# Finite-time average consensus in networked dynamic systems

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***Abstract*— The paper tackles to the finite-time average consensus problem of networked dynamic systems. A dynamic behavior is described by a controlled first order differential equation with and without drift term. The objective of the analysis is to cover a large variety of autonomous systems without restoring their explicit form. For high dimensional multi-system, protocols are proposed to solve the average consensus problem in finite-time. Like an interaction topology, the agreement achievement is analyzed through an undirect fixed graph.The theoretical results are supported by simulations based on multi-unicycle and multi-pendulum systems.**

## I. INTRODUCTION

In the past few years, the cooperative control problem for a group of agents is a popular research topic in decentralized control. Many applications can be found in various area: rendezvous problem of multi-vehicle, control of training, flocking, attitude the synchronization, the fusion of sensors. Is one of the main challenges in cooperative design decentralized control systems, such as some objectives of the group can be achieved. The coherent movement in masses is called consensus. Thus, the problem of consensus plays a central role in study of multi-agent systems. Early works on the consensus problem of multi-agent systems can be found in [4], [5], [6], [7] and [10].

A special case of the consensus problem in multi-agent systems is the finite-time consensus problem, which is sufficiently studied in the literature ([8],[9],[11],[12],[15],[16]. Nevertheless, the finite-time consensus problem that has been solved so far is mostly only for agents with first or second order dynamics, in ([13] et al. and [14] et al.) the authors treated finite-time consensus for nonlinear networked systems where each systems is modeled by drift/driftless systems.

An interesting topic in consensus problem is the average consensus problem means to design a networked interaction protocol such that the states of all the agents converge (asymptotically/ finite time) to the average of their states ([18], [19],[20] and [21]), to name just a few.

In this paper, we investigate the finite-time average consensus problem. Using an undirected fixed graph, the average consensus study for a nonlinear networked systems remains a challenge problem. Further, the research is motivated by the fact that each dynamic system is taken highly nonlinear with/without drift term in the model. Inspired from finite-time stability results presented in [3], [2] and the graph theory [1], nonlinear consensus protocols are proposed throughout the paper.

The paper is organized as follows. Some preliminaries results, the problem statement, and the finite-time average consensus protocol are formulated in section 2. In section 3 one solves a finite-time average consensus of multi-system without drift terms. The finite-time average consensus of multi-system with drift is detailed in section 4. Finally, illustrative examples are presented in section 5.

## II. PRELIMINARIES AND PROBLEM FORMULATION

Throughout the paper, we use $\mathbb{R}$ to denote the set of real number. $\mathbb{R}^n$ is the $n$-dimensional real vector space and $\|.\|$ denotes the Euclidian norm. $\mathbb{R}^{n\times n}$ is the set of $n \times n$ matrices. $diag\{m_1, m_2, ..., m_n\}$ denotes a $n \times n$ diagonal matrix. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The symbol $\otimes$ is the Kronecker product of matrices. We use $sgn(.)$ to denote the signum function. For a scalar $x$, note that $\varphi_\alpha(x) = sgn(x)|x|^\alpha$. We use $\mathbf{x} = (x_1, ..., x_n)^T$ to denote the vector in $\mathbb{R}^n$. Let $\phi_\alpha(\mathbf{x}) = (\varphi_\alpha(x_1), ..., \varphi_\alpha(x_n))^T$, and $\mathbf{1}_n = (1, ..., 1)^T$. The exponent $T$ is the transpose.

### A. Graph theory

In this subsection, we introduce some basic concepts in algebraic graph theory for multi-agent networks. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph, where $\mathcal{V} = \{1, 2, ..., n\}$ is the set of nodes, node $i$ represents the $i$th agent, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair $(i, j)$. $(i, j) \in \mathcal{E}$ if and only if the $i$th agent can send information to the $j$th agent directly.

$A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, where $a_{ij} > 0$ if there is an edge between the $i$th agent and $j$th agent and $a_{ij} = 0$ otherwise. Moreover, if $A^T = A$, then $\mathcal{G}$ is also called an undirected graph. In this paper, we will refer to graphs whose weights take values in the set$\{0, 1\}$ as binary and those graphs whose adjacency matrices are symmetric as symmetric. Let $D = diag\{d_1, ..., d_n\} \in \mathbb{R}^{n\times n}$ be a diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, 1, ..., n$. Hence, we define the Laplacian of the weighted graph

$$L = D - A \in \mathbb{R}^{n\times n}$$

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The undirected graph is called connected if there is a path between any two vertices of the graph. Note that time varying network topologies are not considered in this paper.

