GENERAL RELATIVITY AND SPINORS

Jean Claude Dutailly

GENERAL RELATIVITY AND SPINORS

Jean Claude Dutailly

15th of November 2016

Abstract

This paper exposes, in a comprehensive and consistent way, the Geometry of General Relativity through fiber bundles and tetrads. A review of the concept of motion in Galilean Geometry leads, for its extension to the relativist context, to a representation based on Clifford Algebras. It provides a strong frame-work, and many tools which enable to deal easily with the most general problems, including the extension of the concepts of deformable and rigid solids in RG, at any scale.

It is then natural to represent the moments of material bodies, translational and rotational, by spinors. The results hold at any scale, and their implementation for elementary particles gives the known results with anti-particles, chirality and spin.

This representation, without any exotic assumption, is the gate to a model which encompasses RG and the Gauge Theories, and so to a reconciliation between RG and Particle Theories.

Contents

General Relativity (GR) consists of two theories : one about the Geometry of the Universe, and another about Gravitation. We will deal with the first one in this paper. By Geometry we mean how to represent space and time, and from there the motion of material bodies. GR is reputed to be a difficult theory. Fortunately Mathematics have made huge progresses since 1920, and provide now the tools to deal efficiently with differential geometry. Moreover, quitting the usual and comfortable frame-work of orthonormal frames helps to better understand the full extent of the revision of the concepts implied by Relativity. To extend the concept of motion, both transversal and rotational, to the relativist context requires to give up the schematics of Special Relativity (SR) and the Poincaré's group. The right mathematical representation is through Clifford Algebra, which is at the root of the Spin group, and gives a natural interpretation to the spin. With this new representation of the motion, we can revisit the Kinematics, that is the link between motion, a geometric concept, and forces. The kinematic characteristics of material bodies are then represented through spinors, and we retrieve, for elementary particles, the distinction between particles and anti-particles. The spinors, introduced here through RG, are similar to the spinors of the Standard Model, and are a component of the representation of the state of particles. This topic requires the introduction of Force fields and connections, and is outside the scope of this paper, but is treated in my book "Theoretical Physics".

The first Part is dedicated to the Geometry of the Univers. With 5 assumptions, based on common facts and well known scientific phenomena, we build a comprehensive and simple frame-work, which is in full agreement with the usual GR, but uses more modern mathematical tools.

The second Part is dedicated to the concept of motion. Its representation in Galilean Geometry is actually sophisticated, fully developed for solids, it has been extended to deformable solids and fluids. As it seems that, at the most elementary scale, material bodies have a motion which is also rotational, in the relativist context it is necessary to use a more robust representation, which is provided by Clifford Algebras. Then the motion, translational and rotational, of a material body is represented by an element of the Clifford Algebra, and it can be easily extended to deformable solids, which can be useful in Astro-Physics.

The third Part is dedicated to spinors. Kinematics studies the link between motion and forces acting on material bodies. The kinematic characteristics of material bodies are the mass and the rotational tensor. In the relativist frame-work momenta are represented by Spinors which are vectorial quantities. But the kinematic characteristics of material bodies are defined by 4 scalar parameters, and for elementary particles we retrieve, in a natural way, the distinction between particles and anti-particles.

This paper uses some mathematical definitions or theorems which can be found in more details in my book "Advanced Mathematics for Theoretical Physics"). They are refered to as "Maths.XX".

Part I
THE GEOMETRY OF GENERAL RELATIVITY

Almost all, if not all, measures rely eventually on measures of lengths and times. The concepts of space and time are at the foundation of theories about the geometry of the physical universe, meaning of the container in which live the objects of physics. The issue here is not a model of the Universe, seen in its totality, which is the topic of Cosmology, but a model which tells us how to measure lengths and times, and how to compare measures done by different observers. Such a model is a prerequisite to any physical theory. There are several Geometries used in Physics : Galilean Geometry, Special Relativity (SR) and General Relativity (GR).

1 MANIFOLD STRUCTURE

1.1 The Universe has the structure of a manifold

The first question is how do we measure the location of an event ?

In almost all Physics books the answer will go straight to an orthonormal frame, or in GR to a map with some coordinates ξ_α, often with additional provisions for "inertial frames", before a complicated discourse about light, and quite often trains for the Relativist picture. Actually all these narratives, simply, do not respect the facts.

At small distances it is possible to measure lengths by surveying, and indeed the scientists who established the meter in 1792 based their work on a strict survey along 15 kms. Then it is possible to use an orthonormal frame. But even at small scale, topographers use a set of 3 angles with respect to fixed directions given by staffs, or far enough objects, points in the landscape, or distant stars, combined with one measure of distance. The latter is measured usually by the delay for a signal emitted to rebound on the surface on a distant object. There are small, clever, devices which do that with ultrasound, radars use electromagnetic fields. The speed of the propagation of the signal is taken conventionally fixed and constant. It is assumed to have been measured at small scale, and the results are then extended for larger distances. For not too far away celestial bodies, the distances can be measured using the angles observed at different locations (the parallaxes), the knowledge of the length of the basis of the triangle and some trigonometry. Further away one uses the measure of the luminosity of "standard candles", and eventually the red shift of some specific light waves. This is the meaning of the "cosmic distance ladder" used in Astrophysics. So, measures of spatial location rely essentially on measures of angles, and one measure of distance according to precise protocols based on conventions about the relation between the distance and the phenomenon which is observed. The key is that, on the scale where two methods are applicable, the measures of distances are consistent.

For the temporal location one uses the coincidence with any agreed upon event. For millennia men used the position of celestial bodies for this purpose. Say "See you at Stonehenge at the spring's equinox" and you will be understood. Of course one can use a clock, but the purpose of a clock is to measure elapsed time, so one needs a clock and a starting point, which are agreed upon, to locate an event in time. An observer can locate in time any event which occurs at his place, that is events that he can see directly and, beyond that, the observer accounts for a delay due to the transmission of his perception of the event, based on a convention for the speed of the signal. This speed can be measured itself, for not too far away events, either by a direct communication with a distant observer, or by bouncing a signal on a object at the distant location. But farther away the speed of transmission is set conventionally. Actually the physical support of the signal does not matter much as long as it is efficient, and for the measure of the temporal location, can rely on any convention. There is no need for a physical assumption as the constancy of the speed of light.

The measures of location, in time and space, are so based on conventions. This is not an issue, as long as the protocols are precise, and the measures consistent : the purpose of the measures is to be able to locate efficiently an event. One does that with 3 spatial coordinates, and 1 coordinate for the time, organized in charts combining in a consistent way measures done according to different, agreed upon procedures. The key point is that the charts are compatible : when it is possible to proceed to the measures for the same event by different procedures, there is a way to go from one measure to another. And this enables to extend the range of the chart by applying conventions, such as in the cosmic ladder.

These procedures describe exactly a mathematical object : a manifold. A set of charts covering a domain constitutes an atlas. There are mathematical functions, transition maps, which relate the coordinates of the same point in different charts. A collection of compatible atlas over a set M defines the structure of a manifold. The coordinates represent nothing more than the measures which can be done, and the knowledge of the protocols is sufficient.

This leads to our first proposition :

Proposition 1 *The Universe can be represented by a four dimensional real manifold M*

The charts define over M a topology, deduced from the vector space. The manifold is differentiable (resp. smooth) if the transition maps are differentiable (resp.smooth).

In Galilean Geometry the manifold is the product of $\mathbb{R}$ with a 3 dimensional affine space, and in SR this is a 4 dimensional affine space (affine spaces have a manifold structure).

We will limit ourselves to an area Ω of the universe, which can be large, where there is no singularity, so that one can assume that one chart suffices. We will represent such a chart by a map :

$\varphi_M : \mathbb{R}^4 \to \Omega :: \varphi_M\left(\xi^0, \xi^1, \xi^2, \xi^3\right) = m$

which is assumed to be bijective and smooth, where $\xi = \left(\xi^0, \xi^1, \xi^2, \xi^3\right)$ are the coordinates of m in the chart φ_M.

We will assume that Ω is a relatively compact open in M, then the manifold structure on M is the same as on Ω, and Ω is bounded.

A change of chart is represented by a bijective smooth map (the transition map) :

$\chi : \mathbb{R}^4 \to \mathbb{R}^4 :: \eta^\alpha = \chi^\alpha\left(\xi^0, \xi^1, \xi^2, \xi^3\right)$

such that the new map $\widetilde{\varphi}_M$ and the initial map φ_M locate the same point :

$\widetilde{\varphi}_M\left(\chi^\alpha\left(\xi^0, \xi^1, \xi^2, \xi^3\right), \alpha = 0, ..3\right) = \varphi_M\left(\xi^0, \xi^1, \xi^2, \xi^3\right)$

Notice that there is no algebraic structure on M : $am + bm'$ has no meaning. This is illuminating in GR, but still holds in SR or Galilean Geometry. There is a clear distinction between coordinates, which are scalars depending on the choice of a chart, and the point they locate on the manifold (affine space or not).

The idea that the Universe could be 4 dimensional is not new. The true revolution of Relativity has been to acknowledge that, if the physical universe is 4 dimensional, it becomes necessary to dissociate the abstract representation of the world, the picture given by a mathematical model, from the actual representation of the world as it can be seen through measures. And this dissociation goes through the introduction of a new object in Physics : the observer. Indeed, if the physical Universe is 4 dimensional, the location of a point is absolute : there is a unique material body, in space and time, which can occupy a location. Then, does that mean that past and future exist together ? To avoid the conundrum and all the paradoxes that it entails, the solution is to acknowledge that, if there is a unique reality, actually the reality which is scientifically accessible, because it enables experiments and measures, is specific : it depends on the observer. This does not mean that it would be wrong to represent the reality in its entirety, as it can be done with charts, frames or other abstract mathematical objects. They are necessary to give a consistent picture, and more bluntly, to give a picture that is accessible to our mind. But we cannot identify this abstract representation, common to everybody, with the world as it is. It is common to introduce subtle concepts such as location and velocity through a frame, which is evoked in passing, as if it was obvious, standing somewhere at the disposition of the public. There is nothing like this. I can build my frame, my charts, and from there conceive that it can be extended, and compared to what other Physicists have done. But comparison requires first dissociation.

1.2 The tangent vector space

Spatial locations rely heavily on the measures of angles with respect to fixed directions. At any point there is a set of spatial directions, corresponding to small translations in one of the coordinates. And the time direction is just the translation in time for an observer who is spatially immobile. There is the same construct in Mathematics.

Mathematically at any point of a manifold one can define a set which has the structure of a vector space, with the same dimension as M. The best way to see it is to differentiate the map φ_M with respect to the coordinates (this is close to the mathematical construct). To any vector $u \in \mathbb{R}^4$ is associated the vector $u_m = \sum_{\alpha=0}^3 u^\alpha \partial_\alpha \varphi_M\left(\xi^0, \xi^1, \xi^2, \xi^3\right)$ which is denoted $u_m = \sum_{\alpha=0}^3 u^\alpha \partial\xi_\alpha$.

The basis $(\partial\xi_\alpha)_{\alpha=0}^3$ associated to a chart, called a **holonomic basis**, depends on the chart, but the vector space at m denoted T_mM does not depend on the chart. With this vector space structure one can define a dual space T_mM^* and holonomic dual bases denoted $d\xi^\alpha$ with : $d\xi^\alpha\left(\partial\xi_\beta\right) = \delta^\alpha_\beta$, and any other tensorial structure (see Maths.16).

In the definition of the holonomic basis the tangent space is generated by small displacements along one coordinate, around a point m. So, physically, locally the manifold is close to an affine space with a chosen origin m, and locally GR and SR look the same. This is similar to what we see on Earth : locally it looks flat.

However there are essential distinctions between coordinates, used to measure the location of a point in a chart, and components, used to measure a vectorial quantity with respect to a basis. Points and vectors

are geometric objects, whose existence does not depend on the way they are measured. However a point on a manifold does not have an algebraic structure attached meanwhile a vector belongs to a vector space : one can combine vectors. Some physical properties of objects can be represented by vectors, others cannot, and the distinction comes from the fundamental assumptions of the theory. It is enshrined in the theory itself. From the construct of the tangent space one sees that any quantity defined as a derivative of another physical quantity with respect to the coordinates is vectorial.

The vector spaces T_mM depend on m, and there is no canonical (meaning independent of the choice of a specific tool) procedure to compare vectors belonging to the tangent spaces at two different points. These vectors u_m can be considered as a couple of a location m and a vector u, which can be defined in a holonomic basis or not, and all together they constitute the tangent bundle TM. Notably there is no physical mean to measure a change in the vectors of a holonomic basis with time : it would require to compare $\partial\xi_\alpha$ at two different locations $m, m' \in M$. But, because there are maps to go from the coordinates in a chart to the coordinates in another chart, there are maps which enable to compute the components of vectors in the holonomic bases of different charts, at the same point.

However because the manifolds are actually affine spaces, in SR and Galilean Geometry the tangent spaces at different points share the same structure (which is the underlying tangent vector space), and only in these cases they can be assimilated to $\mathbb{R}^4$. This is the origin of much confusion on the subject, but in the GR context the concepts are clearly differentiated.

A vector field on M is a map : $V : M \to TM :: V(m) = \sum_{\alpha=0}^{3} v^\alpha(m)\,\partial\xi_\alpha$ which associates to any point m a vector of the tangent space T_mM. The vector does not depend on the choice of a basis or a chart, so its components change in a change of chart as : $v^\alpha(m) \to \widetilde{v}^\alpha(m) = \sum_{\beta=0}^{3} [J(m)]^\alpha_\beta\, v^\beta(m)$ where $[J(m)] = \left[\frac{\partial\eta^\alpha}{\partial\xi^\beta}(m)\right]$ is a 4×4 matrix called the jacobian

Similarly a one form on M is a map $\varpi : M \to TM^* :: \varpi(m) = \sum_{\alpha=0}^{3} \varpi_\alpha(m)\, d\xi^\alpha$ and the components change as :

$\varpi_\alpha(m) \to \widetilde{\varpi}_\alpha(m) = \sum_{\beta=0}^{3} [K(m)]^\beta_\alpha\, \varpi_\beta(m)$ and $[K(m)] = [J(m)]^{-1}$

The sets of vector fields, denoted $\mathfrak{X}(TM)$, and of one forms, denoted $\mathfrak{X}(TM^*)$ or $\Lambda_1(M;\mathbb{R})$ are infinite dimensional vector spaces (with pointwise operations).

A **curve** on a manifold is a one dimensional submanifold : this is a geometric structure, and there is a vector space associated to each point of the curve, which is a one dimensional vector subspace of T_mM.

A **path** on a manifold is a map : $p : \mathbb{R} \to M :: m = p(\tau)$ where p is a differentiable map such that $p'(\tau) \neq 0$. Its image is a curve L_p, and p defines a bijection between $\mathbb{R}$ (or any interval of $\mathbb{R}$) and the curve (this is a chart of the curve), the curve is a 1 dimensional submanifold embedded in M. The same curve can be defined by different paths. The tangent is the map : $p'(t) : \mathbb{R} \to T_{p(t)}M :: \frac{dp}{d\tau} \in T_{p(\tau)}L_p$. In a change of parameter of the path : $\widetilde{\tau} = f(\tau)$ (which is a change of chart) for the same point : $m = \widetilde{p}(\widetilde{\tau}) = p(f(\tau))$ the new tangent vector is proportional to the previous one : $\frac{dm}{d\tau} = \frac{d\widetilde{p}}{d\widetilde{\tau}}\frac{d\widetilde{\tau}}{d\tau} \Leftrightarrow \frac{dm}{d\widetilde{\tau}} = \frac{1}{f'}\frac{dm}{d\tau}$

For any smooth vector field there is a collection of smooth paths (the **integrals** of the field) such that the tangent at any point of the curve is the vector field. There is a unique **integral line** which goes through a given point. The **flow** of a vector field V is the map : $\Phi_V : \mathbb{R}\times M \to M :: \Phi_V(\tau, a)$ such that $\Phi_V(., a) : \mathbb{R} \to M :: m = \Phi_V(\tau, a)$ is the integral path going through a and $\Phi_V(., a)$ is a local diffeomorphism :

$$\begin{array}{c} \forall\theta \in \mathbb{R} : \frac{\partial}{\partial\tau}\Phi_V(\tau, a)\,|_{\tau=\theta} = V(\Phi_V(\theta, a)) \\ \forall\tau, \tau' \in \mathbb{R} : \Phi_V(\tau + \tau', a) = \Phi_V(\tau, \Phi_V(\tau', a)) \\ \Phi_V(0, a) = a \\ \forall\tau \in \mathbb{R} : \Phi_V(-\tau, \Phi_V(\tau, a)) = a \end{array} \tag{1}$$

For a given vector field, the parameter τ is defined up to a constant, so *it is uniquely defined* with the condition $\Phi_V(0, a) = a$.

In general the flow is defined only for an interval of the parameter, but this restriction does not exist if Ω is relatively compact.

A map $f : C \to E$ from a curve can be extended to a map $F : \Omega \to E$. So any smooth path can be considered as the integral of some vector field (not uniquely defined), and it is convenient to express a path as the flow of a vector field.

1.3 Fundamental symmetry breakdown

The four coordinates are not equivalent : the measure of the time ξ^0 cannot be done with the same procedures as the other coordinates, and one cannot move along in time : one cannot survey time. This is the fundamental symmetry breakdown.

The time coordinate of an event can be measured, by conventional procedures which relate the time on the clock (whatever it is) of a given observer to the time at which a distant event has occurred. So we assume that *a given observer* can tell if two events A, B occur in his present time (they are simultaneous), and that the relation "two events are simultaneous" is a relation of equivalence between events. Then the observer can label each class of equivalence of events by the time of his clock. Which can be expressed by telling that for each observer, there is a function : $f_o : M \to \mathbb{R} :: f_o(m) = t$ which assigns a time t, *with respect to the clock of the observer*, at any point of the universe (or at least Ω). The points : $\Omega(t) = \{m = f_o(t), m \in \Omega\}$ correspond to the **present** of the observer. No assumption is made about the clock, and different clocks can be used, with the condition that, as for any chart, it is possible to convert the time given by a clock to the time given by another clock (both used by the same observer).

In Galilean Geometry instantaneous communication is possible, so it is possible to define a universal time, to which any observer can refer to locate his position, and the present does not depend on the observer. The manifold M can be assimilated to the product $\mathbb{R} \times \mathbb{R}^3$. The usual representation of material bodies moving in the same affine space is a bit misleading, actually one should say that this affine space $\mathbb{R}^3(t)$ changes continuously, in the same way, for everybody. Told this way, we see that Galilean Geometry relies on a huge assumption about the physical universe.

In Relativist Geometry instantaneous communication is impossible, so it is impossible to synchronize all the clocks. However a given observer can synchronize the clocks which correspond to his present, this is the meaning of the function f_o, whose practical realization does not matter here.

Whenever there is, on a manifold, a map such that f_o, with $f_o'(m) \neq 0$, it defines on M a foliation : there is a collection of hypersurfaces (3 dimensional submanifolds) $\Omega_3(t)$, and the vectors u of the tangent spaces on $\Omega_3(t)$ are such that $f_o'(m)u = 0$, meanwhile the vectors which are transversal to $\Omega_3(t)$ (corresponding to paths which cross the hypersurface only once) are such that $f_o'(m)u > 0$ for any path with t increasing. So there are two faces on $\Omega_3(t)$: one for the incoming paths, and the other one for the outgoing paths. The hypersurfaces $\Omega_3(t)$ are diffeomorphic : they can be deduced from each other by a differentiable bijection, which is the flow of a vector field. Conversely if there is such a foliation one can define a unique function f_o with these properties (Maths.1507[1]). The successions of present "spaces" for any observer is such a foliation, so our representation is consistent. And we state :

Proposition 2 *For any observer there is a function*

$$f_o : M \to \mathbb{R} :: f_o(m) = t \text{ with } f_o'(m) \neq 0 \tag{2}$$

which defines in any area Ω of the Universe a foliation by hypersurfaces

$$\Omega_3(t) = \{m = f_o(t), m \in \Omega\} \tag{3}$$

which represents the location of the events occurring at a given time t on his clock.

An observer can then define a chart of M, by taking the time on his clock, and the coordinates of a point x in the 3 dimensional hypersurfaces $\Omega_3(t)$: it would be some map : $\varphi : \mathbb{R} \times \Omega_3(0) \to M :: m = \varphi(t, x)$ however we need a way to build consistently these spatial coordinates, that is to relate $\varphi(t, x)$ to $\varphi(t', x)$.

2 TRAJECTORIES OF MATERIAL BODIES

The Universe is a container where physical objects live, and the manifold provides a way to measure a location. This is a 4 dimensional manifold which includes the time, but that does not mean that everything is frozen on the manifold : *the universe does not change, but its content changes.* As bodies move in the universe, their representation are paths on the manifold. And the fundamental symmetry breakdown gives a special meaning to the coordinate with respect to which the changes are measured. *Time is not only a parameter to locate an event, it is also a variable which defines the rates of change in the present of an observer.*

2.1 Material bodies and particles

The common definition of a material body in Physics is that of a set of material points which are related. A **material point** is assumed to have a location corresponding to a point of the manifold. According to the relations between material points of the same body we have rigid solids (the distance between two points is constant), deformable solids (the deformation tensor is locally given by the matrix of the transformation of a frame), fluids (the speed of material points are given by a vector field). These relations are expressed in

[1] This theorem, which has far reaching consequences, is new and its proof, quite technical is given in my Mathematics book.

phenomenological laws, they are essential in practical applications. The generalization to Relativity of the concept of solids, or material bodies which have a spatial extension, is an important issue that we address in a following part.

We will consider first in this part material bodies which have no internal structures, or whose internal structure can be neglected, that we will call **particles**. The only property that we will consider here for a particle is its location, given by a geometrical point in the universe. A particle then can be an electron, a nucleus, a molecule, or even a star system, according to the scale of the study.

2.2 World line and proper time

As required in any scientific theory a particle must be defined by its properties, and the first is that it occupies a precise location at any time. The successive locations of the material body define a curve and the particle travels on this curve according to a specific path called its **world line**. Any path can be defined by the flow of a vector such that the derivative with respect to the parameter is the tangent to the curve. The parameter called the **proper time** is then defined uniquely, up to the choice of an origin. The derivative with respect to the proper time is called the **velocity**. By definition *this is a vector,* defined at each point of the curve, and belonging to the tangent space to M. So the velocity has a definition which is independent of any basis.

Observers are assumed to have similarly a world line and a proper time (they have other properties, notably they define a frame).

To sum up :

Proposition 3 *Any particle or observer travels in the universe on a curve according to a specific path , $p : \mathbb{R} \to M :: m = p(\tau)$ called the world line, parametrized by the proper time τ, defined uniquely up to an origin. The derivative of the world line with respect to the proper time is a vector, the velocity, u. So that :*

$$\begin{array}{l} u(\theta) = \frac{dp}{d\tau}|_{\tau=\theta} \in T_{p(\theta)}M \\ p(\tau) = \Phi_u(\tau, a) \ \ with\ a = \Phi_u(0, a) = p(0) \end{array} \tag{4}$$

Observers are assumed to have clocks, that they use to measure their temporal location with respect to some starting point. The basic assumption is the following :

Proposition 4 *For any observer his proper time is the time on his clock.*

So the proper time of a particle can be seen formally as the time on the clock of an observer who would be attached to the particle.

The observer uses the time on his clock to locate temporally any event : this is the purpose of the function f_o and of the foliation $\Omega_3(t)$. The curve on which any particle travels meets only once each hypersurface $\Omega_3(t)$: it is seen only once. This happens at a time t :

$f_o(p(\tau)) = t = f_o(\Phi_u(\tau, a))$

So there is some relation between t and the proper time τ of any particle. It is specific, both to the observer and to the particle. It is bijective and both increases simultaneously, so that : $\frac{d\tau}{dt} > 0$.

The travel of the particle on the curve can be represented by the time of an observer. We will call then this path a **trajectory**.

2.3 Standard Chart

With this assumption each observer can build a chart. On some hypersurface $\Omega_3(0)$ representing the space of the observer at a time $t = 0$ he chooses a chart identifying each point x of $\Omega_3(0)$ by 3 coordinates ξ^1, ξ^2, ξ^3, using the methods to measure spatial locations described previously, and $m = \varphi_o\left(t, \xi^1, \xi^2, \xi^3\right)$ is a chart of the area $\Omega \subset M$ spanned by the $\Omega_3(t)$. Each point $m(t) = \varphi_o\left(t, \xi^1, \xi^2, \xi^3\right)$ corresponds to the trajectory of a material body or of an observer which would stand still at x. We will call this kind of chart a **standard chart** for the observer. It relies on the choice of a chart of $\Omega_3(0)$, that is a set of procedures to measure a spatial location (so several compatible charts can be used) and a clock or any procedure to identify a time. A standard chart is specific to each observer and is essentially fixed.

An observer is not necessarily spatially immobile. But to know his new location he has to proceed to measures which are similar to setting up a chart, with similar protocols, so actually this is a change of chart and it is managed by the relations between old and new coordinates. In order to keep it simple we assume that the standard chart is a chart for an observer who is spatially immobile, and the motion of an observer is a change of observer.

Even if two observers can compare the measures of spatial locations, actually so far we cannot go further : the hypersurfaces $\Omega_3(t)$ are defined by the function f_o and, a priori, are specific to each observer. Moreover

a clock measures the elapsed time. It seems legitimate to assume that, in the procedure, one chooses clocks which run at the same rate. But, to do this, one needs some way to compare this rate, that is a scalar measure of the velocity $\frac{d}{d\tau}p_o(\tau)$. But, as velocities are 4 dimensional vectors, one needs a special scalar product.

The essential feature of proper time is more striking when one considers particles. They should be located at some point of M : they are not spread over all their world line, their location varies along their world line with respect to the parameter τ, their proper time. So their location is definite, but with respect to a parameter τ which is specific to each particle : there is a priori no way to tell where, at some time, are all the particles ! An observer can locate a particle which is in his "present", and so identify specific particles, but this is specific to each observer.

3 METRIC

3.1 Causal structure

The Principle of Causality states that there is some order relation between events. This relation is not total : some events are not related. In the Relativist Geometry it can be stated as a relation between locations in the Universe : a binary relation between two points (A, B).

The function f_o of an observer provides such a relation : it suffices to compare $f_o(A), f_o(B) : B$ follows A if $f_o(B) > f_o(A)$ and is simultaneous to A if $f_o(B) = f_o(A)$. For a relation between points it is natural to look at curves joining the points. For a path $p \in C_1([0,1]; M)$ such that $p(0) = A, p(1) = B$ one can compute $f_o(p(\tau))$. If the function is increasing then one can say that B follows A, and this is equivalent to $f'_o(p(\tau))\frac{dp}{d\tau} > 0$. And we can say that the vector $u = \frac{dp}{d\tau} \in T_{p(\tau)}M$ is future oriented for the observer if $f'_o(p(\tau))u > 0$. We have the same conclusion for any vector at a point $m \in M$ which belongs to one of the hypersurfaces $\Omega_3(t)$ of an observer : if it is transversal it can be oriented towards the future by $f'_o(m)u$, and any curve can be similarly oriented at any point, but the orientation is not necessarily constant.

The classification of the curves which have a constant orientation is a topic of algebraic geometry, but here there is a more interesting issue : the Principle of Causality should be met for any observer. We can study this issue by looking at vectors u at a given point m. The derivative $f'_o(m)$ is just a covector $\lambda \in T_mM^*$. The function : $B : T_mM^* \times T_mM \to \mathbb{R} :: B(\lambda, u) = \lambda(u)$ is continuous in both variables (T_mM^*, T_mM are finite dimensional vector spaces and have a definite topology). For a given λ if $\lambda(u) > 0$ then $\lambda(-u) < 0$, and we have a partition of T_mM in 3 connected components : future oriented vectors $\lambda(u) > 0$, past oriented vectors $\lambda(u) < 0$, null vectors $\lambda(u) = 0$. This partition of T_mM should hold for any observer. The implementation of the Principle of Causality in Relativist Geometry leads to state that, at each point m, there is a set C_+ of vectors future oriented for all observers, and that vectors which do not belong to C_+ are not future oriented for any observer. The opposite set C_- is the set of past oriented vectors. C_+ is a convex open half cone : if for an observer u, v are future oriented, then $\alpha u + (1-\alpha)v$ for $\alpha \in]0,1[$ is future oriented.

For any observer, there is a hyperplan $H_o(m)$ passing by m, which separates C_+, C_- : take $f'_o(m) \in T_mM^*$

$\forall u \in C_-, v \in C_+ : f'_o(m)(u) < 0 < f'_o(m)(v)$

$\Rightarrow \sup_{u\in C_-} f'_o(m)(u) \leq \inf_{v\in C_+} f'_o(m)(v)$

Moreover this hyperplan is tangent to the hypersurface $\Omega_3(t)$ passing by m.

So any observer can choose a basis of T_mM consisting of 3 vectors $(\varepsilon_i)_{i=1}^3$ belonging to $H_o(m)$, that is his "space". Then $f'_o(m)(\varepsilon_i) = 0, i = 1,2,3$ because the vectors are tangent to $\Omega_3(t)$. With any other vector ε_0 as 4th vector of his basis,

$f'_o(m)(u) = f'_o(m)\left(\sum_{i=0}^3 u^i\varepsilon_i\right) = u^0 f'_o(m)(\varepsilon_0)$

To have a consistent result for this function, that is to be able to distinguish a past from a future oriented vector, the observer must choose $\varepsilon_0 \in C_+$, and this choice is always possible by taking his velocity as ε_0.

And this choice can be done in a consistent manner for any observer. Any "physical" basis chosen by an observer is comprised of 3 spatial vectors, which do not belong to C_+ and the 4th vector belong to C_-. This holds for the holonomic basis induced by a standard chart.

The function $B(\lambda, u)$ is defined all over M, does not depend on the observer, it is a bilinear map, so this is a tensor field $B \in TM^* \otimes TM$. In any basis it is expressed at a point by a 4×4 matrix, and this matrix can be considered as the matrix of a bilinear form, from which a symmetric bilinear form can be computed, and so a metric on TM. However we see that there are vectors such that $B(u,u) = 0$. This metric cannot be definite positive.

A manifold is usually not isotropic : not all directions are equivalent. The fundamental symmetry breakdown introduces an anisotropy, specific to each observer, and we see that actually it goes deeper, because it is common to all observers and not all vectors representing a translation in time are equivalent : C_+ is a half cone and not a half space.

So the Principle of Causality leads to assume that there is an additional structure in the Universe. This causal structure is usually defined through the propagation of light : a region B is temporally dependant from a region A if any point of B can be reached from A by a future oriented curve. This is the domain of nice studies (see Wald), but there is no need to involve the light, the causal structure exists at the level of the tangent bundle, its definition does not need the existence of a metric, but clearly leads to assume that there is a metric and that this metric is not definite positive.

3.2 Lorentz metric

A scalar product is defined by a bilinear symmetric form g acting on vectors of the tangent space, at each point of the manifold, thus by a tensor field called a **metric**. In a holonomic basis g reads :

$$g(m) = \sum_{\alpha\beta=0}^{3} g_{\alpha\beta}(m)\, d\xi^{\alpha} \otimes d\xi^{\beta} \text{ with } g_{\alpha\beta} = g_{\beta\alpha} \tag{5}$$

The matrix of g is symmetric and invertible, if we assume that the scalar product is not degenerate. It is diagonalizable, and its eigen values are real. One wants to account for the symmetry breakdown and the causal structure, so these eigen values cannot have all the same sign (a direction is privileged). One knows that the hypersurface $\Omega_3(t)$ are Riemannian : there is a definite positive scalar product (acting on the 3 dimensional vector space tangent to $\Omega_3(t)$), and that transversal vectors correspond to the velocities of material bodies. So there are only two solutions for the signs of the eigen values of $[g(m)]$: either (-,+,+,+) or (+,-,-,-) which provides both a **Lorentz metric**. The scalar product, in an orthonormal basis $(\varepsilon_i)_{i=0}^{3}$ at m reads :

$$\begin{array}{l} \text{signature } (3,1) : \langle u, v\rangle = u^1v^1 + u^2v^2 + u^3v^3 - u^0v^0 \\ \text{signature } (1,3) : \langle u, v\rangle = -u^1v^1 - u^2v^2 - u^3v^3 + u^0v^0 \end{array} \tag{6}$$

Such a scalar product defines by restriction on each hypersurface $\Omega_3(t)$ a positive or a negative definite metric, which applies to spatial vectors (tangent to $\Omega_3(t)$) and provides, up to sign, the usual euclidean metric. So that both signatures are acceptable.

Which leads to :

Proposition 5 *The manifold M representing the Universe is endowed with a non degenerate metric, called the **Lorentz metric**, with signature either (3,1) of (1,3) defined at each point.*

This reasoning is a legitimate assumption, which is consistent with all the other concepts and assumptions, notably the existence of a causal structure, this is not the proof of the existence of such a metric. Such a proof comes from the formula in a change of frames between observers, which can be checked experimentally.

On a finite dimensional, connected, Hausdorff manifold, there is always a definite positive metric. There is no relation between this metric and a Lorentz metric. Not all manifolds can have a Lorentz metric, the conditions are technical but one can safely assume that they are met in a limited region Ω.

A metric is represented at each point by a tensor, whose value can change with the location. One essential assumption of General Relativity is that, meanwhile the container M is fixed, and so the chart and its holonomic basis are fixed geometric representations without specific physical meaning, the metric is a physical object and can vary at each point according to specific physical laws. The well known deformation of the space-time with gravity is expressed, not in the structure of the manifold (which is invariant) but in the value of the metric at each point.

3.3 Gauge group

The existence of a metric implies that, at any point, there are orthonormal bases $(\varepsilon_i)_{i=0}^{3}$ with the property :

Definition 6 $\langle \varepsilon_i, \varepsilon_j\rangle = \eta_{ij}$ *for the signature (3,1) and* $\langle \varepsilon_i, \varepsilon_j\rangle = -\eta_{ij}$ *for the signature (1,3)*

with the matrix $[\eta]$

Notation 7 *In any orthonormal basis* ε_0 *denotes the time vector.*
$\langle \varepsilon_0, \varepsilon_0\rangle = -1$ *if the signature is* $(3,1)$
$\langle \varepsilon_0, \varepsilon_0\rangle = +1$ *if the signature is* $(1,3)$

Notation 8 $[\eta] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *whatever the signature*

An orthonormal basis, at each point, is a geometric **gauge**. The choice of an orthonormal basis depends on the observer : he has freedom of gauge. One goes from one gauge to another by a linear map χ which preserves the scalar product. They constitute a group, called the **gauge group**. These maps are represented by a matrix $[\chi]$ such that :

$$[\chi]^t [\eta] [\chi] = [\eta] \tag{7}$$

The group denoted equivalently $O(3,1)$ or $O(1,3)$, does not depend on the signature (replace $[\eta]$ by -$[\eta]$). $O(3,1)$ is a 6 dimensional Lie group with Lie algebra $o(3,1)$ whose matrices $[h]$ are such that :

$$[h]^t [\eta] + [\eta] [h] = 0 \tag{8}$$

The Lie algebra is a vector space and we will use the basis :

$$[\kappa_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} ; [\kappa_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} ; [\kappa_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\kappa_4] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; [\kappa_5] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; [\kappa_6] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

so that any matrix of $o(3,1)$ can be written :
$[\kappa] = [J(r)] + [K(w)]$ with

$$[J(r)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_3 & r_2 \\ 0 & r_3 & 0 & -r_1 \\ 0 & -r_2 & r_1 & 0 \end{bmatrix} ; [K(w)] = \begin{bmatrix} 0 & w_1 & w_2 & w_3 \\ w_1 & 0 & 0 & 0 \\ w_2 & 0 & 0 & 0 \\ w_3 & 0 & 0 & 0 \end{bmatrix}$$

The exponential of these matrices read (Maths.2.5) :
$\exp[K(w)] = I_4 + \frac{\sinh\sqrt{w^t w}}{\sqrt{w^t w}} K(w) + \frac{\cosh\sqrt{w^t w}-1}{w^t w} K(w)K(w)$

$$\exp[K(w)] = \begin{bmatrix} \cosh\sqrt{w^t w} & w^t \frac{\sinh\sqrt{w^t w}}{\sqrt{w^t w}} \\ w \frac{\sinh\sqrt{w^t w}}{\sqrt{w^t w}} & I_3 + \frac{\cosh\sqrt{w^t w}-1}{w^t w} w w^t \end{bmatrix}$$

$$\exp[J(r)] = I_4 + \frac{\sin\sqrt{r^t r}}{\sqrt{r^t r}} J(r) + \frac{1-\cos\sqrt{r^t r}}{r^r r} J(r)J(r) = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

where R a 3×3 matrix of $O(3)$.

The group $O(3)$ has two connected components : the subgroup $SO(3)$ with determinant 1, and the subset $O_1(3)$ with determinant -1.

$O(3,1)$ has four connected components which can be distinguished according to the sign of the determinant and their projection under the compact subgroup $SO(3) \times \{I\}$.

