Q}(q, \nu, w, t)$ . Moreover, we construct an explicit differential system that characterizes  $M_1$ .

**Theorem 1.** *Let  $q$  be an indeterminate,  $\beta = \nu - 1$  and*

$$D(t, x) = (q\nu + \beta^2)x^2 - q(\nu + 1)x + \beta t(q - 4)(wq + \beta) + q.$$

*There exists a unique triple  $(P(t, x), Q(t, x), R(t, x))$  of polynomials in  $x$  with coefficients in  $\mathbb{Q}[q, \nu, w][[t]]$ , having degree 4, 2 and 2 respectively in  $x$ , such that*

$$\begin{aligned} [x^4]P(t, x) &= 1, & P(0, x) &= x^2(x - 1)^2, \\ [x^2]R(t, x) &= \nu + 1 - w(q + 2\beta), & Q(0, x) &= x(x - 1), \end{aligned} \quad (12)$$

and

$$\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right). \quad (13)$$

*Let  $P_j(t) \equiv P_j$  (resp.  $Q_j, R_j$ ) denote the coefficient of  $x^j$  in  $P(t, x)$  (resp.  $Q(t, x), R(t, x)$ ). The Potts generating function of planar maps,  $M_1 \equiv M(q, \nu, w, t; 1)$ , can be expressed in terms of the series  $P_j$  and  $Q_j$  using  $\tilde{M}_1 := t^2 M_1$  and*

$$12(\beta^2 + q\nu)\tilde{M}_1 + P_3^2/4 + 2t(1 + \nu - w(2\beta + q))P_3 - P_2 + 2Q_0 = 4t(1 + w(3\beta + q)). \quad (14)$$

*An alternative characterization of  $M_1$  is in terms of the derivative of  $\tilde{M}_1$ :*

$$2(\beta^2 + q\nu)\tilde{M}_1' + (1 + \nu - w(2\beta + q))P_3/2 - R_1 = 2 + 2\beta w.$$

*The series  $M_1$  is differentially algebraic, that is, satisfies a non-trivial differential equation with respect to the edge variable  $t$ . The same holds for each series  $P_j, Q_j$  and  $R_j$ .*

Eq. (13) looks like a *partial* differential equation in  $t$  and  $x$ , but it is in fact a system of nine differential equations in  $t$  written in a compact form. Indeed, its numerator reads:

$$2Q_t'PD - QP_t'D - 2QPD_t' = 2R_x'PD - RP_x'D - 2RPD_x'.$$

This is a polynomial in  $x$  of degree 8, and each of its nine coefficients gives a differential equation (with respect to the variable  $t$ ) relating the eleven series  $P_j, Q_j$  and  $R_j$ . For instance, extracting the coefficient of  $x^8$  gives :

$$2Q_2'P_4 - Q_2P_4' = 0. \quad (15)$$

The fact that we only obtain nine equations for eleven unknown series look alarming, but the initial conditions (12) give explicitly the two series  $P_4$  and  $R_2$ . We also observe that (15), combined with the initial conditions (12), implies that  $Q_2 = 1$ . In fact, we will prove that the differential system (13), together with its initial conditions, determines the series  $P_j, Q_j$  and  $R_j$  uniquely. For instance,

$$\begin{aligned} P_0 &= -4t + (q^2w^2 + 16\beta w - 4qw + 8\beta)t^2 \\ &+ 2(-\beta q^2w^3 + q^3w^3 + 2\beta qw^2 - 4q^2w^2 + 16\beta^2w + 4\beta qw + 2\beta^2 - 6qw + 4\beta)t^3 + O(t^4). \end{aligned}$$

Using (14), one can in principle construct a differential equation (DE) for  $M_1$ , but it would probably be very large. However, we will work out some special cases in details, and obtain for instance a DE of order 3 for four-coloured planar maps (Section 8). We predict the order of  $M_1$  to be 5 in general (Section 6.2).

Our solution differs significantly from other published results on the Potts model, in that it ignores the catalytic variable  $y$  (which is set to its natural value 1) and describes the dependence of the series  $M(1)$  in the size variable  $t$ . In contrast, the results of [6, 29] give a precise description of the dependence of  $M(y)$  in  $y$  (in terms of elliptic functions), but the dependence in the size variable seems to remain elusive. This is this dependence that we have characterized, in differential terms.



### 3.1. A TOY EXAMPLE: UNCOLOURED MAPS

In this section, we illustrate on an example how one can derive differential equations, rather than polynomial equations, from equations with one catalytic variable like (10). Our example is Tutte's equation for the enumeration of planar maps [40], counted by the number of edges (variable  $t$ ) and the outer degree (variable  $y$ ):

$$M(t; y) \equiv M(y) = 1 + ty^2M(y)^2 + ty \frac{yM(y) - M_1}{y-1}, \quad (16)$$

where  $M_1$  stands for  $M(1) \equiv M(t; 1)$ . This equation characterizes  $M(y)$  in the ring of formal power series in  $t$ . Indeed, the coefficient of  $t^n$  can be computed by induction on  $n$ , yielding:

$$M(t; y) = 1 + (y + y^2)t + (2y + 2y^2 + 3y^3 + 2y^4)t^2 + O(t^3). \quad (17)$$

The reader can find this expansion, and much more, performed in the accompanying Maple session, available on the web pages of the authors.

The standard method for solving (16) is Brown's quadratic method (see [17], or [27, Section 2.9] for a modern account). It constructs from (16) a polynomial equation satisfied by  $M_1$ . We describe this solution in Section 3.1.1 below. We then present in Section 3.1.2 an alternative solution, which derives from (16) a *system of differential equations* satisfied by  $M_1$ . This gives the spirit of our construction of differential equations for  $q$ -coloured planar maps. The spirit, but not the details: in Section 3.2 we underline a few important differences between the solution of our toy example and that of  $q$ -coloured maps.

**3.1.1. The quadratic method: a polynomial equation for  $M_1$ .** We first form in (16) a perfect square, as if we wanted to solve it for  $M(y)$ :

$$(2ty^2(y-1)M(y) + ty^2 - y + 1)^2 = (y-1 - y^2t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)M_1. \quad (18)$$

Clearly, there exists a unique formal power series in  $t$ , denoted below  $Y$ , that cancels the left-hand side. Its  $n$ th coefficient can be computed by induction, using the expansion (17) of  $M(y)$ :

$$Y = 1 + t + 4t^2 + 25t^3 + O(t^4).$$

Let us denote the right-hand side of (18) by

$$\Delta(y) \equiv \Delta(t; y) = (y-1 - y^2t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)M_1.$$

This is a *quartic polynomial* in  $y$ , with coefficients in  $\mathbb{Q}[[t]]$ . Since  $Y$  cancels the left-hand side of (18), it also cancels its right-hand side. Thus  $Y$  is a root of  $\Delta(y)$ . By differentiating (18) with respect to  $y$ , we see that  $Y$  is also a root of  $\Delta'_y$ , and hence a double root of  $\Delta$ .

Since  $\Delta(y)$  has a double root, its discriminant (with respect to  $y$ ) vanishes. This gives a polynomial equation satisfied by  $M_1$ :

$$27t^2M_1^2 + (1 - 18t)M_1 + 16t - 1 = 0.$$

Of course this can be easily solved, and we obtain the well-known generating function of planar maps counted by edges [40]:

$$M_1 = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2}.$$

**3.1.2. An alternative approach: a differential system.** We now describe another consequence of the fact that  $\Delta(y)$  has a double root, this time in terms of differential equations. The procedure looks cumbersome, compared to the neat fact that the discriminant of  $\Delta(y)$  is zero. But, applied to coloured planar maps, it will yield a system of differential equations that looks much nicer than any polynomial equations we could possibly compute.

Denote  $Z = 1/Y$ . Since  $Y$  is a double root of  $\Delta(y)$ , there exists a polynomial  $P(t; y) \equiv P(y)$  of degree 2 such that

$$\Delta(y) = (1 - yZ)^2 P(y).$$

Since  $Z$  is a formal power series in  $t$ , the same holds for the coefficients of  $P(y)$ . Similarly, there exists a polynomial  $Q(y)$  of degree 2 such that

$$\Delta'_y(y) = (1 - yZ)Q(y), \quad (19)$$

and a polynomial  $R(y)$  of degree 3 such that

$$\Delta'_t(y) = (1 - yZ)R(y). \quad (20)$$

We now want to eliminate  $\Delta$  and  $Z$  in these three equations so as to find a (differential) relation between  $P$ ,  $Q$  and  $R$ . We first eliminate  $Z$ , writing

$$\frac{\Delta'_y{}^2}{\Delta} = \frac{Q^2}{P} \quad \text{and} \quad \frac{\Delta'_t{}^2}{\Delta} = \frac{R^2}{P}. \quad (21)$$

We now differentiate the first equation with respect to  $t$ , and the second one with respect to  $y$ :

$$\frac{2\Delta'_y\Delta''_{y,t}\Delta - \Delta'_y{}^2\Delta'_t}{\Delta^2} = \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right), \quad \frac{2\Delta'_t\Delta''_{y,t}\Delta - \Delta'_t{}^2\Delta'_y}{\Delta^2} = \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right).$$

The ratio of the left-hand sides is  $\Delta'_y/\Delta'_t$ , which, according to (19-20), is also  $Q/R$ . This gives:

$$\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right).$$

Note the striking analogy with our differential system (13) for coloured maps. As in the coloured case, this equation gives, in a compact form, a system of differential equations in  $t$  relating the coefficients of  $P(y)$ ,  $Q(y)$  and  $R(y)$ . We will not discuss which additional properties are needed to characterize them uniquely, but let us express the series  $M_1$  in terms of them. The first equation of (21) reads

$$P\Delta'_y{}^2 = Q^2\Delta.$$

This is a polynomial in  $y$  of degree 8. Extracting the coefficient of  $y^8$  gives the identity

$$64t^2P_2M_1 + 16t^2P_2 - Q_2^2 - 64tP_2 = 0,$$

which determines  $M_1$  in terms of the coefficients  $P_j$  and  $Q_j$  of  $P(y)$  and  $Q(y)$ .

### 3.2. SOME DIFFERENCES BETWEEN THE TOY EXAMPLE AND COLOURED MAPS

Before embarking on the proof of Theorem 1, we want to underline a number of differences between the treatment just applied to uncoloured maps and our treatment of coloured maps below. Let us list the main steps of our approach.

- We take  $q = 2 + 2\cos(2\pi/m)$ , and start from a polynomial equation in one catalytic variable satisfied by  $M(y)$ , obtained in [2]. This is (25).
- This equation has some similarities with (18): in the latter equation, the right-hand side is a polynomial in  $y$  with coefficients in  $\mathbb{Q}[[t]]$ , while in (25), the right-hand side is a polynomial in  $M(y)$  (or rather, in a series  $I(y)$  which contains the same information as  $M(y)$ ), with coefficients in  $\mathbb{Q}(q, \nu, w)[[t]]$ . Hence the roles of  $y$  and  $M(y)$  are exchanged.
- We derive from (25) that a certain polynomial  $\Delta(x)$ , of degree  $2m$  and involving  $m - 2$  unknown series in  $t$ , has  $m - 2$  double roots  $I_1, \dots, I_{m-2}$ . This is Lemma 4.
- We derive our differential system from this property. The counterparts of (19) and (20) do not have left-hand sides  $\Delta'_y$  and  $\Delta'_t$ , but variants of these two polynomials (Proposition 5).

#### 4. DIFFERENTIAL SYSTEM FOR COLOURED PLANAR MAPS

##### 4.1. AN EQUATION WITH ONE CATALYTIC VARIABLE

Our starting point is an equation of [2]. We take  $q = 2 + 2 \cos(2k\pi/m)$ , with  $k$  and  $m$  coprime and  $0 < 2k < m$ . We write  $\beta = \nu - 1$ . We introduce the following notation:

- $I(t, y) \equiv I(q, \nu, w, t; y)$  is a variant of the generating function  $M(y) \equiv M(q, \nu, w, t; y)$  of  $q$ -coloured planar maps defined by (3):

$$I(t, y) = tyqM(y) + \frac{y-1}{y} + \frac{ty}{y-1}. \quad (22)$$

- $N(y, x)$  and  $D(t, x)$  are the following (Laurent) polynomials:

$$N(y, x) = \beta(4 - q)(\bar{y} - 1) + (q + 2\beta)x - q, \quad (23)$$

with  $\bar{y} = 1/y$ , and

$$D(t, x) = (q\nu + \beta^2)x^2 - q(\nu + 1)x + \beta t(q - 4)(wq + \beta) + q. \quad (24)$$

Let  $T_m$  be the  $m$ th Chebyshev polynomial of the first kind, defined by

$$T_m(\cos \theta) = \cos(m\theta).$$

Then there exists  $m+1$  formal power series in  $t$  with coefficients in  $\mathbb{Q}(q, \nu, w)$ , denoted  $C_0(t), \dots, C_m(t)$ , such that

$$D(t, I(t, y))^{m/2} T_m \left( \frac{N(y, I(t, y))}{2\sqrt{D(t, I(t, y))}} \right) = \sum_{r=0}^m C_r(t) I(t, y)^r. \quad (25)$$

This is a combination of Corollary 10 and Lemma 16 from [2]. We call (25) *the invariant equation*, since it is derived from a certain *theorem of invariants*, in Tutte's terminology [51]. We find convenient to denote

$$C(t, x) = \sum_{r=0}^m C_r(t) x^r. \quad (26)$$

Since  $T_m(u)$  is even (resp. odd) in  $u$  if  $m$  is even (resp. odd), the left-hand side of (25) only involves *integer* powers of  $D$ .

As the invariant equation may look intimidating, let us consider an example.

**Example: bipartite maps.** Let us take  $q = 2$  (that is,  $m = 4$  and  $k = 1$ ),  $w = 1$  and  $\nu = 0$ . Then  $M(y)$  is simply the generating function of bipartite maps, counted by the edge number (variable  $t$ ) and the outer degree (variable  $y$ ). Let us show that in this case, the invariant equation (25) coincides with the standard equation obtained by deleting the root-edge.

With our choice of  $q, \nu$  and  $w$ , we have

$$N(y, x) = -2\bar{y}, \quad D(t, x) = x^2 - 2x + 2t + 2,$$

while  $T_m(u) = T_4(u) = 1 - 8u^2 + 8u^4$ . The invariant equation thus reads

$$(I(y)^2 - 2I(y) + 2t + 2)^2 - 8\bar{y}^2(I(y)^2 - 2I(y) + 2t + 2) + 8\bar{y}^4 = \sum_{r=0}^4 C_r I(y)^r.$$

In this equation, let us replace  $I(y)$  by its expression (22) in terms of  $M(y)$  (with  $q = 2$ ), and then expand the equation in the neighbourhood of  $y = 1$ . Saying that the coefficients of  $(y-1)^{-4}, \dots, (y-1)^0$  must vanish gives the values of the series  $C_r$ :

$$C_4 = 1, \quad C_3 = -4, \quad C_2 = 4t, \quad C_1 = 8(1+t) \quad (27)$$

and

$$C_0 = -4 - 40t - 4t^2 + 32t^2M(1).$$

In the invariant equation, let us replace each series  $C_r$  by its expression: we obtain

$$y^2t(y^2 - 1)M(y)^2 + (1 - y^2 + y^2t)M(y) - ty^2M(1) + y^2 - 1 = 0.$$

Equivalently,

$$M(y) = 1 + ty^2 M(y)^2 + ty^2 \frac{M(y) - M(1)}{y^2 - 1},$$

which can be obtained by deleting the root-edge in a bipartite map [40].  $\square$

Let us return to the general case  $q = 2 + 2 \cos(2k\pi/m)$ . By expanding (25) near  $y = 1$ , we can express the series  $C_r(t)$  in terms of the derivatives of  $M$  with respect to  $y$ , evaluated at  $y = 1$ . We will need the expressions of  $C_m, \dots, C_{m-3}$ . Note that the first three are completely explicit in terms of  $q, \nu, t, w$  and  $m$ . The fourth one involves the unknown series  $M(1)$ .

**Lemma 2.** Denote  $\beta = \nu - 1$ ,  $\delta = \beta^2 + q\nu$  and  $\gamma = \frac{q+2\beta}{2\sqrt{\delta}}$ . We have:

$$\begin{aligned} C_m &= \delta^{m/2} \mathbb{T}_m(\gamma), \\ C_{m-1} &= -\frac{q}{4} \delta^{(m-3)/2} \left( 2m(\nu+1)\sqrt{\delta} \mathbb{T}_m(\gamma) + \beta(q-4) \mathbb{T}'_m(\gamma) \right), \\ C_{m-2} &= \frac{1}{8} \delta^{(m-5)/2} \left( 2ma_0\sqrt{\delta} \mathbb{T}_m(\gamma) + q\beta(q-4)a_1 \mathbb{T}'_m(\gamma) \right), \\ C_{m-3} &= \frac{q}{24} \delta^{(m-7)/2} \left( 2mb_0\sqrt{\delta} \mathbb{T}_m(\gamma) + \beta(q-4)b_1 \mathbb{T}'_m(\gamma) \right) + \frac{q(q-4)\beta t^2}{2} M(1) \delta^{(m-1)/2} \mathbb{T}'_m(\gamma), \end{aligned}$$

with

$$\begin{aligned} a_0 &= 2t\beta\delta(q-4)(\beta+wq) + q(m-1)((1+\nu^2)q - 2\beta^2), \\ a_1 &= 2t\delta(\nu+1-w(2\beta+q)) + q(m-1)(\nu+1), \\ b_0 &= 6tw(q-4)\beta\delta((\beta^2-q)m + q(\nu+1)) - 6\beta^2t\delta(m-1)(\nu+1)(q-4) \\ &\quad - q(m-1)(m-2)(\nu+1)((\nu^2-\nu+1)q - 3\beta^2), \\ b_1 &= 6tw\delta(q(m-1)(\nu^2+2\nu+q-3) - (2\beta+q)\delta) \\ &\quad - 6t\delta(m-1)(q(1+\nu^2) - 2\beta^2) + q(m-1)(m-2)(\beta^2 - (\nu^2 + \nu + 1)q). \end{aligned}$$

**Example: bipartite maps (continued).** When  $k = 1$ ,  $m = 4$ ,  $q = 2$ ,  $w = 1$  and  $\nu = 0$ , we have  $\beta = -1$ ,  $\delta = 1$  and  $\gamma = 0$ . Moreover,  $\mathbb{T}_m(\gamma) = 1$  and  $\mathbb{T}'_m(\gamma) = 0$ . We can then check that the above lemma specializes to (27). In particular,  $C_1 = C_{m-3}$  does not involve  $M(1)$  because  $\mathbb{T}'_m(\gamma) = 0$ .  $\square$

*Proof of Lemma 2.* We multiply the invariant equation (25) by  $(y-1)^m$ , so that it becomes non-singular at  $y = 1$ , and expand it around  $y = 1$ . We extract successively the coefficients of  $(y-1)^0, \dots, (y-1)^3$  and obtain in this way expressions of  $C_m, \dots, C_{m-3}$ . For instance, since

$$\begin{aligned} (y-1)I(t, y) &= t + O(y-1), \\ (y-1)N(y, I(t, y)) &= (q+2\beta)t + O(y-1), \\ (y-1)^2 D(t, I(t, y)) &= \delta t^2 + O(y-1), \end{aligned}$$

extracting the coefficient of  $(y-1)^0$  gives

$$\delta^{m/2} t^m \mathbb{T}_m(\gamma) = C_m t^m,$$

from which the expression of  $C_m$  follows. As we extract the coefficients of higher powers of  $(y-1)$ , derivatives of  $\mathbb{T}_m$ , taken at the point  $\gamma$ , occur. We systematically express them in terms of  $\mathbb{T}_m(\gamma)$  and  $\mathbb{T}'_m(\gamma)$  using the differential equation

$$(1-u^2) \mathbb{T}_m''(u) - u \mathbb{T}_m'(u) + m^2 \mathbb{T}_m(u) = 0.$$

Similarly, derivatives of  $(1-y)I(t, y)$  with respect to  $y$  occur, and this is why the expression of  $C_{m-3}$  involves the series  $M(1)$ .  $\blacksquare$

#### 4.2. SOME SPECIAL VALUES OF $y$

The invariant equation (25) reads

$$\text{Rat}(t, y, I(y), C_0, \dots, C_m) = 0 \quad (28)$$

where Rat is the following rational function, with coefficients in  $\mathbb{Q}(q, \nu, w)$ :

$$\text{Rat}(t, y, x, c_0, \dots, c_r) = D(t, x)^{m/2} T_m \left( \frac{N(y, x)}{2\sqrt{D(t, x)}} \right) - \sum_{r=0}^m c_r x^r.$$

Observe that the first term of Rat is the only one that involves  $y$ . In our toy model (Section 3.1), we were dealing with (18), which read  $\text{Rat}(t, y, M(y), M_1) = 0$  with

$$\text{Rat}(t, y, x, c) = (2ty^2(y-1)x + ty^2 - y + 1)^2 - (y-1 - y^2t)^2 + 4ty^2(y-1)^2 - 4t^2y^3(y-1)c.$$

There, the first term was the only one that depended on  $x$ . To solve this toy equation, we considered a series  $Y \equiv Y(t)$  cancelling this term, or equivalently, satisfying

$$\text{Rat}'_x(t, Y, M(Y), M_1) = 0,$$

where the derivative is taken with respect to the third variable of Rat. Similarly, we are going to solve (28) by cancelling  $\text{Rat}'_y$  (this change from  $x$  to  $y$  explains why we wrote in Section 3.2 that the roles of  $y$  and  $M(y)$  are exchanged). More precisely, we focus on

$$T'_m \left( \frac{N(y, I(t, y))}{2\sqrt{D(t, I(t, y))}} \right) = 0.$$

Denoting  $n = \lfloor m/2 \rfloor$ , the set of roots of  $T'_m$  is easily seen to be:

$$\{\cos(j\pi/m), 0 < j < m\} = \{\pm \cos(j\pi/m), j = 1, \dots, n\}. \quad (29)$$

Hence we are interested in the equation

$$N(Y, I(t, Y))^2 = 4 \cos(j\pi/m)^2 D(t, I(t, Y)), \quad (30)$$

for some  $j \in \llbracket 1, n \rrbracket$ .