### B. Some useful lemmas

In order to establish our main results, we need to recall the following Lemmas.

*Lemma 1:* [3]. Consider the system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f(0) = 0$, $\mathbf{x} \in \mathbb{R}^n$, there exist a positive definite continuous and differentiable function $V(\mathbf{x}) : U \subset \mathbb{R}^n \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in ]0, 1[$, and an open neighborhood $U_0 \subset U$ of the origin such that $\dot{V} + c(V(\mathbf{x}))^\alpha \leq 0$, $\mathbf{x} \in U_0 \backslash \{0\}$. Then $V(\mathbf{x})$ converges to zero in finite time. In addition, the finite settling time $T$ satisfies $T \leq \dfrac{V(\mathbf{x}(0))^{1-\alpha}}{c(1-\alpha)}$.

*Lemma 2:* [17] Consider the following system, $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$

$$\dot{x} = g(x) + \tilde{g}(x) \tag{1}$$

where $g(0) = 0$ and $g(x)$ is a continuous homogeneous vector field of degree $d < 0$ with respect to dilation $[\sigma_1, ..., \sigma_n]$, and $\tilde{g}(x) = [\tilde{g}_1(x), ..., \tilde{g}_n(x)]^T \in \mathbb{R}^n$ satisfies $\tilde{g}(0) = 0$. Assume that $x = 0$ is an asymptotically stable equilibrium of the system $\dot{x} = g(x)$. Then $x = 0$ is a globally finite time stable equilibrium of system (1) if $\lim\limits_{\varepsilon \to 0} \dfrac{\tilde{g}(\varepsilon^{\sigma_1} x_1, ..., \varepsilon^{\sigma_n} x_n)}{\varepsilon^{d+\sigma_i}} = 0$, $i = 0, ..., n$, $\forall x \neq 0$, and the stable equilibrium $x = 0$ of system (1) is globally asymptotically stable.

*Lemma 3:* [6]. For a connected undirected graph $\mathcal{G}$, the Laplacian matrix $L$ of $\mathcal{G}$ has the following properties, $\mathbf{x}^T L \mathbf{x} = \frac{1}{2} \sum\limits_{i,j=1}^{n} a_{ij}(x_i - x_j)^2$, which implies that $L$ is positive semi-definite. $0$ is a simple eigenvalue of $L$ and $\mathbf{1}$ is the associated eigenvector. Assume that the eigenvalues of $L$ are denoted by $0, \lambda_2, ..., \lambda_n$ satisfying $0 \leq \lambda_2 \leq ... \leq \lambda_n$. Then the second smallest eigenvalue satisfies $\lambda_2 > 0$. Furthermore, if $\mathbf{1}^T \mathbf{x} = 0$, then $\mathbf{x}^T L \mathbf{x} \geq \lambda_2 \mathbf{x}^T \mathbf{x}$.

*Lemma 4:* [22]. Let $x_1, x_2, ..., x_n \geq 0$ and $0 < p \leq 1$. Then $(\sum\limits_{i=1}^{n} x_i)^p \leq \sum\limits_{i=1}^{n} x_i^p \leq n^{1-p} (\sum\limits_{i=1}^{n} x_i)^p$.

### C. Problem statements

We study the finite-time average consensus of two-types of networked systems. The first type is given by equation (2) which describes a controlled system without drift. The second type is represented by equation (3) which is clearly a controlled linear system with drift. One notes that the matrix $B$ for the two models depends on the system's states.

Consider a group of $N$ high-dimensional agents where each agent's behavior is described by a controlled nonlinear model without drift $\Sigma_1$, as given by dynamic (2), and the system with drift $\Sigma_2$, as shown by dynamic (3), $\forall i \in \mathcal{I} = \{1, ..., N\}$

$$\Sigma_1 : \quad \dot{x}^i = B(x^i) u^i \tag{2}$$

and

$$\Sigma_2 : \quad \dot{x}^i = f^i(x^i) + B(x^i) u^i \tag{3}$$

where $x^i \in \mathbb{R}^n$, and for $1 \leq i \leq N$ $x^i = [x_1^i, x_2^i, ..., x_n^i]^T$, $B(x^i) \in \mathbb{R}^{n \times m}$, the continuous maps $f^i : \mathbb{R}^n \to \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ is the input, and $B(x^i) = [b_{kl}]$ for $1 \leq k \leq n$ and $1 \leq l \leq m$.

