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(PURE) TRANSCENDENCE BASES IN φ-DEFORMED SHUFFLE BIALGEBRAS[1]

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ABSTRACT. Computations with integro-differential operators are often carried out in an associative algebra with unit, and they are essentially non-commutative computations. By adjoining a cocommutative co-product, one can have those operators act on a bialgebra isomorphic to an enveloping algebra. That gives an adequate framework for a computer-algebra implementation via monoidal factorization, (pure) transcendence bases and Poincaré–Birkhoff–Witt bases.

In this paper, we systematically study these deformations, obtaining necessary and sufficient conditions for the operators to exist, and we give the most general cocommutative deformations of the shuffle co-product and an effective construction of pairs of bases in duality. The paper ends by the combinatorial setting of local systems of coordinates on the group of group-like series.

CONTENTS

1. INTRODUCTION

The shuffle product first appeared in 1953 in the work of Eilenberg and Mac Lane [20]. As soon as 1954, Chen used it to express the product of iterated (path) integrals [9], and Ree, building on Friedrichs' criterion, proved that the non-commutative generating series of iterated integrals are exponentials of Lie polynomials, thus connecting the Lie polynomials with the shuffle product [40]. In 1956, Radford proved that the Lyndon words form a (pure)

[1] The present work is part of a series of papers devoted to the study of the renormalization of divergent polyzetas (at positive and at non-positive indices) via the factorization of the non-commutative generating series of polylogarithms and of harmonic sums, and via the effective construction of pairs of dual bases in duality in φ-deformed shuffle algebras. It is a sequel to [14], and its content was presented in several seminars and meetings, including the 74th Séminaire Lotharingien de Combinatoire.

transcendence basis of the shuffle algebra [39]. The latter result is now well understood through the duality between bialgebras and enveloping algebras (see for example [41]), of which the construction in 1958 of the Poincaré–Birkhoff–Witt[2]–Lyndon basis by Chen, Fox and Lyndon [11] and of its dual basis by Schützenberger, via monoidal factorization [42, 41], gave a striking illustration. This pair of dual bases enabled one to factorize the diagonal series in the shuffle bialgebra and, consequently, to proceed combinatorially with the Dyson series [24] or the transport operator [23], which play a leading role in the relations between special functions involved in the theory of quantum groups [29] and in number theory [7].

In 1973, that is, within twenty years of the introduction of the shuffle product, Knutson defined the quasi-shuffle in [30], where it shows up as the inner product of functions on the symmetric groups[3]. This product is very similar to the Rota–Baxter operator introduced by Cartier in 1972, in his study of the so-called Baxter algebras [8]. Although the analogue of Radford's theorem was pointed out by Malvenuto and Reutenauer [33], the factorization of the diagonal series in the quasi-shuffle bialgebra, initiated in [26, 27], has not yet been carried over to more general bialgebras.

Schützenberger's factorization[4] [41] and its extensions have since been applied to the renormalization of the associators [26, 27], to which matter they turned out to be central[5].

The coefficients of these power series are polynomial functions of positive integral multi-indices of Riemann's zeta function[6] [31, 45], and they satisfy quadratic relations [7] which Lyndon words help explicit and explain. The latter relations can be obtained by identifying the local coordinates on a bridge equation connecting the Cauchy and the Hadamard algebras of polylogarithmic functions, and by using the factorization of the non-commutative generating series of polylogarithms [25] and of harmonic sums [26, 27]. This bridge equation is mainly a consequence of the isomorphisms between the algebra of non-commutative generating series of polylogarithms and the shuffle algebra on the one hand, and between the algebra of non-commutative generating series of harmonic sums and the quasi-shuffle algebra on the other hand.

As for the generalization of Schützenberger's factorization to more general bialgebras, the key step, and the main difficulty thereof, is to decompose orthogonally such bialgebras into the Lie algebra generated by its primitive elements and the associated orthogonal ideal, as Ree was able to achieve in the case of the shuffle bialgebra [40], and to construct, whenever possible, the respective bases. In favorable cases, it is to be hoped that those bialgebras are enveloping algebras, so that the Eulerian projectors are convergent and other analytic computations can be performed.

To make that decomposition possible, one first needs to determine the Eulerian projectors by taking the logarithm of the diagonal series and second to insure their convergence. A

[2]From now on, Poincaré–Birkhoff–Witt will be abbreviated to PBW.

[3]In the present paper, that product will be referred to as the quasi-shuffle or as the stuffle product, indifferently.

[4]Also called MSR factorization after the names of Mélançon, Schützenberger and Reutenauer.

[5]These associators, which are formal power series in non-commutative variables, were introduced in quantum field theory by Drinfel'd [13]. The explicit coefficients of the universal associator Φ_{KZ} are polyzetas and regularized polyzetas [31].

[6]These values are usually referred to as MZV's by Zagier [45] and as polyzetas by Cartier [7].

key ingredient is the fact that the diagonal series are group-like and give a host of group-like elements by specialization, so one can use the exponential-logarithm correspondence to compute within a combinatorial Hausdorff group.

To that effect, the present work generalizes the recursive definitions of the shuffle and quasi-shuffle products given by Fliess [22] and Hoffman [28], respectively, to introduce the φ-deformed shuffle product, where φ stands for an arbitrary algebra law. Recent studies on these structures can be found in [16, 35, 36].

These φ-shuffle products interpolate between the classical shuffle and quasi-shuffle products (for $\varphi \equiv 0$ and $\varphi \equiv 1$, respectively), and allow a classification of the associated bialgebras.

This paper is devoted to the combinatorics of φ-deformed shuffle algebras and to the effective constructions of pairs of dual bases. Its organisation is as follows:

- Section 2 is a short reminder of well-known facts about the combinatorics of the q-stuffle product [4], which encompasses the shuffle [41] and the quasi-shuffle products [26, 27].
- In Section 3, we thoroughly investigate algebraic and combinatorial aspects of the φ-deformed shuffle products and explain how to use bases in duality to get a local system of coordinates on the (infinite-dimensional) Lie group of group-like series.

Throughout the paper, we have a particular concern for Lie series and their correspondence with the Hausdorff group.

2. A SURVEY OF SHUFFLE PRODUCTS

For standard definitions and facts pertaining to the (algebraic) combinatorics on words, we refer the reader to the classical books by Lothaire [32] and Reutenauer [41].

Throughout the paper, $\mathbb{K}$ stands for a (unital, associative and commutative) $\mathbb{Q}$-algebra containing a parameter q. In this section, we review the known combinatorics of bases in duality and local coordinates on the infinite-dimensional Lie group of group-like series (Hausdorff group). The parameter q allows for a unified treatment between shuffle ($q = 0$) and quasi-shuffle ($q = 1$) products.

Let $Y = \{y_i\}_{i\geq 1}$ be an alphabet, totally ordered by $y_1 > y_2 > \cdots$. The free monoid and the set of Lyndon words over Y are denoted by Y^* and $\mathcal{L}yn\, Y$, respectively. The unit of Y^* is denoted by 1_{Y^*}. We also write $Y^+ = Y^* \setminus \{1_{Y^*}\}$.

The q-stuffle [4], which interpolates between the shuffle [40], quasi-shuffle [33] (or stuffle) and minus-stuffle products [10], for $q = 0$, 1, and -1, respectively, is defined as follows:

$$u \uplus_q 1_{Y^*} = 1_{Y^*} \uplus_q u = u, \tag{1}$$

$$y_s u \uplus_q y_t v = y_s(u \uplus_q y_t v) + y_t(y_s u \uplus_q v) + q\, y_{s+t}(u \uplus_q v), \tag{2}$$

or its dual co-product, as follows, for any $y_s, y_t \in Y$ and $u, v \in Y^*$,

$$\Delta_{\uplus_q}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*}, \tag{3}$$

$$\Delta_{\uplus_q}(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + q \sum_{s_1+s_2=s} y_{s_1} \otimes y_{s_2}. \tag{4}$$

We now turn to the study of the combinatorial q-stuffle Hopf algebra, which we do by stressing the importance of the Lie elements[7] studied by Ree [40], and show how Schützenberger's factorization extends to this new structure.

The q-stuffle is commutative, associative and unital. With the co-unit defined by $\epsilon(P) = \langle P \mid 1_{Y^*}\rangle$, for $P \in \mathbb{K}\langle Y\rangle$, we get

$$\mathcal{H}_{\uplus_q} = (\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_q}, \epsilon)$$

and

$$\mathcal{H}^{\vee}_{\uplus_q} = (\mathbb{K}\langle Y\rangle, \uplus_q, 1_{Y^*}, \Delta_{\mathtt{conc}}, \epsilon)$$

which are mutually dual bialgebras and, in fact, Hopf algebras because they are $\mathbb{N}$-graded by the weight.