**Lemma 3.** *Let us assume  $k = 1$ , so that  $q = 2 + 2 \cos(2\pi/m)$ . There exist  $m - 2$  distinct formal power series in  $t$ , denoted  $Y_1, \dots, Y_{m-2}$ , that have coefficients in  $\mathbb{R}(\nu, w)$ , constant term 1, and satisfy (30) for some  $j \in \llbracket 1, n \rrbracket$ .*

*Let us denote  $I_i(t) := I(t, Y_i)$ . This is a formal power series in  $t$  with coefficients in  $\mathbb{R}(\nu, w)$ . The  $m - 2$  series  $I_i(t)$  are distinct, and for  $1 \leq i \leq m - 2$ ,*

$$I_i(0) \notin \{0, 1\}, \quad D(t, I_i) \neq 0, \quad \frac{\partial I}{\partial y}(t, Y_i) \neq 0.$$

**Example: one-coloured maps.** We take  $q = 1$ , that is,  $m = 3$  (and  $n = 1$ ). The only relevant value of  $j$  is 1, and we thus want to solve

$$N(Y, I(t, Y))^2 = D(t, I(t, Y)),$$

for  $Y$  a series in  $t$  with constant term 1. We write  $Y = 1 + tZ$ . Using the definitions (23) and (24) of  $N$  and  $D$ , the equation satisfied by the series  $Z$  can be written as

$$Z = \nu + t \text{Pol}(\nu, w, t, Z, M(1 + tZ)),$$

for some polynomial Pol with integer coefficients. This equation allows us to compute by induction on  $n$  the coefficient of  $t^n$  in  $Z$ , and shows the existence and uniqueness of  $Z$  (and  $Y$ ). More precisely,

$$Y = 1 + \nu t + 2\nu^2(w + 1)t^2 + \nu^3(5 + 14w + 6w^2)t^3 + O(t^4).$$

This gives

$$\begin{aligned} I_1(t) &:= I(t, Y) = 1/\nu + (\nu - w - 1)t + O(t^2), \\ D(t, I_1) &= (1 - 1/\nu)^2 + O(t), \\ \frac{\partial I}{\partial y}(t, Y) &= -\frac{1}{\nu^2 t} + O(1), \end{aligned}$$

so that the last three properties of the lemma hold.  $\square$

*Proof of Lemma 3.* We denote  $q_j = 4 \cos(j\pi/m)^2 = 2 + 2 \cos(2j\pi/m)$ . Observe that  $q = q_1$ . We assume that  $m = 2n + 1$  is odd. The proof is similar in the even case.

**Existence of the series  $Y_j$ .** We are looking for series  $Y \equiv Y(t)$  solutions of (30), of the form  $Y = 1 + tZ$  for some series  $Z \equiv Z(t)$ . Multiplying (30) by  $Z^2$  gives the equation  $\Phi_j(t, Z) = 0$ , where

$$\Phi_j(t, z) = z^2 N(1 + tz, I(t, 1 + tz))^2 - q_j z^2 D(t, I(t, 1 + tz)).$$

We are interested in non-zero solutions (since for  $Z = 0$ , we have  $Y = 1$  and  $I(t, Y)$  is not well defined). Observe that  $zI(t, 1 + tz)$  is a formal power series in  $t$  with coefficients in  $\mathbb{R}[\nu, w, z]$ . Hence the same holds for  $zN(1 + tz, I(t, 1 + tz))$  and  $z^2 D(t, I(t, 1 + tz))$ , and finally for  $\Phi_j(t, z)$ . Recall that  $\nu = \beta + 1$ . Expanding  $I$ ,  $N$  and  $D$  at first order in  $t$  gives

$$zI(t, 1 + tz) = 1 + O(t), \quad (31)$$

$$zN(1 + tz, I(t, 1 + tz)) = q + 2\beta - qz + O(t),$$

$$z^2 D(t, I(t, 1 + tz)) = q\beta + q + \beta^2 - q(\beta + 2)z + qz^2 + O(t). \quad (32)$$

The equation  $\Phi_j(t, Z) = 0$  thus reads

$$P_j(Z) + tS_j(t, Z) = 0, \quad (33)$$

where

$$P_j(z) = (q + 2\beta - qz)^2 - q_j (q\beta + q + \beta^2 - q(\beta + 2)z + qz^2)$$

and  $S_j(t, z)$  is a power series in  $t$  with coefficients in  $\mathbb{R}[\nu, w, z]$ . By (33), the coefficient of  $t^0$  in  $Z$ , denoted  $z_0$ , must satisfy  $P_j(z_0) = 0$ . Equivalently,  $z_0 = 1 + \beta v$  with

$$q(q - q_j)v^2 + q(q_j - 4)v + 4 - q_j = 0. \quad (34)$$

This equation has degree 1 in  $v$  if  $q_j = q$ , that is if  $j = 1$ , and degree 2 otherwise. Given the expressions of  $q_j$  and  $q = q_1$ , its roots are found to be

$$v_j^+ = \frac{\sin(j\pi/m)}{2 \cos(\pi/m) \sin((j+1)\pi/m)} \quad \text{and} \quad v_j^- = \frac{\sin(j\pi/m)}{2 \cos(\pi/m) \sin((j-1)\pi/m)},$$

for  $j \neq 1$ . For  $j = 1$  the only root is  $v_1^+ = 1/q$ . The roots  $v_j^+$  and  $v_j^-$  are distinct (we use here the assumption that  $m$  is odd).

Now having fixed  $z_0 = 1 + \beta v$  with  $v = v_j^\pm$ , let us return to (33) and extract from it the coefficient of  $t^p$ , for  $p \geq 1$ . This gives

$$P'_j(z_0)[t^p]Z + (\text{expression involving only } [t^i]Z \text{ for } i < p) = 0.$$

Since  $v$  is not a double root of  $P_j$ , this allows us to compute the coefficient of  $t^p$  in  $Z$  by induction on  $p$ , and thus determines  $Z$  completely. Moreover,  $[t^p]Z$  has a rational expression in  $\nu$  and  $w$  (with real coefficients).

It is easy to see that

$$0 < v_1^+ < \dots < v_n^+ = \frac{1}{2 \cos(\pi/m)} < v_n^- < \dots < v_2^-,$$

so that we have exactly  $(m - 2)$  distinct values  $v_j^\pm$ . Let us denote them  $v_1, \dots, v_{m-2}$ . They give rise to exactly  $(m - 2)$  distinct series  $Y$  satisfying  $Y(0) = 1$  and (30) for some  $j \in \llbracket 1, n \rrbracket$ . We denote them  $Y_1, \dots, Y_{m-2}$ , with  $Y_i = 1 + (1 + \beta v_i)t + O(t^2)$ . We note that  $v_i$  is a real number,

while  $\beta = \nu - 1$  where  $\nu$  is a formal parameter. Hence our  $m - 2$  series  $Y_i$  all differ from the constant 1, and  $I(t, Y_i)$  is well-defined. We have thus proved the first statement of the lemma.

**Properties of  $Y_i$  and  $I_i := I(t, Y_i)$ .** Let us prove that the series  $I_i := I(t, Y_i)$  are distinct. Returning to (31) gives  $I_i = (1 + \beta v_i)^{-1} + O(t)$ . Hence the  $(m - 2)$  constant terms  $I_i(0)$  are distinct since the  $(m - 2)$  numbers  $v_1, \dots, v_{m-2}$  are distinct. Moreover  $I_i(0)$  is obviously non-zero, and it is distinct from 1 because  $v_i \neq 0$ .

Let us now prove that  $D(t, I_i) \neq 0$ . Returning to (32), we find

$$(1 + \beta v_i)^2 D(t, I_i) = \beta^2 (q v_i^2 - q v_i + 1) + O(t) = \left( \beta \frac{\sin(\pi/m)}{\sin((j \pm 1)\pi/m)} \right)^2 + O(t), \quad (35)$$

where the value of the denominator depends on whether  $v_i$  is of the form  $v_j^+$  or  $v_j^-$ . At any rate, this is non-zero.

We finally check that

$$\frac{\partial I}{\partial y}(t, Y_i) = -\frac{1}{t(1 + \beta v_i)^2} + O(1)$$

is non-zero, and this completes the proof of the lemma.  $\blacksquare$

#### 4.3. SOME POLYNOMIALS WITH COMMON ROOTS

We still assume  $q = 2 + 2\cos(2\pi/m)$ . Recall that  $D(t, x)$  and  $C(t, x)$  are polynomials in  $x$  with coefficients in  $\mathbb{Q}(q, \nu, w)[[t]]$ , given by (24) and (26) respectively. Their degrees in  $x$  are 2 and  $m$ , respectively. The coefficients of  $D$  are explicit, but those of  $C$  are unknown, apart from the three leading ones (Lemma 2). We now prove that several polynomials, related to  $C(t, x)$  and  $D(t, x)$ , admit the series  $I_i$  of Lemma 3 as common roots.

**Lemma 4.** *Each of the series  $I_i := I(t, Y_i)$  defined by Lemma 3 satisfies*

$$C(t, I_i)^2 = D(t, I_i)^m, \quad (36)$$

$$D(t, I_i) \frac{\partial C}{\partial x}(t, I_i) = \frac{m}{2} C(t, I_i) \frac{\partial D}{\partial x}(t, I_i), \quad (37)$$

$$D(t, I_i) \frac{\partial C}{\partial t}(t, I_i) = \frac{m}{2} C(t, I_i) \frac{\partial D}{\partial t}(t, I_i). \quad (38)$$

Since  $D(t, I_i) \neq 0$ , it follows from (36) and (37) that  $I_i$  is actually a double root of  $C^2 - D^m$ .

*Proof.* Let us denote

$$G(t, y, x) = \frac{N(y, x)}{2\sqrt{D(t, x)}}.$$

Recall that  $D(t, I_i)$  is a formal power series in  $t$  with coefficients in  $\mathbb{R}(\nu, w)$ . By (35), its constant term is of the form  $\beta^2 r^2 / (1 + \beta v)^2$ , for some real numbers  $r \neq 0$  and  $v$ . So we can define  $\sqrt{D(t, I_i)}$  as a formal power series in  $t$  with coefficients in  $\mathbb{R}(\nu, w)$ .

By construction of the series  $Y_i$ , we have  $G(t, y, I_i) = \pm \cos(j\pi/m)$ , for some  $j \in \llbracket 1, n \rrbracket$ . Given that  $T_m(\cos x) = \cos(mx)$ , each series  $Y_i$  satisfies

$$T_m(G(t, Y_i, I(t, Y_i))) = \varepsilon \quad \text{and} \quad T'_m(G(t, Y_i, I(t, Y_i))) = 0,$$

where  $\varepsilon = \pm 1$ . Hence the invariant equation (25) gives

$$C(t, I_i) = \varepsilon D(t, I_i)^{m/2} \quad (39)$$

from which (36) follows.

Let us now differentiate the invariant equation (25) with respect to  $y$ :

$$\begin{aligned} \frac{m}{2} D(t, I(t, y))^{m/2-1} \frac{\partial D}{\partial x}(t, I(t, y)) \frac{\partial I}{\partial y}(t, y) \mathbb{T}_m(G(t, y, I(t, y))) \\ + D(t, I(t, y))^{m/2} \left( \frac{\partial G}{\partial y}(t, y, I(t, y)) + \frac{\partial G}{\partial x}(t, y, I(t, y)) \frac{\partial I}{\partial y}(t, y) \right) \mathbb{T}'_m(G(t, y, I(t, y))) \\ = \frac{\partial C}{\partial x}(t, I(t, y)) \frac{\partial I}{\partial y}(t, y), \end{aligned}$$

or, if we want to keep things in a reasonable volume,

$$\frac{m}{2} D^{m/2-1} D'_x I'_y \mathbb{T}_m(G) + D^{m/2} (G'_y + G'_x I'_y) \mathbb{T}'_m(G) = C'_x I'_y.$$

When  $y = Y_i$ , then  $\mathbb{T}_m(G) = \varepsilon$ ,  $\mathbb{T}'_m(G) = 0$  and  $I'_y \neq 0$  (by Lemma 3), so that

$$\frac{m}{2} \varepsilon D^{m/2-1} D'_x = C'_x. \quad (40)$$

Combined with (39), this gives (37).

Similarly, differentiating the invariant equation with respect to  $t$  gives

$$\frac{m}{2} D^{m/2-1} (D'_t + D'_x I'_t) \mathbb{T}_m(G) + D^{m/2} (G'_t + G'_x I'_t) \mathbb{T}'_m(G) = C'_t + C'_x I'_t.$$

When  $y = Y_i$ , then  $\mathbb{T}_m(G) = \varepsilon$ ,  $\mathbb{T}'_m(G) = 0$ , and  $C'_x$  is given by (40). Hence

$$\frac{m}{2} \varepsilon D^{m/2-1} D'_t = C'_t.$$

Combined with (39), this gives (38). ■

**Proposition 5.** *Let  $I_i := I(t, Y_i)$  be the  $(m-2)$  series defined in Lemma 3. There exists a triple  $(\widehat{P}(t, x), \widehat{Q}(t, x), \widehat{R}(t, x))$  of polynomials in  $x$  with coefficients in  $\mathbb{R}(\nu, w)[[t]]$ , having degree at most 4, 2 and 2 respectively in  $x$ , such that*

$$C(t, x)^2 - D(t, x)^m = \widehat{P}(t, x) \prod_{i=1}^{m-2} (x - I_i)^2, \quad (41)$$

$$D(t, x) C'_x(t, x) - \frac{m}{2} D'_x(t, x) C(t, x) = \widehat{Q}(t, x) \prod_{i=1}^{m-2} (x - I_i), \quad (42)$$

$$D(t, x) C'_t(t, x) - \frac{m}{2} D'_t(t, x) C(t, x) = \widehat{R}(t, x) \prod_{i=1}^{m-2} (x - I_i). \quad (43)$$

*Proof.* The polynomial  $C(t, x)^2 - D(t, x)^m$  has degree (at most)  $2m$  in  $x$ , and admits each of the  $(m-2)$  distinct series  $I_i$  as a double root (Lemma 4). Thus, there exists a polynomial  $\widehat{P}(t, x)$  of degree at most 4 such that (41) holds. Let us write

$$\widehat{P}(t, x) = (C(t, x)^2 - D(t, x)^m) \prod_{i=1}^{m-2} \frac{1}{I_i^2 (1 - x I_i^{-1})^2}. \quad (44)$$

Recall that  $C$  and  $D$  are polynomials in  $x$  with coefficients in  $\mathbb{R}(\nu, w)[[t]]$ , that the series  $I_i$  belong to  $\mathbb{R}(\nu, w)[[t]]$  and have a non-zero constant term (Lemma 3). Expanding in  $x$  the above expression thus proves that the coefficients of  $\widehat{P}$  also belong to  $\mathbb{R}(\nu, w)[[t]]$ .

Similarly,  $D(t, x) C'_x(t, x) - \frac{m}{2} D'_x(t, x) C(t, x)$  is a polynomial of degree at most  $m$  in  $x$ , since  $D(t, x) C'_x(t, x)$  and  $\frac{m}{2} D'_x(t, x) C(t, x)$  are polynomials of degree  $m+1$  having the same leading coefficient. Hence by (37), there exists a polynomial  $\widehat{Q}(t, x)$  in  $\mathbb{R}(\nu, w)[[t]][x]$ , of degree at most 2, such that (42) holds.

Finally, given that  $D'_t(t, x)$  has degree 0 in  $x$ , and  $C'_t(t, x)$  degree at most  $m-2$  (Lemma 2), the polynomial  $D(t, x) C'_t(t, x) - \frac{m}{2} D'_t(t, x) C(t, x)$  has degree at most  $m$ . Equation (38) implies



the existence of a polynomial  $\widehat{R}(t, x)$  in  $\mathbb{R}(\nu, w)[[t]][x]$ , of degree at most 2, such that (43) holds.

■

#### 4.4. DIFFERENTIAL SYSTEM

We still assume that  $q = 2 + 2\cos(2\pi/m)$ . We will prove that the series  $\widehat{P}$ ,  $\widehat{Q}$  and  $\widehat{R}$  of Proposition 5, suitably normalized into series  $P$ ,  $Q$  and  $R$ , satisfy all equations of Theorem 1. We repeat this theorem for convenience.