*Definition 1:* Given a control-input $u^i$, we say that systems in networks meet a finite-time average consensus if for any system's state initial conditions, there exists some finite time $T$ such that:

$$\lim_{t \to T} \|x^i(t) - \chi(t)\| = 0 \tag{4}$$

for any $i \in \mathcal{I}$, and where $\chi(t) = \dfrac{1}{N} \sum\limits_{j=1}^{N} x^j(t)$.

$\chi(t)$ can be interpreted as the instantaneous consent providing and serves the group objective.
For multi-$\Sigma_1$ and multi-$\Sigma_2$ systems, one might analyze the following consensus protocol.

For $i \in \mathcal{I}$, the consensus protocol candidate is given by,

$$u^i = -C(x^i) \sum_{j=1}^{N} a_{ij} \phi_\alpha(x^i - x^j) \tag{5}$$

where the $a_{ij}$ elements are of the $\mathcal{G}$ adjacency matrix, $\alpha \in ]0, 1[$, and $\phi_\alpha(.)$ is defined in section II. $C(x^i) \in \mathbb{R}^{m \times n}$ is such that the following assumption hold..

*Assumption 1:* $C(x^i)$ is such that the matrix product $B(x^i)C(x^i)$ is positive semi-definite matrix.

Throughout the paper, one denotes by $\tilde{B}(x^i) = B(x^i)C(x^i)$.

*Assumption 2:* For a given control matrix $C(x^i)$, for all $x$, $y \in \mathbb{R}^n$, we assume that

$$\begin{aligned} &(\phi_\alpha(x) - \phi_\alpha(y))^T \tilde{B}(x^i) \phi_\alpha(x - y) \geq \\ &(\phi_\alpha(x) - \phi_\alpha(y))^T \tilde{B}(x^i) (\phi_\alpha(x) - \phi_\alpha(y)) \end{aligned}$$

*Assumption 3:* Consider that

$$g(x^i) = -\sum_{j=1}^{N} a_{ij} \tilde{B}(x^i) \phi_\alpha(\xi^i - \xi^j) \tag{6}$$

is a locally homogeneous vector field of degree $d$ with respect to dilation $[\sigma_1, ..., \sigma_n]$.

## III. The multi-$\Sigma_1$ finite-time average consensus

For finite-time average consensus of multi-$\Sigma_1$ one considers each $\Sigma_1$ vector state to compute the average vector, and the protocol candidate (5) while the interaction topology is an undirected fixed graph. As the matrix $B$ structure is taken identical for each $\Sigma_1$, than one might think to networked homogeneous systems.

*Proposition 1:* Let $\mathcal{G}$ be an undirected and connected graph, under the protocol (5) and Assumptions 1-2-3 the

multi-$\Sigma_1$ achieves a finite-time average consensus in the sense of (4).

**Proof.** We introduce $\xi^i(t) = x^i(t) - \chi(t)$. Due to the fact that $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq N$ (undirected graph) and $\phi_\alpha$ is an odd function, we have,

$$\begin{aligned}\dot{\chi}(t) &= \frac{1}{N}\sum_{i=1}^{N}\dot{x}^i(t)\\ &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\tilde{B}(x^i)\phi_\alpha(x^i - x^j)\\ &= -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\tilde{B}(x^i) - \tilde{B}(x^j)\right)\phi_\alpha(x^i - x^j)\end{aligned}$$

Introducing the protocol (5), we obtain

$$\begin{aligned}\dot{\xi}^i(t) &= \dot{x}^i(t) - \dot{\chi}(t)\\ &= -\sum_{j=1}^{N}a_{ij}\tilde{B}(x^i)\phi_\alpha(x^i - x^j)\\ &+\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\tilde{B}(x^i) - \tilde{B}(x^j)\right)\phi_\alpha(x^i - x^j)\\ &= -\sum_{j=1}^{N}a_{ij}\tilde{B}(x^i)\phi_\alpha(\xi^i - \xi^j)\\ &+\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\tilde{B}(x^i) - \tilde{B}(x^j)\right)\phi_\alpha(\xi^i - \xi^j)\end{aligned}$$

let $\xi(t) = (\xi^1, ..., \xi^N)$, we can write the last equation in the form

$$\dot{\xi}^i(t) = g(\xi^i) + \tilde{g}(\xi) \tag{7}$$

where $g(\xi^i) = -\sum_{j=1}^{N}a_{ij}\tilde{B}(x^i)\phi_\alpha(\xi^i - \xi^j)$ and

$$\tilde{g}(\xi) = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\tilde{B}(x^i) - \tilde{B}(x^j)\right)\phi_\alpha(\xi^i - \xi^j).$$

