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Pseudo-local property of gravity water waves system

Quang-Huy Nguyen

ABSTRACT. By proving a weighted contraction estimate for the flow of gravity water waves, we show that this nonlocal system is in fact pseudo-local in the following sense: locally in time, the dynamic far away from a given bounded region has a small effect on that region (again, in a sense that we will make precise in the article). Our estimate on the flow also implies a new spatial decay property of the waves.

1. Introduction

1.1. The problem. We consider an incompressible, irrotational, inviscid fluid moving in a domain Ω underneath a free surface described by η and above a bottom described by a given function η_* , which is assumed to be bounded and continuous. Namely,

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : \eta_*(x) < y < \eta(t, x)\}.$$

We also denote by Σ the free surface and by Γ the bottom,

$$\Sigma = \{(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\},$$

$$\Gamma = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta_*(x)\}.$$

The velocity field v admits a potential $\phi : \Omega \rightarrow \mathbf{R}$ such that $v = \nabla_{x,y}\phi$ and $\Delta_{x,y}\phi = 0$ in Ω . We introduce the trace of the potential on the surface

$$\psi(t, x) = \phi(t, x, \eta(t, x))$$

and the Dirichlet-Neumann operator

$$(1.1) \quad \begin{aligned} G(\eta)\psi &= \sqrt{1 + |\nabla_x \eta|^2} \left(\frac{\partial \phi}{\partial n} \Big|_{\Sigma} \right) \\ &= (\partial_y \phi)(t, x, \eta(t, x)) - \nabla_x \eta(t, x) \cdot (\nabla_x \phi)(t, x, \eta(t, x)). \end{aligned}$$

Then (see [9]) the gravity water waves system in the Zakharov/Craig–Sulem formulation reads as follows

$$(1.2) \quad \begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi = -\frac{1}{2} |\nabla_x \psi|^2 + \frac{1}{2} \frac{(\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi)^2}{1 + |\nabla_x \eta|^2} - g\eta \end{cases}$$

where g is the acceleration of gravity.

Following [1] we shall consider the vertical and horizontal components of the velocity on the free surface as unknowns which can be expressed in terms of η and ψ :

$$(1.3) \quad B = (v_y)|_{\Sigma} = \frac{\nabla_x \eta \cdot \nabla_x \psi + G(\eta)\psi}{1 + |\nabla_x \eta|^2}, \quad V = (v_x)|_{\Sigma} = \nabla_x \psi - B \nabla_x \eta.$$

Recall also that the Taylor coefficient $a = -\frac{\partial P}{\partial y} \Big|_{\Sigma}$ can be defined in terms of η, ψ, B, V only (see §4.2 in [2] and §4.3.1 in [9]).

The (local) well-posedness theory for gravity water waves (under the formulation (1.2) or the others) in Sobolev spaces $H^s(\mathbf{R}^d)$ has been studied by many authors, for example Yoshihara [17], Wu [14, 15], Lannes [8]; we refer to the recent book of Lannes [9] for a comprehensive survey of the subject. In these works, the waves were assumed to be of infinite extend (and vanish at infinity), that is, there is no restriction on the horizontal direction. However, in reality water waves always propagate in some bounded container (a lake, an ocean, etc) and hence there will be contacts at the "vertical boundary" of the container. A natural question then arises: (Q) can we justify the \mathbf{R}^d -approximation? More precisely, if (1.2) is a good model then it has to satisfy in certain sense the following property: the dynamic at "infinity" has a small effect on bounded regions. Since (1.2) appears to be nonlocal (due to the presence of the Dirichlet-Neumann operator) it is not clear that the above replacement at "infinity" is harmless. We should mention that in the special case of a canal or a rectangle basin where the walls are right vertical, the local theory was considered by Alazard-Burq-Zuily [1], Kinsey-Wu [7], Wu [16]. Our goal in the present paper is to give the following answer to question (Q). Considering a bounded reference domain, we shall prove that in some sense, far away from this reference domain, the dynamic there has a small effect on the reference domain, and the farther it is the smaller the effect is. In other words, this proves that the gravity water waves system enjoys the "pseudo-local property" (the terminology "pseudo" will be clear in our explanation below).

1.2. Main results. We recall first the definition of uniformly local Sobolev spaces (or Kato's sapces) introduced by Kato in [6].

DEFINITION 1.1. *Let $\chi \in C^\infty(\mathbf{R}^d)$ with $\text{supp } \chi \subset [-1, 1]^d$, $\chi = 1$ near $[-\frac{1}{4}, \frac{1}{4}]^d$ such that*

$$(1.4) \quad \sum_{q \in \mathbf{Z}^d} \chi_q(x) = 1, \quad \forall x \in \mathbf{R}^d, \quad \chi_q(x) = \chi(x - q).$$

For $s \in \mathbf{R}$ define $H_{ul}^s(\mathbf{R}^d)$ the space of distributions $u \in H_{loc}^s(\mathbf{R}^d)$ such that

$$\|u\|_{H_{ul}^s(\mathbf{R}^d)} := \sup_{q \in \mathbf{Z}^d} \|\chi_q u\|_{H^s(\mathbf{R}^d)} < +\infty.$$

This definition is independent of the choice of $\chi \in C_0^\infty(\mathbf{R}^d)$ satisfying (1.4) (see Lemma 7.1 in [1]).

NOTATION 1.2. We set for all $\sigma \in \mathbf{R}$,

$$\begin{aligned} \mathcal{H}_{ul}^\sigma &= H_{ul}^{\sigma+\frac{1}{2}}(\mathbf{R}^d) \times H_{ul}^{\sigma+\frac{1}{2}}(\mathbf{R}^d) \times H_{ul}^\sigma(\mathbf{R}^d) \times H_{ul}^\sigma(\mathbf{R}^d), \\ \mathcal{W}^\sigma &= W^{\sigma+\frac{1}{2}, \infty}(\mathbf{R}^d) \times W^{\sigma+\frac{1}{2}, \infty}(\mathbf{R}^d) \times W^{\sigma, \infty}(\mathbf{R}^d) \times W^{\sigma, \infty}(\mathbf{R}^d). \end{aligned}$$

Denote also by $U = (\eta, \psi, B, V)$ the unknown of system (1.2) and by $U^0 = (\eta^0, \psi^0, B^0, V^0)$ its initial value.

The Cauchy theory proved in [1] reads as follows

THEOREM 1.3. *Let $s > 1 + \frac{d}{2}$ and $U^0 \in \mathcal{H}_{ul}^s$ with*

$$(1.5) \quad \inf_{x \in \mathbf{R}^d} (\eta^0(x) - \eta_*(x)) \geq 2h > 0, \quad \inf_{x \in \mathbf{R}^d} a(0, x) \geq 2c > 0.$$

Then there exists $T > 0$ such that the Cauchy problem for (1.2) with datum U^0 has a unique solution

$$U \in L^\infty([0, T], \mathcal{H}_{ul}^s) \cap C^0([0, T], \mathcal{H}_{ul}^r), \quad \forall r < s$$

and

$$(1.6) \quad \inf_{t \in [0, T]} \inf_{x \in \mathbf{R}^d} [\eta(t, x) - \eta_*(x)] \geq h, \quad \inf_{t \in [0, T]} \inf_{x \in \mathbf{R}^d} a(t, x) \geq c.$$

Moreover, for given $h, c > 0$ the existence time T can be chosen uniformly for data belonging to a bounded set of \mathcal{H}_{ul}^s .

Conditions (1.5) mean that initially, the free surface is away from the bottom and the Taylor coefficient is positively away from 0. Then the conclusion (1.6) asserts that these properties are propagated by the waves, locally in time. We shall always consider in the sequel solutions of (1.2) obeying these properties, which for the sake of simplicity is denoted by

$$(1.7) \quad \mathcal{P}_{s, T}(h, c) := \{U = (\eta, \psi, B, V) \in L^\infty([0, T], \mathcal{H}_{ul}^s) \text{ solution to (1.2), satisfying (1.6) and } U|_{t=0} \text{ satisfies (1.5)}\}.$$

Our main result concerning the solution map of the gravity water waves is stated in the following theorem.

THEOREM 1.4. *Let $s > 1 + d/2$, $T > 0$ and two positive constants h, c . Then for every $\lambda \in \mathbf{R}$ there exists a function $\mathcal{K} : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ nondecreasing in each argument, such that*

$$(1.8) \quad \|\langle \cdot \rangle^\lambda (U_1 - U_2)\|_{C([0, T], \mathcal{H}_{ul}^{s-1})} \leq \mathcal{K}(M_1, M_2) \|\langle \cdot \rangle^\lambda (U_1 - U_2)|_{t=0}\|_{\mathcal{H}_{ul}^{s-1}}$$

for all $U_1, U_2 \in \mathcal{P}_{s, T}(h, c)$, provided that the right-hand side is finite, where

$$M_j := \|U_j\|_{L^\infty([0, T], \mathcal{H}_{ul}^s)} < +\infty, \quad j = 1, 2.$$

As a consequence, we have

COROLLARY 1.5. *Let $s > 1 + d/2$; $h, c > 0$ and \mathcal{A} be a bounded set in \mathcal{H}_{ul}^s . Denote by T the uniform existence time of solutions to (1.2) in $\mathcal{P}_{s, T}(h, c)$ with data in \mathcal{A} . Then there exists $0 < T_1 \leq T$ such that the following property holds: for every $\lambda \in \mathbf{R}$ one can find a constant $C > 0$ such that*

$$(1.9) \quad \|\langle \cdot \rangle^\lambda (U_1 - U_2)\|_{C([0, T_1], \mathcal{H}_{ul}^{s-1})} \leq C \|\langle \cdot \rangle^\lambda (U_1 - U_2)|_{t=0}\|_{\mathcal{H}_{ul}^{s-1}},$$

for all $U_j \in \mathcal{P}_{s, T}(h, c)$ with $U_j|_{t=0} \in \mathcal{A}$ and provided that the right-hand side is finite.

In Corollary 1.5 if we take $U_2|_{t=0} = 0$ and use the Sobolev embeddings (see Proposition 2.2, [1])

$$H_{ul}^r(\mathbf{R}^d) \hookrightarrow W^{r - \frac{d}{2}, \infty}(\mathbf{R}^d), \quad r > \frac{d}{2}, \quad r - \frac{d}{2} \notin \mathbf{N},$$

we derive

COROLLARY 1.6. *Let $s > d/2$ and $h, c > 0$. Then for any bounded set \mathcal{A} in \mathcal{H}_{ul}^s , there exists a time $T > 0$ such that: for every $\lambda \in \mathbf{R}$ one can find a constant $C > 0$ such that*

$$(1.10) \quad \|\langle \cdot \rangle^\lambda U\|_{C([0, T], \mathcal{H}_{ul}^{s-1})} \leq C \|\langle \cdot \rangle^\lambda U|_{t=0}\|_{\mathcal{H}_{ul}^{s-1}}$$

for all $U \in \mathcal{P}_{s, T}(h, c)$ with $U|_{t=0} \in \mathcal{A}$ and provided that the right-hand side is finite. Moreover, if $s \geq r > 1 + \frac{d}{2}$ and $r - \frac{d}{2} \notin \mathbf{N}$ it follows that

$$(1.11) \quad \|\langle \cdot \rangle^\lambda U\|_{C([0, T], \mathcal{W}^{r-1-\frac{d}{2}})} \leq C \|\langle \cdot \rangle^\lambda U|_{t=0}\|_{\mathcal{H}_{ul}^{s-1}}.$$

REMARK 1.7. 1. Of course, if $\lambda \leq 0$ then the right-hand sides of (1.8), (1.9), (1.10), (1.11) are automatically finite.

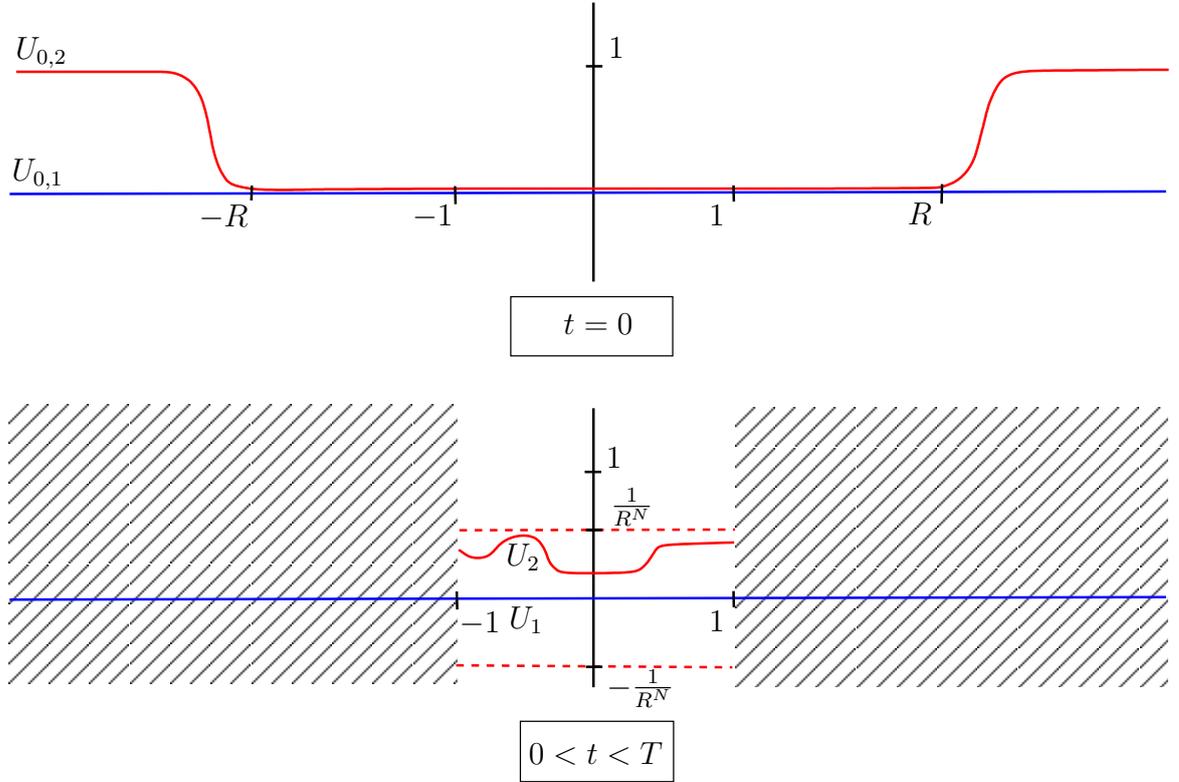
2. For the Camassa-Holm equation, an approximate model in shallow water waves regime, it was proved in [12] with $\lambda > 1/2$ that

$$\|(1+x)^\lambda u\|_{L_t^\infty W_x^{1,\infty}} \lesssim \|(1+x)^\lambda u|_{t=0}\|_{W_x^{1,\infty}}.$$

1.3. Interpretation of the results. 1. The Zakharov system (1.2) appears to be nonlocal, which comes from the fact that the Dirichlet-Neumann operator defined by (1.1) is nonlocal. This can be seen more concretely by considering the case of fluid domain with infinite depth (i.e. $\Gamma = \emptyset$) and free surface at rest (i.e. $\eta = 0$). Then, the Dirichlet-Neumann operators is $G(0) = |D_x|$. However, Corollary 1.5 shows that the system is in fact still weakly local as explained below.