Any matrix of $SO(3,1)$ can be written as the product : $[\chi] = \exp[K(w)] \exp[J(r)]$ (or equivalently $\exp[J(r')] \exp[K(w')]$). So we have the 4 cases :

- $SO_0(3,1)$: with determinant 1: $[\chi] = \exp K(w) \times \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$
- $SO_1(3,1)$: with determinant 1: $[\chi] = \exp K(w) \times \begin{bmatrix} -1 & 0 \\ 0 & -R \end{bmatrix}$
- $SO_2(3,1)$ with determinant = -1: $[\chi] = \exp K(w) \times \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}$
- $SO_3(3,1)$ with determinant = -1: $[\chi] = \exp K(w) \times \begin{bmatrix} 1 & 0 \\ 0 & -R \end{bmatrix}$

where R a 3×3 matrix of $SO(3)$, so that $-R \in O_1(3)$

3.4 Orientation and time reversal

Any finite dimensional vector space is orientable. A manifold is orientable if it is possible to define a consistent orientation of its tangent vector spaces, and not all manifolds are orientable. If it is endowed with a metric then the map : $\det g : M \to \mathbb{R}$ provides an orientation function (its sign changes with the permutation of the vectors of a holonomic basis) and the manifold is orientable.

But on a 4 dimensional vector space one can define other operations, of special interest when the 4 dimensions have not the same properties. For any orthonormal basis $(\varepsilon_i)_{i=0}^3$:

space reversal is the change of basis :

$i = 1, 2, 3 : \widetilde{\varepsilon}_i = -\varepsilon_i$

$\widetilde{\varepsilon}_0 = -\varepsilon_0$

time reversal is the change of basis :

$i = 1, 2, 3 : \widetilde{\varepsilon}_i = \varepsilon_i$

$\widetilde{\varepsilon}_0 = -\varepsilon_0$

These two operations change the value of the determinant, so they are not represented by matrices of $SO(3,1)$:

space reversal matrix : $S = \begin{bmatrix} 1 & 0 \\ 0 & -I_3 \end{bmatrix}$

time reversal matrix : $T = \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}$

$ST = -I_4$

The matrices of the subgroups $SO_k(3,1), k = 1, 2, 3$ are generated by the product of any element of $SO_0(3,1)$ by either S or T.

Is the universe orientable ? Following our assumption, if there is a metric, it is orientable. However one can check for experimental proofs. In a universe where all observers have the same time, the simple existence of stereoisomers which do not have the same chemical properties suffices to answer positively : we can tell to a distant observer what we mean by "right" and "left" by agreeing on the property of a given product. In a space-time universe one needs a process with an outcome which discriminates an orientation. All chemical reactions starting with a balanced mix of stereoisomers produce an equally balanced mix (stereoisomers have the same level of energy). However there are experiments involving the weak interactions (CP violation symmetry in the decay of neutral kaons) which show the required property. So we can state that the 4 dimensional universe is orientable, and then we can distinguish orientation preserving gauge transformations.

A change of gauge, physically, implies some transport of the frame (one does not jump from one point to another) : we have a map : $\chi : R \to SO(3,1)$ such that at each point of the path $p_o : R \to M$ defined on a interval R of $\mathbb{R}$, $\chi(t)$ is an isometry. The path which is followed matters. In particular it is connected. The frame $(\varepsilon_i)_{i=0}^3$ is transported by : $\widetilde{\varepsilon}_i(\tau) = \chi(t)\,\varepsilon_i(0)$. So $\{[\chi(\tau)], t \in R\}$, image of the connected interval R by a continuous map is a connected subset of $SO(3,1)$, and because $\chi(0) = Id$ it must be the component of the identity. So the right group to consider is the **connected component of the identity** $SO_0(3,1)$

3.5 Time like and space like vectors

The causal structure is then fully defined by the metric.

At any point m one can discriminate the vectors $v \in T_m M$ according to the value of the scalar product $\langle v, v \rangle$.

Definition 9 ***Time like*** *vectors are vectors v such that $\langle v, v \rangle < 0$ with the signature (3,1) and $\langle v, v \rangle > 0$ with the signature (1,3)*

Space like *vectors are vectors v such that $\langle v, v \rangle > 0$ with the signature (3,1) and $\langle v, v \rangle < 0$ with the signature (1,3)*

Moreover the subset of time like vectors has two disconnected components (this is no longer true in universes with more than one "time component"). So one can discriminate these components and, in accordance with the assumptions about the velocity of material bodies, it is logical to consider that their velocity is **future oriented**. And one can distinguish gauge transformations which preserve this time orientation.

Definition 10 *We will assume that the future orientation is given in a gauge by the vector ε_0. So a vector u is time like and future oriented if :*

$\langle u, u \rangle < 0, \langle u, \varepsilon_0 \rangle < 0$ with the signature (3,1)

$\langle u, u \rangle > 0, \langle u, \varepsilon_0 \rangle > 0$ with the signature (1,3)

A matrix $[\chi]$ of $SO(3,1)$ preserves the time orientation iff $[\chi]_0^0 > 0$ and this will always happen if $[\chi] = \exp[K(w)]\exp[J(r)]$ that is if $[\chi] \in SO_0(3,1)$.

A gauge transformation which preserves both the time orientation, and the global orientation must preserve also the spatial orientation.

3.6 Velocities have a constant Lorentz norm

The velocity $\frac{dp_o}{d\tau}$ is a vector which is defined independently of any basis, for any observer it is transversal to $\Omega_3(t)$. It is legitimate to say that it is future oriented, and so it must be time-like. One of the basic assumptions of Relativity is that it has a constant length, as measured by the metric, identical for all observers. So it is possible to use the norm of the velocity to define a standard rate at which the clocks run.

Because the proper time of any material body can be defined as the time on the clock of an observer attached to the body this proposition is extended to any particle.

The time is not measured with the same unit as the lengths, used for the spatial components of the velocity. The ratio ξ^i/t has the dimension of a spatial speed. So we make the general assumption that for any observer or particle the velocity is such that $\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2$ where τ is the proper time. Notice that c is a constant, with no specific value. This is consistent with the procedures used to measure the time of events occurring at a distant spatial location. And we sum up :

Proposition 11 *The velocity $\frac{dp}{d\tau}$ of any particle or observer is a time like, future oriented vector with Lorentz norm*

$$\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2 \tag{9}$$

(with signature (3,1) or c^2 with signature (1,3)) where c is a fundamental constant.

3.7 More on the Standard chart of an observer

We have assumed that the observer is spatially immobile, and his chart is built around his location. But with the other assumptions we can define a collection of charts for any point in the hypersurfaces $\Omega_3(t)$.

Theorem 12 *For any observer there is a vector field $\mathbf{O} \in \mathfrak{X}(TM)$ which is future oriented, with length $\langle \mathbf{O}(m), \mathbf{O}(m) \rangle = -1$, normal to $\Omega_3(t)$ and such that : $\mathbf{O}(p_0(t)) = \frac{1}{c}\frac{dp_o}{dt}$ where $\frac{dp_o}{dt}$ is the velocity of the observer at each point of his world line.*

Proof. For an observer the function $f_o : \Omega \to \mathbb{R}$ has for derivative a one form $f'_o(m) \neq 0$ such that $\forall v \in T_m\Omega_3(t) : f'_o(m)\, v = 0$. Using the metric, it is possible to associate to $f'_o(m)$ a vector : $\mathbf{O}(m) = gradf_o : \langle \mathbf{O}(m), v \rangle = f'_o(m)\, v$ which is unique up to a scalar. Thus $\mathbf{O}(m)$ is normal to $\Omega_3(t)$. Along the world line of the observer $\mathbf{O}(m)$ is in the direction of the velocity of the observer. And it is always possible to choose $\mathbf{O}(m)$ such that it is future oriented and with length $\langle \mathbf{O}(m), \mathbf{O}(m) \rangle = -1$ ■

As a consequence :

Theorem 13 *$\Omega_3(t)$ are space like hypersurfaces, with unitary, future oriented, normal $\mathbf{O} \in \mathfrak{X}(TM)$*

Using the vector field $\mathbf{O}$, and any chart φ_Ω of $\Omega(0)$ there is a standard chart associated to an observer.

Definition 14 *The **standard chart** on M of any observer is defined as :*
$\varphi_o : \mathbb{R}^4 \to \Omega :: \varphi_o\left(\xi^0, \xi^1, \xi^2, \xi^3\right) = \Phi_O(ct, x)$
$\xi^0 = ct$, $\varphi_\Omega\left(\xi^1, \xi^2, \xi^3\right) = x$ *in any chart of* $\Omega(0)$

c is required in $\Phi_O(ct, x)$ so that :

$$\xi^0 = ct \tag{10}$$

which makes all the coordinates homogeneous in units [Length].
The holonomic basis associated to this chart is such that :
$\partial\xi_0 = \frac{\partial\varphi_o}{\partial\xi^0} = \frac{1}{c}\frac{\partial}{\partial t}\Phi_{\varepsilon_0}(ct, x) = \mathbf{O}$

$$\mathbf{O}(m) = \partial\xi_0 \tag{11}$$

For any point $m = \varphi_o\left(\xi^0, \xi^1, \xi^2, \xi^3\right) = \Phi_O(ct, x)$ the point x is the point where the integral curve of $\mathbf{O}$ passing by m crosses $\Omega_3(0)$.

So the main characteristic of an observer can be summed in the vector field $\mathbf{O}$ (which is equivalently deduced from the function f_o). From this vector field it is possible to define any standard chart, by choosing

a chart on $\Omega_3(0)$. In this construct the spatial location of the observer does not matter any longer : the only restriction is that he belongs to $\Omega_3(t)$ and he follows a trajectory which is an integral curve of the vector field $\mathbf{O}$: $p_o(t) = \varphi_o(t, x_0)$ for some fixed $x_0 \in \Omega_3(0)$.

According to the principle of locality any measure is done locally : the state of any system at t is represented by the measures done over $\Omega_3(t)$. The system itself can be defined as the "physical content" of $\Omega_3(t)$ and its evolution as the set $\{\Omega_3(t), t \in [0,T]\}$. The physical system itself is observer dependant. The vector field $\mathbf{O}$ defines a special chart, but also the system itself. Two observers who do not share the vector field $\mathbf{O}$ do not perceive the same system. So actually this is a limitation of the Principle of Relativity : it holds but only when the observers agree on the system they study. And of course the observers who share the same $\mathbf{O}$ have a special interest.

3.8 Trajectory and speed of a particle

A particle follows a world line $q(\tau)$, parametrized by its proper time. Any observer sees only one instance of the particle, located at the point where the world line crosses the hypersurface $\Omega_3(t)$ so we have a relation between τ and t. This relation identifies the respective location of the observer and the particle on their own world lines. With the standard chart of the observer it is possible to measure the velocity of the particle at any location, and of course at the location where it belongs to $\Omega_3(t)$.

The trajectory (parametrized by t) of any particle in the standard chart of an observer is :
$q(t) = \Phi_O(ct, x(t)) = \varphi_o\left(ct, \xi^1(t), \xi^2(t), \xi^3(t)\right)$
By differentiation with respect to t :
$\frac{dq}{dt} = c\mathbf{O}(q(t)) + \frac{\partial}{\partial x}\Phi_O(ct, x(t))\frac{\partial x}{\partial t}$
$\frac{\partial}{\partial x}\Phi_O(ct, x(t))\frac{\partial x}{\partial t} = \sum_{\alpha=1}^{3}\frac{d\xi_\alpha}{dt}\partial\xi_\alpha \in T_m\Omega_3(t)$ so is orthogonal to $\mathbf{O}(q(t))$

Definition 15 *The **spatial speed** of a particle on its trajectory with respect to an observer is the vector of* $T_{q(t)}\Omega_3(t)$: $\overrightarrow{v} = \frac{\partial}{\partial x}\Phi_O(ct, x(t))\frac{\partial x}{\partial t} = \sum_{\alpha=1}^{3}\frac{d\xi^\alpha}{dt}\partial\xi_\alpha$

Thus for any particle in the standard chart of an observer :

$$V(t) = \frac{dq}{dt} = c\mathbf{O}(q(t)) + \overrightarrow{v} \tag{12}$$

For the observer in the standard chart we have : $\frac{dp_0}{dt} = c\mathbf{O}(p_0(t)) \Leftrightarrow \overrightarrow{v} = 0$

Notice that the velocity, and the spatial speed, are measured *in the chart of the observer at the point* $q(t)$ *where is the particle*. Because we have defined a standard chart it is possible to measure the speed of a particle located at a point $q(t)$ which is different from the location of the observer. And we can express the relation between τ and t.

Theorem 16 *The proper time* τ *of any particle and the corresponding time of any observer* t *are related by :*

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{\|\overrightarrow{v}\|^2}{c^2}} \tag{13}$$

where $\overrightarrow{v}$ *is the spatial speed of the particle, with respect to the observer and measured in his standard chart. The velocity of the particle is :*

$$\frac{dp}{d\tau} = \frac{1}{\sqrt{1 - \frac{\|\overrightarrow{v}\|^2}{c^2}}}\left(\overrightarrow{v} + c\mathbf{O}(m)\right) \tag{14}$$

Proof. i) Let be a particle A with world line :
$p : \mathbb{R} \to M :: m = p(\tau) = \Phi_u(\tau, a)$ with $a = \Phi_u(0, a) = p(0)$
In the standard chart $\Phi_O(ct, x)$ of the observer O its trajectory is :
$q : \mathbb{R} \to M :: m = q(t) = \Phi_O(ct, x(t))$
So there is a relation between t, τ :
$m = p(\tau) = \Phi_u(\tau, a) = q(t) = \Phi_O(ct, x(t))$
By differentiation with respect to t :
$\frac{d}{dt}q(t) = c\mathbf{O}(p_A(t)) + \overrightarrow{v}$
$\frac{dq}{dt} = \overrightarrow{v} + c\mathbf{O}(m)$
$\frac{dq}{dt} = \frac{dp}{d\tau}\frac{d\tau}{dt}$
$\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2$
$\left\langle \frac{dq}{dt}, \frac{dq}{dt} \right\rangle = -c^2\left(\frac{d\tau}{dt}\right)^2$

$\left\langle \frac{dq}{dt}, \frac{dq}{dt} \right\rangle = \langle \overrightarrow{v}, \overrightarrow{v} \rangle_3 - c^2$ because $\mathbf{O}(m) \perp \Omega_3(t)$
$\| \overrightarrow{v} \|^2 - c^2 = -c^2 \left(\frac{d\tau}{dt} \right)^2$
and because $\frac{d\tau}{dt} > 0 : \frac{d\tau}{dt} = \sqrt{1 - \frac{\| \overrightarrow{v} \|^2}{c^2}}$
ii) The velocity of the particle is :
$\frac{dp}{d\tau} = \frac{dq}{dt} \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{\| \overrightarrow{v} \|^2}{c^2}}} (\overrightarrow{v} + c\mathbf{O}(m))$ ■
As a consequence :

$$\| \overrightarrow{v} \|_3 < c \tag{15}$$

$V(t) = \frac{dp}{dt}$ is the measure of the motion of the particle with respect to the observer : it can be seen as the relative velocity of the particle with respect to the observer. It involves $\overrightarrow{v}$ which has the same meaning as usual, but we see that in Relativity one goes from the 4 velocity $u = \frac{dp}{d\tau}$ (which has an absolute meaning) to the relative velocity $V(t) = \frac{dp}{dt} = \frac{dp}{d\tau} \frac{d\tau}{dt} = u \sqrt{1 - \frac{\| \overrightarrow{v} \|^2}{c^2}}$ by a scalar.

4 FIBER BUNDLES

The location of a particle is absolute : this is the point in the physical Universe that it occupies at some time. Similarly the velocity of a particle or an observer is absolute : in its definition there is no reference to a chart or a frame. It is a vector, which is an intrinsic property of material bodies and particles. It is measured in bases, and the value of its components vary according to precise mathematical rules when one goes from one basis to another. The physical quantity is absolute, but its measure is relative. And this holds for all physical quantities : a measure in itself has no meaning if one does not know how it has been done, the units and the standards used. It is specially important in Relativity because the observers are not interchangeable. The most general mathematical tool to deal with this problem is the fiber bundle, which is a generalization of the concept of vector space tangent to a manifold (see Maths.Part VI).

4.1 General fiber bundle

A fiber bundle, denoted $P(M, F, \pi_P)$, is a manifold P, which is locally the product of two manifolds, the base M and the standard fiber F, with a projection : $\pi_P : P \to M$. The subset of $P : \pi_P^{-1}(m)$ is the fiber over m. It is usually defined over a collection of open subsets of M, patched together, but we will assume that on the area Ω there is only one component (the fiber bundles are assumed to be trivial). A **trivialization** is a map : $\varphi_P : M \times F \to P :: p = \varphi_P(m, v)$ and any element of P is projected on M :
$\forall v \in F : \pi_P(\varphi_P(m, v)) = m$.
So it is similar to a chart, but the arguments are points of the manifolds.
A **section p** on P is defined by a map : $v : M \to F$ and $\mathbf{p}(m) = \varphi_P(m, v(m))$. The set of sections is denoted $\mathfrak{X}(P)$.
A fiber bundle can be defined by different trivializations. In a **change of trivialization** the *same* element p is defined by a different map φ_P : this is very similar to the charts for manifold.
$p = \varphi_P(m, v) = \widetilde{\varphi}_P(m, \widetilde{v})$
and there is a necessary relation between v and $\widetilde{v}$ (m stays always the same) depending on the kind of fiber bundle.

4.2 Vector bundle

If $F = V$ is a vector space then P is a **vector bundle :**
$\varphi_P : M \times F \to P :: \mathbf{X}(m) = (m, \sum_{i=1}^n X_i(m) \varepsilon_i)$
This is a vector of V located at m. The rules in a change of trivialization are such that P has at each point the structure of a vector space :
$w_m = \varphi_P(m, w), w'_m = \varphi_P(m, w'), \alpha, \beta \in \mathbb{R}$:
$\alpha w_m + \beta w'_m = \varphi_P(m, \alpha w + \beta w')$
and a holonomic basis is defined by a basis $(\varepsilon_i)_{i \in I}$ of $V : \varepsilon_i(m) = \varphi_P(m, \varepsilon_i)$
thus vectors of the fiber bundle read :-
$\mathbf{X}(m) = (m, \sum_{i=1}^n X_i(m) \varepsilon_i) = \sum_{i=1}^n X_i(m) \varepsilon_i(m)$
$(\varepsilon_i)_{i \in I}$ plays the same role as the holonomic basis $(\partial \xi_\alpha)_{\alpha=0}^3$ of the tangent bundle TM.
Usually one requires some property of the basis ε_i, for instance it must be orthonormal. The mean to go from one basis to another is provided usually by the action of a group. So they are defined as associated to a principal bundle.

4.3 Principal bundle

If $F = G$ is a Lie group then P is a **principal bundle** : its elements are a value $g(m)$ of G localized at a point m.

p will usually define the basis used to measure vectors, so p is commonly called a gauge. There is a special gauge which can be defined at any point (it will usually be the gauge of the observer) : the **standard gauge,** the element of the fiber bundle such that : $\mathbf{p}(m) = \varphi_P(m, 1)$. *This is not a section* : the standard gauge is arbitrary, it reflects the free will of the observer, and as such is not submitted to any physical law. Its definition, with respect to measures, is done in protocols which document the experiments. There is no such thing as a given, natural, "field of gauges".

A principal bundle $P(M, G, \pi)$ is characterized by the existence of the right action of the group G on the fiber bundle P :

$\rho : P \times G \to P :: \rho(p, g') = \rho(\varphi_P(m, g), g') = \varphi_P(m, g \cdot g')$

which does not depend on the trivialization. So that any $p \in P$ can be written : $p = \varphi_P(m, g) = \rho(\mathbf{p}, g)$ with the standard gauge $\mathbf{p} = \varphi_P(m, 1)$.

A change of trivialization is induced by a map : $\chi : M \to G$ that is by a section $\chi \in \mathfrak{X}(P)$ and :

$p = \varphi_P(m, g) = \widetilde{\varphi}_P(m, \chi(m) \cdot g) = \widetilde{\varphi}_P(m, \widetilde{g}) \Leftrightarrow \widetilde{g} = \chi(m) \cdot g \quad (\chi(m) \text{ acts } \textit{on the left})$

$\chi(m)$ can be identical over M (the change is said to be global) or depends on m (the change is local).

The expression of the elements of a section change as :

$\sigma \in \mathfrak{X}(P) :: \sigma = \varphi_P(m, \sigma(m)) = \widetilde{\varphi}_P(m, \widetilde{\sigma}(m)) \Leftrightarrow \widetilde{\sigma}(m) = \chi(m) \cdot \sigma(m)$

$\sigma(m) = \varphi_P(m, \sigma(m)) = \widetilde{\varphi}_P(m, \chi(m) \cdot \sigma(m))$

A change of trivialization induces a **change of standard gauge** :

$\mathbf{p}(m) = \varphi_P(m, 1) = \widetilde{\varphi}_P(m, \chi(m)) \to \widetilde{\mathbf{p}}(m) = \widetilde{\varphi}_P(m, 1) = \widetilde{\varphi}_P\left(m, \chi(m) \cdot \chi(m)^{-1}\right) = \mathbf{p}(m) \cdot \chi(m)^{-1}$

So changes of trivialization and change of gauge are the same operations.

4.4 Associated fiber bundle

Whenever there is a manifold F, a left action λ of G on F, one can built an **associated fiber bundle** denoted $P[F, \lambda]$ which consists of couples :

$(p, v) \in P \times F$ with the equivalence relation : $(p, v) \sim \left(p \cdot g, \lambda\left(g^{-1}, v\right)\right)$

v belongs to a fixed set, but its value is labeled by the standard which is used and related to a point of a manifold.

It is convenient to define these couples by using the standard gauge on P:

$$(\mathbf{p}(m), v) = (\varphi_P(m, 1), v) \sim \left(\varphi_P(m, g), \lambda\left(g^{-1}, v\right)\right) \tag{16}$$

A standard gauge is nothing more than the use of an arbitrary standard, represented by 1, with respect to which the measure is done. A change of standard gauge in the principal bundle impacts all associated fiber bundles (this is similar to the change of units) :

$$\begin{array}{l} \mathbf{p}(m) = \varphi_P(m, 1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} \\ v_p = (\mathbf{p}(m), v) = (\widetilde{\mathbf{p}}(m), \widetilde{v}) : \widetilde{v} = \lambda(\chi(m), v) \end{array} \tag{17}$$

Similarly for the components of a section :

$\mathbf{v} \in \mathfrak{X}(P[V, \lambda]) :: \mathbf{v}(m) = (\mathbf{p}(m), v(m)) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \lambda(\chi(m), v)\right)$

If F is a vector space V and $[V, \rho]$ a representation of the group G then we have an **associated vector bundle** $P[V, \rho]$ which has locally the structure of a vector space. It is convenient to define a **holonomic basis** $(\varepsilon_i(m))_{i=1}^n$ from a basis $(\varepsilon_i)_{i=1}^n$ of V by : $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$ then any vector of $P[V, \rho]$ reads :

$$v_m = (\mathbf{p}(m), v) = \left(\mathbf{p}(m), \sum_{i=1}^n v^i \varepsilon_i\right) = \sum_{i=1}^n v^i \varepsilon_i(m) \tag{18}$$

A change of standard gauge $\mathbf{p}(m) = \varphi_P(m, 1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$ in the principal bundle impacts all associated vector bundles.

For any vector :

$v_m = (\mathbf{p}(m), v) \sim \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \chi(m)(v)\right)$

Meanwhile the holonomic basis of a vector bundle changes as :

$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \to$

$\widetilde{\varepsilon}_i(m) = (\widetilde{\mathbf{p}}(m), \varepsilon_i) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \varepsilon_i\right)$

$\sim \left(\left(\mathbf{p}(m) \cdot \chi(m)^{-1}\right) \cdot \chi(m), \rho\left(\chi(m)^{-1}\right) \varepsilon_i\right)$

$= \left(\mathbf{p}(m), \rho\left(\chi(m)^{-1}\right)(\varepsilon_i)\right) = \rho\left(\chi(m)^{-1}\right)\varepsilon_i(m)$

so that the components of a vector in the holonomic basis change as :

$v_m = \sum_{i=1}^n v^i \varepsilon_i(m) = \sum_{i=1}^n \widetilde{v}^i \widetilde{\varepsilon}_i(m) = \sum_{i=1}^n \widetilde{v}^i \rho(\chi(m))^{-1} \varepsilon_i(m) \Rightarrow \widetilde{v}^i = \sum_j \left[\rho(\chi(m))\right]^i_j v^j$

The elements of a section stay the same, but their definition changes, meanwhile the holonomic bases are defined by different elements. This is very similar to what we have in any vector space in a change of basis : the vectors of the basis change, the other vectors stay the same, but their components change.

An important point : even if one denotes $\mathbf{v}(m) = \sum_{i=1}^n v^i(m) \varepsilon_i(m)$ actually the vector is measured in a fixed vector space : $\mathbf{v}(m) = (\varphi_P(m,1), v(m))$ where $v(m) = \sum_{i=1}^n v^i(m) \varepsilon_i \in V$. So that the derivatives : $\partial_\alpha \mathbf{v}(m) = (\varphi_P(m,1), \partial_\alpha v(m))$ with $\partial_\alpha v(m) = \sum_{i=1}^n \left(\partial_\alpha v^i(m)\right) \varepsilon_i$.

The *fiber bundle formalism enables to consider the components independently from the basis.* This is possible because the gauge $\mathbf{p}(m) = \varphi_P(m,1)$ is not a section.

Whenever there is a scalar product (bilinear symmetric of Hermitian two form) $\langle\rangle$ on a vector space V, so that (V,ρ) is a *unitary* representation of the group G : $\langle \rho(g) v, \rho(g) v' \rangle = \langle v, v' \rangle$, the scalar product can be extended on the associated vector bundle $P[V,\rho]$:

$$\langle (\mathbf{p}(m), v), (\mathbf{p}(m), v') \rangle_{P[V,\rho]} = \langle v, v' \rangle_W \tag{19}$$

4.5 Standard gauge associated to an observer

Frames and bases are used to measure components of vectorial quantities. Following the Principle of Locality any physical map, used to measure the *components of a vector* at a point m in M, must be done at m, that is in a local frame. Observers belong to $\Omega_3(t)$ and can do measures at any point of $\Omega_3(t)$.

They can measure components of vectors in the holonomic basis $(\partial\xi_\alpha)_{\alpha=0}^3$ given by a chart. This basis changes with the location but the chart is fixed for a given observer.

One property of the observers is that they have freedom of gauge : they can decide to measure the components of vectors in another basis than $(\partial\xi_\alpha)_{\alpha=0}^3$: usually, and this is what we will assume, they choose an orthonormal basis. This can be done by choosing 3 spatial vectors at a point, and we assume that they can extend the choice at any other point of $\Omega_3(t)$). However for the time vector the observer has actually no choice : this is necessarily the vector field $\mathbf{O}$ which is normal to $\Omega_3(t)$) and future oriented, and in the same direction as $\partial\xi_0$.

We will call such orthonormal bases a **Standard gauge**. They are arbitrary, chosen by the observer, with the restriction about the choice of ε_0, and implemented all over $\Omega_3(t)$. They can be defined with respect to the holonomic basis of a chart.

This is equivalent to assume that, *for each observer*, there is a principal bundle $P_o(M, SO_0(3,1), \pi_p)$, a gauge $\mathbf{p}(m) = \varphi_P(m,1)$ and an associated vector bundle $P_o\left[\mathbb{R}^4, \imath\right]$ where $\left(\mathbb{R}^4, \imath\right)$ is the standard representation of $SO_0(3,1)$. It defines at each point an holonomic orthonormal basis : $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$.To sum up :

Proposition 17 *For each observer there is :*

a principal fiber bundle structure $\mathbf{P}_o(M, SO_0(3,1), \pi_p)$ *on* M *with fiber the connected component of identity* $SO_0(3,1)$*, which defines at each point a standard gauge :* $\mathbf{p}(m) = \varphi_P(m,1)$

an associated vector bundle structure $P_o\left[\mathbb{R}^4, \imath\right]$ *where* $\left(\mathbb{R}^4, \imath\right)$ *is the standard representation of* $SO_0(3,1)$*, which defines at any point* $m \in \Omega$ *the standard basis* $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i), i = 0..3$ *where* $\varepsilon_0(m)$ *is orthogonal to the hypersurfaces* $\Omega_3(t)$ *to which* m *belongs.*

4.6 Formulas for a change of observer

Theorem 18 *For any two observers* O, A *the components of the vectors of the standard orthonormal basis of* A*, expressed in the standard basis of* O *are expressed by the following matrix* $[\chi]$ *of* $SO_0(3,1)$*, where* $\overrightarrow{v}$ *is the instantaneous spatial speed of* A *with respect to* O *and* R *a matrix of* $SO(3)$ *:*

$$[\chi] = \begin{bmatrix} \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} & \frac{\frac{v^t}{c}}{\sqrt{1-\frac{\|v\|^2}{c^2}}} \\ \frac{\frac{v}{c}}{\sqrt{1-\frac{\|v\|^2}{c^2}}} & I_3 + \left(\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} - 1\right) \frac{vv^t}{\|v\|^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \tag{20}$$

Proof. Let be :

O be an observer (this will be main observer) with associated vector field $\mathbf{O}$, proper time t and world line $p_o(t)$

A be another observer with associated vector field $\mathbf{A}$, proper time τ

Both observers use their standard chart φ_o, φ_A and their standard orthonormal basis, whose time vector is in the direction of their velocity. The location of A on his world line is the point m such that A belongs to the hypersurface $\Omega_3(t)$

The velocity of A at m :

$\frac{dp_A}{d\tau} = c\mathbf{e}_0(m)$ by definition of the standard basis of A

$\frac{dp_A}{d\tau} = \frac{1}{\sqrt{1-\frac{\|\overrightarrow{v}\|^2}{c^2}}}(\overrightarrow{v} + c\varepsilon_0(m))$ as measured in the standard basis of O

The matrix $[\chi]$ to go from the orthonormal basis $(\varepsilon_i(m))_{i=0}^3$ to $(\mathbf{e}_i(m))_{i=0}^3$ belongs to $SO_0(3,1)$. It reads :

$$[\chi(t)] = \begin{bmatrix} \cosh\sqrt{w^t w} & w^t \frac{\sinh\sqrt{w^t w}}{\sqrt{w^t w}} \\ w\frac{\sinh\sqrt{w^t w}}{\sqrt{w^t w}} & I_3 + \frac{\cosh\sqrt{w^t w}-1}{w^t w} w w^t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

for some $w \in \mathbb{R}^3, R \in SO(3)$

The elements of the first column of $[\chi(t)]$ are the components of $\mathbf{e}_0(m)$, that is of $\frac{1}{c}\frac{dp_A}{d\tau}$ expressed in the basis of O :

$\cosh\sqrt{w^t w} = \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$

$w\frac{\sinh\sqrt{w^t w}}{\sqrt{w^t w}} = \frac{\overrightarrow{v}}{c}\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$

$w = k\overrightarrow{v} \Rightarrow w^t w = k^2\|\overrightarrow{v}\|^2$

which leads to the classic formula with

$w = \frac{v}{\|v\|}\arg\tanh\left\|\frac{v}{c}\right\| = \frac{1}{2}\frac{v}{\|v\|}\ln\left(\frac{c+\|\overrightarrow{v}\|}{c-\|\overrightarrow{v}\|}\right) \sim \frac{1}{2}\frac{v}{\|v\|}\ln\left(1+2\frac{\|\overrightarrow{v}\|}{c}\right) \simeq \frac{v}{c}$ ■

Some key points to understand these formulas :

- They hold *for any observers* O, A, who use their standard orthonormal basis (the time vector is oriented in the direction of their velocity). There is no condition such as inertial frames.

- The points of M where O and A are located can be different, $O \in \Omega_3(\tau), A \in \Omega_3(\tau) \cap \Omega_3(t)$. The spatial speed $\overrightarrow{v}$ is a vector belonging to the space tangent at $\Omega_3(\tau)$ at the location m of A (and not at the location of O at t) and so is the relative speed of A with respect to the point m of M, which is fixed for O.

- The formulas are related to the standard orthonormal bases $(\varepsilon_i(m))_{i=0}^3$ of O and $(\mathbf{e}_i(m))_{i=0}^3$ of A located at the point m of $\Omega_3(t)$ where A is located.

- These formulas apply to the *components of vectors* in the standard orthonormal bases. Except in SR, there is no simple way to deduce from them a relation between the coordinates in the charts of the two observers.

- The formula involves a matrix $R \in SO(3)$ which represents the possible rotation of the spatial frames of O and A, as it would be in Galilean Geometry.

These formulas have been verified with a great accuracy, and *the experiments show that c is the speed of light.*

If we take $\frac{v}{c} \to 0$ we get $[\chi] = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$, that is a rotation of the usual space. The Galilean Geometry is an approximation of SR when the speeds are small with respect to c. Then the velocities are $\frac{d\mu_A}{d\tau} = (\overrightarrow{v} + c\varepsilon_0)$ with a common vector ε_0.

4.7 The Tetrad

4.7.1 The principal fiber bundle

So far we have defined a chart φ_o and a fiber bundle structure P_o for an observer : the construct is based on the trajectory of the observer, and his capability to extend his frame over the hypersurfaces $\Omega_3(t)$. With the formulas above we see how one can go from one observer to another, and thus relate the different fiber bundles P_o. The computations in a change of frame can be done with measures done by the observers, and have been checked experimentally. So it is legitimate to assume that there is a more general structure of principal bundle, denoted $\mathbf{P}_G(M, SO_0(3,1), \pi_G)$,over M. In this representation the bases used by any observer is just a choice of specific trivialization.

Proposition 19 *There is a unique structure of principal bundle* $\mathbf{P}_G(M, SO_0(3,1), \pi_G)$ *with base* M*, standard fiber* $SO_0(3,1)$*. A change of observer is given by a change of trivialization on* P_G*. The standard gauge* $\mathbf{p}(m) = \varphi_G(m,1)$ *is, for any observer, associated to his standard basis* $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$..

Charts on a manifold are a way to locate a point. As such they are arbitrary and fixed. They are only related to the manifold structure. We have defined standard charts and their holonomic bases, which depend on the observer, and are fixed for this observer. And physically one cannot conceive other charts : a physical chart is always the standard chart of some observer.

Standard basis, or standard gauges, are orthonormal, and chosen at any point by the observer. They comprised 4 vectors, called a **tetrad**. The time vector is imposed by the velocity of the observer, but the components of the spatial vectors can be measured in the holonomic basis of a chart.

With the structure of fiber bundle it is possible to compute the impact of a change of gauge. A change of ε_0 is a change of observer. A change of gauge is given by a section χ (global or not) of $\mathbf{P}_G$, the vectors of the standard basis transform according to the matrix $[\chi]$. The operation is associative : the combination of relative motions is represented by the product of the matrices, which is convenient.

The condition for 4 vectors to be orthogonal depends on the metric, which changes with the location. It is proven in Differential Geometry that there is no chart such that its holonomic basis can be orthogonal at each point (the manifolds with this property, which is not assumed for M, are special and said to be parallelizable). This is due to the fact that a metric is an object which is added to the structure of manifold, it does not come with it. And there is no reason why it would be constant [2]. As a consequence an orthonormal basis cannot have fixed components in any chart, even if the observer strives to keep them as fixed as possible. And the components of the tetrad in the - fixed - holonomic chart must change in order to keep the basis orthonormal.

4.7.2 Tetrad

The vectors of a standard basis (the tetrad) can be expressed in the holonomic basis of any chart :

$$\varepsilon_i(m) = \sum_{\alpha=0}^{3} P_i^{\alpha}(m)\,\partial\xi_\alpha \Leftrightarrow \partial\xi_\alpha = \sum_{i=0}^{3} P_\alpha'^{i}(m)\,\varepsilon_i(m) \tag{21}$$

where $[P]$ is a real invertible matrix (which has no other specific property, it does not belong to $SO(3,1)$) and we denote

Notation 20 $[P'] = [P]^{-1} = \left[P_\alpha'^{i}\right]$.

The dual of $(\partial\xi_\alpha)_{\alpha=0}^{3}$ is $(d\xi^\alpha)_{\alpha=0}^{3}$ with the defining relation :
$d\xi^\alpha(\partial\xi_\beta) = \delta_\beta^\alpha$.
The dual $\left(\varepsilon^i(m)\right)_{i=0}^{3}$ is :

$$\varepsilon^i(m) = \sum_{i=0}^{3} P_\alpha'^{i}(m)\,d\xi^\alpha \Leftrightarrow d\xi^\alpha = \sum_{i=0}^{3} P_i^{\alpha}(m)\,\varepsilon^i(m) \tag{22}$$

$\varepsilon^i(m)(\varepsilon_j(m)) = \sum_{\alpha\beta=0}^{3} P_\alpha'^{i} P_j^{\beta} d\xi^\alpha(\partial\xi_\beta) = \sum_{\alpha=0}^{3} P_\alpha'^{i} P_j^{\alpha} = \delta_j^i$
The quantities $(P_i^\alpha(m))_{i=1}^{3}$ (called vierbein) and $\left(P_\alpha'^{i}(m)\right)_{i=1}^{3}$ are one of the variables in any model in GR : as such they replace the metric g.

In the fiber bundle representation the vectors of the tetrad are variables which are vectors $\varepsilon_i \in \mathfrak{X}(TM)$ or covectors $\varepsilon^i \in \mathfrak{X}(TM^*)$. $(\varepsilon_i(m))_{i=0}^{3}$ is the holonomic basis associated to the standard gauge $\mathbf{p}(m) = \varphi_G(m,1)$.