**Theorem 1 (repeated).** *Let  $q$  be an indeterminate,  $\beta = \nu - 1$  and*

$$D(t, x) = (q\nu + \beta^2)x^2 - q(\nu + 1)x + \beta t(q - 4)(wq + \beta) + q. \quad (45)$$

*There exists a unique triple  $(P(t, x), Q(t, x), R(t, x))$  of polynomials in  $x$  with coefficients in  $\mathbb{Q}[q, \nu, w][[t]]$ , having degree 4, 2 and 2 respectively in  $x$ , such that*

$$\begin{aligned} [x^4]P(t, x) &= 1, & P(0, x) &= x^2(x - 1)^2, \\ [x^2]R(t, x) &= \nu + 1 - w(q + 2\beta), & Q(0, x) &= x(x - 1), \end{aligned} \quad (46)$$

and

$$\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right). \quad (47)$$

*Let  $P_j(t) \equiv P_j$  (resp.  $Q_j, R_j$ ) denote the coefficient of  $x^j$  in  $P(t, x)$  (resp.  $Q(t, x), R(t, x)$ ). The Potts generating function of planar maps,  $M_1 \equiv M(q, \nu, t, w; 1)$ , can be expressed in terms of the series  $P_j$  and  $Q_j$  using  $\tilde{M}_1 := t^2 M_1$  and*

$$12(\beta^2 + q\nu)\tilde{M}_1 + P_3^2/4 + 2t(1 + \nu - w(2\beta + q))P_3 - P_2 + 2Q_0 = 4t(1 + w(3\beta + q)). \quad (48)$$

*An alternative characterization of  $M_1$  is in terms of the derivative of  $\tilde{M}_1$ :*

$$2(\beta^2 + q\nu)\tilde{M}_1' + (1 + \nu - w(2\beta + q))P_3/2 - R_1 = 2 + 2\beta w. \quad (49)$$

*The series  $M_1$  is differentially algebraic, that is, satisfies a non-trivial differential equation with respect to the edge variable  $t$ . The same holds for each series  $P_j, Q_j$  and  $R_j$ .*

**A differential equation relating  $\widehat{P}, \widehat{Q}$  and  $\widehat{R}$ .** We first prove that (47) holds for  $\widehat{P}, \widehat{Q}$  and  $\widehat{R}$ . This results from the elimination of  $C$  and  $\prod_i (x - I_i)$  in the three equations of Proposition 5. The derivation is analogous to that of Section 3.1.2. Let us first eliminate the product, in such a way that  $\widehat{Q}^2/(\widehat{P}D^2)$  and  $\widehat{R}^2/(\widehat{P}D^2)$  naturally appear:

$$\frac{(DC'_x - \frac{m}{2}D'_x C)^2}{D^2(C^2 - D^m)} = \frac{\widehat{Q}^2}{\widehat{P}D^2} \quad \text{and} \quad \frac{(DC'_t - \frac{m}{2}D'_t C)^2}{D^2(C^2 - D^m)} = \frac{\widehat{R}^2}{\widehat{P}D^2}.$$

Let us differentiate the first equation with respect to  $t$ , and the second one with respect to  $x$ . The ratio of the two resulting identities is

$$\frac{DC'_x - \frac{m}{2}D'_x C}{DC'_t - \frac{m}{2}D'_t C},$$

which, according to the last two equations of Proposition 5, is  $\widehat{Q}/\widehat{R}$ . This gives

$$\frac{1}{\widehat{Q}} \frac{\partial}{\partial t} \left( \frac{\widehat{Q}^2}{\widehat{P}D^2} \right) = \frac{1}{\widehat{R}} \frac{\partial}{\partial x} \left( \frac{\widehat{R}^2}{\widehat{P}D^2} \right). \quad (50)$$

This equation coincides with (47) but with hats over the letters  $P, Q$  and  $R$ . The series  $P, Q$  and  $R$  occurring in the theorem will simply be normalizations of  $\widehat{P}, \widehat{Q}$  and  $\widehat{R}$  by multiplicative constants independent from  $t$  and  $x$ .

**The leading coefficients of  $\widehat{P}$ ,  $\widehat{Q}$ ,  $\widehat{R}$ .** Proposition 5 gives

$$\begin{aligned} [x^4]\widehat{P}(t, x) &= [x^{2m}] (C(t, x)^2 - D(t, x)^m), \\ [x^2]\widehat{Q}(t, x) &= [x^m] \left( D(t, x)C'_x(t, x) - \frac{m}{2}D'_x(t, x)C(t, x) \right), \\ [x^2]\widehat{R}(t, x) &= [x^m] \left( D(t, x)C'_t(t, x) - \frac{m}{2}D'_t(t, x)C(t, x) \right). \end{aligned}$$

Recall that  $D$  is defined by (45), and that Lemma 2 gives the leading coefficients of  $C(t, x)$ . This allows to determine the leading coefficients of  $\widehat{P}$ ,  $\widehat{Q}$ ,  $\widehat{R}$ :

$$\begin{cases} \widehat{P}_4 := [x^4]\widehat{P}(t, x) &= \delta^m (\mathrm{T}_m(\gamma)^2 - 1) = \frac{q}{m^2}(q/4 - 1)\delta^{m-1} \mathrm{T}'_m(\gamma)^2, \\ \widehat{Q}_2 := [x^2]\widehat{Q}(t, x) &= q\beta(q/4 - 1)\delta^{(m-1)/2} \mathrm{T}'_m(\gamma), \\ \widehat{R}_2 := [x^2]\widehat{R}(t, x) &= q\beta(q/4 - 1)\delta^{(m-1)/2} \mathrm{T}'_m(\gamma)(\nu + 1 - w(q + 2\beta)), \end{cases} \quad (51)$$

where  $\delta = \beta^2 + q\nu$  and  $\gamma = (q + 2\beta)/(2\sqrt{\delta})$ . In the expression of  $\widehat{P}_4$ , we have used the fact that, for all  $x$ ,

$$(u^2 - 1)\mathrm{T}'_m(u)^2 = m^2 (\mathrm{T}_m(u)^2 - 1). \quad (52)$$

Observe that these coefficients are independent of  $t$ . Let us define

$$P(t, x) = \frac{\widehat{P}(t, x)}{\widehat{P}_4}, \quad Q(t, x) = \frac{\widehat{Q}(t, x)}{\widehat{Q}_2} \quad \text{and} \quad R(t, x) = \frac{\widehat{R}(t, x)}{\widehat{R}_2}. \quad (53)$$

We have intentionally normalized  $\widehat{R}(t, x)$  by  $\widehat{Q}_2$  rather than  $\widehat{R}_2$ . It then follows from (50) that  $P$ ,  $Q$ ,  $R$  satisfy the differential system (47). Moreover

$$[x^4]P(t, x) = 1, \quad [x^2]Q(t, x) = 1 \quad \text{and} \quad [x^2]R(t, x) = \nu + 1 - w(q + 2\beta), \quad (54)$$

so that the left-hand side of the initial conditions (46) hold. (We do not give the value of  $[x^2]Q(t, x)$  in the statement of the theorem because it is a consequence of the differential system and the initial conditions, as will be seen in Section 4.6.)

**The case  $t = 0$ .** Let us now establish the right-hand side of the initial conditions (46) by determining the values  $P(0, x)$  and  $Q(0, x)$ . Proposition 5 gives

$$C(0, x)^2 - D(0, x)^m = \widehat{P}(0, x) \prod_{i=1}^{m-2} (x - I_i(0))^2, \quad (55)$$

$$D(0, x)C'_x(0, x) - \frac{m}{2}D'_x(0, x)C(0, x) = \widehat{Q}(0, x) \prod_{i=1}^{m-2} (x - I_i(0)). \quad (56)$$

We will now combine our knowledge of  $C(0, x)$  obtained in [2] with the properties of the series  $I_i(t)$  gathered in Lemma 3 to determine  $\widehat{P}(0, x)$  and  $\widehat{Q}(0, x)$ .

Recall that  $C_r(t)$  denotes the coefficient of  $x^r$  in  $C(t, x)$  (see (26)). The constant term of the series  $C_r(t)$  has been determined in [2, Lemma 16]. With the notation used there,  $C_r(0) = L_r(0; 1)$ , where

$$\sum_{r=0}^m L_r(t; y)x^r = D(t, x)^{m/2} \mathrm{T}_m \left( \frac{N(y, x)}{2\sqrt{D(t, x)}} \right),$$

where  $N$  and  $D$  are given by (23) and (24). Hence

$$C(0, x) \equiv \sum_{r=0}^m C_r(0)x^r = \mathcal{D}^{m/2} \mathrm{T}_m \left( \frac{\mathcal{N}}{2\sqrt{\mathcal{D}}} \right), \quad (57)$$

where  $\mathcal{N} = N(1, x) = (q + 2\beta)x - q$  and  $\mathcal{D} = D(0, x) = (q\nu + \beta^2)x^2 - q(\nu + 1)x + q$ . Thus we first want to factor

$$C(0, x)^2 - D(0, x)^m = \mathcal{D}^m \left( \mathrm{T}_m \left( \frac{\mathcal{N}}{2\sqrt{\mathcal{D}}} \right)^2 - 1 \right).$$

Denote  $n = \lfloor m/2 \rfloor$ . By (52) and (29),

$$T_m(u)^2 - 1 = \frac{1}{m^2}(u^2 - 1)T'_m(u)^2 = 4^{m-1}(u^2 - 1)(u^2)^{1_{m=2n}} \prod_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \left( u^2 - \cos^2 \frac{j\pi}{m} \right)^2. \quad (58)$$

(We have also used the fact that the dominant coefficient of  $T_m$  is  $2^{m-1}$ .) Thus

$$\begin{aligned} C(0, x)^2 - D(0, x)^m &= \frac{1}{4}(\mathcal{N}^2 - 4\mathcal{D})(\mathcal{N}^2)^{1_{m=2n}} \prod_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \left( \mathcal{N}^2 - 4\mathcal{D} \cos^2 \frac{j\pi}{m} \right)^2 \\ &= \frac{q}{4}(q-4)(x-1)^2(\mathcal{N}^2)^{1_{m=2n}} \prod_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \left( \mathcal{N}^2 - 4\mathcal{D} \cos^2 \frac{j\pi}{m} \right)^2. \end{aligned} \quad (59)$$

Let us now compare this to (55). By Lemma 3,  $I_i(0) \neq 1$  for all  $i$ , so that  $(x-1)^2$  is necessarily a factor of  $\widehat{P}(0, x)$ . We will now prove that  $x^2$  is also a factor of  $\widehat{P}(0, x)$ .

Consider the term obtained for  $j=1$  in (59). Using  $4 \cos^2(\pi/m) = q$ , we get

$$\mathcal{N}^2 - 4\mathcal{D} \cos^2 \frac{\pi}{m} = \mathcal{N}^2 - q\mathcal{D} = x\beta(q-4)(q-x(q+\beta)),$$

which has a factor  $x$ . By Lemma 3,  $I_i(0) \neq 0$  for all  $i$ , so that  $x^2$  is necessarily a factor of  $\widehat{P}(0, x)$ . We have proved that  $\widehat{P}(0, x)$ , hence also  $P(0, x)$ , is divisible by  $x^2(x-1)^2$ . Moreover  $P(0, x)$  has degree 4 and constant term 1, thus  $P(0, x) = x^2(x-1)^2$ .

We now wish to determine  $\widehat{Q}(0, x)$ . We have just seen that  $x=0$  and  $x=1$  are double roots of  $C(0, x)^2 - D(0, x)^m$ . So they cancel as well the derivative  $2C(0, x)C'(0, x) - mD(0, x)^{m-1}D'(0, x)$ , and since  $D(0, 0) \neq 0$  and  $D(0, 1) \neq 0$ , they must also cancel  $D(0, x)C'_x(0, x) - \frac{m}{2}D'_x(0, x)C(0, x)$ . Let us now return to the factorisation (56). Since  $I_i(0) \neq 0, 1$ , we see that  $x(x-1)$  must divide  $\widehat{Q}(0, x)$ . Hence it also divides  $Q(0, x)$ . Moreover we have proved above that  $[x^2]Q(0, x) = 1$  so that  $Q(0, x) = x(x-1)$  as stated in the theorem.

#### 4.5. THE POTTS GENERATING FUNCTION OF PLANAR MAPS

We still assume that  $q = 2 + 2 \cos(2\pi/m)$ . Let us prove that the Potts generating function  $M_1$  is related to the  $P_j$ 's and  $Q_j$ 's by (48) and (49). It follows from the first two identities of Proposition 5 that

$$\widehat{P} \left( DC'_x - \frac{m}{2}D'_xC \right)^2 = \widehat{Q}^2 (C^2 - D^m),$$

where all series and polynomials are evaluated at  $(t, x)$ . Using the normalization (53), this gives

$$P \left( DC'_x - \frac{m}{2}D'_xC \right)^2 = qm^2\beta^2(q/4 - 1)Q^2 (C^2 - D^m), \quad (60)$$

since  $\widehat{Q}_2^2 = qm^2\beta^2(q/4 - 1)\widehat{P}_4$  by (51). Recall that  $D$  is defined by (45), that the leading coefficients of  $C$  are given in Lemma 2, and that  $C^2 - D^m$  has degree  $2m$ . Recall also the known values (54) of  $P_4$  and  $Q_2$ . Extracting from (60) the coefficients of  $x^{2m+4}$ ,  $x^{2m+3}$ , and  $x^{2m+2}$  gives:

- for the coefficient of  $x^{2m+4}$ , a tautology, equivalent to (52) taken at  $u = \gamma$ ,
- for the coefficient of  $x^{2m+3}$ , an interesting relation between  $P_3$  and  $Q_1$ , namely

$$P_3 = 2Q_1 + 4t(1 + \nu) - 4tw(2\beta + q), \quad (61)$$

- for the coefficient of  $x^{2m+2}$ , the identity (48). In the calculation we use once again (52) to relate  $T_m(\gamma)$  and  $T'_m(\gamma)$ , as well as (61) to express  $Q_1$  in terms of  $P_3$ .

The second characterization (49) of  $M_1$  is obtained in a similar fashion by combining instead the second and third identities of Proposition 5: they imply

$$R \left( DC'_x - \frac{m}{2} D'_x C \right) = Q \left( DC'_t - \frac{m}{2} D'_t C \right),$$

and extracting the coefficient of  $x^{m+1}$  gives an expression of  $\tilde{M}'_1$  in terms of  $Q_1$  and  $R_1$ , which we transform into (49) using (61).

#### 4.6. UNIQUENESS OF THE SOLUTION

The arguments in this subsection apply whether  $q$  is an indeterminate, or  $q = 2 + 2 \cos(2\pi/m)$ . The differential system of Theorem 1 can be written as

$$2Q'_t PD - QP'_t D - 2QPD'_t = 2R'_x PD - RP'_x D - 2RPD'_x. \quad (62)$$

Since  $P$ ,  $Q$  and  $R$  have respective degree 4, 2 and 2 in  $x$ , this identity relates two polynomials in  $x$  of degree at most 8. Recall that  $P_4 = R_2 = 1$ . Extracting the coefficient of  $x^8$  gives  $Q'_2(t) = 0$ , which, with the initial condition  $Q_2(0) = 1$ , implies  $Q_2(t) = 1$ . Hence the leading coefficients of  $P$ ,  $Q$  and  $R$  (that is, the series  $P_4$ ,  $Q_2$  and  $R_2$ ) are independent of  $t$ , and the left-hand side of (62), as well as its right-hand side, has degree at most 7 in  $x$ . And we are left with eight unknown series.

We denote  $P_{i,j} := [t^i]P_j$ , and similarly for  $Q$  and  $R$ , so that

$$P(t, x) = \sum_{i,j} P_{i,j} t^i x^j.$$

Let  $\mathcal{C}_i$  be the following 8-tuple of coefficients:

$$\mathcal{C}_i = (P_{i,0}, P_{i,1}, P_{i,2}, P_{i,3}; Q_{i,0}, Q_{i,1}; R_{i-1,0}, R_{i-1,1}).$$

The right-hand side of (46) gives us the values of  $P(t, x)$  and  $Q(t, x)$  at  $t = 0$ , so that

$$\mathcal{C}_0 = (0, 0, 1, -2; 0, -1; 0, 0).$$

We will show by induction on  $i \geq 1$  that the differential system determines the eight coefficients of  $\mathcal{C}_i$ , and that these coefficients are rational functions of  $q, \beta, w$ .

For  $i \geq 1$  and  $0 \leq j \leq 7$ , the equation  $\text{Eq}_{i,j}$  obtained by extracting the coefficient of  $t^{i-1}x^j$  in (62) reads

$$\sum_{\substack{i_1+i_2+i_3=i \\ j_1+j_2+j_3=j}} (2i_1 - i_2 - 2i_3) Q_{i_1, j_1} P_{i_2, j_2} D_{i_3, j_3} = \sum_{\substack{i_1+i_2+i_3=i-1 \\ j_1+j_2+j_3=j+1}} (2j_1 - j_2 - 2j_3) R_{i_1, j_1} P_{i_2, j_2} D_{i_3, j_3},$$

where  $D_{i,j} = [t^i x^j] D(t, x)$ . This is a linear equation in the unknowns of  $\mathcal{C}_i$ , of the form

$$\begin{aligned} \sum_{j_1+j_2+j_3=j} (2iQ_{i,j_1}P_{0,j_2}D_{0,j_3} - iQ_{0,j_1}P_{i,j_2}D_{0,j_3}) \\ - \sum_{j_1+j_2+j_3=j+1} (2j_1 - j_2 - 2j_3) R_{i-1, j_1} P_{0, j_2} D_{0, j_3} = K_{i,j}, \end{aligned} \quad (63)$$

where  $K_{i,j}$  is a polynomial in the coefficients of  $\cup_{s < i} \mathcal{C}_s$ , with coefficients in  $\mathbb{Q}[q, \beta, w]$ . It would be convenient if the eight equations  $\text{Eq}_{i,j}$ , for  $j \in \llbracket 0, 7 \rrbracket$ , could define the eight unknown coefficients of  $\mathcal{C}_i$ , but this is not exactly what happens, for two reasons.

First, the equation  $\text{Eq}_{i,0}$  involves none of the coefficients of  $\mathcal{C}_i$ . Indeed, we see on (63), specialized at  $j = 0$ , that the only coefficients of  $\mathcal{C}_i$  that  $\text{Eq}_{i,0}$  may involve are  $Q_{i,0}$ ,  $P_{i,0}$ ,  $R_{i-1,0}$  and  $R_{i-1,1}$ . But they do not occur, because  $P_{0,0} = P_{0,1} = Q_{0,0} = 0$  (this follows from the initial conditions (46)). Hence this equation reads  $K_{i,0} = 0$  and only involves coefficients of  $\cup_{s < i} \mathcal{C}_s$ . We leave it to the reader to check that it is linear in the coefficients of  $\mathcal{C}_{i-1}$ , provided  $i > 2$ .

Then, a similar problem happens with the sum of the eight equations  $\text{Eq}_{i,j}$ , for  $j \in \llbracket 0, 7 \rrbracket$ : it does not involve any of the coefficients of  $\mathcal{C}_i$  either. Indeed, it reads

$$\sum_{j_1, j_2, j_3} (2iQ_{i,j_1}P_{0,j_2}D_{0,j_3} - iQ_{0,j_1}P_{i,j_2}D_{0,j_3}) - \sum_{j_1, j_2, j_3} (2j_1 - j_2 - 2j_3)R_{i-1,j_1}P_{0,j_2}D_{0,j_3} = \sum_{j=0}^7 K_{i,j}.$$

But the left-hand side is the coefficient of  $t^{i-1}$  in

$$2Q'_i(t, 1)P(0, 1)D(0, 1) - Q(0, 1)P'_t(t, 1)D(0, 1) - 2R'_x(t, 1)P(0, 1)D(0, 1) + R(t, 1)P'_x(0, 1)D(0, 1) + 2R(t, 1)P(0, 1)D'_x(0, 1),$$

and this series is zero because  $P(0, 1) = Q(0, 1) = P'_x(0, 1) = 0$  (see the initial conditions (46)). Hence this sum of equations only involves coefficients of  $\cup_{s < i} \mathcal{C}_s$ . We leave it to the reader to check that again, this sum is linear in the coefficients of  $\mathcal{C}_{i-1}$ , provided  $i > 2$ .

These observations lead us to consider the following system of eight equations:

$$\mathcal{S}_i = \left\{ \sum_{j=0}^7 \text{Eq}_{i+1,j}, \text{Eq}_{i+1,0}, \text{Eq}_{i,2}, \text{Eq}_{i,3}, \dots, \text{Eq}_{i,7} \right\}. \quad (64)$$

Solving the system  $\mathcal{S}_1$  gives

$$\mathcal{C}_1 = (-4, 8 - 2wq, 4w(q - \beta) - 2\beta - 4, 2\beta - 2wq; wq + 2\beta + 4, 4w\beta - \beta + wq - 4; 2, wq - \beta - 4). \quad (65)$$

Moreover for  $i > 1$ ,  $\mathcal{S}_i$  is a system of eight linear equations for the eight unknowns of  $\mathcal{C}_i$  in terms of the rational functions in  $\cup_{s < i} \mathcal{C}_s$ . The determinant of this linear system is

$$256 i^6 q^3 \beta^7 w (q - 4) (q\nu + \beta^2)^2,$$

which is non-zero when  $q$  is an indeterminate but also when  $q \neq 0, 4$  is of the form  $2 + 2 \cos(2\pi/m)$ . By induction, this proves that the coefficients  $P_{i,j}, Q_{i,j}, R_{i,j}$  are uniquely determined by the differential system and its initial conditions. Moreover these coefficients lie in  $\mathbb{Q}(q, \beta, w)$ , and their denominators are products of terms  $q, \beta, w, (q - 4)$  and  $(q\nu + \beta^2)$ .

#### 4.7. ABOUT POSSIBLE SINGULARITIES

It remains to prove that the coefficients of the series  $P_j, Q_j$  and  $R_j$  are polynomials in  $q, \nu$  and  $w$ .