Take $s > 1 + d/2$. Let's restrict ourselves to a bounded set \mathcal{A} of \mathcal{H}_{ul}^s and suppose that we are observing a bounded domain, which by translation can be assumed to be centered at the origin, say $\mathcal{O} = B(0, 1)$. Let $U_{0,1}, U_{0,2}$ be two data in \mathcal{A} such that they are identical in a ball $B(0, R)$ and have difference in \mathcal{H}_{ul}^{s-1} of size 1 outside this ball, where $R > 1$ is a given distance. Take a "window" ϕ around our observation region \mathcal{O} , that is, $\phi \in C_0^\infty(B(0, 3/2))$ and $\phi \equiv 1$ in \mathcal{O} . Then by the estimate (1.9) we have for some $T = T(\mathcal{A}) > 0$ and any $N > 0$

$$\begin{aligned} \|\phi(U_1 - U_2)\|_{C([0,T], \mathcal{H}_{ul}^{s-1})} &\leq C_N \|\langle \cdot \rangle^{-N} (U_1 - U_2)\|_{C([0,T], \mathcal{H}_{ul}^{s-1})} \\ &\leq C_N \|\langle \cdot \rangle^{-N} (U_1 - U_2)\|_{C([0,T], \mathcal{H}_{ul}^{s-1})} \\ &\leq C_{N,\mathcal{A}} \|\langle \cdot \rangle^{-N} (U_{0,1} - U_{0,2})\|_{\mathcal{H}_{ul}^{s-1}} \leq C_{N,\mathcal{A}} R^{-N}. \end{aligned}$$



Therefore, under the dynamic governed by system (1.2), a difference of size 1 outside the ball $B(0, R)$ of initial data leads to a difference of size R^{-N} of two solutions in the bounded domain $B(0, 1)$ (see the figures above). When $R \rightarrow +\infty$, the difference

of two solutions tends to 0 at a rate faster than any polynomial. In other words, to some extent, what happens far away has small effect on a given bounded region; moreover this effect becomes smaller and smaller when the distance increases to $+\infty$. This gives us a weakly local property of gravity water waves. This property is indeed dictated by the polynomial decay off the diagonal of the kernel of differential operators in suitable classes, as we shall explain in point 3. below.

2. As a consequence of Corollary 1.6, the estimate (1.11) with $\lambda > 0$ provides a spatial decay property for solutions. In classical Sobolev spaces, solutions always vanish at infinity. On the other hand, Theorem 1.3 gives the existence of solutions in Kato's spaces which can be neither decaying nor periodic. The estimate (1.11) however gives a conclusion for the intermediate case: as long as the datum decays at some algebraic rate (which, physically, is more natural than exponential decay) in Kato's space, the solution decays also, and moreover, at the same rate.

3. Let us explain why the polynomial growth weight $\langle x \rangle^\lambda$ is a reasonable choice in our result. For this purpose, a good way is to look at the linearization of system (1.2) around the rest state $(\eta, \psi) = (0, 0)$ (take $g = 1$):

$$(1.12) \quad \begin{cases} \partial_t \eta - |D_x| \psi = 0, \\ \partial_t \psi + \eta = 0 \end{cases}$$

or equivalently, with $u := \eta + i|D_x|^{\frac{1}{2}}\psi$,

$$\partial_t u + i|D_x|^{\frac{1}{2}}u = 0.$$

Given a datum u_0 at time $t = 0$, this linearized equation has the explicit solution

$$u(t, x) = e^{-it|D_x|^{\frac{1}{2}}}u_0(x) = p(t, D_x)u_0,$$

where the symbol p reads $p(t, \xi) = -it|\xi|^{\frac{1}{2}}$. Then for $\lambda \in \mathbf{R}$ and $0 < T < \infty$, we seek for the following estimate

$$(1.13) \quad \|\langle \cdot \rangle^\lambda u\|_{C([0, T]; H_{ul}^s)} \leq C \|\langle \cdot \rangle^\lambda u_0\|_{H_{ul}^s}.$$

Modifying p at 0 using a suitable cut-off function, we have that p satisfies

$$(1.14) \quad |\partial_\xi^\alpha p(t, \xi)| \leq C_\alpha (1 + \xi)^{-\frac{1}{2}|\alpha|}, \quad \forall \alpha \in \mathbf{N}^d, \quad (t, \xi) \in [0, T] \times \mathbf{R}^d,$$

which is usually denoted by $p \in S_{\frac{1}{2}, 0}^0$. An adaptation of the proof of Proposition 5.3 then implies the estimate (1.13). For simplicity let us consider $s = 0$, we need to show for any fixed $k \in \mathbf{Z}^d$

$$(1.15) \quad A_k := \sum_q \langle \cdot \rangle^\lambda \chi_k P \chi_q \langle \cdot \rangle^{-\lambda} : L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)$$

with norm bounded uniformly in k . Due to the presence of χ_k it suffices to prove $A : L^2(\mathbf{R}^d) \rightarrow L^\infty(\mathbf{R}^d)$. To this end, we call $K(x, y)$ the kernel of the pseudo-differential operator P then the kernel of A_k is

$$\sum_q \langle x \rangle^\lambda \chi_k(x) K(x, y) \chi_q(y) \langle y \rangle^{-\lambda} \sim \sum_q \langle k \rangle^\lambda K(k, q) \langle q \rangle^{-\lambda},$$

where the absolute value of each term in the above series is bounded by $\langle k - q \rangle^{|\lambda|} |K(k, q)|$. On the other hand, from the decay property (1.14) of $\partial_\xi^\alpha p$, the kernel K satisfies

$$|K(k, q)| \leq C_N |k - q|^{-N}, \quad \forall k \neq q.$$

Therefore, for the sequence (1.15) to be convergent, it is reasonable to choose the polynomial weights as above. This argument suggests that the weighted estimate

(1.13) is rather unlikely to hold for the water wave model with surface tension, in which case the propagator is given by

$$u(t, x) = e^{-it|D_x|^{\frac{3}{2}}} u_0(x).$$

This is a pseudo-differential operator with symbol $p(t, \xi) = e^{-it|\xi|^{\frac{3}{2}}}$, whose derivatives do not decay but growth in $\xi \rightarrow +\infty$. This reflects the infinite speed of propagation of the gravity-capillary waves.

1.4. Plan of the proof. To prove Theorem 1.4 we follow essentially the scheme in [1]. The first task is to adapt the paradifferential machinery to Kato's spaces with weights. This is done in Appendix 5, which can be of independent interest for other studies in this framework. Having this in hand, compare to [1] (and also [2]) the main ingredient for the proof of Theorem 1.4 reduces to the study of bound estimates, par-linearization and contraction estimate for the Dirichlet-Neumann operator. These are done in section 2 and 3 below, respectively.

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2. A weighted description for the Dirichlet-Neumann operator

2.1. Definition of the Dirichlet-Neumann operator. In this sections, we drop the time dependence of the domain and work on the domain of the form

$$(2.1) \quad \Omega = \{(x, y) \in \mathbf{R}^{d+1} : \eta_*(x) < y < \eta(x)\}$$

where η_* is a fixed bounded continuous function on \mathbf{R}^d and $\eta \in W^{1,\infty}(\mathbf{R}^d)$. We assume that Ω contains a fixed strip

$$(2.2) \quad \Omega_h := \{(x, y) \in \mathbf{R}^{d+1} : \eta(x) - h \leq y < \eta(x)\}.$$

2.1.1. *Straightening the boundary.* We recall here the change of variables introduced in [2] (see section 3.1.1) to flatten the domain with free boundary (which is in turn inspired by Lannes [8]). Consider the map $(x, z) \mapsto (x, \rho(x, z))$ from $\tilde{\Omega} := \mathbf{R}^d \times (-1, 0)$ to Ω_h determined by

$$(2.3) \quad \rho(x, z) = (1+z)e^{\delta z \langle D_x \rangle} \eta(x) - z \left[e^{-(1+z)\delta \langle D_x \rangle} \eta(x) - h \right] \quad \text{if } (x, z) \in \tilde{\Omega}.$$

For δ small enough this map is a Lipschitz-diffeomorphism from $\tilde{\Omega}$ to Ω_h .

NOTATION 2.1. For any function f defined on Ω , we set

$$(2.4) \quad \tilde{f}(x, z) = f(x, \rho(x, z))$$

then

$$(2.5) \quad \begin{cases} \frac{\partial f}{\partial y}(x, \rho(x, z)) = \frac{1}{\partial_z \rho} \partial_z \tilde{f}(x, z) := \Lambda_1 \tilde{f}(x, z) \\ \nabla_x f(x, \rho(x, z)) = (\nabla_x \tilde{f} - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z \tilde{f})(x, z) := \Lambda_2 \tilde{f}(x, z). \end{cases}$$

2.1.2. *Definition of the Dirichlet-Neumann operator $G(\eta)$.* Let $\psi \in H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)$, we recall how $G(\eta)\psi$ is defined (section 3.1, [1]).

For every $q \in \mathbf{Z}^d$, set $\psi_q = \chi_q \psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$ then one can find $\underline{\psi}_q \in H^1(\Omega)$ such that $\underline{\psi}_q|_{y=\eta(x)} = \psi_q(x)$ and

- (i) $\text{supp } \underline{\psi}_q \subset \{(x, y) : |x - q| \leq 2, \eta(x) - h \leq y \leq \eta(x)\}$
- (ii) $\|\underline{\psi}_q\|_{H^1(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$.

Let $u_q \in H^{1,0}(\Omega) := \{v \in H^1(\Omega), v|_{\Sigma} = 0\}$ be the unique variational solution, to equation $\Delta_{x,y} u_q = -\Delta_{x,y} \underline{\psi}_q$, which is characterized by

$$(2.6) \quad \iint_{\Omega} \nabla_{x,y} u_q(x, y) \cdot \nabla_{x,y} \theta(x, y) dx dy = - \iint_{\Omega} \nabla_{x,y} \underline{\psi}_q(x, y) \cdot \nabla_{x,y} \theta(x, y) dx dy$$

for all $\theta \in H^{1,0}(\Omega)$. The series $u := \sum_{q \in \mathbf{Z}^d} u_q$ is then convergent in

$$H_{ul}^{1,0}(\Omega) := \left\{ v : \sup_{q \in \mathbf{Z}^d} \|\chi_q v\|_{H^1(\Omega)} < +\infty \text{ and } v|_{\Sigma} = 0 \right\}.$$

Finally, $u + \underline{\psi} := \sum_{q \in \mathbf{Z}^d} u_q + \sum_{q \in \mathbf{Z}^d} \underline{\psi}_q$ solves uniquely the elliptic problem

$$(2.7) \quad \Delta_{x,y} \Phi = 0 \text{ in } \Omega, \quad \Phi|_{\Sigma} = \psi, \quad \frac{\partial \Phi}{\partial \nu}|_{\Gamma} = 0,$$

(in the variational sense) and the Dirichlet-Neumann operator is defined by

$$(2.8) \quad \begin{aligned} G(\eta)\psi(x) &= (1 + |\nabla_x \eta|^2)^{\frac{1}{2}} \frac{\partial \Phi}{\partial n}|_{\Sigma} = \left(\frac{\partial \Phi}{\partial y} - \nabla_x \eta \cdot \nabla_x \Phi \right)|_{\Sigma} \\ &= (\Lambda_1 \tilde{\Phi} - \nabla_x \eta \cdot \Lambda_2 \tilde{\Phi})|_{z=0} = (\Lambda_1 \tilde{\Phi} - \nabla_x \rho \cdot \Lambda_2 \tilde{\Phi})|_{z=0}. \end{aligned}$$

2.2. Elliptic regularity with weights. We observe that if u is a solution of the elliptic equation $\Delta u = 0$ on Ω and \tilde{u} is its image via the diffeomorphism (2.3) then

$$(\Lambda_1^2 + \Lambda_2^2) \tilde{u} = 0,$$

which is equivalent to (see equation 3.16, [2])

$$(2.9) \quad (\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z) \tilde{u} = 0,$$

where

$$(2.10) \quad \alpha := \frac{(\partial_z \rho)^2}{1 + |\nabla \rho|^2}, \quad \beta := -2 \frac{\partial_z \rho \nabla_x \rho}{1 + |\nabla_x \rho|^2}, \quad \gamma := \frac{1}{\partial_z \rho} (\partial_z^2 \rho + \alpha \Delta_x \rho + \beta \cdot \nabla_x \partial_z \rho).$$

These coefficients are estimated by

LEMMA 2.2 ([1, Lemma 4.17]). *Let $J = (-1, 0)$. There exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing such that (see Definition (5.1) for the definition of X_{ul}^{μ})*

$$\|\alpha\|_{X_{ul}^{s-\frac{1}{2}}(J)} + \|\beta\|_{X_{ul}^{s-\frac{1}{2}}(J)} + \|\gamma\|_{X_{ul}^{s-\frac{3}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}).$$

Let us denote by \mathcal{L} the linear differential operator

$$(2.11) \quad \mathcal{L} = \partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z$$

and consider the following inhomogeneous initial value problem

$$(2.12) \quad \begin{cases} (\mathcal{L} - \gamma \partial_z) \tilde{u} = F & \text{in } \mathbf{R}^d \times J, \\ \tilde{u}|_{z=0} = \psi. \end{cases}$$

DEFINITION 2.3. Let $w : \mathbf{R}^d \rightarrow \mathbf{R}$ be a function such that $w(x) \neq 0, \forall x \in \mathbf{R}$ and define

$$\begin{cases} r_1 := \frac{\nabla w^{-1}}{w^{-1}}, & r'_1 := \frac{\nabla w}{w}, \\ r_2 := \frac{\Delta w^{-1}}{w^{-1}}, & r'_2 := \frac{\Delta w}{w}. \end{cases}$$

We say that w belongs to the class \mathcal{W} of acceptable weights if r_i, r'_i are in $C_b^\infty(\mathbf{R}^d)$, $i = 1, 2$.

EXAMPLE 2.4. For any $t, s \in \mathbf{R}$, the functions $e^{t\langle x \rangle}, \langle x \rangle^s, e^{t\langle x \rangle} \langle x \rangle^s$ are in class \mathcal{W} .