A change of observer is a change of gauge on the principal bundle

$$\begin{array}{c} \mathbf{p}(m) = \varphi_P(m,1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m)\cdot\chi(m)^{-1} \\ \varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \to \widetilde{\varepsilon}_i(m) = \sum_{j=0}^{3}\left[\chi(m)^{-1}\right]_i^j \varepsilon_j(m) \end{array} \tag{23}$$

In the Standard Chart the 4th vector is always in the direction of the velocity of the observer. So we have :
$\varepsilon_0(p_o(t)) = \partial\xi_0 \Rightarrow P_0'^{i} = \delta_0^i$
$\alpha = 1,2,3 : \frac{\partial}{\partial\xi^\alpha}\varphi_o\left(\xi^0,\xi^1,\xi^2,\xi^3\right) = \partial\xi_\alpha = \frac{\partial}{\partial x}\Phi_{\varepsilon_0}(ct,x)\frac{\partial x}{\partial\xi^\alpha} \in T_m\Omega_3(t) \Rightarrow P_\alpha'^{0} = 0$
and the matrix $[P]$ takes the simpler form :

$$[P] = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}; [Q] = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

[2] Even in an affine space, such as in SR, there is no reason why the metric should be constant. This is an additional assumption in SR.

$[P'] = \begin{bmatrix} 1 & 0 \\ 0 & Q' \end{bmatrix}; [Q'] = \begin{bmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{bmatrix}$

$[Q]\,[Q'] = I_3$

4.7.3 Metric

The scalar product can be computed from the components of the tetrad. By definition :

$g_{\alpha\beta}(m) = \langle \partial\xi_\alpha, \partial\xi_\beta \rangle = \sum_{ij=0}^{3} \eta_{ij} [P']^i_\alpha [P']^j_\beta$

The induced metric on the cotangent bundle is denoted with lifted indexes:

$g^* = \sum_{\alpha\beta} g^{\alpha\beta} \partial\xi_\alpha \otimes \partial\xi_\beta$ and its matrix is $[g]^{-1} : g^{\alpha\beta}(m) = \sum_{ij=0}^{3} \eta^{ij} [P]^\alpha_i [P]^\beta_j$

$$[g]^{-1} = [P][\eta][P]^t \Leftrightarrow [g] = [P']^t [\eta] [P'] \tag{24}$$

It does not depend on the gauge on P_G :

$[\widetilde{g}] = \left[\widetilde{P}'\right]^t [\eta] \left[\widetilde{P}'\right] = [P']^t \left[\chi(m)^{-1}\right]^t [\eta] \left[\chi(m)^{-1}\right] [P'] = [P']^t [\eta] [P']$

In the standard chart of the observer : $g^{00} = -1$.

$[g] = [P']^t [\eta] [P'] = \begin{bmatrix} -1 & 0 \\ 0 & [g]_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & [Q']^t [Q'] \end{bmatrix}$

$[g]^{-1} = [P][\eta][P]^t = \begin{bmatrix} -1 & 0 \\ 0 & [g]_3^{-1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & [Q][Q]^t \end{bmatrix}$

and $[g]_3$ is definite positive.

The metric defines a **volume form** on M. Its expression in any chart is, by definition :

$\varpi_4(m) = \varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 = \sqrt{|\det[g]|}\, d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

$[g] = [P']^t [\eta] [P'] \Rightarrow \det[g] = -(\det[P'])^2 \Rightarrow \sqrt{|\det[g]|} = \det[P']$

$$\varpi_4 = \det[P']\, d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \tag{25}$$

4.7.4 Induced metric

The metric on M induces a metric on any submanifold but it can be degenerated.

On hypersurfaces the metric g_3 is non degenerated if the unitary normal n is such that $\langle n, n \rangle \neq 0$. The induced volume form is :

$\mu_3 = i_n \varpi_4 = \det[P']\, d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3(n)$

For $\Omega_3(t)$ the unitary normal n is ε_0 , the induced metric is Riemannian and the volume form ϖ_3 is :

$\varpi_3 = i_{\varepsilon_0} \varpi_4 = \det[P']\, d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3(\varepsilon_0)$

$= \det[P']\, d\xi^0(\varepsilon_0) \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

$= \det[P']\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

$$\varpi_3 = \det[P']\, d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \tag{26}$$

and conversely :

$\varpi_4 = \varepsilon_0 \wedge \varpi_3 = \det[P']\, d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

ϖ_3 is defined with respect to the coordinates ξ^1, ξ^2, ξ^3 but the measure depends on $\xi^0 = ct$.

For a curve C, represented by any path : $p : \mathbb{R} \to C :: m = p(\theta)$ the condition is $\left\langle \frac{dp}{d\theta}, \frac{dp}{d\theta} \right\rangle \neq 0$. The volume form on any curve defined by a path : $q : \mathbb{R} \to M$ with tangent $V = \frac{dq}{d\theta}$ is $\sqrt{|\langle V, V \rangle|}\, d\theta$. So on the trajectory $q(t)$ of a particle it is

$$\varpi_1(t) = \sqrt{-\langle V, V \rangle}\, dt \tag{27}$$

For a particle there is the privileged parametrization by the proper time, and as $\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2$ the length between two points A, B is :

$\ell_p = \int_{\tau_A}^{\tau_B} \sqrt{-\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle} d\tau = \int_{\tau_A}^{\tau_B} c\, d\tau = c(\tau_B - \tau_A)$

This is an illustration of the idea that all world lines correspond to a travel at the same speed.

4.8 Example : spherical charts

This is a frequent case, which can be implemented easily in our frame-work.

We single out a point $O \in M$

We assume the following :

There is a family $\mathcal{P} \subset C_1(\mathbb{R}; \Omega_3(0))$ of spatial paths : $p : \mathbb{R} \to \Omega_3(0)$ such that :

$\forall p \in \mathcal{P}, \forall \rho \neq \rho' : p(\rho) \neq p(\rho')$ there is no loop and each p is a bijection

$p(0) = O(0)$

$\forall x \in \Omega_3(0)$ there is a unique $p \in \mathcal{P}$ such that : $\exists \rho \in \mathbb{R} : p(\rho) = x$

$\frac{dp}{d\rho} = u(\rho) : \langle u(\rho), u(\rho) \rangle_3 = 1$

Thus the paths constitute a grid, centered in $O(0)$, to locate any point in $\Omega_3(0)$. This is what is done practically by an observer.

Then each path can be identified by the value of $u(0) = v$ and we denote $p(v, \rho) = p(\rho) \in \Omega_3(0)$ which is a chart of $\Omega_3(0)$. Each vector v can be identified by its components in any orthonormal basis at $O(0)$. Let us say :

$v = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ and one can take as coordinates in $\Omega_3(0)$:

$\xi^1 = \rho\cos\phi\cos\theta, \xi^2 = \rho\cos\phi\sin\theta, \xi^3 = \rho\sin\phi$.

The holonomic basis at $x = p(v, \rho)$ is the image of the basis at $O(0)$ by the derivative $p(v, \rho)'|_x$.

Now if we assume that O is an observer, his standard chart is given by

$\varphi_o(t, x) = \varphi_M(ct, \rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi)$.

A path from $A = \varphi_o(\tau_0, x_0)$ to $B = \varphi_o(\tau_1, x_1)$ can be represented by :

$q(\tau) = \varphi_o(\tau, x(\tau))$

$\frac{dq}{d\tau} = c\varepsilon_0 + \frac{dx}{d\tau}$

and its length is :

$\ell(A, B)^2 = \int_{\tau_0}^{\tau_1} \left\langle \frac{dq}{d\tau}, \frac{dq}{d\tau} \right\rangle_M d\tau = \int_{\tau_0}^{\tau_1} \left(c^2\tau^2 - g_3(\tau, x(\tau))\left(\frac{dx}{d\tau}, \frac{dx}{d\tau}\right)\right) d\tau$

The volume measure ϖ_3 reads :

$\varpi_3(x) = \det[P'(\varphi_o(t, x))]\, \rho^2 |\cos\phi| \, d\rho d\theta d\phi$

thus it still depends on t, but acts on variables whose arguments are defined through ρ, θ, ϕ.

No assumption has been made about the "shape" of $\Omega_3(0)$, just that this is a 3 dimensional manifold defined by the chart.

One can assume more, that there is a *physical* spherical symmetry. The physical part of the Geometry is the metric. So we assume that the metric has a symmetry in the following meaning.

There is an action of $SO(3)$ on the vectors v at $O(0)$ (they are defined in an orthonormal basis at $O(0)$) : $v \to [h][v]$ which induces an action on $\mathcal{P}$ and $\Omega_3(0)$:

$R_0 : SO(3) \times \Omega_3(0) \to \Omega_3(0) :: R_0([h]) p(v, \rho) = p([h][v], \rho)$

which can be extended to M :

$R : SO(3) \times M \to M :: R([h]) \varphi_o(t, x) = \varphi_o(t, R_0(g)(x))$

Notice that $SO(3)$ acts only on v and ρ is unchanged.

The geometry is said to be spherically symmetric if R_0 is an isometry. The metric is invariant by R_0 :

$R_0([h])_* g_3 = g_3$

with the push forward $R_0([h])_* g_3$:

$g_3(R_0([h])x)(R_0'([h](x))|_x u_x, R_0'([h](x))|_x v_x) = g_3(x)(u_x, v_x)$

The metric on $\Omega_3(0)$ does not depend on θ, ϕ but depends still on ρ.

Because ε_0 is invariant by this action, it can be extended to M if R_0 is an isometry for any t on $\Omega_3(t)$ .Then the metric, as well as $[P]$, depends only on t, ρ.

A cylindric symmetry can be represented in the same frame-work : the action is then that of a subgroup of $SO(3)$ with a definite axis, which can be taken as one of the vector of the orthonormal basis in $O(0)$.

If this symmetry applies to the whole system (the symmetry of the metric is a prerequisite) then the variables X which have the coordinates as arguments belong to a unitary representation of $SO(3)$ and the simplest is the trivial one : they depend only on t, ρ.

We are free to choose our charts and gauges. So in a problem one can choose to take a particle as the observer, apply the rules above, then the results can be translated for any observer by applying the rules for a change of observer, using the Principle of Relativity. This is the simplest, and most rigorous, way to compute the EM field created by a charged particle.

5 SPECIAL RELATIVITY

All the previous results hold in Special Relativity. This theory, which is still the geometric framework of QTF and Quantum Physics, adds *two* assumptions : the Universe M can be represented as an affine space, and the metric does not depend on the location (these assumptions are independent). As consequences:

- the underlying vector space $\overrightarrow{M}$ (the Minkovski space) is common to all observers : the vectors of all tangent spaces to M belong to $\overrightarrow{M}$

- one can define orthonormal bases which can be freely transported and compared from a location to another

- because the scalar product of vectors does not depend on the location, at each point one can define time-like and space-like vectors, and a future orientation (this condition relates the mathematical and the physical representations, and $\overrightarrow{M}$ is not simply $\mathbb{R}^4$)

- there are fixed charts $\left(O,(\varepsilon_i)_{i=0}^3\right)$, called frames, which consist of an origin (a location O in M : a point) and an orthonormal basis $(\varepsilon_i)_{i=0}^3$. There is necessarily one vector such that $\langle\varepsilon_0,\varepsilon_0\rangle=-1$. It is possible to define, non unique, orthonormal bases such that ε_0 is timelike and future oriented.

- the coordinates of a point m, in any frame $\left(O,(\varepsilon_i)_{i=0}^3\right)$, are the components of the vector OM.

The general results hold and observers can define a standard chart as seen in RG. However this chart is usually not defined by a frame $\left(O,(\varepsilon_i)_{i=0}^3\right)$. Observers can label points which are in their present with their proper time. The role of the function $f(m)=t$ is crucial, because it defines the 3 dimensional hypersurfaces $\Omega(t)$. They are not necessarily hyperplanes, but they must be space like and do not cross each other : a point m cannot belong to 2 different hypersurfaces. These hypersurfaces define the vector field $\varepsilon_0(m)$ to which belongs the velocity of the observer (up to c). In SR one can compare vectors at different points, and usually the vectors $\varepsilon_0(m)$ are different from one location to another. They are identical only if $\Omega(t)$ are hyperplanes normal to a vector ε_0, which implies that the world line of the observer is a straight line, and because the proper time is the parameter of the flow, if the motion of the observer is a translation at a constant spatial speed. These observers are called **inertial**. Notice that this definition is purely geometric and does not involve gravitation or inertia : inertial observers are such that their velocity is a constant vector. *A frame can be associated to an observer only if this is an inertial observer.*

For inertial observers the integral curves are straight lines parallel to ε_0. Any spatial basis $(\varepsilon_i)_{i=1}^3$ of $\Omega(0)$ can be transported on $\Omega(t)$. The standard chart is then similar to a frame in the 4 dimensional affine space $\left(O(0),(\varepsilon_i)_{i=0}^3\right)$ with origin $O(0)$, the 3 spatial vectors $(\varepsilon_i)_{i=1}^3$ and the time vector ε_0.

6 COSMOLOGY

General Relativity has open the way to a "scientific Cosmology", that is the study of the whole Universe and in particular of its evolution, through mathematical models. These theories will never achieve a full scientific status, because they lack one of the key criteria : the possibility to experiment with other universes. They can provide plausible explanations, but not falsifiable ones. This is reflected in the choice of the parameters which are used in the models : one can fine tune them in order to fit with observations, essentially astronomical observations, and represent in a satisfying way "what it is", but not tell "why is it so".

One of the issue of Cosmology is that of the observer, who is an essential part of Relativity. Particles (and galaxies can be considered as particles at this scale) follow world lines. Their location, which is absolute in GR, is precisely defined with respect to a proper time, but this time is specific to each particle. An observer can follow particles which are in his present, and establish a relation between his proper time and that of these particles. A Cosmological model is a model for an observer who would have access to the locations of all the particles of the universe, and indeed the existence of a universal time, which provides a foliation in hypersurfaces analogous to $\Omega_3(t)$, is one of their key component.

A manifold by itself can have some topological properties. It can be compact. It can have holes, defined through homotopy : there is a hole if there are curves in M which cannot be continuously deformed to be reduced to a point. A hole does not imply some catastrophic feature : a doughnut has a hole. Thus it does not imply that the charts become singular. But there are only few purely topological features which can be defined on a manifold, and they are one of the topics of Differential Geometry. In particular a manifold has no shape to speak of.

The metric on M is an addition to the structure of the Universe. It is a mathematical feature from which more features can be defined on M, such that curvature. In GR the metric, and so the curvature of M at a point, depends on the distribution of matter. It is customary (see Wald) to define singularities in the Universe by singularities of geodesics, but geodesics are curves whose definition depends on the metric. A

singularity for the metric, as Black holes or Bing Bang, is not necessarily a singular point for the manifold itself.

From some general reasoning and Astronomical observations, it is generally assumed that the Universe has the structure of a fiber bundle with base $\mathbb{R}$ (a warped Universe) which can be seen as the generalization of M_o, that we have defined above for an observer. Thus there is some universal time (the projection from M to $\mathbb{R}$) and a foliation of M in hypersurfaces similar to $\Omega_3(t)$, which represent the present for the observers who are located on them. The part of the universe on which stands all matter would be a single body moving since the Big Bang (the image of an inflating balloon). So there would not be any physical content before or after this $\Omega_3(t)$ (inside the balloon), but nothing can support this interpretation, or the converse, and probably it will never be.

The Riemannian metric $\varpi_3(t)$ on each $\Omega_3(t)$ is induced by the metric on M, and therefore depends on the universal time t. In the most popular models it comes that the distance between two points on $\Omega_3(t)$, measured by the Riemannian metric, increases with t, and this is the foundation of the narrative about an expanding universe, which is supported by astronomical observations. But, assuming that these models are correct, this needs to be well understood. The change of the metric on $\Omega_3(t)$ makes that the volume form $\varpi_3(t)$ increases, but the hypersurfaces $\Omega_3(t)$ belong to the same manifold M, which does not change with time. The physical universe would be a deformable body, whose volume increases inside the unchanged container. And of course material points do not swell, only the vacuum, which separates material bodies, dilates.

Part II
MOTION

So far we have considered only particles, with no internal structure. The concept of a "material point" which occupies a geometric point, that is with no spatial extension, used to be shocking for many physicists. Actually Mechanics is built around the concept of solids, which can be rigid or deformable, but have an extension, and a particle is seen as an infinitesimal small solid. Solids bring a feature additional to their location, they have an "arrangement", which is represented by an orthonormal basis. As a consequence the motion of a solid encompasses not only a change in its location, but also a rotational motion. Motion, translational and rotational, is a purely geometric concept which is measured by geometric protocols. And we are lead to extend these properties to material points, that is particles : they have a location and an attached orthonormal basis.

The Relativist framework requires a new formalism to represent the motion of a material body, but it is useful to remind how this is done in Galilean Geometry.

7 MOTION IN GALILEAN GEOMETRY

7.1 Rotation in Galilean Geometry

The concept of rotation is well defined in Mathematics : this is the operation which transforms the *orthonormal basis* of a vector space into another. From a physical point of view the rotation is the operation which transforms the orthonormal basis of the observer to an orthonormal basis which is attached to the material body : it measures the *arrangement* of the body with respect to the observer.

The operation belongs to the orthogonal group, in Galilean Geometry to $SO(3)$ and is represented by a matrix R. This is a 3 dimensional Lie group of matrices such that $R^tR = I$. Because of this relation the Lie algebra $so(3) = T_1SO(3)$ is the vector space of 3×3 real antisymmetric matrices. If we take the following matrices as basis of $so(3)$:

$$\kappa_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; \kappa_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \kappa_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then any matrix of $so(3)$ reads :
$\sum_{i=1}^{3} r^i [\kappa_i] = [j(r)]$ with the operator

$$j : \mathbb{R}^3 \to L(\mathbb{R}, 3) :: [j(r)] = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \tag{28}$$

The operator j is very convenient to represent quantities which are rotated [3]. It has many nice algebraic properties and we will use it often.

For any vector u : $\sum_{ij=1}^{3} [j(r)]_j^i u^j \varepsilon_i = \overrightarrow{r} \times \overrightarrow{u}$ with the cross product $\times$.

The group $SO(3)$ is compact, thus the exponential is onto and any matrix of $SO(3)$ can be written as :
$\exp[j(r)] = I_3 + \frac{\sin\sqrt{r^tr}}{\sqrt{r^tr}}[j(r)] + \frac{1-\cos\sqrt{r^tr}}{r^tr}[j(r)][j(r)]$

The vector r is just the components of a vector in a Lie algebra, using a specific basis κ. However there is a natural correspondence between r and geometric characteristics of a rotation.

The axis of rotation is by definition the unique eigen vector of $[g]$ with eigen value 1 and norm 1 in the standard representation of $SO(3)$, it has for components $\begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} / \sqrt{r^tr}$

Similarly one can define the angle θ of the rotation resulting from a given matrix, and $\theta = \sqrt{r^tr}$.

7.2 Rotational motion

We use freely the same word "rotation" for the operation to go from one orthonormal basis to another (the arrangement of a basis with respect to another), and for the motion (the instantaneous rotation around an axis), but they are two distinct concepts and the distinction is essential.

If 2 orthonormal bases (with same origin) are in relative motion, at any time t we have some rotation $R(t) \in SO(3)$ and naturally the instantaneous rotation is defined through the derivative $\frac{dR}{dt}$.

[3] It is similar to the Levi-Civitta tensor ϵ but, in my opinion, much easier to use.

The usual convention is to represent the instantaneous rotational motion by $R(t)^{-1}\frac{dR}{dt} \in so(3)$, which takes as starting point the frame rotated by $R(t)$. Then it can be represented by a single vector : $R(t)^{-1}\frac{dR}{dt} = j(r)$. This choice is not without consequence : in a change of observer, corresponding to $R \to \widetilde{R} = g \times R$: $R(t)^{-1}\frac{dR}{dt}$ does not change : in Galilean Geometry a rotational motion is observer independent. The instantaneous rotational motion can be assimilated to a rotation with constant axis r and rotational speed $\sqrt{r^t r}$: $R(t) = \exp tj(r)$.

So we have a very satisfying representation of geometric rotations : a rotation R can be defined by a single vector, which is simply related to essential characteristics of the transformation, and an instantaneous rotational movement can also be represented by a single vector r. But, as one can see, this model is less obvious than it seems. It relies on the fortuitous fact that the Lie algebra has the same dimension as the Euclidean space (the dimension of $so(n)$ is $\frac{n(n-1)}{2}$) and is compact.

7.3 Spin group

Moreover this mathematical representation is not faithful. The same rotation can be defined equally by the opposite axis, and the opposite angle. This is related to the mathematical fact that $SO(3)$ is not the only group which has $so(3)$ as Lie algebra. The more general group is the Spin group $Spin(3)$ which has also for elements the scalars + 1 and - 1, so that $R(t)$, corresponding to $\left(r, \sqrt{r^t r}\right)$ and $-R(t)$, corresponding to $\left(-r, -\sqrt{r^t r}\right)$ can represent the same physical rotational motion. Actually, the group which should be used to represent rotations in Galilean Geometry is $Spin(3)$, which makes the distinction between the two rotations, and not $SO(3)$. In Physics the distinction matters : in the real world one goes from one point to another along a path, by a continuous transformation which preserves the orientation of a vector, thus the orientation of $\overrightarrow{r}$ is significant. A single vector of $\mathbb{R}^3$ cannot by itself properly identify a physical rotation, one needs an additional parameter which is ± 1 to tell which one of the two orientations of $\overrightarrow{r}$ is chosen, with respect to a direction, the spatial speed on the path.

7.4 Motion of a rigid solid

One can choose any point G, a fixed orthonormal basis $(e_i)_{i=1}^3$ attached to the solid, and represent the arrangement of the rigid solid at a given time as the operation to go from a fixed orthogonal frame $\left(O, (\varepsilon_i)_{i=1}^3\right)$ to $\left(G, (e_i)_{i=1}^3\right)$. It combines a translation D, belonging to the abelian group $\mathcal{T}\left(\mathbb{R}^3\right)$ and a rotation $R \in SO(3)$, and belongs to the group of displacement, which is the semi-direct product $\mathcal{T}\left(\mathbb{R}^3\right) \ltimes SO(3)$. The "semi" implies some relations which makes the structure of the group of displacements more complicated than the direct product $\mathcal{T}\left(\mathbb{R}^3\right) \times SO(3)$.

The motion (translational and rotational) of a rigid solid is then represented by the derivative of the displacement, or more conveniently by the value $\left(\frac{dD}{dt}, R^{-1}\frac{dR}{dt}\right)$ of the corresponding elements in the Lie algebra $T_1\left(\mathcal{T}\left(\mathbb{R}^3\right) \ltimes SO(3)\right)$, which is not the direct product $\left(\mathcal{T}\left(\mathbb{R}^3\right) \times so(3)\right)$. This is convenient because we can represent the motion by two vectors : $\overrightarrow{v}_G = \frac{d\overrightarrow{OG}}{dt}, r$ such as $R^{-1}\frac{dR}{dt} = [j(r)]$, however the formulas are a bit complicated (as can be seen in the law for the composition of speeds for rotating bodies) because the displacement is not a direct product.

So the representation of the motion of a rigid solid in Galilean Geometry implies :

- the location of G and its speed $\overrightarrow{v}_G = \frac{d\overrightarrow{OG}}{dt}$
- the rotation R of $\left(G, (e_i)_{i=1}^3\right)$ and its instantaneous change $R^{-1}\frac{dR}{dt}$

The motion is defined by 6 scalar parameters, or two 3 dimensional vectors.

7.5 Deformable solid

A deformable solid is a material body which keeps some integrity : its material points stay close to each other. It can be conveniently represented as follows.

The body occupies at the time t a compact area $\omega(t) \subset \mathbb{R}^3$. Each material point is identified by its location q at a time $t = 0$. It is assumed that there is a differentiable map : $\phi : \omega(0) \times \mathbb{R} \to \omega(t) :: x = \phi(q,t)$ which gives the location of the material point q at t. The map ϕ is the representation of the continuity of the body.

The orthonormal basis $(\varepsilon_i)_{i=1}^3$ of $\mathbb{R}^3$ at $t = 0$ is transported as : $e_i(q,t) = \phi'_q(q,t)\,\varepsilon_i$ which is usually no longer orthonormal.

By derivation :

$\frac{\partial}{\partial t}e_i(q,t) = \phi^{"}_{qt}(q,t)\,\varepsilon_i = \phi^{"}_{qt}(q,t)\left(\phi'_q(q,t)\right)^{-1} e_i(q,t)$

and the matrix $\gamma = \left[\phi''_{qt}(q,t)\left(\phi'_q(q,t)\right)^{-1}\right]$ is the deformation tensor. It can be decomposed in a symmetric matrix $\frac{1}{2}(\gamma+\gamma^t) = s$ and an antisymmetric matrix $\frac{1}{2}(\gamma-\gamma^t) = j(\omega)$ which measures the torsion. s has real eigen values and represents similitudes in the 3 axes (a "dilation"). $j(\rho)$ can be seen as a rotation with vector ρ (a "shear"), and the deformation tensor is the sum of a shear $j(\rho)$ and a dilation s.

ϕ defines the manifolds $\omega(t) = \phi(.,t)$ embedded in $\mathbb{R}^3$ endowed with the induced metric :

$g_{ij} = \sum_{k=1}^{3}\left[\phi'_q(q,t)\right]_i^k\left[\phi'_q(q,t)\right]_j^k$.

The distance between 2 close elements $\delta q \in T_q\omega(0)$ change as $\sqrt{\sum_{ij} g_{ij}(\delta q)^i(\delta q)^j}$

The volume form is $\varpi = \sqrt{\det g}\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3 = \det\left[\phi'_q(q,t)\right]\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3$ and the volume changes as $\det\left[\phi'_q(q,t)\right]$: the material points which occupy a volume $\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3$ at $t=0$ occupy a volume $\left(\det\left[\phi'_q(q,t)\right]\right)\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3$ at t.

8 MOTION IN RELATIVIST GEOMETRY

8.1 The Poincaré's group

The usual concept of rigid solid, as material body whose material points are at a constant distance, does not hold any more in the Relativist framework. Experiments show that atoms and subatomic particles have kinematic characteristics which look like rotation, and can be measured by quantities which transform according to the rules of $SO(3)$, with some complications, and this leads to the concept of spin. So one needs to incorporate rotations in Relativity, in a way similar to what is done with solids in Newtonian Mechanics, and this leads naturally to look for the Poincaré's group, the semi product of the group $SO(3,1)$ of rotations and of the 4 dimensional translations. This is the simple generalization of the group of displacements of Galilean Geometry. In Special Relativity (and also in QTF) a law is deemed covariant if it is equivariant in a change of frame by the Poincaré's group : this is the implementation of the Principle of Relativity in a representation based on orthogonal frames.

However the use of the Poincaré's group raises several serious issues.

The Poincaré's group represents the operation to go from one orthonormal frame $\left(O,(\varepsilon_i)_{i=0}^3\right)$ to another $\left(A,(e_i)_{i=0}^3\right)$. So its use is valid only in SR, and for inertial observers. It has been considered in GR to use the group of isometries, that is of maps : $f : M \to M$ such that $f'(m) \in \mathcal{L}(T_mM;T_mM)$ preserves the metric. However in Physics, to compare two bases located at different points one does not jump, one follows a path and the path matters : the relativist universe is not isotropic. This amplifies the issue of the spin group and its 2 values ±1.

According to the Principle of Locality the location (O) of the origin of the frame has no physical meaning : we should compare two frames, located at the same point (as we did to prove the formulas to go from one observer to another). A displacement introduces a variable (the translation of the origin O of the frame to go from O to A) which has nothing to do in the matter : in the formulas in a change of observers the spatial speed $\overrightarrow{v}$ is the relative speed with respect to a "copy" of the observer who would be at the same location as the body. Indeed an element of the Poincaré's group is defined by 10 parameters (6 for the Lorentz group and 4 for the translation of the origin), meanwhile 6 suffice in Newtonian Mechanics to define the motion of a solid, and there is no reason why Relativity should add 4 parameters.

A group of displacement is not a direct product of groups, but a semi-direct product, and similarly for the Lie algebras. This introduces complications in Newtonian Mechanics which are amplified in Relativity. The exponential is not surjective for $SO(3,1)$, which is not a compact group. We have $[\chi] = \exp[K(w)]\exp[J(r)]$ where $[K(w)],[J(r)] \in so(3,1)$ thus the derivative $\frac{d\chi}{dt}$ gives a more complicated expression, where $\frac{dw}{dt}, \frac{dr}{dt}$ are mixed with (w,r). In particular appears $\frac{dw}{dt}$, that is the derivative of the spatial speed.

8.1.1 The Spin Bundle

Our purpose is to find an efficient way to represent the motion, translational and rotational in the General Relativity frame-work. We start from 4 facts :

i) We do not need the Poincaré's group : it is defined only in SR and for inertial observers. The origin O of the frame has no physical meaning, the measures should be done at the same location.

ii) The only clear concept of rotation is done by comparing the arrangement of two orthonormal bases, located at the same point. And in the relativist context this requires to consider a group which preserves the Lorentz scalar product.

iii) The right group to consider is the spin group. This holds already in Galilean Geometry, and in Relativist Geometry any observer can distinguish the orientation of the axis of a spatial rotation with respect

to his own velocity. The spin groups $Spin(3,1), Spin(1,3)$ are isomorphic so on this point the signature does not matter.

iv) The convenient tool to compare orthonormal bases at a point is a principal fiber bundle.

We have already assumed the existence of a principal bundle $P_G(M, SO(3,1), \pi)$, so we make the assumption :

Proposition 21 *There is a principal bundle $P_G(M, Spin_0(3,1), \pi_G)$ which has for fiber the connected component of the identity of the Spin group, and for trivialization the map :*
$\varphi_G : M \times Spin_0(3,1) \to P_G :: p = \varphi_G(m,s)$.
The standard gauge used by observers is $\mathbf{p}(m) = \varphi_G(m, \mathbf{1})$

A section $\sigma \in \mathfrak{X}(P_G)$ is defined by a map: $\sigma : M \to Spin(3,1)$ such that : $\sigma(m) = \varphi_G(m, \sigma(m))$ and in a change of gauge :

$$\begin{array}{l} \mathbf{p}(m) = \varphi_G(m,1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \\ \sigma(m) = \varphi_G(m,\sigma) = \widetilde{\varphi}_G(m, \chi(m) \cdot \widetilde{\sigma}) : \widetilde{\sigma} = \chi(m) \cdot \sigma \end{array} \tag{29}$$

8.2 Motion of two orthonormal bases

Orthonormal bases are defined in the vector bundle associated to P_G. The arrangement of an orthonormal basis $(e_i(m))_{i=0..3}$ is measured with respect to the tetrad $(\varepsilon_i(m))_{i=0..3}$ of an observer by an element $[\chi]$ of P_G located at m.

$e_i(m) = \left(\varphi_G(m,1), \sum_{j=0}^{3} [\chi(m)]_i^j \varepsilon_j\right)$

The vectors ε_j are fixed. The motion is given by the derivative

$\frac{d}{dt} e_i(m) = \left(\varphi_G(m,1), \sum_{j=0}^{3} \left[\frac{d}{dt}\chi(m)\right]_i^j \varepsilon_j\right)$

and represented by $\left[\frac{d}{dt}\chi(m)\right]$, that is by the derivatives of the components of $e_i(m)$ in the fixed basis ε_j : the change of the tetrad (that is $\frac{dP}{dt}$) is not involved.

$[\chi] = \exp[K(w)] \exp[J(r)]$

The time axis e_0 is related to the velocity, w is related to the spatial speed $\overrightarrow{v}$, and r to the rotation of the spatial axes.

So, to be consistent, the definition of the motion should involve the derivative of the velocity, that is the spatial acceleration. And, indeed, an observer attached to a material body can measure both a rotational motion and a change in its transversal motion. Assume that, to any material object, whatever its size, is attached an orthonormal basis is not without consequence : this is an extension of the concept of particle, with additional physical properties, which must be accounted for in their representation. In some way it gives relief to the Geometry. Many models in theoretical physics involve a universe with more than 4 dimensions, to account for their physical properties such as charges. One could consider to define a material body by 4 coordinates, corresponding to its location, and 4 additional coordinates for their arrangement. However the arrangement has a meaning only locally, and with respect to a special basis : an orthonormal one (this is the only sensible way to represent a rotation). So actually these properties are closely related to the metric, which is the physical part of the Geometry. The time vector keeps is specificity, it is necessarily oriented as the velocity, but the same restriction does not apply to its derivative.

So with a map : $\mathbb{R} \to P_G :: \varphi_G(q(t), \chi(t))$ the arrangement and the motion can be efficiently represented. The motion depends on two vectors r, w of $\mathbb{R}^3$ and their derivatives. However the relation $[\chi] = [\exp K(w)][\exp J(r)]$ is not convenient, and the group which is involved is the Spin group and not $SO(3,1)$. In order to get a good understanding of this representation and more convenient tools, we need to introduce Clifford Algebras, which are at the root of the Spin groups.

9 CLIFFORD ALGEBRAS

The proofs are given in my book "Theoretical Physics".

9.1 Clifford algebra and Spin groups

A Clifford algebra $Cl(F, \langle\rangle)$ is an algebraic structure, which can be defined on any vector space $(F, \langle\rangle)$ on a field K ($\mathbb{R}$ or $\mathbb{C}$) endowed with a bilinear *symmetric* form $\langle\rangle$. The set $Cl(F, \langle\rangle)$ is defined from K, F and a product, denoted $\cdot$, with the property that for any two *vectors* u, v :

$$\forall u, v \in F : u \cdot v + v \cdot u = 2\langle u, v\rangle \tag{30}$$

A Clifford algebra is then a set which is larger than F : it includes *all vectors* of F, plus scalars, and any linear combinations of products of vectors of F. A Clifford algebra on a n dimensional vector space is a 2^n dimensional vector space on K, and an algebra with $\cdot$. Clifford algebras built on vector spaces on the same field, with same dimension and bilinear form with same signature are isomorphic. On a 4 dimensional real vector space $(F, \langle\rangle)$ endowed with a Lorentz metric there are two structures of Clifford Algebra, denoted $Cl(3,1)$ and $Cl(1,3)$, depending on the signature of the metric, and they are *not* isomorphic. The easiest way to work with a Clifford algebra is to use an orthonormal basis of F.

Notation 22 $(\varepsilon_i)_{i=0}^3$ *is an orthonormal basis with scalar product :* $\langle \varepsilon_i, \varepsilon_i \rangle = \eta_{ii}$

So we have the relation :

$$\varepsilon_i \cdot \varepsilon_j + \varepsilon_j \cdot \varepsilon_i = 2\eta_{ij} \tag{31}$$

A basis of the Clifford algebra is a set which consists of 1 and all ordered products of $\varepsilon_i, i = 0...3$.

Some elements of the Clifford algebra have an inverse for the product, and there are subsets which have a group structure. **Spin(3,1)** is the subset of the Clifford algebra $Cl(3,1)$ with an even number of vectors with scalar product ± 1 : $Spin(3,1) = \{u_1 \cdot u_2 ... \cdot u_{2k}, \langle u_p, u_p \rangle = \pm 1, u_p \in F\}$

The *scalars* ± 1 belong to the groups. The identity element is the scalar 1.

$Spin(3,1)$ and $Spin(1,3)$ are isomorphic.

For any $s \in Spin(3,1)$, the map, called the **adjoint map** :

$$\mathbf{Ad}_s : Cl(3,1) \to Cl(3,1) :: \mathbf{Ad}_s X = s \cdot X \cdot s^{-1} \tag{32}$$

is such that :

$\forall V \in F : \mathbf{Ad}_s V \in F$

$\forall u, v \in F, s \in Pin(3,1) : \langle \mathbf{Ad}_s u, \mathbf{Ad}_s v \rangle_F = \langle u, v \rangle_F$

$\forall s, s' \in Pin(3,1) : \mathbf{Ad}_s \circ \mathbf{Ad}_{s'} = \mathbf{Ad}_{s \cdot s'}$

Ad is distributive with respect to the addition and the product.

Because the action $\mathbf{Ad}_s$ of $Spin(3,1)$ on F gives another vector of F and preserves the scalar product, it can be represented by a 4×4 orthogonal matrix. In any orthonormal basis $(\varepsilon_i)_{i=0}^3$ of F, then $\mathbf{Ad}_s$ is represented by a matrix $[h(s)] \in SO(3,1)$. To two elements $\pm s \in Spin(3,1)$ correspond a unique matrix $[h(s)]$.