By now, it remains to prove that the equilibrium of (7) is finite-time stable, and this is achieved in the subsequent two steps.

**Step1**. First, the goal is to prove the finite time stability of system

$$\dot{\xi}^i(t) = g(\xi^i) \tag{8}$$

Taking the Lyapunov function,

$$V(\xi(t)) = \frac{1}{\alpha+1}\sum_{i=1}^{N}(\xi^i)^T\phi_\alpha(\xi^i) \tag{9}$$

The derivative of $V$ along the solutions of system (8) yields

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N}\left(\phi_\alpha(\xi^i)\right)^T\dot{\xi}^i\\ &= -\sum_{i,j=1}^{N}a_{ij}\left(\phi_\alpha(\xi^i)\right)^T\tilde{B}(x^i)\phi_\alpha(\xi^i - \xi^j)\\ &= -\tfrac{1}{2}\sum_{i,j=1}^{N}a_{ij}\left(\phi_\alpha(\xi^i) - \phi_\alpha(\xi^j)\right)^T\tilde{B}(x^i)\phi_\alpha(\xi^i - \xi^j)\end{aligned}$$

From Assumption 2, the following inequality holds

$$\begin{aligned}\dot{V} &\leq -\tfrac{1}{2}\phi_\alpha^T(\xi)\left(L\otimes\tilde{B}(x^i)\right)\phi_\alpha(\xi)\\ &= -\tfrac{1}{4}\phi_\alpha^T(\xi)\Theta\phi_\alpha(\xi)\end{aligned}$$

where $\Theta = \frac{1}{2}\left(L\otimes\tilde{B}(x^i) + L\otimes\tilde{B}^T(x^i)\right)$.
Let

$$D(x^i) = diag\{0_n, \gamma_2(x^i), ..., \gamma_N(x^i)\}$$

where $0_n = diag\{0, ..., 0\} \in \mathbb{R}^{n\times n}$, and $\forall j = 2, ..., N$ $\gamma_j(x^i) = \lambda_j(L)\varrho_n(x^i)$ with
$\varrho_n(x^i) = diag\{0, \mu_2(x^i), ..., \mu_n(x^i)\} \in \mathbb{R}^{n\times n}$, and where $\mu_2(x^i), ..., \mu_n(x^i)$ are the eigenvalues of the matrix $\tilde{B}(x^i)$ given in increasing order. $\lambda_j(L)$ denotes the $j^{th}$ eigenvalue of $L$. Let them be $\lambda_2(L), ..., \lambda_N(L)$ in increasing order. Since $\mathcal{G}$ is connected (by Lemma 3) $\lambda_2(L) > 0$. Therefore, $\forall x^i$ we have $\lambda_2\mu_2(x^i) > 0$.
Further, since $\Theta$ is symmetric matrix, then there exist an orthogonal matrix $P \in \mathbb{R}^{Nn\times Nn}$ such that $\Theta = P^TD(x^i)P$. Let $z_\alpha = P\phi_\alpha(\xi)$, thus

$$\begin{aligned}\dot{V} &\leq -\tfrac{1}{4}z_\alpha^TDz_\alpha\\ &\leq -\tfrac{1}{4}\lambda_2\mu_1(x^i)\|z_\alpha\|^2\\ &= -\tfrac{1}{4}\lambda_2\mu_1(\xi^i)\|\phi_\alpha(\xi)\|^2\end{aligned}$$

with $\lambda_2\mu_1(x^i) = \min_{\mathbf{z}_\alpha\perp\mathbf{1}_{Nn}}\frac{z_\alpha^TDz_\alpha}{z_\alpha^Tz_\alpha}$.
Let $k = \min_{x^i\in\mathbb{R}^N}\lambda_2\mu_1(x^i) > 0$ and $\xi = \mathbf{1}_N\otimes\xi^i = (\tilde{\xi}_1, ..., \tilde{\xi}_{Nn})^T$, consequently,