Now we fix a weight $w \in \mathcal{W}$ and set $\tilde{v} = w\tilde{u}$. A simple computation shows that \tilde{v} satisfies

$$\mathcal{L}\tilde{v} + (\beta \cdot r_1 - \gamma)\partial_z \tilde{v} + \alpha r_2 \tilde{v} + 2\alpha r_1 \cdot \nabla \tilde{v} = wF.$$

Next, set $\tilde{v}_k = \chi_k \tilde{v}$, then

$$(2.13) \quad \mathcal{L}\tilde{v}_k = \chi_k wF + F_0 + F_1$$

where

$$\begin{cases} F_0 = \alpha \Delta \chi_k \tilde{v} + 2\alpha \nabla \chi_k \cdot \nabla_x \tilde{v} + \beta \cdot \nabla_x \chi_k \partial_z \tilde{v} - \chi_k \beta \cdot r_1 \partial_z \tilde{v} - \chi_k \alpha r_2 \tilde{v} - 2\chi_k \alpha r_1 \cdot \nabla \tilde{v}, \\ F_1 = \chi_k \gamma \partial_z \tilde{v}. \end{cases}$$

Notice that since r_1, r_2 are in $C_b^\infty(\mathbf{R}^d)$ the proof of Lemma 3.18 in [1] still works and we have

LEMMA 2.5. Let $J = (-1, 0)$. There exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing such that for $-\frac{1}{2} \leq \sigma \leq s-1$ with $\sigma + \frac{1}{2} \leq s-1$

$$\sum_{j=0}^1 \|F_j\|_{Y^{\sigma+\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|\nabla_{x,z} \tilde{v}\|_{X_{ul}^\sigma(J)}.$$

(see Definition 5.1 for the definition of Y^μ), where \mathcal{F} depends on w only through the semi-norms of r_i, r'_i , $i = 1, 2$ (in $C_b^\infty(\mathbf{R}^d)$).

Now, applying Proposition 3.19, 3.20 and 3.16 in [1] leads to an elliptic regularity theorem with weights:

THEOREM 2.6. Let $w \in \mathcal{W}, J = (-1, 0)$. Let \tilde{u} be a solution of the problem (2.12) and set $\tilde{v} = w\tilde{u}$. For $-\frac{1}{2} \leq \sigma \leq s-1$ let $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (2.2), $w\psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d), F \in Y_{ul}^\sigma(J)$ and

$$(2.14) \quad \|\nabla_{x,z} \tilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)} < +\infty.$$

Then for every $z \in]-1, 0[$ there exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing, depending only on (s, d) and the semi-norms of $r_i, r'_i, i = 1, 2$ (in $C_b^\infty(\mathbf{R}^d)$) such that

$$\|\nabla_{x,z} \tilde{v}\|_{X_{ul}^\sigma(z_0, 0)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \left\{ \|w\psi\|_{H_{ul}^{\sigma+1}} + \|wF\|_{Y_{ul}^\sigma(J)} + \|\nabla_{x,z} \tilde{v}\|_{X_{ul}^{-\frac{1}{2}}(J)} \right\}.$$

Consequently,

$$\|w\nabla_{x,z} \tilde{u}\|_{X_{ul}^\sigma(z_0, 0)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \left\{ \|w\psi\|_{H_{ul}^{\sigma+1}} + \|wF\|_{Y_{ul}^\sigma(J)} + \|w\nabla_{x,z} \tilde{u}\|_{X_{ul}^{-\frac{1}{2}}(J)} \right\}.$$

REMARK 2.7. We remark that in all the results stated below, the function \mathcal{F} depend on w only through the semi-norms of r_i and r'_i , $i = 1, 2$ in $C_b^\infty(\mathbf{R}^d)$.

To apply Theorem 2.6 we need the following estimate in the low norm $X_{ul}^{-\frac{1}{2}}$. For the proof of this, let us recall the following classical interpolation result

LEMMA 2.8 ([10, Theorem 30]). Let $J = (-1, 0)$ and $\sigma \in \mathbf{R}$. Let $f \in L_z^2(J, H^{\sigma+\frac{1}{2}}(\mathbf{R}^d))$ be such that $\partial_z f \in L_z^2(J, H^{\sigma-\frac{1}{2}}(\mathbf{R}^d))$. Then $f \in C_z^0([-1, 0], H^\sigma(\mathbf{R}^d))$ and there exists an absolute constant $C > 0$ such that

$$\|f\|_{C_z^0([-1,0], H^\sigma(\mathbf{R}^d))} \leq C \|f\|_{L_z^2(J, H^{\sigma+\frac{1}{2}}(\mathbf{R}^d))} + C \|\partial_z f\|_{L_z^2(J, H^{\sigma-\frac{1}{2}}(\mathbf{R}^d))}.$$

PROPOSITION 2.9. Let $J = (-1, 0)$, $\lambda \in \mathbf{R}$ and Φ be the unique solution to (2.7). Then, there exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(2.15) \quad \|\langle x \rangle^\lambda \nabla_{x,z} \tilde{\Phi}\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{\sigma+\frac{1}{2}}(\mathbf{R}^d)}) \|\langle x \rangle^\lambda \psi\|_{H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)}.$$

PROOF. We proceed in two steps.

Step 1. By Lemma 3.6 in [1] one can find a non decreasing function $\mathcal{F}_1 : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that for all $\mu > 0$ satisfying

$$(2.16) \quad \mu \mathcal{F}_1(\|\eta\|_{L^\infty(\mathbf{R}^d)}) \leq 1$$

there exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing such that for all $q \in \mathbf{Z}^d$ we have

$$(2.17) \quad \|e^{\mu\langle x-q \rangle} \nabla_{x,y} u_q\|_{L^2(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Using properties (i) and (ii) above of $\underline{\psi}_q$ (see section 3.1.2), we see that (2.17) also holds for u_q replaced by $\underline{\psi}_q$ for any $\mu > 0$ and thus (2.17) is true for u_q replaced by $\Phi_q = u_q + \underline{\psi}_q$, i.e.,

$$\|e^{\mu\langle x-q \rangle} \nabla_{x,z} \Phi_q\|_{L^2(\Omega)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$$

for any $\mu > 0$ satisfying (2.16).

Using the diffeomorphism (2.3) we deduce that

$$(2.18) \quad \|e^{\mu\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2(\mathbf{R}^d))} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

On the support of χ_k , we have $\langle x \rangle^\lambda e^{\mu\langle x-q \rangle} \sim \langle k \rangle^\lambda e^{\mu\langle k-q \rangle}$. Hence

$$\|\chi_k \langle x \rangle^\lambda \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2(\mathbf{R}^d))} \leq \langle k \rangle^\lambda e^{-\mu\langle k-q \rangle} \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)},$$

from which it follows that

$$\begin{aligned} \|\chi_k \langle x \rangle^\lambda \nabla_{x,z} \tilde{\Phi}\|_{L^2(J, L^2(\mathbf{R}^d))} &\leq \sum_q \|\chi_k \langle x \rangle^\lambda \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2(\mathbf{R}^d))} \\ &\leq \sum_q \langle k \rangle^\lambda e^{-\mu\langle k-q \rangle} \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} \\ &\leq \sum_q \langle k \rangle^\lambda e^{-\mu\langle k-q \rangle} \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \langle q \rangle^{-\lambda} \|\langle x \rangle^\lambda \psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} \\ &\leq \sum_q \langle k-q \rangle^{|\lambda|} e^{-\mu\langle k-q \rangle} \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\langle x \rangle^\lambda \psi_q\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}. \end{aligned}$$

Therefore, we obtain

$$(2.19) \quad \|\chi_k \langle x \rangle^\lambda \nabla_{x,z} \tilde{\Phi}\|_{L^2(J, L^2(\mathbf{R}^d))} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\langle x \rangle^\lambda \psi\|_{H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Step 2. To complete the proof of this lemma, it remains to show for any $k \in \mathbf{Z}^d$ that

$$(2.20) \quad \|\chi_k w \nabla_{x,z} \tilde{\Phi}\|_{L^\infty(J, H^{-\frac{1}{2}}(\mathbf{R}^d))} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|w\psi\|_{H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)},$$

where $w = \langle x \rangle^\lambda$.

By the interpolation Lemma 2.8

$$\|w\chi_k \nabla_x \tilde{\Phi}_q\|_{L^\infty(J, H^{-\frac{1}{2}})} \leq \|w\chi_k \nabla_x \tilde{\Phi}_q\|_{L^2(J, L^2)} + \|w\chi_k \partial_z \nabla_x \tilde{\Phi}_q\|_{L^2(J, H^{-1})}.$$

The second term on the right-hand side is estimated by (2.19), so we need to estimate

$$M := \|w\chi_k \nabla_x \partial_z \tilde{\Phi}_q\|_{L^2(J, H^{-1})}.$$

Notice that for any acceptable weight $\omega \in \mathcal{W}$, there holds with $\tilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ and $\tilde{\chi} = 1$ on $\text{supp } \chi$ that

$$\begin{aligned} \|\omega\chi_k \nabla_x f\|_{H^s(\mathbf{R}^d)} &\leq \|\nabla(\omega\chi_k f)\|_{H^s(\mathbf{R}^d)} + \|\omega \nabla_x \chi_k f\|_{H^s(\mathbf{R}^d)} + \|\nabla_x \omega \chi_k f\|_{H^s(\mathbf{R}^d)} \\ &\leq \|\chi_k \omega f\|_{H^{s+1}(\mathbf{R}^d)} + \|\omega \nabla_x \chi_k f\|_{H^s(\mathbf{R}^d)} + \|r'_1 \tilde{\chi} k\|_{H^s} \|\chi_k \omega f\|_{H^s(\mathbf{R}^d)} \end{aligned}$$

where $r'_1 = \frac{\nabla \omega}{\omega}$ as in Definition 2.3. This implies

$$\|\omega\chi_k \nabla_x f\|_{L^2(J, H^s(\mathbf{R}^d))} \leq C \|\omega f\|_{L^2(J, H^{s+1}(\mathbf{R}^d))_{ul}}.$$

Applying this estimate and (2.19) yields

$$M \leq C \|\langle x \rangle^\lambda \partial_z \tilde{\Phi}_q\|_{L^2(J, L^2)_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|w\psi_q\|_{H^{\frac{1}{2}}}.$$

Finally, to obtain (2.20) we shall prove

$$(2.21) \quad \|w\partial_z \tilde{\Phi}_q\|_{L^\infty(J, H^{-\frac{1}{2}})_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|w\psi_q\|_{H^{\frac{1}{2}}}.$$

Again, by interpolation,

$$\|w\chi_k \partial_z \tilde{\Phi}_q\|_{L^\infty(J, H^{-\frac{1}{2}})} \leq \|w\chi_k \partial_z \tilde{\Phi}_q\|_{L^2(J, L^2)} + \|w\chi_k \partial_z^2 \tilde{\Phi}_q\|_{L^2(J, H^{-1})}.$$

It remains to estimate $A := \|w\chi_k \partial_z^2 \tilde{\Phi}_q\|_{L^2(J, H^{-1})}$. Taking into account the fact that $\tilde{\Phi}_q$ satisfies equation (2.9), we have

$$A \leq A_1 + A_2 + A_3,$$

where by the product rule, Lemma 2.2 and (2.19),

$$\begin{aligned} A_1 &= \|\chi_k w \alpha \Delta \tilde{\Phi}_q\|_{L^2(J, H^{-1})} \leq \|\alpha\|_{L^\infty(J, H^{s-\frac{1}{2}})_{ul}} \|w \Delta \tilde{\Phi}_q\|_{L^2(J, H^{-1})_{ul}} \leq \text{RHS}, \\ A_2 &= \|\chi_k w \beta \partial_z \nabla_x \tilde{\Phi}_q\|_{L^2(J, H^{-1})} \leq \|\beta\|_{L^\infty(J, H^{s-\frac{1}{2}})_{ul}} \|w \partial_z \nabla_x \tilde{\Phi}_q\|_{L^2(J, H^{-1})_{ul}} \leq \text{RHS}, \\ A_3 &= \|\chi_k w \gamma \partial_z \tilde{\Phi}_q\|_{L^2(J, H^{-1})} \leq \|\gamma\|_{L^\infty(J, H^{s-\frac{3}{2}})_{ul}} \|w \partial_z \tilde{\Phi}_q\|_{L^2(J, L^2)_{ul}} \leq \text{RHS}, \end{aligned}$$

where RHS denotes the right-hand side of (2.19). The proof of Proposition 2.9 is complete. \square

Combining Proposition 2.9, Theorem 2.6 and the Poincaré inequality we obtain a weighted estimate for $\tilde{\Phi}$ and its gradient.

COROLLARY 2.10. *Let $\lambda \in \mathbf{R}$ and $\tilde{\Phi}$ be the solution to (2.7). For $-\frac{1}{2} \leq \sigma \leq s-1$ assume that $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (2.2) and $\langle x \rangle^\lambda \psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d)$. Then for any $z_0 \in (-1, 0)$ there exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing such that*

$$\|\langle x \rangle^\lambda \tilde{\Phi}\|_{X_{ul}^{\sigma+1}(z_0, 0)} + \|\langle x \rangle^\lambda \nabla_{x,z} \tilde{\Phi}\|_{X_{ul}^\sigma(z_0, 0)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|\langle x \rangle^\lambda \psi\|_{H_{ul}^{\sigma+1}}.$$

Using Corollary 2.10 one can follow the proof of Theorem 3.10, [1] to derive the following weighted estimate for the Dirichlet-Neumann operator, which can be of independent interest.

COROLLARY 2.11. *Let $\lambda \in \mathbf{R}$. For $-\frac{1}{2} \leq \sigma \leq s-1$ assume that $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (2.2) and $\langle x \rangle^\lambda \psi \in H_{ul}^{\sigma+1}(\mathbf{R}^d)$. Then there exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing such that*

$$\|\langle x \rangle^\lambda G(\eta)\psi\|_{H_{ul}^\sigma} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|\langle x \rangle^\lambda \psi\|_{H_{ul}^{\sigma+1}}.$$

2.3. Parilinearization of the Dirichlet–Neumann operator. We denote by κ the principal symbol of the Dirichlet-Neumann operator:

$$\kappa = \left((1 + |\nabla_x \eta|^2) |\xi|^2 - (\nabla_x \eta \cdot \xi)^2 \right)^{\frac{1}{2}}$$

and define the remainder

$$(2.22) \quad R(\eta)\psi := G(\eta)\psi - T_\kappa \psi.$$

Our aim in this section is to prove the following weighted version of Theorem 3.11 in [1].

THEOREM 2.12. *Let $\lambda \in \mathbf{R}$, then there exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing such that for $0 \leq t \leq s - \frac{1}{2}$, $\eta \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ satisfying (2.2) we have*

$$\|\langle x \rangle^\lambda R(\eta)\psi\|_{H_{ul}^t} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|\langle x \rangle^\lambda \psi\|_{H_{ul}^{t+\frac{1}{2}}}$$

provided that $w\psi \in H_{ul}^{t+\frac{1}{2}}(\mathbf{R}^d)$.