As any algebra $Cl(F, \langle\rangle)$ is a Lie algebra with the bracket :

$\forall X, X' \in Cl(F, \langle\rangle) : [X, X'] = X \cdot X' - X' \cdot X$

which is a bilinear, antisymmetric operation (but not associative) with the Jacobi identity :

$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

The group $Spin(3,1)$ has a Lie algebra $T_1 Spin(3,1)$ which is a subset of the Clifford algebra. Its elements can be written as linear combinations of pairs of elements $\varepsilon_i \cdot \varepsilon_j$.- The map : $\Pi : so(3,1) \to T_1 Spin(3,1)$ is an isomorphism of Lie algebras which reads with any orthonormal basis $(\varepsilon_i)_{i=0}^3$ of F : $\Pi([\kappa]) = \frac{1}{4} \sum_{i,j=0}^3 ([\kappa][\eta])_j^i \varepsilon_i \cdot \varepsilon_j$

so that any element of $T_1 Spin(3,1)$ is the linear combinations of the ordered products of all the four vectors of a basis. With any orthonormal basis and the following choice of basis $(\overrightarrow{\kappa}_a)_{a=1}^6$ of $T_1 Spin(3,1)$ then Π takes a simple form with an adequate ordering of the vectors :

$\Pi([\kappa_1]) = \overrightarrow{\kappa}_1 = \frac{1}{2}\varepsilon_3 \cdot \varepsilon_2$,

$\Pi([\kappa_2]) = \overrightarrow{\kappa}_2 = \frac{1}{2}\varepsilon_1 \cdot \varepsilon_3$,

$\Pi([\kappa_3]) = \overrightarrow{\kappa}_3 = \frac{1}{2}\varepsilon_2 \cdot \varepsilon_1$,

$\Pi([\kappa_4]) = \overrightarrow{\kappa}_4 = \frac{1}{2}\varepsilon_0 \cdot \varepsilon_1$,

$\Pi([\kappa_5]) = \overrightarrow{\kappa}_5 = \frac{1}{2}\varepsilon_0 \cdot \varepsilon_2$,

$\Pi([\kappa_6]) = \overrightarrow{\kappa}_6 = \frac{1}{2}\varepsilon_0 \cdot \varepsilon_3$

where $([\kappa_a])_{a=1}^6$ is the basis of $so(3,1)$ already noticed such that :

$[\kappa] = K(w) + J(r) = \sum_{a=1}^3 r^a [\kappa_a] + w^a [\kappa_{a+3}]$

We will use extensively the convenient (the order of the indices matters) :

Notation 23 *for both* $Cl(3,1), Cl(1,3)$:

$$v(r,w) = \frac{1}{2}\left(w^1\varepsilon_0 \cdot \varepsilon_1 + w^2\varepsilon_0 \cdot \varepsilon_2 + w^3\varepsilon_0 \cdot \varepsilon_3 + r^3\varepsilon_2 \cdot \varepsilon_1 + r^2\varepsilon_1 \cdot \varepsilon_3 + r^1\varepsilon_3 \cdot \varepsilon_2\right) \tag{33}$$

With this notation, whatever the orthonormal basis $(\varepsilon_i)_{i=0}^3$, any element X of the Lie algebras $T_1Spin(3,1)$ or $T_1Spin(1,3)$ reads :

$$X = \upsilon(r,w) = \sum_{a=1}^{3} r^a \overrightarrow{\kappa}_a + w^a \overrightarrow{\kappa}_{a+3} \tag{34}$$

The bracket on the Lie algebra reads :
$[\upsilon(r,w), \upsilon(r',w')]$
$= \upsilon(r,w) \cdot \upsilon(r',w') - \upsilon(r',w') \cdot \upsilon(r,w)$
$= \upsilon(j(r) r' - j(w) w', j(w) r' + j(r) w')$
With signature (1,3) :
$[\upsilon(r,w), \upsilon(r',w')] = -\upsilon(j(r) r' - j(w) w', j(w) r' + j(r) w')$
The map : $\mathbf{Ad} : Spin(3,1) \to \mathcal{L}(T_1Spin(3,1); T_1Spin(3,1))$ is differentiable with respect to g.

$$(\mathbf{Ad}_g X)' = \mathbf{Ad}_g\left(\left[g^{-1} \cdot g', X\right]\right) \tag{35}$$

where $g' = \frac{d}{dx} g(x)$ for x belonging to any manifold.

With these notations, the elements of the Spin group read in both signatures, with the related $a, (w^j, r^j)_{j=1}^3, b$ real scalars and $\varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$:

$$\begin{array}{c} s = a + \upsilon(r,w) + b\varepsilon_5 \\ a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r) \\ ab = -\frac{1}{4} r^t w \\ (a + \upsilon(r,w) + b\varepsilon_5)^{-1} = a - \upsilon(r,w) + b\varepsilon_5 \end{array} \tag{36}$$

The product $s \cdot s$' reads with the operator j introduced previously :
$(a + \upsilon(r,w) + b\varepsilon_5) \cdot (a' + \upsilon(r',w') + b'\varepsilon_5) = a" + \upsilon(r",w") + b"\varepsilon_5$
with :
$a" = aa' - b'b + \frac{1}{4}(w^t w' - r^t r')$
$b" = ab' + ba' - \frac{1}{4}(w^t r' + r^t w')$
and in $Spin(3,1)$:
$r" = \frac{1}{2}(j(r) r' - j(w) w') + a'r + ar' - b'w - bw'$
$w" = \frac{1}{2}(j(w) r' + j(r) w') + a'w + aw' + b'r + br'$
and in $Spin(1,3)$:
$r" = \frac{1}{2}(j(r) r' - j(w) w') + a'r + ar' + b'w + bw'$
$w" = -\frac{1}{2}(j(w) r' + j(r) w') + a'w + aw' + b'r + br'$

There is a scalar product on $Cl(F, \langle\rangle)$ defined by :
$\langle u_{i_1} \cdot u_{i_2} \cdot ... \cdot u_{i_n}, v_{j_1} \cdot v_{j_2} \cdot ... \cdot v_{j_n} \rangle = \langle u_{i_1}, v_{j_1} \rangle \langle u_{i_2}, v_{j_2} \rangle ... \langle u_{i_n}, v_{j_n} \rangle$
It does not depend on the choice of a basis, and any orthonormal basis defined as above is orthonormal. It is invariant by **Ad**.

$$\forall w, w' \in Cl(F, \langle\rangle) : \langle \mathbf{Ad}_s w, \mathbf{Ad}_s w' \rangle_{Cl(E,\langle\rangle)} = \langle w, w' \rangle_{Cl(E,\langle\rangle)} \tag{37}$$

The basis of the Lie algebra is orthogonal.

$$\begin{array}{c} T_1Spin(3,1) : \langle \upsilon(r,w), \upsilon(r',w') \rangle_{Cl} = \frac{1}{4}(r^t r' - w^t w') \\ T_1Spin(1,3) : \langle \upsilon(r,w), \upsilon(r',w') \rangle_{Cl} = -\frac{1}{4}(r^t r' - w^t w') \end{array} \tag{38}$$

We have the identity : $\forall X, Y, Z \in Cl(3,1) : \langle X, [Y,Z] \rangle_{Cl} = \langle [X,Y], Z \rangle_{Cl}$

9.2 Symmetry breakdown

The elements of $SO(3,1)$ are the product of spatial rotations (represented by $\exp J(r)$) and boosts, linked to the speed and represented by $\exp K(w)$. We have similarly a decomposition of the elements of $Spin(3,1)$. But to understand this topic, from both a mathematical and a physical point of view, we need to distinguish the abstract algebraic structure and the sets on which the structures have been defined.

From a vector space $(F, \langle\rangle)$ endowed with a scalar product one can built only one Clifford algebra, which has necessarily the structure $Cl(3,1)$: as a set $Cl(3,1)$ must comprise all the vectors of F. But from any vector subspace of F one can built different Clifford algebras : their algebraic structure depends on the dimension of the vector space, and on the signature of the metric induced on the vector subspace. To have a Clifford algebra structure $Cl(3)$ on F one needs a 3 dimensional vector subspace on which the scalar product

is definite positive, so it cannot include any vector such that $\langle u,u\rangle<0$ (and conversely for the signature $(1,3)$: the scalar product must be definite negative). The subsets of F which are a 3 dimensional vector subspace and do not contain any vector such that $\langle u,u\rangle<0$ are not unique. So we have different subsets of $Cl(3,1)$ with the structure of a Clifford algebra $Cl(3)$, all isomorphic but which do not contain the same vectors. Because the Spin Groups are built from elements of the Clifford algebra, we have similarly isomorphic Spin groups $Spin(3)$, but with different elements. The simplest way to deal with these issues is to fix a vector ε_0.

For a given fixed ε_0, the group $Spin(3)$ can be identified with the subset of $Spin(3,1)$ such that : $\mathbf{Ad}_{s_r}\varepsilon_0 = s_r\cdot\varepsilon_0\cdot s_r^{-1}=\varepsilon_0$ and it reads : $Spin(3)=\left\{s_r=\epsilon\sqrt{1-\frac{1}{4}r^tr}+\upsilon(r,0),r\in\mathbb{R}^3,r^tr\leq 4,\epsilon=\pm1\right\}$

$Spin(3)$ is a compact group, with 2 connected components. The connected component of the identity consist of elements with $\epsilon=1$ and can be assimilated to $SO(3)$.

The quotient space $SW=Spin(3,1)/Spin(3)$ is not a group but a 3 dimensional manifold. Then, for a given vector ε_0, any element $s\in Spin(3,1)$ can be written uniquely (up to sign) : $s=s_w\cdot s_r$ with $s_w\in SW, s_r\in Spin(3)$:

$$\forall s=a+\upsilon(r,w)+b\varepsilon_5\in Spin(3,1):s=\epsilon(a_w+\upsilon(0,w))\cdot\epsilon(a_r+\upsilon(r,0)) \tag{39}$$

with : $a_r=\sqrt{1-\frac{1}{4}r^tr};a_w=\sqrt{1+\frac{1}{4}w^tw}$

$\upsilon(r,0)$ is represented in $so(3,1)$ by a matrix $[J(r)]$ and $\upsilon(0,w)$ by a matrix $[K(w)]$. So we replace the cumbersome formula in a change of gauge $[\chi]=\exp[K(w)]\exp[J(r)]$ by $s=s_w\cdot s_r$.with two elements which are simply related to the velocity (by w) and the rotation (by r). The decomposition depends on the choice of ε_0.

Similarly we have the same decomposition in the Lie algebra.

$T_1Spin(3,1)=L_0\oplus P_0$

where :

$L_0=\left\{\upsilon(r,0),r\in\mathbb{R}^3\right\}\simeq T_1Spin(3),P_0=\left\{\upsilon(0,w),w\in\mathbb{R}^3\right\}\simeq T_1SW.$

The vectors r,w depends on the basis (they are components), however the elements $\upsilon(r,0),\upsilon(0,w)$ depend only on the choice of ε_0 and L_0,P_0 are globally invariant by $Spin(3)$.

The operator : $\mathbf{Ad}:Spin(3,1)\times Cl(3,1)\to Cl(3,1)::\mathbf{Ad}_sX=s\cdot X\cdot s^{-1}$ takes a different matrix form depending on X. With $s=a+\upsilon(r,w)+b\varepsilon_5$:

The action of $Spin(3,1)$ on vectors of F is :

$v=\sum_{i=0}^3v^i\varepsilon_i\to\widetilde{v}=\mathbf{Ad}_sv=\sum_{i=0}^3\widetilde{v}^i\varepsilon_i$

$\widetilde{v}^i=\sum_{j=0}^3[h(s)]_j^iv^j$

$[h(s)]=$

$\begin{bmatrix} a^2+b^2+\frac{1}{4}(r^tr+w^tw) & aw^t-br^t+\frac{1}{2}w^tj(r) \\ aw-br+\frac{1}{2}j(r)w & a^2+b^2+\frac{1}{4}(r^tr+w^tw)+aj(r)+bj(w)+\frac{1}{2}(j(r)j(r)+j(w)j(w)) \end{bmatrix}$

$[h(s)]\in SO(3,1):[h(s)]^t[\eta][h(s)]=[\eta]$.

For a product : $\mathbf{Ad}_s\circ\mathbf{Ad}_{s'}=\mathbf{Ad}_{s.s'}\to[h(s.s')]=[h(s)][h(s')]$

Then if $s=s_w\cdot s_r:[h(s)]=[h(s_w)][h(s_r)]$

If $s=a_w+\upsilon(0,w)$

$[h(s)]=\begin{bmatrix} 2a_w^2-1 & a_ww^t \\ a_ww & 2a_w^2-1+\frac{1}{2}j(w)j(w) \end{bmatrix}$

If $s=a_r+\upsilon(r,0)$

$[h(s)]=\begin{bmatrix} 1 & 0 \\ 0 & 1+a_rj(r)+\frac{1}{2}j(r)j(r) \end{bmatrix}$

$[C(r)]=1+a_rj(r)+\frac{1}{2}j(r)j(r)\in SO(3)$

A change of orthonormal basis of F can be expressed by an action of the Spin group : $\varepsilon_i\to\widetilde{\varepsilon}_i=\mathbf{Ad}_{s^{-1}}\varepsilon_i$

The operator $\mathbf{Ad}_s$ acts on elements $Z\in T_1Spin(3,1)$:

$Z=\sum_{a=1}^6Z_a\overrightarrow{\kappa}_a\to\widetilde{Z}=\sum_{a=1}^6Z_a\mathbf{Ad}_s(\overrightarrow{\kappa}_a)=\sum_{a=1}^6\widetilde{Z}_a\overrightarrow{\kappa}_a$

$Z=\upsilon(X,Y)\to\widetilde{Z}=\upsilon\left(\widetilde{X},\widetilde{Y}\right)$

$\begin{bmatrix}\widetilde{X}\\ \widetilde{Y}\end{bmatrix}=[\mathbf{Ad}_s]\begin{bmatrix}X\\ Y\end{bmatrix}$

where $[\mathbf{Ad}_s]$ is a 6×6 matrix :

$[\mathbf{Ad}_s]=$

$\begin{bmatrix} 1+aj(r)-bj(w)+\frac{1}{2}(j(r)j(r)-j(w)j(w)) & -\left(aj(w)+bj(r)+\frac{1}{2}(j(r)j(w)+j(w)j(r))\right) \\ aj(w)+bj(r)+\frac{1}{2}(j(r)j(w)+j(w)j(r)) & 1+aj(r)-bj(w)+\frac{1}{2}(j(r)j(r)-j(w)j(w)) \end{bmatrix}$

$[\mathbf{Ad}_{s.s'}]=[\mathbf{Ad}_s][\mathbf{Ad}_{s'}]$

9.3 Complex structure on the Clifford algebra

The subspaces L_0, P_0 are crucial in the properties of $T_1 Spin(3,1)$, as seen in the notation $\upsilon(r,w)$. The computations can be made easier by defining on $Cl(3,1)$ and $Cl(1,3)$ a complex structure : the set does not change but it is split in a real and an imaginary part. It is convenient to make computations in the Clifford Algebra.

This is done by a linear map such that : $J^2 = -Id$. Then the product iX is defined as $iX = Xi = J(X)$. Take $J(X) = X \cdot \varepsilon_5$ with $\varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ then $J^2(X) = X \cdot \varepsilon_5 \cdot \varepsilon_5 = -X$. It holds on $Cl(1,3)$ and $Cl(3,1)$.

The distinction between the real and imaginary vector subspaces is done by splitting any orthonormal basis. With $Cl(3,1)$ the real basis consists of 1 and products of the vectors $\varepsilon_i, i = 1,2,3$. With $Cl(1,3)$ the real basis consists of 1,ε_0 and products of ε_0 with the vectors $\varepsilon_i, i = 1,2,3$.

The Clifford algebra becomes a 8 dimensional complex vector space $Cl(3,1)_C$.

Elements of the Lie algebra $T_1 Spin(3,1)$ read :

$\upsilon(r,w) = \sum_{a=1}^3 (r_a + iw_a) \overrightarrow{\kappa}_a = \sum_{a=1}^3 Z^a \overrightarrow{\kappa}_a = Z$

and we have the formulas :

$\forall Z, Z' \in T_1 Spin(3,1)$:

$Z' \cdot Z = -\frac{1}{4} Z^t Z' + \frac{1}{2} j(Z') Z$

$[\upsilon(r,w), \upsilon(r',w')] = j(Z) Z'$

$\langle \upsilon(r,w), \upsilon(r',w') \rangle = Z^t Z' = \frac{1}{4} (r+iw)^t (r'+iw')$

The elements of the Spin group read :

$g = a + \upsilon(r,w) + b\varepsilon_5 = A + Z$

with $A = a + ib, Z = \upsilon(r,w)$

$A^2 = 1 - \frac{1}{4} Z^t Z$

Ad is a complex linear map and its action on the Lie algebra reads :

$[\mathbf{Ad}_g]_C [X]_C = [Ad(Z)] [X]_C = \left(1 + Aj(Z) + \frac{1}{2} j(Z) j(Z)\right) [X]_C$

The derivative of the differentiable map with any argument x : $\sigma : E \to Spin(3,1) :: \sigma(x)$ reads :

$$\begin{array}{c} \frac{\partial \sigma}{\partial x} \cdot \sigma^{-1} = D(Z) \frac{\partial Z}{\partial x} \\ \sigma^{-1} \cdot \frac{\partial \sigma}{\partial x} = D(-Z) \frac{\partial Z}{\partial x} \\ D(Z) = \frac{1}{A} + \frac{1}{2} j(Z) + \frac{1}{4A} j(Z) j(Z) \end{array} \tag{40}$$

Moreover we have the identities :

$\frac{\partial A}{\partial x} = -\frac{1}{4A} Z^t \frac{\partial Z}{\partial x}$

$\frac{\partial Z}{\partial x} = \left(A - \frac{1}{2} j(Z)\right) \frac{\partial \sigma}{\partial x} \cdot \sigma^{-1}$

$D(Z) \frac{\partial Z}{\partial x} = \frac{1}{A} \frac{\partial Z}{\partial x} + \frac{1}{2} \left[Z, \frac{\partial Z}{\partial x}\right] + \frac{1}{4A} \left[Z, \left[Z, \frac{\partial Z}{\partial x}\right]\right]$

$\frac{\partial}{\partial y} \left(\frac{\partial \sigma}{\partial x} \cdot \sigma^{-1}\right) = \left(\frac{\partial}{\partial y} D(Z)\right) \frac{\partial Z}{\partial x} + D(Z) \frac{\partial}{\partial y} \frac{\partial Z}{\partial x}$

$[Ad(Z)] [D(Z)] = D(-Z)$

$D(Z)^{-1} = A - \frac{1}{2} j(Z)$

9.4 Coordinates on the Clifford Algebra

The Clifford Algebra is a vector space, and any element can be represented as a vector with its components in the canonic basis.

The Lie Algebra is a vector subspace, and we have the choice between :

$\upsilon(X_r, X_w) = \sum_{a=1}^3 X_r^a \overrightarrow{\kappa}_a + \sum_{a=4}^6 X_w^{a-3} \overrightarrow{\kappa}_a$

and the complex representation : $Z = \sum_{a=1}^3 Z^a \overrightarrow{\kappa}_a$

The Spin Group is not a vector space, but a 6 dimensional manifold embedded in the Clifford Algebra. Its elements depend on 2 vectors of $\mathbb{R}^3$: r, w but their meaning depend on the chart used.

i) The simplest chart is :

$\sigma : \mathbb{R}^3 \times \mathbb{R}^3 \to Spin(3,1) :: \sigma = a + \upsilon(r,w) + b\varepsilon_5$

with $a^2 - b^2 = 1 + \frac{1}{4} (w^t w - r^t r)$

$ab = -\frac{1}{4} r^t w$

ii) The decomposition :

$\sigma : \mathbb{R}^3 \times \mathbb{R}^3 \to Spin(3,1) :: \sigma = \sigma_w \cdot \sigma_r = (a_w + \upsilon(0,w)) \cdot (a_r + \upsilon(r,0))$

with :

$a_w^2 = 1 + \frac{1}{4} w^t w$

$a_r^2 = 1 - \frac{1}{4} r^t r$

Then σ_w, σ_r are defined up to the sign.

iii) The complex representation :

$\sigma : \mathbb{C}^3 \to Spin(3,1) :: \sigma = A + \sum_{a=1}^3 Z^a \overrightarrow{\kappa}_a$

with : $A^2 = 1 - \frac{1}{4} Z^t Z, Z = r + iw$

The choice of the chart can be fitted to the problem at hand.

10 MOTION IN CLIFFORD ALGEBRAS

10.1 Description of the fiber bundles

From the principal bundle $P_G(M, Spin(3,1), \pi_G)$ other fiber bundles can be defined.

Definition 24 *The vector bundle TM defined through the tetrad of an observer is $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right] : \varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$*

In a change of observer :

$$\begin{array}{l} \mathbf{p}(m) = \varphi_G(m,1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : (\mathbf{p}(m), u) \sim \left(\widetilde{\mathbf{p}}(m), \mathbf{Ad}_{\chi(m)} u\right) \\ \varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \to \widetilde{\varepsilon}_i(m) = \mathbf{Ad}_{\chi(m)^{-1}} \varepsilon_i(m) = \sum_{j=0}^{3} \left[h\left(\chi(m)^{-1}\right)\right]_i^j \varepsilon_j(m) \end{array} \tag{41}$$

The formulas are the same as previously, the relation between $\varepsilon_i(m), \widetilde{\varepsilon}_i(m)$ is just explicit with **Ad**. In $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$ the components of vectors are measured in orthonormal bases.

$\varepsilon_0(m) = (\mathbf{p}(m), \varepsilon_0)$ is the 4th vector both in the Clifford algebra and in the tangent space $T_m M$. It corresponds to the velocity of the observer : $\varepsilon_0(q_o(t)) = \frac{1}{c}\frac{dq_o}{dt}$ is fixed along his world line.

The Lorentz scalar product on $\mathbb{R}^4$ is preserved by **Ad** thus it can be extended to $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$.

The gauge of an observer is defined by his tetrad : it is the physical link between the abstract fiber bundle P_G and the measures involving P_G.

Because M is endowed with the structure of the principal bundle P_G, there is a structure of **Clifford bundle** $Cl(TM)$: a structure of Clifford algebra $Cl((T_m M, g(m)))$ at each point $m \in M$, whose elements are defined through products of vectors $\varepsilon_i(m)$, and it is isomorphic to $Cl(3,1)$ (Maths.2106). Pointwise the Clifford product holds with the usual properties, and with the vectors defined in the tetrad.

Definition 25 *The **Clifford bundle** $Cl(TM)$ is the associated vector bundle $P_G[Cl(3,1), \mathbf{Ad}]$ defined through the basis $(\varepsilon_i(m))_{i=0}^3$.*

A basis of $Cl(T_m M)$ is given by 1 and ordered products of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$. It changes as $\widetilde{\varepsilon}_i(m) = \mathbf{Ad}_{\chi(m)^{-1}} \varepsilon_i(m)$ and the components change as $\left[\mathbf{Ad}_{\chi(m)}\right] X$, the matrix $\left[\mathbf{Ad}_{\chi(m)}\right]$ depending on X.

$X(m) = (\varphi_G(m,1), X) \sim \left(\varphi_G\left(m, \chi(m)^{--1}\right), \mathbf{Ad}_{\chi(m)} X\right)$

The observer uses the frame $\left(O, (\varepsilon_i)_{i=0}^3\right)$ to measure the components of vectors of TM. The breakdown, specific to each observer, comes from the distinction of his present, and is materialized in his standard basis by the vector $\varepsilon_0(m)$.This choice leads to a split of the Spin group between the spatial rotations, represented by $Spin(3)$, and the homogeneous space $SW = Spin(3,1)/Spin(3)$.

Any section $\sigma \in \mathfrak{X}(P_G)$ can be decomposed, for a given vector field ε_0 and a fixed $\epsilon = \pm 1$, in two sections : $\epsilon\sigma_w \in \mathfrak{X}(P_W), \epsilon\sigma_r \in \mathfrak{X}(P_R)$ with $\sigma(m) = \epsilon\sigma_w(m) \cdot \epsilon\sigma_r(m)$

10.2 Motion of a particle

10.2.1 Arrangement of the particle

The fundamental assumption is the existence of an orthonormal basis $(e_i)_{i=0}^3$ attached to the particle. At each point it is measured in the vector bundle $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$. The basis $(e_i)_{i=0}^3$ is deduced from the tetrad $(\varepsilon_i)_{i=0}^3$ of the observer by an element $\sigma \in Spin(3,1)$ such that : $e_i = \mathbf{Ad}_\sigma \varepsilon_i$ and we define the *arrangement* of the particle with respect to the observer O by σ.

The velocity of the particle reads in the tetrad at each point :

$V = \frac{dq}{dt} = c\varepsilon_0 + \overrightarrow{v} = \sum_{j=0}^3 U^j \varepsilon_j$ with $U^j = \sum_{\alpha=0}^3 P_\alpha^{\prime j} V^\alpha$

Because the velocity V of the particle is proportional to e_0 we have :

$V = \sqrt{-\langle V, V\rangle} e_0 \Leftrightarrow U = \sqrt{-\langle V, V\rangle} \mathbf{Ad}_\sigma \varepsilon_0$

and $\langle V, \varepsilon_0\rangle_{TM} = -c = \sqrt{-\langle V, V\rangle}\, \langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{TM}$

$\langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{TM}$ is the scalar product in the tetrad, so $\langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{TM} = \langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{Cl}$ and does not depend on the metric. Notice that $\mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0$ are both vectors in the fixed vector space $\mathbb{R}^4$

$\Rightarrow \sqrt{-\langle V, V\rangle} = -\frac{c}{\langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{Cl}}$

In a change of gauge :

$\mathbf{p}(m) = \varphi_G(m,1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \sigma \to \widetilde{\sigma} = \chi(m) \cdot \sigma$

$(\mathbf{p}(m), e_i) \sim \left(\widetilde{\mathbf{p}}(m), \mathbf{Ad}_{\chi(m)} e_i\right) = \left(\widetilde{\mathbf{p}}(m), \mathbf{Ad}_{\chi(m)} \mathbf{Ad}_\sigma \varepsilon_i\right) = \left(\widetilde{\mathbf{p}}(m), \mathbf{Ad}_{\widetilde{\sigma}} \varepsilon_i\right)$

$$\begin{array}{c} e_i = \mathbf{Ad}_\sigma \varepsilon_i \\ V = \frac{dq}{dt} = c\varepsilon_0 + \overrightarrow{v} = -\frac{c}{\langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0 \rangle_{Cl}} \mathbf{Ad}_\sigma \varepsilon_0 \end{array} \tag{42}$$

With the chart : $\sigma = \sigma_w \cdot \sigma_r = \epsilon(a_w + \upsilon(0,w)) \cdot \epsilon(a_r + \upsilon(r,0))$ with $\epsilon = \pm 1$
$\frac{U}{\sqrt{-\langle V,V\rangle}} = e_0 = \mathbf{Ad}_{\sigma_w \cdot \sigma_r} \varepsilon_0 = \mathbf{Ad}_{\sigma_w} \mathbf{Ad}_{\sigma_r} \varepsilon_0 = \mathbf{Ad}_{\sigma_w} \varepsilon_0$ because $\sigma_r \in T_1 Spin(3)$
so : $\frac{U}{\sqrt{-\langle V,V\rangle}} = \mathbf{Ad}_{\sigma_w} \varepsilon_0$
The matrix of $\mathbf{Ad}_{\sigma_w}$ is :
$[h(\sigma_w)] = \begin{bmatrix} 2a_w^2 - 1 & a_w w^t \\ a_w w & 2a_w^2 - 1 + \frac{1}{2} j(w) j(w) \end{bmatrix}$
$\frac{U}{\sqrt{-\langle V,V\rangle}} = \left(2a_w^2 - 1\right)\varepsilon_0 + \sum_{i=1}^3 a_w w_i \varepsilon_i = \left(2a_w^2 - 1\right)\varepsilon_0 + a_w w$
with $w = \sum_{i=1}^3 w_i \varepsilon_i$
$V = c\varepsilon_0 + \overrightarrow{v} \Rightarrow \sqrt{-\langle V,V\rangle} = \frac{c}{2a_w^2 - 1}$
$V = \frac{c}{2a_w^2-1} \mathbf{Ad}_{\sigma_w} \varepsilon_0 = c \sum_{\alpha=0}^3 \left(\mathbf{P}_0^\alpha(q(t)) + \frac{a_w}{2a_w^2-1} \sum_{i=1}^3 w_i(t) \mathbf{P}_i^\alpha(q(t))\right) \partial\xi_\alpha$
$\overrightarrow{v} = 0 \Leftrightarrow w = 0$
V is determined by σ_w only. Meanwhile σ is uniquely defined by $(e_i)_{i=0}^3$, σ_w is defined up to the sign. In all cases we have $a_w \sum_{i=1}^3 w_i \varepsilon_i = a_w \overrightarrow{w}$ in the same direction as the spatial velocity, but this can be achieved either by $\overrightarrow{w}$ in the same direction as the spatial velocity and $a_w > 0$ or by $-\overrightarrow{w}$ and $-a_w$. σ_r is similarly defined up to the sign.

From the formula above V has the dimension of a spatial speed, and w is unitless, by the use of the universal constant c, which provides a natural standard.

$$V = c\left(\varepsilon_0 + \sum_{a=1}^3 \frac{a_w}{2a_w^2-1} w_a \varepsilon_a\right) \tag{43}$$

With the complex chart :
$\sigma = A + \sum_{a=1}^3 Z^a \overrightarrow{\kappa}_a = a + \upsilon(r,w) + b\varepsilon_5$
The matrix $[h(\sigma)]$ has been given previously and :
$\mathbf{Ad}_\sigma \varepsilon_0 = [h(s)] = \left(a^2 + b^2 + \frac{1}{4}(r^t r + w^t w)\right)\varepsilon_0 + \left(aw - br + \frac{1}{2} j(r) w\right)$
$a^2 + b^2 + \frac{1}{4}(r^t r + w^t w) = A\overline{A} + \frac{1}{4} Z^t \overline{Z}$
$aw - br + \frac{1}{2} j(r) w = -\operatorname{Im} A\overline{Z} - \frac{1}{4} \operatorname{Im} j(Z) \overline{Z}$
$\mathbf{Ad}_\sigma \varepsilon_0 = \left(A\overline{A} + \frac{1}{4} Z^t \overline{Z}\right)\varepsilon_0 - \operatorname{Im}\left(A + \frac{1}{4} j(Z)\right)\overline{Z}$
$V = c\varepsilon_0 + \overrightarrow{v} = \sqrt{-\langle V,V\rangle} \mathbf{Ad}_\sigma \varepsilon_0$
$\sqrt{-\langle V,V\rangle} = \frac{c}{A\overline{A} + \frac{1}{4} Z^t \overline{Z}}$

$$V = c\left(\varepsilon_0 - \frac{1}{A\overline{A} + \frac{1}{4} Z^t \overline{Z}} \operatorname{Im}\left(A + \frac{1}{4} j(Z)\right)\overline{Z}\right) \tag{44}$$

$\overrightarrow{v} = 0 \Leftrightarrow w = 0 \Leftrightarrow A = a_r, Z = r$
$r = 0 \Leftrightarrow A = a_w, Z = iw$
$A = a_w a_r - i\frac{1}{4}(w^t r)$
$Z = a_w r + i\left(a_r - \frac{1}{2} j(r)\right) w$

10.2.2 Motion

The tetrad attached to the particle is defined in the tetrad of the observer, and the motion is defined by derivation with respect to a fixed observer. A continuous motion is such that the map : $\sigma : \mathbb{R} \to Spin(3,1)$ with respect to the time t of the observer is smooth. From the definitions above (remember that the vectors are defined in a fixed vector space) :
$\forall i = 0..3 : e_i = \mathbf{Ad}_\sigma \varepsilon_i$
$\frac{de_i}{dt} = \frac{d}{dt} \mathbf{Ad}_\sigma \varepsilon_i = \mathbf{Ad}_\sigma \left[\sigma^{-1} \cdot \frac{d\sigma}{dt}, \varepsilon_i\right] = \left[\frac{d\sigma}{dt} \cdot \sigma^{-1}, \mathbf{Ad}_\sigma \varepsilon_i\right] = \left[\frac{d\sigma}{dt} \cdot \sigma^{-1}, e_i\right]$
$V = \sqrt{-\langle V,V\rangle} \mathbf{Ad}_\sigma \varepsilon_0 = \sqrt{-\langle V,V\rangle} e_0$
$\frac{dV}{dt} = \frac{d}{dt}\sqrt{-\langle V,V\rangle} e_0 + \sqrt{-\langle V,V\rangle} \frac{de_0}{dt}$
$= \left(\frac{1}{\sqrt{-\langle V,V\rangle}} \frac{d}{dt}\sqrt{-\langle V,V\rangle}\right)\sqrt{-\langle V,V\rangle} e_0 + \sqrt{-\langle V,V\rangle}\left[\frac{d\sigma}{dt} \cdot \sigma^{-1}, e_0\right]$
$\frac{dV}{dt} = \left(\frac{1}{\sqrt{-\langle V,V\rangle}} \frac{d}{dt}\sqrt{-\langle V,V\rangle}\right) V + \left[\frac{d\sigma}{dt} \cdot \sigma^{-1}, V\right]$

$\sqrt{-\langle V,V\rangle} = -\frac{c}{\langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{Cl}}$
$\frac{1}{\sqrt{-\langle V,V\rangle}} \frac{d}{dt}\sqrt{-\langle V,V\rangle} = \frac{1}{\langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{Cl}} \left\langle \frac{d}{dt}\mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0 \right\rangle_{Cl} = \frac{1}{\langle \mathbf{Ad}_\sigma \varepsilon_0, \varepsilon_0\rangle_{Cl}} \left\langle \left[\frac{d\sigma}{dt}\cdot\sigma^{-1}, \mathbf{Ad}_\sigma \varepsilon_0\right], \varepsilon_0 \right\rangle_{Cl}$
$= \frac{1}{c}\left\langle \left[\frac{d\sigma}{dt}\cdot\sigma^{-1}, V\right], \varepsilon_0 \right\rangle_{Cl}$
$\frac{dV}{dt} = \frac{V}{c}\left\langle \left[\frac{d\sigma}{dt}\cdot\sigma^{-1}, V\right], \varepsilon_0 \right\rangle_{Cl} + \left[\frac{d\sigma}{dt}\cdot\sigma^{-1}, V\right]$

$\delta_R\sigma = \frac{d\sigma}{dt}\cdot\sigma^{-1}$ is the right logarithmic derivative, and $\delta_L\sigma = \sigma^{-1}\cdot\frac{d\sigma}{dt}$ is the left logarithmic derivative. They both belong to $T_1 Spin(3,1)$ and are related by $\mathbf{Ad}_\sigma : \delta_R\sigma = \mathbf{Ad}_\sigma \delta_L\sigma \Leftrightarrow \delta_L\sigma = \mathbf{Ad}_{\sigma^{-1}}\delta_R\sigma$. And we define the motion (both translational and rotational) of the particle by $\frac{d\sigma}{dt}\cdot\sigma^{-1}$.

$$\begin{array}{c} \frac{d\sigma}{dt}\cdot\sigma^{-1} = \upsilon(X_r, X_w) \\ \forall i = 0..3 : \frac{de_i}{dt} = [\upsilon(X_r, X_w), e_i] \\ \frac{dV}{dt} = \frac{V}{c}\left\langle [\upsilon(X_r, X_w), V], \varepsilon_0 \right\rangle_{Cl} + [\upsilon(X_r, X_w), V] \end{array} \tag{45}$$

With $\sigma = \sigma_w\cdot\sigma_r = \epsilon(a_w + \upsilon(0,w))\cdot\epsilon(a_r + \upsilon(r,0))$
$\frac{d\sigma}{dt}\cdot\sigma^{-1} = \upsilon(X_r, X_w)$ with
$X_r = -\frac{1}{2}j(w)\frac{dw}{dt} + \left[1 - \frac{1}{2}j(w)j(w)\right]\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt}$
$X_w = \frac{1}{a_w}\left(1 - \frac{1}{4}j(w)j(w)\right)\frac{dw}{dt} + [a_w j(w)]\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt}$
and the inverse relation reads, with some computation :
$\frac{dr}{dt} = \left(\frac{1}{a_r} - \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\left(X_r + \frac{1}{2}\frac{1}{a_w}j(w)X_w\right)$
$\frac{dw}{dt} = -j(w)X_r + \left(a_w - \frac{1}{4a_w}j(w)j(w)\right)X_w$
$\sqrt{-\langle V,V\rangle} = \frac{c}{2a_w^2 - 1}$
$\frac{dV}{dt} = cX_w + \left(j(X_r) - (X_w^t v)\frac{1}{c}\right)v$ with, in the tetrad : $V = c\varepsilon_0 + v$

$$\begin{array}{c} \frac{d\sigma}{dt}\cdot\sigma^{-1} = \upsilon(X_r, X_w) \\ X_r = -\frac{1}{2}j(w)\frac{dw}{dt} + \left[1 - \frac{1}{2}j(w)j(w)\right]\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt} \\ X_w = \frac{1}{a_w}\left(1 - \frac{1}{4}j(w)j(w)\right)\frac{dw}{dt} + [a_w j(w)]\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt} \\ \frac{dV}{dt} = cX_w + \left(j(X_r) - (X_w^t v)\frac{1}{c}\right)v \end{array} \tag{46}$$

With the complex formalism : $\sigma = A + \sum_{a=1}^3 Z^a \overrightarrow{\kappa}_a$
$\sigma = A + Z$
$\frac{d\sigma}{dt}\cdot\sigma^{-1} = D(Z)\frac{dZ}{dt} = Y_r + iY_w$
$[D(Z)]^{-1} = A - \frac{1}{2}j(Z)$
$\frac{dZ}{dt} = [D(Z)]^{-1}(Y_r + iY_w) = \left(A - \frac{1}{2}j(Z)\right)(Y_r + iY_w)$

$$\begin{array}{c} \frac{d\sigma}{dt}\cdot\sigma^{-1} = D(Z)\frac{dZ}{dt} = Y_r + iY_w \\ \frac{dZ}{dt} = \left(A - \frac{1}{2}j(Z)\right)(Y_r + iY_w) \\ \frac{dV}{dt} = cY_w + \left(j(Y_r) - (Y_w^t v)\frac{1}{c}\right)v \end{array} \tag{47}$$

10.2.3 Spin

The spin is a rotational motion. The spatial basis of the particle is deduced from the spatial tetrad by a rotation of $SO(3)$:

$$[h(\sigma_r)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 + a_r j(r) + \frac{1}{2}j(r)j(r) \end{bmatrix}$$

and the rotational motion can be defined as : $\frac{d\sigma_r}{dt}\cdot\sigma_r^{-1} \in T_1 Spin(3)$.