$$\dot{V} \leq -\frac{k}{4}\sum_{i=1}^{Nn}|\varphi_\alpha(\tilde{\xi}_i)|^2 \leq -\frac{k}{4}\sum_{i=1}^{Nn}|\tilde{\xi}_i|^{2\alpha} \leq -\frac{k}{4}(\sum_{i=1}^{Nn}|\tilde{\xi}_i|^{\alpha+1})^{\frac{2\alpha}{\alpha+1}} \tag{10}$$

which permits to write

$$\dot{V} \leq -\frac{k}{4}(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}} \tag{11}$$

Since $0 < \frac{2\alpha}{\alpha+1} < 1$ and $\frac{k}{4}(\alpha+1)^{\frac{2\alpha}{\alpha+1}} > 0$, by Lemma 1 the above differential equation (8) shows that $V$ reaches zero in finite time.
**Step2**. From Assumption 3, the vector field $g(\xi^i)$ is homogeneous of degree $d$ which is negative due to the fact that $\xi = 0$ is a finite time stable equilibrium of $\dot{\xi}^i = g(\xi^i)$. Moreover, it is straightforward to prove that $\lim_{\varepsilon\to 0}\frac{\tilde{g}(\varepsilon^{\sigma_1}x_1, ..., \varepsilon^{\sigma_n}x_n)}{\varepsilon^{d+\sigma_i}} = 0$. Then by Lemma 2, system (7) is finite time stable. Thus, as a result the multi-$\Sigma_1$ dynamic system with the protocol (5) solve a finite-time average consensus. This ends the proof.

■

## IV. The multi-$\Sigma_2$ finite-time average consensus

The multi-$\Sigma_2$ in network is based on dynamic (3) while the consensus protocol candidate is given by (5). Recall that the $\Sigma_2$ dynamic as given by (3) is currently present in controlled autonomous systems. Further, the drift term will be considered as linear with respect to the $\Sigma_2$'s states. Note that $f^i$ in (3) can be different for each $\Sigma_2$ dynamic systems. Than one might think to networked heterogeneous systems.

**Case1**: Linear drift term

we consider equation (3) and let $f^i(x^i) \equiv \tilde{A}x^i$, system (3) becomes

$$\Sigma_{2,1}: \quad \dot{x}^i = \tilde{A}x^i + B(x^i)u^i \tag{12}$$

where $\tilde{A} \in \mathbb{R}^{n\times n}$ with $\tilde{A} = [\tilde{a}_{p,q}]_{1\leq p,q\leq n}$.

*Proposition 2:* Let $\mathcal{G}$ be an undirected and connected graph. Under the protocol (5) and Assumption 1-2-3 the multi-$\Sigma_{2,1}$ achieves a finite-time average consensus in the sense of (4).

**Proof**. One introduces $\xi(t) = x^i(t) - \chi(t)$. The goal is to rewrite equation (12) in closed loop depending on $\xi^i$ and to prove that $\xi$ converges to zero in finite time.
Since $a_{ij} = a_{ji}$ and $\phi_\alpha$ is an odd function, then we have

$$\begin{aligned}\dot{\chi} &= \frac{1}{N}\sum_{i=1}^{N}(\tilde{A}x^i + B(x^i)u^i)\\ &= \frac{1}{N}\sum_{i=1}^{N}\tilde{A}x^i + \frac{1}{N}\sum_{i=1}^{N}B(x^i)u^i\\ &= \frac{1}{N}\sum_{i=1}^{N}\tilde{A}x^i\\ &-\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\tilde{B}(x^i)-\tilde{B}(x^j)\right)\phi_\alpha(x^i-x^j)\end{aligned}$$