We will use three identities that we will establish later. Their proofs do not assume anything on the singularities of the coefficients of  $P_j, Q_j$  and  $R_j$ . The first one is the characterization (49) of  $\tilde{M}_1$ , established in Section 4.8:

$$2(\beta^2 + q\nu)\tilde{M}'_1 + (1 + \nu - w(2\beta + q))\bar{P}_3/2 - \bar{R}_1 = 2 + 2\beta w. \quad (66)$$

The other two are established in Section 6.2:

$$\beta(wq + \beta)(q - 4)Q_0 + q(\beta + 2)R_0 + 2(\beta(q - 4)(wq + \beta)t + q)R_1 = 2\beta(q - 4)(wq - 2)(wq + \beta)t + 2q(wq - 2), \quad (67)$$

$$\beta(wq + \beta)(q - 4)Q_1 - 2(\beta^2 + q\beta + q)R_0 - q(\beta + 2)R_1 = 2\beta(q - 4)(2\beta w + wq - \beta - 2)(wq + \beta)t - 2q(\beta qw - 2\beta w + wq - \beta - 2). \quad (68)$$

Let us now prove by induction on  $i$  that the coefficients of  $\mathcal{C}_i$  have no singularity at  $q = 4$  or  $q = -\beta^2/\nu$ . This holds for  $i = 0$  and  $i = 1$ . From (66), we derive that this holds for  $R_{i-1,1}$  if it holds for  $P_{i-1,3}$ , which we assume by the induction hypothesis. Now if we remove the last equation from the system  $\mathcal{S}_i$  (given by (64)), we obtain seven polynomial equations between the coefficients of

$\cup_{s \leq i} \mathcal{C}_s$ . Once the values of  $\mathcal{C}_1$  are known, they are linear in  $P_{i,0}, P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,0}, Q_{i,1}, R_{i-1,0}$ , with determinant:

$$-128i^6 q^2 \beta^4 w (6q^4 \nu^3 + \beta q^3 (\beta^4 + 21\beta^3 + 36\beta^2 + 11\beta - 6) + \beta^3 q^2 (3\beta^3 + 34\beta^2 + 42\beta + 12) + \beta^5 q (3\beta^2 + 24\beta + 16) + \beta^7 (\beta + 6)).$$

The last factor is irreducible, and this determinant contains no factor  $(q-4)$  nor  $(q\nu + \beta^2)$ . This proves that the coefficients in  $\mathcal{C}_i$  are not singular at  $q = 4$  nor  $q = -\beta^2/\nu$ .

Let  $\text{Eq}_i^{(1)}, \text{Eq}_i^{(2)}, \text{Eq}_i^{(3)}$  be the equations obtained by extracting the coefficient of  $t^i$  in the equations (66), (67) and (68). The system

$$\left\{ \sum_{j=0}^7 \text{Eq}_{i+1,j}, \text{Eq}_{4,i}, \dots, \text{Eq}_{7,i}, \text{Eq}_i^{(1)}, \text{Eq}_i^{(2)}, \text{Eq}_i^{(3)} \right\}$$

relates polynomially the coefficients  $P_{s,j}, Q_{s,j}, R_{s,j}$  for  $s \leq i$  and the coefficients (in  $t$ ) of the series  $M_1(t)$ . For  $i > 1$ , it is linear in  $P_{i,0}, \dots, P_{i,3}, Q_{i,0}, Q_{i,1}, R_{i,0}, R_{i,1}$ , with determinant

$$-16i^5 \beta^5 (q-4)w(q\nu + \beta^2)^5,$$

and this excludes singularities at  $q = 0$ .

Observe that the sum of (67) and (68) is divisible by  $\beta$ . If we consider now the system

$$\left\{ \text{Eq}_{i+1,0}, \text{Eq}_{4,i}, \dots, \text{Eq}_{7,i}, \text{Eq}_i^{(1)}, \text{Eq}_i^{(2)}, \beta^{-1}(\text{Eq}_i^{(2)} + \text{Eq}_i^{(3)}) \right\},$$

in the same unknowns as before, we obtain the determinant

$$16i^5 q^2 (q-4)w(q\nu + \beta^2)^5,$$

proving this time that the coefficients are not singular at  $\beta = 0$ .

Finally, to rule out poles at  $w = 0$ , we resort to a different argument. First, we return to the case  $q = q_m := 2 + 2\cos(2\pi/m)$  studied from Section 4.1 to Section 4.5. A first observation is that the series  $C_r$  involved in the invariant equation (25) are series in  $t$  with coefficients in  $\mathbb{Q}[q, \nu, w]$ . This follows from the proof of Lemma 16 in [2], using the fact that the objects denoted by  $L_r(t; y)$  belong to  $\mathbb{Q}[q, \nu, w, t, 1/y]$ . A second observation is that the series  $Y_i$  of Lemma 3 satisfy  $Y_i = 1 + t(1 + \beta v) + O(t^2)$  where  $v \in \mathbb{R}^*$ , and that their coefficients belong to  $\mathbb{R}(\nu)[w]$ . This follows easily from the proof of Lemma 3, since  $P'_j(z_0)$  does not depend on  $w$ . Consequently,  $I_i := I(t, Y_i)$  is a series in  $t$ , with a non-zero constant term *that does not depend on  $w$* , and its other coefficients lie in  $\mathbb{R}(\nu)[w]$ . Using these two observations, (44) now implies that  $\hat{P}(t, x)$  is a polynomial in  $x$  with coefficients in  $\mathbb{R}(\nu)[w][[t]]$ . By (53), the same holds for  $P(t, x)$ . Similarly, the polynomials  $Q(t, x)$  and  $R(t, x)$  have coefficients in  $\mathbb{R}(\nu)[w][[t]]$ .

In the next subsection, we prove (without assuming polynomiality of the coefficients of the series  $P_j, Q_j$  and  $R_j$ ), that these series, once specialized at  $q = q_m$ , are indeed the coefficients of the polynomials  $P(t, x), Q(t, x)$  and  $R(t, x)$  constructed for  $q = q_m$ . Hence, if one of these series had a pole at  $w = 0$ , this would remain the case for infinitely many values  $q_m$ , and this contradicts the fact that  $P(t, x), Q(t, x)$  and  $R(t, x)$  have coefficients in  $\mathbb{R}(\nu)[w][[t]]$ .

#### 4.8. CONCLUSION OF THE PROOF

We can now conclude the proof of Theorem 1. Let  $q$  be an indeterminate. We have proved in Section 4.6 that the differential system (47) and the initial conditions (46) define the series  $P_j, Q_j$  and  $R_j$  uniquely as formal power series in  $t$ , and that their coefficients lie in  $\mathbb{Q}[q, \beta, w]$ .

Let us temporarily denote these series by  $\bar{P}_j, \bar{Q}_j$  and  $\bar{R}_j$ , to avoid confusion with the series denoted  $P_j, Q_j$  and  $R_j$  above, which depend on a specific value of  $q$  of the form  $q_m := 2 + 2\cos(2\pi/m)$ . Specializing the indeterminate  $q$  to  $q_m$  in the series  $\bar{P}_j, \bar{Q}_j$  and  $\bar{R}_j$  gives series in  $t$  satisfying the differential system and its initial conditions. But we have also proved that this system has a unique power series solution when  $q = q_m$ . Thus  $\bar{P}_j$  evaluated at  $q = q_m$  coincides with  $P_j$ , and similar statements relate the series  $Q_j$  and  $R_j$  and their barred versions. We have

proved in Subsection 4.5 that when  $q = q_m$ , the Potts generating function  $M_1$  is related to the  $P_j$ 's,  $Q_j$ 's and  $R_j$ 's by (48) and (49). This means that when  $q = q_m$ ,

$$12(\beta^2 + q\nu)\tilde{M}_1 + \bar{P}_3^2/4 + 2t(1 + \nu - w(2\beta + q))\bar{P}_3 - \bar{P}_2 + 2\bar{Q}_0 = 4t(1 + w(3\beta + q))$$

and

$$2(\beta^2 + q\nu)\tilde{M}_1' + (1 + \nu - w(2\beta + q))\bar{P}_3/2 - \bar{R}_1 = 2 + 2\beta w.$$

Since all series involved in these identities have polynomial coefficients in  $q$ , and coincide for infinitely many values of  $q$ , they must hold for  $q$  an indeterminate.

Let us finally prove that  $M_1$  is differentially algebraic. This follows from the uniqueness of the solution of our differential system in terms of power series, via an approximation theorem due to Denef and Lipshitz [20, Thm. 2.1]. This theorem generalizes to differential systems one of Artin's approximation theorems for algebraic systems, and implies, in our context, that each of the series  $P_j$ ,  $Q_j$  and  $R_j$  is differentially algebraic. The expression of  $\tilde{M}_1 = t^2 M_1$  given by (48), and the fact that differentially algebraic series form a ring, implies that  $M_1$  is also differentially algebraic.

## 5. DIFFERENTIAL SYSTEM FOR COLOURED TRIANGULATIONS

We now consider triangulations, and more generally *near-triangulations*, which are planar maps in which every non-root face has degree 3. We weight these maps by the number of vertices (variable  $w$ ), the degree of the root-face ( $y$ ), and by their Potts polynomial (divided by  $q$ ). We denote by  $T(q, \nu, w; y) \equiv T(y)$  the associated generating function:

$$T(q, \nu, w; y) = \frac{1}{q} \sum_M P_M(q, \nu) w^{v(M)} y^{\text{drf}(M)},$$

where the sum runs over all near-triangulations. We ignore the number of edges, which would be redundant: a near-triangulation with  $v$  vertices and outer degree  $d$  has  $3v - d - 3$  edges. In our first paper on coloured maps [2], we counted edges (with a variable  $t$ ) rather than vertices, using a generating function  $Q(q, \nu, t; x, y)$  involving two catalytic variables  $x$  and  $y$ . It is related to  $T(y)$  by:

$$T(q, \nu, w; y) \equiv T(y) = w Q(q, \nu, w^{1/3}; 0, w^{1/3}y).$$

Our objective is to establish a differential system for the Potts generating function of near-triangulations of outer degree 1, denoted by  $T_1 \equiv T_1(w)$ . Note that this is the coefficient of  $y$  in  $T(y)$ . More generally, we write

$$T(q, \nu, w; y) = \sum_{d \geq 0} T_d(w) y^d,$$

hoping that no confusion arises with the  $m$ th Chebyshev polynomial  $T_m$ . The root-edge of near-triangulations counted by  $T_1$  is a loop. Its deletion gives a near-triangulation of outer degree 2, and thus

$$T_1 = \nu T_2,$$

an identity that will be useful later. The expansion of  $T_1$  at order 3 reads

$$\begin{aligned} T_1 &= \nu(q - 1 + \nu)w^2 + \nu((q - 1)(q - 2 + 2\nu) + \nu^2(q - 1 + \nu^2) \\ &\quad + 2\nu(q - 1 + \nu)(q - 1 + \nu^2) + \nu^2(q - 1 + \nu^2)^2) w^3 + O(w^4), \end{aligned}$$

as illustrated in Figure 3.

Our differential system for near-triangulations is very similar to the one obtained for general maps (Theorem 1), but a bit simpler.

**Theorem 6.** *Let  $q$  be an indeterminate,  $\beta = \nu - 1$  and*

$$D(w, x) = q\nu^2 x^2 + \beta(4\beta + q)x + q\beta\nu(q - 4)w + \beta^2.$$

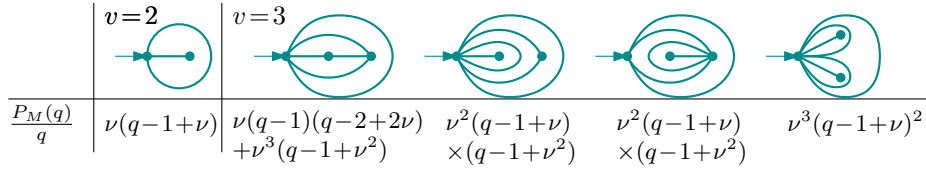


FIGURE 3. The rooted near-triangulations of outer degree 1 with  $v = 2$  and  $v = 3$  vertices, and their Potts polynomials (divided by  $q$ ).

There exists a unique triple  $(P(w, x), Q(w, x), R(w, x))$  of polynomials in  $x$  with coefficients in  $\mathbb{Q}[q, \nu][[w]]$ , having degree 3, 2 and 1 respectively in  $x$ , such that

$$[x^3]P(w, x) = 1, \quad P(0, x) = x^2(x + 1/4), \quad (69)$$

$$Q(0, x) = x(2\nu x + 1),$$

and

$$\frac{1}{R} \frac{\partial}{\partial x} \left( \frac{R^2}{PD^2} \right) = \frac{1}{Q} \frac{\partial}{\partial w} \left( \frac{Q^2}{PD^2} \right). \quad (70)$$

Let  $P_j(w) \equiv P_j$  (resp.  $Q_j, R_j$ ) denote the coefficient of  $x^j$  in  $P(w, x)$  (resp.  $Q(w, x), R(w, x)$ ). The Potts generating function of near-triangulations of outer degree 1, denoted by  $T_1(w) \equiv T_1$ , can be expressed in terms of these series using

$$20\nu^2 q T_1 - 4\nu^2 P_1 + 4\nu Q_0 + (Q_1 - 1)(Q_1 + \nu - 3) + 2\nu(q\nu - 24\beta - 6q)w = 0. \quad (71)$$

An alternative characterization of  $T_1$  is

$$2\nu q T_1' = R_1 - q(\beta - 1) + 8\beta. \quad (72)$$

The series  $T_1$  is differentially algebraic, that is, satisfies a non-trivial differential equation with respect to the vertex variable  $w$ . The same holds for each series  $P_j, Q_j$  and  $R_j$ .

Eq. (70) is, in compact form, a system of eight differential equations in  $w$  relating the 9 series  $P_j, Q_j, R_j$ . Since  $P_3$  is given explicitly, we have in fact as many equations as unknown series. We will see that  $Q_2 = 2\nu$ . In fact, we will prove that (70), combined with the initial conditions (69), determines uniquely all series  $P_j, Q_j$ , and  $R_j$ . For instance,

$$P_0 = -\beta w + \beta(8q + q(q-12)\beta/4 - (q+6)\beta^2 - 3\beta^3)w^2 + O(w^3).$$

We expect  $T_1$  to satisfy a differential equation of order 4 (Section 6.2). We will work out in details several special cases in which  $T_1$  satisfies a second order DE (Sections 7 to 9).

The proof of the above theorem is similar to that of Theorem 1, and we mostly tell where these proofs differ, without giving details otherwise.

### 5.1. AN EQUATION WITH ONE CATALYTIC VARIABLE

As in the case of general planar maps, we begin with an equation for the series  $T(y)$ , taken from [2]. We assume that  $q = 2 + 2\cos(2k\pi/m)$  with  $k$  and  $m$  coprime and  $0 < k < 2m$ . We still write  $\beta = \nu - 1$ . We introduce the following notation:

- $I(w, y) \equiv I(q, \nu, w; y)$  is a variant of the generating function  $T(q, \nu, w; y)$ :

$$I(w, y) = yqT(q, \nu, w; y) - \frac{1}{y} + \frac{1}{y^2},$$

- $N(y, x)$  and  $D(w, x)$  are the following (Laurent) polynomials:

$$N(y, x) = \beta(4 - q)\bar{y} + q\nu x + \beta(q - 2),$$

with  $\bar{y} = 1/y$ , and

$$D(w, x) = q\nu^2 x^2 + \beta(4\beta + q)x + q\beta\nu(q - 4)w + \beta^2. \quad (73)$$

We still denote by  $T_m$  the  $m$ th Chebyshev polynomial of the first kind.



**Proposition 7.** *There exist  $m+1$  formal power series in  $w$  with coefficients in  $\mathbb{Q}(q, \nu)$ , denoted  $C_0(w), \dots, C_m(w)$ , such that*

$$D(w, I(w, y))^{m/2} T_m \left( \frac{N(y, I(w, y))}{2\sqrt{D(w, I(w, y))}} \right) = \sum_{r=0}^m C_r(w) I(w, y)^r. \quad (74)$$

*Proof.* This follows from Corollary 12 and Lemma 19 of [2]. One must prove that the series in  $t$  denoted  $C_r$  in [2] (which coincide with the series  $C_r$  of the present paper, with  $t = w^{1/3}$ ), are not only series in  $t$  but in fact series in  $t^3$ . This can be done by following carefully the proof of Lemma 19 of [2], using the fact that the series denoted  $K(ty)$  is a series in  $t^3$ . This implies that the series denoted  $t^j S_{i,j}$  and  $t^j T_{i,j}$  are also series in  $t^3$ , and one concludes using Eq. (93) of [2].

■

By expanding the invariant equation (74) near  $y = 0$ , we can express the series  $C_r$  in terms of the derivatives of  $T$  with respect to  $y$ , evaluated at  $y = 0$ . We will need the following expressions of  $C_m, \dots, C_{m-3}$ . The first three are explicit in terms of  $q, \nu$  and  $w$ , but the last one involves the series  $T'_y(0) \equiv T_1$ .

**Lemma 8.** *Denote  $\beta = \nu - 1$ , and recall that  $q = 2 + 2 \cos(2k\pi/m)$ . We have:*

$$\begin{aligned} \frac{(-1)^k C_m}{(q\nu^2)^{m/2}} &= 1, \\ \frac{(-1)^k C_{m-1}}{(q\nu^2)^{m/2-1}} &= \frac{m\beta}{2} (4\beta + q + \beta m(q-4)), \\ \frac{(-1)^k C_{m-2}}{(q\nu^2)^{m/2-2}} &= \frac{m\beta}{24} \left( 12q^2\nu^3 w(q-4) + 48\nu^2\beta(m-1) + \beta(m-1)(q-4) \times \right. \\ &\quad \left. (6q + 12 + 6\beta(mq+4) + \beta^2(m^2q - 4m^2 + mq + 20m - 12)) \right), \\ \frac{(-1)^k C_{m-3}}{(q\nu^2)^{m/2-3}} &= -\frac{1}{2} m^2 q^3 \nu^4 \beta^2 (q-4) T_1 \\ &\quad + \frac{m(m-1)\beta^2}{720} \left( 180q^2\nu^3 w(q-4) \left( (q-4)\beta m + 8\beta + 2q \right) + 960\beta\nu^3(m-2) \right. \\ &\quad + 720\beta\nu^2(m-2)(q-4)(\beta m - \beta + 2) + \beta(m-2)(q-4)^2 \left( \beta^3 m^3 (q-4) \right. \\ &\quad + 3\beta^2 m^2 (\beta q + 16\beta + 5q) + \beta m (2\beta^2 q - 8\beta^2 + 15\beta q + 360\beta + 60q + 180) \\ &\quad \left. \left. - 180\beta^2 + 360\beta + 60q + 300 \right) \right), \end{aligned}$$

where  $T_1$  is the Potts generating function for near-triangulations with outer degree 1.

**Remark.** When comparing this lemma with its counterpart for general planar maps, Lemma 2, we observe that we have no term  $T_m(\gamma)$  nor  $T'_m(\gamma)$  here. This is because

$$\gamma = \lim_{y \rightarrow 0} \frac{N(y, I(w, y))}{2\sqrt{D(w, I(w, y))}} = \frac{\sqrt{q}}{2} = \cos(k\pi/m),$$

so that  $T_m(\gamma) = (-1)^k$  and  $T'_m(\gamma) = 0$ .