Meanwhile for the decomposition of σ we have the choice of ϵ and $(r,w) \sim (-r,-w)$, the rotational motion $\frac{d\sigma_r}{dt}\cdot\sigma_r^{-1}$ does not depend on ϵ, but introduces a new factor with the derivative. In Galilean Geometry the convention is that $-\rho$ represents the opposite spin, with the same axis. In the Relativist framework, one can distinguish the two rotations, because there is always a privileged direction (that of the velocity). One can distinguish the two spin elements $\pm\sigma_r$ (which correspond to the same matrix of $SO(3,1)$) and differentiate the rotational motion from its opposite by fixing ϵ. If we impose that $\overrightarrow{w}$ is in the direction of $\overrightarrow{v}$, then $+\rho$ and $-\rho$ represent spinning with the same axis, but opposite rotations, or equivalently, to keep the usual convention, rotations with opposite axis. These opposite rotational motions are usually called polarization (spin "up" or "down").

In Galilean Geometry two opposite rotational motions are the image of each other in a space inversion (a symmetry with respect to a plan). In the Relativist Framework such an operation is a symmetry with

respect to a *spatial* vector (and not the space inversion which is a symmetry with respect to $\Omega_3(t)$). And actually this is done through the choice of an orientation for $\overrightarrow{w}$.

The vector $r(t) \in \mathbb{R}^3$, however the characteristic of the spin is $\frac{d\sigma_r}{dt} \cdot \sigma_r^{-1} = \upsilon(X_r, 0) \in T_1 Spin(3)$ and we have seen that $\upsilon(X_r, 0)$ does not depend on the choice of a spatial basis. So we have the known paradox : we have a quantity, the spin, which looks like a rotation, which can be measured as a rotation, but is not related to a precise basis, even if its measure is done in one !

10.2.4 Estimates

It is useful to have estimates for w, using the spatial speed.

Let us denote : $x = 1 - \frac{\left\|\overrightarrow{v}\right\|^2}{c^2}$

With the representation $\sigma = \sigma_w \cdot \sigma_r = (a_w + \upsilon(0, w)) \cdot (a_r + \upsilon(r, 0))$

$a_w = \epsilon\sqrt{\frac{1}{2}\left(\frac{1}{\sqrt{x}} + 1\right)} = \epsilon\frac{1}{\sqrt{2}}x^{-1/4}\left(1 + \sqrt{x}\right)^{1/2}$

$w = \epsilon\sqrt{2}\left(1 - \frac{\left\|\overrightarrow{v}\right\|^2}{c^2} + \sqrt{1 - \frac{\left\|\overrightarrow{v}\right\|^2}{c^2}}\right)^{-1/2}\frac{\overrightarrow{v}}{c}$

Usually $\frac{\left\|\overrightarrow{v}\right\|^2}{c^2} \ll 1$ and we have the estimates :

$a_w \simeq \epsilon\left(1 + \frac{1}{8}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2}\right)$

$w \simeq \epsilon\left(1 + \frac{3}{8}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2}\right)\frac{\overrightarrow{v}}{c}$

$V \simeq c\left(\varepsilon_0 + \epsilon\left(1 - \frac{3}{8}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2}\right)\overrightarrow{w}\right)$

$A \simeq \epsilon a_r\left(1 + \frac{1}{8}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2}\right) - i\frac{1}{4}\epsilon r^t\frac{\overrightarrow{v}}{c}$

$Z \simeq \epsilon\left(1 + \frac{1}{8}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2}\right)r + i\left(a_r - \frac{1}{2}j(r)\right)\epsilon\frac{\overrightarrow{v}}{c}$

The derivative of w is given by the formula :

$\frac{dw}{dt} = \left[\left(\frac{2a_w^2+1}{4a_w^3}\right)j\left(\frac{\overrightarrow{v}}{c}\right)j\left(\frac{\overrightarrow{v}}{c}\right) + \left(\frac{2a_w^2+1}{4a_w^3}\right)\frac{\left\|\overrightarrow{v}\right\|^2}{c^2} + \frac{2a_w^2-1}{a_w}\right]\left(\frac{d}{dt}\frac{\overrightarrow{v}}{c}\right)$

$\frac{dw}{dt} \simeq \left(1 + \frac{9}{8}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2} + \frac{3}{4}j\left(\frac{\overrightarrow{v}}{c}\right)j\left(\frac{\overrightarrow{v}}{c}\right)\right)\left(\frac{d}{dt}\frac{\overrightarrow{v}}{c}\right)$

$X_r \simeq -\frac{1}{2}\left(1 + \frac{3}{4}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2}\right)j\left(\frac{\overrightarrow{v}}{c}\right)\left(\frac{d}{dt}\frac{\overrightarrow{v}}{c}\right) + \left[1 - \frac{1}{2}j\left(\frac{\overrightarrow{v}}{c}\right)j\left(\frac{\overrightarrow{v}}{c}\right)\right]\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt}$

$X_w \simeq \left(1 + \frac{\left\|\overrightarrow{v}\right\|^2}{c^2} - \frac{1}{2}j\left(\frac{\overrightarrow{v}}{c}\right)j\left(\frac{\overrightarrow{v}}{c}\right)\right)\left(\frac{d}{dt}\frac{\overrightarrow{v}}{c}\right) + \left(1 + \frac{1}{2}\frac{\left\|\overrightarrow{v}\right\|^2}{c^2}\right)j\left(\frac{\overrightarrow{v}}{c}\right)\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt}$

The rotation of the spatial basis is usually measured by a matrix $R(t) \in SO(3)$ based on a vector $\rho \in \mathbb{R}^3$:

$R(t) = \exp j(\rho(t)) = I_3 + \frac{\sin\sqrt{\rho^t\rho}}{\sqrt{\rho^t\rho}}[j(\rho)] + \frac{1-\cos\sqrt{\rho^t\rho}}{\rho^t\rho}[j(\rho)][j(\rho)]$

$[h(\sigma_r)] = \begin{bmatrix} 1 & 0 \\ 0 & I_3 + a_r j(r) + \frac{1}{2}j(r)j(r) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R(t) \end{bmatrix}$

thus :

$I_3 + \frac{\sin\sqrt{\rho^t\rho}}{\sqrt{\rho^t\rho}}[j(\rho)] + \frac{1-\cos\sqrt{\rho^t\rho}}{\rho^t\rho}[j(\rho)][j(\rho)] = 1 + a_r j(r) + \frac{1}{2}j(r)j(r)$

$I_3 + a_r j(r) + \frac{1}{2}j(r)j(r)$ has for eigen vector r with eigen value 1

$\exp j(\rho(t))$ has for eigen vector ρ with eigen value 1

thus $r = \lambda\rho$

The sign of a_r is fixed by ϵ, that is the orientation of w.

$\lambda = \epsilon\sqrt{2\frac{1-\cos\sqrt{\rho^t\rho}}{\rho^t\rho}}$

$a_r = \epsilon\frac{\sin\sqrt{\rho^t\rho}}{\sqrt{2\left(1-\cos\sqrt{\rho^t\rho}\right)}}$

$r = \epsilon\sqrt{2\frac{1-\cos\sqrt{\rho^t\rho}}{\rho^t\rho}}$

And :

$\sigma_r = \epsilon\left(\frac{\sin\sqrt{\rho^t\rho}}{\sqrt{2\left(1-\cos\sqrt{\rho^t\rho}\right)}} + \upsilon\left(\sqrt{2\frac{1-\cos\sqrt{\rho^t\rho}}{\rho^t\rho}}, 0\right)\right)$

10.2.5 Jet representation

The motion of the particle is then fully represented by maps :

$\mathbb{R} \to M :: q(t)$

$\mathbb{R} \to Spin(3,1) :: \sigma(t)$

$\mathbb{R} \to T_1 Spin(3,1) :: \upsilon(X_r(t), X_w(t))$

$X_r(t), X_w(t)$ are independent maps and, in a continuous motion : $\upsilon(X_r(t), X_w(t)) = \frac{d\sigma}{dt} \cdot \sigma^{-1}$

This can be seen as a map : $\mathbb{R} \to J^1 P_G :: (q(t), \sigma(t), \upsilon(X_r(t), X_w(t)))$ with the 1st jet prolongation of the fiber bundle P_G (Maths.26). Both $\sigma(t), \upsilon(X_r(t), X_w(t))$ belong to the Clifford algebra $Cl(TM)$ located at $q(t)$.

The trajectory is defined by the set of differential equations :

$q(t) = \varphi_o\left(\xi^0, \xi^1, \xi^2, \xi^3\right)$

$\frac{dq}{dt} = \frac{d\xi^\alpha}{dt} = V\left(\varphi_o\left(\xi^0, \xi^1, \xi^2, \xi^3\right)\right)$

$q(0) = a$

The arrangement of the particle is given by :

$\forall i = 0..3 : e_i = \mathbf{Ad}_\sigma \varepsilon_i$

All these components involve only $\sigma(t)$ and not its derivative and change as :

$\frac{de_i}{dt} = [\upsilon(X_r(t), X_w(t)), e_i]$

$\frac{dV}{dt} = \frac{V}{c}\langle[\upsilon(X_r, X_w), V], \varepsilon_0\rangle_{Cl} + [\upsilon(X_r, X_w), V]$

10.2.6 Example

Let be a particle moving on a circle in a plane around some point O. We can take for the observer a spherical chart (even in the GR context) as defined before :

$\xi^1 = \rho\cos\phi\cos\theta, \xi^2 = \rho\cos\phi\sin\theta, \xi^3 = \rho\sin\phi, \xi^0 = ct$

We assume that $\rho = Ct, \phi = Ct = 0, \theta(t)$ with $\omega = \frac{d\theta}{dt} = Ct$

$\overrightarrow{v} = \rho\omega(-\sin\theta, \cos\theta, 0)$

$\|\overrightarrow{v}\|^2 = \sum_{\alpha\beta=1}^3 g_{\alpha\beta} v^\alpha v^\beta$

If on $\Omega_3(t) : g_{\alpha\beta} \simeq \delta^\alpha_\beta : \|\overrightarrow{v}\|^2 = \rho^2\omega^2$

$a_w = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\frac{\rho^2\omega^2}{c^2}}}\right)} \simeq \left(1 + \frac{1}{8}\frac{\rho^2\omega^2}{c^2}\right)$

$w = \frac{1}{\sqrt{\frac{1}{2}\sqrt{1-\frac{\rho^2\omega^2}{c^2}}\left(1+\sqrt{1-\frac{\rho^2\omega^2}{c^2}}\right)}}\frac{\rho\omega}{c}(-\sin\theta, \cos\theta, 0) = w_0(-\sin\theta, \cos\theta, 0)$

with $w_0 = \frac{1}{\sqrt{\frac{1}{2}\sqrt{1-\frac{\rho^2\omega^2}{c^2}}\left(1+\sqrt{1-\frac{\rho^2\omega^2}{c^2}}\right)}}\frac{\rho\omega}{c} \simeq \left(1 + \frac{1}{8}\frac{\rho^2\omega^2}{c^2}\right)\frac{\rho\omega}{c}$

$\frac{d\sigma_w}{dt} \cdot \sigma_w^{-1} = \upsilon(-\frac{1}{2}j(w)\frac{dw}{dt}, \frac{1}{a_w}\left(1 - \frac{1}{4}j(w)j(w)\right)\frac{dw}{dt})$

$\frac{dw}{dt} = -w_0\omega\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}$

$\frac{d\sigma_w}{dt} \cdot \sigma_w^{-1} = -\omega w_0 \upsilon\left(\frac{1}{2}w_0\begin{bmatrix}0\\0\\1\end{bmatrix}, \frac{1}{a_w}\left(1 + \frac{1}{4}w_0^2\right)\begin{matrix}\cos\theta\\ \sin\theta\\ 0\end{matrix}\right)$

$\simeq -\frac{\rho\omega^2}{c}\left(1 + \frac{1}{4}\frac{\rho^2\omega^2}{c^2}\right)\upsilon\left(\frac{1}{2}\frac{\rho\omega}{c}\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}\right)$

The local tetrad of the observer in $\Omega_3(t)$ is $(\partial\xi_\alpha)_{\alpha=1}^3$. Let us assume that the orthonormal frame attached to the particle has for components (their axis of rotation is orthogonal to the plan xy) :

$e_1 = \begin{bmatrix}\cos\zeta(t)\\ \sin\zeta(t)\\ 0\end{bmatrix}, e_1 = \begin{bmatrix}-\sin\zeta(t)\\ \cos\zeta(t)\\ 0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$[R(t)] = \begin{bmatrix}\cos\zeta(t) & -\sin\zeta(t) & 0\\ \sin\zeta(t) & \cos\zeta(t) & 0\\ 0 & 0 & 1\end{bmatrix}$

$a_r = \epsilon\sqrt{\frac{1}{2}(1+\cos\zeta)}$

$r = \begin{bmatrix}0\\0\\\epsilon\sqrt{2}\sqrt{1-\cos\zeta}\end{bmatrix}$

$\sigma_r = \epsilon\left(\sqrt{\frac{1}{2}(1+\cos\zeta)} + \upsilon\left((0,0,\sqrt{2}\sqrt{1-\cos\zeta}),0\right)\right)$
If $\zeta(t) = \omega_r t$
$\frac{dr}{dt} = \begin{bmatrix} 0 \\ 0 \\ \epsilon\sqrt{2}(1-\cos\zeta)^{-1/2}(\sin\zeta)\omega_r \end{bmatrix}$
$\frac{d\sigma_r}{dt}\cdot\sigma_r^{-1} = \upsilon(\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt}, 0) = 2\omega_r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
$\mathbf{Ad}_{\sigma_w}\left(\frac{d\sigma_r}{dt}\cdot\sigma_r^{-1}\right)$
$= \upsilon\left(\left[1-\frac{1}{2}j(w)j(w)\right]\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt}, [a_w j(w)]\left(\frac{1}{a_r} + \frac{1}{2}j(r) + \frac{1}{4a_r}j(r)j(r)\right)\frac{dr}{dt}\right)$
The motion is then represented by :
$\upsilon\left(\left(\omega_r\left(2+\frac{\rho^2\omega^2}{c^2}\right) - \frac{1}{2}\frac{\rho^2\omega^3}{c^2}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{\rho\omega}{c}(2\omega_r - \omega)\left(1+\frac{1}{4}\frac{\rho^2\omega^2}{c^2}\right)\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}\right)$

10.2.7 Periodic Motions

Periodic motions are of special interest because they can be seen as stable motions. A periodic, continuous, motion is given by a map :

$\sigma : \mathbb{R} \to Spin(3,1) :: \sigma(t) = A(t) + Z(t)$ where $Z(t+T) = Z(t)$ for some fixed period. Then $A^2(t+T) = 1 - \frac{1}{4}Z(t+T)^t Z(t+T) = A^2(t)$

Z can be written :
$Z(t) = \sum_{n\in\mathbb{Z}} \widehat{Z}(n)\exp in\omega t$ with $\widehat{Z}(n) = \frac{1}{T}\int_0^T Z(t)\exp(-in\omega t)\,dt$ and $\omega = \frac{2\pi}{T}$
$Z(0) = \sum_{n\in\mathbb{Z}} \widehat{S}(n)$
$A(t) = \sum_{n\in\mathbb{Z}} \widehat{A}(n)\exp in\omega t$ with $\widehat{A}(n) = \frac{1}{T}\int_0^T A(t)\exp(-in\omega t)\,dt$
$A(t)^2 = 1 - \frac{1}{4}Z(t)^t Z(t)$
$Z(t)^t Z(t) = \sum_{n\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} \widehat{Z}(n-p)^t \widehat{Z}(p)\exp in\omega t = 4\left(1 - \sum_{n\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} A(n-p)A(p)\exp in\omega t\right)$
$n \neq 0 : \sum_{p\in\mathbb{Z}} \widehat{Z}(n-p)^t \widehat{Z}(p) - 4A(n-p)A(p) = 0$
$\sum_{p\in\mathbb{Z}} \widehat{Z}(-p)^t \widehat{Z}(p) - 4A(-p)A(p) = 4$

We do not have necessarily $\widehat{Z}(n)^t \widehat{Z}(n) = 4\left(1 - \widehat{A}(n)^2\right)$ so $\widehat{A}(n) + \widehat{Z}(n)$ does not necessarily belong to $Spin(3,1)$.

We have similarly :
$\frac{d\sigma}{dt}\cdot\sigma^{-1} = \delta Z(t) = \sum_{n\in\mathbb{Z}} \widehat{\delta Z}(n)\exp in\omega t$
with $\widehat{\delta Z}(n) = \frac{1}{T}\int_0^T \delta Z(t)\exp(-in\omega t)\,dt$
In a continuous motion :
$\delta Z(t) = D(Z(t))\frac{dZ}{dt}$
$\sum_{n\in\mathbb{Z}} \widehat{\delta Z}(n)\exp in\omega t = i\omega D(Z(t))\sum_{n\in\mathbb{Z}} n\widehat{Z}(n)\exp in\omega t$
$\widehat{\delta Z}(n) = i\omega n D(Z(t))\widehat{Z}(n)$

The periodicity is assumed with respect to the time and the orthonormal basis of an observer, whatever it is : there is no assumption about the evolution of the tetrad $[P_i^\alpha]$ with t, but of course to be consistent we must assume that the physical conditions at $q(t+T)$ are the same as at $q(t)$. Anyway the path followed by the particle cannot be a loop in Relativity. For a bonded particle the motion sums up to a rotational motion, that is to its spin.

10.3 Motion of material bodies

It is possible to extend the concept of deformable solid to the framework of RG.

10.3.1 Representation of trajectories by sections of the fiber bundle

In the previous formula :
$V = \frac{dq}{dt} = c\varepsilon_0 + \overrightarrow{v} = -\frac{c}{\langle \mathbf{Ad}_\sigma\varepsilon_0, \varepsilon_0\rangle_{Cl}}\mathbf{Ad}_\sigma\varepsilon_0$
the tangent to the curve with path : $q : \mathbb{R} \to M :: q(t)$ is defined through a map : $\sigma : \mathbb{R} \to P_G :: \sigma(t) = \varphi_G(q(t), \sigma(t))$.

This can be extended to a section : $\sigma :: M \to P_G :: \sigma(m) = \varphi_G(m, \sigma(m))$
For any function : $f : M \to \mathbb{R} :: f(m)$ the map :
$W : M \to P_G\left[\mathbb{R}^4, \mathbf{Ad}\right] :: W(m) = f(m)\mathbf{Ad}_{\sigma(m)}\varepsilon_0(m) = \left(\mathbf{p}(m), f(m)\mathbf{Ad}_{\sigma(m)}\varepsilon_0\right)$ is well defined.

$U(m) = f(m)\,\mathbf{Ad}_{\sigma(m)}\varepsilon_0 \in \mathbb{R}^4$ that is a fixed vector space : $U(m) = f(m)\,[h(\sigma(m))]\,\varepsilon_0$ with the matrix $[h(\sigma(m))] \in SO(3,1)$.

This is a section of $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right] : W \in \mathfrak{X}\left(P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]\right)$.

In a change of gauge :

$\mathbf{p}(m) = \varphi_G(m,1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \sigma(m) \to \widetilde{\sigma} = \chi(m) \cdot \sigma(m)$

$(\mathbf{p}(m), W(m)) \sim \left(\widetilde{\mathbf{p}}(m), \mathbf{Ad}_{\chi(m)} W(m)\right) =$

$\left(\widetilde{\mathbf{p}}(m), f(m)\,\mathbf{Ad}_{\chi(m)}\mathbf{Ad}_{\sigma(m)}\varepsilon_0\right) = \left(\widetilde{\mathbf{p}}(m), f(m)\,\mathbf{Ad}_{\widetilde{\sigma}(m)}\varepsilon_0\right)$

$W(m)$ defines a vector field on TM by the projection :

$\pi_G : P_G \to M :: \pi(\varphi_G(m,g)) = m$

$V(m) = \pi'(m)\,W(m) = f(m)\,[h(\sigma(m))]\,\varepsilon_0(m) = f(m)\sum_{i=0}^{3}[h(\sigma(m))]_0^i\,[P(m)]_i^\alpha\,\partial\xi_\alpha(m)$

This vector field defines integral curves : $q : \mathbb{R} \to M :: q(\tau) = \Phi_V(\tau, a)$ passing through $a \in M$ fixed. And by definition :

$\frac{dq}{dt} = V(q(\tau)) = f(q(\tau))\,\mathbf{Ad}_{\sigma(q(\tau))}\varepsilon_0(q(\tau))$.

$\langle V(q(\tau)), V(q(\tau))\rangle_{TM} = \left\langle f(q(\tau))\,\mathbf{Ad}_{\sigma(q(\tau))}\varepsilon_0(q(\tau)), f(q(\tau))\,\mathbf{Ad}_{\sigma(q(\tau))}\varepsilon_0(q(\tau))\right\rangle_{TM}$

$= f(q(\tau))^2\,\langle \varepsilon_0(q(\tau)), \varepsilon_0(q(\tau))\rangle_{TM} = -f(q(\tau))^2$

because $\mathbf{Ad}$ preserves the scalar product.

$\langle V(q(\tau)), \varepsilon_0\rangle_{TM} = f(m)\left\langle \mathbf{Ad}_{\sigma(q(\tau))}\varepsilon_0(q(\tau)), \varepsilon_0(q(\tau))\,.\right\rangle_{TM} = f(m)\left\langle \mathbf{Ad}_{\sigma(q(\tau))}\varepsilon_0, \varepsilon_0\right\rangle_{Cl}$

$= f(m)\left\langle \left(2a_w^2 - 1\right)\varepsilon_0, \varepsilon_0\right\rangle_{Cl} = -f(m)\left(2a_w^2 - 1\right)$

$\left(2a_w^2 - 1\right) = 2\left(1 + \frac{1}{4}w^t w\right) - 1 = 1 + \frac{1}{4}w^t w > 0$

thus the path is future oriented if $f(m) > 0$.

If we take : $f(m) = c$ then V is a field of world lines, for any observer.

And with : $f(m) = \sqrt{-\langle V(q(t)), V(q(t))\rangle_{TM}}$ then it defines a field of trajectories of particles, which do not cross, as measured by any observer (who defines t).

Theorem 26 *Any section $\sigma \in \mathfrak{X}(P_G)$ defines, for any positive function $f \in C_\infty(\Omega; \mathbb{R}_+)$ and observer, a vector field $V \in \mathfrak{X}\left(P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]\right)$ by :*

$$V(m) = f(m)\,\mathbf{Ad}_{\sigma(m)}\varepsilon_0(m)$$

which can represent a field of world lines for $f(m) = c$, and, for any observer, a field of trajectories with $f(m) = \sqrt{-\langle V(q(t)), V(q(t))\rangle_{TM}}$.

Remarks :

i) There is a *unique* vector field : $V(m) = f(m)\,\mathbf{Ad}_{\sigma(m)}\varepsilon_0(m)$ does not depend on the decomposition σ_r, σ_w. The spatial speed is the vector : $v = \epsilon\sqrt{1 + \frac{1}{4}w^t w}\sum_{i=1}^{3} w_i\varepsilon_i$ so the definition of w (that is of one of the two σ_w) depends of ϵ : $\epsilon = +1$ if $\sum_{i=1}^{3} w_i\varepsilon_i$ is in the direction of the spatial speed, and $\epsilon = -1$ is in the opposite direction.

ii) All this is defined with respect to an observer, who fixes $\varepsilon_0(m)$ .The speed is measured with the time t of the observer.

iii) Any map $\sigma : \mathbb{R} \to P_G$ is projected on M as a curve, which is not necessarily time like.

Conversely

Theorem 27 *For any time like, future oriented vector field $V \in \mathfrak{X}(TM)$ there is a section $\sigma_w \in \mathfrak{X}(P_W)$, defined up to sign, such that :*

$V(q(\tau)) = \sqrt{-\langle V(q(\tau)), V(q(\tau))\rangle_{TM}}\mathbf{Ad}_{\sigma_w(q(\tau))}\varepsilon_0(q(\tau))$ *along the integral curves of V.*

Proof. The parameter τ of the path along the integral curves is defined up to an additive constant. So $V(q(\tau)), f(q(\tau)) = \sqrt{-\langle V(q(\tau)), V(q(\tau))\rangle_{TM}}$ are well defined for any integral curve by the condition $\Phi_V(0, a) = a$.

In the associated bundle $P_G\left[\mathbb{R}^4, Ad\right]$:

$V(q(\tau)) = (p(q(\tau)), U(q(\tau)))$ with $U(q(\tau)) = \sum_{j=0}^{3} U_j\varepsilon_j$

The condition $U(q(\tau)) = f(q(\tau))\,\mathbf{Ad}_{\sigma(q(\tau))}\varepsilon_0$ reads in coordinates : $\sigma_w = (a_w + \upsilon(0, w))$

$[\mathbf{Ad}_{\sigma_w}] = [h(\sigma_w)] = \begin{bmatrix} 2a_w^2 - 1 & a_w w^t \\ a_w w & 2a_w^2 - 1 + \frac{1}{2}j(w)\,j(w) \end{bmatrix}$

$\frac{U}{f} = \left(2a_w^2 - 1\right)\varepsilon_0 + a_w\sum_{j=1}^{3} w_j\varepsilon_j$

$\frac{U_0}{f} = 2a_w^2 - 1$

$\langle U, \varepsilon_0\rangle = \langle V, \varepsilon_0(q(\tau))\rangle = -\frac{U_0}{f} < 0 \Rightarrow \frac{U_0}{f} + 1 > 0$

$a_w = \epsilon\sqrt{\frac{1}{2}\left(\frac{U_0}{f}+1\right)}$

$i=1,2,3 : w_i = \frac{1}{f}U_i/a_w$

In a change of gauge on P_G :

$V(m) = f(m)(\mathbf{p}(m), \mathbf{Ad}_{\sigma_w}\varepsilon_0) = f(m)\left(\mathbf{p}(m)\cdot\chi^{-1}, \mathbf{Ad}_{\widetilde{\sigma}_w}\varepsilon_0\right)$

$= f(m)\left(\mathbf{p}(m), \mathbf{Ad}_{\chi^{-1}}\mathbf{Ad}_{\widetilde{\sigma}_w}\varepsilon_0\right) = f(m)\left(\mathbf{p}(m), \mathbf{Ad}_{\chi^{-1}\cdot\widetilde{\sigma_w}}\varepsilon_0\right)$

$\chi^{-1}\cdot\widetilde{\sigma_w} = \sigma_w \Leftrightarrow \widetilde{\sigma_w} = \chi\cdot\sigma_w$ ■

A field of world lines is represented by a vector field future oriented such that : $\langle V(m), V(m)\rangle_{TM} = -c^2$

For a given observer, the trajectories of particles whose trajectories do not cross (such as a beam of particles) can be represented by a vector field, future oriented, such that $\langle V(m), V(m)\rangle_{TM} < 0$.

So they can both be represented by a section $\sigma_w \in \mathfrak{X}(P_W)$, up to sign.

$V = c\varepsilon_0 + \overrightarrow{v} = \left(2a_w^2 - 1\right)\varepsilon_0 + a_w\overrightarrow{w}$

If $a_w > 0$ then $\overrightarrow{w}$ is oriented as $\overrightarrow{v}$, and has the opposite direction if $a_w < 0$.

If V is past oriented ($u_0 < 0$) or null ($\langle V, V\rangle = 0$) there is no solution :

$2a_w^2 - 1 = \frac{1}{2}(u_0 - 1) < -\frac{1}{2} \Rightarrow a_w^2 < 1$ and $a_w^2 \neq 1 + \frac{1}{4}w^t w$

10.3.2 Representation of material bodies in GR

In Mechanics a material body is comprised of "material points" that is elements of matter whose location is a single geometric point, and change with time in a consistent way : their trajectories do not cross, and the body keeps some cohesion, which is represented by a deformation tensor for deformable solids. So a material body can be represented in GR by a field of vectors U, future oriented with length $\langle U, U\rangle = -c^2$, such that, at some time 0, the particles are all together in a compact subset $\varpi(0)$ of a 3 dimensional space like submanifold.

The proper time τ is, up to an additive constant, a characteristic of the vector field so, if the time $\tau = 0$ is defined by $\varpi(0)$, at any given time τ the location of the body itself is $\varpi(\tau) = \{\Phi_U(\tau, a), a \in \omega(0)\}$.

Definition 28 *A **material body** B is defined by a field of vectors U, future oriented with length $\langle U, U\rangle = -c^2$, and a compact subset $\varpi(0)$ of a 3 dimensional space like submanifold. The body is located at its proper time τ on the set $\varpi(\tau) = \{\Phi_U(\tau, a), a \in \omega(0)\}$ diffeomorphic to $\varpi(0)$.*

The definition of a material body is intrinsic : the vector field U and the submanifold $\varpi(0)$ do not depend on a chart or an observer.

The area swept by the body is $\widehat{\omega} = \{\Phi_U(\tau, a) : \tau \in \mathbb{R}, a \in \omega(0)\}$, which is a manifold with chart $\Phi_U(\tau, a)$.

For any observer O the material body is seen at the time t as the set $\widehat{\omega} \cap \Omega_3(t)$. The material points are not labeled by the same location : for the observer their trajectories is $\Phi_V(t, x)$ with the vector field $V : U = \frac{1}{\sqrt{-\langle V,V\rangle}}V$ and $x \in \Omega_3(0)$. So actually *the characterization of a material body is observer dependant* : they do not see the same body. It coincides with B only for the observers such that $\tau = t \Rightarrow \varepsilon_0(m) = U(m)$. Conversely given a material body there is a family of observers $\mathcal{B}$ for which $\varpi(t) = \varpi(\tau)$ and one says that they are "attached" to the material body (they are not necessarily physically on the material body, but their velocities must be identical).

This general definition applies to solids, in the usual meaning, but also to fluids, which are composed of material points which travel along trajectories which do not cross.

A material point $x \in \omega(0)$ is transported along the integral curve of U : $\Phi_U(\tau, x) \in \omega(\tau)$. Let $(e_i(\tau, x))_{i=0}^3$ be a tetrad attached to the material point $x \in \omega(0)$. The vector $e_i(\tau, x)$ is transported as : $\Phi'_{Ux}(\tau, x)\, e_i(0, x)$ along the integral curve. If U is not a Kiling vector field the set $(e_i(\tau, x))_{i=0}^3$ is not an orthonormal basis. So, to be consistent, we have to assume that, in addition to the vector field U, there is a map which transports the tetrad of the material points : that is a section $\sigma \in \mathfrak{X}(P_G)$. Then the vector field U can be defined from σ as above.

10.3.3 Representation of a deformable solid by a section of P_G

To any section of P_G and any observer $O(\varepsilon_0)$ is associated a unique field V of trajectories. Then to any compact subset $\widehat{\varpi}$ of $\Omega_3(0)$ is attached a material body $\mathcal{B}$, with the same proper time t as O. The tetrads of O are arbitrary, but $\sigma(m)$ defines at each point of $\mathcal{B}$ an orthonormal basis, whose arrangement with respect to O is given by $\sigma(m)$:

$e_i(m) = \mathbf{Ad}_{\sigma(m)}\varepsilon_i(m)$

The trajectories of the material points are given by $V = \frac{c}{2a_w^2-1}\mathbf{Ad}_{\sigma(m)}\varepsilon_0(m)$: they do not have necessarily the same spatial speed : $\langle V, V\rangle = -\left(\frac{c}{2a_w^2-1}\right)^2$.

There are 2 solutions for the decomposition of σ, however, to be consistent with the definition for a single particle, $\overrightarrow{w}$ must be oriented in the direction of the spatial speed $\overrightarrow{v}$, then the spin is defined as above, the orientation of the axis gives the rotational speed.

In Galilean Geometry the deformation tensor is defined by the change $\frac{\partial}{\partial t}e_i(q,t)$ of $e_i(q,t)$ with respect to $e_i(q,t)$. The equivalent in our framework is $\frac{d\sigma}{dt}\cdot\sigma^{-1}=\sum_{\alpha=0}^{3}V^\alpha\partial_\alpha\sigma\cdot\sigma^{-1}$ whose matrix is

$[\upsilon(X_r,X_w)]=[K(X_w)]+[J(X_r)]\in so(3,1)$:

$[K(X_w)]=\begin{bmatrix}0 & X_w^t\\ X_w & 0\end{bmatrix},[J(X_r)]=\begin{bmatrix}0 & 0\\ 0 & j(X_r)\end{bmatrix}$

The deformation tensor has a symmetric ($[K(X_w)]$) and an antisymmetric ($[J(X_r)]$) part, as the usual deformation tensor.

And we can state :

Definition 29 *A **deformable solid** is a material body, represented by a vector field U future oriented with $\langle U,U\rangle=-c^2$, a compact subset $\varpi(0)$ of a 3 dimensional space like submanifold and a section $\sigma\in\mathfrak{X}(P_G)$ such that : $U(m)=\mathbf{Ad}_{\sigma(m)}\varepsilon_0(m)$ for the observers ε_0 attached to the body.*

Conversely a section $\sigma\in\mathfrak{X}(P_G)$ and a compact subset ω of M defines for any observer a deformable solid. It belongs at t to the subsets $\{\varphi_O(x,t),x\in\omega\cap\Omega_3(0)\}$.

The motion of a material body is then defined by a section of the 1st jet prolongation of P_G :

$\sigma\in\mathfrak{X}(P_G)\to J^1\sigma=\left(m,\sigma(m),\partial_\alpha\sigma\cdot\sigma^{-1},\alpha=0...3\right)\in J^1Cl(TM)$

By definition the motion is continuous : $\upsilon(X_{r\alpha},X_{w\alpha})=\partial_\alpha\sigma\cdot\sigma^{-1}$.

The field of trajectories of the material points is :

$V=\frac{dq}{dt}=c\varepsilon_0+\overrightarrow{v}=-\frac{c}{\langle\mathbf{Ad}_\sigma\varepsilon_0,\varepsilon_0\rangle_{Cl}}\mathbf{Ad}_\sigma\varepsilon_0$

$\forall i,\alpha=0..3$:

$\partial_\alpha e_i=[\upsilon(X_{r\alpha},X_{w\alpha}),e_i]$

$\partial_\alpha V=\frac{V}{c}\langle[\upsilon(X_{r\alpha},X_{w\alpha}),V],\varepsilon_0\rangle_{Cl}+[\upsilon(X_{r\alpha},X_{w\alpha}),V]$

and along the trajectories : $\frac{d}{dt}=\sum_{\alpha=0}^{3}V^\alpha\partial_\alpha$

It is usually more convenient to define a deformable solid from the point of view of an observer $\mathcal{B}$ attached to the solid : $\varepsilon_0=U$, and do the computations in a spherical system of coordinates. Then proceed to a change of gauge to represent the motion of the solid from the point of view of another observer. It requires just the map $\chi:\mathcal{B}\to O\in Spin(3,1)$. The composition of motions in the GR framework is thus easy.

The arrangement of each individual particle, represented by σ, is not necessarily identical. A **rigid solid** can be defined as a solid such that the motion is identical at each point :

$\forall x\in\omega(0):\frac{d\sigma}{dt}\cdot\sigma^{-1}(\Phi_V(t,x))=\upsilon(Y_r(t),Y_w(t))$

$\Leftrightarrow\sigma(\Phi_V(t,x))=s(t)\cdot\sigma(\Phi_V(0,x))$ with $s(t)\in Spin(3,1)$

and $s(t)$ represents the arrangement of the rigid solid with respect to the observer. Then the deformation tensor depends only on t.

Part III
KINEMATICS

Fields acts on particles by forces which change the motion of particles, according to kinematic characteristics of these particles. They are expressed as mass and inertial tensors, from which are defined translational and rotational momenta. Newtonian Mechanics has developed a comprehensive and sophisticated theory of Kinematics, and Analytic Mechanics has provided much of the initial framework for QM. Relativity introduces a totally new concept of motion, which is now absolute in a quadridimensional universe, and the usual concept of rigid solid does not hold any longer. If the usual concepts of Kinematics can more or less be fitted to Special Relativity, General Relativity requires a totally new approach, with spinors, which have been introduced, by a very different way, in the.Quantum Theory of Fields.

As we have done for the Geometric concepts, it is useful to rediscover the main concepts of Kinematics in Newtonian Mechanics.

11 USUAL REPRESENTATIONS

11.1 In Newtonian Mechanics

Motion and momentum are two different, but related, physical quantities. They are measured by different protocols. Momenta can be computed but actually this is the change in the value of the momenta which is measured, through inertial forces which express the resistance of a material body to change its motion.

As for motion, there is a translational momentum and a rotational momentum, to which are associated linear forces (or "forces") and torques.

The balance of energy exchanged by a material body with the forces exercised on it is then expressed by the kinetic energy, and there are a translational and a rotational kinetic energy.

The picture is clear for rigid solids, but can be extended to deformable solids, which are of a greater interest because they can be defined in the relativist context.

11.1.1 Translational Momentum

To a material point with mass m and speed $\overrightarrow{v} = \frac{dq}{dt}$ is associated the translational momentum $\overrightarrow{p} = m\overrightarrow{v}$. And the Fundamental Law of Mechanics states the relation $\overrightarrow{F} = \frac{d\overrightarrow{p}}{dt}$ between a force exercised on the material point and the change of its momentum. The assumption that m is a scalar constant leads then to a direct relation between the force and the motion. So a *change* of motion can be measured (by accelerometers as in smartphones) without any measure of the motion, even by an observer attached to the material body. And if $\overrightarrow{F} = 0$ then the momentum is constant.