Consequently,

$$\begin{aligned}\dot{\xi}^i &= \tilde{A}\xi^i - \sum_{j=1}^{N}a_{ij}\tilde{B}(x^i)\phi_\alpha(\xi^i-\xi^j)\\ &+\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\tilde{B}(x^i)-\tilde{B}(x^j)\right)\phi_\alpha(\xi^i-\xi^j)\end{aligned}$$

keeping the same steps of the previous proof, we introduce

$$\dot{\xi}^i = h(\xi^i) + \tilde{h}(\xi)$$

where $h(\xi^i) = \tilde{A}\xi^i - \sum_{j=1}^{N}a_{ij}\tilde{B}(x^i)\phi_\alpha(\xi^i-\xi^j)$ and

$$\tilde{h}(\xi) = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left(\tilde{B}(x^i)-\tilde{B}(x^j)\right)\phi_\alpha(\xi^i-\xi^j).$$

As $\tilde{h}(\xi) = \tilde{g}(\xi)$, then it remains to prove the finite-time stability of the system

$$\dot{\xi}^i = h(\xi^i) \tag{13}$$

Using the Lyapunov function (9), the time derivative of $V(\xi)$ along the networked system trajectories (13), is given by

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N}\phi_\alpha^T(\xi^i)\dot{\xi}^i\\ &= \sum_{i=1}^{N}\phi_\alpha^T(\xi^i)\tilde{A}\xi^i - \sum_{i,j=1}^{N}a_{ij}\phi_\alpha^T(\xi^i)\tilde{B}(x^i)\phi_\alpha(\xi^i-\xi^j)\\ &\leq \|\tilde{A}\|_\infty\sum_{i=1}^{N}\phi_\alpha^T(\xi^i)\xi^i - \frac{k}{4}(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}}\\ &\leq \|\tilde{A}\|_\infty V(\xi(t)) - \tfrac{k}{4}(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}}\\ &\leq -V(\xi(t))^{\frac{2\alpha}{\alpha+1}}[\tfrac{k}{4}(\alpha+1)^{\frac{2\alpha}{\alpha+1}} - \|\tilde{A}\|_\infty(V(\xi(t)))^{\frac{1-\alpha}{\alpha+1}}]\end{aligned}$$

where $\|\tilde{A}\|_\infty = \max_{1\leq p\leq n}\sum_{q=1}^{n}|\tilde{a}_{pq}| > 0$. Since $\frac{1-\alpha}{\alpha+1} > 0$ and $V$ is continuous function which takes 0 at the origin ($\xi = 0$), there exists an open neighborhood $\Omega$ of the origin such that the last inequality (14) yields to

$$\dot{V}(\xi(t)) \leq -\frac{k}{8}(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V(\xi(t))^{\frac{2\alpha}{\alpha+1}} \tag{14}$$

by Lemma 1, $V$ reaches zero in finite time. Therefore $\xi^i = 0$ is a finite-time stable equilibrium of system (13). We may follow step 2 of the previous analysis to end the proof.

■

**Case2:** Nonlinear drift term

In this case, we consider the system (3) and the drift term as nonlinear and we assume that $f^i$ is a convex vector field.

*Proposition 3:* Let $\mathcal{G}$ be an undirected graph. Under the protocol (5), a networked system multi-$\Sigma_2$ based on (3) meets a finite-time average consensus.

**Proof**. As $f^i$ is assumed to be convex, we have

$$f^i(x^i) - \frac{1}{N}\sum_{i=1}^{N}f^i(x^i) \leq f^i(x^i) - f^i(\frac{1}{N}\sum_{i=1}^{N}x^i)$$

Moreover $f^i$ is locally Lipschitz function in an open set $\Omega \subset \mathbb{R}^n$ containing $\xi$. Therefore,

$$\begin{aligned}\|f^i(x^i) - \tfrac{1}{N}\sum_{i=1}^{N}f^i(x^i)\| &\leq \|f^i(x^i) - f^i(\chi)\|\\ &\leq c\|\xi^i\|\end{aligned}$$

where $c > 0$ is the Lipschitz's constant. Now, for convenience the Lyapunov function is given by (9) and the following inequality is obtained

$$\begin{aligned}\dot{V}(\xi(t)) &= \sum_{i=1}^{N}(\phi_\alpha(\xi^i))^T\dot{\xi}^i\\ &\leq c\sum_{i=1}^{N}\phi_\alpha^T(\xi^i)\xi^i - \frac{k}{4}(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V^{\frac{2\alpha}{\alpha+1}}\\ &\leq -V(\xi(t))^{\frac{2\alpha}{\alpha+1}}[\tfrac{k}{4}(\alpha+1)^{\frac{2\alpha}{\alpha+1}} - c(V(\xi(t)))^{\frac{1-\alpha}{\alpha+1}}]\end{aligned}$$