PROOF. Let us fix a real number $t \in [0, s - \frac{1}{2}]$. By definition of the Dirichlet-Neumann operator, one has

$$G(\eta)\psi = h_1 \partial_z \tilde{\Phi} - h_2 \cdot \nabla_x \tilde{\Phi} \Big|_{z=0}, \quad h_1 = \frac{1 + |\nabla_x \rho|^2}{\partial_z \rho}, \quad h_2 = \nabla_x \rho.$$

Let A and a be the two symbols of class $\Gamma_{\frac{1}{2}}^1(\mathbf{R}^d \times J)$ given in Lemma 3.20, [1]. We set

$$\tilde{g}_k = (\partial_z - T_A)(\chi_k w \tilde{\Phi}), \quad h_j|_{z=0} = h_j^0, \quad j = 1, 2, \quad A|_{z=0} = A_0, \quad a|_{z=0} = a_0.$$

Then we can write

$$(2.23) \quad \begin{aligned} \chi_k w G(\eta)\psi &= h_1^0 (\partial_z (\chi_k w \tilde{\Phi}))|_{z=0} - \chi_k w h_2^0 \nabla_x \psi \\ &= h_1^0 (\partial_z (\chi_k w \tilde{\Phi}))|_{z=0} - \chi_k h_2^0 \nabla_x (w\psi) + \chi_k h_2^0 \psi \nabla w \\ &= h_1^0 \tilde{g}_k|_{z=0} + h_1^0 [T_{A_0}, \chi_k](w\psi) + \chi_k (h_1^0 T_{A_0} - h_2^0 \cdot \nabla)(w\psi) + \chi_k h_2^0 \psi \nabla w. \end{aligned}$$

Let $\tilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ and $\tilde{\chi} = 1$ on $\text{supp } \chi$, we have

$$\chi_k w G(\eta)\psi = B_1 + B_2,$$

where

$$B_1 = \tilde{\chi}_k h_1^0 \tilde{g}_k|_{z=0} + \tilde{\chi}_k h_1^0 [T_{A_0}, \chi_k](w\psi) + \chi_k (h_1^0 T_{A_0} - h_2^0 \cdot \nabla)(w\psi), \quad B_2 = \chi_k h_2^0 \psi \nabla w.$$

The proof of Theorem 4.11, [1] shows that

$$B_1 = \chi_k T_\kappa(w\psi) + R$$

with the remainder R satisfies

$$\|R\|_{H^t} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|w\psi\|_{H_{ul}^{t+\frac{1}{2}}}.$$

On the other hand, since $h_2^0 \in H^{s-\frac{1}{2}}(\mathbf{R}^d)_{ul}$ with norm bounded by $\mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}})$ and $t \leq s - \frac{1}{2}$ the product rule yields

$$(2.24) \quad \|B_2\|_{H^t} = \|\chi_k h_2^0 \psi \nabla w\|_{H^t} = \|\chi_k h_2^0 \psi w r_1'\|_{H^t} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|w\psi\|_{H_{ul}^t}.$$

This completes the proof of Theorem 2.12. \square

2.4. A weighted estimate for Φ . We use the elliptic regularity theorem 2.6 to prove a weighted estimate for Φ -solution to (2.7), which will be used later in proving contraction estimate for Dirichlet-Neumann operator.

LEMMA 2.13. *Let $\lambda \in \mathbf{R}$. With $\mu > 0$ satisfying (2.16) and Φ_q, ψ_q as in section 2.1.2 there exists a non-decreasing function \mathcal{F} independent of q such that*

$$\sum_{k \in \mathbf{Z}^d} \|\langle x \rangle^\lambda e^{\frac{\mu}{2}\langle x-q \rangle} \chi_k \nabla_{x,z} \tilde{\Phi}_q\|_{L^\infty(J \times \mathbf{R}^d)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|\langle x \rangle^\lambda \psi_q\|_{H^s}.$$

PROOF. Set $w = \langle x \rangle^\lambda$, $\lambda \in \mathbf{R}$. We remark that $w\psi_q \in H^s(\mathbf{R}^d)$ for every $q \in \mathbf{Z}^d$ provided that $\psi \in H_{ul}^s(\mathbf{R}^d)$. It is clear that

$$\|w e^{\frac{\mu}{2}\langle x-q \rangle} \chi_k \nabla_{x,z} \tilde{\Phi}_q\|_{L^\infty(J \times \mathbf{R}^d)} \leq e^{-\frac{\mu}{4}\langle k-q \rangle} \|w e^{3\mu/4\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^\infty(J \times \mathbf{R}^d)}.$$

Consider the weight $\langle x \rangle^\lambda e^{3\mu/4\langle x-q \rangle} \in \mathcal{W}$ which has semi-norms independent of q . Applying Theorem 2.6 to $\tilde{\Phi}_q$ (with $\sigma = s - 1$) and taking into account Remark 2.7, we may estimate

$$\begin{aligned} & \sum_{k \in \mathbf{Z}^d} \|w e^{\frac{\mu}{2}\langle x-q \rangle} \chi_k \nabla_{x,z} \tilde{\Phi}_q\|_{L^\infty(J \times \mathbf{R}^d)} \\ & \leq \sum_{k \in \mathbf{Z}^d} e^{-\frac{\mu}{4}\langle k-q \rangle} \|w e^{3\mu/4\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^\infty(J \times \mathbf{R}^d)} \\ & \leq \|w e^{3\mu/4\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^\infty(J, H^{s-1}(\mathbf{R}^d))_{ul}} \\ & \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \left\{ \|w e^{3\mu/4\langle x-q \rangle} \psi_q\|_{H_{ul}^s} + \|w e^{3\mu/4\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{X_{ul}^{-\frac{1}{2}}(J)} \right\} \\ & \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \left\{ \|w\psi_q\|_{H^s} + \|w e^{3\mu/4\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{X_{ul}^{-\frac{1}{2}}(J)} \right\}. \end{aligned}$$

Remark that in the first inequality, we have used the trivial fact that $\sum_{k \in \mathbf{Z}^d} e^{-\frac{\mu}{4}\langle k-q \rangle}$ is finite and independent of q .

To complete the proof we need to prove that

$$(2.25) \quad \|w e^{3\mu/4\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|w\psi_q\|_{H^{\frac{1}{2}}}.$$

However, using interpolation inequality as in step 2 of the proof of Proposition 2.9, it suffices to show that

$$(2.26) \quad \|w e^{3\mu/4\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2)_{ul}} \leq \mathcal{F}(\|\eta\|_{H_{ul}^{s+\frac{1}{2}}}) \|w\psi_q\|_{H^{\frac{1}{2}}}.$$

Indeed, by virtue of (2.18) one can estimate

$$\begin{aligned} \|\chi_p \langle x \rangle^\lambda e^{\frac{3\mu}{4}\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2)} & \lesssim e^{-\frac{\mu}{4}\langle p-q \rangle} \langle p \rangle^\lambda \|\chi_p e^{\mu\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2)} \\ & \lesssim e^{-\frac{\mu}{4}\langle p-q \rangle} \langle p \rangle^\lambda \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\chi_q \psi\|_{H^{\frac{1}{2}}} \\ & \lesssim e^{-\frac{\mu}{4}\langle p-q \rangle} \langle p \rangle^\lambda \langle q \rangle^{-\lambda} \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\langle x \rangle^\lambda \chi_q \psi\|_{H^{\frac{1}{2}}} \\ & \lesssim \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|\langle x \rangle^\lambda \chi_q \psi\|_{H^{\frac{1}{2}}}, \end{aligned}$$

which is the desired bound. \square

3. Weighted contraction for the Dirichlet-Neumann operator

The main ingredient in proving contraction for the Dirichlet-Neumann operator is the contraction estimate for solutions to the elliptic problem (2.7). The key idea then is to compare the two variational solutions after changing the variable Φ_j to $\tilde{\Phi}_j$ as in (2.4). However, after straightening the fluid domains by the diffeomorphism (2.3), the new domains will depend on their upper surface. To overcome this, we use a slightly different diffeomorphism as follows.

Given $\eta_* \in C_b^0(\mathbf{R}^d)$ and $h > 0$, there exists $\tilde{\eta} \in C_b^\infty(\mathbf{R}^d)$ such that

$$(3.1) \quad \eta_*(x) < \tilde{\eta}(x) < \eta_*(x) + \frac{h}{3}, \quad \forall x \in \mathbf{R}^d.$$

Then, because $\eta_j > \eta_* + h$ we set

$$\begin{cases} \Omega_{1,j} &= \{(x, y) : x \in \mathbf{R}^d, \eta_j(x) - \frac{h}{3} < y < \eta_j(x)\}, \\ \Omega_{2,j} &= \{(x, y) : x \in \mathbf{R}^d, \tilde{\eta}(x) \leq y \leq \eta_j(x) - \frac{h}{3}\}, \\ \Omega_{3,j} &= \{(x, y) : x \in \mathbf{R}^d, \eta_*(x) < y < \tilde{\eta}(x)\}, \\ \Omega_j &= \Omega_{1,j} \cup \Omega_{2,j} \cup \Omega_{3,j}, \end{cases}$$

and

$$\begin{cases} \tilde{\Omega}_1 &= \mathbf{R}_x^d \times (-1, 0)_z, \\ \tilde{\Omega}_2 &= \mathbf{R}_x^d \times [-2, -1]_z, \\ \tilde{\Omega}_3 &= \{(x, z) \in \mathbf{R}^d \times (-\infty - 2) : z + 2 + \tilde{\eta}(x) > \eta_*(x)\}, \\ \tilde{\Omega} &= \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \tilde{\Omega}_3. \end{cases}$$

Remark that $\tilde{\Omega}$ depends on η_* , h but not on η_j . Thus, we can define

$$(3.2) \quad \rho_j(x, z) = \begin{cases} \rho_{1,j}(x, z) = (1+z)e^{\delta z \langle D_x \rangle} \eta_j(x) - z \left[e^{-(1+z)\delta \langle D_x \rangle} \eta_j(x) - \frac{h}{3} \right], & \text{in } \tilde{\Omega}_1, \\ \rho_{2,j}(x, z) = (2+z) \left[e^{\delta(z+1) \langle D_x \rangle} \eta_j(x) - \frac{h}{3} \right] - (1+z)\tilde{\eta}, & \text{in } \tilde{\Omega}_2, \\ \rho_{3,j}(x, z) = z + 2 + \tilde{\eta}(x), & \text{in } \tilde{\Omega}_3. \end{cases}$$

LEMMA 3.1. *The mappings $(x, z) \mapsto (x, \rho_j(x, z))$ are Lipschitz diffeomorphisms from $\tilde{\Omega}$ to Ω_j .*

PROOF. Observe first that $\rho_{k,j}$ are Lipschitz for $k = 1, 2, 3$; $j = 1, 2$. Clearly, $(x, z) \mapsto (x, \rho_{3,j}(x, z))$ are diffeomorphisms from $\tilde{\Omega}_3$ to $\Omega_{3,j}$. The same property holds for $\rho_{1,j}$ as in (2.3). We now prove it for $\rho_{2,j}$. Notice first that

$$\rho_{2,j}(-1, x) = \eta_j - \frac{h}{3}, \quad \rho_{2,j}(-2, x) = \tilde{\eta}.$$

Compute now

$$\begin{aligned} \partial_z \rho_{2,j} &= e^{\delta(z+1) \langle D_x \rangle} \eta_j(x) - \frac{h}{3} - (2+z)\delta e^{\delta(z+1) \langle D_x \rangle} \langle D_x \rangle \eta_j - \tilde{\eta} \\ &= e^{\delta(z+1) \langle D_x \rangle} \eta_j(x) - \eta_j(x) - (2+z)\delta e^{\delta(z+1) \langle D_x \rangle} \langle D_x \rangle \eta_j + \eta_j(x) - \tilde{\eta} - \frac{h}{3}. \end{aligned}$$

By writing $e^{\delta(z+1) \langle D_x \rangle} \eta_j - \eta_j = \delta(z+1) \int_0^1 e^{r\delta(z+1) \langle D_x \rangle} \langle D_x \rangle \eta_j dr$ we deduce that

$$\begin{aligned} &\left\| e^{\delta(z+1) \langle D_x \rangle} \eta_j - \eta_j \right\|_{L^\infty(\mathbf{R}^d)} + \left\| (2+z)\delta e^{\delta(z+1) \langle D_x \rangle} \langle D_x \rangle \eta_j \right\|_{L^\infty(\mathbf{R}^d)} \\ &\leq C\delta \|\eta_j\|_{W^{1,\infty}(\mathbf{R}^d)} \leq \frac{h}{6} \end{aligned}$$

for $\delta > 0$ small enough. On the other hand, thanks to (2.2) and (3.1) it holds that

$$\eta_j - \tilde{\eta} - \frac{h}{3} = (\eta_j - \eta_*) + (\eta_* - \tilde{\eta}) - \frac{h}{3} > h - \frac{h}{3} - \frac{h}{3} = \frac{h}{3}$$

and thus $\partial_z \rho_{2,j} \geq \frac{h}{3} - \frac{h}{6} = \frac{h}{6}$ in $\tilde{\Omega}_2$. Therefore, we can conclude that $(x, z) \mapsto (x, \rho_{2,j}(x, z))$ are diffeomorphisms from $\tilde{\Omega}_2$ to $\Omega_{2,j}$. \square

With the functions ρ_j above we denote for every $f : \Omega \rightarrow \mathbf{R}$

$$(3.3) \quad \tilde{f}_j(x, z) = f(x, \rho_j(x, z))$$

and as in (2.5) we define the differential operators $\Lambda^j = (\Lambda_1^j, \Lambda_2^j)$. Hereafter, $J = (-2, 0)$.

LEMMA 3.2. *We have $\Lambda^1 - \Lambda^2 = \wp \partial_z = (\wp_1, \wp_2) \partial_z$ with $\wp = 0$ for $z < -2$ and*

$$(3.4) \quad \|w \wp\|_{L^2(J, L^2(\mathbf{R}^d))_{ul}} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}}}) \|w(\eta_1 - \eta_2)\|_{H_{ul}^{\frac{1}{2}}}.$$

PROOF. By definition, one gets

$$\begin{aligned} \wp_1 &= \frac{\partial_z(\rho_2 - \rho_1)}{\partial_z \rho_1 \partial_z \rho_2}, \\ \wp_2 &= -\frac{\nabla_x(\rho_2 - \rho_1)}{\partial_z \rho_1} - \nabla_x \rho_2 \frac{\partial_z(\rho_2 - \rho_1)}{\partial_z \rho_1 \partial_z \rho_2} \end{aligned}$$

so in $\tilde{\Omega}_3$, $\wp = 0$. To obtain (3.4) one writes

$$\|w \wp\|_{L^2(J, L^2(\mathbf{R}^d))_{ul}} \leq \|w \wp\|_{L^2((-1, 0), L^2(\mathbf{R}^d))_{ul}} + \|w \wp\|_{L^2((-2, -1), L^2(\mathbf{R}^d))_{ul}}$$

to use definition (3.2), the fact that $\tilde{\eta} \in C_b^\infty(\mathbf{R}^d)$ and the $\frac{1}{2}$ -smoothing effect of the Poisson kernel, which is Lemma 5.5 applied with $r = 1$. \square

Let us recall here the Poincare inequality proved in [1] (cf. Remark 3.2)

LEMMA 3.3. *Let*

$$H^{1,0}(\Omega_j) = \{u \in L^2(\Omega_j) : \nabla_{x,y} u \in L^2(\Omega_j) \text{ and } u|_{y=\eta_j(x)} = 0\}.$$

Then there exists a constant $C > 0$ depending on $\|\eta\|_{L^\infty(\mathbf{R}^d)} + \|\eta_\|_{L^\infty(\mathbf{R}^d)}$ such that*

$$\iint_{\Omega_j} \alpha(x) |u(x, y)|^2 dx dy \leq C \iint_{\Omega_j} \alpha(x) |\nabla_{x,y} u(x, y)| dx dy$$

for all $u \in H^{1,0}(\Omega_j)$, $\alpha \in C_b^\infty(\mathbf{R}^d)$, $\alpha \geq 0$ and C is independent of α .