For a system of material points the picture is more complicated, because actually the forces are localized quantities : they should be represented, not by a single vector $\overrightarrow{F}$, but by a couple $\left(q, \overrightarrow{F}\right)$. However Galilean Geometry has the special feature (because it is represented as an affine space) that one can define a center of mass G for any system of material points : $\left(\sum_a m_a\right)\overrightarrow{OG} = \sum_a m_a\overrightarrow{OM}_a$. Then the system is equivalent to a particle of mass $\sum_a m_a$ located at G and the sum $\overrightarrow{F}_G = \sum_a \overrightarrow{F}_a$, exercised at G, has a physical meaning. And the Law of Mechanics can be written :

$\sum_a \frac{d\overrightarrow{p}_a}{dt} = \frac{d\overrightarrow{p}_G}{dt} = \overrightarrow{F}_G$

11.1.2 Torque

Another consequence of the localization of the forces is the existence of torques, similar to forces, but which are distinct physical quantities.

For a force $\left(M_a, \overrightarrow{F_a}\right)$ the torque is defined with respect to any point fixed O by $\tau_a(O) = \overrightarrow{OM_a} \times \overrightarrow{F_a}$ with the cross product. $\tau_a(O)$ reads : $\tau_a(O) = j\left(\overrightarrow{OM_a}\right)\overrightarrow{F_a} = j\left(\overrightarrow{F_a}\right)\overrightarrow{M_aO}$ so this is actually an operator, acting on O, with an antisymmetric matrix, which can then be represented by a vector of $\mathbb{R}^3$ with the usual convention. As with the translational momentum, the rotational momentum is defined by :

$\Gamma_a(O) = j\left(\overrightarrow{OM_a}\right)\overrightarrow{p_a} = m_a j\left(\overrightarrow{OM_a}\right)\left(\frac{d}{dt}\overrightarrow{OM}_a\right)$

$\frac{d}{dt}\Gamma_a(O) = j\left(\frac{d}{dt}\overrightarrow{OM_a}\right)\overrightarrow{p_a} + j\left(\overrightarrow{OM_a}\right)\frac{d}{dt}\overrightarrow{p_a} = \tau_a(O)$

Because in Galilean Geometry one can define a center of mass :

$\Gamma_a(O) = j\left(\overrightarrow{OG}\right)\overrightarrow{p_a} + j\left(\overrightarrow{GM_a}\right)\overrightarrow{p_a}$

$\sum_a \Gamma_a(O) = j\left(\overrightarrow{OG}\right)\sum_a \overrightarrow{p_a} + \sum_a j\left(\overrightarrow{GM_a}\right)\overrightarrow{p_a} = \sum_a j\left(\overrightarrow{GM_a}\right)\overrightarrow{p_a}$
$= \sum_a j\left(\overrightarrow{GM_a}\right) m_a\left(\frac{d}{dt}\overrightarrow{OG} + \frac{d}{dt}\overrightarrow{GM}_a\right) = \sum_a j\left(\overrightarrow{GM_a}\right) m_a \frac{d}{dt}\overrightarrow{GM}_a = \sum_a \Gamma_a(G)$
and one can define a total torque :
$\tau = \sum_a \tau_a(O) = \sum_a \tau_a(G)$
For a rigid solid :
$\overrightarrow{G(t)M_a(t)} = R(t)\overrightarrow{X}_a$ with $\overrightarrow{X}_a = Ct$
$\sum_a \Gamma_a(G) = \sum_a m_a j\left(\overrightarrow{GM_a}\right)\frac{d}{dt}\overrightarrow{GM_a} = \sum_a m_a j\left(R(t)\overrightarrow{X}_a\right)\frac{dR}{dt}\overrightarrow{X}_a$
$= \sum_a m_a j\left(R(t)\overrightarrow{X}_a\right) R(t) R(t)^{-1}\frac{dR}{dt}\overrightarrow{X}_a$
$= R(t)\sum_a m_a j\left(\overrightarrow{X}_a\right) R(t)^{-1}\frac{dR}{dt}\overrightarrow{X}_a$
$= R(t)\sum_a m_a j\left(\overrightarrow{X}_a\right) j(r(t))\overrightarrow{X}_a = -R(t)\sum_a m_a j\left(\overrightarrow{X}_a\right) j\left(\overrightarrow{X}_a\right) r(t)$
$[J] = -\sum_a m_a j\left(\overrightarrow{X}_a\right) j\left(\overrightarrow{X}_a\right)$ is a fixed symmetric matrix, the rotational tensor, and $[J]\,r(t)$ is the rotational momentum.[4]
$\sum_a \Gamma_a(G) = R(t)[J]\,r(t)$
and $\sum_a \tau_a(G) = \frac{dR}{dt}[J]\,r(t) + R[J]\frac{dr}{dt} = R\left(j(r)[J]\,r + [J]\frac{dr}{dt}\right)$

11.1.3 Kinetic energy

Mechanical Energy is defined as the work done by a force along a path : $W = \int_{q_1}^{q_2}\left\langle \overrightarrow{F}, \overrightarrow{dq}\right\rangle$ thus with $\overrightarrow{F} = \frac{d\overrightarrow{p}}{dt}$:

$W = \int_{t_1}^{t_2}\frac{1}{m}\left\langle \overrightarrow{p}, \frac{d\overrightarrow{p}}{dt}\right\rangle dt = \frac{1}{2}\int_{t_1}^{t_2}\frac{1}{m}\frac{d}{dt}\left\langle \overrightarrow{p}, \overrightarrow{p}\right\rangle dt$ which leads to the definition of the variation of kinetic energy : $\delta K = \frac{1}{m}\left\langle \overrightarrow{p}, \overrightarrow{\delta p}\right\rangle$, that is the energy that the body exchanges with the exterior in a change $\overrightarrow{\delta p}$ of momentum, and the kinetic energy $K = \frac{1}{2m}\left\langle \overrightarrow{p}, \overrightarrow{p}\right\rangle$ when, in a continuous motion, $\overrightarrow{\delta p} = \frac{d\overrightarrow{p}}{dt}$. It is defined with respect to an observer, as well as $\overrightarrow{p}$.

Kinetic energy being a scalar one can sum the kinetic energy related to the translational momentum of a set of material points :
$K = \sum_a \frac{1}{2m_a}\left\langle \overrightarrow{p_a}, \overrightarrow{p_a}\right\rangle$
$= \sum_a \frac{1}{2}m_a\left\langle \frac{d}{dt}\overrightarrow{OG} + \frac{d}{dt}\overrightarrow{GM}_a, \frac{d}{dt}\overrightarrow{OG} + \frac{d}{dt}\overrightarrow{GM}\right\rangle$
$= \sum_a \frac{1}{2}m_a\left\langle \frac{d}{dt}\overrightarrow{OG}, \frac{d}{dt}\overrightarrow{OG}\right\rangle + 2\sum_a \frac{1}{2}m_a\left\langle \frac{d}{dt}\overrightarrow{OG}, \frac{d}{dt}\overrightarrow{GM_a}\right\rangle + \sum_a \frac{1}{2}m_a\left\langle \frac{d}{dt}\overrightarrow{GM}_a, \frac{d}{dt}\overrightarrow{GM_a}\right\rangle$
$= \frac{1}{2}M\left\|\overrightarrow{v_G}\right\|^2 + \sum_a \frac{1}{2}m_a\left\langle \frac{d}{dt}\overrightarrow{GM}_a, \frac{d}{dt}\overrightarrow{GM_a}\right\rangle$
For a solid :
$\overrightarrow{G(t)M_a(t)} = R(t)\overrightarrow{X}_a$
$\left\langle \frac{d}{dt}\overrightarrow{GM}_a, \frac{d}{dt}\overrightarrow{GM_a}\right\rangle = \left\langle \frac{dR}{dt}\overrightarrow{X}_a, \frac{dR}{dt}\overrightarrow{X_a}\right\rangle = \left\langle R^{-1}\frac{dR}{dt}\overrightarrow{X}_a, R^{-1}\frac{dR}{dt}\overrightarrow{X_a}\right\rangle = \left\langle j(r)\overrightarrow{X}_a, j(r)\overrightarrow{X_a}\right\rangle$
$= \left\langle j\left(\overrightarrow{X}_a\right) r, j\left(\overrightarrow{X}_a\right) r\right\rangle = [r]^t\left[j\left(\overrightarrow{X}_a\right)\right]^t\left[j\left(\overrightarrow{X}_a\right)\right][r] = -[r]^t\left[j\left(\overrightarrow{X}_a\right)\right]\left[j\left(\overrightarrow{X}_a\right)\right][r]$
$\sum_a \frac{1}{2}m_a\left\langle \frac{d}{dt}\overrightarrow{GM}_a, \frac{d}{dt}\overrightarrow{GM_a}\right\rangle = \frac{1}{2}[r]^t[J][r]$
$K = \frac{1}{2}M\left\|\overrightarrow{v_G}\right\|^2 + \frac{1}{2}[r]^t[J][r]$
And the variation of rotational kinetic energy is :
$\frac{d}{dt}\left(\frac{1}{2}[r]^t[J][r]\right) = \frac{1}{2}\left[\frac{dr}{dt}\right]^t[J][r] + \frac{1}{2}[r]^t[J]\left[\frac{dr}{dt}\right]$
The torque on the solid :
$\tau(G) = \sum_a \tau_a(G) = \frac{d}{dt}(R(t)[J]\,r(t)) = R\left(j(r)[J]\,r + [J]\frac{dr}{dt}\right)$
$[J]\frac{dr}{dt} = R^t[\tau(G)] - j(r)[J]\,r$
$\frac{1}{2}\left[\frac{dr}{dt}\right]^t[J][r] = \frac{1}{2}\left([\tau(G)]^t R + [r]^t[J]\,j(r)\right)[r]$
$\frac{1}{2}[r]^t[J]\left[\frac{dr}{dt}\right] = \frac{1}{2}[r]^t\left([R]^t[\tau(G)] - j(r)[J][r]\right)$
$\frac{d}{dt}\left(\frac{1}{2}[r]^t[J][r]\right) = \frac{1}{2}[\tau(G)]^t[R][r] + \frac{1}{2}[r]^t[J]\,j(r)[r] + \frac{1}{2}[r]^t[R]^t[\tau(G)] - \frac{1}{2}[r]^t\,j(r)[J][r]$
$\frac{d}{dt}\left(\frac{1}{2}[r]^t[J][r]\right) = [r]^t[R]^t[\tau(G)]$ is the work done by the torque $\tau(G)$
$\delta K = \frac{1}{m}\left\langle \overrightarrow{p}, \overrightarrow{\delta p_G}\right\rangle + [r]^t[R]^t[\delta\Gamma(G)]$

[4] Matrices like $j(X)\,j(X)$ have negative eigen values, so the minus sign induces positive momenta along the eigen vectors.

The representation and computations above rely heavily on the existence of a center of mass, the fact that $SO(3)$ has the same dimension as the space $\mathbb{R}^3$, and on the properties of solids. However the definitions can be extended to deformable solids.

11.1.4 Density

Material bodies consist of material points, so it is natural to introduce a density μ, seen as the number of identical material points at the same location x at the time t : $\mu(x,t)$. With a volume form $\varpi_3 = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ the density defines a measure $\mu\varpi_3$ such that the mass of the material body in an area Ω at t is $M(t) = \int_\Omega \mu(x,t)\,\varpi_3(x)$.

In the model of deformable solid introduced previously a material point is labeled by its position q at $t=0$ and its position at t is given by a differentiable map : $X(q,t) = \phi(q,t)$. A basis, e_i attached at q, orthonormal at $t=0$ is transported by $\phi'_q(q,t) : e_i(q,t) = \phi'_q(q,t)\,e_i(q,0)$. It is no longer orthonormal at t and defines a metric $g_{ij}(q,t) = \langle e_i(q,t), e_j(q,t)\rangle$ and a volume form $\varpi(q,t) = \sqrt{\det g}\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 = \det \phi'_q(q,t)\,\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$.

$\varpi(q,t)$ is just the push forward of ϖ_3 by ϕ. The material points which occupy a volume $\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ at $t=0$ occupy a volume $\det \phi'_q(q,t)\,\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ at t. Then the conservation of mass, which is equivalent to the conservation of the number of particles, leads to :

$\frac{\partial}{\partial t}\left(\mu(q,t)\det \phi'_q(q,t)\right) = 0$

$\frac{\partial \mu}{\partial t}\det \phi'_q + \mu\left(\det \phi'_q\right) Tr\left(\left[\frac{\partial}{\partial t}\phi'_q\right]\left[\phi'_q\right]^{-1}\right) = 0$

The trajectories of the particles are : $\frac{\partial}{\partial t}X(q,t) = \frac{\partial}{\partial t}\phi(q,t)$

Let us define : $V : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 :: \overrightarrow{V}(x,t)$ such that : $\overrightarrow{V}(X(q,t),t) = \frac{\partial}{\partial t}X(q,t)$. It is called the "flow velocity".

$\Rightarrow \frac{\partial^2 \phi_i}{\partial t \partial q_k}(q,t) = \frac{\partial V_i}{\partial q_k} = \sum_{j=1}^3 \frac{\partial V_i}{\partial x_j}\frac{\partial x_j}{\partial q_k} = \sum_{j=1}^3 \frac{\partial V_i}{\partial x_j}\frac{\partial \phi_j}{\partial q_k}$

$\Rightarrow \left[\frac{\partial}{\partial t}\phi'_q\right] = \left[\frac{\partial V}{\partial x}\right]\left[\frac{\partial \phi}{\partial q}\right]$

$Tr\left(\left[\frac{\partial}{\partial t}\phi'_q\right]\left[\phi'_q\right]^{-1}\right) = Tr\left[\frac{\partial V}{\partial x}\right] = div\overrightarrow{V}$

and we get the continuity equation : $\frac{\partial \mu}{\partial t} + \mu div\overrightarrow{V} = 0$.

The reasoning is done usually for fluids but it holds for a deformable solid.

11.1.5 Stress tensor, Energy-momentum tensor

The motion of each material point of a deformable solid can be represented by :

- a translation, given by $\frac{dq}{dt} = \frac{\partial}{\partial t}\phi(q,t) = \overrightarrow{V}(X(q,t),t)$
- a deformation of its orthonormal basis $(e_i(q,t))_{i=1}^3$ given by : $\frac{\partial}{\partial t}e_i(q,t) = [\gamma(q,t)]\,e_i(q,t)$ with the deformation tensor $[\gamma] = \left[\phi"_{qt}(q,t)\right]\left[\phi'_q(q,t)\right]^{-1} = [\partial_x V]$ which can be decomposed in a symmetric part $[s] = \frac{1}{2}\left([\gamma] + [\gamma]^t\right)$ and an antisymmetric part $[j(\rho)] = \frac{1}{2}\left([\gamma] - [\gamma]^t\right)$

The usual momentum of the material point is :

$\overrightarrow{p}(q,t) = \mu(q,t)\frac{dq}{dt} = \mu(q,t)\overrightarrow{V}(X(q,t),t)$.

The deformation of the solid is the effect of forces, or conversely the solid opposes forces to its deformation. From :

$\frac{d}{dt}\left(\mu(q,t)V(X(q,t),t)\right) = \frac{d\mu}{dt}V + \mu\left([\partial_x V][\phi'_t] + [\partial_t V]\right) = \frac{d\mu}{dt}V + \mu\left(([s] + [j(\rho)])V + [\partial_t V]\right)$

one can identify :

- a force corresponding to a variation of the translational momentum : $\mu\left[\partial_t \overrightarrow{V}\right]$
- the forces, similar to a pressure (they act symmetrically), opposed to the variation of the volume :

$\frac{d\mu}{dt}V + \mu[s]V = \mu([s] - divV)V$

- a torque $\mu[j(\rho)]V$

The variation of the kinetic energy can be computed as above.

$\frac{dK}{dt} = \frac{1}{2}\frac{1}{\mu}\left\langle \overrightarrow{p}(q,t), \frac{d}{dt}\overrightarrow{p}(q,t)\right\rangle = \frac{1}{2}[V]^t\left\{\mu[\partial_t V] + \mu([s] - divV)V + \mu[j(\rho)]V\right\}$

$= \frac{1}{2}\mu[V]^t[\partial_t V] + \frac{1}{2}\mu[V]^t[s][V] - \frac{1}{2}\mu(divV)[V]^t[V] + \mu\frac{1}{2}[V]^t[j(\rho)]V$

and we have a kinetic energy corresponding to the rotational momentum $\mu\frac{1}{2}[j(V)]\rho$.

This is usually written with a "stress tensor" $T = T^i_j \varepsilon^j \otimes \varepsilon_i$ such that the forces, on the surface $d\sigma$ with normal $\overrightarrow{n}$, opposing the deformation (the "stress"), are $\overrightarrow{dF} = T(\overrightarrow{n}) = \sum_{i,j=1}^3 [T]^i_j [n]^j \varepsilon_i d\sigma$.

By considering a small volume Ω with border $\partial\Omega$:

- the sum of the stress on Ω is :

$\int_{\partial\Omega}\sum_{i,j=1}^3 [T]^i_j [n]^j \varepsilon_i d\sigma = \int_{\partial\Omega}\left\langle \sum_{i=1}^3 [T]^i \varepsilon_i, \overrightarrow{n}\right\rangle d\sigma = \int_\Omega \overrightarrow{dF_v}\varpi_3$ that is a force by unit of volume $\overrightarrow{dF}_v = \sum_{j=1}^3 div\left(\sum_{i=1}^3 [T]^i_j \varepsilon_i\right)\varepsilon_j$

- the torque with respect to the origin :

$\tau(O) = \int_{\partial\Omega} \overrightarrow{X} \times \sum_{i,j=1}^{3} [T]_j^i [n]^j \varepsilon_i d\sigma = \int_{\partial\Omega} \left\langle j(X) \sum_{i=1}^{3} [T]^i \varepsilon_i, \overrightarrow{n} \right\rangle d\sigma = \int_\Omega d\tau\varpi_3$

with the fixed orthonormal basis ε_j

$d\tau = \sum_{i,j=1}^{3} div\left(j(X) \sum_{i=1}^{3} [T]_j^i \varepsilon_i\right) \varepsilon_j = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \left(\left[j(X)[T]_j\right]\right)^i \varepsilon_j$

$= \sum_{i,j=1}^{3} j(\varepsilon_i)[T]_j \varepsilon_j + j(X) \sum_{i,j=1}^{3} \left[\left[\frac{\partial}{\partial x_i} T\right]_j^i\right] \varepsilon_j$

$= \sum_{i,j=1}^{3} j(\varepsilon_i)[T]_j^i \varepsilon_j + \overrightarrow{X} \times \overrightarrow{dF}_v$

Thus there is an elementary torque located at X equal to $\sum_{i,j=1}^{3} j(\varepsilon_i)[T]_j^i \varepsilon_j$

11.1.6 Symmetries

Symmetries have a meaning only for rigid solid. A solid presents a symmetry if it looks the same for a class of observers : a rotation of the spatial basis of the observer, belonging to a subgroup of $SO(3)$ gives the same measure. So we can have spherical symmetry (the whole of $SO(3)$) or cylindrical symmetry (rotations with a fixed axis). However there is an intriguing question : which is the measure involved ? It can be a visual look, that is the shape of the body, and it refers to the value of a density function μ at the border of ω, the 3 dimensional volume of the body. But a more meaningful criterion is a kinematic symmetry, which can be measured by the rotational momentum of the solid :

$\sum_a \Gamma_a(G) = R(t)[J] r(t)$

In a change of observer given by a global rotation $g \in SO(3)$:

$R \to \widetilde{R} = gR$,

$R^{-1} \frac{dR}{dt} = j(r) \to \widetilde{R}^{-1} \frac{d\widetilde{R}}{dt} = j(r)$

$\sum_a \Gamma_a(G) \to \widetilde{R(t)}[J] r(t) = g \sum_a \Gamma_a(G)$

Notice that the instantaneous motion (represented by r) does not change. There is a symmetry if :

$gR(t)[J] r(t) = R(t)[J] r(t)$

that is if $R(t)[J] r(t)$ is an eigen vector of g, the only real eigen vector of g is given by the axis with the eigen value 1. The matrix $[J]$ is symmetric, and has 3 orthogonal eigen vectors r_a, with real eigen values λ_a. If the motion is a constant rotation with axis one of this eigen vectors r_a, then $R(t)[J] r(t) = \lambda_a (\exp tr_a) r_a = \lambda_a r_a$ and there is a symmetry for $g = \exp r_a$.

11.1.7 Energy momentum tensor

A more general way to deal with these issues is with the "Energy-Momentum" tensor, which comes from the Principle of Least Action. A system represented by variables $z^i(m), i = 1...N$ defined over a manifold with coordinates $(\xi_\alpha)_{\alpha=1}^3$, and their first partial derivatives $z_\alpha^i(m)$ is endowed with a scalar lagrangian such that the equilibrium is reached when the functional $\int_\Omega L\left(z^i(m), z_\alpha^i(m)\right) \varpi(m)$ is stationary. Then the quantity :

$T = \sum_{i\alpha\beta} \frac{\partial L}{\partial z_\alpha^i} z_i^\beta \partial\xi_\alpha \otimes d\xi^\beta$

is a tensor, called the Energy-Momentum tensor. The Lagrangian has the meaning of the energy of the system, and a change $\delta z^i = \sum_\beta z_\beta^i \delta v^\beta$ of the variables $z^i(m)$ along $\delta v = \sum_{\beta=0}^{3} v^\beta \delta\xi_\beta$ changes the energy by $\delta\ell = \int_\Omega div(T(\delta v))\varpi$ so that $T(\delta v)$ can be seen as a reaction of the system to a change by δv, that is as a force. Then the quantities $\Pi_i = \sum_{\alpha\beta} \frac{\partial L}{\partial z_\alpha^i} z_i^\beta \partial\xi_\alpha \otimes d\xi^\beta$ are the momenta associated to the scalar variable z^i. They are the generalized definition of the translational and rotational momenta, as they apply for any motion. If $z = \left(z^i\right)_{i=1}^n$ are the components of a vector in some vector space E then there is a momentum Π_z expressed as a tensor valued in the dual space E^*.

To sum up, in Newtonian Mechanics :

i) The kinematic of a material body is represented by a translational momentum and a rotational momentum, which are distinct and read : $\overrightarrow{p} = m\overrightarrow{v}; \Gamma = R(t)[J] r$

ii) Each momentum is related to the motion, and overall the kinematic characteristics of a solid are represented by 7 independent scalars (the mass and 6 parameters for $[J]$).

iii) The momenta can be computed, but this is the change in the momenta which is measured, through the inertial forces.

iv) The representation of the momenta by vectors of $\mathbb{R}^3$ is conventional. If it is natural for $\overrightarrow{p}$, the vector $R(t)[J] r$ has no direct relation with a physical basis $\overrightarrow{\varepsilon}_i$.

v) For deformable solids and systems the definition of momenta is less straightforward and comes from the identification of the forces, inertial and external, acting on the body. The representation of momenta and forces is given through the lagrangian.

vi) The conservation of the momentum in the transformation of a system is only a special case of the laws of the transformation, meanwhile the conservation of energy is just the balance of the energy exchanged by its different components.

11.2 Usual representations in the relativist framework

11.2.1 Translational momentum

The translational momentum is defined as the 4 dimensional vector : $P = mV$. It depends on the observer, and here it means the choice of the time t in the derivative $V = \frac{dq}{dt}$.

In the relativist context location and motion are absolute. So there is an intrinsic definition of the momentum, *for an observer who is attached to the particle* with the proper time and velocity $u = \frac{dq}{d\tau}$: $p = mu$. Then, if we take the same definition for the kinetic energy, with respect to this observer, it is constant : $K = \frac{1}{m}\langle p, p\rangle = -mc^2$.

For any other observer :

$P = mV = p\frac{d\tau}{dt} = p\sqrt{1 - \frac{\|\overrightarrow{v}\|^2}{c^2}}$

$-\frac{1}{m}\langle P, P\rangle = -\left(1 - \frac{\|\overrightarrow{v}\|^2}{c^2}\right)mc^2 = \left(1 - \frac{\|\overrightarrow{v}\|^2}{c^2}\right)K \leq K$

$p = m\frac{1}{\sqrt{1-\frac{\|\overrightarrow{v}\|^2}{c^2}}}\left(c\varepsilon_0\left(q\left(t\right)\right) + \overrightarrow{v}\right)$ is a 4 dimensional vector. However the common practice is to distinguish its spatial and time components. The spatial part : $\overrightarrow{p_r} = m\frac{\overrightarrow{v}}{\sqrt{1-\frac{\|\overrightarrow{v}\|^2}{c^2}}}$ which is similar to the usual translational momentum, and $m\frac{c}{\sqrt{1-\frac{\|\overrightarrow{v}\|^2}{c^2}}}$ is then related to the energy E, defined by :

$E = c^2 m\frac{1}{\sqrt{1-\frac{\|\overrightarrow{v}\|^2}{c^2}}} = \langle Pc, \varepsilon_0\rangle = mc\left\langle \frac{dq}{d\tau}, \varepsilon_0\right\rangle$

$\Rightarrow E^2 = c^2\left\|\overrightarrow{p}_r\right\|^2 + m^2c^4$ which is just $\langle pc, pc\rangle = -m^2c^4 = c^2\left\|\overrightarrow{p}_r\right\|^2 - E^2$

The advantages of this expression is that for small speed it gives :

$E = c^2 m\frac{1}{\sqrt{1-\frac{\|\overrightarrow{v}\|^2}{c^2}}} \simeq c^2 m\left(1 + \frac{1}{2}\frac{\|\overrightarrow{v}\|^2}{c^2}\right) = \frac{1}{2}m\left\|\overrightarrow{v}\right\|^2 + mc^2$

and it can be adapted to massless particles such as photons.

The total energy of the particle E has one part corresponding to a kinetic energy and another one to an "energy at rest". So keep the principle of conservation of energy leads to accept that mass itself can be transformed into energy, according to the famous relation $E = mc^2$.

However it mixes two concepts - momentum and energy - which are usually seen as distinct and are measured by different protocols.

The definition of rigid solid of Newtonian Mechanics does not extend to the relativist Geometry, and there is no satisfying definition for the rotational momentum.

In GR what is considered is the energy-momentum tensor T, which is a key part of the Einstein equation. There is no general formula to specify T, only phenomenological laws. The most usual are based on the behavior of dust clouds, including sometimes thermodynamic components.

11.2.2 The Dirac's equation

In writing $pc = (c\overrightarrow{p_r}, E)$ the energy E and p_r are two separate quantities which can be measured [5]. In the usual interpretation of QM to E and p_r are associated operators acting on scalar wave functions ψ.

In common QM, "quantization" is just an operation where mathematical symbols are substituted to other symbols. Starting from : $E^2 = c^2\left\|\overrightarrow{p}_r\right\|^2 + m^2c^4$ the "minimal substitution rule" : $E \to i\hbar\frac{\partial}{\partial t}; p_{r\alpha} \to -i\hbar\partial_\alpha$ gives the Klein-Gordon equation : $\left(\Box + m^2\right)\psi = 0$ which, checked for the spectrum of Hydrogen, provides wrong results.

In order to have first order derivatives Dirac proposed another equation, starting from $E = \sqrt{c^2\left\|\overrightarrow{p}_r\right\|^2 + m^2c^4}$ assuming that :

$E = A.p_r + Bm$ the substitution gives : $i\hbar\frac{\partial\psi}{\partial t} = \left(Ai\hbar\nabla + B\mu\right)\psi$

But this is possible only if ψ is a vectorial quantity (and no longer a scalar function). Moreover to be Lorentz equivariant A, B must be 4×4 complex matrices, built from a set of matrices with the relation : $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}I_4$. The wave functions ψ are then vectors, belonging to a 4 dimensional complex vector space E, which is the representation of the Lorentz group acting through the matrices γ. They are called spinors.

[5] Actually only the change of momentum and energy can be physically measured.

The Dirac's equation then reads :
$i\frac{\partial\psi}{\partial t} = -i\sum_{\alpha=1}^{3}\gamma_\alpha\frac{\partial\psi}{\partial\xi_\alpha} + m\gamma_0\psi$
and can be seen as a propagation equation for ψ or, in the usual QM, as a substitute for the Schrödinger equation. Its eigen values correspond to the energy. Its eigen vectors, which provides a basis for observables quantities, correspond to "plane waves" :

with positive energies : $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\exp(-imt), \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\exp(-imt)$

with negative energies : $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\exp(imt), \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\exp(imt)$

The existence of the last two solutions leads to antiparticles. The proof of their existence has not closed the issue of the interpretation of these solutions, the most common being that antiparticles are "holes" in a sea of virtual particles, and that they moved backwards in time.

$\psi(t,x)$ is such that $\rho = \psi(t,x)^*\psi(t,x)$ gives the probability to find the particle at (t,x). Then the Dirac's currents $j^a = \overline{\psi}^t\gamma_a\psi$ gives the probability to find the particle in $\xi^a, a = 1,2,3$ and the solutions of the Dirac's equation meet the continuity equation :
$\frac{\partial\rho}{\partial t} + \sum_{\alpha=1}^{3}\frac{\partial j^a}{\partial\xi_\alpha} = 0$
The scheme has been extended to account for the action of the fields, and leads to the standard model. But its construction is totally abstract, and is justified only by the results that it provides, through complicated computations.

So in the Relativist context we have two representations, checked at the opposite scales, which procede from totally different principles. And this is at the core of the belief that QM and GR are not conciliable.

12 MOMENTA REVISITED

Our purpose is to find an efficient way to represent the kinematic characteristics of particles and material bodies in the framework of General Relativity. We focus on the properties assigned to momenta of material bodies :

- momenta are physical quantities, related to the motion but distinct, and a change in the value of the momenta can be physically measured through the inertial forces, by specific protocols;
- they are computed from their properties. A particle is defined not only by its location and transversal motion, but also by an orthonormal basis attached to it, with its rotational motion. So we must consider translational and rotational momenta.
- they must be expressed in a format which is equivariant with respect to the Lorentz group $Spin(3,1)$.
- momenta are localized quantities : a momentum is defined at each location $q(t)$ of the particle.
- momenta are expressed by vectorial quantities : the linear combination of momenta at the same point has a physical meaning (such as in a collision).
- in a continuous motion the momenta are related to the motion by some fixed relation.
- for a free particle, which is not submitted to any force, the momenta are constant along its world line.

We will naturally look for a fiber bundle representation. It should be a vector bundle associated to a principal bundle based on $Spin(3,1)$, and the natural choice is $P_G[E,\gamma]$, with some vector space E and action γ. E is the vector space in which are represented forces and torques. In Newtonian Mechanics they are represented by 2 distinct vectors in $\mathbb{R}^3$, but, at least for the torque, this is just a convention. So we are quite free in the choice of the vector space E. It is legitimate to look for two vectors in the Minkovski space, or a vector in a complex 4 dimensional space.

The motion $\frac{d\sigma}{dt}\cdot\sigma^{-1}$ is represented in the Lie Algebra. The derivative $\gamma'(1)$ provides a representation of the Lie algebra $T_1Spin(3,1)$ but with the bracket as internal operation, which has little interest here, so we look for a representation (E,γ) of the Clifford algebra itself. This is consistent with the assessment that the Clifford bundle $Cl(TM)$ is the right framework to represent arrangement and motion of material bodies.

The motion is represented by $(q,\sigma,\upsilon(X_r,X_w))$ in $J^1Cl(TM)$.

The momentum is represented in the associated vector bundle $P_G[E,\gamma]$

The motion comes, in a continuous motion, from the derivation of the arrangement σ. It is then natural to consider a quantity $S = \gamma(\sigma)S_0 \in E$ representing the state of the particle, with a fixed vector S_0 representing the kinematic characteristics of the body, from which the momentum is computed by derivation.

12.1 Representation of the Clifford Algebra

12.1.1 Principles

A geometric representation (E,γ) of a Clifford algebra is an isomorphism $\gamma : Cl \to L(E;E) :: [\gamma(X)]$ where $[\gamma(X)]$ is the matrix of an endomorphism of E, represented in some basis. All the operations in the Clifford algebra (multiplication by a scalar, sum, Clifford product) are reproduced on the matrices. A representation is fully defined by the family of matrices, the generators, $(\gamma_i)_{i=0}^3$, representing each vector $(\varepsilon_i)_{i=0}^3$ of an orthonormal basis. The choice of these matrices is not unique : the only condition is that $[\gamma_i][\gamma_j]+[\gamma_j][\gamma_i]=2\eta_{ij}[I]$ and any family of matrices deduced by conjugation $\widetilde{\gamma}_j = M\gamma_j M^{-1}$ with a fixed matrix M gives an equivalent representation. An element of the Clifford algebra is then represented by a linear combination of generators :

$\gamma(X) = \gamma\left(\sum_{\{i_1...i_r\}} X^{i_1...i_r}\varepsilon_{i_1}\cdot...\cdot\varepsilon_{i_r}\right) = \sum_{\{i_1...i_r\}} X^{i_1...i_r}\gamma_{i_1}...\gamma_{i_r}$

A Clifford algebra has, up to isomorphism, a unique faithful algebraic irreducible representation in an algebra of matrices. As can be expected the representations depend on the signature :

for $Cl(3,1)$ this is $\mathbb{R}(4)$ the 4×4 real matrices (the corresponding spinors are the Majorana spinors), acting on a 4 dimensional vector space;

for $Cl(1,3)$ this is $H(2)$ the 2×2 matrices with quaternionic elements.

So the choice of a representation raises the issue of the signature. However the vector space E upon which are represented the momenta can be a 4 dimensional complex vector space. The representation of complex Clifford algebras are on complex vector spaces. Moreover some Clifford algebras present a specific feature : they are the direct sum of two subalgebras which can be seen as algebras of left handed and right handed elements. This property depends on the existence of an element ω, which exists in any complex algebra, but not in $Cl(1,3), Cl(3,1)$. As chirality is a defining feature of particles and of the rotational motion, this is an additional argument for using a complex Clifford algebra.

The first step is to expand $Cl(1,3), Cl(3,1)$ into $Cl(\mathbb{C},4)$.

12.1.2 Complexification of real Clifford algebras

We have seen how to introduce a complex structure on the Clifford algebra. There is another method, more usual, by extending the set such that the operations hold with complex numbers (Maths.6.5.2). One starts by he complexification of the vector space F : it is enlarged by all vectors of the form iu : $F_{\mathbb{C}} = F\oplus iF$. The real scalar product is extended to a complex bilinear form $\langle\rangle_{\mathbb{C}}$, with the signature $(+++{+})$[6], any orthonormal basis $(\varepsilon_j)_{j=0}^3$ of F is an orthonormal basis of $F_{\mathbb{C}}$ with complex components. There is a complex Clifford algebra $Cl(F_{\mathbb{C}},\langle\rangle)$ which is the complexified of $Cl(F,\langle\rangle)$. In $Cl(F_{\mathbb{C}},\langle\rangle)$ the product of vectors is :

$\forall u,v\in F_{\mathbb{C}} : u\odot v + v\odot u = 2\langle u,v\rangle_{\mathbb{C}}$

with the bilinear *symmetric* form $\langle u,v\rangle_{\mathbb{C}}$ of signature $(+\,+\,+\,+)$. $Cl(3,1)$ and $Cl(1,3)$ have the same complexified algebraic structure $Cl(\mathbb{C},4)$. Any orthonormal basis of $Cl(3,1)$ or $Cl(1,3)$ is an orthonormal basis of $Cl(\mathbb{C},4)$ and : $\varepsilon_i\odot\varepsilon_j+\varepsilon_j\odot\varepsilon_i = 2\delta_{ij}$ and $\varepsilon_0\odot\varepsilon_0 = +1$

$Cl(3,1)$ and $Cl(1,3)$ are real vector subspaces of $Cl(\mathbb{C},4)$.

There are real algebras morphisms (injective but not surjective) from the real Clifford algebras to $Cl(\mathbb{C},4)$.

With the signature (3,1) let us choose as above a vector $\varepsilon_0\in F$ such that $\varepsilon_0\cdot\varepsilon_0=-1$.

Let us define the map :

$\widetilde{C} : (F,\langle\rangle)\to Cl(\mathbb{C},4) ::$

$\widetilde{C}(u) = (u+\langle\varepsilon_0,u\rangle_F\varepsilon_0) - i\langle\varepsilon_0,u\rangle_F\varepsilon_0 = u+\langle\varepsilon_0,u\rangle_F(\varepsilon_0-i\varepsilon_0)$

(this is just the map : $\widetilde{C}(\varepsilon_j)=\varepsilon_j, j=1,2,3; \widetilde{C}(\varepsilon_0)=i\varepsilon_0$)

$\widetilde{C}(u)\odot\widetilde{C}(v)+\widetilde{C}(v)\odot\widetilde{C}(u)$

$= (u+\langle\varepsilon_0,u\rangle_F(\varepsilon_0-i\varepsilon_0))\odot(v+\langle\varepsilon_0,v\rangle_F(\varepsilon_0-i\varepsilon_0))$

$+(v+\langle\varepsilon_0,v\rangle_F(\varepsilon_0-i\varepsilon_0))\odot(u+\langle\varepsilon_0,u\rangle_F(\varepsilon_0-i\varepsilon_0))$

$= u\odot v+\langle\varepsilon_0,v\rangle_F u\odot(\varepsilon_0-i\varepsilon_0)+\langle\varepsilon_0,u\rangle_F(\varepsilon_0-i\varepsilon_0)\odot v$

$+\langle\varepsilon_0,u\rangle_F\langle\varepsilon_0,v\rangle_F(\varepsilon_0-i\varepsilon_0)\odot(\varepsilon_0-i\varepsilon_0)$

$+v\odot u+\langle\varepsilon_0,u\rangle_F v\odot(\varepsilon_0-i\varepsilon_0)+\langle\varepsilon_0,v\rangle_F(\varepsilon_0-i\varepsilon_0)\odot u$

$+\langle\varepsilon_0,v\rangle_F\langle\varepsilon_0,u\rangle_F(\varepsilon_0-i\varepsilon_0)\odot(\varepsilon_0-i\varepsilon_0)$

$= 2\langle u,v\rangle_{\mathbb{C}}+2\langle\varepsilon_0,v\rangle_F\langle u,\varepsilon_0-i\varepsilon_0\rangle_{\mathbb{C}}+2\langle\varepsilon_0,u\rangle_F\langle\varepsilon_0-i\varepsilon_0,v\rangle_{\mathbb{C}}$

$+2\langle\varepsilon_0,u\rangle_F\langle\varepsilon_0,v\rangle_F\langle\varepsilon_0-i\varepsilon_0,\varepsilon_0-i\varepsilon_0\rangle_{\mathbb{C}}$

$= 2\langle u+\langle\varepsilon_0,u\rangle_F(\varepsilon_0-i\varepsilon_0), v+\langle\varepsilon_0,v\rangle_F(\varepsilon_0-i\varepsilon_0)\rangle_{\mathbb{C}}$

[6]Actually the signature of a bilinear symmetric form is defined for real vector space, but the meaning will be clear for the reader. We will always work here with bilinear form and not hermitian form.