Since $\frac{1-\alpha}{\alpha+1} > 0$ and $V$ is continuous function which takes 0 at the origin ($\xi = 0$), there exists an open neighborhood $\Omega$ of the origin such that

$$\dot{V}(\xi(t)) \leq -\frac{k}{8}(\alpha+1)^{\frac{2\alpha}{\alpha+1}}V(\xi(t))^{\frac{2\alpha}{\alpha+1}} \tag{15}$$

Then, we may conclude that the multi-$\Sigma_2$ system issues from (3) associated to the protocol (5) leads to a finite-time average consensus. This ends the proof.

■

## V. Illustrative examples

Two illustrative examples are considered where the multi-unicycle represents a networked system modeled by (2) (driftless) and the multi-pendulum implies a networked multi-model of type (3) (with drift). Each associated protocol is deduced from (5). The results of the finite-time average consensus are obtained under the undirected graph of figure Fig. 1,

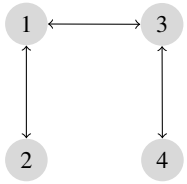


Fig. 1. $\mathcal{G}$ for a system with 4 agents.

### A. Finite-time average consensus for multi-unicycle

Consider $N$ wheeled mobile robots (unicycles) where the $i^{th}$ nonholonomic kinematic model is as:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_i \\ w_i \end{pmatrix} \quad i = 1, ..., N \quad (16)$$

where $(x_i, y_i, \theta_i)$ denotes the position and the orientation in an inertial frame. The inputs $u_i$ and $w_i$ are the linear and angular velocities, respectively. Let $B = \begin{pmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \end{pmatrix}$

Based on Proposition 1, the finite-time average consensus problem can be achieved through the following protocol

$$u_i = -\sum_{j=1}^{N} a_{ij}\varphi_\alpha(x_i - x_j)\cos\theta_i - \sum_{j=1}^{N} a_{ij}\varphi_\alpha(y_i - y_j)\sin\theta_i \quad (17)$$

$$w_i = \sum_{j=1}^{N} a_{ij}\varphi_\alpha(x_i - x_j)\sin\theta_i - \sum_{j=1}^{N} a_{ij}\varphi_\alpha(y_i - y_j)\cos\theta_i \quad (18)$$

where $\varphi_\alpha$ is defined in section II and $a_{ij}$ are associated to the graph in Fig. 1. The simulation results are limited to $N = 4$ that integrate the following initial conditions $(x_1, y_1, \theta_1)(t = 0) = (14, 2, \pi)$, $(x_2, y_2, \theta_2)(t = 0) = (-4, 2, -\frac{\pi}{2})$, $(x_3, y_3, \theta_3)(t = 0) = (10, 8, \frac{\pi}{2})$ and $(x_4, y_4, \theta_4)(t = 0) = (-10, -8, 0)$. The numerical simulations are performed using (16) and protocols (17)-(18). The results of figures Fig. 2-3 evolve according to the developed theoretical results of multi-$\Sigma_1$. The common value is also the average trajectory of the unicycles. The $\|(x_i, y_i) - (ave(x_i), ave(y_i))\|$ converges in finite-time to zero as show in figure Fig.4.

### B. Finite-time average consensus in multi-pendulum dynamics

Consider a set of $N$ pendulum with the following model

$$\ddot{\theta}_i = -\frac{g}{l_i}\sin(\theta_i) - \frac{\psi_i}{m_i l_i}\dot{\theta}_i + u_i \quad (19)$$