THEOREM 3.4. *Let $\psi_j \in H_{ul}^{\frac{1}{2}}(\mathbf{R}^d)$ and $\Phi_j, j = 1, 2$ be the unique solution in $H_{ul}^1(\Omega_j)$ of the problem*

$$\Delta_{x,y} \Phi_j = 0 \text{ in } \Omega, \quad \Phi_j|_{\Sigma} = \psi_j, \quad \frac{\partial \Phi_j}{\partial \nu}|_{\Gamma} = 0.$$

Set $\eta = \eta_1 - \eta_2$, $\psi = \psi_1 - \psi_2$, $\tilde{\Phi} = \tilde{\Phi}_1 - \tilde{\Phi}_2$ where $\tilde{\Phi}_j$ is the image of Φ_j as in (3.3). Then for every $\lambda \in \mathbf{R}$, $w = \langle \cdot \rangle^\lambda$ there exists a nonnegative function \mathcal{F} such that

$$(3.5) \quad \|w \nabla_{x,z} \tilde{\Phi}\|_{X_{ul}^{-\frac{1}{2}}(J)} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}}}) \left(\|w \eta\|_{H_{ul}^{s-\frac{1}{2}}} \|\psi_2\|_{H_{ul}^s} + \|w \psi\|_{H_{ul}^{\frac{1}{2}}} \right).$$

For the proof of this result, we shall apply Lemma 2.13 for $\tilde{\Phi}_q$. However, $\tilde{\Phi}_q$ here is the image of Φ_q via the diffeomorphism corresponding to one of ρ_j defined by (3.2) instead of (2.3). We want the same result as Lemma 2.13 in this situation. To have this, we notice that on $J = (-2, 0)$, ρ_j is comprised of two functions $\rho_{1,j}$ for $z \in (-1, 0)$ and $\rho_{2,j}$ for $z \in (-2, -1]$. The function $\rho_{1,j}$ possesses the same properties as ρ does and so does $\rho_{2,j}$ since $\tilde{\eta} \in C_b^\infty \subset H_{ul}^\infty$. Therefore, we obtain

LEMMA 3.5. *Let $\lambda \in \mathbf{R}$, $w = \langle \cdot \rangle^\lambda$ and $\Phi_{j,q}, \psi_{j,q}, j = 1, 2, q \in \mathbf{Z}$ as in section 2.1.2. There exists \mathcal{F}_1 non-decreasing such that: if $0 < \mu \mathcal{F}_1(\|\eta_j\|_{H_{ul}^s}) \leq 1$ then one can find a non-decreasing function \mathcal{F} independent of q such that*

$$\sum_{k \in \mathbf{Z}^d} \|\langle x \rangle^\lambda e^{\frac{\mu}{2}\langle x-q \rangle} \chi_k \nabla_{x,z} \tilde{\Phi}_{j,q} \|_{L^\infty(J \times \mathbf{R}^d)} \leq \mathcal{F}(\|\eta_j\|_{H_{ul}^{s+\frac{1}{2}}}) \|\langle x \rangle^\lambda \psi_{j,q}\|_{H^s}.$$

PROOF. (of Theorem 3.4) For simplicity in notations we shall denote $\mathcal{F} = \mathcal{F}(\|\eta_1\|_{H_{ul}^{s+\frac{1}{2}}}, \|\eta_2\|_{H_{ul}^{s+\frac{1}{2}}})$ which may change from line to line. We proceed in the following steps.

Step 1. Let $\tilde{\Phi}_{j,q} = u_{j,q} + \underline{\psi}_{j,q}$ where $u_{j,q}$ is the variational solution characterized by (2.6). After changing the variables, (2.6) becomes

$$\int_{\tilde{\Omega}} \Lambda^j \tilde{\Phi}_{j,q} \Lambda^j \theta J_j dX = 0, \quad \forall \theta \in H^{1,0}(\tilde{\Omega}), \quad j = 1, 2$$

with the Jacobian $J_j = |\partial_z \rho_j| = \partial_z \rho_j$ ($\partial_z \rho_j$ is a.e. positive in $\tilde{\Omega}$).

Set $\tilde{\Phi}_q = \tilde{\Phi}_{1,q} - \tilde{\Phi}_{2,q}$, $\underline{\psi}_q = \underline{\psi}_{1,q} - \underline{\psi}_{2,q}$ and choose

$$\theta = e^{2\delta g_\varepsilon} (\tilde{\Phi}_q - \underline{\psi}_q) \in H^{1,0}(\tilde{\Omega})$$

where $g_\varepsilon = \frac{\langle x-q \rangle}{1+\varepsilon\langle x-q \rangle}$. It follows that

$$\left| \int_{\tilde{\Omega}} \Lambda^1 \tilde{\Phi}_q \Lambda^1 \theta J_1 dX \right| \leq \sum_{j=1}^3 A_j,$$

$$\begin{cases} A_1 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) \tilde{\Phi}_{2,q} \Lambda^1 \theta J_1| dX, \\ A_2 = \int_{\tilde{\Omega}} |\Lambda^2 \tilde{\Phi}_{2,q} (\Lambda^1 - \Lambda^2) \theta J_1| dX, \\ A_3 = \int_{\tilde{\Omega}} |\Lambda^2 \tilde{\Phi}_{2,q} \Lambda^2 \theta (J_1 - J_2)| dX. \end{cases}$$

By Lemma (3.2) we know that $\Lambda^1 - \Lambda^2 = 0$ in $\tilde{\Omega}_3$. Likewise, $J_1 - J_2 = \partial_z \rho_1 - \partial_z \rho_2 = 0$ in $\tilde{\Omega}_3$. Consequently, with $\tilde{\Omega}_0 = \mathbf{R}^d \times J$ we have $A_j, j = 1, 2, 3$ are equal to the corresponding integrals over $\tilde{\Omega}_0$.

Step 2. (Estimate for A_1) First of all, we remark that

$$(3.6) \quad \Lambda^j (e^{2\delta g_\varepsilon} U) = e^{2\delta g_\varepsilon} \Lambda^j U + (0, U) 2\delta e^{2\delta g_\varepsilon} \nabla g_\varepsilon.$$

Using Lemma 3.2 and formula (3.6) with $j = 1, U = \tilde{\Phi}_q - \underline{\psi}_q$ one can write

$$\begin{aligned} A_1 &= \int_{\tilde{\Omega}_0} e^{2\delta g_\varepsilon} |\partial_z \tilde{\Phi}_{2,q} \Lambda^1 (\tilde{\Phi}_q - \underline{\psi}_q) J_1| dX + 2\delta \int_{\tilde{\Omega}_0} e^{2\delta g_\varepsilon} |\nabla g_\varepsilon \partial_z \tilde{\Phi}_{2,q} (\tilde{\Phi}_q - \underline{\psi}_q) J_1| dX \\ &:= A_{1,1} + A_{1,2}. \end{aligned}$$

Since $\|J_j\|_{L_{x,z}^\infty} \leq \mathcal{F}$, we may estimate

$$\begin{aligned} A_{1,1} &\leq \mathcal{F} \int_{\tilde{\Omega}_0} e^{2\delta g_\varepsilon} |\partial_z \tilde{\Phi}_{2,q} \Lambda^1 (\tilde{\Phi}_q - \underline{\psi}_q)| dX \\ &\leq \mathcal{F} \|\partial_z \tilde{\Phi}_{2,q}\|_{L^2(\tilde{\Omega}_0)} \|e^{\delta g_\varepsilon} \Lambda^1 (\tilde{\Phi}_q - \underline{\psi}_q)\|_{L^2(\tilde{\Omega}_0)}. \end{aligned}$$

On the other hand, there holds

$$(3.7) \quad \begin{aligned} \|f_1 f_2\|_{L^2(J, L^2(\mathbf{R}^d))} &\leq \sum_k \|\tilde{\chi}_k f_1 \chi_k f_2\|_{L^2(J, L^2(\mathbf{R}^d))} \\ &\leq \|f_1\|_{L^2(J, L^2(\mathbf{R}^d)_{ul})} \sum_k \|\chi_k f_2\|_{L^\infty(J \times \mathbf{R}^d)}. \end{aligned}$$

Now we choose $\delta > 0$ such that

$$(3.8) \quad \delta \mathcal{F}_1(\|\eta_2\|_{H_{ul}^{s+\frac{1}{2}}}) \leq \frac{1}{2}$$

then the condition of Lemma 3.5 is fulfilled with $\mu = 2\delta$. It then follows from (3.7) and Lemma 3.2 that

$$(3.9) \quad \begin{aligned} \|\wp e^{\delta g_\varepsilon} \partial_z \tilde{\Phi}_{2,q}\|_{L^2(\tilde{\Omega}_0)} &\leq \|w \wp w^{-1} e^{\delta \langle x-q \rangle} \partial_z \tilde{\Phi}_{2,q}\|_{L^2(J, L^2(\mathbf{R}^d))} \\ &\leq \|w \wp\|_{L^2(J, L^2(\mathbf{R}^d)_{ul})} \sum_k \|\chi_k w^{-1} e^{\delta \langle x-q \rangle} \partial_z \tilde{\Phi}_{2,q}\|_{L^\infty(J \times \mathbf{R}^d)} \\ &\leq \mathcal{F} \|w \eta\|_{H_{ul}^{\frac{1}{2}}} \|w^{-1} \psi_{2,q}\|_{H^s}. \end{aligned}$$

Therefore,

$$A_{1,1} \leq \mathcal{F} \|e^{\delta g_\varepsilon} \Lambda^1(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega}_0)} \|w \eta\|_{H_{ul}^{\frac{1}{2}}} \|w^{-1} \psi_{2,q}\|_{H^s}.$$

For $A_{1,2}$ we have

$$A_{1,2} \leq 2\delta \mathcal{F} \|e^{\delta g_\varepsilon} \wp_2 \partial_z \tilde{\Phi}_{2,q}\|_{L^2(\tilde{\Omega}_0)} \|e^{\delta g_\varepsilon}(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega}_0)}.$$

The first L^2 -norm on the right-hand side is already estimated by (3.9). For the second term on the right-hand side, one applies the Poincare inequality in Lemma 3.3 and changes the variables to derive

$$(3.10) \quad \|e^{\delta g_\varepsilon}(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega}_0)} \leq \mathcal{F} \|e^{\delta g_\varepsilon} \partial_z(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega})} \leq \mathcal{F} \|e^{\delta g_\varepsilon} \Lambda^1(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega})}$$

from which we deduce that $A_{1,2}$ satisfies the same estimate as $A_{1,1}$ does and hence, so does A_1 , i.e.,

$$(3.11) \quad A_1 \leq \mathcal{F} \|e^{\delta g_\varepsilon} \Lambda^1(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega}_0)} \|w \eta\|_{H_{ul}^{\frac{1}{2}}} \|w^{-1} \psi_{2,q}\|_{H^s}.$$

Step 3. (Estimates for A_2 , A_3) By Lemma 3.4 we have

$$(\Lambda^1 - \Lambda^2)\theta = \wp e^{2\delta g_\varepsilon} \partial_z(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q).$$

It follows that

$$A_2 \leq \mathcal{F} \|\wp e^{\delta g_\varepsilon} \Lambda^2 \tilde{\Phi}_{2,q}\|_{L^2(\tilde{\Omega}_0)} \|e^{\delta g_\varepsilon} \partial_z(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega}_0)}.$$

Using the definition of Λ^2 and the same method as in (3.9) one obtains that the first term is also bounded by the right-hand side of (3.9). On the other hand, it is easy to see the second term is bounded by $\mathcal{F} \|e^{\delta g_\varepsilon} \Lambda^1(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)\|_{L^2(\tilde{\Omega}_0)}$. Therefore, A_2 also satisfies the bound (3.11).