Let $\mathcal{D}_Y$ be the diagonal series over $\mathcal{H}_{\uplus_q}$, i.e.,

$$\mathcal{D}_Y = \sum_{w \in Y^*} w \otimes w. \tag{5}$$

Then[8]

$$\log \mathcal{D}_Y = \sum_{w \in Y^+} w \otimes \pi_1(w), \tag{6}$$

where π_1 is the extended Eulerian projector[9] over $\mathcal{H}_{\uplus_q}$, defined by (see [4])

$$\pi_1(w) = w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k \in Y^+} \langle w \mid u_1 \uplus_q \cdots \uplus_q u_k\rangle u_1 \ldots u_k. \tag{7}$$

Let $\{\Pi_l\}_{l \in \mathcal{L}yn\,Y}$ be defined by

$$\begin{cases} \Pi_y = \pi_1(y), & \text{for } y \in Y,\\ \Pi_l = [\Pi_s, \Pi_r], & \text{for the standard factorization } (s,r) \text{ of } l \in \mathcal{L}yn\,Y - Y. \end{cases} \tag{8}$$

Then it forms a basis of the Lie algebra of primitive elements of $\mathcal{H}_{\uplus_q}$ (see [4]).

Let $\{\Pi_w\}_{w \in Y^*}$ be defined, for any $w \in Y^*$ such that $w = l_1^{i_1} \ldots l_k^{i_k}$ with $l_1 > \ldots > l_k$ and $l_1 \ldots, l_k \in \mathcal{L}yn\,Y$, by

$$\Pi_w = \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k}. \tag{9}$$

Then, by the PBW theorem, the set $\{\Pi_w\}_{w \in Y^*}$ is a basis of $\mathbb{K}\langle Y\rangle$ (see [4]).

[7]Following Ree [40], the Lie elements contain the non-commutative power series which are Lie series (as the Chen non-commutative generating series of iterated integrals), i.e., they are group-like for the co-product of the shuffle.

[8]The diagonal series lives in $\mathbb{K}\langle\langle Y^* \otimes Y^*\rangle\rangle \simeq (\mathbb{K}\langle Y\rangle \otimes \mathbb{K}\langle Y\rangle)^*$.

[9]In fact, π_1 is a projector which maps $\mathcal{H}_{\uplus_q}$ onto the space of its primitive elements $Prim(\mathcal{H}_{\uplus_q})$, see Lemma 7.

Let $\{\Sigma_w\}_{w\in Y^*}$ be the family dual[10] to $\{\Pi_w\}_{w\in Y^*}$ in the quasi-shuffle algebra. Then $\{\Sigma_w\}_{w\in Y^*}$ freely generates the quasi-shuffle algebra, and the subset $\{\Sigma_l\}_{l\in\mathcal{L}yn\,Y}$ forms a transcendence basis of $(\mathbb{K}\langle Y\rangle, \,\text{ш}_q, 1_{Y^*})$. The Σ_w can be obtained as follows (see [4]):

$$\begin{cases}\Sigma_y = y, & \text{for } y \in Y,\\ \Sigma_l \;= \displaystyle\sum_{(!)} \frac{q^{i-1}}{i!} y_{s_{k_1}+\cdots+s_{k_i}} \Sigma_{l_1\cdots l_n}, & \text{for } l = y_{s_1}\ldots y_{s_k} \in \mathcal{L}yn\,Y,\\ \Sigma_w = \dfrac{\Sigma_{l_1}^{\text{ш}_q i_1} \,\text{ш}_q \cdots \,\text{ш}_q \Sigma_{l_k}^{\text{ш}_q i_k}}{i_1!\cdots i_k!}, & \text{for } w = l_1^{i_1}\ldots l_k^{i_k},\end{cases} \tag{10}$$

and $l_1 \succ_{lex} \cdots \succ_{lex} l_k \in \mathcal{L}yn\,Y$. In the second expression of (10), the sum (!) is taken over all subsequences $\{k_1,\ldots,k_i\} \subset \{1,\ldots,k\}$ and all Lyndon words $l_1 \succeq_{lex} \cdots \succeq_{lex} l_n$ such that $(y_{s_1},\ldots,y_{s_k}) \overset{*}{\Leftarrow} (y_{s_{k_1}},\ldots,y_{s_{k_i}}, l_1,\ldots,l_n)$, where $\overset{*}{\Leftarrow}$ denotes the transitive closure of the relation on standard sequences, denoted by $\Leftarrow$ (see [4]).

In this case, since $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$ are multiplicative, we get the q-extended Schützenberger's factorization as follows (see [4]):

$$\mathcal{D}_Y = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}yn\,Y}^{\searrow} \exp(\Sigma_l \otimes \Pi_l). \tag{11}$$

This series, in the factorized form, encompasses a large part of the combinatorics of Dyson's functional expansions in quantum field theory [18, 34]. It is the infinite-dimensional analogue of the theorem of Wei and Norman [2, 43, 44].

3. Algebraic aspects of φ-shuffle bialgebras

From now on, we will work with an alphabet $Y = \{y_i\}_{i\in I}$ with I an arbitrary index set[11], which needs not be totally ordered unless we write it explicitly.

3.1. **First properties.** Let us consider the following recursion in order to construct a map

$$Y^* \times Y^* \longrightarrow \mathbb{K}\langle Y\rangle. \tag{12}$$

i) For any $w \in Y^*$,

$$(\mathrm{Init}) \qquad 1_{Y^*} \,\text{ш}_\varphi w = w \,\text{ш}_\varphi 1_{Y^*} = w. \tag{13}$$

ii) For any $a, b \in Y$ and $u, v \in Y^*$,

$$(\mathrm{Rec}) \qquad au \,\text{ш}_\varphi bv = a(u \,\text{ш}_\varphi bv) + b(au \,\text{ш}_\varphi v) + \varphi(a,b)(u \,\text{ш}_\varphi v), \tag{14}$$

where φ is an arbitrary mapping defined by its structure coefficients

$$\varphi : Y \times Y \longrightarrow \mathbb{K}Y, \tag{15}$$

$$(y_i, y_j) \longmapsto \sum_{k\in I} \gamma_{y_i,y_j}^{y_k}\, y_k. \tag{16}$$

The following proposition guarantees the existence of a unique bilinear law on $\mathbb{K}\langle Y\rangle$ satisfying (Init) and (Rec).

[10]The duality pairing is given by $\langle u \mid v\rangle = \delta_{u,v}$, for $u, v \in Y^*$.

[11]The indexing is one-to-one, i.e., there is no repetition.

Proposition 1 ([14]). *The recursion* (Rec) *together with the initialization* (Init) *defines a unique mapping*

$$\stuffle_{\varphi} : Y^* \times Y^* \longrightarrow \mathbb{K}\langle Y\rangle,$$

which can, at once, be extended by linearity as a law

$$\stuffle_{\varphi} : \mathbb{K}\langle Y\rangle \otimes \mathbb{K}\langle Y\rangle \longrightarrow \mathbb{K}\langle Y\rangle.$$

The space $\mathbb{K}\langle Y\rangle$ endowed with the law $\stuffle_{\varphi}$ is an algebra (with unit $1_{\mathbb{K}\langle Y\rangle}$ by definition). It will be called the φ-shuffle algebra. In full generality, this algebra need not be associative or commutative if φ is not so. In the next example, we give a table of well known laws which can be defined according to this pattern (in which φ is reasonable).

Example 1. Below, a summary table of φ-deformed cases found in the literature is given. The last case (infiltration product) comes from computer science (see [37, 38, 15])

Name	(recursion) Formula	φ
Shuffle	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$
Quasi-shuffle Stuffle	$x_iu \stuffle x_jv = x_i(u \stuffle x_jv) + x_j(x_iu \stuffle v) + x_{i+j}(u \stuffle v)$	$\varphi(x_i, x_j) = x_{i+j}$
Min-shuffle	$x_iu \minishuffle x_jv = x_i(u \minishuffle x_jv) + x_j(x_iu \minishuffle v) - x_{i+j}(u \minishuffle v)$	$\varphi(x_i, x_j) = -x_{i+j}$
Muffle	$x_iu \muffle x_jv = x_i(u \muffle x_jv) + x_j(x_iu \muffle v) + x_{i\times j}(u \muffle v)$	$\varphi(x_i, x_j) = x_{i\times j}$
q-stuffle	$x_iu \stuffle_q x_jv = x_i(u \stuffle_q x_jv) + x_j(x_iu \stuffle_q v) + qx_{i+j}(u \shuffle v)$	$\varphi(x_i, x_j) = qx_{i+j}$
q-stuffle (character)	$x_iu \shuffle_q x_jv = x_i(u \shuffle_q x_jv) + x_j(x_iu \shuffle_q v) + q^{i\times j}x_{i+j}(u \shuffle v)$	$\varphi(x_i, x_j) = q^{i\times j}x_{i+j}$
$\mathtt{LDIAG}(1, q_s)$ non-crossed, non-shifted	$au * bv = a(u * bv) + b(au * v) + q_s^{\lvert a\rvert\lvert b\rvert}(a.b)(u * v)$	$\varphi(a, b) = q_s^{\lvert a\rvert\lvert b\rvert}(a.b)$ $(a.b)$ assoc.
B-shuffle	$au \shuffle_B bv = a(u \shuffle_B bv) + b(au \shuffle_B v) + \langle a, b\rangle(u \shuffle_B v)$	$\varphi(a, b) = \langle a, b\rangle = \langle b, a\rangle$
Semigroup--shuffle	$x_tu \shuffle_{\perp} x_sv = x_t(u \shuffle_{\perp} x_sv) + x_s(x_tu \shuffle_{\perp} v) + x_{t\perp s}(u \shuffle_{\perp} v)$	$\varphi(x_t, x_s) = x_{t\perp s}$
q-Infiltration	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b}a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b}a$

Now we set forth the first properties of $\stuffle_{\varphi}$ (see [21]): *associativity, commutativity* and *dualizability*.