*Proof.* We multiply the invariant equation (74) by  $y^{2m}$ , so that it becomes not singular at  $y = 0$ , and expand it around  $y = 0$ . We extract successively the coefficients of  $y^0, y^2, y^4$  and  $y^6$  and obtain in this way expressions of  $C_m, \dots, C_{m-3}$  (the coefficients of odd powers of  $y$  do not give

more information). For instance, since

$$\begin{aligned} y^2 I(w, y) &= 1 + O(y), \\ y^2 N(y, I(w, y)) &= q\nu + O(y), \\ y^4 D(w, I(w, y)) &= q\nu^2 + O(y), \end{aligned}$$

extracting the coefficient of  $y^0$  gives

$$q^{m/2} \nu^m \mathbb{T}_m(\sqrt{q}/2) = C_m,$$

and the expression of  $C_m$  follows since  $\mathbb{T}_m(\sqrt{q}/2) = (-1)^k$ . As we extract the coefficients of higher powers of  $y$ , derivatives of  $\mathbb{T}_m$ , taken at the point  $\sqrt{q}/2 = \cos(k\pi/m)$ , occur. We can find explicit expressions for them in terms of  $q$ ,  $k$  and  $m$  using  $\mathbb{T}'_m(\sqrt{q}/2) = 0$  and the differential equation

$$(1 - u^2) \mathbb{T}''_m(u) - u \mathbb{T}'_m(u) + m^2 \mathbb{T}_m(u) = 0.$$

Similarly, derivatives of  $y^2 I(w, y)$  with respect to  $y$  occur, and this is why the expression of  $C_{m-3}$  involves the series  $T_1$ .  $\blacksquare$

## 5.2. SOME SPECIAL VALUES OF $y$

We now work out the counterpart of Lemma 3. We still denote  $n = \lfloor m/2 \rfloor$ .

**Lemma 9.** *Let  $q = 2 + 2 \cos(2\pi/m)$ , and denote  $\beta = \nu - 1$ . There exist  $m - 2$  distinct formal power series in  $w$ , denoted  $Y_1, \dots, Y_{m-2}$ , that have constant term  $1 + O(\beta)$  as  $\beta \rightarrow 0$  and satisfy*

$$N(Y, I(w, Y))^2 = 4 \cos(j\pi/m)^2 D(w, I(w, Y)), \quad (75)$$

for some  $j \in \llbracket 1, n \rrbracket$ . Their coefficients are algebraic functions of  $\beta$  over  $\mathbb{R}$ .

Let us denote  $I_i(w) := I(w, Y_i)$ . This is a formal power series in  $w$  with coefficients in the algebraic closure of  $\mathbb{R}(\beta)$ , denoted by  $\overline{\mathbb{R}(\beta)}$ . The  $(m - 2)$  series  $I_i$  are distinct, and for  $1 \leq i \leq m - 2$ ,

$$I_i(0) \notin \{0, -1/4, -1/(2\nu)\}, \quad D(w, I_i) \neq 0, \quad \frac{\partial I}{\partial y}(w, Y_i) \neq 0.$$

The key difference with Lemma 3 is that the coefficients of the series  $Y_i$  do not have their coefficients in  $\mathbb{R}(\beta)$ , but in its algebraic closure. Let us illustrate this by an example.

**Example: bicoloured triangulations.** We take  $q = 2$ , that is,  $m = 4$  (and  $n = 2$ ). The relevant values of  $j$  are 1 and 2, and we want to solve

$$N(Y, I(w, Y))^2 = 2D(w, I(w, Y))$$

for  $j = 1$ , and

$$N(Y, I(w, Y)) = 0$$

for  $j = 2$ . Using the definitions of  $N$  and  $D$ , this reads, for  $j = 1$ ,

$$4Y^3 (2\beta Y - 2\beta + Y - 2) T(Y) - 4Y^3 (1 + \beta) w + (Y - 2) (\beta Y^2 - 2\beta Y + 2\beta - 2Y + 2) = 0,$$

and for  $j = 2$ ,

$$2Y^3 (1 + \beta) T(Y) + \beta + 1 - Y = 0.$$

Recall that we are interested in series  $Y(w)$  with constant term  $1 + O(\beta)$ . We find one such solution for each of the above equation, which satisfy

$$\begin{aligned} Y_1(w) &= \frac{\beta + 1 - \sqrt{1 - \beta^2}}{\beta} + O(w), \\ Y_2(w) &= 1 + \beta + O(w). \end{aligned}$$

$\square$

*Proof.* We denote  $q_j = 4 \cos(j\pi/m)^2 = 2 + 2 \cos(2j\pi/m)$ . Observe that  $q = q_1$ . We assume that  $m = 2n + 1$  is odd. The proof is similar in the even case.

**Existence of the series  $Y_i$ .** We are looking for series  $Y = Y(w)$  solutions of (75). Multiplying (75) by  $Y^4$ , we obtain the equation  $\Phi_j(w, Y) = 0$ , where

$$\Phi_j(w, y) = y^4 N(y, I(w, y))^2 - q_j y^4 D(w, I(w, y)).$$

Observe that  $y^2 I(w, y)$  is a formal power series in  $w$  with coefficients in  $\mathbb{R}[\nu, y]$ . Hence the same holds for  $y^2 N(y, I(w, y))$  and  $y^4 D(w, I(w, y))$ , and finally for  $\Phi_j(w, y)$ . Expanding  $I$ ,  $N$  and  $D$  at first order in  $w$  gives:

$$y^2 I(w, y) = 1 - y + O(w), \quad (76)$$

$$y^2 N(y, I(w, y)) = q\nu - y(q + 2\beta(q - 2)) + (q - 2)\beta y^2 + O(w),$$

$$y^4 D(w, I(w, y)) = q\nu^2 - 2q\nu^2 y + (q + 3q\beta + (q + 4)\beta^2) y^2 - \beta(4\beta + q) y^3 + \beta^2 y^4 + O(w). \quad (77)$$

The equation  $\Phi_j(w, Y) = 0$  thus reads

$$P_j(Y) + wS_j(w, Y) = 0, \quad (78)$$

where

$$P_j(y) = (q\nu - y(q + 2\beta(q - 2)) + (q - 2)\beta y^2)^2 - q_j(q\nu^2 - 2q\nu^2 y + (q + 3q\beta + (q + 4)\beta^2) y^2 - \beta(4\beta + q) y^3 + \beta^2 y^4)$$

and  $S_j(w, y)$  is a power series in  $w$  with coefficients in  $\mathbb{R}[\nu, y]$ . In particular, the coefficient of  $w^0$  in  $Y$ , denoted  $y_0$ , must satisfy  $P_j(y_0) = 0$ . The roots of this quartic polynomial can be seen as Puiseux series in  $\beta$  with coefficients in  $\mathbb{C}$  (see for instance [39, Ch. 6]). Let us focus on the roots that are finite at  $\beta = 0$  and have constant term 1. Using Newton's polygon method, we find that they read  $y_0 = 1 + \beta v + O(\beta^3)$ , where  $v$  must satisfy the equation (34) that we studied when constructing the series  $Y_i$  for general planar maps. Thus when  $j > 1$  we find for  $P_j$  two distinct roots with constant term 1, of the form  $y_0 = 1 + \beta v_j^\pm + O(\beta^3)$ , and only one such root  $y_0 = 1 + \beta v_1^+ + O(\beta^3)$  when  $j = 1$  (with  $v_1^+ = 1/q$ ).

Now having fixed one root  $y_0$  of  $P_j$ , let us return to (78) and extract from it the coefficient of  $w^p$ , for  $p \geq 1$ . This gives

$$P'_j(y_0)[w^p]Y + (\text{expression involving only } [w^i]Y \text{ for } i < p) = 0.$$

Since  $y_0$  is not a double root of  $P_j$ , this allows us to compute the coefficient of  $w^p$  in  $Y$  by induction on  $p$ , and thus determines  $Y$  completely. Moreover,  $[w^p]Y$  is an algebraic function of  $\beta$  over  $\mathbb{R}$ .

As argued in the proof of Lemma 3, the  $(m - 2)$  values  $v_1^+, \dots, v_n^+, v_2^-, \dots, v_n^-$  are distinct. They give rise to  $(m - 2)$  distinct series  $Y(w)$  satisfying  $Y(0) = 1 + O(\beta)$  and (75) for some  $j \in \llbracket 1, n \rrbracket$ . We denote them  $Y_1, \dots, Y_{m-2}$  with  $Y_i(0) = 1 + \beta v_i + O(\beta^3)$ , where as before,

$$\{v_1, \dots, v_{m-2}\} = \{v_1^+, \dots, v_n^+, v_2^-, \dots, v_n^-\}.$$

Clearly these series are non-zero, and we have proved the first statement of the lemma.

**Properties of  $Y_i$  and  $I_i := I(w, Y_i)$ .** Let us prove that the series  $I_i$  are distinct. Returning to (76) gives

$$I_i(0) = -\beta v_i + O(\beta^2).$$

This proves that the series  $I_i$  are distinct (since the  $v_i$ 's are distinct), and also that  $I_i(0)$  is distinct from 0,  $-1/4$  and  $-1/(2\nu)$ .

Let us now prove that  $D(w, I_i(w)) \neq 0$ . Returning to (77), we find that:

$$D(0, I_i(0)) = \beta^2(qv_i^2 - qv_i + 1) + O(\beta^3).$$

Comparing with (35) shows that this is non-zero.

We finally check that

$$\frac{\partial I}{\partial y}(0, Y_i(0)) = -1 + 4\beta v_i + O(\beta^2)$$

is non-zero, as stated in the lemma. ■

### 5.3. SOME POLYNOMIALS WITH COMMON ROOTS

We still assume that  $q = 2 + 2 \cos(2\pi/m)$ . We denote as before:

$$C(w, x) = \sum_{r=0}^m C_r(w) x^r.$$

Thus  $C(w, x)$  is a polynomial in  $x$  with coefficients in  $\mathbb{R}(\nu)[[w]]$ . This is also true of  $D(w, x)$  (see (73)). The coefficients of  $D$  are explicit, but those of  $C$  are unknown. Here is now the counterpart of Lemma 4.

**Lemma 10.** *Each of the series  $I_i := I(w, Y_i)$  defined by Lemma 9 satisfies*

$$\begin{aligned} C(w, I_i)^2 &= D(w, I_i)^m, \\ D(w, I_i) \frac{\partial C}{\partial x}(w, I_i) &= \frac{m}{2} C(w, I_i) \frac{\partial D}{\partial x}(w, I_i), \\ D(w, I_i) \frac{\partial C}{\partial w}(w, I_i) &= \frac{m}{2} C(w, I_i) \frac{\partial D}{\partial w}(w, I_i). \end{aligned}$$

The first two identities imply that  $I_i$  is actually a double root of  $C^2 - D^m$ .

The proof is identical to that of Lemma 4, with the variable  $t$  replaced by  $w$ .

**Proposition 11.** *Let  $I_i := I(w, Y_i)$  be the  $(m-2)$  series defined in Lemma 9. There exists a triple  $(\widehat{P}(w, x), \widehat{Q}(w, x), \widehat{R}(w, x))$  of polynomials in  $x$  with coefficients in  $\overline{\mathbb{R}(\beta)}[[w]]$ , having degree at most 3, 2 and 1 respectively in  $x$ , such that*

$$\begin{aligned} C(w, x)^2 - D(w, x)^m &= \widehat{P}(w, x) \prod_{i=1}^{m-2} (x - I_i)^2, \\ D(w, x) C'_x(w, x) - \frac{m}{2} D'_x(w, x) C(w, x) &= \widehat{Q}(w, x) \prod_{i=1}^{m-2} (x - I_i), \\ D(w, x) C'_w(w, x) - \frac{m}{2} D'_w(w, x) C(w, x) &= \widehat{R}(w, x) \prod_{i=1}^{m-2} (x - I_i). \end{aligned}$$

*Proof.* The proof is almost the same as that of Proposition 5, with the variable  $t$  replaced by  $w$ . The only difference is that here the polynomial  $C(w, x)^2 - D(w, x)^m$  has degree at most  $2m-1$  (instead of  $2m$ ), and the polynomial  $D(w, x) C'_w(w, x) - \frac{m}{2} D'_w(w, x) C(w, x)$  has degree at most  $m-1$  (instead of  $m$ ). This is because the coefficient of  $x^{2m}$  in  $C(w, x)^2$  and  $D(w, x)^m$  is  $q^m \nu^{2m}$  (by Lemma 8), and the coefficient of  $x^m$  in  $D(w, x) C'_w(w, x)$  and  $\frac{m}{2} D'_w(w, x) C(w, x)$  is  $-\frac{m}{2} \beta(q-4) q^{m/2+1} \nu^{m+1}$  (again, by Lemma 8). ■

### 5.4. DIFFERENTIAL SYSTEM

We still assume that  $q = 2 + 2 \cos(2\pi/m)$ .

**A differential equation relating  $\widehat{P}$ ,  $\widehat{Q}$  and  $\widehat{R}$ .** Starting from Proposition 11, one first proves that

$$\frac{1}{\widehat{Q}} \frac{\partial}{\partial w} \left( \frac{\widehat{Q}^2}{\widehat{P} D^2} \right) = \frac{1}{\widehat{R}} \frac{\partial}{\partial x} \left( \frac{\widehat{R}^2}{\widehat{P} D^2} \right).$$

This argument is the same as in Section 4.4. The above equation coincides with (70), but with hats over the letters  $P, Q$  and  $R$ . The series  $P, Q$  and  $R$  occurring in Theorem 6 will be normalizations of  $\widehat{P}, \widehat{Q}$  and  $\widehat{R}$  by multiplicative constants, independent from  $w$  and  $x$ .

**The leading coefficients of  $\widehat{P}$ ,  $\widehat{Q}$ ,  $\widehat{R}$ .** Proposition 11 gives

$$\begin{aligned} [x^3]\widehat{P}(w, x) &= [x^{2m-1}] (C(w, x)^2 - D(w, x)^m), \\ [x^2]\widehat{Q}(w, x) &= [x^m] \left( D(w, x)C'_x(w, x) - \frac{m}{2}D'_x(w, x)C(w, x) \right), \\ [x^1]\widehat{R}(w, x) &= [x^{m-1}] \left( D(w, x)C'_w(w, x) - \frac{m}{2}D'_w(w, x)C(w, x) \right). \end{aligned}$$

Recall that  $D$  is given explicitly by (73), and that Lemma 8 gives the leading coefficients of  $C(w, x)$ . This allows to us determine the leading coefficients of  $\widehat{P}$ ,  $\widehat{Q}$ ,  $\widehat{R}$ :

$$\begin{cases} \widehat{P}_3 := [x^3]\widehat{P}(w, x) = m^2 \beta^2 (q-4)q^{m-1}\nu^{2m-2}, \\ \widehat{Q}_2 := [x^2]\widehat{Q}(w, x) = \frac{m^2 \beta^2}{2} (q-4)q^{m/2}\nu^m, \\ \widehat{R}_1 := [x^1]\widehat{R}(w, x) = \frac{m^2 \beta^2}{2} (q-4)q^{m/2}\nu^m \left( qT'_1 + \frac{q(\beta-1) - 8\beta}{2\nu} \right). \end{cases} \quad (79)$$

Let us define

$$P(w, x) = \frac{\widehat{P}(w, x)}{\widehat{P}_3}, \quad Q(w, x) = \frac{2\nu\widehat{Q}(w, x)}{\widehat{Q}_2} \quad \text{and} \quad R(w, x) = \frac{2\nu\widehat{R}(w, x)}{\widehat{Q}_2}. \quad (80)$$

Then  $P$ ,  $Q$ ,  $R$  satisfy (70) and moreover,

$$[x^3]P(w, x) = 1, \quad [x^2]Q(w, x) = 2\nu, \quad [x^1]R(w, x) = 2\nu qT'_1 + q(\beta-1) - 8\beta. \quad (81)$$

In particular, the first of the initial conditions (69) holds. (We do not give  $Q_2$  explicitly in the statement of the theorem, as its value follows from the differential system and the initial conditions.)

**The case  $w = 0$ .** It remains to determine the values  $P(0, x)$  and  $Q(0, x)$ . Proposition 11 gives

$$C(0, x)^2 - D(0, x)^m = \widehat{P}(0, x) \prod_{i=1}^{m-2} (x - I_i(0))^2, \quad (82)$$

$$D(0, x)C'_x(0, x) - \frac{m}{2}D'_x(0, x)C(0, x) = \widehat{Q}(0, x) \prod_{i=1}^{m-2} (x - I_i(0)). \quad (83)$$

We will now combine our knowledge of  $C(0, x)$  obtained in [2] with the properties of the series  $I_i(w)$  gathered in Lemma 9 to determine  $\widehat{P}(0, x)$  and  $\widehat{Q}(0, x)$ .

The series  $C(0, x)$  can be determined using results in [2, Lemma 19] (with  $t = w^{1/3}$ ). There, it is proved that

$$C(0, x^2 - x) \equiv \sum_{r=0}^m C_r(0)(x^2 - x)^r = \mathcal{D}^{m/2} \mathbb{T}_m \left( \frac{\mathcal{N}}{2\sqrt{\mathcal{D}}} \right),$$

where

$$\mathcal{N} = N(1/x, x^2 - x) = \beta(4 - q)x + q\nu(x^2 - x) + \beta(q - 2)$$

and

$$\mathcal{D} = D(0, x^2 - x) = q(1 + \beta)^2(x^2 - x)^2 + \beta(4\beta + q)(x^2 - x) + \beta^2.$$

Recall the expression (58) of  $\mathbb{T}_m(u)^2 - 1$ . Thus (??) gives

$$\begin{aligned} C(0, x^2 - x)^2 - D(0, x^2 - x)^m &= \frac{1}{4}(\mathcal{N}^2 - 4\mathcal{D})(\mathcal{N}^2)^{\mathbb{1}_{m=2n}} \prod_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \left( \mathcal{N}^2 - 4\mathcal{D} \cos^2 \frac{j\pi}{m} \right)^2 \\ &= \widehat{P}(0, x^2 - x) \prod_{i=1}^{m-2} (x^2 - x - I_i(0))^2 \end{aligned} \quad (84)$$

by (82). We want to prove that  $x^2(x+1/4)$  is a factor of  $\widehat{P}(0, x)$ , or equivalently, that  $x^2(x-1)^2(x-1/2)^2$  is a factor of  $\widehat{P}(0, x^2-x)$ . Since by Lemma 9,  $I_i(0) \notin \{0, -1/4\}$ , it suffices to prove that  $x^2(x-1)^2(x-1/2)^2$  divides the right-hand side of (84). Using the above values of  $\mathcal{N}$  and  $\mathcal{D}$ , we first find a factor  $(x-1)^2$  in  $(\mathcal{N}^2-4\mathcal{D})$ . Then, using  $4\cos^2(\pi/m) = q$ , we find that the term obtained for  $j=1$  contains a factor  $(x-1/2)^2$ . Finally, using  $4\cos^2(2\pi/m) = (q-2)^2$ , we find that the term obtained for  $j=2$  contains a factor  $x^2$ , provided  $m \geq 5$ . When  $m=3$ , that is,  $q=1$ , a factor  $x^2$  is found in the term obtained for  $j=1$ . When  $m=4$ , that is  $q=2$ , a factor  $x^2$  comes out from the term  $\mathcal{N}^2$ .

We now wish to prove that  $x(2\nu x+1)$  divides  $\widehat{Q}(0, x)$ . We have just seen that  $x=0$  is a double root of  $\widehat{P}(0, x)$ . Using the same argument as in Section 4.4, this implies that  $x$  divides  $\widehat{Q}(0, x)$ . For the remaining factor, we have to prove that  $(2\nu(x^2-x)+1)$  divides  $\widehat{Q}(0, x^2-x)$ . Let us combine (83) (taken at  $x^2-x$ ) with (??). We obtain:

$$\begin{aligned} \widehat{Q}(0, x^2-x) \prod_{i=1}^{m-2} (x^2-x-I_i(0)) &= D(0, x^2-x)C'_x(0, x^2-x) - \frac{m}{2}D'_x(0, x^2-x)C(0, x^2-x) \\ &= \frac{\mathcal{D}^{1+m/2}}{2x-1} \frac{\partial}{\partial x} \left( \frac{\mathcal{N}}{2\sqrt{\mathcal{D}}} \right) \Gamma'_m \left( \frac{\mathcal{N}}{2\sqrt{\mathcal{D}}} \right). \end{aligned}$$

A factor  $(2\nu(x^2-x)+1)$  is found in  $\frac{\partial}{\partial x} \left( \frac{\mathcal{N}}{2\sqrt{\mathcal{D}}} \right)$ . Lemma 9 ensures that  $I_i(0) \neq -1/(2\nu)$ , and thus this must be a factor of  $\widehat{Q}$ .