$= 2\left\langle \widetilde{C}(u), \widetilde{C}(v) \right\rangle_{\mathbb{C}}$

As a consequence, by the universal property of Clifford algebras, there is a unique real algebra morphism $C : Cl(3,1) \to Cl(\mathbb{C},4)$ such that $\widetilde{C} = C \circ \jmath$ where $\jmath$ is the canonical injection $(F, \langle\rangle) \to Cl(3,1)$. We will denote for simplicity $\widetilde{C} = C$. The image $C(Cl(3,1))$ is a real subalgebra of $Cl(\mathbb{C},4)$, which can be identified with $Cl(3,1)$ so it does not depend on the choice of ε_0 (but the map C depends on ε_0).

Similarly with $\widetilde{C}'(\varepsilon_j) = i\varepsilon_j, j = 1,2,3; \widetilde{C}'(\varepsilon_0) = \varepsilon_0$ we have a real algebra morphism $C' : Cl(1,3) \to Cl(\mathbb{C},4)$ and $C'(Cl(1,3))$ is a real subalgebra of $Cl(\mathbb{C},4)$. Moreover $C'(\varepsilon_j) = -i\eta_{jj}C(\varepsilon_j)$.

12.1.3 Chirality

In $Cl(\mathbb{C},4)$ the special element is : $\omega = \pm\varepsilon_0 \odot \varepsilon_1 \odot \varepsilon_2 \odot \varepsilon_3 \in Spin(\mathbb{C},4)$. Thus there is a choice and we will use : $\omega = \varepsilon_0 \odot \varepsilon_1 \odot \varepsilon_2 \odot \varepsilon_3$.

$\omega \odot \omega = 1$ and the map : $f : Cl(\mathbb{C},4) \to Cl(\mathbb{C},4) :: f(X) = \omega \odot X$ is linear and has for eigen values ± 1. There are two eigen spaces, which are subalgebras :

$Cl(\mathbb{C},4) = Cl^R(\mathbb{C},4) \oplus Cl^L(\mathbb{C},4)$:

$Cl^R(\mathbb{C},4) = \{X \in Cl(\mathbb{C},4) : \omega \odot X = X\}$,

$Cl^L(\mathbb{C},4) = \{X \in Cl(\mathbb{C},4) : \omega \odot X = -X\}$

denoted : $Cl_\epsilon(\mathbb{C},4), \epsilon = \pm 1$

For the representation (E,γ) of $Cl(4,\mathbb{C})$:

$\gamma(\omega)\gamma(\omega) = \gamma(1) = I$ and we have similarly : $E = E^R \oplus E^L$ with

$E^R = \{S \in E : \gamma(\omega) S = S\}, E^L = \{S \in E : \gamma(\omega) S = -S\}$

and the projections : $\gamma_\epsilon(S) = \frac{1}{2}(S + \epsilon(\omega) S)$.

For any homogeneous element

$X = v_1 \odot v_2 ... \odot v_k \in Cl(\mathbb{C},4) : \omega \odot X = (-1)^k X \odot \omega$

$\Rightarrow \gamma(\omega)\gamma(X) = (-1)^k \gamma(X)\gamma(\omega)$

$\gamma(\omega)\gamma(X) S = (-1)^k \gamma(X)\gamma(\omega) S$

If $\gamma(\omega) S = \epsilon S : \gamma(\omega)\gamma(X) S = \epsilon(-1)^k \gamma(X) S$. Thus for k even $\gamma(X)$ preserves both E^R, E^L, for k odd $\gamma(X)$ exchanges E^R, E^L.

12.1.4 The choice of the representation γ

A representation is defined by the choice of its generators γ_i, and any set of generators conjugate by a fixed matrix gives an equivalent representation. We can specify the generators by the choice of a basis $(e_i)_{i=1}^4$ of E . The previous result leads to a practical choice. Let e_1, e_2 be a basis of E^R and e_3, e_4 a basis of E^L. Then :

$\gamma(\omega) = \gamma_R - \gamma_L = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$

Denote : $\gamma_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}$ with four 2×2 complex matrices $j = 0...3$.

$\gamma(\omega)\gamma(\varepsilon_j) = -\gamma(\varepsilon_j)\gamma(\omega)$ which imposes the condition :

$\begin{bmatrix} A_j & -B_j \\ C_j & -D_j \end{bmatrix} = -\begin{bmatrix} A_j & B_j \\ -C_j & -D_j \end{bmatrix} \Rightarrow \gamma_j = \begin{bmatrix} 0 & B_j \\ C_j & 0 \end{bmatrix}$

The defining relations : $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk} I_4$ lead to :

$\begin{bmatrix} B_jC_k + B_kC_j & 0 \\ 0 & C_jB_k + C_kB_j \end{bmatrix} = 2\delta_{jk} I_4$

$j \neq k : B_jC_k + B_kC_j = C_jB_k + C_kB_j = 0$

$j = k : B_jC_j = C_jB_j = I_2 \Leftrightarrow C_j = B_j^{-1}$

thus $(\gamma_i)_{i=0}^3$ is fully defined by a set $(B_i)_{i=0}^3$ of 2×2 complex matrices

$\gamma_j = \begin{bmatrix} 0 & B_j \\ B_j^{-1} & 0 \end{bmatrix}$

meeting : $j \neq k : B_jB_k^{-1} + B_kB_j^{-1} = B_j^{-1}B_k + B_k^{-1}B_j = 0$

which reads :

$B_jB_k^{-1} = -\left(B_jB_k^{-1}\right)^{-1} \Leftrightarrow \left(B_jB_k^{-1}\right)^2 = -I_2$

$B_j^{-1}B_k = -\left(B_j^{-1}B_k\right)^{-1} \Leftrightarrow \left(B_j^{-1}B_k\right)^2 = -I_2$

Let us define : $k = 1,2,3 : M_k = -iB_kB_0^{-1}$

The matrices $(M_k)_{k=1}^3$ are such that :

$M_k^2 = -\left(B_jB_0^{-1}\right)^2 = -I_2$

$M_jM_k + M_kM_j = -B_jB_0^{-1}B_kB_0^{-1} - B_kB_0^{-1}B_jB_0^{-1}$

$= -\left(-B_jB_k^{-1}B_0 - B_kB_j^{-1}B_0\right)B_0^{-1}$

$= B_jB_k^{-1} + B_kB_j^{-1} = 0$

that is $k=1,2,3: M_jM_k+M_kM_j=2\delta_{jk}I_2$
Moreover : $\gamma(\omega)=\gamma_0\gamma_1\gamma_2\gamma_3 \Rightarrow$
$B_0B_1^{-1}B_2B_3^{-1}=I_2$
$B_0^{-1}B_1B_2^{-1}B_3=-I_2$
with $B_k=iM_kB_0, B_k^{-1}=-iB_0^{-1}M_k^{-1}$
$B_0\left(-iB_0^{-1}M_1^{-1}\right)(iM_2B_0)\left(-iB_0^{-1}M_3^{-1}\right)=I_2=-iM_1^{-1}M_2M_3^{-1}$
$B_0^{-1}(iM_1B_0)\left(-iB_0^{-1}M_2^{-1}\right)(iM_3B_0)=-I_2=iB_0^{-1}M_1M_2^{-1}M_3B_0$
which reads :
$iM_2=-M_1M_3=M_3M_1$
$-M_1^{-1}M_3^{-1}=iM_2^{-1}\Leftrightarrow iM_2=M_3M_1$
$M_2M_3+M_3M_2=0=iM_1M_3M_3+M_3M_2\Leftrightarrow iM_1=-M_3M_2=M_2M_3$
$M_1M_2+M_2M_1=0=iM_3M_2M_2+M_2M_1\Rightarrow iM_3=-M_2M_1=M_1M_2$
The set of 3 matrices $(M_k)_{k=1}^3$ has the multiplication table :
$$\begin{bmatrix} 1\backslash 2 & M_1 & M_2 & M_3 \\ M_1 & I & iM_3 & -iM_2 \\ M_2 & -iM_3 & I & iM_1 \\ M_3 & iM_2 & -iM_1 & I \end{bmatrix}$$
which is the same as the set of Pauli's matrices :

$$\sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix};\sigma_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix};\sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix};\sigma_0=\begin{bmatrix}1&0\\0&1\end{bmatrix} \tag{48}$$

There is still some freedom in the choice of the γ_i matrices by the choice of B_0 and the simplest is : $B_0=-iI_2\Rightarrow B_k=\sigma_k$
Moreover, because scalars belong to Clifford algebras, one must have the identity matrix I_4 and $\gamma(z)=zI_4$
Thus :

$$\gamma_0=\begin{bmatrix}0&-i\sigma_0\\i\sigma_0&0\end{bmatrix};\gamma_1=\begin{bmatrix}0&\sigma_1\\\sigma_1&0\end{bmatrix};\gamma_2=\begin{bmatrix}0&\sigma_2\\\sigma_2&0\end{bmatrix};\gamma_3=\begin{bmatrix}0&\sigma_3\\\sigma_3&0\end{bmatrix}; \tag{49}$$

The matrices γ_j are then unitary and Hermitian :

$$\gamma_j=\gamma_j^*=\gamma_j^{-1} \tag{50}$$

which is extremely convenient.
We will use the following :

Notation 30 $j=1,2,3:\widetilde{\gamma}_j=\begin{bmatrix}\sigma_j&0\\0&\sigma_j\end{bmatrix}$

$j\neq k,l=1,2,3:\gamma_j\gamma_k=-\gamma_k\gamma_j=i\epsilon(j,k,l)\widetilde{\gamma}_l$
$j=1,2,3:\gamma_j\gamma_0=-\gamma_0\gamma_j=i\begin{bmatrix}\sigma_j&0\\0&-\sigma_j\end{bmatrix}=i\gamma_5\widetilde{\gamma}_j$

12.1.5 Representation of the real Clifford Algebras

Notice that the choice of the matrices is done in $Cl(\mathbb{C},4)$, so it is independent of the signature. We get the representations of the real algebras by the matrices $\gamma C(\varepsilon_j)$ and $\gamma C'(\varepsilon_j)$

$$\begin{array}{l} Cl(3,1):\gamma C(\varepsilon_j)=\gamma_j, j=1,2,3; \gamma C(\varepsilon_0)=i\gamma_0; \gamma C(\varepsilon_5)=i\gamma_5 \\ Cl(1,3):\gamma C'(\varepsilon_j)=i\gamma_j, j=1,2,3; \gamma C'(\varepsilon_0)=\gamma_0; \gamma C'(\varepsilon_5)=\gamma_5 \end{array} \tag{51}$$

However, because C is a real, and not a complex map : $\gamma C(\lambda X)\neq\lambda\gamma C(X)$ if $\lambda\in\mathbb{C}$.
The representation that we have chosen here is not unique and others, equivalent, would hold. However the defining relations are rather strong and the choices which give manageable matrices are limited. In the Standard Model the representation of $Cl(1,3)$ is by the matrices : $\widetilde{\gamma}_0=i\gamma_0, \widetilde{\gamma}_j=\gamma_j, j=1,2,3$ and $\widetilde{\gamma}_5=-i\widetilde{\gamma}_0\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3$.

Invariant vector subspaces

(E,γ) is a faithful, and thus irreducible, representation of $Cl(4,\mathbb{C})$, and because $C(Cl(3,1)), C'(Cl(1,3))$ are real subalgebras of $Cl(4,\mathbb{C})$, the set of vectors of E which are invariant by γC is the set invariant by $\gamma C(\varepsilon_j), j=0..3$ and similarly with $\gamma C'$.

Let be the vector subspaces :

$E_\epsilon = \left\{ \begin{bmatrix} S_R \\ S_L \end{bmatrix} \in E : S_L = \epsilon i S_R = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^2 \right\}, \epsilon = \pm 1$

then :

with $Cl(3,1)$

$i\gamma_0 \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} i\epsilon S_R \\ -S_R \end{bmatrix} = \begin{bmatrix} i\epsilon S_R \\ \epsilon i (i\epsilon S_R) \end{bmatrix}$

$\gamma_j \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} i\epsilon\sigma_j S_R \\ \sigma_j S_R \end{bmatrix} = \begin{bmatrix} i\epsilon S_R \sigma_j \\ -i\epsilon (i\epsilon\sigma_j S_R) \end{bmatrix}$

$S_0 \in E_\epsilon \Rightarrow \gamma_0 C(\varepsilon_0) S_0 \in E_\epsilon, j=1,2,3 : \gamma C(\varepsilon_j) S_0 \in E_{-\epsilon}$

with $Cl(1,3)$

$\gamma_0 \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} \epsilon S_R \\ iS_R \end{bmatrix} = \begin{bmatrix} \epsilon S_R \\ i\epsilon (\epsilon S_R) \end{bmatrix}$

$i\gamma_j \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} 0 & i\sigma_j \\ i\sigma_j & 0 \end{bmatrix} \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} -\epsilon\sigma_j S_R \\ i\sigma_j S_R \end{bmatrix} = \begin{bmatrix} -\epsilon\sigma_j S_R \\ -\epsilon i (-\epsilon\sigma_j S_R) \end{bmatrix}$

$S_0 \in E_\epsilon \Rightarrow \gamma_0 C'(\varepsilon_0) S_0 \in E_\epsilon, j=1,2,3 : \gamma C'(\varepsilon_j) S_0 \in E_{-\epsilon}$

So the set $E_0 = E_+ \cup E_-$ is globally invariant by both $Cl(3,1), Cl(1,3)$. It is not a vector space.

12.1.6 Expression of the matrices

Complex notation with the Dirac's matrices

With complex vector spaces the following notation is very convenient.

Define, for any $z \in \mathbb{C}^3$:

Notation 31 $\sum_{a=1}^{3} z_a \sigma_a = \sigma(z)$ *with* $z \in \mathbb{C}^3$

$\sigma(z) = \begin{bmatrix} z_3 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 \end{bmatrix} \in sl(\mathbb{C}, 2)$

Then we have the identities :

$(\sigma(z))^* = \sigma(\overline{z})$

$\sigma(z)\sigma(z') = \sigma(j(z)z') + z^t z' \sigma_0$

$\sigma(z)\sigma(z') - \sigma(z')\sigma(z) = \sigma(j(z)z') - \sigma(j(z')z) = 2\sigma(j(z)z')$

$\sigma(z')\sigma(z)\sigma(z') = \left((z')^t z'\right)\sigma(z)$

$\sigma(z) = k\sigma_0, k \in \mathbb{C} \Rightarrow z, k = 0$

$\det \sigma(z) = -z^t z$

$\sigma(z)^{-1} = \frac{1}{z^t z}\sigma(z)$

Representations of the elements of the Lie algebras

In Cl(3,1) :

$$\gamma C(\upsilon(r,w)) = -\frac{1}{2}i \begin{bmatrix} \sigma(r+iw) & 0 \\ 0 & \sigma(r-iw) \end{bmatrix} = -\frac{1}{2}i \begin{bmatrix} \sigma(Z) & 0 \\ 0 & \sigma(\overline{Z}) \end{bmatrix} \tag{52}$$

In Cl(1,3) :

$$\gamma C'(\upsilon(r,w)) = \frac{1}{2}i \begin{bmatrix} \sigma(r-iw) & 0 \\ 0 & \sigma(r+iw) \end{bmatrix} \tag{53}$$

Representations of the elements of the Spin group

$\gamma C(a + \upsilon(r,w) + b\varepsilon_5) = aI_4 + \gamma C(\upsilon(r,w)) + b\gamma_5$

In Cl(3,1) :

$\gamma C(a + \upsilon(r,w) + b\varepsilon_5) = \begin{bmatrix} a + ib - \frac{1}{2}i\sigma(r+iw) & 0 \\ 0 & a - ib - \frac{1}{2}i\sigma(r-iw) \end{bmatrix} = \begin{bmatrix} A - \frac{1}{2}i\sigma(Z) & 0 \\ 0 & \overline{A} - \frac{1}{2}i\sigma(\overline{Z}) \end{bmatrix}$

In Cl(1,3) :

$$\gamma C'\left(a+\upsilon\left(r,w\right)+b\varepsilon_5\right)=\begin{bmatrix} a-ib+\frac{1}{2}i\sigma\left(r-iw\right) & 0 \\ 0 & a+ib+\frac{1}{2}i\sigma\left(r+iw\right)\end{bmatrix}=\begin{bmatrix} \overline{A}-\frac{1}{2}i\sigma\left(\overline{Z}\right) & 0 \\ 0 & A-\frac{1}{2}i\sigma\left(Z\right)\end{bmatrix}$$

12.2 Scalar product of Spinors

We need a scalar product on E, preserved by a gauge transformation, that is by $Spin(3,1), Spin(1,3)$.

Theorem 32 *The only scalar products on E, preserved by* $\{\gamma C\left(\sigma\right), \sigma \in Spin(3,1)\}$ *are* $G=\begin{bmatrix} 0 & k\sigma_0 \\ \overline{k}\sigma_0 & 0\end{bmatrix}$ *with* $k \in \mathbb{C}$

Proof. It is represented in the basis of E by a 4×4 Hermitian matrix G such that : $G=G^*$
$\forall s \in Spin(3,1) : \left[\gamma C\left(s\right)\right]^* G\left[\gamma C\left(s\right)\right]=G$
or $\forall s \in Spin(1,3) : \left[\gamma C'\left(s\right)\right]^* G\left[\gamma C'\left(s\right)\right]=G$
$\left[\gamma C\left(s\right)\right]^*\left[G\right]=\left[G\right]\left[\gamma C\left(s\right)\right]^{-1}=\left[G\right]\left[\gamma C\left(s^{-1}\right)\right]$
$\left[\gamma C\left(s\right)\right]=\gamma C\left(A+Z\right)=\begin{bmatrix} A\sigma_0-\frac{1}{2}i\sigma\left(Z\right) & 0 \\ 0 & \overline{A}\sigma_0-\frac{1}{2}i\sigma\left(\overline{Z}\right)\end{bmatrix}$
$\left[\gamma C\left(s\right)\right]^*=\begin{bmatrix} \overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right) & 0 \\ 0 & A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\end{bmatrix}$
$\left[\gamma C\left(s^{-1}\right)\right]=\gamma C\left(A-Z\right)=\begin{bmatrix} A\sigma_0+\frac{1}{2}i\sigma\left(Z\right) & 0 \\ 0 & \overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\end{bmatrix}$
$G=\begin{bmatrix} M & P \\ P^* & N\end{bmatrix}$, with $M=M^*, N=N^*$
$\left[G\right]\left[\gamma C\left(s\right)\right]^{-1}=\left[G\right]\left[\gamma C\left(s^{-1}\right)\right] \Leftrightarrow$
$\begin{bmatrix} \left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)M & \left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)P \\ \left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)P^* & \left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)N\end{bmatrix}$
$=\begin{bmatrix} M\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right) & P\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right) \\ P^*\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right) & N\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)\end{bmatrix}$
We must have the identities, $\forall Z$:
$\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)M=M\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)$
$\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)P=P\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)$
$\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)P^*=P^*\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)$
$\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)N=N\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)$
Let us consider first $s \in Spin\left(3\right) \Leftrightarrow A, Z \in \mathbb{R}$
The conditions read
$\sigma\left(Z\right)M=M\sigma\left(Z\right)$
$\sigma\left(Z\right)P=P\sigma\left(Z\right)$
$\sigma\left(Z\right)P^*=P^*\sigma\left(Z\right)$
$\sigma\left(Z\right)N=N\sigma\left(Z\right)$
The only matrices which commute with all Dirac matrices are scalar, thus :
$M=m\sigma_0, N=n\sigma_0, P=p\sigma_0$
$G=\begin{bmatrix} m\sigma_0 & p\sigma_0 \\ \overline{p}\sigma_0 & n\sigma_0\end{bmatrix}$, with $m,n \in \mathbb{R}$
Then for $s \in Spin\left(3,1\right)$ the conditions become :
$\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)m=m\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)$
$\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)p=p\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)$
$\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)\overline{p}=\overline{p}\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)$
$\left(A\sigma_0+\frac{1}{2}i\sigma\left(Z\right)\right)n=n\left(\overline{A}\sigma_0+\frac{1}{2}i\sigma\left(\overline{Z}\right)\right)$
$\overline{A}m=mA \Rightarrow m=0$
$An=n\overline{A} \Rightarrow n=0$
The only solution is :
$G=\begin{bmatrix} 0 & k\sigma_0 \\ \overline{k}\sigma_0 & 0\end{bmatrix}$ ■

The scalar product will never be definite positive, so we can take $k=-i$ that is $G=\gamma_0$. And it is easy to check that it works also for the signature (1,3).

Any vector of E reads :
$S=\sum_{i=1}^{4} S^i e_i=\begin{bmatrix} S_R \\ S_L\end{bmatrix}$ with 2 vectors $S_R, S_L \in \mathbb{C}^2$

The scalar product of two vectors S, S' of E is then:

$$\langle S, S' \rangle_E = [S]^* [\gamma_0] [S'] = i \left([S_L]^* [S'_R] - [S_R]^* [S'_L] \right) \tag{54}$$

It is not definite positive but :
$[S_L]^* [S_R] = \left([S_L]^* [S_R] \right)^t = [S_R]^t \overline{[S_L]} = \overline{\left([S_R]^* [S_L] \right)}$
$\Rightarrow \langle S, S \rangle_E = i \left([S_L]^* [S_R] - [S_R]^* [S_L] \right) = i \left([S_L]^* [S_R] - \overline{[S_L]^* [S_R]} \right) = -2 \operatorname{Im} \left([S_L]^* [S_R] \right)$

$$\langle S, S \rangle_E = -2 \operatorname{Im} \left([S_L]^* [S_R] \right) \tag{55}$$

And if $S \in E_\epsilon : S_L = \epsilon i S_R : \langle S, S \rangle_E = -2 \operatorname{Im} \left(-\epsilon i [S_R]^* [S_R] \right) = 2\epsilon [S_R]^* [S_R]$ thus the scalar product is definite positive on E_+ and definite negative on E_-. These two vector spaces are Hilbert spaces.

12.2.1 Norm on the space of spinors

E_ϵ are Hilbert spaces, so normed vector spaces. More generally there is a norm on E : $\|S\| = \sqrt{[S]^* [S]}$
It has the properties :
$\|S\| \geq 0$
$\|S\| = 0 \Rightarrow S = 0$
$\|kS\| = |k| \|S\|$
$\|S + S'\| \leq \|S\| + \|S'\|$

13 THE SPINOR REPRESENTATION OF MOMENTA

13.1 The Spinor bundle

Because M is endowed with the structure of the principal bundle P_G, there is a structure of spin bundle (Maths.2110), an associated vector bundle $P_G [E, \gamma C]$ such that at each point of M, any element of $Cl(3,1)$ acts on the vectors of $P_G [E, \gamma C]$ through γC.

Definition 33 *The **Spinor bundle** is the associated vector bundle $P_G [E, \gamma C]$*

Its elements S are **spinors**. They are measured by observers in the standard gauge defined through the holonomic basis : $\mathbf{e}_i (m) = (\mathbf{p}(m), e_i)$.

In a change of gauge the holonomic basis becomes :

$$\begin{array}{c} \mathbf{p}(m) = \varphi_G (m, 1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \\ \mathbf{e}_i (m) = (\mathbf{p}(m), e_i) \to \widetilde{\mathbf{e}}_i (m) = \gamma C \left(\chi(m)^{-1} \right) \mathbf{e}_i (m) \\ (\mathbf{p}(m), S) \sim (\widetilde{\mathbf{p}}(m), \gamma C (\chi(m)) S) \end{array} \tag{56}$$

A jet in $J^1 P_G [E, \gamma C]$ is represented by : $j^1 S = (m, S, \delta S)$ where $S, \delta S \in E$ and change as in $P_G [E, \gamma C]$.

The scalar product on E is preserved by γC thus it can be extended to $P_G [E, \gamma C]$ and to the space of sections $\mathfrak{X} (P_G [E, \gamma C])$ by :
$\langle \mathbf{S}, \mathbf{S}' \rangle = \int_\Omega \langle \mathbf{S}(m), \mathbf{S}'(m) \rangle_E \varpi_4 (m)$

13.2 Definition of the Momenta

13.2.1 Definition

Proposition 34 *The momenta of a particle are represented in the first jet extension $J^1 P_G [E, \gamma C]$.*
Along any trajectory by a map $j^1 S : \mathbb{R} \to J^1 P_G [E, \gamma C] :: j^1 S(t) = (q(t), S(t), \delta S(t))$
$S(t), \delta S(t) \in P_G [E, \gamma C]$ are located at $q(t)$.
In a continuous motion $j^1 S$ is a the first jet prolongation of a map :
$S : \mathbb{R} \to J^1 P_G [E, \gamma C] :: \left(q(t), S(t), \frac{dS}{dt}(t) \right)$

Momenta and motion are two distinct concepts. The maps :
$j^1 \sigma : \mathbb{R} \to J^1 P_G :: (q(t), \sigma(t), \upsilon (X_r, X_w))$
$j^1 S : \mathbb{R} \to J^1 P_G [E, \gamma C] :: (q(t), S(t), \delta S(t))$
are a priori distinct. The main physical assumption is that there is a relation between the motion and the momentum. In the usual representations the relation is given, for the translational momentum by a scalar, the mass, and for the rotational momentum by a matrix, the inertial tensor. Because we assume that to any particle is associated an orthonormal basis, the momentum requires more than a scalar.

For any observer $\mathbf{p}(q(t)) = \varphi_G(q(t), 1)$ the motion of the body is along the trajectory : $\left(q(t), \sigma(t), \frac{d\sigma}{dt} \cdot \sigma^{-1}\right)$.

The state of the particle is : $(\mathbf{p}(q(t)), S(t))$ and we assume that $\exists S_0 \in E : S(t) = \gamma C(\sigma(t)) S_0$

In a continuous motion, the observer measures the change, through inertial forces :

$\frac{d}{dt}S(t) = \gamma C\left(\frac{d}{dt}\sigma(t)\right) S_0 = \gamma C\left(\frac{d}{dt}\sigma(t) \cdot \sigma(t)^{-1}\right) \gamma C(\sigma(t)) S_0 = \gamma C\left(\frac{d}{dt}\sigma(t) \cdot \sigma(t)^{-1}\right) S(t)$

And we generalize as : for a, not necessarily continuous, motion $(q(t), \sigma(t), \upsilon(X_r, X_w))$ the momenta follow :

$(q(t), S(t) = \gamma C(\sigma(t)) S_0, \delta S(t) = \gamma C(\upsilon(X_r, X_w)) S(t))$

Proposition 35 *For any particle there is a fixed differential operator $\mathcal{M}$ which relates the motion and the momenta :*

$$\begin{array}{c} \mathcal{M} : J^1 Cl(TM) \to J^1 P_G[E, \gamma C] :: \\ \mathcal{M}(q(t), \sigma(t), \upsilon(X_r, X_w)) = (q, S = \gamma C(\sigma) S_0, \delta S = \gamma C(\upsilon(X_r, X_w)) S) \end{array} \tag{57}$$

where $S_0 \in E$ is a fixed vector called the ***inertial spinor****.*

In a change of gauge :

$\mathbf{p}(m) = \varphi_G(m, 1) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \sigma \to \widetilde{\sigma} = \chi \cdot \sigma$

$\upsilon(X_r, X_w) \to \upsilon\widetilde{(X_r, X_w)} = \mathbf{Ad}_{\chi(m)} \upsilon(X_r, X_w)$

$S \to \widetilde{S} = \gamma C(\chi(m)) S$

$\delta S \to \widetilde{\delta S} = \gamma C(\chi(m)) \delta S$

$(q(t), S(t), \delta S(t)) \to \left(q(t), \widetilde{S}(t), \widetilde{\delta S}(t)\right)$

and :

$\widetilde{S} = \gamma C(\chi) S = \gamma C(\chi) \gamma C(\sigma) S_0 = \gamma C(\chi \cdot \sigma) S_0 = \gamma C(\widetilde{\sigma}) S_0$

$\widetilde{\delta S} = \gamma C(\chi) \delta S = \gamma C(\chi) \gamma C(\upsilon(X_r, X_w)) S = \gamma C(\chi) \gamma C(\upsilon(X_r, X_w)) \gamma C\left(\chi^{-1}\right) \widetilde{S}$

$= \gamma C(\mathbf{Ad}_\chi \upsilon(X_r, X_w)) \widetilde{S} = \gamma C\left(\upsilon\widetilde{(X_r, X_w)}\right) \widetilde{S}$

So S_0 *does not change* : this is an intrinsic property of the particle, which is measured by an observer through $S = \gamma C(\sigma) S_0$. And $\sigma = 1$ for an observer attached to the particle.

The spinor, which characterizes the momenta is $S = \gamma C(\sigma) S_0$.

The change of momentum, equal to the inertial forces (corresponding to the derivatives of the momenta), is $\delta S = \gamma C(\upsilon(X_r, X_w)) S$

$\delta S_R = \sum_{\alpha=0}^{3} \gamma C(\upsilon(X_r, 0)) S$ is the equivalent of a change of rotational momentum or an inertial torque.

$\delta S_T = \sum_{\alpha=0}^{3} \gamma C(\upsilon(0, X_w)) S$ is the equivalent of a change of translational momentum or a translational inertial force.

13.2.2 Forces, torques and Spinors

i) (E, γ) is a faithful (bijective) representation of $Cl(4, \mathbb{C})$ and $(E, \gamma C)$ is a faithful representation of $Cl(3, 1)$:

$\forall X, X' \in Cl(3, 1), S \in E : \gamma C(X) S = \gamma C(X') S \Leftrightarrow \gamma C(X - X') S = 0 \Leftrightarrow X = X'$

As a consequence there is no symmetries : it would imply that, for s belonging to a subgroup of $Spin(3, 1)$: $\gamma C(s) S = S \Leftrightarrow \gamma C(s) \gamma C(\sigma) S_0 = \gamma C(\sigma) S_0 \Leftrightarrow \gamma C\left(\sigma^{-1} \cdot s \cdot \sigma\right) S_0 = S_0$

But we will see that the kinematic characteristics are actually defined by a scalar (the mass) and a 3 dimensional real vector (the inertial vector), and symmetries are defined through this vector.

ii) The motion is represented in the real Clifford algebra. It is legitimate to assume that S_0 belongs to a subset which is invariant by $Cl(3, 1)$ (or similarly by $Cl(1, 3)$). So we can state :

Proposition 36 *For particles the inertial spinor S_0 belongs to the set of vectors :*

$E_0 = \left\{ \begin{bmatrix} S_R \\ S_L \end{bmatrix} \in E : S_L = \pm i S_R \right\}$

Then $\forall s \in Cl(3, 1) : \gamma C(s) S_0 \in E_0$ and idem for $Cl(1, 3)$ because the set is globally invariant.

iii) A vector of E, with 4 complex components, can represent :

either a combination of a translational and rotational momentum (S)

or a combination of force and torque (δS).

Forces and torques are measured through the change of motion of known particles.

The action of the fields is represented by a differential operator acting on $j^1 S$:

$D_F : J^1 P_G[E, \gamma C] \to J^1 P_G[E, \gamma C]$

The relation $\sigma \to S$ through S_0 is the mathematical expression of the continuity of the particle. The condition : $S(t) = \gamma C(\sigma(t)) S_0$ provides differential equations with respect to σ which give the motion. Their solutions depend on the value of S_0, which enables to estimate S_0.

The vectors e_i of the basis of E have no universal physical meaning : it depends on the system. Actually forces and torques are identified by the change of motion with which they are associated, that is by $\upsilon(X_r, X_w)$ and not to vectors of the basis ε_i as in Newtonian Mechanics : forces correspond to $\upsilon(0, X_w)$ and torques to $\upsilon(X_r, 0)$. And the identification of the axes e_i can be done, for a rigid solid, through the inertial vector as we will see.

13.3 Mass and Kinetic Energy

13.3.1 Mass

The scalar product is invariant by the action of γ, thus :

$\langle S(t), S(t)\rangle = \langle \gamma C(\sigma(t)) S_0, \gamma C(\sigma(t)) S_0\rangle = \langle S_0, S_0\rangle = -2\,\mathrm{Im}\left([S_L]^* [S_R]\right)$

By similarity with $\langle P, P\rangle = -M_p^2 c^2$ it is then natural to state that $\langle S_0, S_0\rangle$ represents the square of the mass of the particle, up to a constant depending on the units.

With the proposition above : $[S_L] = \epsilon i [S_R] \Rightarrow \langle S_0, S_0\rangle = 2\epsilon [S_R]^* [S_R]$

This quantity can be positive or negative. We will come back on this issue later and define the mass "at rest" of the particle by :

$$M_p = \sqrt{|\langle S_0, S_0\rangle|} = \sqrt{2\left|\mathrm{Im}\left([S_L]^* [S_R]\right)\right|} = \sqrt{2[S_R]^* [S_R]} \tag{58}$$

Then S_R reads :

$S_R = \frac{M_p}{\sqrt{2}} \begin{bmatrix} a \\ b \end{bmatrix}$ and $1 = \left(|a|^2 + |b|^2\right)$

It is customary to represent the polarization of the plane wave of an electric field by two complex quantities (the Jones vector) :

$E_x = E_{0x} e^{i\alpha_x}$

$E_y = E_{0y} e^{i\alpha_y}$

where (E_{0x}, E_{0y}) are the components of a vector E_0 along the axes x, y.

So we can write similarly :

$$S_R = \frac{M_p}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos\alpha_0 \\ e^{i\alpha_2} \sin\alpha_0 \end{bmatrix} \tag{59}$$

13.3.2 Kinetic Energy

$\frac{d}{dt}\langle S(t), S(t)\rangle = 0 = \left\langle \frac{d}{dt} S(t), S(t)\right\rangle + \left\langle S(t), \frac{d}{dt} S(t)\right\rangle$ thus $\left\langle S(t), \frac{d}{dt} S(t)\right\rangle$ is pure imaginary.

The variation of the kinetic energy is defined in Newtonian Mechanics as :

$\delta K = \frac{1}{m}\left\langle \overrightarrow{p}, \overrightarrow{\delta p_G}\right\rangle + [r]^t [R]^t [\delta\Gamma(G)]$

It involves both the present state of momentum and its evolution. The natural generalization is :

$\delta K = \frac{1}{M_p}\frac{1}{i}\langle S, \delta S\rangle = \frac{1}{M_p}\frac{1}{i}\langle \gamma C(\sigma) S_0, \gamma C(\upsilon(X_r, X_w)) \gamma C(\sigma) S_0\rangle = \frac{1}{M_p}\frac{1}{i}\langle S_0, \gamma C(\mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w)) S_0\rangle$

In a continuous motion along the trajectory :

$\upsilon(X_r, X_w) = \frac{d\sigma}{dt} \cdot \sigma^{-1}$

$\frac{dK}{dt} = \frac{1}{M_p}\frac{1}{i}\left\langle \gamma C(\sigma) S_0, \gamma C\left(\frac{d\sigma}{dt} \cdot \sigma^{-1}\right) \gamma C(\sigma) S_0\right\rangle = \frac{1}{M_p}\frac{1}{i}\left\langle S_0, \gamma C\left(\sigma^{-1} \cdot \frac{d\sigma}{dt}\right) S_0\right\rangle$

$$\begin{array}{c} \delta K = \frac{1}{M_p}\frac{1}{i}\langle S, \delta S\rangle = \frac{1}{M_p}\frac{1}{i}\langle S_0, \gamma C(\mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w)) S_0\rangle \\ \frac{dK}{dt} = \frac{1}{M_p}\frac{1}{i}\left\langle S_0, \gamma C\left(\sigma^{-1} \cdot \frac{d\sigma}{dt}\right) S_0\right\rangle \end{array} \tag{60}$$

The scalar product does not depend on the observer, however in a continuous motion the observer is involved in the definition of t.

13.3.3 Inertial vector

Let us denote $[S_0] = \begin{bmatrix} S_R \\ S_L \end{bmatrix}$, $Z \in T_1 Spin(3,1)$ in the complex formalism.