Definition 1 ([14]). *A law* $\mu : \mathbb{K}\langle Y\rangle \otimes \mathbb{K}\langle Y\rangle \to \mathbb{K}\langle Y\rangle$ *is said to be* dualizable *if there exists a (linear) mapping*

$$\Delta_{\mu} : \mathbb{K}\langle Y\rangle \longrightarrow \mathbb{K}\langle Y\rangle \otimes \mathbb{K}\langle Y\rangle$$

(necessarily unique) such that the dual mapping

$$\Big(\mathbb{K}\langle Y\rangle \otimes \mathbb{K}\langle Y\rangle\Big)^* \longrightarrow \mathbb{K}\langle\langle Y\rangle\rangle$$

restricts [12] *to* μ. *Or, equivalently,*

$$\text{for all } u,v,w \in Y^* : \qquad \langle \mu(u \otimes v) \mid w\rangle = \langle u \otimes v \mid \Delta_\mu(w)\rangle^{\otimes 2}.$$

Theorem 1 ([14])**.** *We have:*

(1) *The law* $⧢_\varphi$ *is associative (respectively commutative) if and only if the extension* $\varphi : \mathbb{K}Y \otimes \mathbb{K}Y \longrightarrow \mathbb{K}Y$ *is so.*

(2) *Let* $\gamma_{x,y}^z := \langle \varphi(x,y) \mid z\rangle$ *be the structure constants of* φ. *Then* $⧢_\varphi$ *is dualizable if and only if* φ *is also dualizable, that is to say, there exists a map* $\delta : \mathbb{K}Y \longrightarrow \mathbb{K}Y \otimes \mathbb{K}Y$ *such that for all* $x,y,z \in X$ *we have*

$$\langle \varphi(x,y) \mid z\rangle = \langle x \otimes y \mid \delta(z)\rangle \ .$$

This map δ *is given by*[13]

$$\delta(z) = \sum_{x,y \in Y} \gamma_{x,y}^z \, x \otimes y \ .$$

For the proof of the theorem we need the following auxiliary result.

Lemma 1 ([14])**.** *Let* Δ *be the morphism* $\mathbb{K}\langle Y\rangle \longrightarrow A\langle\langle Y^* \otimes Y^*\rangle\rangle$ *defined on the letters by*[14]

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m \in I} \gamma_{y_n,y_m}^{y_s} y_n \otimes y_m \ .$$

Then

(1) *for all* $u,v,w \in Y^*$, $\langle u ⧢_\varphi v \mid w\rangle = \langle u \otimes v \mid \Delta(w)\rangle^{\otimes 2}$.

(2) *for all* $w \in Y^+$, $\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in Y^+} \langle \Delta(w) \mid u \otimes v\rangle u \otimes v$.

Proof of Theorem 1 (sketch). The theorem follows by application of items (1) and (2) in Lemma 1. $\square$

If φ is associative (which is fulfilled in all cases of Table 1), we extend φ to Y^+ by the universal property of the free semigroup Y^+,

$$\begin{cases} \varphi(x) = x, & \text{for } x \in Y, \\ \varphi(xw) = \varphi(x, \varphi(w)), & \text{for } x \in Y \text{ and } w \in Y^+, \end{cases} \tag{17}$$

and we extend the definition of the structure constants accordingly: for $x_1 \ldots x_l \in Y^+$,

$$\gamma_{x_1 \ldots x_l}^y = \langle y \mid \varphi(x_1 \ldots x_l)\rangle = \sum_{t_1,\ldots,t_{l-2} \in Y} \gamma_{x_1,t_1}^y \gamma_{x_2,t_2}^{t_1} \ldots \gamma_{x_{l-1},x_l}^{t_{l-2}}. \tag{18}$$

Note that the fact that φ is dualizable can be rephrased as:

$$\text{for all } y \in Y : \ \{w \in Y^2 | \gamma_w^y \neq 0\} \text{ is finite.} \tag{19}$$

[12] through the pairings $\langle - \mid - \rangle$.

[13] Note that all these conditions are equivalent to the fact that $(\gamma_{x,y}^z)_{x,y,z \in Y}$ satisfies:

$$\text{for all } z \in Y : \ \#\{(x,y) \in Y^2 | \gamma_{x,y}^z \neq 0\} < +\infty \ .$$

[14] If φ is dualizable, this expression can be written

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \delta(y_s) \ .$$

In this case, it can be checked immediately that, for an arbitrarily fixed $N \geq 1$,

$$\text{for all } y \in Y : \{w \in Y^N | \gamma_w^y \neq 0\} \text{ is finite,} \tag{20}$$

but, by no means, we have in general that

$$\text{for all } y \in Y : \{w \in Y^+ | \gamma_w^y \neq 0\} \text{ is finite.} \tag{21}$$

Remark 1. i) Condition (21) is strictly stronger than (19) as the example of any group law on Y, with $|Y| \geq 2$ and finite, shows.

ii) Non-dualizable laws occur with the alphabet $Y_{\mathbb{Z}} = \{y_j\}_{j\in\mathbb{Z}}$ and the stuffle on it ($\varphi(y_i, y_j) = y_{i+j}$). This alphabet naturally appears in the theory of polylogarithms at negative integers in [17] where another non-dualizable law (called $\top$) arises. See also Example 2 below.

Definition 2. *An associative law φ on $\mathbb{K}Y$ will be said to be* moderate *if and only if it fulfils condition* (21)*.*

Let us now state the structure theorem from [14].

Theorem 2 ([14])**.** *Let us suppose that φ is dualizable and associative. We still denote its dual co-multiplication by*

$$\Delta_{\uplus_\varphi} : \mathbb{K}\langle Y\rangle \longrightarrow \mathbb{K}\langle Y\rangle \otimes \mathbb{K}\langle Y\rangle.$$

Then $\mathcal{B}_\varphi = (\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_\varphi}, \varepsilon)$ is a bialgebra. If, moreover, φ is commutative, the following conditions are equivalent:

1. *$\mathcal{B}_\varphi$ is an enveloping bialgebra.*
2. *$\mathcal{B}_\varphi$ is isomorphic to $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ as a bialgebra.*
3. *For all $y \in Y$, the following series is a polynomial*

$$(P) \qquad y + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\ldots,x_l \in Y} \langle y \mid \varphi(x_1 \ldots x_l)\rangle\, x_1 \ldots x_l\, .$$

4. *φ is moderate.*

Proof (sketch). 4 $\Longrightarrow$ 3) Obvious.

3 $\Longrightarrow$ 2) One first constructs an endomorphism of $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*})$ sending each letter $y \in Y$ to the polynomial form (P) and then proves that it is an automorphism of AAU[15] which sends $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_\varphi}, \varepsilon)$ to $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$.

2 $\Longrightarrow$ 1) Because $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ is an enveloping bialgebra.

1 $\Longrightarrow$ 4) Observe that, for each letter $y \in Y$, we have

$$\langle \Delta_{\uplus_\varphi}^{(n-1)}(y) \mid x_1 \otimes x_2 \otimes \cdots \otimes x_n\rangle = \gamma^y_{x_1\ldots x_l}\, . \qquad \square$$

Example 2. (1) The muffle product (see Table 1), which determines the product of Hurwitz polyzetas with rational centers and correspond to $\varphi(x_i, x_j) = x_{i.j}$ for $i, j \in \mathbb{Q}_+^*$, is not dualizable ($\gamma^1_{n,1/n} = 1$ for all $n \geq 1$).

[15] Abbreviation for associative algebra with unit.

(2) The q-infiltration bialgebra (see again Table 1) has its origin in computer science [37, 38] and appears as a generic solution in [15]. It provides a bialgebra

$$\mathcal{H}_{q-\text{inflitr}} = (\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{X^*}, \Delta_{\uparrow_q}, \epsilon)$$

($q \in \mathbb{K}$) based on a φ which is an associative, commutative and dualizable law, but, if $Y \neq \emptyset$ this law is moderate only if and only if q is nilpotent in the $\mathbb{Q}$-algebra $\mathbb{K}$. Indeed, for all $x \in Y$, $(1+qx)$ is group-like and it has an inverse in $\mathbb{K}\langle X\rangle$ if and only if q is nilpotent. In this case the antipode is the involutive antiautomorphism defined on the letters by

$$S(x) = \frac{-x}{1+qx}.$$

3.2. **Structural properties.** Here, we only assume that φ is associative.