### 5.5. THE POTTS GENERATING FUNCTION OF TRIANGULATIONS

It follows from the first two identities of Proposition 11 that

$$\widehat{P} \left( DC'_x - \frac{m}{2}D'_xC \right)^2 = \widehat{Q}^2 (C^2 - D^m).$$

Using the normalisation (80), this gives

$$P \left( DC'_x - \frac{m}{2}D'_xC \right)^2 = \frac{qm^2\beta^2}{4}(q/4-1)Q^2(C^2 - D^m), \quad (85)$$

since  $\widehat{Q}_2^2 = qm^2\beta^2(1+\beta)^2(q/4-1)\widehat{P}_3$  by (79).

Recall that the leading coefficients of  $C$  are given in Lemma 8, and that  $C^2 - D^m$  has degree  $2m-1$ . Recall also the known values (81) of  $P_3$  and  $Q_2$ . Extracting the coefficients of  $x^{2m+3}$ ,  $x^{2m+2}$ , and  $x^{2m+1}$  in (85) first gives (for the coefficient of  $x^{2m+3}$ ) a tautology, then (for the coefficient of  $x^{2m+2}$ ) an interesting relation between  $P_2$  and  $Q_1$ , namely

$$4\nu P_2 = 4Q_1 + \nu - 4, \quad (86)$$

and finally (for the coefficient of  $x^{2m+1}$ ) and expression of  $T_1$  which we transform into (71) thanks to (86).

The expression (72) of  $T'_1$  was obtained in (81).

### 5.6. UNIQUENESS OF THE SOLUTION

We now want to show that the differential system of Theorem 6, together with its initial conditions, uniquely defines the nine series  $P_0, P_1, P_2, P_3, Q_0, Q_1, Q_2, R_0, R_1$ , whether  $q$  is an indeterminate or a real number of the form  $2+2\cos(2\pi/m)$ . The proof parallels the case of general planar maps given in Section 4.6 (with  $t$  replaced by  $w$ ), and is actually a bit simpler. We merely sketch the main steps.

The system can be written as in (62):

$$2Q'_wPD - QP'_wD - 2QP'D'_w = 2R'_xPD - RP'_xD - 2RPD'_x. \quad (87)$$

Both sides are polynomials of degree at most 7 in  $x$ . Recall that  $P_3 = 1$ . Extracting the coefficient of  $x^7$  gives  $Q'_2(w) = 0$ , which, with the initial condition  $Q_2(0) = 2\nu$ , implies  $Q_2(w) = 2\nu$ . We are left with seven equations and seven unknown series.

We denote  $P_{i,j} = [w^i]P_j$ , and similarly for  $Q$  and  $R$ . Let  $\mathcal{C}_i = (P_{i,0}, P_{i,1}, P_{i,2}; Q_{i,0}, Q_{i,1}; R_{i-1,0}, R_{i-1,1})$ . The right-hand side of (69) gives

$$\mathcal{C}_0 = (0, 0, 1/4; 0, 1; 0, 0).$$

The next steps are the same as in Section 4.6. For  $i \geq 1$  and  $j \in \llbracket 0, 6 \rrbracket$ , we denote by  $\text{Eq}_{i,j}$  the equation obtained by extracting the coefficient of  $w^{i-1}x^j$  in (87). Again, the equation  $\text{Eq}_{i,0}$  only involves series from  $\cup_{s < i} \mathcal{C}_s$  because  $P_{0,0} = P_{0,1} = Q_{0,0}$  as in the case of general maps. However, we do not have here the second difficulty that came from the factor  $(x-1)$  in  $P(0, x)$ ,  $P'_x(0, x)$  and  $Q(0, x)$ , and we consider the system

$$\mathcal{S}_i = \{\text{Eq}_{i+1,0}, \text{Eq}_{i,1}, \text{Eq}_{i,2}, \dots, \text{Eq}_{i,6}\}.$$

Solving the system  $\mathcal{S}_1$  for the unknowns of  $\mathcal{C}_1$  gives

$$\mathcal{C}_1 = (-\beta, \beta^2 + \beta q/2 - 4\beta, 4\beta^2 + 2\beta q - 2q; 8\beta + 2q - 2\beta^2 + \beta q, 4\beta^3 + 2q\beta^2 + 4\beta^2 - 2q; -2\beta, \beta q - 8\beta - q).$$

Then for  $i > 1$ ,  $\mathcal{S}_i$  is a system of seven linear equations for the seven unknowns in  $\mathcal{C}_i$ , the determinant of which is:

$$i^5 q^3 \beta^7 (q-4) (\beta+1)^4 (\beta-1)^3 (4\beta^2 - q) / 2.$$

We conclude as in Section 4.6.

From this point on, we argue as in Section 4.8 to prove that, for  $q$  an indeterminate, the series  $P_j$ ,  $Q_j$  and  $R_j$  are differentially algebraic and satisfy (71) and (72). It then follows from (71) that  $T_1$  is differentially algebraic too.

### 5.7. ABOUT POSSIBLE SINGULARITIES

It remains to prove that the coefficients of the series  $P_j, Q_j, R_j$  are polynomials in  $q$  and  $\nu$ .

First, we note that (72) implies that this is true for  $R_1$ . Consider the system of (only) six equations:

$$\{\text{Eq}_{i+1,0}, \text{Eq}_{i,1}, \dots, \text{Eq}_{i,5}\}.$$

It relates polynomially the coefficients in  $\cup_{s \leq i} \mathcal{C}_s$ . For  $i > 1$ , it is linear in  $P_{i,0}, P_{i,1}, P_{i,2}, Q_{i,0}, Q_{i,1}, R_{i-1,0}$  once the values  $P_{1,0}$  and  $R_{0,0}$  are known, with determinant:

$$-i^5 q \beta^9 (\beta-1) (q^3 - 4q^2(\beta^2 - \beta + 1) + 2q(\beta-1)^2(\beta^2 - 4\beta + 1) - 16\beta^2(\beta-1)^2).$$

This determinant contains no factor  $(q-4)$ ,  $(\beta+1)$  nor  $(4\beta^2 - q)$ . By induction on  $i$ , we conclude that the denominators of the coefficients of  $P_j, Q_j$  and  $R_j$  consist of factors  $q, \beta$  and  $(\beta-1)$ .

To show that these factors do not occur, we use two identities that we will prove in Section 6.1 (without assuming polynomiality of the coefficients...):

$$\nu q(q-4)Q_0 - (4\beta + q)R_0 + 2(\nu q w(q-4) + \beta)R_1 = 2\beta(q-4)(\nu q w(q-4) + \beta)$$

and

$$\nu \beta q(q-4)Q_1 - 2\nu^2 q R_0 + \beta(4\beta + q)R_1 = 2(q-4)\beta^2(4\beta + q).$$

Let  $\text{Eq}'_i$  and  $\text{Eq}''_i$  be obtained by extracting the coefficient of  $w^i$  in these equations. Consider the system

$$\{\text{Eq}_{i+1,0}, \text{Eq}_{i,1}, \text{Eq}_{i,2}, \text{Eq}_{i,3}, \text{Eq}'_i, \text{Eq}''_i\}.$$

It relates polynomially the coefficients  $P_{s,j}, Q_{s,j}$  and  $R_{s,j}$  for  $s \leq i$ . For  $i > 1$ , it is linear in  $P_{i,0}, P_{i,1}, P_{i,2}, Q_{i,0}, Q_{i,1}, R_{i,0}$ , with determinant

$$-2i^4 q \beta^{10} (\beta+1)(q-4)(4\beta + q).$$

We have thus ruled out factors  $(\beta-1)$ . To rule out the factor  $q$ , we note that  $\text{Eq}''_i$  is a multiple of  $q$  (because by (72), all coefficients  $R_{i,1}$ , for  $i > 0$ , are multiples of  $q$ ). So if we replace the last equation of the previous system by  $\text{Eq}''_i/q$ , the factor  $q$  disappears from the determinant.

We are left with the factor  $\beta$ . This time we form the system

$$\{\text{Eq}_{i,3}, \text{Eq}_{i,4}, \text{Eq}_{i,5}, \text{Eq}_{i,6}, \text{Eq}'_i, \text{Eq}''_i\},$$

which is linear in the unknowns  $P_{i,0}, P_{i,1}, P_{i,2}, Q_{i,0}, Q_{i,1}, R_{i,0}$ , with determinant

$$-2i^4 q^5 (q-4)(\beta-1)(\beta+1)^{10}(4\beta^2-q).$$

There is no factor  $\beta$ , and we conclude that the coefficients of  $P_j, Q_j$  and  $R_j$  belong to  $\mathbb{Q}[q, \nu]$ .

## 6. SIMPLIFYING THE SYSTEM: FIVE NON-DIFFERENTIAL EQUATIONS

In this section, we derive five equations between the unknown series  $P_j, Q_j$  and  $R_j$ , which do not involve their derivatives (however, one of them involves the derivative of  $M_1$  or  $T_1$ , depending on the family of maps we consider).

Since the system obtained for triangulations is a bit simpler than for general maps (the degrees of  $P$  and  $R$  being smaller), we begin with this case.

### 6.1. TRIANGULATIONS

We start from the system of Theorem 6. It involves nine series in  $t$  denoted  $P_0, \dots, P_3, Q_0, \dots, Q_2, R_0, R_1$ . One of them is given explicitly in the theorem:  $P_3 = 1$ . Moreover, we have derived from the system in Section 5.6 that  $Q_2 = 2\nu$ . This leaves us with seven unknown series, related by seven differential equations. These equations are the coefficients of  $x^0, \dots, x^6$  in (87).

We are going to derive three linear (non-differential) relations between the unknown series by letting  $x$  tend to  $\infty$  or to one of the two roots (in  $x$ ) of  $D(w, x)$ . Extracting the coefficient of  $x^6$  from (87) gives

$$Q'_1 = \nu P'_2.$$

Using the initial conditions (69), we obtain

$$\nu P_2 = Q_1 + \nu/4 - 1. \quad (88)$$

This identity was already obtained by expanding (85) in the proof of Theorem 6 (see (86)).

Let us now specialize (87) at the two roots  $\delta_1(w)$  and  $\delta_2(w)$  of  $D(w, x)$ . These roots, seen as series in  $w$ , satisfy:

$$\delta_{1,2}(w) = \beta \frac{\pm \sqrt{(4-q)(4\beta^2-q)} - 4\beta - q}{2q\nu^2} + O(w),$$

with  $\beta = \nu - 1$  as above. Since  $P(0, x) = x^2(x + 1/4)$ , the  $\delta_i$ 's are not roots of  $P$ , and thus taking the limit  $x \rightarrow \delta_i$  in (87) gives, for  $i = 1, 2$ ,

$$Q(w, \delta_i(w))D'_w(w, \delta_i(w)) = R(w, \delta_i(w))D'_x(w, \delta_i(w)).$$

Since  $D$  and its roots are explicit, this is a linear system in  $Q_0, Q_1, R_0$  and  $R_1$ , symmetric in  $\delta_1$  and  $\delta_2$ . Solving for  $Q_0$  and  $Q_1$  gives

$$\nu q(q-4)Q_0 - (4\beta+q)R_0 + 2(\nu qw(q-4) + \beta)R_1 = 2\beta(q-4)(\nu qw(q-4) + \beta) \quad (89)$$

and

$$\nu\beta q(q-4)Q_1 - 2\nu^2 qR_0 + \beta(4\beta+q)R_1 = 2(q-4)\beta^2(4\beta+q). \quad (90)$$

Let us finally recall the two characterizations of  $T_1$  obtained in Theorem 6:

$$20\nu^2 qT_1 - 4\nu^2 P_1 + 4\nu Q_0 + (Q_1 - 1)(Q_1 + \nu - 3) + 2\nu(q\nu - 24\beta - 6q)w = 0. \quad (91)$$

and

$$2\nu qT'_1 = R_1 - q(\beta - 1) + 8\beta. \quad (92)$$

We call Eqs. (88-92) *the five generic non-differential equations*: they involve the derivative  $T'_1$ , but no derivative of the other unknown series  $P_j, Q_j$  and  $R_j$ . And of course they hold for indeterminates  $q$  and  $\nu$ .

**Prediction of the order of  $T_1$ .** Using (in this order), equations (88), (92), (90), (89) and (91), we obtain expressions of  $Q_1, R_1, R_0, Q_0$  and finally  $P_1$  as linear combinations of  $P_2, T_1$  and  $T'_1$



(with coefficients in  $\mathbb{Q}(q, \nu, w)$ ). We only give the connection between  $Q_0$ ,  $P_2$  and  $T_1$ , which we will use later:

$$2(4\nu^3 q^2 w + \beta(4\beta^2 - q)) T_1' / \nu + 2q\nu Q_0 - \beta(q + 4\beta) P_2 = (4\beta + q)(4q\nu w - \beta/4). \quad (93)$$

Recall that  $T_1 = \nu T_2$  where  $T_2$  counts near-triangulations with outer degree 2.

We have thus got rid of five unknown series, but introduced a new one, namely  $T_1$ . Reporting the expressions of  $Q_1, R_1, R_0, Q_0$  and  $P_1$  in our differential system gives a system of three equations in  $P_0, P_2$  and  $T_1$ , of order 1 in  $P_0$  and  $P_2$  and order 2 in  $T_1$ . Thus the field generated over  $\mathbb{Q}(q, \nu, w)$  by  $P_0, P_2, T_1$  and their derivatives has transcendence degree at most 4, and we expect  $T_1$  to satisfy a differential equation of order at most 4. We have been able to eliminate  $P_0$  and to obtain two differential equations in  $P_2$  and  $T_1$ : one of order 1 in  $P_2$  and 2 in  $T_1$ , the other of respective orders 2 and 3. Going further with the elimination seems to require heavy computer algebra, and we have failed to obtain an explicit equation for  $T_1$ . We study in Sections 7 to 9 three special cases, of combinatorial interest, where the order of  $T_1$  is only 2.

## 6.2. GENERAL PLANAR MAPS

We now proceed similarly with the differential system of Theorem 1. It involves eleven series in  $t$  denoted  $P_0, \dots, P_4, Q_0, \dots, Q_2, R_0, \dots, R_2$ . Two of them are given explicitly:  $P_4 = 1$  and  $R_2 = \nu + 1 - w(q + 2\beta)$ . We have also seen in Section 4.6 that  $Q_2 = 1$ . This leaves us with eight unknown series, related by a system of eight differential equations.

As in the case of triangulations, we can derive three linear (non-differential) relations between them by considering the system (62) as  $x$  approaches  $\infty$ , or one of the two roots (in  $x$ ) of the quadratic polynomial  $D(t, x)$ . The derivation is the same as in the previous subsection, and we simply give the resulting three identities.

The first one was already obtained in the proof of Theorem 1 by expanding (60) in  $x$  (see (61)):

$$P_3 = 2Q_1 + 4t(1 + \nu) - 4tw(2\beta + q). \quad (94)$$

The other two are the counterparts of (89) and (90), and they read:

$$\begin{aligned} \beta(wq + \beta)(q - 4)Q_0 + q(\beta + 2)R_0 + 2(\beta(q - 4)(wq + \beta)t + q)R_1 = \\ 2\beta(q - 4)(wq - 2)(wq + \beta)t + 2q(wq - 2), \end{aligned} \quad (95)$$

$$\begin{aligned} \beta(wq + \beta)(q - 4)Q_1 - 2(\beta^2 + q\beta + q)R_0 - q(\beta + 2)R_1 = \\ 2\beta(q - 4)(2\beta w + wq - \beta - 2)(wq + \beta)t - 2q(\beta qw - 2\beta w + wq - \beta - 2). \end{aligned} \quad (96)$$

Let us finally recall the two characterizations of  $\tilde{M}_1 = t^2 M_1$  obtained in Theorem 1:

$$12(\beta^2 + q\nu)\tilde{M}_1 + P_3^2/4 + 2t(1 + \nu - w(2\beta + q))P_3 - P_2 + 2Q_0 = 4t(1 + w(3\beta + q)), \quad (97)$$

and

$$2(\beta^2 + q\nu)\tilde{M}_1' + (1 + \nu - w(2\beta + q))P_3/2 - R_1 = 2 + 2\beta w. \quad (98)$$

Again, we call Eqs. (94-98) *the five generic non-differential equations*: they involve the derivative  $\tilde{M}_1'$ , but no derivative of the other unknown series  $P_j, Q_j$  and  $R_j$ . And of course they hold when  $q, \nu$  and  $w$  are indeterminates.

**Prediction of the order of  $M_1$ .** Using (in this order) the five equations (94), (98), (96), (95), and (97), we can express  $Q_1, R_1, R_0, Q_0$ , and finally  $P_2$  as polynomials in  $P_3, \tilde{M}_1$  and  $\tilde{M}_1'$  (with coefficients in  $\mathbb{Q}(q, \nu, w, t)$ ). We thus get rid of five unknown series (but introduce a new one, namely  $\tilde{M}_1$ ). Reporting these five expressions in our differential system gives a system of four equations in  $P_0, P_1, P_3$  and  $\tilde{M}_1$ , of order 1 in  $P_0, P_1, P_3$  and order 2 in  $\tilde{M}_1$ . We thus expect  $\tilde{M}_1$  to satisfy a differential equation of order at most 5. We study in Sections 8 and 10 two special cases, of combinatorial or physical interest, where the order is only 3.

7. PROPERLY COLOURED TRIANGULATIONS:  $\nu = 0$ 

We specialize the differential system obtained for triangulations to  $\nu = 0$ , and recover the solution of the problem studied by Tutte between 1973 and 1984. Recall that, for generic values of  $q$  and  $\nu$ , we have  $T_1 = \nu T_2$ , where  $T_i$  counts  $q$ -coloured triangulations with outer degree  $i$ . Thus  $T_1$  vanishes when  $\nu = 0$ , and we focus on  $T_2$  instead, as Tutte did.

**Theorem 12 (Tutte [50]).** *Let  $T_2 \equiv T_2(q, w)$  denote the generating function of properly  $q$ -coloured near-triangulations of outer degree 2, counted by the number of vertices (as before, the root-vertex is coloured in a prescribed colour). This series is characterized by*

$$2(1 - q)w + (w + 10T_2 - 6wT_2')T_2'' + (4 - q)(20T_2 - 18wT_2' + 9w^2T_2'') = 0,$$

with the initial conditions  $T_2 = O(w^2)$ . Equivalently,  $T_2 = \sum_{n \geq 2} a_n(q)w^n$ , where  $a_n \equiv a_n(q)$  is given by  $a_2 = q - 1$  and for  $n > 0$ ,

$$(n + 1)(n + 2)a_{n+2} = (q - 4)(3n - 1)(3n - 2)a_{n+1} + 2 \sum_{i=1}^n i(i + 1)(3n - 3i + 1)a_{i+1}a_{n+2-i}.$$

**Remark.** Take a properly coloured near-triangulation of outer degree 2, having at least one finite face (that is, having at least three vertices). If we delete its root-edge, we obtain a properly coloured triangulation with the same number of vertices. This means that when  $\nu = 0$ ,

$$T_2 = (q - 1)w^2 + T_3,$$

where  $T_3$  counts properly  $q$ -coloured triangulations (with the root-vertex coloured in a prescribed colour). The series  $H$  occurring in (1) is  $qT_2$ , and Tutte's differential equation (1) is equivalent to the above theorem.

*Proof.* We specialize Theorem 6 to  $\nu = 0$ , that is  $\beta = -1$ . The polynomial  $D(w, x)$  becomes independent of  $w$  and has degree one in  $x$ :

$$D(w, x) = (4 - q)x + 1. \quad (99)$$

Let us now specialize the five generic non-differential identities (88-92). The first three give

$$Q_1 = 1, \quad R_0 = 2 \quad \text{and} \quad R_1 = 2(4 - q). \quad (100)$$

Observe that  $R(w, x) = 2D(w, x)$ . The identities (91) and (92) then become tautologies at  $\nu = 0$ , and we still need to relate  $T_2$  to the series  $Q_j$  and  $P_j$ . We use for that the equation (93). Once specialized at  $\nu = 0$ , it reads:

$$2T_2' + P_2 = 1/4. \quad (101)$$

This will replace the identity (92) relating  $T_1'$  and  $R_1$ . We proceed without deriving a counterpart of (91).