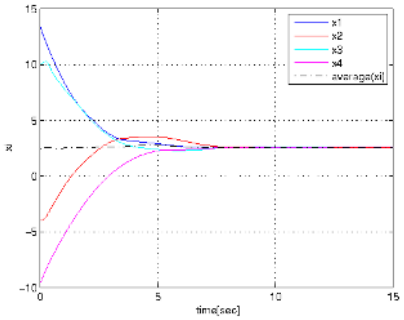


Fig. 2. Average consensus of position $x_i$ for 4 unicycles as multi-$\Sigma_1$

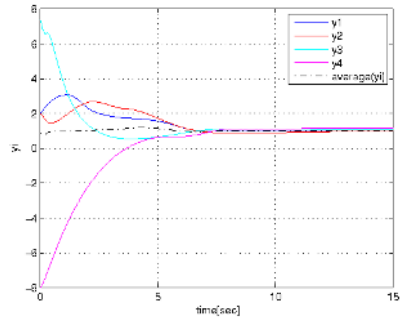


Fig. 3. Average consensus of position $y_i$ for 4 unicycles as multi-$\Sigma_1$

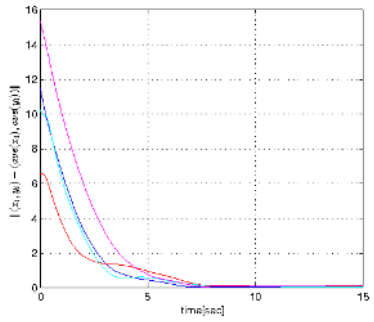


Fig. 4. Convergence of $\|(x_i, y_i) - (ave(x_i), ave(y_i))\|$

where $m_i$, $g_i$, $l_i$ and $\psi_i$ are positive constants. For this system the drift term issues from the first order differential form (see (3)) is

$$f^i(\theta_i, \dot{\theta}_i) = \begin{pmatrix} \dot{\theta}_i \\ -\frac{g}{l_i}\sin(\theta_i) - \frac{\psi_i}{m_i l_i}\dot{\theta}_i \end{pmatrix}$$

we can easily check the convexity condition for the drift term $f^i$. Following to the subsequent theoretical analysis (see Proposition 3), taking $C = (1 \;\; 1)$, a protocol that solves the finite-time average consensus for multi-pendulum is as

$$u_i = -\sum_{j=1}^{N} a_{ij}(\varphi_\alpha(\theta_i - \theta_j) + \varphi_\alpha(\dot{\theta}_i - \dot{\theta}_j)) \quad (20)$$

This set of $N = 4$ pendulums is analyzed. As heterogenous multi-system, the 4 pendulum parameters aren't similar. Thus, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ and $m_4 = 4$ (Kg). The standard gravity vector is $g = 9.8(\text{m.s}^{-2})$, the lengths $l_i = 1$ (m) and the coefficient $\psi_i = 0.1$ $(\text{Kg.m}^2.\text{s}^{-1})$. Initial conditions are such that $\theta_i = (-0.8, 0.4, 1, 2, 1.6)$ (rad) and $\dot{\theta}_i = (0, 0, 0, 0)$ (rad.s$^{-1}$).

Clearly from figures in Fig. 5-6, the synchronization toward the average trajectory of 4 pendulums in angular positions and velocities are obtained. It is important to note that the average is time-varying and the multi-system of pendulums is heterogeneous with respect to the proposed physical parameters. This confirm the theoretical results of Proposition 3.

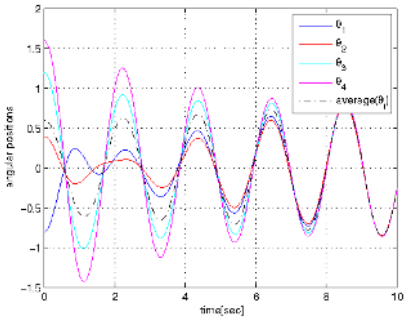


Fig. 5. The time-varying average of angular positions consent by 4 pendulums.

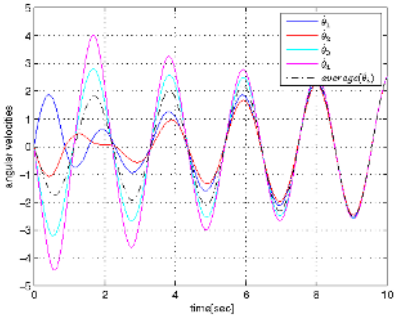


Fig. 6. The time-varying average of angular velocities consent by 4 pendulums.

## VI. Conclusion

In this work, a controlled dynamic model of networked autonomous systems is formulated by two-types of nonlinear and continuous first-order differential equations. Some protocols are proposed and sufficient conditions are achieved covering high dimensional networked homogeneous dynamics. The theoretical results solve finite-time average consensus problems of dynamic systems in the form multi-$\Sigma_1$ and multi-$\Sigma_2$, and they are supported by simulations based on multi-unicycle and multi-pendulum systems. For the future, it is interesting to treat the case of networked heterogeneous multi-system as multi-$\Sigma_1$-$\Sigma_2$.

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