The bialgebra

$$\mathcal{H}^{\vee}_{\uplus_\varphi} = (\mathbb{K}\langle Y\rangle, \uplus_\varphi, 1_{Y^*}, \Delta_{\mathtt{conc}}, \epsilon) \tag{22}$$

is a Hopf algebra because it is co-nilpotent[16]. Its antipode can be computed by $a(1_{Y^*}) = 1$ and, for $w \in Y^+$,

$$a_{\uplus_\varphi}(w) = \sum_{k\geq 1}(-1)^{-k} \sum_{\substack{u_1,\ldots,u_k\in Y^+\\ u_1\ldots u_k=w}} u_1 \uplus_\varphi \cdots \uplus_\varphi u_k. \tag{23}$$

Due to the finite number of decompositions of any word $u_1\ldots u_k = w \in Y^+$ into factors $u_1,\ldots,u_k \in Y^+$, we can, at this very early stage, define an endomorphism $\Phi(S)$ of $\mathbb{K}\langle Y\rangle$ as follows:

$$\Phi(S)[w] = \sum_{k\geq 1} a_k \sum_{\substack{u_1,\ldots,u_k\in Y^+\\ u_1\ldots u_k=w}} u_1 \uplus_\varphi \cdots \uplus_\varphi u_k, \tag{24}$$

associated to any univariate formal power series $S = a_1X + a_2X^2 + a_3X^3 + \cdots$. The case of

$$\log(1+X) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} X^k \tag{25}$$

will be of particular importance. It reads here in the style of formula (23),

$$\check{\pi}_1(w) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{\substack{u_1,\ldots,u_k\in Y^+\\ u_1\ldots u_k=w}} u_1 \uplus_\varphi \cdots \uplus_\varphi u_k. \tag{26}$$

[16]The law $\Delta_{\mathtt{conc}}$, dual to the concatenation is, of course, defined by

$$\Delta_{\mathtt{conc}}(w) = \sum_{uv=w} u \otimes v.$$

The corresponding n-fold $\Delta^+_{\mathtt{conc}}$ ($\Delta^+ = \Delta$ minus the primitive part) reads

$$\Delta^{+(n-1)}_{\mathtt{conc}}(w) = \sum_{\substack{u_1u_2\cdots u_n=w\\ u_i\in Y^+}} u_1 \otimes u_2 \otimes \cdots \otimes u_n,$$

from which it is clear that $\Delta^{+(n-1)}_{\mathtt{conc}}(w) = 0$ for $n > |w|$.

This $\check{\pi}_1 \in \mathrm{End}(\mathbb{K}\langle Y\rangle)$ has an adjoint $\pi_1 \in \mathrm{End}(\mathbb{K}\langle\langle Y\rangle\rangle)$ which reads

$$\pi_1(S) = \sum_{w\in Y^*} \langle S \mid \check{\pi}_1(w)\rangle w \tag{27}$$

$$= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle S \mid u_1 \,\text{⧢}_\varphi \cdots \,\text{⧢}_\varphi u_k\rangle \, u_1 \ldots u_k. \tag{28}$$

It is an easy exercise to check that the family in the sums of (27) is summable[17]. It is easy to check that the dominant term of all terms in a ⧢_φ product is the corresponding $\sqcup\!\sqcup$ product. This explains why we still have the theorem of Radford.

Theorem 3 (RADFORD'S THEOREM)**.** *When φ is commutative, the associative and commutative algebra with unit $(\mathbb{K}\langle Y\rangle, \text{⧢}_\varphi, 1_{Y^*})$ is a polynomial algebra. More precisely, the morphism $\beta : \mathbb{K}[\mathcal{L}yn\, Y] \to (\mathbb{K}\langle Y\rangle, \text{⧢}_\varphi, 1_{Y^*})$ defined by $\beta(l) = l$ for $l \in \mathcal{L}yn\, Y$ is an isomorphism. In other words, the family*

$$\left(l_1^{\text{⧢}_\varphi i_1} \,\text{⧢}_\varphi \cdots \,\text{⧢}_\varphi l_k^{\text{⧢}_\varphi i_k}\right)_{\substack{k\geq 0,\ \{l_1,l_2,\ldots,l_k\}\subset \mathcal{L}yn\, Y \\ (i_1,i_2,\ldots,i_k)\in(\mathbb{N}_+)^k}}$$

is a linear basis of $\mathbb{K}\langle Y\rangle$.

Proof. One checks that

$$l_1^{\text{⧢}_\varphi i_1} \,\text{⧢}_\varphi \cdots \,\text{⧢}_\varphi l_k^{\text{⧢}_\varphi i_k} = l_1^{\sqcup\!\sqcup i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, l_k^{\sqcup\!\sqcup i_k} + \sum_{|v|<\sum_{1\leq j\leq k} i_j|l_j|} c_v\, v.$$

The result follows. $\square$

The theorem of Radford is important in the classical cases because it is the left-hand side of Schützenberger's factorization in which we have the move

PBW $\to$ Radford;

see [12] for a discussion of the converse.

Lemma 2 (φ-EXTENDED FRIEDRICHS' CRITERION)**.** *We denote[18] by*

$$\Delta_{\text{⧢}_\varphi} : \mathbb{K}\langle\langle Y\rangle\rangle \to \mathbb{K}\langle\langle Y^* \otimes Y^*\rangle\rangle$$

the dual of ⧢_φ applied to series, i.e., defined by

$$\Delta_{\text{⧢}_\varphi}(S) = \sum_{u,v\in Y^*} \langle S \mid u \,\text{⧢}_\varphi\, v\rangle\, u \otimes v.$$

Let now $S \in \mathbb{K}\langle\langle Y\rangle\rangle$. Then we have:

1. *If $\langle S \mid 1_{Y^*}\rangle = 0$ then S is primitive (i.e., $\Delta_{\text{⧢}_\varphi}(S) = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S$)[19] if and only if we have $\langle S \mid u \,\text{⧢}_\varphi v\rangle = 0$ for any u and $v \in Y^+$.*
2. *If $\langle S \mid 1_{Y^*}\rangle = 1$, then S is group-like (i.e., $\Delta_{\text{⧢}_\varphi}(S) = S \otimes S$)[20] if and only if we have $\langle S \mid u \,\text{⧢}_\varphi v\rangle = \langle S \mid u\rangle\langle S \mid v\rangle$ for any u and $v \in Y^*$.*

[17]A family of (simple, double, etc.) series is summable if it is locally finite (see [14] for a complete development).

[18]As in the classical case, $\Delta_{\text{⧢}_\varphi}$ is a $\mathtt{conc}$-morphism as can be seen by transposition of the fact that $\Delta_{\mathtt{conc}}$ is a ⧢_φ-morphism.

[19]Tensor products of linear forms.

[20]idem

Proof. The expected equivalences are due to the following facts:

$$\Delta_{\text{ш}_\varphi}(S) = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S - \langle S \mid 1_{Y^*}\rangle 1_{Y^*} \otimes 1_{Y^*} + \sum_{u,v \in Y^+} \langle S \mid u \,\text{ш}_\varphi v\rangle u \otimes v,$$

$$\Delta_{\text{ш}_\varphi}(S) = \sum_{u,v \in Y^*} \langle S \mid u \,\text{ш}_\varphi v\rangle u \otimes v \quad \text{and} \quad S \otimes S = \sum_{u,v \in Y^*} \langle S \mid u\rangle \langle S \mid v\rangle u \otimes v. \qquad \square$$

Lemma 3. *Let $S \in \mathbb{K}\langle\langle Y\rangle\rangle$ be such that $\langle S \mid 1_{Y^*}\rangle = 1$. Then S is group-like if and only if*[21] $\log(S)$ *is primitive.*

Proof. Since $\Delta_{\text{ш}_\varphi}$ and the maps $T \mapsto T \otimes 1_{Y^*}, T \mapsto 1_{Y^*} \otimes T$ are continuous homomorphisms, then, if $\log S$ is primitive, we have (see Lemma 2(1))

$$\Delta_{\text{ш}_\varphi}(\log S) = \log S \otimes 1_{Y^*} + 1_{Y^*} \otimes \log S,$$

and, since $\log S \otimes 1_{Y^*}$ and $1_{Y^*} \otimes \log S$ commute, we get successively

$$\begin{aligned}
\Delta_{\text{ш}_\varphi}(S) &= \Delta_{\text{ш}_\varphi}(\exp(\log S)) \\
&= \exp(\Delta_{\text{ш}_\varphi}(\log S)) \\
&= \exp(\log S \otimes 1_{Y^*}) \exp(1_{Y^*} \otimes \log S) \\
&= (\exp(\log S) \otimes 1_{Y^*})(1_{Y^*} \otimes \exp(\log S)) \\
&= S \otimes S.
\end{aligned}$$

This means, together with $\langle S \mid 1_{Y^*}\rangle$, that S is group-like. The converse can be obtained in the same way. $\square$

Remark 2. i) In fact, Lemma 3 establishes a nice log-exp correspondence for the Lie group of group-like series.

ii) Through the canonical pairing $\langle - \mid - \rangle : \mathbb{K}\langle\langle Y\rangle\rangle \otimes \mathbb{K}\langle Y\rangle \to \mathbb{K}$, we have $\mathbb{K}\langle\langle Y\rangle\rangle \simeq (\mathbb{K}\langle Y\rangle)^*$. Group-like (respectively primitive) series are in bijection with characters (respectively infinitesimal characters) of the algebra $(\mathbb{K}\langle Y\rangle, \text{ш}_\varphi, 1_{Y^*})$.