We now go back to the differential system (70), which simplifies into

$$2 \frac{\partial}{\partial x} \left( \frac{1}{P} \right) = \frac{1}{QD} \frac{\partial}{\partial w} \left( \frac{Q^2}{P} \right). \quad (102)$$

This is a rational expression in  $x$ . Given the form (99) of  $D$ , we find convenient to expand the numerator of (102) in powers of  $X = x + 1/(4 - q)$  rather than  $x$ . We write accordingly:

$$P(w, x) = X^3 + \tilde{P}_2 X^2 + \tilde{P}_1 X + \tilde{P}_0, \quad Q(w, x) = X + \tilde{Q}_0, \quad R(w, x) = 2(4 - q)X,$$

where the  $\tilde{P}_j$ 's and  $\tilde{Q}_0$  are series in  $w$  with rational coefficients in  $q$ . Since we have taken into account the values (100) of  $Q_1$ ,  $R_0$  and  $R_1$ , we have only four unknown series left. The identity (101) gives

$$\tilde{P}_2 = \frac{q + 8}{4(q - 4)} - 2T_2'. \quad (103)$$

The following initial conditions are translated from (69):

$$\tilde{P}_0(0) = \frac{q}{4(q - 4)^3}, \quad \tilde{P}_1(0) = \frac{2 + q}{4(q - 4)^2}, \quad \tilde{P}_2(0) = \frac{8 + q}{4(q - 4)}, \quad \tilde{Q}_0(0) = \frac{1}{q - 4}. \quad (104)$$

Expanding in  $X$  the numerator of (102) gives a system of four differential equations between the four series  $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2$  and  $\tilde{Q}_0$ :

$$\begin{aligned} 2\tilde{P}_0\tilde{Q}'_0 - \tilde{Q}_0\tilde{P}'_0 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_1 - \tilde{Q}_0\tilde{P}'_1 - \tilde{P}'_0 + 2(4-q)\tilde{P}_1 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_2 - \tilde{Q}_0\tilde{P}'_2 - \tilde{P}'_1 + 4(4-q)\tilde{P}_2 &= 0, \\ 2\tilde{Q}'_0 - \tilde{P}'_2 + 6(4-q) &= 0. \end{aligned}$$

The first and last equations are readily solved. With the initial conditions (104), the last one gives

$$\begin{aligned} \tilde{Q}_0 &= \frac{1}{2}\tilde{P}_2 + 3(q-4)w - \frac{q}{8(q-4)}, \\ &= -T'_2 + 3(q-4)w + \frac{1}{q-4}, \end{aligned} \tag{105}$$

thanks to (103), while the first one gives

$$4(q-4)\tilde{P}_0 = q\tilde{Q}_0^2 = q\left(-T'_2 + 3(q-4)w + \frac{1}{q-4}\right)^2. \tag{106}$$

Reporting the expressions (103) and (105) of  $\tilde{P}_2$  and  $\tilde{Q}_0$  in the third differential equation gives an equation in  $\tilde{P}_1$  and  $T_2$  that we can integrate (using the initial conditions (104)):

$$\tilde{P}_1 = (T'_2)^2 + T'_2\left(6(q-4)w - \frac{q+4}{2(q-4)}\right) + 10(4-q)T_2 + \frac{w}{2}(q+8) + \frac{2+q}{2(q-4)^2}.$$

Reporting this, as well as (105) and (106) in the second differential equation finally gives Tutte's equation for the generating function of properly  $q$ -coloured near-triangulations of outer degree 2. The rest of the proof is straightforward.  $\blacksquare$

## 8. FOUR COLOURS: $q = 4$

Several signs suggest that the case of four colours should be simpler. First, this is visible on the differential equation and recurrence relation of Theorem 12. Then, the polynomial  $D(t, x)$ , given by (45) (or its counterpart  $D(w, x)$  given by (73) for triangulations) becomes a square when  $q = 4$ , independent of the size variable  $t$  (or  $w$ ). We use this simplification to derive a differential equation for the generating function of four-coloured maps, of a relatively small order (3 for general maps, 2 for triangulations).

### 8.1. TRIANGULATIONS

**Theorem 13.** *Let  $T_1 \equiv T_1(\nu, w)$  be the generating function of four-coloured near-triangulations of outer degree 1, counted by the number of vertices ( $w$ ) and the number of monochromatic edges ( $\nu$ ) (as before, the root-vertex is coloured in a prescribed colour). This series is characterized by the initial conditions  $T_1 = \nu(3 + \nu)w^2 + O(w^3)$  and the following differential equation of order 2 and degree 6, which we write in terms of  $S = 2T_1 - w$ :*

$$P(S, S', S'') = 0$$

with

$$\begin{aligned}
P(X, Y, Z) = & 48\beta^4\nu^2w^2Y^4Z^2 + 3\beta^5\alpha wY^4Z^2 + 10240\nu^4\beta^2w^3Y^3Z^2 - 5\beta^5\alpha XY^3Z^2 \\
& - 8\beta^4w(10X\nu^2 - \alpha^2)Y^3Z^2 + 768\nu^2\alpha\beta^3w^2Y^3Z^2 + 589824\nu^6w^4Y^2Z^2 + 49152\nu^4\beta\alpha w^3Y^2Z^2 \\
& - 15\beta^4\alpha^2XY^2Z^2 - 288\beta^2\nu^2w^2(80X\nu^2 - 3\alpha^2)Y^2Z^2 - 6\alpha\beta^3w(280X\nu^2 - \alpha^2)Y^2Z^2 \\
& + 15\alpha\beta^3(40X\nu^2 - \alpha^2)XYZ^2 - 138240\nu^4\beta\alpha w^2XYZ^2 - 2160\nu^2\beta^2\alpha^2wXYZ^2 \\
& - 6144\nu^4w^3(320X\nu^2 + \alpha^2)YZ^2 + 5\beta^2(160X\nu^2 - \alpha^2)(20X\nu^2 + \alpha^2)XZ^2 \\
& + 80\nu^2w^2(128X\nu^2 + \alpha^2)(160X\nu^2 - \alpha^2)Z^2 + \beta\alpha(160X\nu^2 - \alpha^2)(560X\nu^2 + \alpha^2)wZ^2 \\
& + 24\nu^2\beta^4wY^5Z - 40\beta^4\nu^2XY^4Z + 40\beta^3\nu^2\alpha wY^4Z + 3584\nu^4\beta^2w^2Y^4Z + 98304ZY^3\nu^6w^3 \\
& + 1536\nu^4\beta\alpha w^2Y^3Z - 24\nu^2\beta^2w(320X\nu^2 - \alpha^2)Y^3Z - 60\nu^2\alpha\beta^3XY^3Z - 8\beta\alpha\nu^2w(160X\nu^2 - \alpha^2)Y^2Z \\
& + 4\nu^2(160X\nu^2 - \alpha^2)(5\beta^2X - 256\nu^2w^2)Y^2Z + 4096\nu^6w^2Y^4 + 128\nu^4\alpha\beta Y^4w + 160X\nu^4\beta^2Y^4,
\end{aligned}$$

where we have written  $\beta = \nu - 1$  and  $\alpha = \nu - 2$ .

*Proof.* We specialize Theorem 6 to  $q = 4$ . The polynomial  $D(w, x)$  becomes a square, and is independent of  $w$ :

$$D(w, x) = (2\nu x + \beta)^2.$$

We use extensively the notation  $\beta = \nu - 1$ ,  $\alpha = \nu - 2$ . Let us specialize to  $q = 4$  the five non-differential identities (88-92). Equation (88) holds verbatim. The identities (89) and (90) both specialize to

$$2\nu R_0 = \beta R_1,$$

so that  $R(w, x)$  is a multiple of  $\sqrt{D(w, x)} = 2\nu x + \beta$ . However, we can also derive a second relation from (89) and (90): if we first eliminate  $R_0$  between them, and then set  $q = 4$ , we obtain

$$(16\nu^2w + \alpha\beta)R_1 - 4\nu\beta Q_1 + 8\nu^2Q_0 + 4\nu\beta^2 = 0. \quad (107)$$

We also have the two relations (91) and (92) relating  $T_1$  to the other unknown series.

We now go back to the differential system (70). We will expand its numerator in powers of  $X = x + \beta/(2\nu)$ . We write accordingly

$$P(w, x) = X^3 + \tilde{P}_2X^2 + \tilde{P}_1X + \tilde{P}_0, \quad Q(w, x) = 2\nu X^2 + \tilde{Q}_1X + \tilde{Q}_0, \quad R(w, x) = R_1X,$$

where the  $\tilde{P}_j$ 's and  $\tilde{Q}_j$  are series in  $w$  with rational coefficients in  $\nu$ . Note that we have six unknown series instead of seven because  $R(w, x)$  is a multiple of  $X$  (in other words,  $\tilde{R}_0 = 0$ ). The remaining four generic identities, namely (88), (107), (91) and (92), translate as follows:

$$\tilde{Q}_1 = \nu\tilde{P}_2 - 3\alpha/4, \quad (108)$$

$$8\nu^2\tilde{Q}_0 = -(16\nu^2w + \alpha\beta)R_1, \quad (109)$$

$$\nu\tilde{P}_1 = 10\nu(2T_1 - w) + \tilde{Q}_0 + \nu\tilde{P}_2^2/4 + 3\alpha\tilde{P}_2/8 + 5\alpha^2/(64\nu), \quad (110)$$

$$R_1 = 4\nu(2T_1' - 1). \quad (111)$$

The following initial conditions are translated from (69):

$$\tilde{P}_0(0) = -\frac{\beta^2\alpha}{16\nu^3}, \quad \tilde{P}_1(0) = \frac{\beta(\alpha + \beta)}{4\nu^2}, \quad \tilde{Q}_0(0) = \frac{\beta\alpha}{2\nu}, \quad \tilde{Q}_1(0) = -\alpha - \beta. \quad (112)$$

Expanding in  $X$  the numerator of (70) gives a system of five differential equations in the six series  $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{Q}_0, \tilde{Q}_1, R_1$ :

$$\begin{aligned} 2\tilde{P}_0\tilde{Q}'_0 - \tilde{Q}_0\tilde{P}'_0 + 2\tilde{P}_0R_1 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_1 + 2\tilde{P}_0\tilde{Q}'_1 - \tilde{P}'_0\tilde{Q}_1 - \tilde{P}'_1\tilde{Q}_0 + 3\tilde{P}_1R_1 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_2 + 2\tilde{P}_1\tilde{Q}'_1 - \tilde{P}'_1\tilde{Q}_1 - \tilde{P}'_2\tilde{Q}_0 - 2\nu\tilde{P}'_0 + 4\tilde{P}_2R_1 &= 0, \\ 2\tilde{Q}'_1\tilde{P}_2 - \tilde{Q}_1\tilde{P}'_2 - 2\nu\tilde{P}'_1 + 2\tilde{Q}'_0 + 5R_1 &= 0, \\ \tilde{Q}'_1 - \nu\tilde{P}'_2 &= 0. \end{aligned}$$

We will need to inject an additional equation (namely (109)) to complete this system.

The fifth (and last) equation of the system is solved by (108).

The fourth equation is a consequence of (108), (110) and (111).

In the first equation of the system, let us replace  $\tilde{R}_1$  by its expression in terms of  $\tilde{Q}_0$  derived from (109). Using the initial conditions (112), the resulting equation (in  $\tilde{P}_0$  and  $\tilde{Q}_0$ ) can now be integrated into:

$$4\nu(16\nu^2w + \alpha\beta)\tilde{P}_0 = -\beta\tilde{Q}_0^2. \quad (113)$$

Let us finally eliminate  $\tilde{R}_1$  between the first and second equations of the system. The resulting equation can be integrated into

$$2\tilde{Q}_1 = \frac{\tilde{P}_1\tilde{Q}_0}{\tilde{P}_0} + c\sqrt{\tilde{P}_0},$$

for some constant  $c$ . Using the initial conditions (112), this constant is found to be zero, so that

$$2\tilde{Q}_1\tilde{P}_0 = \tilde{P}_1\tilde{Q}_0. \quad (114)$$

This is as far as we have been able to go in the integration of the system. At this stage, we have seven unknown series (the  $\tilde{P}_j$ 's,  $\tilde{Q}_j$ 's,  $R_1$  and  $T_1$ ) related by one differential equation (the third one in the system), and six “non-differential” equations (namely (108–111), (113) and (114)) in which the only derivative is  $T'_1$ . Using in the following order (108), (111), (109), (110), and (113), we can now express  $\tilde{Q}_1, R_1, \tilde{Q}_0, \tilde{P}_1, \tilde{P}_0$  in terms of  $\tilde{P}_2, T'_1$  and  $T_1$ . Plugging these expressions in the remaining non-differential equation (114) and in the remaining differential equation gives a pair of differential equations between  $\tilde{P}_2$  and  $T_1$ . Given the form of (110) and (111), it makes sense to write them in terms of  $S = 2T_1 - w$ . They read:

$$\begin{aligned} 4(4\beta\nu\tilde{P}_2 + 128\nu^2w + 5\alpha\beta)S' &= 16\nu^2\tilde{P}_2^2 + 24\nu\alpha\tilde{P}_2 + 640\nu^2S + 5\alpha^2, \\ 2(16\nu^2w + \alpha\beta)(4\nu\tilde{P}_2 + 3\alpha - 2\beta S')S'' - 32\nu^2\beta S'^2 \\ + 8\nu((16\nu^2w + \alpha\beta)\tilde{P}'_2 + 4\nu^2\tilde{P}_2 - 3\nu\alpha)S' - (12\nu^2\alpha\tilde{P}_2 + 320\nu^3S + 7\alpha^2\nu)\tilde{P}'_2 &= 0. \end{aligned}$$

We finally eliminate  $\tilde{P}_2$  as follows: we differentiate the first equation above to obtain a total of three *polynomial* equations in  $S, S', S'', \tilde{P}_2$  and  $\tilde{P}'_2$ , from which we eliminate  $\tilde{P}_2$  and  $\tilde{P}'_2$  using resultants. This yields the differential equation of the theorem.

To finish, one checks that the recurrence relation that underlies the differential equation for  $T_1$  determines all coefficients of this series once we prescribe the first three coefficients of  $T_1$ . ■

## 8.2. GENERAL PLANAR MAPS

**Theorem 14.** *Let  $M_1 \equiv M_1(\nu, w; t)$  be the generating function of four-coloured planar maps, counted by vertices ( $w$ ), edges ( $t$ ) and monochromatic edges ( $\nu$ ) (as before, the root-vertex is coloured in a prescribed colour). This series is characterized by the initial conditions  $M_1 = w + w(\nu + 3w + \nu w)t + O(t^2)$  and a differential equation of order 3 and degree 11.*

The differential equation can be seen in the Maple session accompanying this paper, on the authors' web pages.

*Proof.* We specialize Theorem 1 to  $q = 4$ . The polynomial  $D(t, x)$  is again a perfect square:

$$D(t, x) = ((\nu + 1)x - 2)^2.$$

Equation (94) still holds of course. The identities (95) and (96) both specialize to

$$(\nu + 1)R_0 + 2R_1 + 4(1 - 2w) = 0.$$

Given that  $R_2 = (\nu + 1)(1 - 2w)$  (this is one of the initial conditions in Theorem 1), this means that  $R(t, x)$  has a factor  $\sqrt{D} = (\nu + 1)x - 2$ . As in the case of triangulations, we can derive a second identity from (95) and (96) by eliminating  $R_1$  between them:

$$\begin{aligned} (\beta - (\beta + 2)^2(\beta + 4w)t)R_0 + (\beta + 2)(\beta + 4w)Q_0 + 2(\beta + 4w)Q_1 = \\ 4(2w - 1)(\beta + 2)(\beta + 4w)t - 8w, \end{aligned} \quad (115)$$

with  $\beta = \nu - 1$ . The two characterizations (97) and (98) of  $\tilde{M}_1 = t^2M_1$  still hold.

We now go back to the differential system (47). We will expand its numerator in powers of  $X = x - 2/(\nu + 1)$  instead of  $x$ . We write accordingly

$$P(t, x) = X^4 + \tilde{P}_3X^3 + \tilde{P}_2X^2 + \tilde{P}_1X + \tilde{P}_0, \quad Q(t, x) = X^2 + \tilde{Q}_1X + \tilde{Q}_0,$$

and

$$R(t, x) = (\nu + 1)(1 - 2w)X^2 + \tilde{R}_1X,$$

where the  $\tilde{P}_j$ ,  $\tilde{Q}_j$  and  $\tilde{R}_j$  are series in  $t$  with rational coefficients in  $\nu$  and  $w$ . We have taken into account the fact that  $R$  is a multiple of  $X$ . The remaining four identities, namely (94), (115), (97) and (98), translate as follows:

$$\tilde{Q}_1 = \tilde{P}_3/2 + 2(2w - 1)(\beta + 2)t, \quad (116)$$

$$2((\beta + 2)^2(\beta + 4w)t - \beta)\tilde{R}_1 + (\beta + 2)^2(\beta + 4w)\tilde{Q}_0 = 0, \quad (117)$$

$$\tilde{P}_2 = 12(\beta + 2)^2\tilde{M}_1 + 2\tilde{Q}_0 + \tilde{P}_3^2/4 - 2(2w - 1)(\beta + 2)t\tilde{P}_3 - 12tw\beta - 12t, \quad (118)$$

$$\tilde{R}_1 = 2(\beta + 2)^2\tilde{M}'_1 - (2w - 1)(\beta + 2)\tilde{P}_3/2 - 2\beta w - 2. \quad (119)$$

Eq. (118) actually results from the elimination of  $\tilde{Q}_1$  between (97) and (116).

The following initial conditions are translated from (46):

$$\tilde{P}_0(0) = \frac{4\beta^2}{(2 + \beta)^4}, \quad \tilde{P}_1(0) = \frac{4\beta(\beta - 2)}{(2 + \beta)^3}, \quad \tilde{Q}_0(0) = -\frac{2\beta}{(2 + \beta)^2}, \quad \tilde{Q}_1(0) = \frac{2 - \beta}{2 + \beta}. \quad (120)$$

Expanding in  $X$  the numerator of (47) gives a system of (only) six equations in the seven unknown series  $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{Q}_0, \tilde{Q}_1, \tilde{R}_1$ :

$$\begin{aligned} 2\tilde{P}_0\tilde{Q}'_0 - \tilde{Q}_0\tilde{P}'_0 + 2\tilde{P}_0\tilde{R}_1 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_1 + 2\tilde{P}_0\tilde{Q}'_1 - \tilde{P}'_0\tilde{Q}_1 - \tilde{P}'_1\tilde{Q}_0 + 3\tilde{P}_1\tilde{R}_1 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_2 + 2\tilde{P}_1\tilde{Q}'_1 - \tilde{P}'_1\tilde{Q}_1 - \tilde{P}'_2\tilde{Q}_0 - \tilde{P}'_0 + 4\tilde{P}_2\tilde{R}_1 - (2w - 1)(\beta + 2)\tilde{P}_1 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_3 + 2\tilde{P}_2\tilde{Q}'_1 - \tilde{P}'_2\tilde{Q}_1 - \tilde{P}'_3\tilde{Q}_0 - \tilde{P}'_1 + 5\tilde{P}_3\tilde{R}_1 - 2(2w - 1)(\beta + 2)\tilde{P}_2 &= 0, \\ 2\tilde{Q}'_1\tilde{P}_3 - \tilde{Q}_1\tilde{P}'_3 - \tilde{P}'_2 + 2\tilde{Q}'_0 + 6\tilde{R}_1 - 3(2w - 1)(\beta + 2)\tilde{P}_3 &= 0, \\ 2\tilde{Q}'_1 - \tilde{P}'_3 - 4(2w - 1)(\beta + 2) &= 0. \end{aligned}$$

We will need another equation (namely (117)) to complete this system.

The sixth (and last) equation of the system is solved by (116).

The fifth equation is a consequence of (116), (118) and (119).

In the first equation of the system, we now replace  $\tilde{R}_1$  by its expression in terms of  $\tilde{Q}_0$  derived from (117). The resulting equation (in  $\tilde{P}_0$  and  $\tilde{Q}_0$ ) can now be integrated into:

$$((\beta + 2)^2(\beta + 4w)t - \beta)\tilde{P}_0 + \beta\tilde{Q}_0^2 = 0. \quad (121)$$

We have used the initial conditions (120) to determine the integration constant.