For A_3 one uses the formula (3.6) to get $A_3 \leq A_{3,1} + A_{3,2}$ with

$$\begin{aligned} A_{3,1} &= \int_{\tilde{\Omega}_0} e^{2\delta g_\varepsilon} |\Lambda^2 \tilde{\Phi}_{2,q} \Lambda^2(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)(J_1 - J_2)| dX, \\ A_{3,2} &= \delta \int_{\tilde{\Omega}_0} e^{2\delta g_\varepsilon} |\nabla g_\varepsilon \Lambda_2^2 \tilde{\Phi}_{2,q}(\tilde{\Phi}_q - \underline{\tilde{\psi}}_q)(J_1 - J_2)| dX. \end{aligned}$$

First, $A_{3,2}$ is estimated by $\|(J_1 - J_2)e^{\delta g_\varepsilon} \Lambda^2 \tilde{\Phi}_{2,q}\|_{L^2(\tilde{\Omega}_0)} \|e^{\delta g_\varepsilon} (\tilde{\Phi}_q - \tilde{\psi}_q)\|_{L^2(\tilde{\Omega}_0)}$. The second term is estimated by (3.10) and the first term is estimated as in (3.9) with \wp replaced by $J_1 - J_2$ which satisfies $\|w(J_1 - J_2)\|_{L^2(J, L^2_{ul})} \leq \mathcal{F} \|w\eta\|_{H_{ul}^{\frac{1}{2}}}$. Similarly,

$$A_{3,1} \leq \|(J_1 - J_2)e^{\delta g_\varepsilon} \Lambda^2 \tilde{\Phi}_{2,q}\|_{L^2(\tilde{\Omega}_0)} \|e^{\delta g_\varepsilon} \Lambda^2 (\tilde{\Phi}_q - \tilde{\psi}_q)\|_{L^2(\tilde{\Omega}_0)}.$$

We only need to study the second term on the right-hand side. With $u := \tilde{\Phi}_q - \tilde{\psi}_q$ one has $\Lambda_1^2 u = \frac{\partial_z \rho_1}{\partial_z \rho_2} \Lambda_1^1 u$ which implies $\|e^{\delta g_\varepsilon} \Lambda_1^2 u\|_{L^2(\tilde{\Omega}_0)} \leq \mathcal{F} \|e^{\delta g_\varepsilon} \Lambda_1^1 u\|_{L^2(\tilde{\Omega}_0)}$. On the other hand,

$$\Lambda_2^2 u = \nabla_x u - \frac{\nabla_x \rho_2}{\partial_z \rho_2} \partial_z u = \Lambda_2^1 u + \left(\frac{\nabla_x \rho_1}{\partial_z \rho_1} - \frac{\nabla_x \rho_2}{\partial_z \rho_2} \right) \partial_z \rho_1 \left(\frac{1}{\partial_z \rho_1} \partial_z u \right).$$

Hence, $\|e^{\delta g_\varepsilon} \Lambda_2^2 u\|_{L^2(\tilde{\Omega}_0)} \leq \mathcal{F} \|e^{\delta g_\varepsilon} \Lambda^1 u\|_{L^2(\tilde{\Omega}_0)}$ and $\|e^{\delta g_\varepsilon} \Lambda^2 u\|_{L^2(\tilde{\Omega}_0)} \leq \mathcal{F} \|e^{\delta g_\varepsilon} \Lambda^1 u\|_{L^2(\tilde{\Omega}_0)}$. In conclusion, we have proved that: for any (small) $\delta > 0$ satisfying (3.8), there holds

$$(3.12) \quad \left| \int_{\tilde{\Omega}} \Lambda^1 \tilde{\Phi}_q \Lambda^1 \theta J_1 dX \right| \leq \mathcal{F} \|e^{\delta g_\varepsilon} \Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q)\|_{L^2(\tilde{\Omega}_0)} \|w\eta\|_{H_{ul}^{\frac{1}{2}}} \|w^{-1} \psi_{2,q}\|_{H^s}.$$

Step 4. Next, in view of (3.6) we write

$$(3.13) \quad \begin{aligned} & \int_{\tilde{\Omega}} \Lambda^1 \tilde{\Phi}_q \Lambda^1 \theta J_1 dX \\ &= \int_{\tilde{\Omega}} e^{2\delta g_\varepsilon} \Lambda^1 \tilde{\Phi}_q \Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q) J_1 dX + 2\delta \int_{\tilde{\Omega}} \Lambda_2^1 \tilde{\Phi}_q \cdot (\tilde{\Phi}_q - \tilde{\psi}_q) e^{2\delta g_\varepsilon} \nabla g_\varepsilon J_1 dX \\ &= \int_{\tilde{\Omega}} e^{2\delta g_\varepsilon} |\Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q)|^2 J_1 dX + \int_{\tilde{\Omega}} e^{2\delta g_\varepsilon} \Lambda^1 \tilde{\psi}_q \Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q) J_1 dX \\ &\quad + 2\delta \int_{\tilde{\Omega}} \Lambda_2^1 (\tilde{\Phi}_q - \tilde{\psi}_q) (\tilde{\Phi}_q - \tilde{\psi}_q) e^{2\delta g_\varepsilon} \nabla g_\varepsilon J_1 dX \\ &\quad + 2\delta \int_{\tilde{\Omega}} \Lambda_2^1 \tilde{\psi}_q (\tilde{\Phi}_q - \tilde{\psi}_q) e^{2\delta g_\varepsilon} \nabla g_\varepsilon J_1 dX := B_1 + B_2 + B_3 + B_4. \end{aligned}$$

From the estimate (3.10) one has

$$(3.14) \quad |B_3| \leq \delta \mathcal{F}_2(\|\eta_1\|_{H_{ul}^{s+\frac{1}{2}}}) \|e^{\delta g_\varepsilon} \Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q)\|_{L^2(\tilde{\Omega})}^2$$

where $\mathcal{F}_2 : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a non decreasing function. Likewise,

$$(3.15) \quad |B_4| \leq \delta \mathcal{F}_2(\|\eta_1\|_{H_{ul}^{s+\frac{1}{2}}}) \|e^{\delta g_\varepsilon} \Lambda^1 \tilde{\psi}_q\|_{L^2(\tilde{\Omega})} \|\Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q) e^{\delta g_\varepsilon}\|_{L^2(\tilde{\Omega})}.$$

Finally, it is clear that

$$(3.16) \quad |B_2| \leq \mathcal{F}_2(\|\eta_1\|_{H_{ul}^{s+\frac{1}{2}}}) \|e^{\delta g_\varepsilon} \Lambda^1 \tilde{\psi}_q\|_{L^2(\tilde{\Omega})} \|\Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q) e^{\delta g_\varepsilon}\|_{L^2(\tilde{\Omega})}.$$

Now, remark that there exists a constant c_0 depend only on h such that $|J_1| \geq c_0$. Choose $\delta > 0$ satisfying

$$(3.17) \quad \delta \left\{ \mathcal{F}_1(\|\eta_2\|_{H_{ul}^{s+\frac{1}{2}}}) + \mathcal{F}_2(\|\eta_1\|_{H_{ul}^{s+\frac{1}{2}}}) \right\} = \min\left(\frac{c_0}{2}, \frac{1}{2}\right).$$

A combination of (3.12)-(3.16) yields

$$\left\| e^{\delta g_\varepsilon} \Lambda^1 (\tilde{\Phi}_q - \tilde{\psi}_q) \right\|_{L^2(\tilde{\Omega})} \leq \mathcal{F} \left\{ \|w\eta\|_{H_{ul}^{\frac{1}{2}}} \|w^{-1} \psi_{2,q}\|_{H^s} + \|e^{\delta g_\varepsilon} \Lambda^1 \tilde{\psi}_q\|_{L^2(\tilde{\Omega})} \right\}.$$

Step 5. Now, letting $\varepsilon \rightarrow 0$ and taking into account properties (i), (ii) of $\underline{\psi}_q$ in section 2.1.2 lead to

$$(3.18) \quad \begin{aligned} \|e^{\delta\langle x-q \rangle} \Lambda^1 \tilde{\Phi}_q\|_{L^2(\tilde{\Omega})} &\leq \mathcal{F} \left\{ \|w\eta\|_{H_{ul}^1} \|w^{-1}\psi_{2,q}\|_{H^s} + \|e^{\delta\langle x-q \rangle} \Lambda^1 \tilde{\underline{\psi}}_q\|_{L^2(\tilde{\Omega})} \right\} \\ &\leq \mathcal{F} \left\{ \|w\eta\|_{H_{ul}^1} \|w^{-1}\psi_{2,q}\|_{H^s} + \|\psi_q\|_{H^{\frac{1}{2}}} \right\}. \end{aligned}$$

Hence

$$\|e^{\delta\langle x-q \rangle} \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(\tilde{\Omega})} \leq \mathcal{F} \left\{ \|w\eta\|_{H_{ul}^1} \|w^{-1}\psi_{2,q}\|_{H^s} + \|\psi_q\|_{H^{\frac{1}{2}}} \right\}.$$

Consequently,

$$(3.19) \quad \begin{aligned} \|\chi_p w \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2(\mathbf{R}^d))} &\leq e^{-\delta\langle p-q \rangle} \langle p \rangle^\lambda \mathcal{F} \left\{ \|w\eta\|_{H_{ul}^{\frac{1}{2}}} \|w^{-1}\psi_{2,q}\|_{H^s} + \|\psi_q\|_{H^{\frac{1}{2}}(\Omega)} \right\} \\ &\leq e^{-\delta\langle p-q \rangle} \langle p \rangle^\lambda \langle q \rangle^{-\lambda} \mathcal{F} \left\{ \|w\eta\|_{H_{ul}^{\frac{1}{2}}} \|\psi_{2,q}\|_{H^s} + \|w\psi_q\|_{H^{\frac{1}{2}}} \right\}. \end{aligned}$$

Finally, we get

$$(3.20) \quad \begin{aligned} \|\chi_p w \nabla_{x,z} \tilde{\Phi}\|_{L^2(J, L^2(\mathbf{R}^d))} &\leq \sum_q \|\chi_p w \nabla_{x,z} \tilde{\Phi}_q\|_{L^2(J, L^2(\mathbf{R}^d))} \\ &\leq \mathcal{F} \left\{ \|w\eta\|_{H_{ul}^{\frac{1}{2}}} \|\psi_2\|_{H_{ul}^s} + \|w\psi\|_{H_{ul}^{\frac{1}{2}}} \right\}. \end{aligned}$$

Step 6. It remains to prove that $\|\chi_p w \nabla_{x,z} \tilde{\Phi}\|_{L^\infty(J, H^{-\frac{1}{2}}(\mathbf{R}^d))}$ is bounded by the right hand side of (3.5).

The estimate of $\|\chi_p w \nabla_x \tilde{\Phi}\|_{L^\infty(J, H^{-\frac{1}{2}}(\mathbf{R}^d))}$ follows from (3.20) and the interpolation Lemma 2.8. By the same lemma, for $\|\chi_p w \partial_z \tilde{\Phi}\|_{L^\infty(J, H^{-\frac{1}{2}}(\mathbf{R}^d))}$ it remains to estimate

$$\|\chi_p w \partial_z^2 \tilde{\Phi}\|_{L^2(J, H^{-1}(\mathbf{R}^d))}.$$

For this purpose we use equation (3.24) below, satisfied by $\tilde{\Phi}$ to have

$$(3.21) \quad \begin{aligned} \|\chi_p w \partial_z^2 \tilde{\Phi}\|_{L^2(J, H^{-1}(\mathbf{R}^d))} &\leq \|\chi_p w \alpha_1 \Delta \tilde{\Phi}\|_{L^2(J, H^{-1}(\mathbf{R}^d))} + \|\chi_p w \beta_1 \cdot \nabla \partial_z \tilde{\Phi}\|_{L^2(J, H^{-1}(\mathbf{R}^d))} \\ &\quad + \|\chi_p w \gamma_1 \partial_z \tilde{\Phi}\|_{L^2(J, H^{-1}(\mathbf{R}^d))} + \|\chi_p w F\|_{L^2(J, H^{-1}(\mathbf{R}^d))}. \end{aligned}$$

Because $-1 < s - 2$, the estimate (3.26) applied with $f = \psi_2$ implies the desired estimate for $\|\chi_p w F\|_{L^2(J, H^{-1}(\mathbf{R}^d))}$. Concerning the other terms, the product rule (2.13), [2] gives

$$(3.22) \quad \begin{aligned} \|\chi_p w \alpha_1 \Delta \tilde{\Phi}\|_{L^2(J, H^{-1})} &\leq \|\chi_p w \alpha_1\|_{L^\infty(J, H^{s-\frac{1}{2}})} \|\Delta \tilde{\Phi}\|_{L^2(J, H^{-1})} \\ \|\chi_p w \beta_1 \cdot \nabla \partial_z \tilde{\Phi}\|_{L^2(J, H^{-1})} &\leq \|\chi_p w \beta_1\|_{L^\infty(J, H^{s-\frac{1}{2}})} \|\nabla \partial_z \tilde{\Phi}\|_{L^2(J, H^{-1})} \\ \|\chi_p w \gamma_1 \partial_z \tilde{\Phi}\|_{L^2(J, H^{-1})} &\leq \|\chi_p w \gamma_1\|_{L^\infty(J, H^{s-\frac{3}{2}})} \|\partial_z \tilde{\Phi}\|_{L^2(J, L^2)}. \end{aligned}$$

Then, thanks to (3.20) we are left with the estimates for the first term on the right-hand side of the above inequalities. Again, this is done along the same line as in the proof of Lemma 2.2 noticing that $\tilde{\eta} \in C_b^\infty \subset H_{ul}^\infty$. This completes the proof. \square

We are now in position to derive the weighted estimate for the Dirichlet-Neumann operator:

THEOREM 3.6. *Assume that $s > 1 + \frac{d}{2}$. Then for every $\lambda \in \mathbf{R}$, $w = \langle \cdot \rangle^\lambda$ there exists $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ non decreasing such that for all $\eta_1, \eta_2 \in H_{ul}^{s+\frac{1}{2}}(\mathbf{R}^d)$ and $f \in H_{ul}^s(\mathbf{R}^d)$ we have*

$$\|w[G(\eta_1) - G(\eta_2)]f\|_{H_{ul}^{s-\frac{3}{2}}} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}}}) \|w(\eta_1 - \eta_2)\|_{H_{ul}^{s-\frac{1}{2}}} \|f\|_{H_{ul}^s}.$$

PROOF. We have by definition

$$(3.23) \quad G(\eta_j)f = \left(\frac{1 + |\nabla_x \rho_j|^2}{\partial_z \rho_j} \partial_z \tilde{\Phi}_j - \nabla_x \rho_j \nabla_x \tilde{\Phi}_j \right) \Big|_{z=0}.$$

Set $\tilde{\Phi} = \tilde{\Phi}_1 - \tilde{\Phi}_2$ then

$$(3.24) \quad (\partial_z^2 + \alpha_1 \Delta_x + \beta_1 \cdot \nabla \partial_z - \gamma_1 \partial_z) \tilde{\Phi} = F$$

where

$$F = \{(\alpha_2 - \alpha_1) \Delta_x + (\beta_2 - \beta_1) \cdot \nabla \partial_z - (\gamma_2 - \gamma_1) \partial_z\} \tilde{\Phi}_2.$$

We fix $z_0 \in (-1, 0)$ and set $I_0 = (z_0, 0)$. We first prove that

$$(3.25) \quad \|w \nabla_{x,z} \tilde{\Phi}\|_{X_{ul}^{s-\frac{3}{2}}(I_0)} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}}}) \|w(\eta_1 - \eta_2)\|_{H_{ul}^{s-\frac{1}{2}}} \|f\|_{H_{ul}^s}.$$

Thanks to Proposition 5.5 and the product rule we see easily that

$$(3.26) \quad \|wF\|_{L^2(J, H^{s-2})_{ul}} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H_{ul}^{s+\frac{1}{2}} \times H_{ul}^{s+\frac{1}{2}}}) \|w(\eta_1 - \eta_2)\|_{H_{ul}^{s-\frac{1}{2}}} \|f\|_{H_{ul}^s}.$$

Since $\tilde{\Phi}|_{z=0} = 0$, with the aid of Theorem 2.6 (which is applicable since $\rho_{1,j}$ and ρ in (2.3) have exactly the same form), the proof of (3.25) reduces to estimate $\|w \nabla_{x,z} \tilde{\Phi}\|_{X_{ul}^{-\frac{1}{2}}(I_0)}$. This is a consequence of Theorem 3.4 applied with $\psi_1 = \psi_2 = f$ and the fact that $I_0 \subset J$.