Lemma 4. (1) *Group-like series form a group (for concatenation).*
(2) *The space* $\mathrm{Prim}(\mathbb{K}\langle\langle Y\rangle\rangle)$ *is a Lie algebra (for the bracket derived from concatenation).*

Proof. As in the classical case. $\square$

We extend the transposition process in the same way as in Lemma 2 and note, for $n \geq 1$, that

$$\Delta^{(n-1)}_{\text{ш}_\varphi} : \mathbb{K}\langle\langle Y\rangle\rangle \to \mathbb{K}\langle\langle (Y^*)^{\otimes n}\rangle\rangle, \tag{29}$$

the dual of $\text{ш}_\varphi^{(n-1)}$ applied to series, i.e., defined by

$$\Delta^{(n-1)}_{\text{ш}_\varphi}(S) = \sum_{u_1,u_2,\ldots u_n \in Y^*} \langle S \mid u_1 \,\text{ш}_\varphi \cdots \text{ш}_\varphi u_n\rangle\, u_1 \otimes \cdots \otimes u_n\ . \tag{30}$$

[21]For any $h \in \mathbb{K}\langle\langle Y\rangle\rangle$, if $\langle h \mid 1_{Y^*}\rangle = 0$, we define

$$\log(1_{Y^*} + h) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} h^n \quad \text{and} \quad \exp(h) = \sum_{n \geq 0} \frac{h^n}{n!},$$

and we have the usual formulas $\log(\exp(h)) = h$ and $\exp(\log(1_{Y^*} + h)) = 1_{Y^*} + h$.

We will use the following lemma several times, which gives the combinatorics of products of primitive series (and the polynomials).

Lemma 5 (HIGHER ORDER CO-MULTIPLICATIONS OF PRODUCTS)**.** *Let us consider the language $\mathcal{M}$ over the alphabet $\mathcal{A} = \{a_1, a_2, \ldots, , a_m\}$,*

$$\mathcal{M} = \{w \in \mathcal{A}^* \mid w = a_{j_1} \ldots a_{j_{|w|}}, j_1 < \cdots < j_{|w|}, 1 \le |w| \le m\},$$

and the morphism

$$\begin{aligned} \mu : \mathbb{K}\langle\mathcal{A}\rangle &\longrightarrow \mathbb{K}\langle\langle Y\rangle\rangle, \\ a_i &\longmapsto S_i, \end{aligned}$$

where $S_1, \ldots, S_m$ are primitive series in $\mathbb{K}\langle\langle Y\rangle\rangle$. Then

$$\Delta_{\uplus_\varphi}^{(n-1)}(S_1 \ldots S_m) = \sum_{\substack{w_1,\ldots,w_n \in \mathcal{M} \\ |w_1|+\cdots+|w_n|=m \\ a_1\cdots a_m \in \mathrm{supp}(w_1 ⧢ \ldots ⧢ w_n)}} \mu(w_1) \otimes \cdots \otimes \mu(w_n).$$

Proof (sketch). Let $\mathcal{S} = (S_1, \ldots, S_m)$ be this family of primitive series and, for $I = \{i_1, \ldots, i_k\} \subset [1 \cdots m]$ in increasing order, let us write $\mathcal{S}[I]$ for the product $S_{i_1} \cdots S_{i_k}$. Then we have

$$\Delta_{\uplus_\varphi}^{(n-1)}(S_1 \ldots S_m) = \sum_{I_1+\cdots+I_n=[1\cdots m]} \mathcal{S}[I_1] \otimes \cdots \otimes \mathcal{S}[I_n].$$

Setting $w_i = (a_1 a_2 \ldots a_m)[I]$, one gets the expected result. □

Lemma 6 (PAIRING OF PRODUCTS)**.** *Let $S_1, \ldots, S_m$ be primitive series in $\mathbb{K}\langle\langle Y\rangle\rangle$, and let $P_1, \ldots, P_n$ be* proper[22] *polynomials in $\mathbb{K}\langle Y\rangle$. Then one has in general*

$$\langle P_1 \uplus_\varphi \cdots \uplus_\varphi P_n \mid S_1 \ldots S_m\rangle = \sum_{\substack{w_1,\ldots,w_n \in \mathcal{M} \\ |w_1|+\cdots+|w_n|=m \\ a_1\cdots a_m \in \mathrm{supp}(w_1 ⧢ \ldots ⧢ w_n)}} \prod_{i=1}^{n} \langle P_i \mid \mu(w_i)\rangle.$$

In particular, we have the following exhaustive list of circumstances:

(1) *If $n > m$, then $\langle P_1 \uplus_\varphi \cdots \uplus_\varphi P_n \mid S_1 \ldots S_m\rangle = 0$.*
(2) *If $n = m$, then*

$$\langle P_1 \uplus_\varphi \cdots \uplus_\varphi P_n \mid S_1 \ldots S_n\rangle = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \langle P_i \mid S_{\sigma(i)}\rangle.$$

(3) *If $n < m$, then one has the general form in which every product in the sum contains at least a factor $\langle P_i \mid \mu(w_i)\rangle$ with $|w_i| \ge 2$.*

Proof. It is a consequence of Lemma 5 through the equality

$$\langle P_1 \uplus_\varphi \cdots \uplus_\varphi P_n \mid S_1 \ldots S_n\rangle = \langle P_1 \otimes \cdots \otimes P_n \mid \Delta_{\uplus_\varphi}^{(n-1)}(S_1 \ldots S_n)\rangle .$$ □

[22] i.e., polynomials without constant term; see [1].

In the sequel, we assume that φ is associative and dualizable.

Now, we have the following two structures:

$$\mathcal{H}_{\uplus_\varphi} = (\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_\varphi}, \epsilon), \tag{31}$$
$$\mathcal{H}^\vee_{\uplus_\varphi} = (\mathbb{K}\langle Y\rangle, \uplus_\varphi, 1_{Y^*}, \Delta_{\mathtt{conc}}, \epsilon), \tag{32}$$

which are mutually dual[23] bialgebras. The bialgebra $\mathcal{H}_{\uplus_\varphi}$ need not be a Hopf algebra, *even if $\Delta_{\uplus_\varphi}$ is cocommutative* (see Example 2.2).

Now, let us consider

$$\mathcal{I} := \mathrm{span}_{\mathbb{K}}\{u \uplus_\varphi v\}_{u,v\in Y^+}, \tag{33}$$
$$\mathbb{K}_+\langle Y\rangle := \{P \in \mathbb{K}\langle Y\rangle \mid \langle P \mid 1_{Y^*}\rangle = 0\}, \tag{34}$$
$$\mathcal{P} := \mathrm{Prim}(\mathcal{H}_{\uplus_\varphi}) = \{P \in \mathbb{K}\langle Y\rangle \mid \Delta^+_{\uplus_\varphi}(P) = 0\}, \tag{35}$$

where

$$\Delta^+_{\uplus_\varphi}(P) = \Delta_{\uplus_\varphi}(P) - (P \otimes 1_{Y^*} + 1_{Y^*} \otimes P) + \langle P \mid 1_{Y^*}\rangle 1_{Y^*} \otimes 1_{Y^*}\,. \tag{36}$$

Remark 3. At this stage (φ not necessarily moderate), it can happen that $\mathrm{Prim}(\mathcal{H}_{\uplus_\varphi}) = \{0\}$. This is for example the case with the q-infiltration bialgebra on one letter at $q = 1$,

$$\mathcal{H}_{\uplus_\varphi} = (\mathbb{K}[x], \mathtt{conc}, 1_{x^*}, \Delta_{\uparrow_1}, \epsilon),$$

and, more generally, when q is not nilpotent[24].

We can also endow $\mathrm{End}(\mathbb{K}\langle Y\rangle)$, the $\mathbb{K}$-module of endomorphisms of $\mathbb{K}\langle Y\rangle$, with the *convolution* product defined by

$$\text{for all } f, g \in \mathrm{End}(\mathbb{K}\langle Y\rangle), \qquad f \star g = \mathtt{conc} \circ (f \otimes g) \circ \Delta_{\uplus_\varphi}, \tag{37}$$
$$\text{i.e., for all } P \in \mathbb{K}\langle Y\rangle, \qquad (f \star g)(P) = \sum_{u,v\in Y^*} \langle P \mid u \uplus_\varphi v\rangle f(u)g(v). \tag{38}$$

Then $\mathrm{End}(\mathbb{K}\langle Y\rangle)$ becomes a $\mathbb{K}$-associative algebra with unity (AAU), its unit being $e = 1_{\mathbb{K}\langle Y\rangle} \circ \epsilon$.

It is convenient to represent every $f \in \mathrm{End}(\mathbb{K}\langle Y\rangle)$ by its graph, a double series which reads

$$\Gamma(f) = \sum_{w\in Y^*} w \otimes f(w). \tag{39}$$

This representation is faithful and, by direct computation, one gets

$$\Gamma(f)\Gamma(g) = \Gamma(f \star g), \tag{40}$$

where the multiplication of double series is performed by the stuffle on the left and the concatenation on the right.

Definition 3. *Here t is a real parameter. Let us define define*

$$\mathcal{D}_Y := \Gamma(\mathrm{Id}_{\mathbb{K}\langle Y\rangle}) = \sum_{w\in Y^*} w \otimes w, \quad \mathcal{H}aus_Y := \log \mathcal{D}_Y, \quad \sigma_Y(t) := \exp(t\,\mathcal{H}aus_Y).$$

From now on, we assume that φ is associative, commutative and dualizable.

[23]This duality is separating; see [5].

[24]Recall that q is an element of the ring $\mathbb{K}$ (see example 2.2).

Lemma 7 (π_1 IS A PROJECTOR ON THE PRIMITIVE SERIES)**.** *The endomorphism π_1 is a projector, the image of which is exactly the space of primitive series,* $\mathrm{Prim}(\mathbb{K}\langle\langle Y\rangle\rangle)$.