$\gamma C(Z)[S_0] = -\frac{i}{2}\begin{bmatrix} \sigma(Z) & 0 \\ 0 & \sigma\left(\overline{Z}\right) \end{bmatrix}\begin{bmatrix} S_R \\ S_L \end{bmatrix}$

$\langle S_0, \gamma C(Z) S_0\rangle = -\frac{i}{2}\begin{bmatrix} S_R^* & S_L^* \end{bmatrix}\begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}\begin{bmatrix} \sigma(Z) S_R \\ \sigma\left(\overline{Z}\right) S_L \end{bmatrix}$

$= \frac{1}{2}\left(-S_R^* \sigma\left(\overline{Z}\right) S_L + S_L^* \sigma(Z) S_R\right)$

$S_L^* \sigma(Z) S_R = (S_L^* \sigma(Z) S_R)^t = S_R^t [\sigma(Z)]^t \overline{S_L} = \overline{\overline{S}_R^t \overline{[\sigma(Z)]}^t S_L} = \overline{S_R^* \sigma(\overline{Z}) S_L}$
$\langle S_0, \gamma C(Z) S_0 \rangle = \frac{1}{2}\left(-\overline{S_L^* \sigma(Z) S_R} + S_L^* \sigma(Z) S_R\right) = i \operatorname{Im} S_L^* \sigma(Z) S_R$

Denote the vector : $k \in \mathbb{C}^3 : k^a = S_L^* \sigma_a S_R$ then $S_L^* \sigma(Z) S_R = \sum_{a=1}^3 Z^a S_L^* \sigma_a S_R = k^t Z$. And one can check that : $k^t k = \left([S_L]^* [S_R]\right)^2$
$\langle S_0, \gamma C(Z) S_0 \rangle = i \operatorname{Im} k^t Z$

For $a = 1,2,3$ take $\upsilon(X_r, X_w) = \overrightarrow{\kappa_a}$
$\langle S_0, \gamma C(\overrightarrow{\kappa_a}) S_0 \rangle_E = i \operatorname{Im} k^a = -\frac{1}{2} i \langle S_0, \widetilde{\gamma}_a S_0 \rangle_E$
$\operatorname{Im} k^a = -\frac{1}{2} \langle S_0, \widetilde{\gamma}_a S_0 \rangle_E$
Take $\upsilon(X_r, X_w) = \overrightarrow{\kappa_{a+3}}$
$\langle S_0, \gamma C(\overrightarrow{\kappa_{a+3}}) S_0 \rangle_E = \langle S_0, \gamma C(i\overrightarrow{\kappa_a}) S_0 \rangle_E = i \operatorname{Im} i k^a = i \operatorname{Re} k^a = \frac{1}{2} i \langle S_0, \gamma_0 \gamma_a S_0 \rangle_E$
$\operatorname{Re} k^a = \frac{1}{2} \langle S_0, \gamma_0 \gamma_a S_0 \rangle_E$
$k^a = \frac{1}{2} \langle S_0, \gamma_0 \gamma_a S_0 \rangle_E + i\left(-\frac{1}{2} \langle S_0, \widetilde{\gamma}_a S_0 \rangle_E\right) = \frac{1}{2} \langle S_0, (\gamma_0 \gamma_a - i\widetilde{\gamma}_a) S_0 \rangle_E$
$k^a = \frac{1}{2} \langle S_0, (\gamma_0 \gamma_a - i\widetilde{\gamma}_a) S_0 \rangle_E$ corresponds to the Dirac's current.

With $S_R = \frac{M_p}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos\alpha_0 \\ e^{i\alpha_2} \sin\alpha_0 \end{bmatrix}, S_L = i\epsilon S_R$:
$k^a = S_L^* \sigma_a S_R = -i\epsilon S_R^* \sigma_a S_R$
$k = -i\epsilon \frac{M_p^2}{2} \begin{bmatrix} (\sin 2\alpha_0) \cos(\alpha_1 - \alpha_2) \\ -(\sin 2\alpha_0) \sin(\alpha_1 - \alpha_2) \\ \cos 2\alpha_0 \end{bmatrix} = -i\epsilon \frac{M_p^2}{2} k_0$
with $k_0^t k_0 = 1$

Then $\delta K = \frac{1}{M_p} \frac{1}{i} \langle S_0, \gamma C(\mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w)) S_0 \rangle = \frac{1}{M_p} \operatorname{Im} k^t \mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w)$
In a continuous motion :
$\upsilon(X_r, X_w) = \frac{d\sigma}{dt} \cdot \sigma^{-1} \Leftrightarrow \sigma^{-1} \cdot \frac{d\sigma}{dt} = \mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w)$
$\frac{dK}{dt} = \frac{1}{M_p} \operatorname{Im} k^t \left(\sigma^{-1} \cdot \frac{d\sigma}{dt}\right)$
If $S_0 \in E_0 : \delta K = -\epsilon \frac{M_p}{2} k_0^t \operatorname{Im} i \mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w) = -\epsilon \frac{M_p}{2} k_0^t \operatorname{Re} \mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w)$
We sum up the results :

$$\begin{array}{c} a = 1,2,3 : k^a = S_L^* \sigma_a S_R = \frac{1}{2} i \langle S_0, (\gamma_0 \gamma_a - \widetilde{\gamma}_a) S_0 \rangle_E \\ \delta K = \frac{1}{M_p} \frac{1}{i} \langle S_0, \gamma C(\mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w)) S_0 \rangle = \frac{1}{M_p} \operatorname{Im} k^t \mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w) \\ S_0 \in E_0 : k = -i\epsilon \frac{M_p^2}{2} k_0 \\ k_0 = \begin{bmatrix} (\sin 2\alpha_0) \cos(\alpha_1 - \alpha_2) \\ -(\sin 2\alpha_0) \sin(\alpha_1 - \alpha_2) \\ \cos 2\alpha_0 \end{bmatrix}; k_0^t k_0 = 1 \\ \delta K = -\epsilon \frac{M_p}{2} k_0^t \operatorname{Re} \mathbf{Ad}_{\sigma^{-1}} \upsilon(X_r, X_w) \end{array} \tag{61}$$

The vector k, that we will call the **inertial vector**, does not depend on the motion of the particle. In a change of gauge S_0 does not change, so $k^a = S_L^* \sigma_a S_R$ does not change. k and $\langle S_0, S_0 \rangle$ characterize the kinematic features of the material body. They are defined by 7 independent parameters, as we have in Newtonian Mechanics, and 4 when $S_0 \in E_0$. Two material bodies such that $S_0' = e^{i\alpha} S_0$ with $\alpha \in \mathbb{R}$ have the same kinematic characteristics.

13.4 Momenta of Deformable Solids

13.4.1 Spinor Fields

A section of P_G can represent the motion of particles whose trajectories do not cross and have similar behavior. And a section of $P_G[E, \gamma C]$ can represent the kinematic characteristics of identical particles.

Definition 37 *A **Spinor field** is a section* $\mathbf{S} \in \mathfrak{X}\left(J^1 P_G[E, \gamma C]\right)$ *which represents the kinematics characteristics of a particle.* $S = (m, S(m), \delta_\beta S(m), \beta = 0..3)$

From a Mathematical point of view the condition is that there is a section $J^1\sigma \in \mathfrak{X}\left(J^1 P_G\right)$ and an inertial spinor S_0 such that :
$S(m) = \gamma C(\sigma(m)) S_0, \delta_\alpha S(m) = \gamma C(\upsilon(X_{r\alpha}(m), X_{w\alpha}(m))) S(m)$. A necessary condition is that : $\langle S(m), S(m) \rangle_E = Ct.$

From a Physical point of view such a section represents particles which have the same kinematics characteristics and whose trajectories do not cross. As a consequence the motion is continuous and $\upsilon\left(X_{r\alpha}\left(m\right),X_{w\alpha}\left(m\right)\right)=\partial_{\alpha}\sigma\cdot\sigma^{-1}$.

Conversely a vector $S_0\in E$ and a section $J^1\sigma\in\mathfrak{X}\left(J^1P_G\right)$ defines a spinor field.

13.4.2 Density

With a population of similar particles represented by a spinor field it is natural to consider a density of particles, that is a function $\mu:M\to\mathbb{R}$ such that $\mu\left(m\right)$ represents the number of identical particles located at the same point. Then for any observer the conservation of the number of particles implies that :

$\mathcal{N}\left(t\right)=\int_{\Omega(t)}\mu_3\left(t,x\right)\varpi_3=Ct$

which can be written :

$\int_{\Omega(t)}i_V\left(\mu\varpi_4\right)=Ct$

where V is the vector field representing the trajectories, as it is deduced from $\sigma:V=-\frac{c}{\langle\mathbf{Ad}_{\sigma}\varepsilon_0,\varepsilon_0\rangle_{Cl}}\mathbf{Ad}_{\sigma}\varepsilon_0$.

Consider the manifold $\Omega\left([t_1,t_2]\right)$ with borders $\Omega\left(t_1\right),\Omega\left(t_2\right)$:

$\mathcal{N}\left(t_2\right)-\mathcal{N}\left(t_1\right)=\int_{\partial\Omega([t_1,t_2])}i_V\left(\mu\varpi_4\right)=\int_{\Omega([t_1,t_2])}d\left(i_V\mu\varpi_4\right)$

$d\left(i_V\mu\varpi_4\right)=\pounds_V\left(\mu\varpi_4\right)-i_Vd\left(\mu\varpi_4\right)=\pounds_V\left(\mu\varpi_4\right)-i_V\left(d\mu\wedge\varpi_4\right)-i_V\mu d\varpi_4=\pounds_V\left(\mu\varpi_4\right)$

$\mathcal{N}\left(t_2\right)-\mathcal{N}\left(t_1\right)=\int_{\omega([t_1,t_2])}\pounds_V\left(\mu\varpi_4\right)$

with the Lie derivative $\pounds$. The conservation of the number of particles is equivalent to the condition $\pounds_V\left(\mu\varpi_4\right)=0$.

$\pounds_V\mu\varpi_4$

$=\frac{d\mu}{dt}\varpi_4+\mu\pounds_V\varpi_4$

$=\frac{d\mu}{dt}\varpi_4+\mu\left(divV\right)\varpi_4$

$=\left(\frac{d\mu}{dt}+\mu\left(divV\right)\right)\varpi_4$

and we retrieve the usual **continuity equation** :

$$\frac{d\mu}{dt}+\mu divV=0 \tag{62}$$

Without a density we should have also the conservation of $\langle S_0,S_0\rangle$ in any spinor field. The previous demonstration can be done with $\langle S_0,S_0\rangle=\mu$ and leads to $divV=0$. So actually to have a physical meaning it is necessary to add a density to a spinor field.

Let us define :

$T:TM\otimes J^1P_G\left[E,\gamma C\right]\to\mathbb{R}::$

$T\left(\sum_{\alpha}U^{\alpha}\partial\xi_{\alpha}\right)=\frac{1}{i}\sum_{\alpha,\beta=0}^{3}\mu\left\langle S,U^{\alpha}\upsilon\left(X_{r\beta},X_{w\beta}\right)\right\rangle$

$=-\frac{1}{2}\mu\epsilon k_0^t\,\mathrm{Im}\left(\sum_{\alpha,\beta=0}^{3}\frac{U^{\alpha}}{c}\left(\sigma^{-1}\cdot\partial_{\beta}\sigma\right)\right)$

T is a tensor : its action is linear, and the result does not depend on the chart or the gauge. It gives the resistance of the particle to change its motion by $\sigma^{-1}\cdot\partial_{\beta}\sigma$ in the direction U^{α}.This is the energy-momentum tensor of the Spinor field.

The trace $Tr\left(T\right)$ of the tensor T is the tensor :

$Tr\left(T\right)\left(\sum_{\alpha}U^{\alpha}\partial\xi_{\alpha}\right)=\frac{1}{i}\sum_{\alpha=0}^{3}\mu\left\langle S,U^{\alpha}\upsilon\left(X_{r\alpha},X_{w\alpha}\right)\right\rangle$

that is the kinetic energy (up to a constant).

Take $\upsilon\left(X_{r\alpha},X_{w\alpha}\right)=\upsilon\left(0,\delta_{\alpha}w\right)$

$Tr\left(T\right)\left(\sum_{\alpha}U^{\alpha}\partial\xi_{\alpha}\right)=-\frac{1}{2}\mu\epsilon k_0^t\sum_{\alpha=0}^{3}\frac{U^{\alpha}}{c}\delta_{\alpha}w$

can be seen as the pressure of the flow of matter in the spatial direction $\delta_{\alpha}w$.

13.4.3 Spinor field for a deformable solid

One can define, for any observer, a deformable solid by a section $\sigma\in P_G$. The particles travel on trajectories V defined by σ_w. Adding a density μ, and an inertial spinor S_0, then, because S is valued in the vector space E, the integral : $\int_{\omega(t)}\mu\left(m\right)S\left(m\right)\varpi_3\left(t,m\right)$ where $\omega\left(t\right)=\Phi_V\left(\omega,t\right)$ and ω is a compact subset of $\Omega\left(0\right)$, makes sense.

$S\left(t\right)=\gamma C\left(\int_{\omega(t)}\sigma\left(m\right)\mu\left(m\right)\varpi_3\left(m\right)\right)S_0$

$\Gamma=\int_{\omega(t)}\sigma\left(m\right)\mu\left(m\right)\varpi_3\left(m\right)\in Cl\left(3,1\right)$

We have several cases of interest.

If the solid is rigid : $\sigma\left(\Phi_V\left(t,x\right)\right)=s\left(t\right)\cdot g\left(\Phi_V\left(0,x\right)\right)$ with $s\left(t\right)\in Spin\left(3,1\right)$. Then

$\int_{\omega(t)}\sigma\left(m\right)\mu\left(m\right)\varpi_3\left(m\right)=s\left(t\right)\int_{x\in\omega}g\left(x\right)\mu\left(\left(\Phi_V\left(t,x\right)\right)\right)\varpi_3\left(\Phi_V\left(t,x\right)\right)$

and $S\left(t\right)=\gamma C\left(s\left(t\right)\right)S_B\left(t\right)$ with $S_B\left(t\right)=\gamma C\left(\int_{x\in\omega}g\left(x\right)\mu\left(\left(\Phi_V\left(t,x\right)\right)\right)\varpi_3\left(\Phi_V\left(t,x\right)\right)\right)S_0$.

The variation of $S_B(t)$ can be computed as above :

$S_B(t_2) - S_B(t_1) = \int_{\omega([t_1,t_2])} \gamma C(\pounds_V(g\mu\varpi_4)) S_0 = \int_{\omega([t_1,t_2])} \gamma C\left(\frac{dg\mu}{dt} + g\mu(divV)\right)\varpi_4 S_0$

$= \int_{\omega([t_1,t_2])} \gamma C\left(g\left(\frac{d\mu}{dt} + \mu(divV)\right)\right)\varpi_4 S_0$

With the continuity equation : $S_B(t) = Ct$ and $S(t) = \gamma C(s(t)) S_B$.

The solid can be replaced by a particle moving along one integral curve of the vector field V with spinor $S(t) = \gamma C(s(t)) S_B$. This is the generalization of the rule of Newtonian Mechanics.

$S_B = S_B(0) = \gamma C\left(\int_{x\in\omega} g(x)\mu(x)\varpi_3(x)\right) S_0$

The computation of the integral $\Gamma = \int_{x\in\omega} g(x)\mu(x)\varpi_3(x) \in Cl(3,1)$ can be done in any chart, adjusted for the symmetries of the solid. And if $S_0 \in E_0$ then $S_B \in E_0$. However Γ does not necessarily belong to $Spin(3,1)$.

In the general case the deformation tensor is $\partial_\alpha\sigma \cdot \sigma^{-1}$. This is a 1 form on M valued in $T_1 Spin(3,1)$.

The stress tensor is then : $\gamma C(\partial_\alpha\sigma) S_0 \otimes d\xi^\alpha = \gamma C\left(\partial_\alpha\sigma \cdot \sigma^{-1}\right)\gamma C(\sigma) S_0 \otimes d\xi^\alpha$. This is a 1 form on M valued in E. On a trajectory $\delta U = \sum_{\alpha=0}^3 \delta U^\alpha \partial\xi_\alpha$ the inertial forces, similar to stress forces, which preserve the integrity of the solid are :

$\delta F = \sum_{\alpha=0}^3 \delta U^\alpha \delta_\alpha S \in E$.

We still have $S(t) = \gamma C(\Gamma(t)) S_0 \in E_0$ if $S_B \in E_0$.

13.4.4 Symmetries

Symmetries have a meaning only for rigid solids. As in Newtonian Mechanics they are kinematic symmetries, related to the momentum of the material body. For a rigid solid : $S_B(t) = Ct$ and $S(t) = \gamma C(s(t)) S_B$ so that there is an inertial vector defined by $S_B \in E_0$:

$k^a_{0B} = \frac{1}{i}\epsilon\frac{1}{M_B^2}\langle S_B, (\gamma_0\gamma_a - i\widetilde{\gamma}_a) S_B\rangle_E$ with $k^t_{0B} k_{0B} = 1$

The Dirac's current $(\gamma_0\gamma_a - i\widetilde{\gamma}_a) S_B$ can be identified with the flow of matter in the 3 spatial directions corresponding to $\gamma_a = \gamma(\varepsilon_a)$.

For any rigid solid in Newtonian Mechanics there is an inertial tensor, represented by a symmetric matrix $[J]$ with 3 orthogonal eigen vectors and real eigen values λ_a. So we can say that they correspond to the 3 vectors ε_a and $k^a_{0B} = \frac{1}{\sqrt{\sum_{a=1}^3 \lambda_a^2}}\lambda_a$.

The symmetries are, as in Newtonian Mechanics, related to the eigen vectors of $[J]$.

In all practical applications this is the vector k_0 which is involved, the basis $(e_i)_{i=1}^4$ and the inertial spinor S_0 are only used to identify the forces and torques, and this is done in conventional bases depending on the problem, as required (that is in relation with physical measures).

Classic Mechanics provides efficient and simpler tools, and the use of spinors would be just pedantic in common problems. However this approach can be used at any scale. It can be used to study the deformation of nuclei, atoms or molecules. At the other end it can be useful in Astrophysics, where trajectories of stars systems or galaxies are studied. The spinor can account for the rotational momentum of the bodies, which is significant and contributes to the total kinetic energy of the system.

However Spinors have been introduced for elementary particles in the frame-work of Quantum Mechanics, and we need to tell how they can be quantized.

14 SPINORS OF ELEMENTARY PARTICLES

14.1 Quantization of spinors

Quantum Mechanics proper can be summed up in a collections of axioms about the representation of a system, in which states are vectors of a Hilbert spaces, observables are operators. I will not come back on the meaning of these axioms, the reader can find a comprehensive presentation of the theory in my book. Let us just say that these axioms are actually mathematical theorems, which can be proved for any system represented by variables which meet some precise (but quite general) conditions.

The vector space E is normed, and E_ϵ are Hilbert spaces. In a model involving a particle the spinor is represented by a map : $J^1 S : [0,T] \subset \mathbb{R} \to J^1 P_G[E, \gamma C]$ for some $S_0 \in E_0$. The state of the system is actually this map (and not only its value at each t). It is assumed to be such that $\int_0^T \max(\|S(t)\|, \|\delta S(t)\|) dt < \infty$ then it belongs to a separable, infinite dimensional, separable, Fréchet space F and the theorems of QM apply. The space F is isomorphic to an open subset of a Hilbert space H, and to each *map* $J^1 S$ is associated a vector of this Hilbert space.

The vector space F is invariant under the global action :

$\lambda : Spin(3,1) \times F \to F :: \lambda(g,S)(t) = \gamma C(g) S(t)$

so (F,λ) is a representation of the group $Spin(3,1)$. One can say that S and $\lambda(g,S)$ represent the same state of the system. The action preserves the scalar product, so H is a unitary representation of $Spin(3,1)$ and by restriction of $SO(3,1)$. The only irreducible unitary representations of $SO(3,1)$ are infinite dimensional and parametrized by a scalar, which we can assume is the mass, and a signed integer.

The rotational momentum $\gamma C(\upsilon(X_r(t),0))S_0$ belongs to a vector space invariant by the action of $Spin(3)$, which is a compact group. Its observables belong to finite dimensional representations, parametrized by a half integer $j \in \frac{1}{2}\mathbb{N}$.

14.2 Periodic states

We have seen that a periodic motion can be represented by a map :

$\sigma : \mathbb{R} \to Spin(3,1) :: \sigma(t) = A(t) + Z(t)$ where $Z(t+T) = Z(t)$ for some fixed period

with :

$Z(t) = \sum_{n\in\mathbb{Z}} \widehat{Z}(n) \exp in\omega t$ with $\widehat{Z}(n) = \frac{1}{T}\int_0^T Z(t)\exp(-in\omega t)\,dt$ and $\omega = \frac{2\pi}{T}$

$Z(0) = \sum_{n\in\mathbb{Z}} \widehat{S}(n)$

$A(t) = \sum_{n\in\mathbb{Z}} \widehat{A}(n) \exp in\omega t$ with $\widehat{A}(n) = \frac{1}{T}\int_0^T A(t)\exp(-in\omega t)\,dt$

$A(t)^2 = 1 - \frac{1}{4}Z(t)^t Z(t)$

The spinor is then :

$S(t) = \gamma C(\sigma(t))S_0 = \sum_{n\in\mathbb{Z}} \widehat{S}(n)\exp in\omega t$ with $\widehat{S}(n) = \frac{1}{T}\int_0^T S(t)\exp(-in\omega t)\,dt$

$\widehat{S}(n) = \gamma C\left(\widehat{A}(n) + \widehat{Z}(n)\right)S_0$

By derivation :

$\frac{dS}{dt} = \sum_{n\in\mathbb{Z}} in\omega \widehat{S}(n)\exp in\omega t$

we have necessarily the relation :

$\widehat{\frac{dS}{dt}}(n) = in\omega\widehat{S}(n)$

and $\frac{dS}{dt}|_{t=0} = \sum_{n\in\mathbb{Z}} in\omega\widehat{S}(n)$

The average energy on the trajectory is : $\frac{1}{M_p}\frac{1}{T}\int_0^T \frac{1}{i}\left\langle S(t), \frac{d}{dt}S(t)\right\rangle dt$

The variables belong to a Hilbert space H with scalar product :

$\langle Y_1, Y_2\rangle_H = \frac{1}{T}\int_0^T \langle Y_1(t), Y_2(t)\rangle_E\,dt = \sum_{n\in Z}\left\langle \widehat{Y_1}(n), \widehat{Y_2}(n)\right\rangle_E$

Thus :

$\frac{1}{T}\int_0^T \frac{1}{i}\left\langle S(t), \frac{d}{dt}S(t)\right\rangle dt = \sum_{n\in Z}\left\langle \widehat{S}(n), \widehat{\frac{dS}{dt}}(n)\right\rangle = \sum_{n\in Z} in\omega\left\langle \widehat{S}(n), \widehat{S}(n)\right\rangle$

$\left\langle \widehat{S}(n), \widehat{S}(n)\right\rangle$ can be computed with : $S_0 = \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix}$ and one gets :

$\left\langle \widehat{S}(n), \widehat{S}(n)\right\rangle = M_p^2\left(\left(\mathrm{Re}\,\widehat{A}(n)\right)^2 - \left(\mathrm{Im}\,\widehat{A}(n)\right)^2 + \frac{1}{4}\left(\left(\mathrm{Re}\,\widehat{Z}(n)^t\widehat{Z}(n)\right)^2 - \left(\mathrm{Im}\,\widehat{Z}(n)^t\widehat{Z}(n)\right)^2\right)\right)$

$\frac{dS}{dt}|_{t=0} = \sum_{n\in\mathbb{Z}} in\omega\widehat{S}(n) \Rightarrow \sum_{n\in Z}\left\langle in\omega\widehat{S}(n), in\omega\widehat{S}(n)\right\rangle = \omega^2\sum_{n\in Z} n^2\left\langle \widehat{S}(n), \widehat{S}(n)\right\rangle < \infty$

$\Rightarrow \sum_{n\in Z} n\left\langle \widehat{S}(n), \widehat{S}(n)\right\rangle S_R^* S_R < \infty$

$\frac{1}{M_p}\frac{1}{T}\int_0^T \frac{1}{i}\left\langle S(t), \frac{d}{dt}S(t)\right\rangle dt$

$= \omega M_p \sum_{n\in Z} n\left(\left(\mathrm{Re}\,\widehat{A}(n)\right)^2 - \left(\mathrm{Im}\,\widehat{A}(n)\right)^2 + \frac{1}{4}\left(\left(\mathrm{Re}\,\widehat{Z}(n)^t\widehat{Z}(n)\right)^2 - \left(\mathrm{Im}\,\widehat{Z}(n)^t\widehat{Z}(n)\right)^2\right)\right)$

The average kinetic energy is proportional to the frequency.

14.3 Spinors for elementary particles

14.3.1 Particles and Anti-particles

The inertial spinor is a starting point in the identification of "elementary particles", that is the ultimate constituent of matter.

The first natural requisite is that $S_0 \in E_0$. The value of ϵ is related to a choice of a basis of E_ϵ. In the usual cases ϵ is purely conventional, however for elementary particles one can expect that it is related to some fundamental feature of matter. The logical explanation is that the value of ϵ distinguishes particles and antiparticles. The mass is $M_p^2 = \epsilon 2\left[S_R\right]^*\left[S_R\right]$. Do antiparticles have negative mass ? The idea of a negative mass is still controversial. Dirac considered that antiparticles move backwards in time and indeed a negative mass combined with the first Newton's law seems to have this effect. But here the world line of the particle is defined by σ_w, and there is no doubt about the behavior of an antiparticle : it moves towards the future. The mass at rest M_p is somewhat conventional, the defining relation is $\langle S_0, S_0\rangle = \epsilon 2M_p^2$ so we can choose any sign for M_p, and it seems more appropriate to take $M_p > 0$ both for particles and antiparticles.

The inertial spinor of particles is then :

$S_0 = \frac{M_p}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1}\cos\alpha_0 \\ e^{i\alpha_2}\sin\alpha_0 \\ ie^{i\alpha_1}\cos\alpha_0 \\ ie^{i\alpha_2}\sin\alpha_0 \end{bmatrix}$
and of antiparticles :
$S_0 = \frac{M_p}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1}\cos\alpha_0 \\ e^{i\alpha_2}\sin\alpha_0 \\ -ie^{i\alpha_1}\cos\alpha_0 \\ -ie^{i\alpha_2}\sin\alpha_0 \end{bmatrix}$
It is characterized by 4 parameters : $M_p, \alpha_0, \alpha_1, \alpha_2$.

14.3.2 Chirality

In the Spinor representation particles have both a left S_L and a right S_R part, which are linked but not equal. We have one of the known features of elementary particles : chirality. The representation (E, γ) has been chosen because of this property. If the real Clifford algebras leave invariant E_0, some of their elements exchange E_ϵ and $E_{-\epsilon}$.

$S_0 \in E_\epsilon \Rightarrow \gamma_0 C(\varepsilon_0) S_0 \in E_\epsilon, j = 1,2,3 : \gamma C(\varepsilon_j) S_0 \in E_{-\epsilon}$ with the same property in $Cl(1,3)$.

So E_ϵ is preserved by $X \in T_1 Spin(3), \sigma \in Spin(3)$.

Space reversal is the operation :

$u = u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3 \to u^0\varepsilon_0 - u^1\varepsilon_1 - u^2\varepsilon_2 - u^3\varepsilon_3$

corresponding to $s = \varepsilon_0, s^{-1} = -\varepsilon_0$ in $Cl(3,1)$, $s^{-1} = \varepsilon_0$ in $Cl(1,3)$ so it preserves E_ϵ.

Time reversal is the operation :

$u = u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3 \to -u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3$

corresponding to $s = \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$, with $s^{-1} = \varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_1$ in $Cl(3,1)$, $s^{-1} = \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ in $Cl(1,3)$ so it exchanges E_ϵ and $E_{-\epsilon}$.

These results are consistent with what is checked in Particles Physics, and the Standard Model. However the latter does not consider both signatures. This feature does not allow to distinguish one signature as more physical than the other.

14.3.3 Inertial vector

The inertial vector is : $k = -i\epsilon\frac{M_p}{2}k_0 = -i\epsilon\frac{M_p}{2}\begin{bmatrix} (\sin 2\alpha_0)\cos(\alpha_1 - \alpha_2) \\ -(\sin 2\alpha_0)\sin(\alpha_1 - \alpha_2) \\ \cos 2\alpha_0 \end{bmatrix}$. Particles and antiparticles with the same parameters $M_p, \alpha_0, \alpha_1, \alpha_2$ have opposite inertial vectors, and so opposite momenta and kinematic behaviors.

Particles whose inertial vectors differ by a complex scalar of module 1 have the same kinematic behavior. This is the starting point for the idea of rays in QM.

14.3.4 Spin

$Spin(3)$ preserves E_ϵ, then $(E_\epsilon, \gamma C), (E_\epsilon, \gamma C')$ are representations of $Spin(3)$. Moreover the scalar product is definite positive or negative and preserved by $Spin(3)$ so we have unitary representations, which are isomorphic to one of the classic representations $\left(P^j, D^j\right)$ with $j \in \frac{1}{2}\mathbb{N}$. Actually elementary particles have a spin $\frac{1}{2}$, the first in line as we could assume, and we retrieve the "particles of spin $\frac{1}{2}$".

The natural representation is by a periodic motion : the particle spins at a constant rotational speed. Because the spatial spin is quantized, the rotational motion is itself quantized. The average kinetic energy is proportional to the frequency. The axis of rotation can change (by the action of $Spin(3)$). Moreover the spin can take the opposite value, corresponding to $\upsilon(X_r, 0) \to \upsilon(-X_r, 0)$. This is a discontinuous process (because the spin is quantized, it cannot take intermediate values) which requires an external action and entails a change of kinetic energy.

To each particle corresponds an antiparticle with the same mass. And particles show polarization characteristics similar to waves. The picture is similar to the Dirac's spinors, with different definitions of the γ matrices.

14.3.5 Charge

Assume that we study a system comprising of unknown particles $p = 1...N$. The modeling of their kinematic characteristics leads naturally to assume that these particles belong to some spinor fields : $S_p \in \mathfrak{X}(P_G[E, \gamma C])$ with different, unknown, inertial spinor.S_0.

What the quantization theorem tells us is that the solutions must be found in maps : $S_p : \Omega \to E$ which can be sorted out by the value of k, their inertial vector, but they belong also to classes of maps characterized by $z \in \mathbb{Z}$. One can assume that the signed integer z is related to a charge. But we see that any particle which has the same inertial vector k belongs to a definite class characterized by the same z : these particles have the same behavior in a field. This is the starting point for the representation of charged particles and we can guess that the inertial vector is more than a kinematic feature. Actually it is also a magnetic moment.

14.4 Composite particles and Atoms

Stable combinations of elementary particles are represented by the tensorial product of the spinors. Then the motion is represented in the universal enveloping algebra U of $T_1 Spin\,(3,1)$. This is a vector space, built from *tensorial powers* of the Lie algebra $T_1 Spin\,(3,1)$, such that the elements of the form : $X \otimes Y - Y \otimes X - [X,Y] \sim 0$. Any representation of the Lie algebra can be extended to a representation of its universal enveloping algebra. So the kinematic behavior of composite particles can be represented in U for the motion and (E,Γ) for the spinors, with the action Γ defined by the ordered products of the matrices γ_i.

When the internal structure can be neglected, nuclei, atoms or molecules can be represented by a single spinor, as a deformable solid. And the natural assumption about their motion is that they have a periodic rotational motion, which is then quantized.

Part IV
CONCLUSION

In this paper we have built a comprehensive and consistent Geometry of General Relativity starting from the way one proceeds to measures, some general principles of Physics, and the concepts of space, time, material bodies and their motion, with their characteristic properties. We have not started from scratch, but from the usual, well known and proven formalism of Galilean Geometry, without any exotic assumption. Relativity extends the frame-work, it does not negate it. And it leads to uncover some troubling facts which were actually already present in Galilean Geometry. Exploring the concept of motion, we have seen that the idea of an orthonormal frame is actually present in our perception and understanding of the motion of a material body. We are so well used to deal with rotation that we forget two significant features : it is a property of material bodies, and it adds 3 parameters to characterize, geometrically, a material body, even in Galilean Geometry. Observers use a tetrad, but actually a tetrad is attached to any material body, and it must be seen as a property of matter, whatever the scale. The tetrad is orthonormal, and thus defined with respect to the metric, which is of physical nature. As well as particles travel with constant velocity, the tetrad attached to a material body must adjust (in a chart), to adapt to a changing metric. This is where the use of the tetrad formalism finds all its worth, compared to the usual computations with banal charts : it has a physical meaning, and is closer to the way measures are done.

The right way to deal with a metric is by principal bundles. But the representation of the concept of motion leads to see the Clifford bundle as the natural, and physical, framework to represent any change in the geometric state of a material body, be it its location or its arrangement. The Clifford bundle replaces the tangent bundle TM as the true physical domain where any change in the geometric characteristics of material bodies occurs.

Moreover the motion is essentially characterized by two vectors $r, w \in \mathbb{R}^3$ which have a clear physical meaning, and are related to the 6 parameters used in Galilean Geometry. With all the tools of Clifford Algebra, it is then easy to work on and compute all the geometric problems in RG, even problems involving rotation which would have been intractable in the usual framework. Actually in the most part of the computations one can forget the chart, and the $\partial\xi_\alpha, d\xi^\beta$ which have been the nightmare of Physicists.

The concept of deformable solid or rigid solid of Newtonian Mechanics can be extended to the relativist context,.which enables to consider material bodies at any scale, and should be useful in Astro-Physics.

Starting from fiber bundles, the introduction of Spinors is then natural, and we retrieve the known results of the Quantum Thory of Fields. Moreover the link between the spinor and the kinematic characteristics of particles is obvious, and can be the starting point to the incorporation of the gravitational field, following the principle of equivalence. The state of a particle is then the tensorial product of a spinor and a vector which carries the charges. The introduction of connections, as in any gauge field theory, leads then to a full model which is the expected merger of Quantum Mechanics and General Relativity, and is studied in my book on Theoretical Physics.

Part V
BIBLIOGRAPHY

H.Araki *Mathematical theory of quantum fields* Oxford Science Publications (2000)

N.Ashby *Relativity in the global positioning system* Living Review in Relativity 6,(2003),1

J.C. Baez, M.Stay *Physics, Topology, Logic and Computation: A Rosetta Stone* arXiv 0903.0340 (2009)

A.Bird *Philosophy of science* Rootledge (1998)

R.Coquereaux *Clifford algebra, spinors and fundamental interactions : twenty years after* arXiv:math-ph/0509040v1 (16 sep 2005)

B.d'Espagnat *Reality and the physicist* Cambridge University Press (1989)

P.A.M.Dirac *The principles of Quantum Mechanics* Oxford Science Publications (1958)

J.C.Dutailly *Mathematics for theoretical physics* (2016) Amazon E-book

J.C.Dutailly *Theoretical Physics* (2016) Amazon

J.C.Dutailly *Estimation of the probability of transition between phases* CNRS (http://hal.archives-ouvertes.fr/hal-01075940, 20 october 2014)

Francis C.E.H *A construction of full QED using finite dimensional Hilbert space* EJTP 10 N°28 (2013)

Francis C.E.H *The Hilbert space of conditional clauses* arXiv:1205-4607 (2013)

G.Giachetta,L.Mangiarotti, G.Sardanashvily *Advanced classical field theory* World Scientific (2009)

M.Guidry *Gauge fields theories* Wiley (1991)

D.W.Henderson *Infinite dimensional manifolds are open subsets of Hilbert spaces* (1969) Internet paper

J.M.Jauch *Foundations of Quantum Mechanics* AddisonWesley (1968)

A.W.Knapp *Lie groups beyond an introduction* Birkhäuser (2005)

I.Kolar, P.Michor, J.Slovak *Natural operations in differential geometry* Springer-Verlag (1991)

A.Garrett Lisi *Clifford bundle formulation of BF gravity generalized to the standard model* arXiv:gr-qc 0511120v2 (21 nov 2005)

R.Haag *Local quantum physics* 2nd Ed.Springer (1991)

G.Mackey *The mathematical fundations of Quantum Mechanics* W.A.Benjamin (1963)

M.A.Naimark *Linear representations of the Lorentz group* Pergamon (1964)

E.Negut *On intrinsic properties of relativistic motions* Revue Roumaine des Sciences Techniques (1990).

J.von Neumann *Mathematical Foundations of Quantum Mechanics*, Beyer, R. T., trans., Princeton Univ. Press. (1996 edition)

Y.Ohnuki *Unitary representations of the Poincaré group and relativistic wave equations* World Scientific (1988)

R.Omnès *The interpretation of quantum mechanics* Princeton (1994)

P.J.E.Peebles *Principles of physical cosmology* Princeton series in Physics (1993)

S.Pokorski *Gauge fields theories* Cambridge (2000)

T.Schücker *Forces from Conne's geometry* arXiv:hep/th/0111236v3 (26 march 2007)

Mesgun Sehbatu *The standard model and beyond* (1992)

A.Smola, A.Gretton, L.Song, B.Schölkop *A Hilbert Space Embedding for Distributions*

H.Torunczyk *Characterizing Hilbert spaces topology* Fundamental mathematica (1981)

R.M.Wald *General Relativity* University of Chicago (1984)

S.Weinberg *The quantum theory of fields* Cambridge University Press (1995)

S.Weinberg *Dreams of a Final Theory* Pantheon Books (1992)

H.Weyl *The theory of groups and quantum mechanics* Dover (1931 / 1950)