Next, to obtain the bound for $\|\nabla_{x,z} \tilde{\Phi}\|_{H_{ul}^{s-\frac{3}{2}}}$ at $z = 0$ we shall use the argument in step 5. of the proof of Theorem 3.4 (by virtue of Lemma 2.8). Then, we only need to estimate $\|\partial_z^2 \tilde{\Phi}\|_{L^2(I_0, H_{ul}^{s-2})}$, which follows by using equation (3.24). Finally, using (3.23) and Proposition 5.3, we conclude the proof of Theorem 3.6. \square

REMARK 3.7. Theorem 3.4 is also a crucial ingredient in proving contraction of the remainder R appearing in the reformulation of water waves system-equation (4.4) in Proposition 4.2, [1]. Notice that our estimate (3.5) is sufficient for this purpose because

$$\|w(\psi_1 - \psi_2)\|_{H_{ul}^{\frac{1}{2}}} \leq \|w(\psi_1 - \psi_2)\|_{H_{ul}^{s-1}}$$

owing to the fact that $s > 1 + \frac{d}{2}$.

4. Proof of the main results

4.1. Proof of Theorem 1.4. The contraction estimate in Theorem 1.4 was proved in [2] (see Theorem 5.1) for classical Sobolev spaces and then in [1] for Kato's spaces. Both use the following scheme:

- 1) study the Dirichlet-Neumann operator: bound estimates and parilinearization
- 2) contraction estimate for the Dirichlet-Neumann operator
- 3) parilinearization of the difference equations (after reformulation)
- 4) estimates for the good unknown
- 5) back to the original unknowns.

Here, we shall follow the same scheme as above. The first two items are the real new points in our problem and have been studied in Section 2 and 3. For the last three items we need a para-differential machinery in Kato's paces with weights and this is established in Appendix 5. The key point in this machinery is that: whenever we estimate $S(u, v)$ in weighted norms, where S is an operator of two variables, we are always able to shift the weight to u or v . Having this in hand, items 3), 4), 5) follow line by line those in [1] and [2]: one only need to replace $\|\cdot\|_{H_{ul}^\sigma}$ or $\|\cdot\|_{H^\sigma}$ by $\|\langle x \rangle^\lambda \cdot\|_{H_{ul}^\sigma}$ in the relevant estimates. We conclude the proof.

4.2. Proof of Corollary 1.5. We need to show how (1.8) implies (1.9). To this end, it suffices to prove that there exist $0 < T_1 \leq T$ and $N > 0$ (both are independent of U_j) such that

$$(4.1) \quad \|U_j\|_{L^\infty([0, T_1], \mathcal{H}_{ul}^{s_j})} \leq N, \quad j = 1, 2.$$

For simplicity in notations, we shall drop the index j and define

$$M_\sigma(\tau) = \|U\|_{L^\infty([0, \tau], \mathcal{H}_{ul}^\sigma)}, \quad \forall \tau \in [0, T].$$

Let us recall the a priori estimate derived in [1]: for any $1 + d/2 < \sigma \leq s$ and $\mathcal{T} > 0$ one can find a non decreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(4.2) \quad M_s(\mathcal{T}) \leq \mathcal{F}(M_\sigma(0) + \mathcal{T}M_\sigma(\mathcal{T}))(M_s(0) + \mathcal{T}M_s(\mathcal{T})).$$

Fix $s_0 \in (1 + \frac{d}{2}, s)$. Since U is a solution to the gravity waters system in $C^0([0, T], \mathcal{H}_{ul}^{s_0})$, the estimate (4.2) gives for some non decreasing $\mathcal{F}_1 : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ (independent of U)

$$M_{s_0}(\tau) \leq \mathcal{F}_1(M_{s_0}(0) + \tau M_{s_0}(\tau)), \quad \forall \tau \in [0, T].$$

According to Theorem 1.3 the solution U is continuous in time with value in $\mathcal{H}_{ul}^{s_0}$ since $s_0 < s$. Consequently, $M_{s_0}(\cdot)$ is continuous in τ and the standard argument then gives the existence of $T_0 \in (0, T]$ and $N > 0$, both are independent of U , such that

$$(4.3) \quad M_{s_0}(\tau) \leq N, \quad \forall \tau \in [0, T_0].$$

Applying again the estimate (4.2) with $\sigma = s_0 < s$ we get for some non-decreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ (independent of U)

$$M_s(\tau) \leq \mathcal{F}(M_{s_0}(0) + \tau M_{s_0}(\tau))(M_s(0) + \tau M_s(\tau)), \quad \forall \tau \in [0, T].$$

By (4.3), this implies

$$M_s(\tau) \leq \mathcal{F}(N(1 + T_0))(M_s(0) + \tau M_s(\tau)), \quad \forall \tau \in [0, T_0].$$

Now, let $T_1 \in (0, T_0]$ satisfying

$$T_1 \mathcal{F}(N(1 + T_0)) \leq \frac{1}{2}$$

one deduces

$$M_s(T_1) \leq 2\mathcal{F}(N(1 + T_0))M_s(0),$$

which concludes the proof.

5. Appendix: Paradifferential calculus in Kato's spaces with weights

In this section, we adapt the paradifferential machinery for the presence of weights which can be of independent interest. The proofs of these results follow those in [1] but we need to take some care (so we only present the proof whenever it is necessary). We recall first various spaces which will be used in the sequel.

DEFINITION 5.1. Let $p \in [1, +\infty]$, $J = (z_0, 0)$, $z_0 < 0$ and $\sigma \in \mathbf{R}$.

1. The space $L^p(J, H^\sigma(\mathbf{R}^d))_{ul}$ is defined as the space of measurable functions u from $J_z \times \mathbf{R}_x^d$ to \mathbf{C} such that

$$\|u\|_{L^p(J, H^\sigma(\mathbf{R}^d))_{ul}} := \sup_{q \in \mathbf{Z}^d} \|\chi_q u\|_{L^p(J, H^\sigma(\mathbf{R}^d))} < +\infty.$$

2. We set

$$X_{ul}^\sigma(J) = L^\infty(J, H^\sigma(\mathbf{R}^d))_{ul} \cap L^2(J, H^{\sigma+\frac{1}{2}}(\mathbf{R}^d))_{ul}$$

$$Y_{ul}^\sigma(J) = L^1(J, H^\sigma(\mathbf{R}^d))_{ul} + L^2(J, H^{\sigma-\frac{1}{2}}(\mathbf{R}^d))_{ul}$$

endowed with their natural norms.

The same spaces without subscript "ul" are defined for classical Sobolev spaces.

Notice that $L^\infty(J, H^\sigma(\mathbf{R}^d))_{ul} = L^\infty(J, H_{ul}^\sigma(\mathbf{R}^d))$.

NOTATION 5.2. (i) To avoid repeating, $\tilde{\chi}$ denotes a function in $C_0^\infty(\mathbf{R}^d)$ such that $\tilde{\chi} = 1$ on the support of χ in definition 1.1.

(ii) For $t \in \mathbf{R}$, we denote $\lceil t \rceil$ the smallest integer strictly greater than or equal t .

5.1. Weighted continuity of pseudo-differential operators. In [1], the authors proved the continuity of pseudo-differential operators on the framework of L^2 based uniformly local Sobolev spaces. Here, we perform similar results with the presence of the weight $\langle x \rangle^\lambda$. We denote by $S_{1,0}^m$ the set of symbols $p \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ such that

$$|D_\xi^\alpha D_x^\beta p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|} \quad \forall \alpha, \beta \in \mathbf{N}^d, \forall (x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d.$$

PROPOSITION 5.3. Let P be a pseudo-differential operator whose symbol p belongs to $S_{1,0}^m$. Then for any $\lambda, s \in \mathbf{R}$, there exists $C > 0$ such that

$$\|\langle x \rangle^\lambda P u\|_{H_{ul}^s} \leq C \|\langle x \rangle^\lambda u\|_{H_{ul}^{s+m}},$$

provided that the right hand side is finite.

PROOF. We write

$$(5.1) \quad \langle x \rangle^\lambda \chi_k P u = \sum_{|k-q| \leq 2} \langle x \rangle^\lambda \chi_k P \chi_q u + \sum_{|k-q| > 3} \langle x \rangle^\lambda \chi_k P \chi_q u := A + \sum_{|k-q| > 3} B_{k,q}.$$

Since $\chi_q u = (\chi_q \langle x \rangle^\lambda u) (\tilde{\chi}_q \langle x \rangle^{-\lambda}) \in H^{s+m}(\mathbf{R}^d)$, we have from the classical theory that

$$\begin{aligned} \|\langle x \rangle^\lambda \chi_k P \chi_q u\|_{H^s} &\leq \langle k \rangle^\lambda \|P \chi_q u\|_{H^s} \leq \langle k \rangle^\lambda \|\chi_q u\|_{H^{s+m}} \leq \langle k \rangle^\lambda \langle q \rangle^{-\lambda} \|\langle x \rangle^\lambda \chi_q u\|_{H^{s+m}} \\ &\leq \langle k - q \rangle^{|\lambda|} \|\langle x \rangle^\lambda \chi_q u\|_{H^{s+m}} \leq \|\langle x \rangle^\lambda u\|_{H_{ul}^{s+m}}, \end{aligned}$$

provided $|k - q| \leq 2$. Thus,

$$A \leq C \|\langle x \rangle^\lambda u\|_{H_{ul}^{s+m}}.$$

To bound the second part, we fix $n_0 \in \mathbf{N}$, $n_0 \geq s$. We shall prove

$$(5.2) \quad \|D_x^\alpha B_{k,q}\|_{L^2(\mathbf{R}^d)} \leq \frac{C}{\langle k - q \rangle^{d+1}} \|u\|_{H_{ul}^{s+m}}, \quad |\alpha| \leq n_0$$

which implies the desired estimate for $\sum_{|k-q|>3} B_{k,q}$.

By the presence of χ_k , $\|D_x^\alpha B_{k,q}\|_{L^2(\mathbf{R}^d)} \leq C \|D_x^\alpha B_{k,q}\|_{L^\infty(\mathbf{R}^d)}$. We have

$$D_x^\alpha B_{k,q}(x) = \langle D_x^\alpha K(x, \cdot), \chi_q u \rangle$$

with

$$(5.3) \quad K(x, y) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{i(x-y)\xi} p(x, \xi) d\xi \chi_k(x) \langle x \rangle^\lambda \tilde{\chi}_q(y).$$

Fix $n_1 \in \mathbf{N}$, $n_1 \geq -(s+m)$ and $\beta \in \mathbf{N}^d$, $|\beta| \leq n_1$. Let $\gamma \in \mathbf{N}^d$ be such that $|\gamma| = N$ with

$$(5.4) \quad N \geq \max(m + n_0 + n_1 + d + 1, |\lambda| + d + 1).$$

Multiplying $D_x^\alpha D_y^\beta K(x, y)$ by $(x-y)^\gamma$ and integrating by parts with a remark that $|x-y| \geq \delta|k-q|$ (for some $\delta > 0$) on the support of $\chi_k(x) \langle x \rangle^\lambda \tilde{\chi}_q(y)$, we obtain

$$\left| D_x^\alpha D_y^\beta K(x, y) \right| \leq \frac{C_{\beta,d,\lambda}}{\langle k-q \rangle^N} \langle k \rangle^\lambda \sum_{|\beta_1| \leq |\beta|} \left| \partial^{\beta_1} \tilde{\chi}_q(y) \right|.$$

It follows that

$$(5.5) \quad \begin{aligned} |D_x^\alpha B_{k,q}(x)| &\leq \|D_x^\alpha K(x, \cdot)\|_{H^{-(s+m)}} \|\chi_q u\|_{H^{s+m}} \\ &\leq \frac{C}{\langle k-q \rangle^N} \langle k \rangle^\lambda \|\chi_q u\|_{H^{s+m}} \\ &\leq \frac{C}{\langle k-q \rangle^N} \langle k \rangle^\lambda \langle q \rangle^{-\lambda} \|\chi_q \langle x \rangle^\lambda u\|_{H^{s+m}} \\ &\leq \frac{C}{\langle k-q \rangle^N} \langle k-q \rangle^{|\lambda|} \|\langle x \rangle^\lambda u\|_{H_{ul}^{s+m}} \\ &\leq \frac{C}{\langle k-q \rangle^{d+1}} \|\langle x \rangle^\lambda u\|_{H_{ul}^{s+m}} \end{aligned}$$

which proves (5.2). \square

In a particular case the proof above gives the following more precise result.

PROPOSITION 5.4. *Let $m \in \mathbf{R}$, $h(\xi) = \tilde{h}(\frac{\xi}{|\xi|}) |\xi|^m \psi(\xi)$ where $\tilde{h} \in C^\infty(\mathbf{S}^{d-1})$ and $\psi \in C^\infty(\mathbf{R}^d)$ is such that $\psi(\xi) = 1$ if $|\xi| \geq 1$, $\psi(\xi) = 0$ if $|\xi| \leq \frac{1}{2}$. Then for every $\lambda, s \in \mathbf{R}$ and*

$$(5.6) \quad r > [m] + [s] + [m+s] + [|\lambda|] + \frac{3d}{2} + 1,$$

there exists a constant C such that

$$\|\langle x \rangle^\lambda h(D_x) u\|_{H_{ul}^s(\mathbf{R}^d)} \leq C \|\tilde{h}\|_{H^r(\mathbf{S}^{d-1})} \|\langle x \rangle^\lambda u\|_{H_{ul}^{s+m}(\mathbf{R}^d)}.$$

Remark that the condition on r above comes from the choice of N in (5.4), plus $d/2 + \varepsilon$ derivatives from Sobolev embedding. Next, tracking the proof of Lemma 7.10 in [1] and Proposition 5.3 above, we easily obtain the following proposition.

PROPOSITION 5.5. *Let $r > 0$, and $\lambda, m \in \mathbf{R}$. Let $p \in S_{1,0}^r(\mathbf{R}^d)$, $a \in S_{1,0}^m(\mathbf{R}^d)$ be two symbols with constant coefficients. We assume that there exists $c_0 > 0$ such that for all $\xi \in \mathbf{R}^d$ we have $p(\xi) \geq c_0 |\xi|^r$. Then for all $s \in \mathbf{R}$ and $I = [0, T]$, one can find a positive constant C such that*

$$(5.7) \quad \|\langle x \rangle^\lambda e^{-tp(D)} a(D) u\|_{L^\infty(I, H^s)_{ul}} + \|\langle x \rangle^\lambda e^{-tp(D)} a(D) u\|_{L^2(I, H^{s+\frac{\varepsilon}{2}})_{ul}} \leq C \|\langle x \rangle^\lambda u\|_{H_{ul}^{s+m}}.$$

5.2. Para-differential calculus with weights. Assuming the theory of para-differential calculus for classical Sobolev spaces (see [11]) and for uniformly local Sobolev spaces (see [1]), we present in this section such a theory with the presence of weights.

Given $m \in \mathbf{R}$, $\rho \geq 0$ we denote by $\Gamma_\rho^m(\mathbf{R}^d)$ the class of symbols of order m and by T_a the associated para-differential operator as in Definition 7.15, [1]. In particular, $\dot{\Gamma}_\rho^m(\mathbf{R}^d)$ denotes the subspace of $\Gamma_\rho^m(\mathbf{R}^d)$ which consists of symbols $a(x, \xi)$ homogeneous of degree m with respect to ξ .