Proof (sketch). The proof follows the lines of [41] with the difference that $\pi_1(w)$ might not be a polynomial and the operator defined in Lemma 2 is not a genuine co-product. The diagonal series $\mathcal{D}_Y$ (when considered as a series in $(\mathbb{K}\langle\langle Y\rangle\rangle)\langle\langle Y\rangle\rangle$, the coefficient ring, $\mathbb{K}\langle\langle Y\rangle\rangle$, being endowed with the $\uplus_\varphi$ product) is group-like in the sense of Lemma 2. Then, using

$$\log(\mathcal{D}_Y) = \sum_{w\in Y^*} w \otimes \pi_1(w)$$

(which can be established by summability of the family $(w\otimes\pi_1(w))_{w\in Y^*}$; but remember that the $\pi_1(w)$ are, in general, series[25]), one gets that $\pi_1(w)$ is a primitive series for all w. Now, from

$$\pi_1(S) = \sum_{w\in Y^*} \langle S \mid w\rangle \pi_1(w),$$

one has $\pi_1(S) \in \mathrm{Prim}(\mathbb{K}\langle\langle Y\rangle\rangle)$. Conversely, from Friedrichs' criterion, one gets $\pi_1(S) = S$ if $S \in \mathrm{Prim}(\mathbb{K}\langle\langle Y\rangle\rangle)$. □

In the remainder of the paper, we suppose that φ is moderate (and still dualizable, associative and commutative).

Definition 4 (PROJECTORS, [41])**.** *Let* $\mathrm{I}_+ : \mathbb{K}\langle Y\rangle \longrightarrow \mathbb{K}\langle Y\rangle$ *be the linear mapping defined by*

$$\mathrm{I}_+(1_{Y^*}) = 0, \quad \textit{and} \quad \textit{for all } w \in Y^+,\ \mathrm{I}_+(w) = w.$$

One defines[26]

$$\pi_1 := \log(e + \mathrm{I}_+) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \mathrm{I}_+{}^{\star n}, \quad \textit{where} \quad \mathrm{I}_+{}^{\star n} := \mathtt{conc}_{n-1} \circ \mathrm{I}_+{}^{\otimes n} \circ \Delta_{\uplus_\varphi}^{(n-1)}.$$

It follows immediately that

$$\exp(\pi_1) = e + \sum_{n\geq 1} \frac{1}{n!}\pi_1^{\star n} = \sum_{n\geq 0} \pi_n, \tag{41}$$

where $e = 1_{\mathbb{K}\langle Y\rangle} \circ \epsilon$ is the orthogonal complement of I_+ and neutral for the convolution product. The π_n so obtained is called the n-th Eulerian projector.

Lemma 8. *The endomorphism $\check{\pi}_1$ defined in* (26) *is the adjoint of π_1. One has*

$$\check{\pi}_1 = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \uplus_\varphi^{(n-1)} \circ \mathrm{I}_+{}^{\otimes n} \circ \Delta_{\mathtt{conc}}^{(n-1)}.$$

Proof. Immediate. □

[25] In greater detail, this equality amounts to checking the summability of the family

$$\Big(\frac{(-1)^{k-1}}{k} w \otimes \langle w \mid u_1 \uplus_\varphi \cdots \uplus_\varphi u_k\rangle u_1\cdots u_k\Big)_{\substack{w\in Y^*,\ k\geq 1\\ u_1,\ldots,u_k\in Y^+}}$$

(which is immediate) and rearranging the sums.

[26] The series below are summable because the family $(\mathrm{I}_+{}^{\star n})_{n\geq 0}$ is locally nilpotent (see [14] for complete proofs). Note that this definition gives the same result as the computation of the adjoint of $\check{\pi}_1$ given in (27).

Proposition 2. (GRAPH OF π_1, VALUES AND ITS EXPONENTIAL AS RESOLUTION OF UNITY)**.**

(1) *For all Y and φ (moderate, associative, commutative and dualizable), one has*

$$\log \mathcal{D}_Y = \sum_{w \in Y^+} w \otimes \pi_1(w) = \sum_{w \in Y^+} \check{\pi}_1(w) \otimes w.$$

(2) *Let $P \in \mathbb{K}\langle Y\rangle$ be a primitive polynomial, for $\Delta_{\sqcup\!\sqcup_\varphi}$. Then*

$$\pi_1(P) = P, \quad \textit{for all } k, n \in \mathbb{N}_+, \quad \pi_n(P^k) = \delta_{k,n} P^k.$$

(3) *One has*

$$\mathrm{Id}_{\mathbb{K}\langle Y\rangle} = e + I_+ = \sum_{n \geq 0} \pi_n. \tag{42}$$

Equation (42) *is a resolution of identity with mutually orthogonal summands.*

(4) *We have*

$$\mathbb{K}_+\langle Y\rangle = \mathcal{P} \overset{\perp}{\oplus} \mathcal{I} = \mathcal{P} \oplus \Big(\bigoplus_{n \geq 2} \pi_n(\mathbb{K}\langle Y\rangle) \Big).$$

Proof. The only statement which cannot be proved through an isomorphism with the shuffle algebra is the first equality of the point (4). The fact that $\mathcal{P} \cap \mathcal{I} = \{0\}$ comes from Friedrichs' criterion, and $\mathcal{P} + \mathcal{I} = \mathbb{K}_+\langle Y\rangle$ is a consequence of the fact (seen again through any isomorphism with the shuffle algebra) that

$$(\mathcal{H}_{\sqcup\!\sqcup_\varphi})_+ = \mathrm{span}_{\mathbb{K}}\big(\bigcup_{n \geq 1} (P_1 \sqcup\!\sqcup_\varphi \cdots \sqcup\!\sqcup_\varphi P_n)_{P_i \in \mathrm{Prim}(\mathcal{H}_{\sqcup\!\sqcup_\varphi})} \big). \qquad \square$$

Remark 4. (1) The first equality of Proposition 2.(4), i.e.,

$$\mathbb{K}_+\langle Y\rangle = \mathcal{P} \overset{\perp}{\oplus} \mathcal{I},$$

is known as the theorem of Ree [40].

(2) The projector on $\mathcal{P}$ parallel to $\mathcal{I}$ is not in general in the descent algebra (see [19]). This proves that, although they are isomorphic, the spaces $\mathcal{I}$ and $\bigoplus_{n \geq 2} \pi_n(\mathbb{K}\langle Y\rangle)$ are, in general, not identical.

Proposition 2.(1) leads to the following corollary.

Corollary 1. *We have $\pi_1(1_{Y^*}) = \check{\pi}_1(1_{Y^*}) = 0$ and, for all $w \in Y^+$,*

$$\pi_1(w) = w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \langle w \mid u_1 \sqcup\!\sqcup_\varphi \cdots \sqcup\!\sqcup_\varphi u_k \rangle u_1 \ldots u_k,$$

$$\check{\pi}_1(w) = w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \langle w \mid u_1 \ldots u_k \rangle u_1 \sqcup\!\sqcup_\varphi \cdots \sqcup\!\sqcup_\varphi u_k.$$

In particular, $\pi_1(1_{Y^*}) = \check{\pi}_1(1_{Y^*}) = 0$, for any $y \in Y$, $\check{\pi}_1(y) = y$, and

$$\pi_1(y) = y + \sum_{l \geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1 \ldots x_l \in Y^*} \gamma^y_{x_1, \ldots, x_l} \, x_1 \ldots x_l.$$

Remark 5. We already knew that, as soon as φ is associative, $\check{\pi}_1(w)$ is a polynomial. Here, because φ is moderate, dualizable, and associative, $\pi_1(w)$ is also a polynomial, and because φ is commutative, it is primitive.

Proposition 3. *We have:*

(1) *The expression of $\sigma_Y(t)$ is given by*

$$\sigma_Y(t) = \sum_{n\geq 0} t^n \sum_{w\in Y^*} w \otimes \pi_n(w) = \sum_{n\geq 0} t^n \sum_{w\in Y^*} \check{\pi}_n(w) \otimes w,$$

where $\check{\pi}_n$ is the adjoint of π_n. These are given by $\pi_n(1_{Y^*}) = \check{\pi}_n(1_{Y^*}) = \delta_{0,n}$ and, for all $w \in Y^+$,

$$\pi_n(w) = \frac{1}{n!} \sum_{u_1,\ldots,u_n\in Y^+} \langle w \mid \check{\pi}_1(u_1) ⧢_\varphi \cdots ⧢_\varphi \check{\pi}_1(u_n)\rangle \pi_1(u_1)\ldots\pi_1(u_n),$$

$$\check{\pi}_n(w) = \frac{1}{n!} \sum_{u_1,\ldots,u_n\in Y^+} \langle w \mid \pi_1(u_1)\ldots\pi_1(u_n)\rangle \check{\pi}_1(u_1) ⧢_\varphi \cdots ⧢_\varphi \check{\pi}_1(u_n)).$$

(2) *For any $w \in Y^*$, we have*

$$\begin{aligned} w &= \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 ⧢_\varphi \cdots ⧢_\varphi u_k\rangle \pi_1(u_1)\ldots\pi_1(u_k) \\ &= \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1\ldots u_k\rangle \check{\pi}_1(u_1) ⧢_\varphi \cdots ⧢_\varphi \check{\pi}_1(u_k). \end{aligned}$$