Let us finally eliminate  $\tilde{R}_1$  between the first and second equations of the system. The resulting equation can be integrated into

$$2\tilde{Q}_1 = \frac{\tilde{P}_1\tilde{Q}_0}{\tilde{P}_0} + c\sqrt{\tilde{P}_0},$$

for some constant  $c$ . Using the initial conditions (120), this constant is found to be zero, and

$$2\tilde{Q}_1\tilde{P}_0 = \tilde{P}_1\tilde{Q}_0. \quad (122)$$

This is as far as we have been able to go in the integration of the system. Using (in this order) (116), (119), (117), (118), (121), and (122), we can now express  $\tilde{Q}_1$ ,  $\tilde{R}_1$ ,  $\tilde{Q}_0$ ,  $\tilde{P}_2$ ,  $\tilde{P}_0$  and finally  $\tilde{P}_1$  in terms of  $\tilde{P}_3$ ,  $\tilde{M}'_1$  and  $\tilde{M}_1$ . Plugging these expressions in the remaining equations of the system (the third and fourth) gives a pair of differential equations between  $\tilde{P}_3$  and  $\tilde{M}_1$ . Both have order 1 in  $\tilde{P}_3$  and order 2 in  $\tilde{M}_1$ . Denoting  $\lambda = 2 + \beta$  and  $p = \lambda^2 t(\beta + 4w)$ , they read:

$$\begin{aligned} & 16\lambda^2(p - \beta) \left( t\tilde{M}'_1 - 3\tilde{M}_1 \right) \tilde{M}_1'' - p\tilde{P}_3^2\tilde{M}_1'' + 4\lambda t(2w - 1)(p - 3\beta)\tilde{P}_3\tilde{M}_1'' \\ & \quad + 16t \left( t\lambda^2(2\beta^2 w + \beta(2w + 1)^2 + 8w) - 2\beta(\beta w + 1) \right) \tilde{M}_1'' + 8\lambda^2 p (\tilde{M}'_1)^2 \\ & \quad - 4t\lambda(2w - 1)(2p - 3\beta)\tilde{P}'_3\tilde{M}'_1 + p \left( \tilde{P}'_3 - 2\lambda(2w - 1) \right) \tilde{P}_3\tilde{M}'_1 + 12\lambda(2w - 1)(p - \beta)\tilde{P}'_3\tilde{M}_1 \\ & \quad - 8t\lambda^2(4\beta w^2 + 2w(\beta^2 + 2\beta + 4) + \beta)\tilde{M}'_1 - tw\lambda^2 \left( \tilde{P}_3 + 4\lambda t(2w - 1) \right) \tilde{P}'_3 \\ & \quad \quad \quad + 2tw\lambda^3(2w - 1)\tilde{P}_3 + 8tw\lambda^2(1 + w\beta) = 0, \end{aligned}$$

and

$$\begin{aligned} & \left( p\tilde{P}_3/\lambda - 4t(2w - 1)(p - 2\beta) \right) \tilde{M}_1'' \\ & \quad + p \left( \tilde{P}'_3/\lambda + 12w - 6 \right) \tilde{M}'_1 - 3p/t \left( \tilde{P}'_3/\lambda + 4w - 2 \right) \tilde{M}_1 + 2tw\lambda\tilde{P}'_3 = 0. \end{aligned}$$

By eliminating  $\tilde{P}_3$  between them, we obtain a differential equation of order 3 for  $\tilde{M}_1$  (we first eliminate  $\tilde{P}'_3$  between the two equations to obtain one equation between  $\tilde{P}_3$ ,  $\tilde{M}_1$ ,  $\tilde{M}'_1$  and  $\tilde{M}_1''$ ; then we follow the steps described at the end of Section 8.1).

To finish, one checks that the recurrence relation that underlies the differential equation for  $\tilde{M}_1$  determines all coefficients of this series once the coefficients of  $t^2$  and  $t^3$  in  $\tilde{M}_1$  are prescribed.

■

## 9. SPANNING FORESTS OF CUBIC MAPS: $q = 0$

When  $q = 0$ , the series  $T_1$  has an interesting combinatorial interpretation in terms of spanning forests, as discussed in Section 2.3. Moreover, the polynomial  $D(w, x)$  involved in the differential system for triangulations simplifies greatly, and this will allow us to characterize  $T_1$  by a second order equation. We say a map is *near-cubic* if all non-root vertices have degree 3.

**Theorem 15.** *Let  $G(\beta, w)$  be the generating function of near-cubic planar maps with a root-vertex of degree 1, equipped with a spanning forest, counted by the number of faces ( $w$ ) and the number of connected components of the forest, minus one ( $\beta$ ). This series is characterized by the initial conditions  $G = O(w^2)$  and the following differential equation, written in terms of  $W = 2G - w/\beta$ :*

$$\begin{aligned} 0 = & \left( 3\beta^4 w W'^4 - \beta^3(5W\beta - \beta w + w)W'^3 + 4(\beta + 1)(5W\beta - \beta w + w)^2 \right) W'' \\ & - 48\beta^2 w(\beta + 1)W'^3 + 8\beta(\beta + 1)(5W\beta - \beta w + w)W'^2 + 4(\beta^2 - 1)(5W\beta - \beta w + w)W'. \end{aligned}$$

**Remark.** Theorem 15 has been rederived in [10] using a simpler, and more combinatorial, argument.

*Proof.* Let us apply (8) to the set  $\mathcal{M}$  of near-triangulations with outer degree 1. We obtain:

$$T_1|_{q=0} = \frac{1}{\beta}G(\beta, w\beta), \quad (123)$$

where  $\beta = \nu - 1$  and  $G(\beta, w)$  is defined above.

Hence let us specialize Theorem 6 to  $q = 0$ . The polynomial  $D(w, x)$  has degree 1 in  $x$ , and is independent of  $w$ :

$$D(w, x) = \beta^2(1 + 4x).$$

Let us now specialize the five generic non-differential equations (88-92). Equation (88) holds verbatim. The identities (89) and (90) give  $R_0 = -2\beta$  and  $R_1 = -8\beta$ , so that:

$$R(w, x) = -2\beta(1 + 4x) = -2D(w, x)/\beta. \quad (124)$$

Equation (91) does not involve  $T_1$ , but it expresses  $P_1$  in terms of  $Q_0$  and  $Q_1$ :

$$-48\nu\beta w - 4P_1\nu^2 + 4\nu Q_0 + (Q_1 - 1)(Q_1 + \nu - 3) = 0. \quad (125)$$

Finally, (92) gives again  $R_1 = -8\beta$ , and we still need to relate  $T_1$  (or equivalently  $T_2 = T_1/\nu$ ) to the series  $P_j$  and  $Q_j$ . This can be done by specializing (93) to  $q = 0$ :

$$8\beta T_1'/\nu - 4P_2 + 1 = 0. \quad (126)$$

We now go back to the differential system (70). We will expand its numerator in powers of  $X = x + 1/4$  instead of  $x$ . We write accordingly

$$P(w, x) = X^3 + \tilde{P}_2 X^2 + \tilde{P}_1 X + \tilde{P}_0, \quad Q(w, x) = 2\nu X^2 + \tilde{Q}_1 X + \tilde{Q}_0, \quad R(w, x) = -8\beta X,$$

where the  $\tilde{P}_j$ 's and  $\tilde{Q}_j$  are series in  $w$  with rational coefficients in  $\nu$ . Note that we only have five unknown series instead of seven, because  $R$  is completely determined (see (124)). The remaining three generic identities, namely (88), (125) and (126), give

$$\tilde{Q}_1 = \nu\tilde{P}_2 + 1 - \nu/2, \quad (127)$$

$$4\nu\tilde{Q}_0 = 4\nu^2\tilde{P}_1 + 2\nu^2\tilde{P}_2 - \tilde{Q}_1^2 - 4\beta\tilde{Q}_1 + 48\nu\beta w - (7\nu - 6)(\nu - 2)/4, \quad (128)$$

and

$$2\nu\tilde{P}_2 = 4\beta T_1' - \nu. \quad (129)$$

We will need the values of several series at  $w = 0$ , which follow from the initial conditions (69):

$$\tilde{P}_0(0) = 0, \quad \tilde{P}_1(0) = 1/16, \quad \tilde{P}_2(0) = -1/2, \quad \tilde{Q}_0(0) = (\nu - 2)/8. \quad (130)$$

Expanding in  $X$  the numerator of (70) gives a system of five differential equations between the five series  $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{Q}_0$  and  $\tilde{Q}_1$ :

$$\begin{aligned} 2\tilde{P}_0\tilde{Q}'_0 - \tilde{Q}_0\tilde{P}'_0 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_1 + 2\tilde{P}_0\tilde{Q}'_1 - \tilde{P}'_0\tilde{Q}_1 - \tilde{P}_1\tilde{Q}'_0 - 8\beta\tilde{P}_1 &= 0, \\ 2\tilde{Q}'_0\tilde{P}_2 + 2\tilde{P}_1\tilde{Q}'_1 - \tilde{P}'_1\tilde{Q}_1 - \tilde{P}_2\tilde{Q}'_0 - 2\nu\tilde{P}'_0 - 16\beta\tilde{P}_2 &= 0, \\ 2\tilde{Q}'_1\tilde{P}_2 - \tilde{Q}_1\tilde{P}'_2 - 2\nu\tilde{P}'_1 + 2\tilde{Q}'_0 - 24\beta &= 0, \\ \tilde{Q}'_1 - \nu\tilde{P}'_2 &= 0. \end{aligned}$$

The fifth (and last) equation is solved by (127).

The fourth one is a consequence of (128) and (127).

The first one is readily integrated into  $\tilde{P}_0 = c\tilde{Q}_0^2$ , for some constant  $c$ . The initial conditions (130) give  $c = 0$ : in other words,  $\tilde{P}_0 = 0$  and  $P(w, x)$  is a multiple of  $D(w, x)$ .

In the third equation, let us replace  $\tilde{P}_0$  by 0, then  $\tilde{Q}_0$  by its expression derived from (128), and  $\tilde{Q}_1$  by its expression (127). Finally, let us introduce the primitive  $S$  of  $\tilde{P}_2$ :

$$S := \int \tilde{P}_2 dw = \frac{2\beta}{\nu}T_1 - \frac{w}{2} \quad (131)$$



(the second equality follows from (129), with  $T_1(0) = 0$ ). The resulting equation (in  $\tilde{P}_1$  and  $S$ ) can now be integrated into:

$$-2(2\nu S' + \nu - 2)\tilde{P}_1 + \nu S'^3 + (\nu - 2)S'^2/2 + 48w\beta S' - 80\beta S = 0. \quad (132)$$

We have used the initial conditions (130) to determine the integration constant.

This is as far as we have been able to integrate the system. We can now express  $\tilde{P}_1, \tilde{P}_2, \tilde{Q}_1, \tilde{Q}_0$  in terms of  $S$  and  $S'$  using (in this order) (132), (131), (127), and (128). We plug these values in the second equation of the system, together with  $\tilde{P}_0 = 0$ , and this gives us a second order equation for  $S$ , or, equivalently, for  $T_1$ :

$$P(T_1, T_1', T_1'') = 0$$

with

$$\begin{aligned} P(X, Y, Z) = & 48\beta^4 wZY^4 + 8\beta^3 w(\beta - 8)ZY^3 - 80\beta^4 XZY^3 - 12\beta^2 w(\beta - 2)ZY^2 + 120\beta^3 XZY^2 \\ & + 6\beta^2 wZY - 60\beta^2 XZY + 400\beta^3 \nu ZX^2 + 10\beta ZX - 80\nu\beta^2 w(\nu + 3)ZX + 4\beta(\nu + 3)^2 \nu w^2 Z \\ & - \nu wZ - 192\nu w\beta^3 Y^3 - 16\nu\beta^2 w(\beta - 14)Y^2 + 160\beta^3 \nu XY^2 \\ & - 4\nu\beta w(\beta^2 - \beta + 16)Y + 40\nu\beta^2(\beta - 5)XY + 2\nu^2 w\beta - 20\nu\beta(\beta - 3)X. \end{aligned}$$

We now return to (123) to derive an equation for  $G(\beta, w)$ , or equivalently for  $W = 2G - w/\beta$ .

To finish, one checks that the recurrence relation that underlies the differential equation for  $G$  determines all coefficients of this series once we prescribe  $G = O(w^2)$ .  $\blacksquare$

## 10. THE SELF-DUAL POTTS MODEL ON GENERAL PLANAR MAPS

To finish, we derive a third order differential equation for the self-dual Potts model defined at the end of Section 2.3.

**Theorem 16.** *Let  $S \equiv S(\beta, t)$  be the generating function for the self-dual Potts model on planar maps. More precisely, let*

$$S(\beta, t) = \beta t^2 M(\beta^2, \beta + 1, \beta^{-1}, t; 1) = \sum_M t^{e(M)+2} \sum_{S \subseteq E(M)} \beta^{2c(S)+e(S)-1-\nu(M)},$$

where the first sum runs over all rooted planar maps  $M$ . This series is characterized by the initial conditions  $S = t^2 + O(t^3)$  and the following differential equation of order 3 and degree 4:

$$P(S, S', S'', S''') = 0,$$

where

$$\begin{aligned} P(X, Y, Z, T) = & 2p(\lambda\beta Y - 2)(2\beta Y - 1)(3\lambda\beta X - \beta t - 4t)T - \beta^2 p^2(3\lambda\beta X - \beta t - 4t)Z^3 \\ & - p\beta(-2\lambda\beta^2 pY^2 + \beta(16\lambda t - \beta - 6)Y + 24\lambda^2\beta^2 XY + 6\lambda\beta(\beta - 10)X - 2\lambda\beta t + 48t + 2)Z^2 \\ & + 2(\lambda\beta Y - 2)(2\lambda\beta^2 pY^2 - \beta(8\beta^2 t + 160\beta t + \beta + 288t - 10)Y + 48\lambda^2\beta^2 XY - 48\lambda\beta^2 X \\ & + 8\beta^2 t + 80\beta t + \beta + 64t - 4)Z - 8(2\beta Y - 1)(\lambda Y - 1)(\lambda\beta Y - 2)^2, \end{aligned}$$

with  $\lambda = \beta + 2$  and  $p = 8\lambda t - 1$ .

*Proof.* We specialize Theorem 1 to  $q = \beta^2$ ,  $w = 1/\beta$ . The argument of Section 4.6 proves that in this case as well, the differential system and its initial conditions define uniquely the series  $P_j, Q_j$  and  $R_j$ . Observe that one of the initial conditions reads  $R_2 = 0$ , so that  $R(t, x)$  has degree 1 in  $x$ . This is the key simplification in the self-dual Potts model.

The polynomial  $D(t, x)$  does not simplify drastically. With the notation of the theorem, it reads:

$$D(x, t) = \lambda\beta^2 X^2 + (\beta - 2)p\beta^2/4,$$

with  $X = x - 1/2$ . We write accordingly

$$P(t, x) = \tilde{P}(t, X) = X^4 + \tilde{P}_3 X^3 + \tilde{P}_2 X^2 + \tilde{P}_1 X + \tilde{P}_0, \quad (133)$$

$$Q(t, x) = \tilde{Q}(t, X) = X^2 + \tilde{Q}_1 X + \tilde{Q}_0, \quad \text{and} \quad R(t, x) = \tilde{R}(t, X) = \tilde{R}_1 X + \tilde{R}_0. \quad (134)$$

The initial conditions at  $t = 0$  are even functions of  $X$ :

$$P(0, x) = (X^2 - 1/4)^2, \quad Q(0, x) = X^2 - 1/4.$$

The other two initial conditions read  $\tilde{P}_4 = 1$  and  $\tilde{R}_2 = 0$ , and we have already taken them into account in (133) and (134). But then it is easy to check that  $(\tilde{P}(t, -X), \tilde{Q}(t, -X), -\tilde{R}(t, -X))$  is another solution of the differential system, satisfying the same initial conditions. The uniqueness of the solution implies that  $\tilde{P}$  and  $\tilde{Q}$  are even functions of  $X$ , while  $\tilde{R}$  must be odd. Hence

$$\tilde{P}_1 = \tilde{P}_3 = \tilde{Q}_1 = \tilde{R}_0 = 0,$$

and we are left with four unknown series only.

With these values, the generic equations (94) and (96) automatically hold, and the other three reduce to:

$$\begin{cases} 4\lambda\tilde{Q}_0 + p(\tilde{R}_1 - \beta + 2) & = 0, \\ 12\beta\lambda S + 2\tilde{Q}_0 - \tilde{P}_2 - 4(\beta + 4)t & = 0, \\ -2\beta\lambda S' + \tilde{R}_1 + 4 & = 0, \end{cases} \quad (135)$$

where  $S = \beta\tilde{M}_1 = \beta t^2 M_1$  is the series defined in the theorem.

Expanding in  $X$  the numerator of (47) gives a system of four equations in the four series  $\tilde{P}_0$ ,  $\tilde{P}_2$ ,  $\tilde{Q}_0$ ,  $\tilde{R}_1$ , corresponding to the coefficients of  $X^{2i}$  for  $i$  from 0 to 3. The other coefficients vanish due to the parity properties.

In these four equations, let us replace  $\tilde{P}_2$ ,  $\tilde{Q}_0$  and  $\tilde{R}_1$  by their expressions in terms of  $S$  derived from (135). The fourth equation then vanishes. The three others are found to be linearly dependent (over  $\mathbb{Q}(\beta)$ ), and we consider those corresponding to the coefficients of  $X^0$  and  $X^4$ :

$$4(p\beta S'' + 4\lambda\beta S' - 8)\tilde{P}_0 = p(2\beta S' - 1)\tilde{P}'_0$$

and

$$4\tilde{P}'_0 = (2\beta^2 p^2 S' - p\beta(48\lambda\beta S - 8\beta t - 48t - 1))S'' + 8\lambda\beta^2 p S'^2 - 4\beta(\beta + 6)pS' + 64\beta t + 128t - 8.$$

It remains to eliminate  $\tilde{P}_0$  to obtain the differential equation of the theorem. To conclude, one checks that the underlying recurrence relation on the coefficients of  $S(t)$  determines all of them once the coefficient of  $t^2$  is prescribed.  $\blacksquare$

## 11. FINAL COMMENTS

At the moment, our solution does not solve the important question of locating phase transitions and finding critical exponents of the Potts model on planar maps. Partial results are known, for instance for the Ising model [8, 32, 2], for maps equipped with a spanning forest [2], and for properly coloured triangulations [35]. The latter work exploits Tutte's differential equation (1). Moreover, a parametrized description of the critical value of  $\nu$  (depending on  $q$ ) for planar maps is given in [6], for  $0 < q < 4$ . The analysis builds on the catalytic variable that records the degree of the outer face rather than on the size variable, but this should not influence the location of the critical point. Also the results of [23] involve the dependence of the series in this catalytic variable.

It is natural to ask if the Potts generating function might also satisfy a *linear* differential equation. This would make the analysis of critical points much easier, since finding the location and nature of the singularities would then be (almost) automatic [24]. However, this possibility has been ruled out (for triangulations) by an asymptotic argument in [10].

The second order equations that we have obtained for triangulations in special cases (Sections 7 to 9) probably deserve a further treatment. Do they fit in Painlevé's classification?

Finally, our proofs are very heavy, and it would be nice to establish differential algebraicity by simpler, and ideally more combinatorial means. This has been done in several special cases

(Ising [13, 14], spanning trees [34], spanning forests [10], bipolar orientations [5, 9]). Two particularly attractive problems are three-coloured maps (known to be algebraic [2]) and properly coloured triangulations (Tutte’s recurrence relation (1)).

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O. BERNARDI: BRANDEIS UNIVERSITY, DEPARTMENT OF MATHEMATICS, 415 SOUTH STREET, WALTHAM, MA 02453, USA

*E-mail address:* [bernardi@brandeis.edu](mailto:bernardi@brandeis.edu)

M. BOUSQUET-MÉLOU: CNRS, LABRI, UNIVERSITÉ DE BORDEAUX, 351 COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE

*E-mail address:* [mireille.bousquet@labri.fr](mailto:mireille.bousquet@labri.fr)