To deal with the weight $\langle \cdot \rangle^\lambda$, for any symbol $a \in \Gamma_\rho^m$ and $s, \lambda \in \mathbf{R}$, let us define the semi-norm

$$(5.8) \quad M_\rho^m(a, s)_\lambda = \sup_{|\alpha| \leq I(d, \lambda, s, m)} \sup_{|\xi| \geq \frac{1}{2}} \|(1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{R}^d)},$$

where $I_\rho(d, m, s)_\lambda$ is the smallest even integer strictly greater than

$$(5.9) \quad [m] + [s] + [m + s] + [|\lambda|] + \frac{5d}{2} + 2.$$

If a is a symbol independent of ξ , the associated operator T_a is called a paraproduct and we have the formal decomposition of Bony

$$au = T_a u + T_u a + R(a, u).$$

5.2.1. *Symbolic calculus.* The following technical lemmas will be used in proving the results on symbolic calculus.

LEMMA 5.6. *Let $\lambda, \mu \in \mathbf{R}$ and $N \geq |\lambda| + d + 1$. Then there exists $C > 0$ such that*

$$(5.10) \quad \sup_{x \in \mathbf{R}^d} \|\langle x \rangle^\lambda \langle x - \cdot \rangle^{-N} u\|_{H^\mu(\mathbf{R}^d)} \leq C \|\langle \cdot \rangle^\lambda u\|_{H_{ul}^\mu(\mathbf{R}^d)}$$

provided that the right hand side is finite.

PROOF. We write

$$\langle x \rangle^\lambda \langle x - y \rangle^{-N} \chi_q(y) u(y) = \langle x \rangle^\lambda \langle y \rangle^{-\lambda} \frac{1}{\langle x - q \rangle^N} \frac{\langle x - q \rangle^N}{\langle x - y \rangle^N} \tilde{\chi}_q(y) \langle y \rangle^\lambda \chi_q(y) u(y).$$

Since the function $y \mapsto \frac{\langle x - q \rangle^N}{\langle x - y \rangle^N} \tilde{\chi}_q(y)$ belongs to $W^{\infty, \infty}(\mathbf{R}^d)$ with semi-norms uniformly bounded (independently of x and q), we deduce that

$$\begin{aligned} \|\langle x \rangle^\lambda \langle x - \cdot \rangle^{-N} u\|_{H^\mu(\mathbf{R}^d)} &\leq \sum_{q \in \mathbf{Z}^d} \|\langle x \rangle^\lambda \langle x - \cdot \rangle^{-N} \chi_q u\|_{H^\mu} \leq C_N \sum_{q \in \mathbf{Z}^d} \frac{\langle x \rangle^\lambda \langle q \rangle^{-\lambda}}{\langle x - q \rangle^N} \|\langle \cdot \rangle^\lambda u\|_{H_{ul}^\mu} \\ &\leq C_N \sum_{q \in \mathbf{Z}^d} \frac{\langle x - q \rangle^{|\lambda|}}{\langle x - q \rangle^N} \|\langle \cdot \rangle^\lambda u\|_{H_{ul}^\mu} \leq C'_N \|\langle \cdot \rangle^\lambda u\|_{H_{ul}^\mu}. \end{aligned}$$

□

Combining this lemma and the proof of Lemma 7.13, [1], we obtain

LEMMA 5.7. *Let $\lambda \in \mathbf{R}$ and set $w = \langle x \rangle^\lambda$. Let $\chi \in C_0^\infty(\mathbf{R}^d)$ and $\tilde{\chi} \in C_0^\infty(\mathbf{R}^d)$ be equal to one on the support of χ . Let $\psi, \theta \in \mathcal{S}(\mathbf{R}^d)$. For every $m, \sigma \in \mathbf{R}$ there exists a constant $C > 0$ such that*

$$(5.11) \quad \sum_{j \geq -1} \|w \chi_k \psi(2^{-j} D) ((1 - \tilde{\chi}_k) u) \theta(2^{-j} D) v\|_{H^m(\mathbf{R}^d)} \leq C \|wu\|_{H_{ul}^\sigma(\mathbf{R}^d)} \|v\|_{L^\infty(\mathbf{R}^d)}.$$

For every $m, \sigma, t \in \mathbf{R}$ one can find a constant $C > 0$ such that

$$(5.12) \quad \sum_{j \geq -1} \|w \chi_k \psi(2^{-j}D)((1 - \tilde{\chi}_k)u)\theta(2^{-j}D)v\|_{H^m(\mathbf{R}^d)} \leq C \|wu\|_{H_{ul}^\sigma(\mathbf{R}^d)} \|v\|_{H_{ul}^t(\mathbf{R}^d)}$$

and

$$(5.13) \quad \sum_{j \geq -1} \|w \chi_k \psi(2^{-j}D)((1 - \tilde{\chi}_k)u)\theta(2^{-j}D)\tilde{\chi}_k v\|_{H^m(\mathbf{R}^d)} \leq C \|wu\|_{H_{ul}^\sigma(\mathbf{R}^d)} \|\tilde{\chi}_k v\|_{H^t(\mathbf{R}^d)}.$$

REMARK 5.8. It follows easily from the proof of the above lemma that the same estimates as in (5.11), (5.12) and (5.13) hold if on the left-hand sides 2^{-j} is replaced by 2^{-j-j_0} where $j_0 \in \mathbf{Z}$ is fixed. We shall use this remark to deal with paraproduct estimates.

It turns out that the symbolic calculus with weights possesses the same features as in the usual setting.

THEOREM 5.9. Let $m, m' \in \mathbf{R}$, $\rho \geq 0$, $\lambda \in \mathbf{R}$ and $w = \langle \cdot \rangle^\lambda$.

(i) If $a \in \Gamma_0^m(\mathbf{R}^d)$, then for all $\mu \in \mathbf{R}$, there exist a constant $C > 0$ such that

$$\|w T_a u\|_{H_{ul}^\mu(\mathbf{R}^d)} \leq C M_0^m(a, \mu)_\lambda \|wu\|_{H_{ul}^{\mu+m}(\mathbf{R}^d)}.$$

(ii) If $a \in \Gamma_\rho^m(\mathbf{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ then, for all $\mu \in \mathbf{R}$, there exist a constant $C > 0$ such that

$$\begin{aligned} & \|w(T_a T_b - T_{a\#b})u\|_{H_{ul}^\mu(\mathbf{R}^d)} \\ & \leq C \left(M_\rho^m(a, \mu)_\lambda M_0^{m'}(b, \mu)_\lambda + M_0^m(a, \mu)_\lambda M_\rho^{m'}(b, \mu)_\lambda \right) \|wu\|_{H_{ul}^{\mu+m+m'-\rho}(\mathbf{R}^d)} \end{aligned}$$

where

$$a\#b := \sum_{|\alpha| < \rho} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi).$$

(iii) Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$ with $\rho \in [0, 1]$ and denote by $(T_a)^*$ the adjoint operator of T_a and by \bar{a} the complex conjugate of a . Then for all $\mu \in \mathbf{R}$ there exists a constant $C > 0$ such that

$$\|w((T_a)^* - T_{\bar{a}})u\|_{H_{ul}^\mu(\mathbf{R}^d)} \leq C M_\rho^m(a, \mu)_\lambda \|wu\|_{H_{ul}^{\mu+m-\rho}(\mathbf{R}^d)}.$$

PROOF. We give the proof for the first assertion only since these three points are proved along the same lines. For simplicity we shall consider symbols in $\dot{\Gamma}_\rho^m(\mathbf{R}^d)$.

Step 1. Consider first the case where a is a bounded function and write

$$\chi_k w T_a u = \chi_k w T_a(\tilde{\chi}_k u) + \chi_k w T_a((1 - \tilde{\chi}_k)u).$$

The classical theory gives

$$\begin{aligned} \|\chi_k w T_a(\tilde{\chi}_k u)\|_{H^\mu} & \leq C \langle k \rangle^\lambda \|a\|_{L^\infty} \|\tilde{\chi}_k u\|_{H^\mu} \leq C \langle k \rangle^\lambda \|a\|_{L^\infty} \langle k \rangle^{-\lambda} \|\tilde{\chi}_k w u\|_{H^\mu} \\ & \leq C \|a\|_{L^\infty} \|wu\|_{H_{ul}^\mu}. \end{aligned}$$

The estimate for the second term follows immediately from (5.11).

Step 2. Next we consider the case $a(x, \xi) = b(x)h(\xi)$ where $h(\xi) = |\xi|^m \tilde{h}(\frac{\xi}{|\xi|})$ with $\tilde{h} \in C^\infty(\mathbf{S}^{d-1})$. Then directly from the definition we have $T_a = T_b(\psi h)(D_x)$ and the desired estimate in (i) follows from Step 1 and Proposition 5.4.

Step 3. Finally, for the general case we introduce $(\tilde{h}_\nu)_{\nu \in \mathbf{N}^*}$ an orthonormal basis of $L^2(\mathbf{S}^{d-1})$ consisting of eigenfunctions of the (self-adjoint) Laplace Beltrami operator $\Delta_\omega = \Delta_{\mathbf{S}^{d-1}}$ on $L^2(\mathbf{S}^{d-1})$ and argue as in the proof of Theorem 7.16, [1]. It is here where we use the special choice of the number of derivatives in (5.9). \square

5.2.2. Paraproducts.

PROPOSITION 5.10. *Let $w = \langle \cdot \rangle^\lambda$, $\lambda \in \mathbf{R}$ and s_0, s_1, s_2 be such that $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$. Then there exists $C > 0$ such that*

$$\|wT_a u\|_{H_{ul}^{s_0}} \leq C \min \left\{ \|a\|_{H_{ul}^{s_1}} \|wu\|_{H_{ul}^{s_2}}, \|wa\|_{H_{ul}^{s_1}} \|u\|_{H_{ul}^{s_2}} \right\}.$$

PROOF. We write

$$(5.14) \quad \chi_k w T_a u = \chi_k w T_a (1 - \tilde{\chi}_k) u + \chi_k w T_{\tilde{\chi}_k a} \tilde{\chi}_k u + \chi_k w T_{(1 - \tilde{\chi}_k) a} \tilde{\chi}_k u.$$

By the classical result, we have

$$(5.15) \quad \|\chi_k w T_{\tilde{\chi}_k a} \tilde{\chi}_k u\|_{H^{s_0}} \lesssim \langle k \rangle^\lambda \|T_{\tilde{\chi}_k a} \tilde{\chi}_k u\|_{H^{s_0}} \lesssim \langle k \rangle^\lambda \|\tilde{\chi}_k a\|_{H^{s_1}} \|\tilde{\chi}_k u\|_{H^{s_2}} \lesssim \|a\|_{H_{ul}^{s_1}} \|wu\|_{H_{ul}^{s_2}}.$$

On the other hand, applying (5.12) gives

$$\|\chi_k w T_a (1 - \tilde{\chi}_k) u\|_{H^{s_0}} \lesssim \|a\|_{H_{ul}^{s_1}} \|wu\|_{H_{ul}^{s_2}}$$

and it follows from (5.13) (applied with $\lambda = 0$) that

$$\begin{aligned} \|\chi_k w T_{(1 - \tilde{\chi}_k) a} \tilde{\chi}_k u\|_{H^{s_0}} &\lesssim \langle k \rangle^\lambda \|\chi_k T_{(1 - \tilde{\chi}_k) a} \tilde{\chi}_k u\|_{H^{s_0}} \\ &\lesssim \langle k \rangle^\lambda \|a\|_{H_{ul}^{s_1}} \|\tilde{\chi}_k u\|_{H^{s_2}} \lesssim \|a\|_{H_{ul}^{s_1}} \|wu\|_{H_{ul}^{s_2}}. \end{aligned}$$

Consequently, we obtain

$$\|wT_a u\|_{H_{ul}^{s_0}} \leq C \|a\|_{H_{ul}^{s_1}} \|wu\|_{H_{ul}^{s_2}}.$$

Now if instead of (5.2.2), we decompose

$$\chi_k w T_a u = \chi_k w T_{(1 - \tilde{\chi}_k) a} u + \chi_k w T_{\tilde{\chi}_k a} \tilde{\chi}_k u + \chi_k w T_{\tilde{\chi}_k a} (1 - \tilde{\chi}_k) u$$

then we get

$$\|wT_a u\|_{H_{ul}^{s_0}} \leq C \|wa\|_{H_{ul}^{s_1}} \|u\|_{H_{ul}^{s_2}}.$$

The proof is complete. \square

PROPOSITION 5.11. *Let $w = \langle \cdot \rangle^\lambda$, $\lambda \in \mathbf{R}$ and two functions $a \in H_{ul}^{s_1}(\mathbf{R}^d)$, $u \in H_{ul}^{s_2}(\mathbf{R}^d)$ with $s_1 + s_2 > 0$ then we have*

(i)

$$(5.16) \quad \|wR(a, u)\|_{H_{ul}^{s_1 + s_2 - \frac{d}{2}}(\mathbf{R}^d)} \leq C \|a\|_{H_{ul}^{s_1}(\mathbf{R}^d)} \|wu\|_{H_{ul}^{s_2}(\mathbf{R}^d)}.$$

(ii) *If in addition $s_0 \leq s_1$ and $s_0 < s_1 + s_2 - \frac{d}{2}$ then there exists a constant $C > 0$ such that*

(5.17)

$$\|w(a - T_a)u\|_{H_{ul}^{s_0}(\mathbf{R}^d)} \leq C \min \left\{ \|a\|_{H_{ul}^{s_1}(\mathbf{R}^d)} \|wu\|_{H_{ul}^{s_2}(\mathbf{R}^d)}, \|wa\|_{H_{ul}^{s_1}(\mathbf{R}^d)} \|u\|_{H_{ul}^{s_2}(\mathbf{R}^d)} \right\}.$$

PROOF. (i) By definition, we have (for some cut-off function φ)

$$R(a, u) = \sum_{j \geq -1} \sum_{|k-j| \leq 1} \varphi(2^{-j} D) a \cdot \varphi(2^{-k} D) u.$$

We write $a = \tilde{\chi}_k a + (1 - \tilde{\chi}_k) a$, $u = \tilde{\chi}_k u + (1 - \tilde{\chi}_k) u$ so that

$$\chi_k w R(a, u) = \chi_k w R(\tilde{\chi}_k a, \tilde{\chi}_k u) + \chi_k w S_k(a, u).$$

The first term is estimated by the same method as (5.15) with the use of Theorem 2.9 (i) in [2]. The remainder $w\chi_k S_k(a, u)$ is estimated by using (5.12) and (5.13).

(ii) is a direct consequence of (i) and Proposition 5.10. \square

REMARK 5.12. We remark that with the methods in the proofs above, the commutator estimate in Lemma 7.20, [1] still holds for uniformly local Sobolev spaces with the weight $\langle \cdot \rangle^\lambda$.

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