In particular, for any $y_s \in Y$, we have $y_s = \check{\pi}_1(y_s)$ and

$$y_s = \sum_{k\geq 1} \frac{1}{k!} \sum_{y_{s_1},\ldots,y_{s_k}\in Y} \gamma^{y_s}_{y_{s_1},\ldots,y_{s_k}} \pi_1(y_{s_1})\ldots\pi_1(y_{s_k}).$$

Proof. Direct computation. □

Applying the tensor product[27] of isomorphisms of algebras[28] $\alpha\otimes\mathrm{Id}_Y$ to the diagonal series $\mathcal{D}_Y$, we obtain a group-like element, and then computing the logarithm of this element (or equivalently, applying $\alpha\otimes\mathrm{Id}_Y$ to $\mathcal{H}aus_Y$) we obtain $\mathcal{S}$ which is, by Lemma 3, primitive:

$$\mathcal{S} = \sum_{w\in Y^*} \alpha(w)\ \pi_1(w) = \sum_{w\in Y^*} \alpha\circ\check{\pi}_1(w)\ w. \tag{43}$$

Lemma 9. *For any $w \in Y^+$, one has $\pi_1(w) \in \mathrm{Prim}(\mathbb{K}\langle Y\rangle)$.*

Proof. Immediate from Lemma 7. □

A primitive projector, $\pi : \mathbb{K}\langle Y\rangle \longrightarrow \mathbb{K}\langle Y\rangle$, is defined in the same way as a Lie projector by the three following conditions:

$$\pi\circ\pi = \pi,\quad \pi(1_{Y^*}) = 0,\quad \pi(\mathbb{K}\langle Y\rangle) = \mathrm{Prim}(\mathbb{K}\langle Y\rangle) = \mathcal{P}. \tag{44}$$

For example, π_1 defined in Definition 4 (see also Proposition 2) is a primitive projector which will be used to construct bases of $\mathcal{P}$ and its enveloping algebra (see Theorem 5 below). Another example of a primitive projector is the orthogonal projector on $\mathcal{P}$ attached to the decomposition in Remark 4.

[27] Extended to series.

[28] In order to clarify the ideas at this point, the reader can also take the alphabet duplication isomorphism

$$\text{for all } \bar{y} \in \bar{Y},\ \bar{y} = \alpha(y),$$

and use $\{w\}_{w\in\bar{Y}^*}$ as a basis for $\mathbb{K}\langle\bar{Y}\rangle$.

Now, for the remainder of the paper, let $\mathcal{Y} = \{y_w\}_{w\in Y^+}$ (respectively $\mathcal{Y}_1 = \{y_x\}_{x\in Y}$) be a copy of Y^+ (respectively Y).

Let us then equip $\mathbb{K}\langle\mathcal{Y}\rangle$ and $\mathbb{K}\langle\mathcal{Y}_1\rangle$ with $\bullet$ (the concatenation so denoted to be distinguished from the concatenation within Y^+) and $\sqcup\!\sqcup$ (or equivalently by $\Delta_\bullet$ and $\Delta_{\sqcup\!\sqcup}$).

Thus, the Hopf algebras $(\mathbb{K}\langle\mathcal{Y}\rangle, \bullet, 1_{\mathcal{Y}^*}, \Delta_{\sqcup\!\sqcup}, \epsilon_{\mathcal{Y}^*})$ and $(\mathbb{K}\langle\mathcal{Y}_1\rangle, \bullet, 1_{\mathcal{Y}^*}, \Delta_{\sqcup\!\sqcup}, \epsilon_{\mathcal{Y}_1^*})$ are connected, $\mathbb{N}$-graded, non-commutative and co-commutative bialgebras, and hence enveloping bialgebras (in fact, they are free algebras but specially indexed to match our purpose).

Now we can state the following

Theorem 4 (NEW LETTERS AS IMAGES)**.** *Let $\pi : \mathbb{K}\langle Y\rangle \longrightarrow \mathbb{K}\langle Y\rangle$ be a primitive projector. Let ψ_π be the* conc*-morphism defined by*

$$\begin{array}{rcl}\psi_\pi : \mathbb{K}\langle\mathcal{Y}\rangle & \longrightarrow & \mathbb{K}\langle Y\rangle,\\ y_w & \longmapsto & \psi_\pi(y_w) = \pi(w).\end{array}$$

Then ψ_π is surjective and a Hopf morphism.

Moreover, $\ker\psi_\pi = \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, where $\mathcal{J}_1$ and $\mathcal{J}_2$ are the two-sided ideals of $\mathbb{K}\langle\mathcal{Y}\rangle$ generated by

$$S_1 = \{y_u - y_{\pi(u)}\}_{u\in Y^+} \text{ and } S_2 = \{y_u \bullet y_v - y_v \bullet y_u - y_{[\pi(u),\pi(v)]}\}_{u,v\in Y^+},$$

respectively, where the indexing of the alphabet has been extended by linearity to polynomials, i.e.,

$$y_P := \sum_{w\in Y^+} \langle P \mid w\rangle\, y_w.$$

Proof. The fact that ψ_π is surjective is due to $\pi(\mathbb{K}\langle Y\rangle) = \mathcal{P}$ and to the fact that any enveloping algebra (here $\mathcal{H}_{\sqcup\!\!\sqcup_\varphi}$) is generated by its primitive elements. The fact that ψ_π is a Hopf morphism is due to a general property of enveloping algebras: *if a morphism of AAU between two enveloping algebras sends the primitive elements of the first to primitive elements of the second, then it is a Hopf morphism.*

Let now $(p_i)_{i\in J}$ be an ordered (J is endowed with a total ordering $\prec_J$) basis[29] of $\mathcal{P} = \mathrm{Prim}(\mathbb{K}\langle Y\rangle)$, and let us recall that $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ denotes the two sided ideal generated by the elements $\mathcal{J}_i$ (itself generated by $S_i,\ i = 1, 2$).

First, we observe that the elements of $S_1 \cup S_2$ are in the kernel of ψ_{π_1}, and then $\mathcal{J} \subset \ker\psi_{\pi_1}$.

On the other hand, for $u_1, u_2, \ldots, u_n \in Y^+$, one has

$$y_{u_1} \bullet y_{u_2} \bullet \cdots \bullet y_{u_n} \equiv y_{\pi(u_1)} \bullet y_{\pi(u_2)} \bullet \cdots \bullet y_{\pi(u_n)} \mod \mathcal{J} \tag{45}$$

(in fact they are even equivalent $\mathrm{mod}\ \mathcal{J}_1$), which amounts to saying that $\mathbb{K}\langle\mathcal{Y}\rangle = \mathcal{J} + {<}\mathcal{P}{>}$, where ${<}\mathcal{P}{>}$ is the space "generated by $\mathcal{P}$", in fact, generated by

$$\bigsqcup_{n\geq 0} \{y_{p_{i_1}} \bullet \cdots \bullet y_{p_{i_n}}\}_{i_j\in J}.$$

Now, by recurrence over the number of inversions, one can show, using S_2, that

$$y_{p_{i_1}} \bullet \cdots \bullet y_{p_{i_n}} \equiv y_{p_{\sigma(i_1)}} \bullet \cdots \bullet y_{p_{\sigma(i_n)}} \mod \mathcal{J}, \tag{46}$$

where $\sigma \in \mathfrak{S}_n$ is such that $\sigma(i_1) \succ_J \sigma(i_2) \succ_J \cdots \succ_J \sigma(i_n)$ (large order reordering).

[29]With the properties of φ here, the bialgebra $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\!\sqcup_\varphi}, \epsilon)$ is isomorphic to $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$ in which the module of primitive elements is free, thus $\mathcal{P} = \mathrm{Prim}(\mathbb{K}\langle Y\rangle)$ is free.

Let $\mathcal{C}$ be the space generated by the elements

$$\{y_{p_{j_1}} \bullet \cdots \bullet y_{p_{j_n}}\}_{\substack{j_1 \succ_J j_2 \succ_J \cdots \succ_J j_n \\ n \geq 0}} . \tag{47}$$

By (45) and (46), we get $\mathcal{J} + \mathcal{C} = \mathbb{K}\langle \mathcal{Y} \rangle$.

Now, due to the PBW theorem, the family of images

$$\Big(\Phi_{\pi_1}(y_{p_{j_1}} \bullet y_{p_{j_2}} \bullet \cdots \bullet y_{p_{j_n}})\Big)_{\substack{j_1 \succ_J j_2 \succ_J \cdots j_n \\ n \geq 0}} \tag{48}$$

is a basis of $\mathbb{K}\langle Y \rangle$, which proves that $\Phi_{\pi_1}|_{\mathcal{C}}\colon \mathcal{C} \to \mathbb{K}\langle Y \rangle$ is an isomorphism and completely proves the claim. □

We now suppose that the alphabet Y is totally ordered.

Definition 5. (1) *Let $\{\Pi_l\}_{l \in \mathcal{L}yn\, Y}$ and $\{\Pi_w\}_{w \in Y^*}$ be the families of elements of $\mathcal{P}$ and $\mathbb{K}\langle Y \rangle$, respectively, obtained as follows:*

$$\begin{aligned} \Pi_{y_k} &= \pi_1(y_k), && \text{for } k \geq 1, \\ \Pi_l &= [\Pi_s, \Pi_r], && \text{for } l \in \mathcal{L}yn\, X, \text{ standard factorization of } l = (s, r), \\ \Pi_w &= \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k}, && \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 \succ_{lex} \cdots \succ_{lex} l_k, l_1 \ldots, l_k \in \mathcal{L}yn\, Y. \end{aligned}$$

(2) *Let $\{\Sigma_w\}_{w \in Y^*}$ be the family of the φ-deformed quasi-shuffle algebra obtained by duality with $\{\Pi_w\}_{w \in Y^*}$:*

$$\text{for all } u, v \in Y^*, \qquad \langle \Sigma_v \mid \Pi_u \rangle = \delta_{u,v}.$$

A priori, the $\{\Sigma_w\}_{w \in Y^*}$ could be series. We prove first that, in this context, they are polynomials.

Proposition 4 (Adjoint of ϕ_{π_1})**.** *Let ϕ_{π_1}, be the* conc*-endomorphism of algebras defined on the letters as follows:*

$$\begin{aligned} \phi_{\pi_1} : \mathbb{K}\langle Y \rangle &\longrightarrow \mathbb{K}\langle Y \rangle, \\ y_k &\longmapsto \phi_{\pi_1}(y_k) = \pi_1(y_k). \end{aligned}$$

Then ϕ_{π_1} is an automorphism with the following properties:

(1) *This automorphism is such that, for every $l \in \mathcal{L}yn\, Y$,*

$$\phi_{\pi_1}(P_l) = \Pi_l,$$

where P_l are the polynomials calculated with the mechanism of Definition 5, setting $\varphi \equiv 0$ (or, equivalently, by (8) with $q = 0$), i.e., within the shuffle algebra $(\mathbb{K}\langle Y \rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$.

(2) *This automorphism has an adjoint $\phi_{\pi_1}^{\vee}$ within $\mathbb{K}\langle Y \rangle$ which reads, on the words $w \in Y^*$,*

$$\phi_{\pi_1}^{\vee}(w) = \sum_{k \geq 0} \sum_{y_{i_1} \cdots y_{i_k} \in Y} \langle w \mid \pi_1(y_{i_1}) \cdots \pi_1(y_{i_k}) \rangle \; y_{i_1} y_{i_2} \cdots y_{i_k}.$$

(3) *In the style of Definition 4, one has*

$$\phi_{\pi_1} = e + \sum_{k\geq 1} \mathtt{conc}^{(k-1)} \circ (\pi_1 \circ I_1)^{\otimes k} \circ \Delta_{\mathtt{conc}}^{(k-1)},$$
$$\phi_{\pi_1}^{\vee} = e + \sum_{k\geq 1} \mathtt{conc}^{(k-1)} \circ (I_1 \circ \check{\pi}_1)^{\otimes k} \circ \Delta_{\mathtt{conc}}^{(k-1)},$$

where I_1 is the projector on $\mathbb{K}Y$ parallel to $\bigoplus_{n\neq 1}(\mathbb{K}\langle Y\rangle)_n$.

(4) *For all $w \in Y^*$, $\Sigma_w = (\phi_{\pi_1}^{\vee})^{-1}(S_w)$.*

Proof (sketch). It was proved in Theorem 2 that the endomorphism ϕ_{π_1} is an isomorphism. The recursions used to construct Π_l and P_l prove that $\phi_{\pi_1}(P_l) = \Pi_l$, and then $\phi_{\pi_w}(P_l) = \Pi_w$ for every word w. Now the expression of ϕ_{π_1} is a direct consequence of the definition of ϕ_{π_1}. This implies at once the expression of $\phi_{\pi_1}^{\vee}$ and the fact that $\phi_{\pi_1}^{\vee} \in \mathrm{End}(\mathbb{K}\langle Y\rangle)$. The last equality comes from the following

$$\delta_{u,v} = \langle \Pi_u \mid \Sigma_v\rangle = \langle \phi_{\pi_1}(P_u) \mid \Sigma_v\rangle = \langle P_u \mid \phi_{\pi_1}^{\vee}(\Sigma_v)\rangle,$$

which shows that, for all $w \in Y^*$, $\phi_{\pi_1}^{\vee}(\Sigma_w) = S_w$ and the claim follows. $\square$

We can now state the following result.

Theorem 5. (1) *The family $\{\Pi_l\}_{l\in\mathcal{L}yn\,Y}$ forms a basis of $\mathcal{P}$.*

(2) *The family $\{\Pi_w\}_{w\in Y^*}$ is a linear basis of $\mathbb{K}\langle Y\rangle$.*

(3) *The family $\{\Sigma_w\}_{w\in Y^*}$ is a linear basis of the φ-shuffle algebra.*

(4) *The family $\{\Sigma_l\}_{l\in\mathcal{L}yn\,Y}$ forms a pure transcendence basis of $(\mathbb{K}\langle Y\rangle, \uplus_{\varphi}, 1_{Y^*})$.*

The first terms of these families, for the q-stuffle (see (8) and (10)) can be found in [3].

3.3. **Local coordinates by φ-extended Schützenberger factorization.** We have observed very early (φ needs only to be associative) that the set of group-like series (for $\Delta_{\uplus_{\varphi}}$) forms a (infinite-dimensional Lie) group (see Lemmas 3 and 4), its Lie algebra is the (Lie) algebra of Lie series, and we have a nice log-exp correspondence (see Lemma 3). We will see in this paragraph that, when φ possesses all the "good" properties (moderate, dualizable, associative and commutative), we have an analogue of the Wei–Norman theorem [2, 43, 44] which gives a system of local coordinates for every finite-dimensional (real or complex) Lie group. Let us recall it here.

Theorem 6 ([2, 43, 44])**.** *Given a (finite-dimensional) Lie group G (real $\mathbf{k} = \mathbb{R}$ or complex $\mathbf{k} = \mathbb{C}$), its Lie algebra $\mathfrak{g}$, and a basis $B = (b_i)_{1\leq i\leq n}$ of $\mathfrak{g}$, there exists a neighbourhood W of 1_G (in G) and n local coordinate analytic functions*

$$W \to \mathbf{k},\ (f_i)_{1\leq i\leq n}$$

such that, for all $g \in W$, we have

$$g = \prod_{1\leq i\leq n}^{\rightarrow} e^{t_i(g)b_i} = e^{t_1(g)b_1}e^{t_2(g)b_2}\ldots e^{t_n(g)b_n}.$$

Now, we have seen that, if φ is moderate, dualizable, associative and commutative,

$$\mathcal{H}_{\uplus_{\varphi}} = (\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_{\varphi}}, \epsilon) \tag{49}$$

is isomorphic to the shuffle bialgebra algebra $(\mathbb{K}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \epsilon)$, therefore one can construct bases $\{\Pi_w\}_{w\in Y^*}$; $\{\Sigma_w\}_{w\in Y^*}$ of $\mathbb{K}\langle Y\rangle$ with the following properties:

(1) the restricted family $\{\Pi_l\}_{l\in\mathcal{L}yn\,Y}$ is a basis of $\mathcal{P}=\mathrm{Prim}(\mathbb{K}\langle Y\rangle)$;
(2) the whole basis is constructed by decreasing concatenation (see Definition 5) and hence of type PBW;
(3) they are in duality $\langle \Pi_u \mid \Sigma_v\rangle = \delta_{u,v}$;
(4) due to these three properties, we have

$$\Sigma_w = \frac{\Sigma_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, \Sigma_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1!\cdots i_k!}, \qquad \text{for } w = l_1^{i_1}\ldots l_k^{i_k}. \tag{50}$$

Now, within the algebra of double series (whose support is $\mathbb{K}^{Y^*\otimes Y^*}$) endowed with the law $\sqcup\!\sqcup_\varphi \hat{\otimes}\mathtt{conc}$, M.-P. Schützenberger (see [41]) gave the beautiful formula

$$\sum_{w\in Y^*} w\otimes w = \prod_{l\in\mathcal{L}yn\,Y}^{\searrow} e^{\Sigma_l \hat{\otimes} P_l}, \tag{51}$$

which can be used to provide a system of local coordinates on the *Hausdorff group*, i.e., the group of series in $\mathbb{K}\langle\langle Y\rangle\rangle$ which are group-like for $\Delta_{\sqcup\!\sqcup_\varphi}$. Indeed, due to the fact that for a group-like S, $(S\hat{\otimes}\mathrm{Id})$ is compatible with the law of the double algebra, we get[30], applying the operator $(S\hat{\otimes}\mathrm{Id})$ to (51),

$$S = (S\hat{\otimes}\mathrm{Id})(\sum_{w\in Y^*} w\hat{\otimes}w) = \prod_{l\in\mathcal{L}yn\,Y}^{\searrow} e^{\langle S|\Sigma_l\rangle\; P_l}, \tag{52}$$

which is the perfect analogue of the theorem of Wei and Norman for the Hausdorff group (group of group-like series).

4. Conclusion

In this paper, we have systematically studied the deformations of the shuffle product by addition of a superposition term. Fortunately, this study provides necessary and sufficient conditions for the objects (antipode, Ree ideal, bases in duality) and operators (infinite convolutional series, primitive projectors) to exist together with their consequences. We have established a local system of coordinates for the (infinite-dimensional) Lie group of group-like series. This system is the perfect analogue of the well-known theorem of Wei and Norman which holds for every finite-dimensional Lie group.

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